AUTOMATIC CONTROL THEORY (EML 4312)

Catalog Data: EML 4312 Automatic Control Theory.
Credit 3.
Feedback control systems; stability analysis; graphical methods. Applications with emphasis on hydraulic, pneumatic and electromechanical devices.
Prerequisite: EGN 3321

Textbook: "Control System Engineering" by Palm, John Wiley & Sons

Reference: "Automatic Control Systems" by Kuo, Benjamin C., Prentice-Hall (1984)

"Feedback & Control Systems" by DiStefano, Studderud, and Williams, Schaum's Outline Series, McGraw-Hill, 1984

Coordinator: Cesar Levy, Assistant Professor of Mechanical Engineering

Goals: To familiarize Mechanical Engineering students with the principles of analysis and design of feedback control for mechanical systems.

Prerequisites by Topic:

1. Kinematics of particles & rigid bodies
2. Free body diagrams
3. newton's Laws of Motion
4. Work and Energy methods
5. Fundamentals of mechanical vibrations

Topics:

1. Introduction to feedback control systems practical demonstrations (2 classes)
2. Modeling of components-mechanical systems (2 classes)
3. Modeling of components-electrical systems (2 classes)
4. Modeling of components-fluid & thermal systems (2 classes)
5. Classical solution of systems reponse (2 classes)
6. Laplace Techniques (2 classes)
7. Control representations and block diagram algebra (2 classes)
8. Steady state operations, errors and initial values (2 classes)

EML 4312 (continued)

9. Transient responses and stability criteria, Routh, etc. (2 classes)
10. Frequency response and stability Nyquist plots (2 classes)
11. Nyquist plots (2 classes)
12. Bode Plots - gain and phase margins (4 classes)
13. Root locus techniques (4 classes)
14. Tests (2 classes)

Computer Usage: Packaged software is used by the students to complete their homework assignments.

Estimated Content:

Engineering Science: 100%
Engineering Design: 0%

EML 4312
Automatic Control Systems

Summer 1986

REVISED Tentative Course Outline

Lecture No.		Topic	Assignment	
1	5/ 7/86	Course Introduction		
M 2	5/12/86	Modelling of Dynamic Systems	Ch. 1 & 2	
3	5/14/86	"		
M 4	5/19/86	"		
5	5/21/86	"	HW-Set #1 Due	
M 6	5/26/86	No Class - Memorial Day		
7	5/28/86	First-Order System Response	Ch 3	
M 8	6/ 2/86	"	HW-Set #2 Due	
9	6/ 4/86	Second-Order System Response		
M 10	6/ 9/86	Routh-Hurwitz Stability		
11	6/11/86	Nonlinear Models & Linearization	HW-Set #3 Due	
M 12	6/16/86	Laplace Transforms & Transfer Functions	Ch 4	
13	6/18/86	"	HW-Set #4 Due	
M 14	6/23/86	*** Test #1 ***		
15	6/25/86	Feedback Control Systems	Ch 6	
M 16	6/30/86	"		
17	7/ 2/86	Control System Design	Ch 7	#5 Due
M 18	7/ 7/86	"		#6 Due
19	7/ 9/86	Root Locus	Ch 8	
M 20	7/14/86	Root Locus		#7 Due
21	7/16/86	Nyquist Stability	Ch 9	
M 22	7/21/86	"		
23	7/23/86	System Compensation & Bode Design	Ch 9	
M 24	7/28/86	"		
25	7/30/86	Modern Control Theory	Ch 5	
M 26	8/ 4/86	*** Test #2 ***		
27	8/ 6/86	Modern Control Theory		
M 28	8/11/86	Digital Control	Ch 10 & 11	
29	8/13/86	Digital Control		
Finals Week		Final Exam		

EML 4312
Automatic Control Theory

Spring 1988

Tentative Course Outline

Lecture No.		Topic	Assignment	
1		Course Introduction		
2		System Modelling	Chapters 1 & 2	
3				
4				
5	21 Jan.			HW # 1 DUE
6		First Order Response	Chapter 3	
7	28 Jan.			HW # 2 DUE
8		Second Order Response		
9	4 Feb.	Routh-Hurvitz Stability		HW # 3 DUE
10		Nonlinear Models and Linearity	Chapter 4	
11		Laplace Transforms and Transfer Fns		
12	16 Feb.			HW # 4 DUE
13		Feedback Control Systems	Chapter 6	
14	23 Feb	****** First Exam *********		
15		Feedback Control Systems		
16	1 Mar.	Control System Design	Chapter 7	HW # 5 DUE
17	3 Mar.			HW # 6 DUE
18		Root Locus	Chapter 8	
19	17 Mar.			HW # 7 DUE
20		Nyquist Stability	Chapter 9	
21				
22		System Compensation and Bode Plots		
23	31 Mar.			HW # 8 DUE
24		Modern Control Theory	Chapter 5	
25	7 Apr.			HW # 9 DUE
26	12 Apr	******* Second Exam ************		
27		Digital Control	Chapters 10 & 11	
28	14 Apr.			HW # 10 DUE
29		Review		

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Automatic Control Systems

Spring 1988

Course Information

TEXT

Palm, William J., Control Systems Engineering, John Wiley & Sons, New York, 1986.

INSTRUCTOR

Professor C. Levy
Office: VH-165
Phone: 554-3439
Office Hours: M: 1500 - 1630, F: 1000-1230
T-Th: 1600-1800

GRADING

Your final grade is based on 100%, computed as follows:

Test #1	20%
Test #2	20%
Final Exam	40%
Homework & Quizzes	20%

The cut-offs for letter grades will go no higher than:
90% - 100% = A; 80% - 89% = B; 70% - 79% = C; 60% - 69% = D.

TESTS

Tests will be given in class durnig the class sessions noted on the course outline. No make-up tests will be given. If you miss an exam you will recieve a zero for that grade.

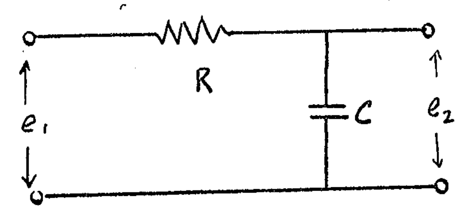
HOMEWORK

Homework is due at the beginning of lecture! No late homework will be accepted. Notice that the homework assigned in this course will be equal in weight to one of the semester tests. All homework is to be done on engineering pad paper, one problem per page. See example.

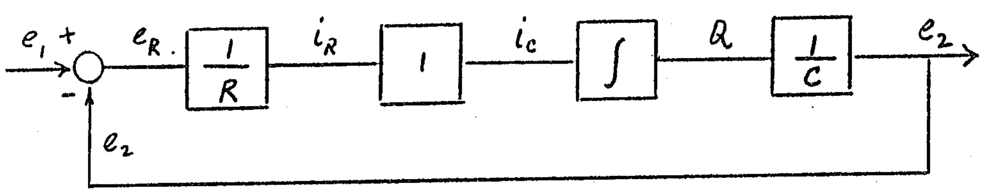
QUIZZES

To keep you up-to-date on your reading and homework assignments, short unannounced quizzes may be given at any time. Be forewarned!

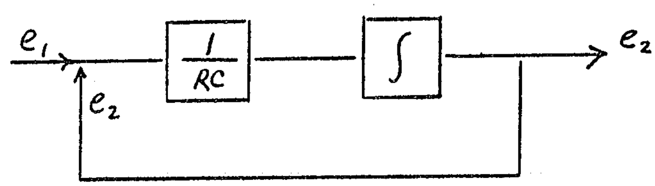
DERIVE A MODEL FOR THE CIRCUIT SHOWN. THE INPUT VOLTAGE IS e_1 AND THE OUTPUT VOLTAGE IS e_2.



USING THE BLOCK DIAGRAM METHOD OF PALM:



SIMPLIFIED,



FROM THIS BLOCK DIAGRAM

$$e_2 = \frac{1}{RC} \int (e_1 - e_2)\, dt$$

OR

$$\boxed{RC \frac{d}{dt} e_2 + e_2 = e_1}$$

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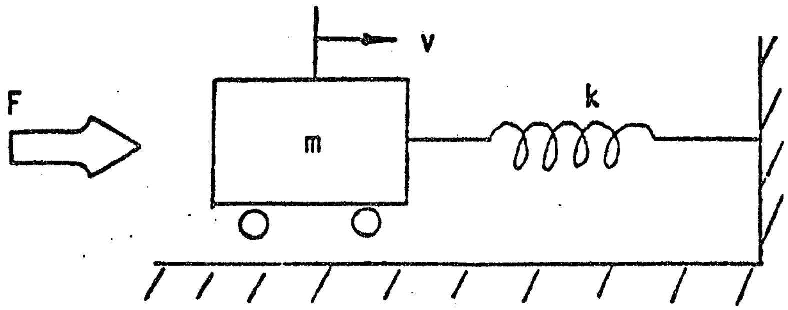
GIVEN: A brief statement of the problem in your own words.



Sketches are helpful in organizing your thoughts and notations. They help your approach to the problem.

m =
k =
F =

List of Known Data

FIND: List all of the required answers

SOLUTION: Basic Principles or Equations

List of Assumptions

Calculations

- Begin with Governing Equations
- Simplify in Symbol Form
- Substitute Numbers

Answer (with units)

Draw a box around it for identification

Ask Yourself:

- Does it seem reasonable?
- Are the units correct?

- USE PENCIL -- WORK IT ONCE -- DO NOT RECOPY
- ONLY ONE PROBLEM PER PAGE
- DO NOT WRITE ON THE BACK

BE NEAT !!

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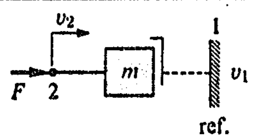
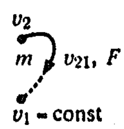
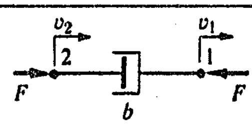
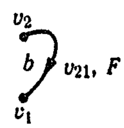
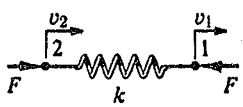
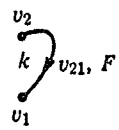
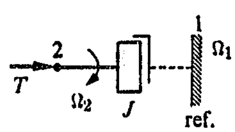
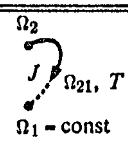
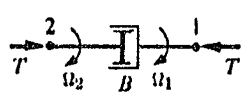
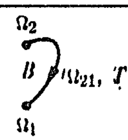
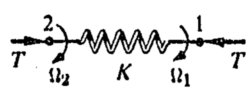
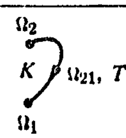
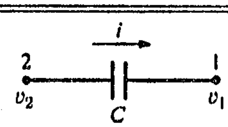
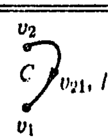
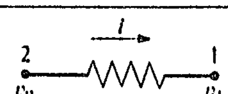
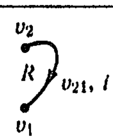
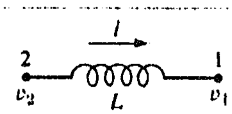
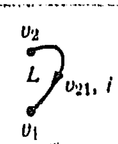
EML 4312 Automatic Controls

Summary of Ideal Mechanical and Electrical Elements

Name	Symbol	Linear graph	Describing equations	Energy or power
Mass			$F = m\frac{dv_2}{dt}$ $v_{21} = \frac{1}{m}\int_0^t F\,dt + (v_{21})_0$	$\mathcal{E}_k = \frac{1}{2} m v_2^2$ $\mathcal{E}_k \geq 0$
Damper			$F = bv_{21}$ $v_{21} = \frac{1}{b} F$	$\mathcal{P} = bv_{21}^2$ $\mathcal{P} \geq 0$
Spring			$F = k\int_0^t v_{21}\,dt + F_0$ $v_{21} = \frac{1}{k}\frac{dF}{dt}$	$\mathcal{E}_p = \frac{1}{2}\frac{F^2}{k}$ $\mathcal{E}_p \geq 0$
Inertia			$T = J\frac{d\Omega_2}{dt}$ $\Omega_{21} = \frac{1}{J}\int_0^t T\,dt + (\Omega_{21})_0$	$\mathcal{E}_k = \frac{1}{2} J\Omega_2^2$ $\mathcal{E}_k \geq 0$
Rotational damper			$T = B\Omega_{21}$ $\Omega_{21} = \frac{1}{B} T$	$\mathcal{P} = B\Omega_{21}^2$ $\mathcal{P} \geq 0$
Rotational spring			$T = K\int_0^t \Omega_{21}\,dt + T_0$ $\Omega_{21} = \frac{1}{K}\frac{dT}{dt}$	$\mathcal{E}_p = \frac{1}{2}\frac{T^2}{K}$ $\mathcal{E}_p \geq 0$
Capacitance			$i = C\frac{dv_{21}}{dt}$ $v_{21} = \frac{1}{C}\int_0^t i\,dt + (v_{21})_0$	$\mathcal{E}_e = \frac{1}{2} C v_{21}^2$ $\mathcal{E}_e \geq 0$
Resistance			$i = \frac{1}{R} v_{21}$ $v_{21} = Ri$	$\mathcal{P} = \frac{1}{R} v_{21}^2$ $\mathcal{P} \geq 0$
Inductance			$i = \frac{1}{L}\int_0^t v_{21}\,dt + i_0$ $v_{21} = L\frac{di}{dt}$	$\mathcal{E}_m = \frac{1}{2} Li^2$ $\mathcal{E}_m \geq 0$

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Summary of Energy Storage and Dissipation for Ideal Elements

A-type storage	*T*-type storage	*D*-type dissipation
Mass $\mathcal{E}_k = \frac{1}{2}mv_2^2$	Spring $\mathcal{E}_p = \frac{1}{2}\frac{F^2}{k}$	Damper $\mathcal{P} = bv_{21}^2 = \frac{F^2}{b}$
Inertia $\mathcal{E}_k = \frac{1}{2}J\Omega_2^2$	Spring $\mathcal{E}_p = \frac{1}{2}\frac{T^2}{K}$	Damper $\mathcal{P} = B\Omega_{21}^2 = \frac{T^2}{B}$
Capacitance $\mathcal{E}_e = \frac{1}{2}C(v_{21})^2$	Inductance $\mathcal{E}_m = \frac{1}{2}Li^2$	Resistance $\mathcal{P} = \frac{(v_{21})^2}{R} = Ri^2$

Power and Energy Transferred at a Connecting Point

Mechanical-translational	Mechanical-rotational	Electrical
$F \longrightarrow \bullet \longleftarrow F$, v, x	$T \longrightarrow \odot \longleftarrow T$, Ω, Θ	$i \longrightarrow \bullet \longrightarrow i$, v, λ
$\mathcal{P} = Fv$	$\mathcal{P} = T\Omega$	$\mathcal{P} = iv$
$\mathcal{E} = \int_{t_a}^{t_b} Fv\,dt$	$\mathcal{E} = \int_{t_a}^{t_b} T\Omega\,dt$	$\mathcal{E} = \int_{t_a}^{t_b} iv\,dt$

Generalized Describing Equations for Lumped Ideal Elements

A-type elements	*T*-type elements	*D*-type elements
Mass, inertia, capacitance	Spring, inductance	Damper, resistance
$f = C\frac{dv_{21}}{dt}$	$v_{21} = L\frac{df}{dt}$	$v_{21} = Rf$
$v_{21} = \frac{1}{C}\int_0^t f\,dt + (v_{21})_0$	$f = \frac{1}{L}\int_0^t v_{21}\,dt + f_0$	$f = \frac{1}{R}v_{21}$
$\mathcal{E}_a = \frac{1}{2}C(v_{21})^2$	$\mathcal{E}_t = \frac{1}{2}Lf^2$	$\mathcal{P} = \frac{(v_{21})^2}{R}$
Linear graph $\overset{2}{\frown}_{1}$ v_{21}, f		

LECTURE # 1 - INTRODUCTION

- DYNAMIC SYSTEMS CAN BE CLASSIFIED AS STABLE OR UNSTABLE
- IN GROSS SENSE STABILITY CAN BE DEFINED AS

UNSTABLE STABLE NEUTRAL

- FOR UNSTABLE SYSTEMS OUR JOB IS TO CONTROL THE INSTABILITY & BRING SYSTEM UNDER CONTROL

  EXAMPLES:

  - HEATING UNIT WITHOUT FEED BACK
  - F-16

- BUT BEFORE ONE CAN CONTROL THE SYSTEM
  - MODEL THE SYSTEM
  - ANALYZE THE MODEL

- ANALYSIS, MODELING & CONTROL HAVE INTERESTED ENGINEERS FOR A LONG TIME
- WITH ADVENT OF COMPUTERS THIS SUBJECT HAS BECOME IMPORTANT SINCE IT ALLOWS US TO LOOK AT MORE COMPLEX SYSTEMS
  - BEFORE COMPUTER - ONLY THE MOST SIMPLIFIED MODELS

- WE CAN BETTER USE LIMITED MATERIALS & ENERGY TO DESIGN } REQUIRES MORE DETAILED MODEL
- CAN USE COMPUTERS FOR MEASUMENT & CONTROL, DATA ANALYSIS & DECISION MAKING (i.e. MICROPROCESSORS TO CALIBRATE THEMSELVES)

MODELLING OF DYNAMIC SYSTEMS

- OBJECTIVE OF ENGINEERING ANALYSIS OF PHYSICAL SYSTEMS

  $\Longrightarrow$ PREDICTION OF BEHAVIOR

- REAL SYSTEMS ARE TOO COMPLEX FOR EXACT ANALYSIS

  SO WE MAKE SIMPLIFYING ASSUMPTIONS BASED ON JUDGEMENT / EXPERIENCE

- MODEL IS IDEALIZED VERSION OF THE SYSTEM

  WITH BEHAVIOR THAT APPROX. THE IMPORTANT BEHAVIOR OF REAL SYSTEM

- NEGLECT EFFECTS THAT HAVE SMALL IMPACT ON PROPERTIES TO BE STUDIED
  - EXAMPLES
    - MECHANICAL VIBRATIONS OF AN ELECTRICAL CIRCUIT
    - GRAVITATIONAL PULL OF NEARBY MASSES COMPARED TO EARTH

DFN THE PROCESS OF SIMPLIFYING A PHYSICAL SYSTEM TO OBTAIN A MATHEMATICALLY SOLVABLE SITUATION IS CALLED MODELLING

DFN THE RESULTING SIMPLIFIED VERSION OF THE SYSTEM IS CALLED THE MATHEMATICAL MODEL.

- THE ACCURACY OF THE MODEL DEPENDS ON THE VALIDITY OF THE ASSUMPTIONS

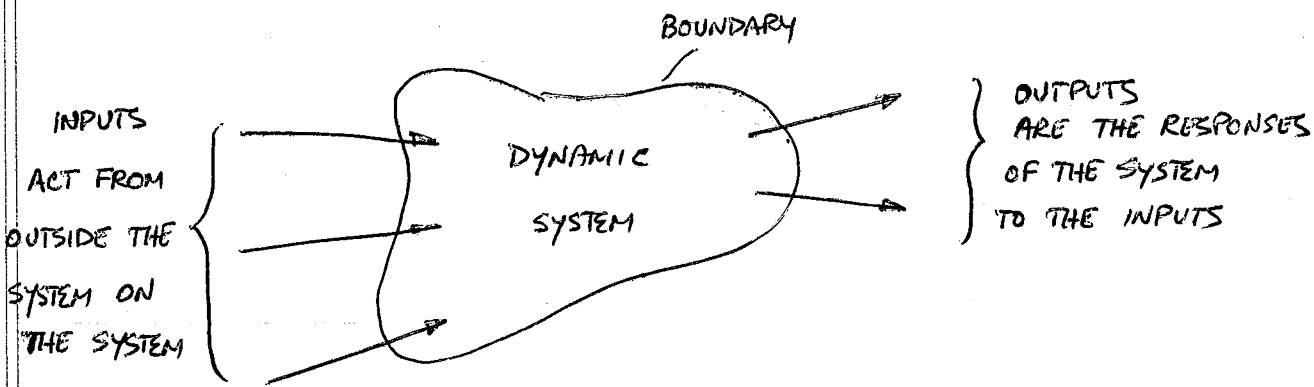
A MATHEMATICAL ANALYSIS IS USELESS, NO MATTER HOW CLEVER OR COMPLETE, IF BASED ON A MODEL THAT DOESN'T CORRECTLY REPRESENT THE BEHAVIOR OF THE PHYSICAL SYSTEM

OFTEN - EXPERIMENTATION WITH THE REAL SYSTEM IS REQUIRED

DFN A SYSTEM IS A COLLECTION OF MATTER, PARTS OR COMPONENTS WHICH ARE INCLUDED INSIDE A SPECIFIED, OFTEN ARBITRARY BOUNDARY

DEN A DYNAMIC SYSTEM – A SYSTEM WHERE ONE OR MORE QUANTITIES OF THE SYSTEM CHANGE W/ TIME

• THE SYSTEM'S BOUNDARY IS CHOSEN FOR CONVENIENT CONCEPTUAL SEPARATION OF THE SYSTEM FROM ITS SURROUNDING ENVIRONMENT



• DEFINE A SYSTEM, ALONG W/ INPUTS & OUTPUTS, TO UNDERSTAND THE CAUSE & EFFECT RELATIONSHIPS AT WORK IN THE REAL PHYSICAL SYSTEMS – WE WANT TO KNOW

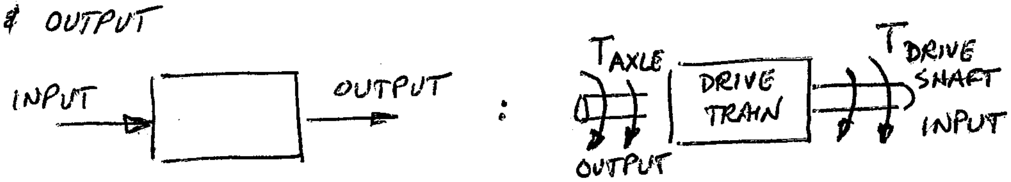
(1) WHAT CHANGES IN WHAT INPUTS CAUSE WHAT CHANGES IN WHAT OUTPUTS

(2) CAN SYSTEM OUTPUT BE REGULATED OR CONTROLLED BY CONTROLLING SYSTEM INPUTS.

(3) HOW CAN THE CAUSE & EFFECT RELATIONSHIP BETWEEN INPUT & OUTPUT BE CONTROLLED

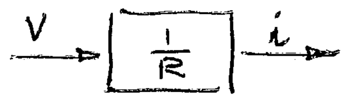
BLOCK DIAGRAMS – SYSTEMS APPROACH

• MODEL THE SYSTEM AS A SERIES OF ELEMENTS REPRESENTED BY BLOCKS EACH AFFECTED BY INPUT & OUTPUT

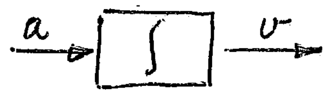


• ANALYSIS FOCUSES ON THE CONNECTION BETWEEN THESE ELEMENTS THESE CONNECTIONS DETERMINE SYSTEM BEHAVIOR

EXAMPLES



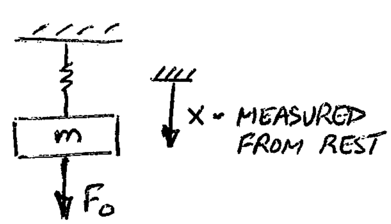
RESISTOR RELATING VOLTAGE TO CURRENT
PHYSICAL ELEMENT $i = V/R$



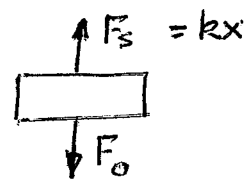
PROCESS - INTEGRATOR RELATES ACCEL. TO VELOCITY $v = \int a\,dt$

- WHENEVER OUTPUT IS TIME INTEGRAL OF INPUT THIS IS INTEGRAL CAUSALITY
- INTEGRAL CAUSALITY CONSTITUTES A BASIC FORM OF CAUSALITY FOR AL PHYSICAL SYSTEMS

EXAMPLE

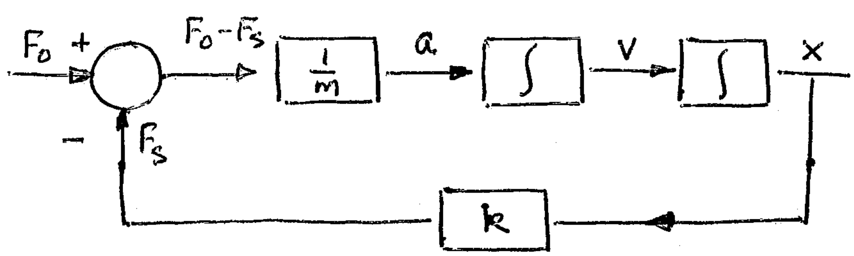


DYNAMIC FBD ON MASS IS



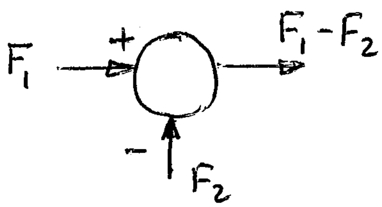
$$\Sigma \bar{F} = F_0 - F_s = ma = m\ddot{x}$$

INPUT



$F_0 = m\ddot{x} + F_s = m\ddot{x} + kx$ relates input F_0 to output x

- THIS IS A BLOCK DIAGRAM OF THE SYSTEM NOT A CONTROL DIAGRAM



COMPARATOR REPRESENTS THE ELEMENT THAT INTRODUCES ADDITION & SUBTRACTION

BLOCKS ONLY ALLOW FOR MULT, DIV & $\int$

STATIC & DYNAMIC SYSTEMS

DFN STATIC ELEMENT - ONE WHOSE OUTPUT AT ANY TIME DEPENDS ONLY ON INPUT AT THAT TIME

• <u>DFN</u> DYNAMIC ELEMENT - ONE WHOSE OUTPUT DEPENDS ON PRESENT AND PAST INPUTS

EXAMPLE: STATIC - RESISTOR - CURRENT DEPENDS ONLY ON PRESENT APPLIED VOLTAGE $i = \frac{V}{R}$

DYNAMIC - CAPACITOR - CURRENT DEPENDS ON RATE OF CHANGE OF APPLIED VOLTAGE $i = C\frac{dV}{dt}$

$$V = \frac{1}{C}\int i\,dt + V_0$$

• <u>DFN</u> A <u>DYNAMIC SYSTEM</u> IS ONE THAT HAS AT LEAST ONE DYNAMIC ELEMENT

• AS WE HAVE SAID OUR JOB IS TO MODEL, ANALYZE & CONTROL A SYSTEM
  • WE'VE BRIEFLY DISCUSSED MODELLING

• <u>DFN</u> ANALYSIS - PREDICTING THE BEHAVIOR OF THE SYSTEM FROM THE DEVELOPED MATH MODEL

• <u>DFN</u> CONTROL - PROCESS OF DELIBERATELY INFLUENCING THE BEHAVIOR OF A SYSTEM TO ACHIEVE DESIRED RESULT

THERMOSTAT

AUTO PILOT, CRUISE CONTROL

• CONTROLLER SHOULD

(1) BRING OPERATING CONDITIONS TO DESIRED VALUE

(2) MAINTAIN DESIRED CONDITIONS IN THE PRESENCE OF VARIATIONS OF THE EXTERNAL CONDITIONS (DISTURBANCES)

<u>TYPES OF MODELS</u> LESSON #2

• STATIC ELEMENTS - LEAD TO ALGEBRAIC EQNS
• DYNAMIC ELEMENTS - " " DIFFERENTIAL EQNS
• LUMPED PARAMETER - NON-SPACIALLY DEPENDENT BUT TIME DEPENDENT
• DISTRIBUTED PARAMETER - SPACIAL & TIME DEPENDENCE MORE COMPLEX INVOLVE PDE'S

EXAMPLE POINT MASS VS DISTRIBUTED MASS (1) VS (2) EQS OF MOTION

- NONLINEAR - WHEN DEPENDENT VARIABLES OR DERIVATIVES INVOLVE POWERS OR TRANSCENDAL FNS: $\frac{d^2y}{dt^2} + y^2 = 0$

- LINEAR - WHEN DEPENDENT VAR. & DERIV'S ARE TO THE FIRST POWER

$$\ddot{y} + 5\dot{y} + 6y = 0$$

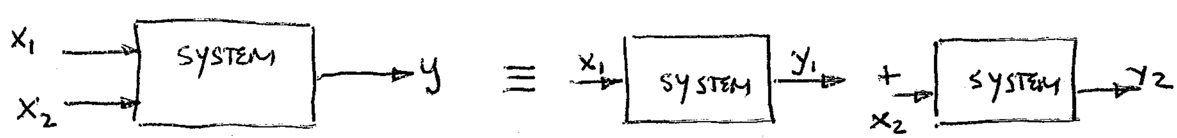
  - CAN APPLY SUPERPOSITION PRINCIPLE
- TIME INVARIANT - MODELS W/ CONSTANT COEFF'S (STATIONARY)
- TIME VARIANT (NONSTATIONARY) - MODELS W/ NON CONST COEFFS, $\ddot{y} + c(t)y = c$
- DISCRETE - GOVERNING EQNS ARE REPLACED BY DIFFERENCE EQNS
- CONTINUOUS TIME - " " ARE DIFFERENTIAL OR ALGEBRAIC EQNS
- DETERMINISTIC - MODELS COEFFS/INPUTS ARE KNOWN
- STOCHASTIC - MODELS COEFFICIENTS/INPUTS HAVE AN UNCERTAINTY

- FOR EASE OF ANALYSIS USE LUMPED & STATIC ELEMENTS
- FOR MORE REALISM USE DISTRIBUTED & DYNAMIC ELEMENTS
- ENGINEERS TRY TO SIMPLIFY MODELS TO MAKE ANALYSIS EASIER
  - ONE WAY TO DO THIS IS TO DEVELOP A LINEAR MODEL
    - THIS ALLOWS FOR USE OF <u>PRINCIPLE OF SUPERPOSITION</u>

<u>DFN</u> - THE RESPONSE $y(t)$ OF A <u>LINEAR SYSTEM</u> DUE TO SEVERAL INPUTS $x_1(t)$, $x_2(t)$, $x_3(t)$ .... ACTING SIMULTANEOUSLY IS EQUAL TO THE SUM OF THE RESPONSES OF EACH INPUT, IF EACH WERE TO ACT ALONE.

IF $y_i(t)$ is response due to $x_i(t)$ THEN

$$y(t) = \sum y_i(t)$$



WHERE $y_1 + y_2 = y$

- YOU CAN TELL A SYSTEM IS LINEAR IF WHEN YOU MULTIPLY THE INPUT BY A CONSTANT THEN THE <u>OUTPUT</u> WILL BE MULTIPLIED BY SAME CONSTANT.

<u>EXAMPLE OF LINEARITY</u>: $y = f(x)$ $\qquad x \rightarrow \boxed{f(x)} \xrightarrow{y}$

and now if $x = ax_1 + bx_2$ & $y_1 = f(x_1)$ $\quad y_2 = f(x_2)$

THEN f is LINEAR IF $y = f(ax_1 + bx_2) = af(x_1) + bf(x_2) = ay_1 + by_2$

$$\xrightarrow{x} \boxed{m} \rightarrow y = mx$$

$$\xrightarrow{x} \boxed{D} \rightarrow y = Dx = \frac{d}{dt}x$$

$$\xrightarrow{x} \boxed{\int} \rightarrow y = \int x\,dt$$

$$\xrightarrow{x} \boxed{(\ )^2} \rightarrow y = x^2$$ is not linear ; $y = \sin x$ not linear

- CAN EXTEND THIS DEFN TO FNS OF MORE THAN ONE VARIABLE
- SUPPOSE WE HAVE A SYSTEM THAT IS DESCRIBED BY

$$\frac{d^2y}{dt^2} + 4x\frac{dy}{dt} + 2x^2 = 0$$ WHERE $x(t)$ is THE INPUT
$y(t)$ is THE OUTPUT

IF IT IS LINEAR $\Rightarrow$ IF INPUT IS REPLACED BY cx THEN OUTPUT IS REPLACED BY cy

$$\frac{d^2 cy}{dt^2} + 4(cx)\frac{d\,cy}{dt} + 2(cx)^2 = 0$$

or $\quad c\ddot{y} + 4c^2x\dot{y} + 2c^2x^2 \neq c\left[\ddot{y} + 4x\dot{y} + 2x^2\right] = 0$

- IF YOU ARE GIVEN AN ODE, LOOK AT THE <u>HOMOGENEOUS</u> EQN TO DETERMINE LINEARITY

- IS $\frac{d^2}{dt^2}y + 4t\frac{dy}{dt} + 4y = \sin t$ LINEAR?

- FOR DYNAMIC SYSTEMS – no "real" physical system, can be described by a linear constant coefficent differential equation. HOWEVER IN A LIMITED RANGE OF OPERATION THEY CAN BE APPROXIMATED BY SUCH MODELS

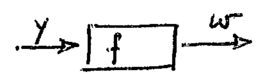
LINEARIZING

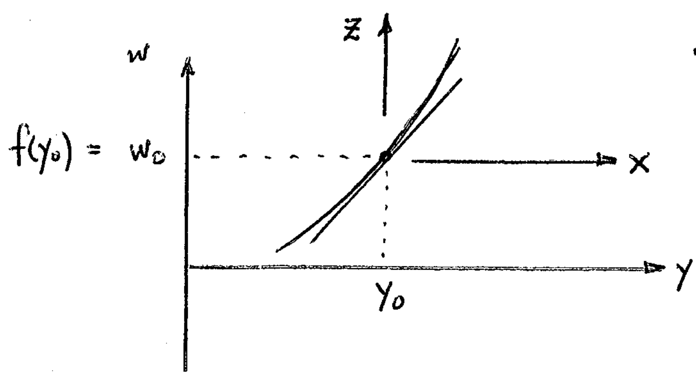
- BECAUSE OF THE USEFULNESS OF SUPERPOSITION, WE TRY TO OBTAIN THE LINEAR MODELS. HOW DO WE DO THIS? BY LINEARIZING ABOUT AN OPERATING PT.
  - TO DO THIS WE USE TAYLOR SERIES (APPENDIX A PALM)

- CONSIDER THE STATIC ELEMENT W/ INPUT/OUTPUT MODEL

$$w = f(y)$$

y - input
w - output
operating pt (y_0, w_0)





$$w = f(y) = f(y_0) + \left(\frac{df}{dy}\right)\bigg|_{y=y_0} (y-y_0) + \left(\frac{d^2f}{dy^2}\right)\bigg|_{y=y_0} \frac{(y-y_0)^2}{2} + \cdots$$

IF y is close to y_0 THEN

$$w \cong f(y_0) + \left(\frac{df}{dy}\right)\bigg|_{y=y_0} (y-y_0)$$

$\left(\frac{df}{dy}\right)\bigg|_{y=y_0}$ is slope at operating point = constant

$$w \cong w_0 + C(y-y_0) \; ;$$ now LET $w - w_0 = z$, $y - y_0 = x$

$$\therefore \quad z = Cx$$

THUS WE HAVE REPLACED THE CURVE $w = f(y)$

BY A LINEAR FN <u>MEASURED FROM THE OPERATING PT</u>

EXAMPLE – ANGULAR OSCILLATION OF A PENDULUM

$$\frac{dy}{dt} + C\sin y = 0$$ note $\frac{dy}{dt}$ is linear but $\sin y$ isn't

WE CAN LINEARIZE $\sin y$ ABOUT THE OPERATING PT $y=0$

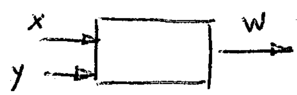
$$\sin y = y - y^3/3! + y^5/5! - \dots$$

FOR y close to $y=0$

$$\frac{dy}{dt} + C\sin y \Rightarrow \frac{dy}{dt} + Cy = 0$$ THIS IS A LINEAR ODE

- LINEARIZING FOR A FN OF 2 VARIABLES x, y are inputs w is output

$$w = f(x,y)$$



LET THE OPERATING PT BE $w = w_o$ WHEN $x = x_o, y = y_o$

$$w = f(x,y) \cong f(x_o, y_o) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_o\\y=y_o}} (x - x_o) + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_o\\y=y_o}} (y - y_o)$$

if $x - x_o = X$ $\quad a = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_o\\y=y_o}}$

$y - y_o = Y$

$w - w_o = Z$ $\quad b = \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_o\\y=y_o}}$

$$Z = aX + bY$$

- WE CAN USE THIS PROCESS FOR FNS OF n-VARIABLES

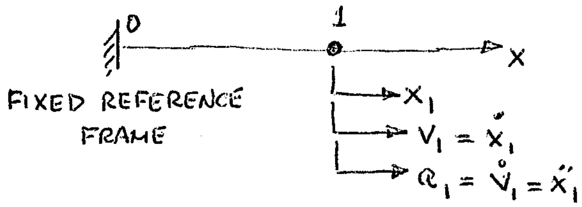
→ WILL DISCUSS LATER ON IN COURSE ←

DYNAMIC SYSTEM ELEMENTS – MODELLING

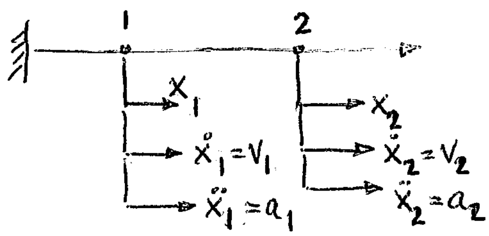
- REAL PHYSICAL SYSTEMS ARE CONTROLLED BY FLOW, STORAGE AND EXCHANGE OF VARIOUS FORMS OF ENERGY
- MODEL OR DYNAMIC SYSTEMS ARE MADE UP OF BASIC BUILDING BLOCKS THAT IDEALIZE THE ESSENTIAL PHYSICAL PHENOMENA OF REAL SYSTEM
- BUILDING BLOCKS WE WILL STUDY – SIMPLE MECHANICAL & ELECTRICAL SYSTEM ELEMENTS THAT STORE, DISSIPATE & TRANSFORM ENERGY

MECHANICAL ELEMENTS IN TRANSLATION

- LOOK AT SIMPLE TRANSLATIONAL MOTION OF A PARTICLE FROM DYNAMICS



DISPLACEMENT IN DIRECTION SHOWN MEASURED FROM REFERENCE PT 0



SPRINGS, DAMPERS, MASSES IN GENERAL WILL HAVE MOTIONS THE REQUIRE KNOWLEDGE OF RELATIVE MOTION

THUS $x_{21} = x_2 - x_1$

$v_{21} = v_2 - v_1 = \dot{x}_{21}$

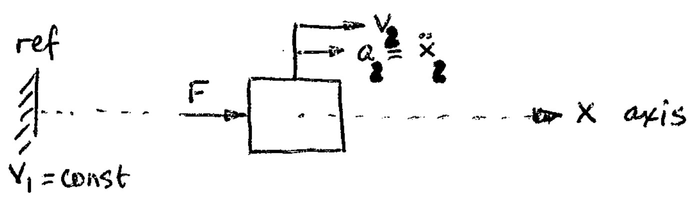
$a_{21} = a_2 - a_1 = \ddot{x}_{21}$

NOTE THAT THIS IS TRUE FOR CASE WHERE EITHER FRAME OF REF. IS STATIONARY OR TRANSLATING ONLY.

CAN DEFINE $\mathbb{P} = \vec{F} \cdot \vec{v}$ and work $= \int \vec{F} \cdot \vec{v}\, dt$

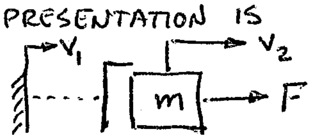
PURE TRANSLATIONAL MASS

- NEWTON'S SECOND LAW $F = \dfrac{dp}{dt}$ $\quad p = \text{momentum} = mv$
- FOR THE IDEAL MASS CASE $\dfrac{dp}{dt} = m\dfrac{dv}{dt} = m\dfrac{d^2x}{dt^2} = ma$



- a is absolute acceleration
- $\dfrac{dv_2}{dt} = \dfrac{dv_{21}}{dt}$ IF

- V_1 IS THE VELOCITY OF THE FRAME OF REF & IT IS EITHER 0 or a const
- SYMBOLIC REPRESENTATION IS



- PURE TRANSLATIONAL MASS – ALLOWS FOR NO ROTATION, FRICTION OR ELASTICITY
  - PARTICLES OF BODY ARE RIGIDLY CONNECTED
  - $V_2 <<$ SPEED OF LIGHT, $V_1 = \text{const}$ or 0
  - F is resultant force acting on mass

THUS $\vec{p} = \int_0^t \vec{F}dt + \vec{p}(t=0)$ & $\vec{v}_{21} = \frac{1}{m}\int_0^t \vec{F}dt + \vec{v}_{21}(t=0)$

ELEMENTAL EQUATION

- Power delivered to or removed from mass

$$\mathbb{P} = \vec{F} \cdot \vec{V}$$

- Work done on the mass to move it from pt 1 to pt 2

$$W = \int_{x_1}^{x_2} \vec{F} \cdot d\vec{x}_{21} = \int_{t_1}^{t_2} \vec{F} \cdot \vec{V}_{21}\, dt = \int_{p_1}^{p_2} \vec{V}_{21} \cdot d\vec{p}$$

- MASS STORES WORK AS A FORM OF ENERGY ASSOCIATED WITH VELOCITIES OF MASS PARTICLES (KINETIC ENERGY $= E_k$)
- FOR A NEWTONIAN MASS $\vec{p} = m\vec{V}_2 = m\vec{V}_{21}$ or $d\vec{p} = m\, d\vec{V}_{21}$

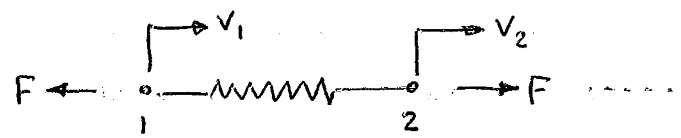
$E_k = W = \frac{1}{2} m V_2^2 \geq 0$ ALSO $E_k = \frac{1}{2m} p^2$

- SIGN OF $\vec{V}$ IS IMPORTANT AS IT TELL US HOW TO RETRIEVE STORED ENERGY

PURE TRANSLATIONAL SPRING – A ZERO MASS MECHANICAL ELEMENT WHICH DEFORMS BY STEADY AMOUNTS WHEN LOADED BY STEADY FORCES

- SINCE NO MASS $\Rightarrow$ $\Sigma\vec{F} = \vec{0}$ AND

SYMBOLIC REPRESENTATION



$x_{21} = $ fn of F

$$\bar{V}_{21} = \frac{d\bar{x}_{21}}{dt} \quad ; \quad \bar{x}_{21} = \int_0^t \bar{V}_{21}\, dt + \bar{x}_{21}(t=0)$$

- SINCE WE HAVE EQUILIBRIUM OF FORCES <u>NOTE</u> F IS THOUGHT OF AS FLOWING <u>THROUGH</u> THE SPRING

- FOR AN IDEAL SPRING : $\bar{x}_{21} = \frac{\bar{F}}{k}$ $\therefore$ $\bar{v}_{21} = \frac{1}{k}\frac{d\bar{F}}{dt}$

or $\bar{F} = k\int_0^t \bar{v}_{21}\, dt + \bar{F}(t=0)$

Work done on the spring by the forces.

$$W = \int \vec{F} \cdot \vec{v}_{21}\, dt = \int \vec{F} \cdot d\vec{x}_{21}$$

FOR AN IDEAL SPRING. $d\vec{x}_{21} = \frac{1}{k} d\vec{F}$

$$E_p = W = \frac{1}{2} k x_{21}^2 = \frac{1}{2k} F^2 \geq 0$$

- THIS WORK IS STORED AS POTENTIAL ENERGY IN THE SPRING

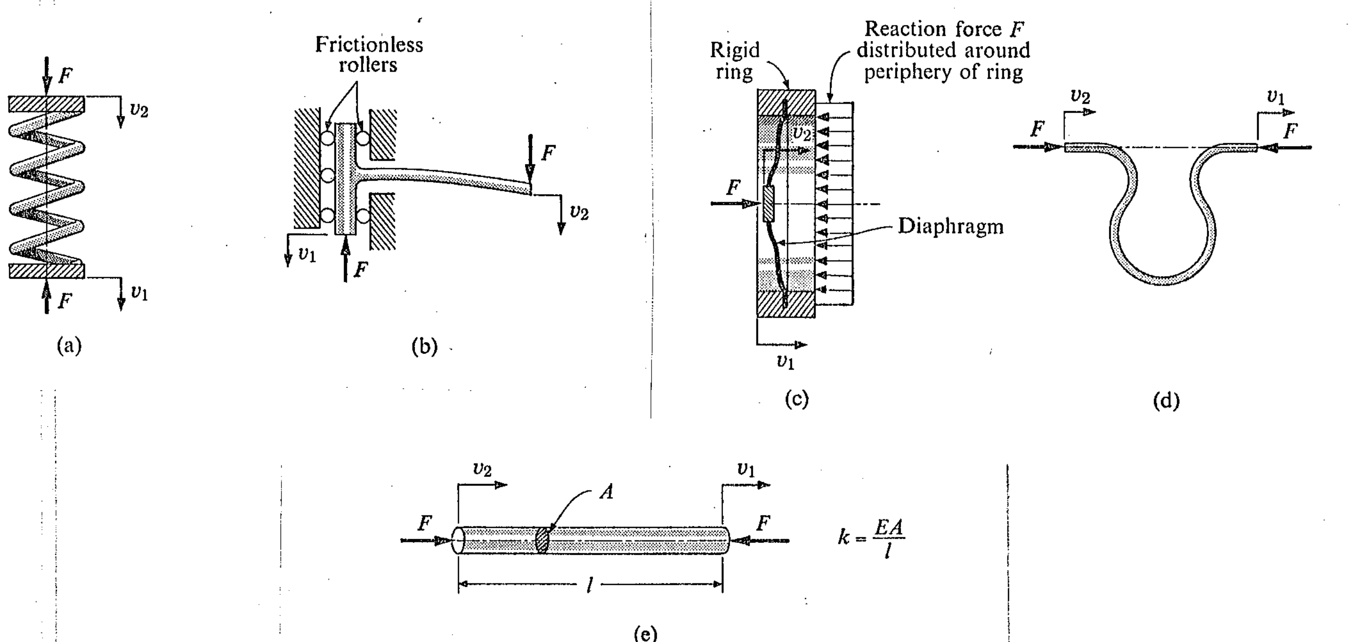
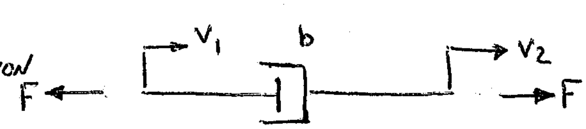


FIG. 2–8. Typical translational springs. (a) Coil spring. (b) Cantilever beam. (c) Corrugated circular diaphragm. (d) Curved bar. (e) Uniform circular bar.

LESSON #3

- PURE TRANSLATIONAL DAMPER – HAS NO MASS OR SPRING EFFECTS, REPRESENTS ONLY THE EFFECTS OF RESISTANCE TO <u>RATE</u> OF DEFORMATION

SYMBOLIC REPRESENTATION



$$\Sigma \vec{F} = \vec{0}$$

$$F = fn \text{ of } v_{21}$$

FOR AN IDEAL DAMPER

$$F = b v_{21} = b \dot{x}_{21}$$

FOR A GIVEN CHANGE IN POSITION $x_{21} = \frac{1}{b} \int F\, dt + x_{21}(t=0)$

WORK DONE $W = \int \vec{F} \cdot d\vec{v}_{21} = \int \vec{F} \cdot \vec{v}_{21}\, dt$

Power $\mathbb{P} = \vec{F} \cdot \vec{v}_{21}$

FOR AN IDEAL DAMPER

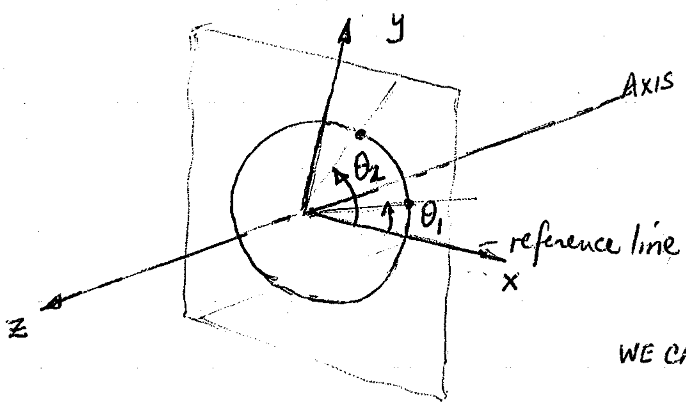
$$\mathbb{P} = b v_{21}^2 \geq 0$$

$$W = b \int v_{21}^2 \, dt$$

- NOTE WORK DONE INCREASE W/ TIME & CANNOT BE REMOVED
- DAMPER DISSIPATES THE WORK DONE OR ENERGY DELIVERED
- POWER FLOW IS NOT REVERSIBLE & ALWAYS FLOWS INTO DAMPER
- DAMPER UNLIKE MASS & SPRING, BUT IMPORTANT

ROTATIONAL MECHANICAL SYSTEMS

- FOR SYSTEMS THAT UNDERGO ROTATIONAL MOTION (TURBINES, GALVANOMETERS, PUMPS ETC) WE NEED TO DEFINE IDEAL ROTARY MECHANICAL ELEMENTS WHICH ARE THE COUNTERPARTS OF THE TRANSLATIONAL ELEMENTS
-



$$\theta_{21} = \theta_2 - \theta_1$$

$$\omega_{21} = \dot{\theta}_2 - \dot{\theta}_1 = \omega_2 - \omega_1$$

$$\alpha_{21} = \alpha_2 - \alpha_1 = \dot{\omega}_2 - \dot{\omega}_1 = \ddot{\theta}_2 - \ddot{\theta}_1$$

WE CAN DEFINE $W = \int \vec{T} \cdot \vec{\omega} \, dt$, WORK DONE ON THE ELEMENT AND $\mathbb{P} = \vec{T} \cdot \vec{\omega}$

PURE ROTATIONAL MASS

$J = mr^2$ POINT MASS rotating at distance r from axis of rotation

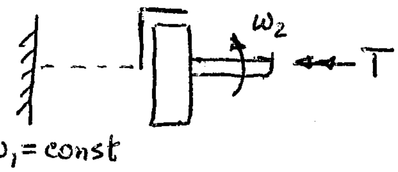
$J = \sum m_k r_k^2$ for n masses

$J = \int r^2 dm$ for distributed mass

units $kg \cdot m^2$ or $lb \cdot in \cdot sec^2$

CAN DEFINE THE ANGULAR MOMENTUM h SO THAT $T = \frac{dh}{dt} = \dot{h}$

SYMBOLICALLY



direction of T determined by rh screw rule

- <u>IDEAL ROTATIONAL MASS</u> IS A <u>COLLECTION OF IDEAL TRANSLATIONAL MASSES</u> RIGIDLY CONNECTED TOGETHER OR CONSTRAINED TO ROTATE AT ω_2 ABOUT A FIXED AXIS

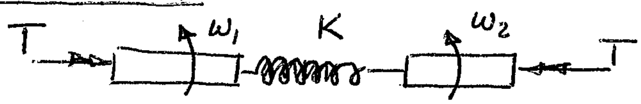
THUS $\quad \vec{h} = J\vec{\omega}_2 \quad , \qquad$ Since $\quad \vec{T} = \dfrac{d\vec{h}}{dt} = J\dot{\vec{\omega}}_2 = J\vec{\alpha}_2 = J\vec{\alpha}_{21}$

since $\vec{\alpha}_1 = \vec{0}$

- $E_k = \frac{1}{2} J \omega_2^2 = \frac{1}{2} \frac{h^2}{J} \qquad n \cdot m \;\text{ or }\; lb \cdot in$

- THIS ENERGY CAN BE RECOVERED. INDEPENDENT OF T

- <u>ROTATIONAL SPRING</u>



- HAS NO INERTIA & TRANSMITS TORQUE

$$\theta_{21} = \frac{1}{K} T$$

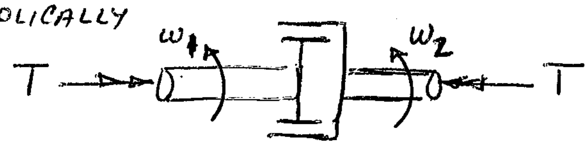
$$\omega_{21} = \omega_2 - \omega_1 = \frac{d\theta_{21}}{dt} = \frac{1}{K}\frac{dT}{dt} \quad \Rightarrow \quad T = K\int_0^t \omega_{21}\, dt + T(t=0)$$

- $E_P = \int \vec{T} \cdot \vec{\omega}_{21}\, dt = \int \vec{T} \cdot d\vec{\theta}_{21} = \frac{1}{2K} T^2 = \frac{1}{2} K \theta_{21}^2 \geq 0$

- ENERGY ALWAYS POSITIVE & RECOVERABLE BY AFFECTING THE TORQUE

- <u>ROTATIONAL DAMPER</u>
  - ENCOUNTERED IN BEARINGS OR SURFACES OF ROTATING ELEMENTS IMMERSED IN FLUIDS (AIR IS A FLUID ALSO !!)
- PROVIDES RESISTANCE TO THE RATE OF RELATIVE ANGULAR MOTION

SYMBOLICALLY



- HAS NO MASS OR COMPLIANCE
- FOR THE IDEAL DAMPER $T = B\omega_{21}$

$$\mathbb{P} = \vec{T} \cdot \vec{\omega}_{21} = B\omega_{21}^2 \geq 0$$

$B \begin{cases} n \cdot m \cdot sec/rad \\ lb \cdot in \cdot sec/rad \end{cases}$

- DISSIPATES ENERGY LIKE ITS LINEAR COUNTERPART

- IN ALL THESE CASES WE HAVE LINEARIZED

$$p = \frac{mv_2}{\sqrt{1 - (v/c)^2}} \qquad v_2 \ll c$$

$\left.\begin{matrix} k \\ b \end{matrix}\right\}$ LINEARIZED ABOUT OPERATING RANGE

$$h = rp \qquad v_2 \ll c$$

$\left.\begin{matrix} K \\ B \end{matrix}\right\}$ LINEARIZED ABOUT OPERATING RANGE

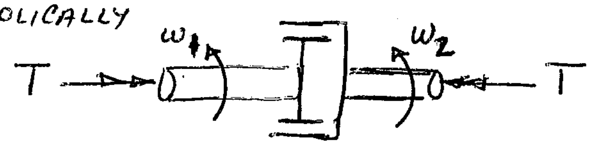
MECHANICAL TRANSFORMERS

- THIRD CLASS OF ELEMENTS THAT DO NOT STORE OR DISSIPATE ENERGY BUT TRANSFORMES ENERGY – PURE TRANSFORMERS
- MAGNITUDE OF FORCE (OR TORQUE) OR VELOCITY (OR ANGULAR VELOCITY) ARE CHANGED SO THAT NET POWER FLOW INTO THE ELEMENT IS ZERO.

EXAMPLE : SIMPLE GEAR TRAIN (ROTATIONAL TRANSFORMER)

- MODEL AS 2 CIRCLES ROLLING W/O SLIPPAGE.

SYMBOLICALLY



- HAS NO MASS OR COMPLIANCE
- FOR THE IDEAL DAMPER $T = B\omega_{21}$

$$\mathbb{P} = \vec{T} \cdot \vec{\omega}_{21} = B\omega_{21}^2 \geqslant 0 \qquad B\begin{cases} n \cdot m \cdot sec/rad \\ lb \cdot in \cdot sec/rad \end{cases}$$

- DISSIPATES ENERGY LIKE ITS LINEAR COUNTERPART

- IN ALL THESE CASES WE HAVE LINEARIZED

$$p = \frac{m v_2}{\sqrt{1-(v/c)^2}} \qquad v_2 \ll c$$

$$\left.\begin{matrix} k \\ b \end{matrix}\right\} \text{ LINEARIZED ABOUT OPERATING RANGE}$$

$$h = rp \qquad v_2 \ll c$$

$$\left.\begin{matrix} K \\ B \end{matrix}\right\} \text{ LINEARIZED ABOUT OPERATING RANGE}$$

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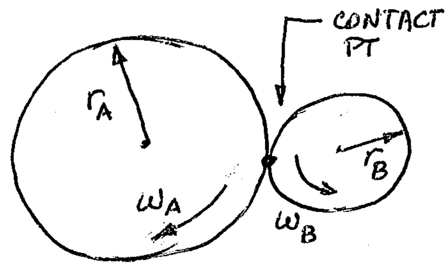


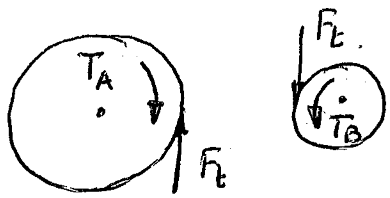
ASSUME $\omega_{\text{FRAME OF REF}} = 0$

PITCH RADII r_A/r_B ARE EQUAL $\frac{N_A}{N_B}$

AT CONTACT $V_A = V_B$ or $\omega_A r_A = \omega_B r_B$

$\therefore \quad \frac{\omega_B}{\omega_A} = \frac{r_A}{r_B} = \frac{N_A}{N_B} =$ GEAR RATIO

N_A, N_B : # TEETH



AT CONTACT PT $\sum F = 0$

$$\left.\begin{aligned} F_t r_A &= T_A \\ -F_t r_B &= T_B \end{aligned}\right\} \quad \frac{T_B}{T_A} = \frac{-r_B}{r_A} = \frac{-1}{\text{GEAR RATIO}}$$

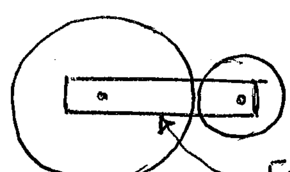
- IF FRICTION BETWEEN GEAR TOOTH SURFACES IS ZERO
- IF GEARS HAVE NO CHANGE IN ANGULAR MOMENTUM
- IF BEARING TORQUE IS NEGLIGIBLE & ITS FRICTION IS ZERO

$$\mathbb{P} = \bar{T}_A \cdot \bar{\omega}_A + \bar{T}_B \cdot \bar{\omega}_B = T_A \omega_A + (-T_A/GR)(\omega_A \cdot GR) = 0$$

- FOR THE CASE WHERE THE FRAME OF REFERENCE IS MOVING

$$\frac{\omega_{B1}}{\omega_{A1}} = \frac{r_A}{r_B} = \frac{N_A}{N_B} = \frac{-T_A}{T_B} = \text{GEAR RATIO} \qquad \text{BUT } \mathbb{P} = 0$$

where $\omega_{B1} = \omega_B - \omega_1 \qquad \omega_{A1} = \omega_A - \omega_1$



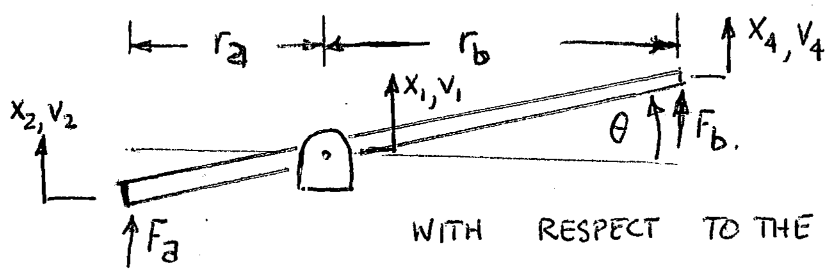
Frame of reference is moving w/ speed ω_1

THUS AN IDEAL OR PURE ROTATIONAL TRANSFORMER STORES OR DISSIPATES NO ENERGY & DETERMINES ω_B GIVEN ω_A. The IDEAL TRANSFORMER

HAS A LINEAR RELATION BETWEEN OUTPUT (ω_{B1}) & INPUT (ω_{A1}): GEAR RATIO

- USES
  - (1) CHANGE SPEED OF MOTOR OR OTHER HIGH POWER SOURCE
  - (2) OBTAIN NON-UNIFORM MOTION FROM UNIFORM ONE
  - (3) ACHIEVE MECHANICAL ADVANTAGE

A <u>PURE TRANSLATIONAL TRANSFORMER</u> - no mass, infinitely stiff
no friction



WITH RESPECT TO THE PIVOT PT VELOCITY

$$\frac{V_{41}}{V_{21}} = \frac{r_b \cancel{\omega}}{-r_a \cancel{\omega}} = n \qquad \text{(LEVER RATIO)}$$

ALSO SINCE IT IS MASSLESS Σ moments must be 0.

$$\therefore \quad F_a r_a = F_b r_b \quad \text{or} \quad F_b / F_a = \frac{r_a}{r_b} = -\frac{1}{n}$$

AN IDEAL LINEAR TRANSFORMER HAS RATIO n BEING CONSTANT AND ALSO HAS ZERO POWER FLOW.

<u>MECHANICAL TRANSDUCERS</u>

DFN - A DEVICE THAT TRANSFORMS ENERGY OR POWER IN ONE FORM INTO ENERGY OR POWER IN ANOTHER

DFN - A PURE TRANSDUCER DOES THIS WITHOUT ENERGY LOSS OR STORAGE, & NET POWER FLOW INTO THE TRANSDUCER IS ZERO

<u>EXAMPLES</u>

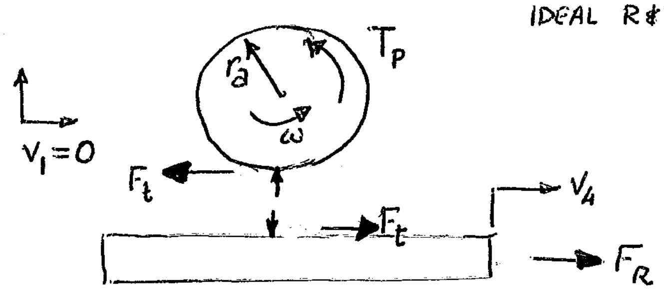
ELECTRIC MOTORS (ELECTRICAL TO MECH.)

GAS TURBINES (FLUID TO MECH)

THERMOCOUPLES (THERMAL TO ELECTRICAL)

EXAMPLE OF RACK & PINION

- RACK IS A GEAR OF INFINITE RADIUS



IDEAL R&P
- ASSUME NO MOMENTUM, $J, m = 0$
- FRICTION TORQUES $= 0$
- GEARS ARE PERFECT & RIGID
- PINION ROTATES ONLY
- RACK TRANSLATES ONLY

$r_a \omega = V_4 \Rightarrow \dfrac{V_4}{\omega} = r_a = n$ TRANSDUCTION RATIO

$F_t = -F_R \qquad T_p = F_t r_a \qquad \therefore \quad F_R / T_p = \dfrac{-1}{r_a} = -\dfrac{1}{n}$

A <u>PURE</u> ROTARY TO TRANSLATORY ENERGY TRANDUCER PRODUCES A DISPLACEMENT (VELOCITY) WHICH IS A SINGLE VALUED FN OF THE ROTARY DISPLACEMENT (OR ANGULAR VELOCITY). NET POWER $= 0$

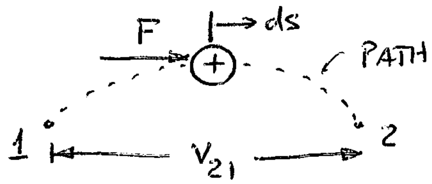
AN <u>IDEAL</u> R-TO-T TRANDUCER HAS ROTARY MOTION PROPORTIONAL TO TRANSLATORY MOTION

ELECTRICAL SYSTEM ELEMENTS

WE WILL FIND THE SAME CONCEPTS OF WORK, ENERGY & POWER USEFUL IN STUDYING MECHANICAL SYSTEMS ALSO HAVE THEIR COUNTERPARTS IN ELECTRICAL

RECALL VOLTAGE — ELECTRIC FIELD POTENTIAL
CURRENT — FLOW OF ELECTRIC CHARGE

<u>DFN</u> VOLTAGE (OR POTENTIAL DIFFERENCE) V_{21} BETWEEN TWO PTS — THE WORK DONE IN CARRYING A UNIT POSITIVE CHARGE FROM PT 1 TO PT 2



$$V_{21} = \int \bar{F} \cdot d\bar{s}$$

IF $V_{21} > 0 \Rightarrow V_2 > V_1$ PT 2 IS AT A HIGHER POTENTIAL

FROM THE VOLTAGE DFN

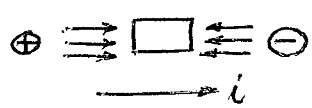
THE AMOUNT OF WORK DONE WHEN AN INFINITESIMAL CHARGE dq IS MOVED FROM PT 1 TO 2

$$dW_{21} = V_{21}\, dq \quad \text{or} \quad V_{21} = \frac{dW_{21}}{dq}$$

$$[V_{21}] = \text{VOLTS} = \frac{\text{JOULES}}{\text{COUL.}} = \frac{\text{WATTS} \cdot \text{SEC}}{\text{COULOMB}}$$

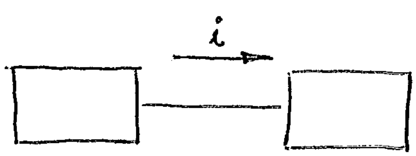
DFN CURRENT IS THE RATE OF FLOW OF CHARGE ACROSS A GIVEN AREA

$$i = \frac{dq}{dt} \qquad [i] = \text{AMP} = \frac{\text{COULOMB}}{\text{SEC.}}$$



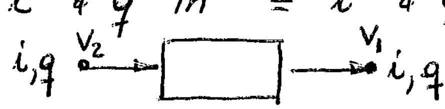
CONVENTION FOR ⊕ CURRENT

- NEWTON'S 3rd LAW HAS AN EQUIVALENT – PRINCIPLE OF CONSERVATION OF ELECTRIC CHARGE : 1 NET CHARGE LEAVING ONE ELEMENT MUST ENTER ADJACENT ELEMENT



: 2 CHARGE ENTERING ELEMENT — CHARGE LEAVING ELEMENT = CHANGE OF NET CHARGE STORED IN THE ELEMENT

- WE WILL RESTRICT OURSELVES TO CASES WHERE NO ACCUMULATION OCCURS IN AN ELEMENT
- THUS i & q in $=$ i & q OUT LIKE DAMPERS, SPRINGS



- ELECTRICAL ELEMENTS TRANSMIT CURRENT AS MECHANICAL ELEMENTS TRANSMIT FORCE

POWER, WORK & ENERGY

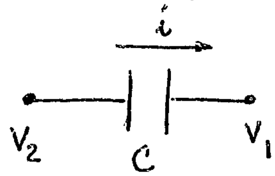
$$\mathbb{P} = \text{rate of change of work} = \frac{dW_{21}}{dt} = V_{21}\frac{dq}{dt} = i\,V_{21} \qquad \text{(WATTS)}$$

$$\text{electrical energy} = \int \mathbb{P}\,dt = \int i V_{21}\,dt = \int V_{21}\,dq \qquad \text{(joules)}$$

ELECTRICAL ELEMENTS

→ CAPACITANCE (STORES ENERGY IN AN ELECTRIC FIELD BUT NOT CHARGE)

LIKE MASS (STORES ENERGY IN GRAVITATIONAL FIELD)



VOLTAGE DROP IN DIRECTION OF CURRENT FLOW

IDEAL CAPACITANCE

$$q = C V_{21} \qquad [C] \text{ - farads} = \frac{\text{amp}\cdot\text{sec}}{\text{Volt}}$$

$$i = C\frac{dV_{21}}{dt}$$

$[\,p = m v_{21}\,]$

$\left[\,F = m\frac{dv_{21}}{dt}\,\right]$

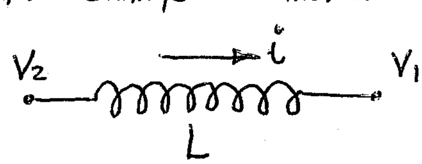
- ZERO RESISTANCE TO FLOW
- DIELECTRIC HAS INFINITE RESISTANCE TO FLOW
- MAGNETIC FIELD EFFECTS ARE ZERO

$$V_{21} = \frac{1}{C}\int_0^t i\,dt + V_{21}(t=0)$$

$$E_e = \int V_{21}\,dq = \frac{1}{2}C\,V_{21}^2 \geq 0 \qquad \left[E_k = \frac{1}{2} m v_{21}^2\right]$$

→ INDUCTANCE (STORES ENERGY IN A MAGNETIC FIELD) LIKE SPRING

- CURRENT FLOW INDUCES MAGNETIC FIELD
- CHANGE IN CURRENT, INTENSITY OF MAGNETIC FIELD CHANGES
- LENZ'S LAW - CHANGE IN MAGNETIC FIELD INDUCES VOLTAGE DIFFERENCES



$$V_{21} = \frac{d\lambda_{21}}{dt} \qquad \lambda_{21} \text{ - FLUX LINKAGE} \qquad [\lambda_{21}] \text{ - WEBERS - VOLTS}\cdot\text{SE}$$

$$\lambda_{21} = \int_0^t V_{21}\,dt + \lambda_{21}(t=0) \qquad \left[x_{21} = \int_0^t v_{21}\,dt + x_{21}(t=0)\right]$$

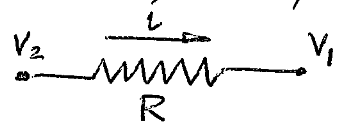
AN IDEAL INDUCTANCE $\lambda_{21} = L i$ $[L]$ - HENRIES $= \frac{\text{VOLT}\cdot\text{SEC}}{\text{AMP}}$

$[F = k x_{21}]$ or $i = \frac{1}{L}\lambda_{21} \Rightarrow i = \frac{1}{L}\int_0^t V_{21}\,dt + i(t=0)$

$$E_m = \int V_{21}\, i\, dt = \int i\, d\lambda_{21} = \frac{1}{2L}\lambda_{21}^2 = \frac{1}{2} i^2 L \geq 0 \qquad \left[E_p = \frac{1}{2}\frac{F^2}{k}\right]$$

ENERGY STORED BY VIRTUE OF CURRENT. ENERGY IS RECOVERABLE. NOT DIRECTLY DEPENDENT ON V_{21}. L DEPENDENT ON GEOMETRY & MATERIAL

<u>RESISTANCE</u> (DISSIPATES ENERGY) LIKE DAMPER



$i = \frac{1}{R} V_{21}$ $[F = b v_{21}]$

RESISTORS EXHIBIT NO CAPACITANCE OR INDUCTANCE

$$\mathbb{P} = i V_{21} = \frac{1}{R} V_{21}^2 = i^2 R > 0 \text{ WATTS} \qquad [\mathbb{P} = b v_{21}^2]$$

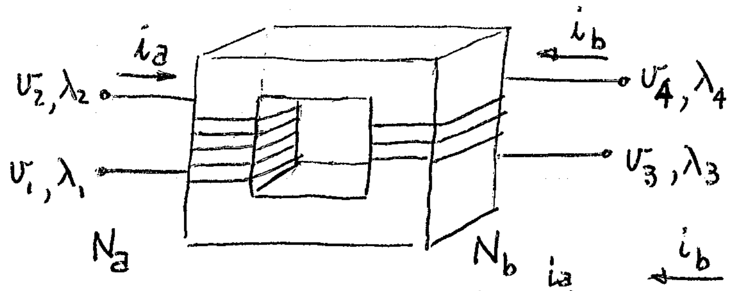
POWER FLOWS INTO A RESISTOR BUT NEVER OUT - ENERGY IS DISSIPATED ... POWER IS TRANSFORMED TO HEAT

LESSON #4

<u>TRANSFORMERS</u> (TRANSFORM ENERGY w/o DISSIPATION)

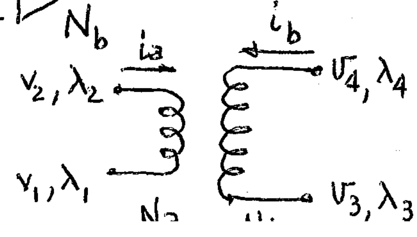
- PURE ELECTRICAL TRANSFORMER - DOESN'T STORE OR DISSIPATE ENERGY
- DETERMINES OUTPUT VOLTAGE DIFF. AS A FN OF INPUT VOLTAGE DIFF.

- ASSUME TWO COILS OF N_a, N_b TURNS WOUND AROUND A MAGNET WHOSE PERMEABILITY IS LARGE IN COMPARISON TO AIR



- ASSUME MAGNETIC FLUX Φ IS NOT LOST
- FARADAY'S LAW

$$\lambda_{21} = \Phi N_a$$
$$\lambda_{43} = \Phi N_b$$



$\lambda_{43}/\lambda_{21} = N_b/N_a$

since V is related to λ $\Rightarrow$ $\frac{V_{43}}{V_{21}} = \frac{N_b}{N_a} = n$ turns ratio

SINCE NO net POWER IN AN IDEAL TRANSFORMER THEN

$$i_a V_{21} = -i_b V_{43} \Rightarrow \frac{i_b}{i_a} = -\frac{V_{21}}{V_{43}} = \frac{-1}{n}$$

ELECTROMECHANICAL TRANSDUCERS

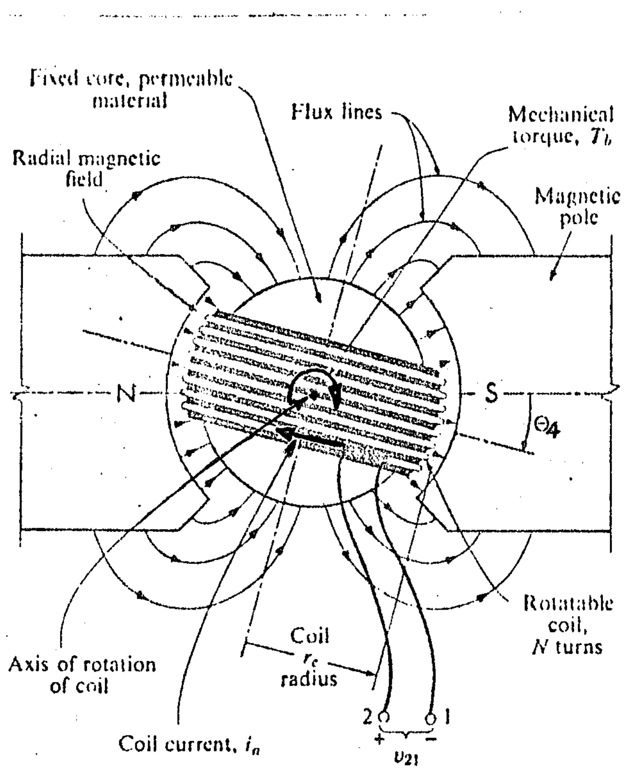


FIG. 2-38. Schematic diagram of rotary electromagnetic electrical-to-mechanical transducer.

- CONVERTS ELECTRICAL CURRENT TO MECHANICAL TORQUE
- VOLTAGE INTO ANGULAR VELOCITY
- A RADIAL MAGNETIC FIELD OF STRENGTH B (WEBERS/M^2) IS ESTABLISHED BETWEEN MAGNET
- MOVABLE N-TURN COIL ROTATES ABOUT AXIS
- AS CURRENT PASSES THROUGH COIL, MAGNETIC FIELD PRODUCES TORQUE
- BIOT-SAVART LAW GIVES

$$T_b = -2N\ell\, B\, r_c\, i_a = -\frac{1}{n} i_a$$

$2N\ell$ - LENGTH OF CONDUCTOR

r_c - COIL RADIUS

n - TRANSDUCTION RATIO OR ELECTROMECHANICAL COUPLING CONST

- IF COIL MOVES W/ $\omega_4 = \dot{\theta}_4$, VOLTAGE DROP IS INDUCED ACROSS TERMINALS
- BY FARADAY'S LAW

$$v_{21} = N\frac{d\phi}{dt} = 2N\ell\, B\, r_c\, \omega_4 = \frac{\omega_4}{n}$$

$N\phi$ = FLUX LINKAGE

NET POWER IS ZERO.

- WE HAVE NOTED SIMILARITIES IN ELECTRICAL / MECHANICAL ELEMENT EQS
- BASIS FOR COMPARISON IDEAS OF THROUGH & ACROSS VARIABLES

- IN OUR LUMPED PARAMETER ANALYSES FOR MASS, SPRING, DAMPER, INDUCTANCE CAPACITANCE, RESISTANCE

**

- NOTE RELATIONSHIPS BETWEEN A THROUGH VARIABLE (WHOSE VALUE DOESN'T CHANGE AT THE TERMINAL ENDS OF AN IDEAL ELEMENT) AND AN ACROSS VARIABLE (WHOSE VALUE IS GIVEN AS A RELATIVE VALUE OR DIFFERENCE BETWEEN TERMINALS)

  THROUGH VARIABLE – FORCE, TORQUE, CURRENT

  ACROSS VARIABLE – VELOCITY, ANGULAR VELOCITY, VOLTAGE DROP

- IN A PHYSICAL SYSTEMS TO MEASURE THROUGH VARIABLE MUST SEVER THE SYSTEM AT SOME POINT TO MEASURE FORCE, TORQUE, CURRENT
  - REMEMBER AMMETER MUST BE PLACE IN SERIES
  - FORCES & TORQUES – FREE BODY DIAGRAMS

- FOR ACROSS VARIABLES NO NEED TO SEVER OR TO ENTER SYSTEM
  - REMEMBER VOLTMETER PLACED IN PARALLEL
  -

- THERMAL & FLUID SYSTEMS

  THROUGH – FLUID FLOW, HEAT FLOW $\underline{\underline{Q}} = \rho V A$, $\underline{\underline{q}} = k \Delta T$

  ACROSS – PRESSURE DROP, TEMPERATURE DIFFERENCE

- A CONVENIENT SYMBOL TO RELATE THROUGH & ACROSS VARIABLES IS THE LINEAR GRAPH REPRESENTATION.

- TERM LINEAR MEANS GRAPH IS DEFINED BY LINE SEGMENT
- NOT TO BE CONFUSED W/ LINEAR RELATIONSHIP OF IDEAL ELEMENT

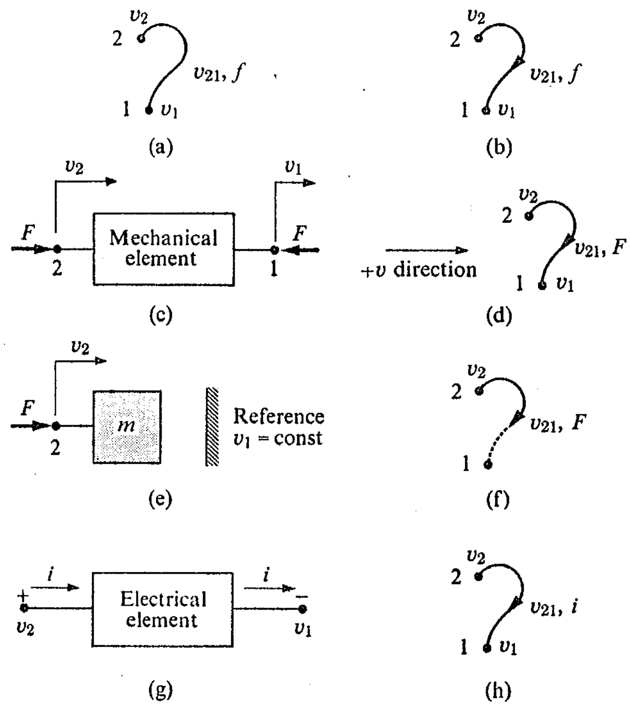


FIG. 2–39. Linear graph representation of pure mechanical and electrical system elements. (a) Linear graph. (b) Oriented linear graph. (c) Nonmass mechanical element. (d) Oriented graph. (e) Translational mass. (f) Oriented graph. (g) Electrical element. (h) Oriented graph.

- THE NODES INDICATE ACROSS-VARIABLE OF ELEMENT
- LINE INDICATES THE CONTINUITY OF THE THROUGH VARIABLE
- ARROW INDICATES ORIENTATION OF BOTH POSITIVE ACROSS & THROUGH VARIABLE
- $+V_{21}$ CAUSES $+f$
- $\Rightarrow$ POWER FLOWS INTO ELEMENT WHEN V_{21} & f ARE + $(\mathbb{P} = \bar{f} \cdot \bar{V}_{21})$
- MASS IS SPECIAL SINCE ONE TERMINAL IS THE FIXED REF. FRAME AND FORCE NOT PHYSICALLY TRANSMITTED TO THAT TERMINAL

POWER & ENERGY

POWER IS THE PRODUCT OF THRU VARIABLE & ACROSS VARIABLE

$$\mathbb{P} = \bar{f} \cdot \bar{V}_{21}$$

$$E = \int \mathbb{P}\, dt = \int \bar{f} \cdot \bar{V}_{21}\, dt$$

		EXAMPLES
MASS, INERTIA (J), CAPACITANCE	A-TYPE ELEMENTS	$E = \frac{1}{2} m V_2^2$
SPRINGS & INDUCTANCE	T-TYPE ELEMENTS	$E = \frac{1}{2} \frac{F^2}{k}$
DAMPERS & RESISTANCE	D-TYPE ELEMENTS	$\mathbb{P} = b V_{21}^2$

A-TYPE ELEMENTS STORE ENERGY BY MEANS OF THEIR ACROSS VARIABLES : V_{21}, ω_{21}

T-TYPE " " " " " " " THROUGH " : F, T, i

TABLE 2-2
Power and Energy Transferred at a Connecting Point

Mechanical-translational	Mechanical-rotational	Electrical
v, x $F \longrightarrow \bullet \longleftarrow F$	Ω, Θ $T \longrightarrow \odot \longleftarrow T$	v, λ $i \longrightarrow \bullet \longrightarrow i$
$\mathcal{P} = Fv$	$\mathcal{P} = T\Omega$	$\mathcal{P} = iv$
$\mathcal{E} = \int_{t_a}^{t_b} Fv\,dt$	$\mathcal{E} = \int_{t_a}^{t_b} T\Omega\,dt$	$\mathcal{E} = \int_{t_a}^{t_b} iv\,dt$

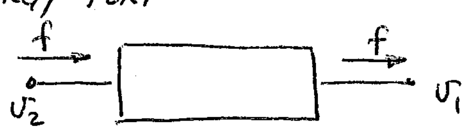
TABLE 2-3
Summary of Energy Storage and Dissipation for Ideal Elements

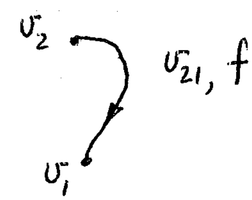
A-type storage	*T*-type storage	*D*-type dissipation
Mass $\mathcal{E}_k = \frac{1}{2}mv_2^2$	Spring $\mathcal{E}_p = \frac{1}{2}\frac{F^2}{k}$	Damper $\mathcal{P} = bv_{21}^2 = \frac{F^2}{b}$
Inertia $\mathcal{E}_k = \frac{1}{2}J\Omega_2^2$	Spring $\mathcal{E}_p = \frac{1}{2}\frac{T^2}{K}$	Damper $\mathcal{P} = B\Omega_{21}^2 = \frac{T^2}{B}$
Capacitance $\mathcal{E}_e = \frac{1}{2}C(v_{21})^2$	Inductance $\mathcal{E}_m = \frac{1}{2}Li^2$	Resistance $\mathcal{P} = \frac{(v_{21})^2}{R} = Ri^2$

DEFINE f = GENERALIZED THROUGH VARIABLE (FORCE, TORQUE, CURRENT)
v = GENERALIZED ACROSS VARIABLE (VELOC, ω, VOLTAGE)
h = INTEGRATED THROUGH VARIABLE $f = dh/dt$ (mom., h, q)
x = INTEGRATED ACROSS VARIABLE $v = \frac{dx}{dt}$ (disp, θ, FLUX LINKAGE)

POWER $\mathbb{P}$ INTO AN ELEMENT $= \bar{f} \cdot \bar{v}_{21}$
ENERGY $E = \int \mathbb{P}\,dt = \int f \cdot v_{21}\,dt$

AN ELEMENT WHICH RELATES f TO v_{21} IS CALLED A TWO TERMINAL SINGLE ENERGY PORT





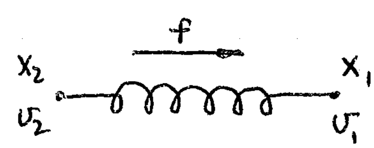
GENERALIZED INDUCTANCE (T-TYPE STORAGE)

FORM: $x_{21} = Lf$ $\qquad v_{21} = L\frac{df}{dt}$

SPRING $L = \frac{1}{k}$, f-FORCE
T. SPRING $L = \frac{1}{K}$; f-TORQUE
ELECTRICAL L-INDUCT; f-CURREN.

$E = \frac{1}{2}\frac{1}{L}(x_{21})^2 = \frac{1}{2}Lf^2$

COIL SYMBOL REPRESENTS T-TYPE ELEMENT



GENERALIZED CAPACITANCE (A-TYPE ENERGY STORAGE)

$$f = C\,\frac{dv_{21}}{dt}$$

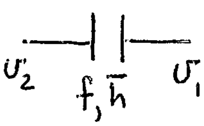
MASS C = mass, f = force

ROTATIONAL $C = J$; f = torque $v_{21} = \omega_2$

ELECT. C = CAPACITANCE ; f = CURRENT

$$E = \frac{1}{2} C (v_{21})^2 = \frac{1}{2C} \bar{h}^2$$

SYMBOL FOR GENERALIZED CAPACITANCE IS SAME AS THAT FOR ELECT.



GENERALIZED RESISTANCE (D-TYPE DISSIPATORS)

$$f = \frac{1}{R} v_{21}$$

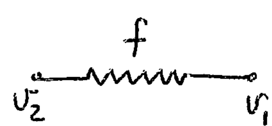
MECH − $\frac{1}{R} = b$ f = force

TORS. − $\frac{1}{R} = B$ f = TORQUE $V_{21} = \omega_{21}$

ELECT. − R RESISTANCE f = CURRENT

$$\mathbb{P} = \frac{1}{R} V_{21}^2$$

ELECTRICAL RESISTANCE SYMBOL



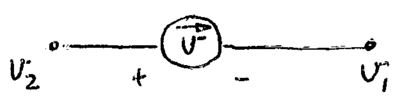
IS SYMBOL FOR GEN. RESISTANCE

DFN - TWO TERMINAL <u>SOURCE</u> IS A DEVICE CAPABLE OF DELIVERING ENERGY CONTINUOUSLY TO A SYSTEM.

TWO LIMITING CASES - IF SOURCE'S ACROSS-VARIABLE IF A SPECIFIED FN OF TIME (A-TYPE SOURCE) OR IF SOURCE'S THROUGH VARIABLE IS A SPECIFIED FN OF TIME (T-TYPE SOURCE)

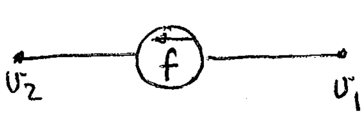
FOR IDEAL SOURCES

$v_{21} = f_n(t)$



ARROW INDICATES DIRECTION OF ACROSS VARIABLE DROP; INDEP OF f

$f = f_n(t)$



f is a fn of time ; independent of THE VALUE OF v_{21} ACROSS THE TERMINALS, AND IN DIRECTION OF ARROW

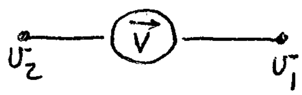
A TYPE SOURCES — EXAMPLES: BATTERY (OCEAN DEPTH PRODUCES ΔP)

T TYPE SOURCES — TORQUE, FORCE (HYDRAULIC PUMP - FLUID FLOW)

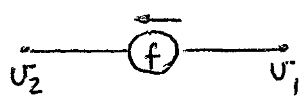
FOR AN A-TYPE SOURCE : f IS KNOWN AS CONJUGATE VARIABLE

T-TYPE SOURCE : v_{21} IS KNOWN AS CONJUGATE VARIABLE

** SIGN OF CONJUGATE VARIABLE IS POSITIVE FOR A SOURCE WHEN POWER FLOWS INTO THE SOURCE.



($-f$ is conj variable)

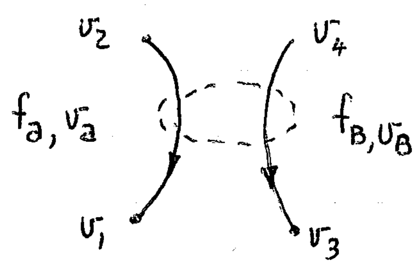


($-v_{21}$ is conj variable)

NOTE if f & v_{21} are of opposite signs $\mathbb{P} = \bar{f}\cdot\bar{v}_{21} < 0$ & $E < 0$

LINEAR GRAPHS FOR TRANSFORMERS

TRANSFORM ACROSS VARIABLES INTO ACROSS VARIABLES, THRU VARIABLES INTO THRU VARIABLES



$v_B = n v_a$

$f_B = -\frac{1}{n} f_a$

$v_a = v_{21}$ $v_B = v_{43}$

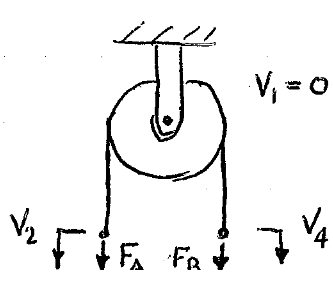
f_a & f_B ARE + IN DIRECTION OF ARROWS

v_a, v_B ARE + IF $V_2 > V_1$, $V_4 > V_3$

n IS TRANSFORMATION RATIO

THIS IS AN EXAMPLE OF A FOUR TERMINAL - TWO ENERGY PORT SYSTEM

EXAMPLE OF MECHANICAL TRANSFORMER

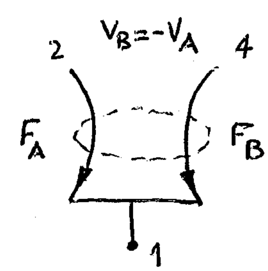


INEXTENSIBLE CORD

FRICTIONLESS PULLEY
NO INERTIA

$$V_B = V_{41} = -V_{21} = -V_A$$

$$n = -1$$



SYSTEMS ANALOGY

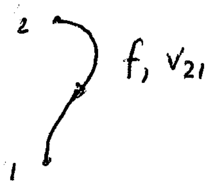
- WE HAVE STUDIED TRANSLATIONAL, ROTATIONAL, ELECTRICAL SYSTEMS BECAUSE OF SIMILARITY IN BEHAVIOR
- SIMILARITY IS DUE TO TWO FACTORS IN LUMPED ELEMENT SYSTEM!

(1) • ELEMENTAL EQUATIONS

(2) • SIMILARITY OF PHYSICAL LAWS GOVERNING THE RELATIONSHIPS OF VARIABLE

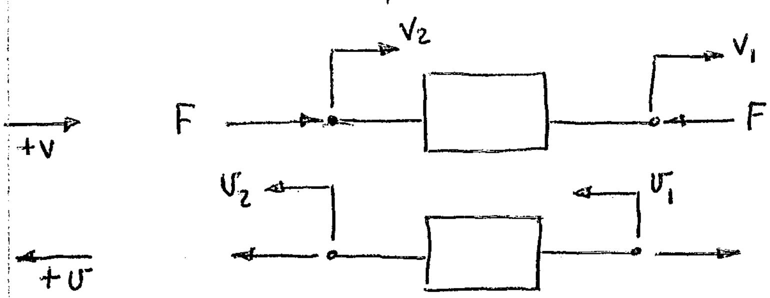
- SIMILAR COMPATABILITY & CONTINUITY RELATIONSHIPS

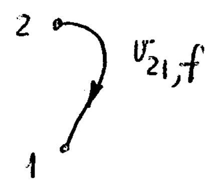
- HAVE USED THIS TO DEFINE GENERALIZED APPROACH TO THE ANALYSIS
- (2) HAS LED TO THE LINEAR GRAPHS WHICH ARE AN AID TO VISUALIZATION AND UNDERSTANDING OF PHYSICAL SYSTEMS
- LINEAR GRAPHS - GENERAL METHOD OF DIAGRAMMING LUMPED PHYSICAL SYSTEMS
  - SHOWS SYSTEM TOPOLOGY - INTERCONNECTEDNESS OF SYSTEM ELEMENTS
  - MAY BE USED FOR ANY SYSTEM WHETHER OR NOT COMPOSED OF IDEAL ELEMEN
  - LINEAR GRAPH IS ORIENTED (DIRECTED GRAPH); LINES OF GRAPH ARE CALLED BRANCHES, WHICH HAVE DIRECTION TO INDICATE REFERENCE COND.



- ARROW SHOWS ACROSS VARIABLE DROP & POSITIVE THRU VARIABLE FLOW
- IF ARROW IS REVERSED SIGN OF f, V_{21} ARE REVERSED

- $P = f \cdot V_{21}$ is + WHEN IT FLOWS INTO ELEMENT
- NOTE ARROW DOES NOT CONTAIN ANY INFO ABOUT DIRECTION OF ACTUAL SYSTEM. ONLY DEFINES ACROSS VARIABLE & THROUGH VARIABLE TO BE USED

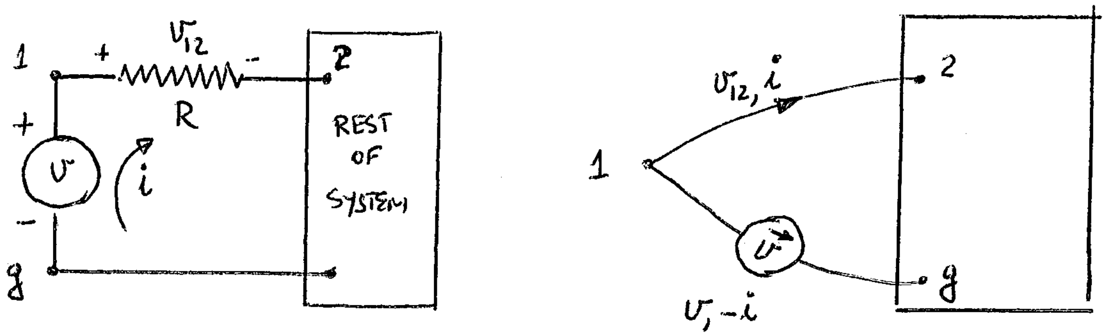




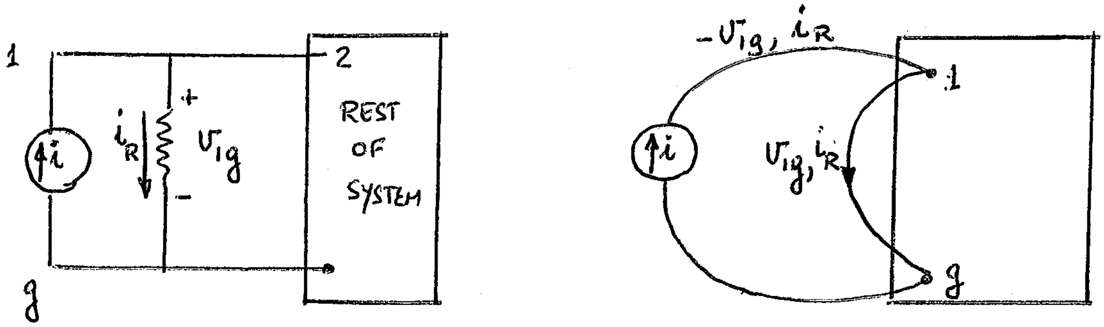
- DOES NOT INDICATE TENSION OR COMPRESSION EITHER

NOTE: IN A COMPLEX SYSTEM W/ SEVERAL MASSES, INERTIAS, CAPACITANCES ALL VELOCITIES MUST BE MADE WRT A COMMON REFERENCE (GROUND)

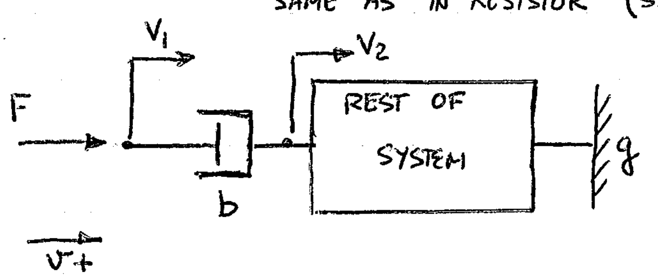
MODELLING OF SOURCES - ALSO KNOWN AS <u>ACTIVE ELEMENTS</u>, ALL OTHER ELEMENTS ARE PASSIVE

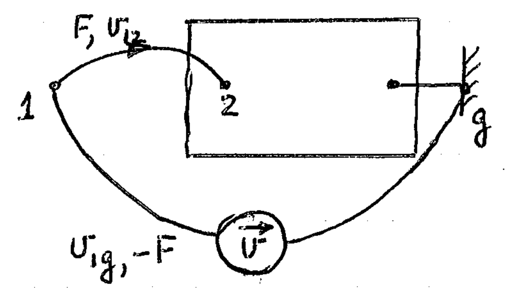


A-TYPE SOURCE : NOTE VOLTAGE DROP v GOES FROM + TO - ; CONJUGATE VARIABLE IS $-i$



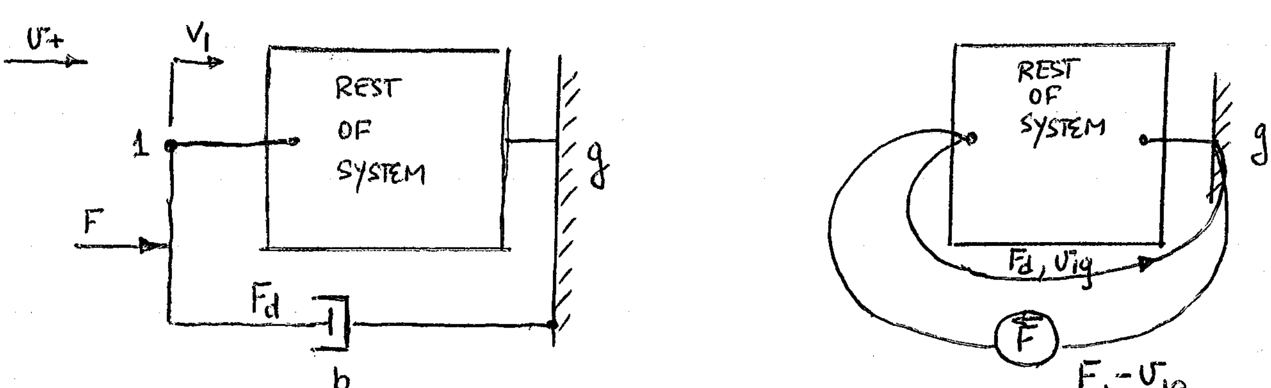
T-TYPE SOURCE : NOTE VOLTAGE DROP GOES FROM + TO - ; CONJUGATE VARIABLE IS $-V_{1g}$ SAME AS IN RESISTOR (SINCE PARALLEL CIRCUIT)





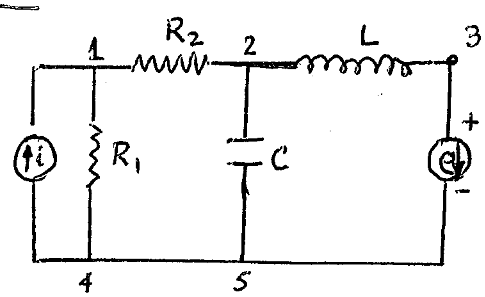
<u>VELOCITY SOURCE</u>

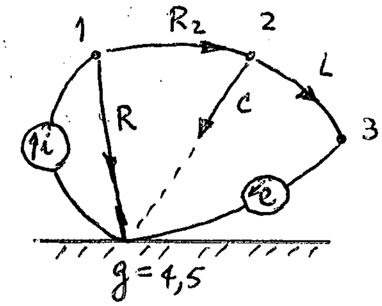
A-TYPE SOURCE : NOTE RELATIVE VELOCITY BETWEEN 1 & g ; CONJUGATE VARIABLE IS $-F$

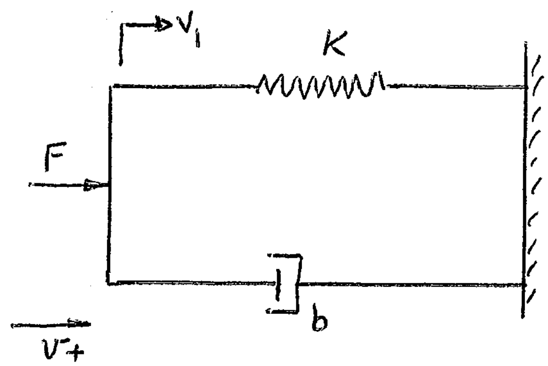


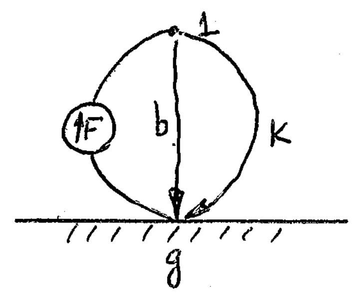
T-TYPE SOURCE:

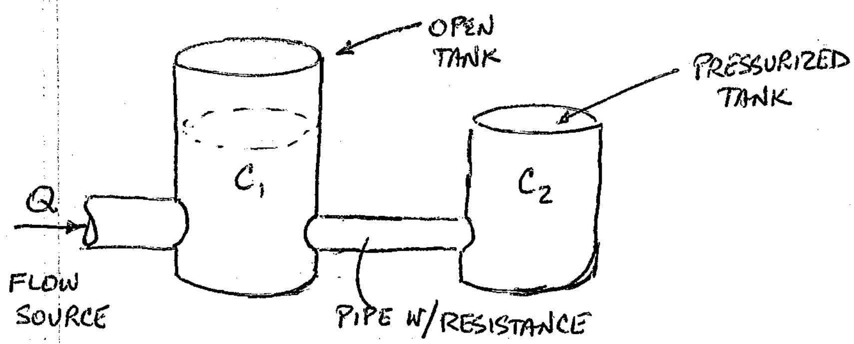
DEFN A LINEAR GRAPH OF A COMPLETE SYSTEM IS A SYSTEM GRAPH

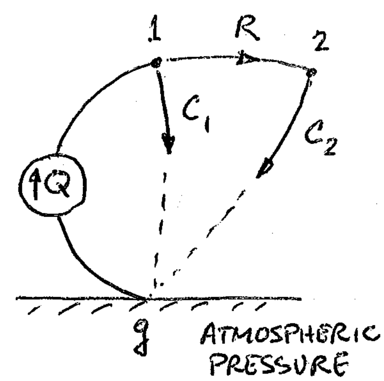












HOW TO DRAW LINEAR GRAPH

[1] IDENTIFY LUMPED ELEMENTS (WILL REQUIRE MODELLING)

NUMBER OF ELEMENTS INCLUDING SOURCES = NO. OF BRANCHES IN GRAPH

[2] ESTABLISH A REFERENCE PT (GROUND) AS A REFERENCE FOR MEASUREMENT OF ALL ACROSS VARIABLES

- IN TRANSLATIONAL, ROTATIONAL, FLUID FLOW SYSTEMS - INERTIAL FRAME OF REFERENCE

[3] AT EACH 2-TERMINAL ELEMENT AN ACROSS VARIABLE EXISTS

EACH ACROSS VARIABLE IS MEASURED WRT THE REFERENCE PT

$\therefore\ V_2 = V_{2g} = V_2 - V_g$. AT EACH NODE (OR VERTEX) REPRESENTS AN ACROSS VAR.

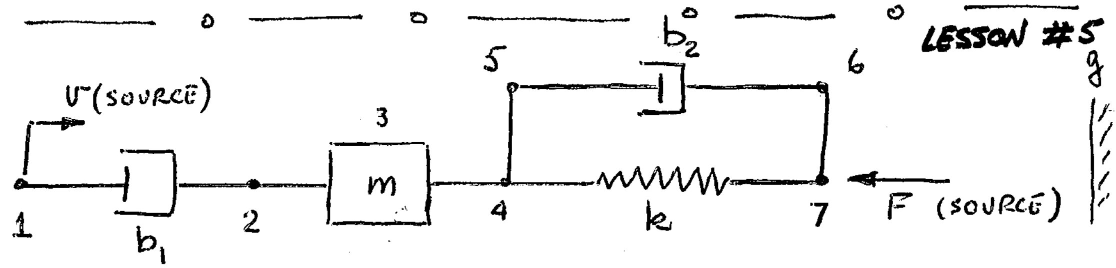
[4] FOR EACH 2-TERMINAL ELEMENT, A BRANCH IS INSERTED. ALL INERTIAL BRANCHES ARE INSERTED BETWEEN THEIR NODE & THE REFERENCE VERTEX

[5] EACH SOURCE GETS A BRANCH

[6] ORIENT THE SYSTEM GRAPH. THIS ESTABLISHES REFERENCE CONDITIONS OF THE THRU AND ACROSS VARIABLES

LESSON #5

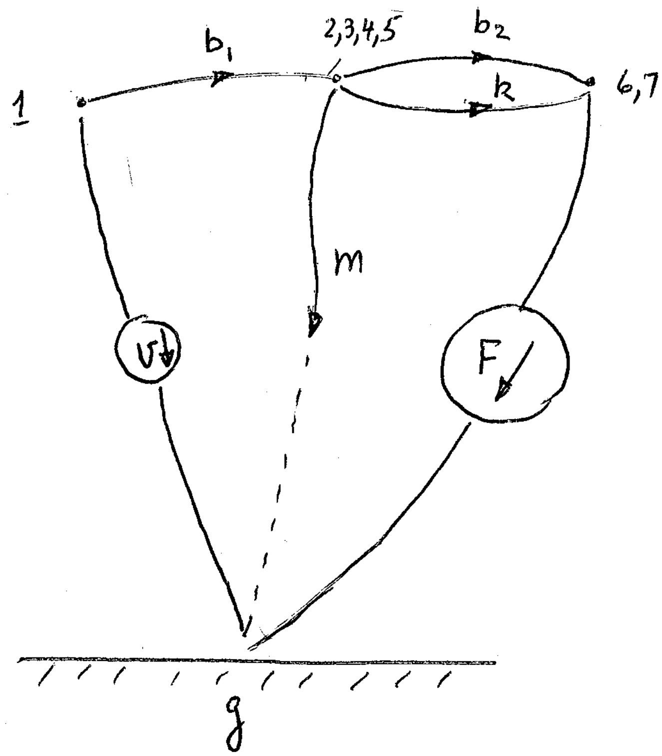
EXAMPLE



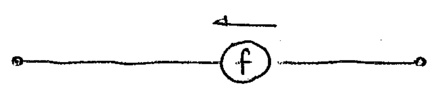
$\longrightarrow + v$

- HOW MANY ELEMENTS & WHAT ARE THEY ?
- WHAT IS THE REFERENCE PT ?
- HOW MANY <u>DISTINCT</u> VELOCITIES ARE THERE ?
- WHERE DO YOU INSERT BRANCHES ?
- WHAT DO YOU DO WITH THE MASS ?
- WHAT ABOUT THE SOURCES ?
- WHAT ABOUT ORIENTATION ?

REMEMBER – ASSIGNMENT OF ORIENTATION TO THE BRANCHES OF THE GRAPH (WHEN RELATED TO THE CHOSEN POSITIVE DIRECTION FOR VELOCITY) IMPLIES THE CHOICE OF REFERENCE CONDITIONS FOR THE VARIOUS FORCES AND VELOCITIES IN THE SYSTEM

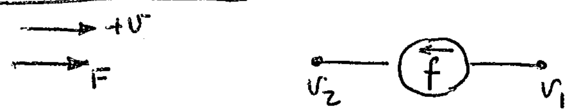


SINCE F ACTS TO OPPOSE VELOCITY SOURCE, BY OUR CONVENTION

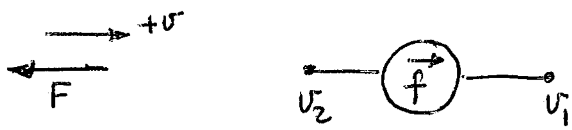


F ACTS FROM NODE 7 TO GROUND

FOR T-TYPE SOURCES : IF FORCE IS IN DIRECTION OF + VELOCITY



IF FORCE IS IN OPPOSITE DIRECTION OF + VELOCITY



- GENERALIZED CONTINUITY AND COMPATABILITY

  THE CONTINUITY AND COMPATABILITY REQ'TS ARE STATEMENTS OF PHYSICAL RELATIONSHIPS THAT MUST HOLD FOR VARIABLES OF A CONNECTED SET OF LUMPED ELEMENTS BECAUSE OF THE WAY THEY ARE CONNECTED. THESE RELATIONS DEAL W/ THE STRUCTURE OF SYSTEM.

- CONTINUITY - VERTEX (OR NODE) LAW

  FOR AN ORIENTED SYSTEM GRAPH THE ALGEBRAIC SUM OF THE THRU VARIABLES ENTERING A NODE IS ZERO

  - Σ THRU VARIABLES ENTERING NODE = Σ THRU VARIABLES LEAVING

ELECTRICAL SYSTEMS - CONSERVATION OF CHARGE OR KIRCHOFF'S CURRENT LAW

MECHANICAL SYSTEMS - CONSERVATION OF MOMENTUM OR NEWTON'S 1 & 3rd LAWS

FLUID SYTEMS - CONSERVATION OF MATTER

THERMAL SYSTEMS - CONSERVATION OF ENERGY

- APPLIES TO A CLOSED VOLUME WHICH CUTS THE SYSTEM GRAPH
- CAN ALSO BE APPLIED AT EACH VERTEX WITHIN THIS CLOSED VOLUME HOWEVER THIS IS NOT NECESSARY
- GIVEN A SYSTEM GRAPH OF 2-TERMINAL ELEMENTS WITH n VERTICE ONLY $(n-1)$ VERTEX EQNS ARE LINEARLY INDEPENDENT.

- FOR SYSTEMS W/ MULTITERMINAL ELEMENTS HAVING p parts NOT CONNECTED THERE ARE $(n-p)$ LINEAR INDEPENDENT VERTEX EQNS

EXAMPLE: IF A TRANSFORMER ELEMENT IS IN THE SYSTEM IT WILL SPLIT THE SYSTEM GRAPH INTO 2 PARTS $\Rightarrow$ $(n-2)$ VERTEX EQNS

COMPATIBILITY OR PATH LAW

ELECTRICAL - POTENTIAL IS A SCALAR (KIRCHHOFF'S VOLTAGE LAW)

MECHANICAL - DISTANCE IS A SCALAR

FLUID - PRESSURE IS A SCALAR

THERMAL - TEMPERATURE IS SCALAR

CAN ADD AS YOU MOVE FROM PT - PT

- ALGEBRAIC SUM OF THE ACROSS VARIABLES AROUND ANY CLOSED PATH IS ZERO.
- GIVEN A SYSTEM GRAPH OF 2-TERMINAL ELEMENTS WITH n VERTICES AND b branches; ONLY $b-(n-1)$ OF THE PATH EQNS ARE LINEARLY INDEPENDENT. IF THERE ARE P PARTS; THERE ARE

$b-(n-p)$ LINEARLY INDEPENDENT EQNS

- FORMULATION OF SYSTEM EQNS
- A SUFFICIENT SET OF EQNS FOR DETERMINING THE SYSTEM EQUATION FOR ANY OUTPUT OF ANY SYSTEM (LINEAR OR NONLINEAR) IS OBTAINED BY USING A SET OF

$(n-1)$ LINEARLY INDEPENDENT VERTEX EQNS

$(b-n+1)$ LINEARLY INDENDENT PATH EQNS

$b-s$ ELEMENTAL EQNS

n IS NO OF VERTICES, b IS NO OF BRANCHES, S IS NO OF SOURCES

- FOR A SYSTEM GRAPH WITH b branches OF WHICH S BRANCHES ARE SOURCES $\Rightarrow$ $(2b-s)$ UNKNOWNS. WHY?

FOR EACH NONSOURCE BRANCH : 2 UNKNOWNS - THRU & ACROSS VARIABLE

FOR EACH SOURCE : 1 UNKNOWN - CONJUGATE VARIABLE

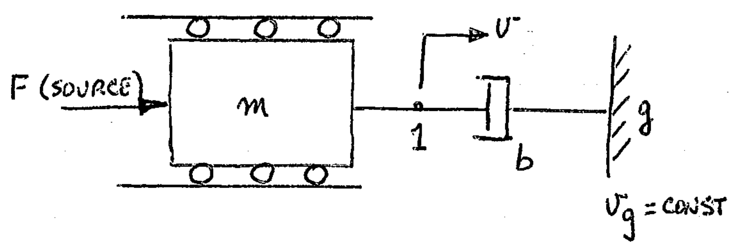
NOTE $\underset{\text{VERTEX}}{(n-1)} + \underset{\text{PATH}}{(b-n+1)} + \underset{\text{ELEMENTAL}}{(b-s)} = 2b-s$

- THUS THIS FORMS A NECESSARY & SUFFICIENT SET ALLOWING US TO ELIMINATE ALL BUT ONE PARTICULAR VARIABLE.

NONLINEAR SYSTEMS

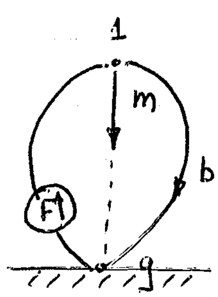
- THIS FORMS A SUFFICIENT SET FOR NONLINEAR SYSTEMS, BUT IT IS NOT ALWAYS POSSIBLE TO ELIMINATE ALL BUT ONE VARIABLE.
- THE ENTIRE SET OF EQNS MAY NOT BE NECESSARY TO DETERMINE RESPONSE

EXAMPLE



NOTE: MASS & DAMPER HAVE SAME VELOCITY

- APPLIED FORCE DOESN'T ACCELERATE MASS, BUT $F-F_b$.



ELEMENTAL EQNS $(b-s) = 3-1 = 2$

$$F_m = m\frac{dv_{1g}}{dt} = m\frac{dv_1}{dt} \qquad (1)$$

$$F_b = b\,v_{1g} = b\,v_1 \qquad (2)$$

CONTINUITY (VERTEX LAW) $n-1 = 2-1 = 1$

$$F = F_m + F_b \qquad (3)$$

COMPATIBILITY (PATH LAW) $b-n+1 = 3-2+1 = 2$

REMEMBER - V IS CONJ. VAR. OF F : $-v + v_{1g} = 0$ (PATH ALONG F & m) (4)

$-v_{1g} + v_{1g} = 0$ (PATH ALONG m & b) (5)

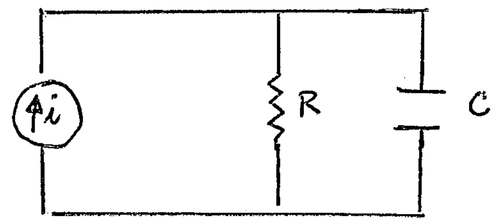
FROM (1) & (2) INTO (3)

$$F = F_m + F_b = m\frac{dv_1}{dt} + b\,v_1$$

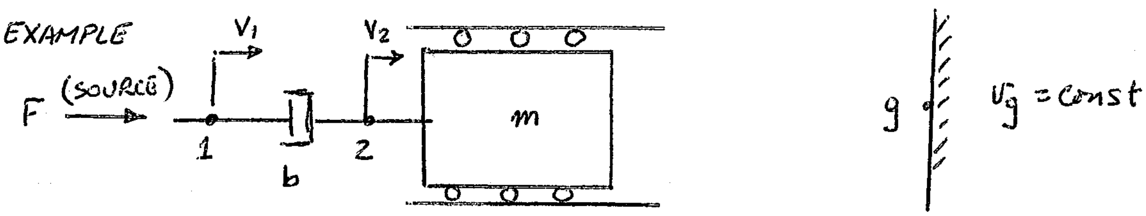
THIS ALLOWS USE TO FIND V_1

AND $x_1 = \int_0^t v_1 d\tilde{t} + x_1(t=0)$

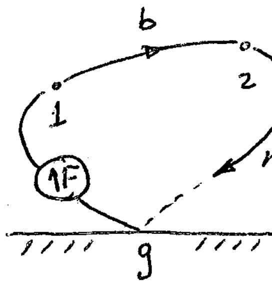
EXERCISE: COMPARE THIS TO:



EXAMPLE



NOTE: FORCE IS TRANSMITTED THRU DAMPER OR SERIES CONNECTION



ELEMENTAL EQNS. $(b-s) = 3-1 = 2$

$$F_m = m\frac{dv_{2g}}{dt} = m\frac{dv_2}{dt}$$

$$F_b = b\,v_{12}$$

CONTINUITY (VERTEX) $n-1 = 3-1 = 2$

$$\left.\begin{aligned} F - F_b &= 0 \\ F_b - F_m &= 0 \end{aligned}\right\} F = F_b = F_m$$

COMPATIBILITY (PATH LAW) $b-n+1 = 3-3+1$

REMEMBER: $-v$ is CONJ VAR OF F:

$$-v + v_{12} + v_{2g} = 0$$

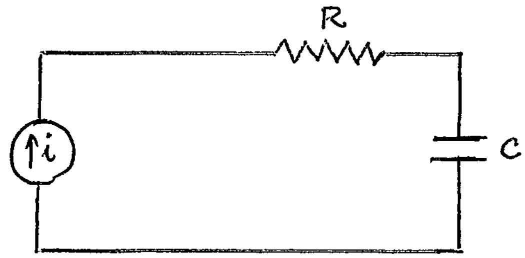
DIFFERENTIATE COMPATIBILITY: $$-\frac{dv}{dt} + \frac{dv_{12}}{dt} + \frac{dv_{2g}}{dt} = 0$$

$$-\frac{dv}{dt} + \frac{1}{b}\frac{dF_b}{dt} + \frac{F_m}{m} = 0$$

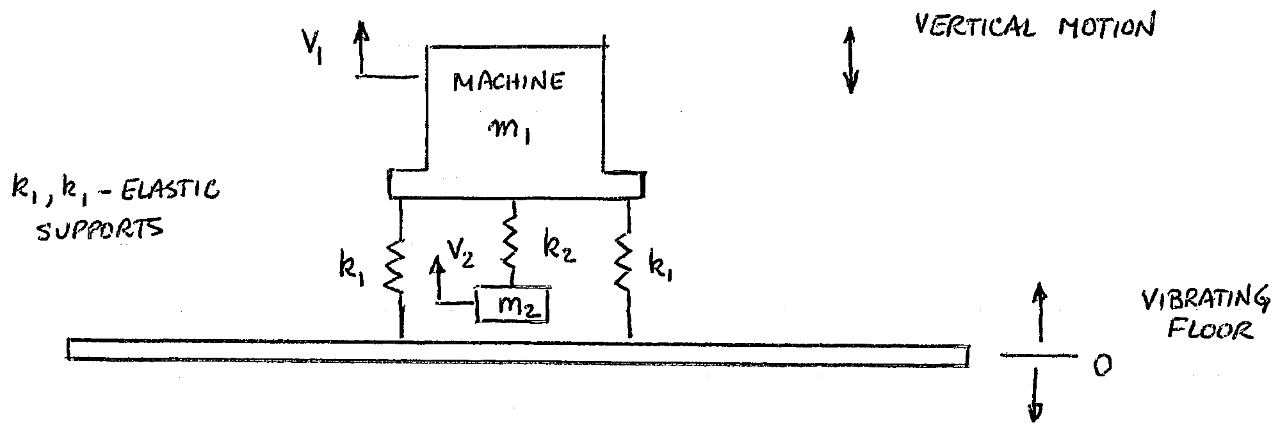
$$\frac{dv}{dt} = \frac{1}{b}\frac{dF}{dt} + \frac{F}{m}$$

KNOWING F AS FN OF t, WE CAN FIND v AS FN OF t

EXERCISE COMPARE THIS TO

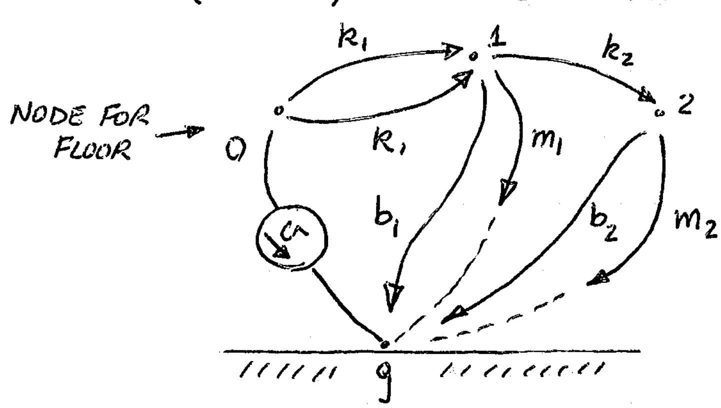


VIBRATION ABSORBER



- SYSTEM IS EXCITED BY THE VIBRATING FLOOR - WHICH IS MODELLED AS AN IDEAL VELOCITY SOURCE
- VIBRATION OF MACHINE IS EXCESSIVE - AS DESIGNERS OUR JOB IS TO DESIGN A VIBRATION ABSORBER (k_2, m_2) TO REDUCE AMPLITUDE OF VIBRATION (x_1) OF THE SYSTEM

CHOOSE (GROUND) AS INERTIAL FRAME NOT FLOOR !!



- HERE - b_1, b_2 are AIR DAMPING (ASSUMED LINEAR)
- CHOOSE INPUT AS VELOCITY OF FLOOR V, OUTPUT V_1

ELEMENTAL EQNS $(b-s) = 8-1=7$

$k_1 X_{01} = F_{K1}$ (2) $\qquad F_{b_1} = b_1 V_{1g} = b_1 V_1$

$k_2 X_{12} = F_{K2}$ $\qquad F_{b_2} = b_2 V_{2g} = b_2 V_2$

$F_{m_1} = m_1 \frac{dV_{1g}}{dt} = m_1 \frac{dV_1}{dt}$ $\qquad F_{m_2} = m_2 \frac{dV_{2g}}{dt} = m_2 \frac{dV_2}{dt}$

CONTINUITY (VERTEX) $n-1 = 4-1=3$

node 0 : $-F + 2F_{K1} = 0$

1 : $2F_{K1} = F_{b1} + F_{m1} + F_{K2}$ (1)

2 : $F_{K2} = F_{b2} + F_{m2}$ (2)

COMPATIBILITY EQNS (PATH) $b-n+1 = 8-4+1 = 5$

$-V + V_{01} + V_{1g} = 0 \Rightarrow V = V_{01} + V_1$ or $V_{01} = V - V_1$ (3)

$V_{01} - V_{01} = 0$ IDENTITY

$-V_{1g} + V_{1g} = 0$ IDENTITY

$-V_{1g} + V_{12} + V_{2g} = 0 \Rightarrow V_{12} = V_1 - V_2$ (4)

$-V_{2g} + V_{2g} = 0$ IDENTITY

- IF WE DIFFERENTIATE (1) : $2\frac{dF_{K1}}{dt} = \frac{dF_{b1}}{dt} + \frac{dF_{m1}}{dt} + \frac{dF_{K2}}{dt}$

$$2k_1 V_{01} = b_1 \frac{dV_1}{dt} + m_1 \frac{d^2V_1}{dt^2} + k_2 V_{12} \quad (5)$$

- USE (3) & (4) IN (5) TO FIND V_2

- $$V_2 = \frac{1}{K_2}\left[(2k_1+k_2)V_1 + b_1\dot{V}_1 + m_1\ddot{V}_1 - 2k_1 V\right] \quad (6)$$

IF WE DIFFERENTIATE (2)

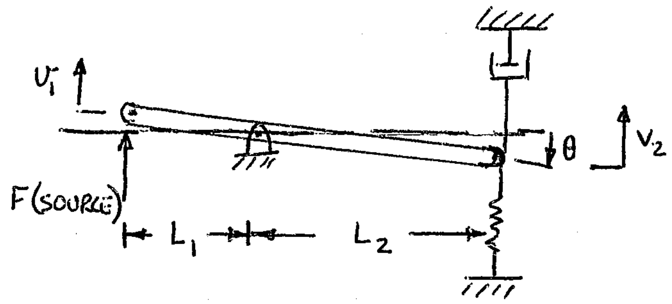
$$dF_{K2}/dt = dF_{b2}/dt + \frac{dF_{m2}}{dt} \Rightarrow k_2 V_{12} = b_2 \frac{dV_2}{dt} + m_2 \frac{d^2V_2}{dt^2} \quad (7)$$

• KNOWING (4) & (6) PLACED INTO (7) LEADS TO

$$m_1 m_2 \frac{d^4 V_1}{dt^4} + (m_1 b_2 + m_2 b_1)\frac{d^3 V_1}{dt^3} + \left[m_1 k_2 + b_1 b_2 + m_2(2k_1 + k_2)\right]\frac{d^2 V_1}{dt^2}$$

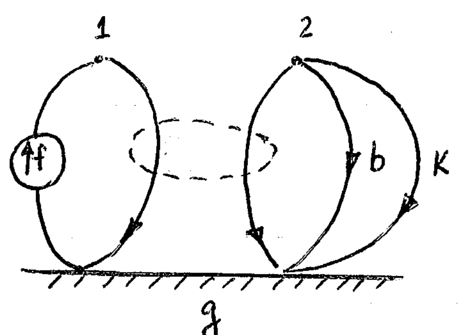
$$+ \left[b_1 k_2 + b_2(2k_1 + k_2)\right]\frac{dV_1}{dt} + 2k_1 k_2 V_1 =$$

$$2m_2 k_1 \frac{d^2 V}{dt^2} + 2k_1 b_2 \frac{dV}{dt} + 2k_1 k_2 V$$



FOR LEVERED SYSTEM

F is input, θ output



ELEMENTAL EQNS $b-s = 5-1 = 4$

$F_K = K x_{2g} = K x_2$ (3

$F_b = b\, V_{2g} = b V_2$ (3′)

$F_2/F_1 = -\frac{1}{n}$

$V_{2g}/V_{1g} = n$

LET F_1 & F_2 BE THE THRU VARIABLES AND 1 & 2 RESPECTIVELY OF THE LEVER

CONTINUITY (VERTEX) $n-p = 4-2=2$

$f - f_1 = 0 \quad \Rightarrow \quad F = F_1 \qquad (1)$

$f_2 + f_b + F_K = 0 \quad \Rightarrow \quad F_b + F_K = -F_2 \qquad (2)$

COMPATIBILITY OR PATH $b-n+p = 5-4+2 = 3$

$-v + v_{1g} = 0 \quad \Rightarrow \quad v = v_1$

$-v_{2g} + v_{2g} = 0$ (2 TIMES) IDENTITIES

FROM (1) & (2) $\quad (F_b + F_K)/F = -F_2/F_1 = \frac{1}{n} \quad \Rightarrow \quad F_b + F_K = \frac{1}{n}F \qquad (4)$

BUT $n = -L_2/L_1$ and $x_2 = -L_2\theta \Rightarrow V_2 = -L_2\dot{\theta} \qquad (5)$

THUS FROM (3), (3′) INTO (4) $\Rightarrow K x_2 + b V_2 = -\frac{L_1}{L_2}F$, WITH (5)

$$K L_2^2 \theta + b L_2^2 \dot{\theta} = L_1 F$$

FIRST ORDER SYSTEMS

- ASSUME WE KNOW HOW TO MODEL & HAVE DETERMINED SYSTEMS ODE
- WE START LOOKING AT SOLUTION METHODS TO THE MODEL'S DIFF EQ.
  - WE INVESTIGATE FIRST & 2^{ND} ORDER SYSTEMS INITIALLY AND HIGHER ORDER SYSTEMS LATER
- WE LOOK AT LINEAR MODELS FIRST THEN NON LINEAR
- WE WILL WANT TO DETERMINE STABILITY OF THE SYSTEM

LINEAR FIRST ORDER CONSTANT COEFF O.D.E , HOMOGENEOUS (NO INPUTS)

$$\frac{dy}{dt} - ry = 0 \qquad \dot{y} - ry = 0$$

- GENERAL SOLUTION IS $y = Ae^{st}$ where A & S are const.

$$(s-r)Ae^{st} = 0 \quad \text{or} \quad r = s \qquad \text{CHARACTERISTIC EQN}$$

$$y(t) = Ae^{rt} \qquad \text{SOLUTION TO ODE.}$$

- A DETERMINED BY INITIAL CONDITION

ASSUME $y(t=t_0) = y_0$

$$y_0 = Ae^{rt_0} \quad \text{or} \quad A = y_0 e^{-rt_0}$$

$$y(t) = y_0 e^{r(t-t_0)}$$

- THIS IS KNOWN AS FREE RESPONSE OF SYSTEM SINCE NO INPUTS

- FOR $t > t_0$ & $r > 0$ y INCREASES TO ∞
  - $t > t_0$ $r = 0$ y REMAINS AT y_0
  - $t > t_0$ $r < 0$ y DECREASES TO 0

- WE ARE ONLY INTERESTED IN OBSERVING SYSTEM RESPONSE AFTER INITIAL TIME. THIS IS EQUIVALENT TO LETTING $t_0 = 0$

$$y(t) = y_0 e^{rt} \qquad y_0 = y(t=0)$$

- IF $r < 0$ DEFINE $\tau = -\frac{1}{r}$ TIME CONSTANT

$$y = y_o e^{-t/\tau}$$

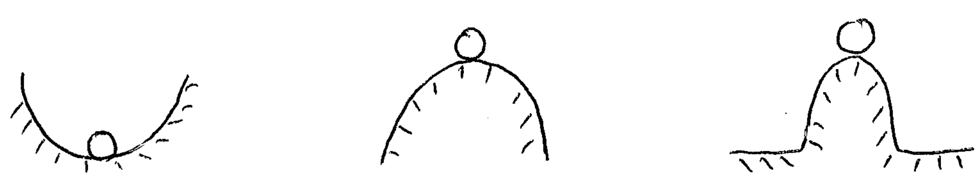
- TIME CONSTANT IS IMPORTANT SINCE WHEN $t = \tau$ $y/y_o = e^{-1}$
  - THE RESPONSE WILL DECREASE TO 37% OF ITS INITIAL VALUE

EQUILIBRIUM AND STABILITY

- EQUILIBRIUM OF A SYSTEM - WHEN THERE ARE NO CHANGES IN MODEL'S VARIABLE & THIS OCCURS WHEN ALL THE TIME DERIVATIVES = 0

$$\frac{dy}{dt} - ry = 0 - ry = 0 \qquad \text{ONLY SOL'N } y = 0 \text{ IF } r \neq 0$$

- THUS $y = 0$ IS THE EQUIL. POINT OF SYSTEM
- THUS A SYSTEM WILL REMAIN AT ITS EQUILIBRIUM PT <u>UNLESS</u> SOME CAUSE (OTHER THAN THOSE ALREADY DESCRIBED BY THE MODEL) DISPLACES THE SYSTEM FROM ITS EQUILIBRIUM VALUE
- <u>EXAMPLE</u> - MASS & SPRING SYSTEM: EQUILIBRIUM IS δ_{ST} OF SPRING
- IF THE SYSTEM IS NOW DISTURBED SLIGHTLY
  - IF IT RETURNS TO ITS EQUIL. POINT AND REMAINS THERE - STABLE
  - IF IT DOESN'T RETURN & CONTINUES TO MOVE FURTHER - UNSTABLE
  - IF IT DOESN'T RETURN BUT REMAINS WHERE IT IS - NEUTRAL



- LOCAL STABILITY - RESTRICTION TO SMALL EXCURSIONS FROM EQUIL
- GLOBAL STABILITY - NO SUCH RESTRICTIONS

THUS $y = y_o e^{rt}$ IS STABLE FOR $r < 0$

UNSTABLE FOR $r > 0$

NEUTRALLY STABLE FOR $r = 0$

*** • LOCAL STABILITY PROPERTIES FOR LINEAR MODEL ARE DETERMINED BY THE ROOTS OF THE CHARACTERISTIC EQN.

• A LINEAR MODEL IS GLOBALLY STABLE IF IT IS LOCALLY STABLE

• FOR NONLINEAR MODELS – IT CAN BE LOCALLY STABLE BUT GLOBALLY NOT

<u>RULE</u> – AN EQUILIBRIUM OF A FIRST-ORDER MODEL IS GLOBALLY STABLE IF AND ONLY IF THE ROOT OF THE CHARACTERISTIC EQUATION s IS NEGATIVE; IT IS NEUTRALLY STABLE IF $S=0$ AND UNSTABLE IF $S>0$.

• ESTIMATION OF r IN $\dot{y} = ry$

• IF FOR A SYSTEM r CANNOT BE COMPUTED FROM BASIC PRINCIPLES

• IF y WAS MEASURED AT VARIOUS TIMES, ESTIMATE r

$$\frac{dy}{dt} = ry \quad \Rightarrow \quad \frac{dy}{y} = r\,dt$$

$$\ln y - \ln y_0 = rt \quad \text{or} \quad \ln y = \ln y_0 + rt$$

• PLOT y VS. t ON SEMILOG PAPER ⇒ THE SLOPE IS r

STEP RESPONSE OF FIRST-ORDER SYSTEMS

• REMEMBER THE LEVER PROBLEM

$$\dot{\theta} + \frac{K}{b}\theta = \frac{L_1}{bL_2^2}F$$

• REMEMBER THE MASS-DAMPER PROBLEM W/ FORCE F APPLIED TO MASS

$$\dot{V}_1 + \frac{b}{m}V_1 = \frac{F}{m}$$

• THESE HAVE FORM

$$\dot{y} = ry + Mv$$

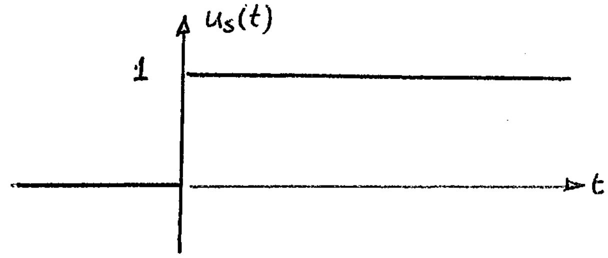
y IS OUTPUT OR RESPONSE

v IS INPUT TO SYSTEM

• WHEN $v = 0$ WE HAVE EQN WE HAVE JUST STUDIED

• WANT TO STUDY BEHAVIOR OF $\dot{y} = ry + Mv$ FOR COMMONLY ENCOUNTERED INPUT FNS.

• THE SIMPLEST INPUT IS A STEP FN $u_s(t) = U$



$$u_s(t) = \begin{cases} 0 & t \leq 0 \\ 1 & t > 0 \end{cases}$$

• THIS REPRESENTS AN INPUT THAT IS SWITCHED ON IN A TIME INTERVAL WHICH IS MUCH SMALLER THAN THE TIME CONSTANT OF THE SYSTEM

• SOLUTION FOR $r = -\frac{1}{\tau}$: $\dot{y} = -\frac{1}{\tau} y + M u_s(t)$

FOR $t > 0$ $u_s(t) = 1$

$$\dot{y} + \frac{1}{\tau} y = M \qquad \text{INHOMOGENEOUS } 1^{ST} \text{ ORDER LODE}$$

• FROM YOUR ORDINARY DIFFERENTIAL EQNS CLASS

$$y = y_h + y_p \qquad \underline{\text{PRINCIPLE OF SUPERPOSITION}}$$

WHERE y_h SATISFIES $\dot{y} + \frac{1}{\tau} y = 0$ $\quad y = Ae^{-t/\tau}$

y_p IS ANY SOLUTION THAT SATISFIES THE INHOMOGENEOUS CASE

CHOOSE $y_p = \text{constant}$ $\therefore \dot{y}_p = 0$

THUS $0 + \frac{D}{\tau} = M$ OR $D = M\tau$

• $$y = Ae^{-t/\tau} + M\tau \qquad A \text{ is a constant}$$

• APPLY $y(t=0) = y_0$

$$y_0 = A + M\tau \quad \text{or} \quad A = y_0 - M\tau$$

$$y = (y_0 - M\tau)e^{-t/\tau} + M\tau \quad = \quad \text{TRANSIENT + STEADY STATE}$$

• NOTE – AS $t \to \infty$ $y \to M\tau$

FOR EQUILIBRIUM $\dot{y} = 0 \Rightarrow \frac{1}{\tau} y = M$ or $y = M\tau$

• AS $t \to \infty$ TRANSIENT PART DIES OFF & LEAVES THE STEADY STATE

• STEADY STATE PART IS THE EQUILIBRIUM PT OF SYSTEM

- LETS REWRITE THIS:

$$y = (y_0 - M\tau)e^{-t/\tau} + M\tau = y_0 e^{-t/\tau} + M\tau\,(1 - e^{-t/\tau})$$

= FREE + FORCED RESPONSE

- FORCED RESPONSE IS DUE TOTALLY TO THE INPUT
- NOTE THAT FOR A STABLE SYSTEM - FREE RESPONSE IS PART OF TRANSIENT

- FORCED RESPONSE MAY APPEAR IN BOTH

- LOOK AT THE NEUTRALLY STABLE CASE $r=0$

$$\dot{y} = M u_s(t) = M \quad \text{FOR } t > 0$$

$$y(t) = \int_0^t M\,d\tilde{t} + y(t=0)$$

$$= Mt + y_0$$

- as $t \to \infty$ $\quad y \to \infty$

- LOOK AT THE UNSTABLE CASE $r > 0$

$$\dot{y} - ry = M u_s(t) = M \quad \text{FOR } t > 0$$

- $y = y_P + y_h$
- y_P SATISFIES $\dot{y} - ry = 0 \qquad y = Ae^{rt}$
- y_h : ANY SOLN TO THE INHOMOGENEOUS ODE

LET y_p = constant $\quad \dot{y}_p = 0$

$0 - rD = M \quad$ or $\quad D = -\frac{M}{r}$

- $y = Ae^{rt} - \frac{M}{r}$
- IF $y(t=0) = y_0 \implies y_0 = A - \frac{M}{r}$ or $A = y_0 + \frac{M}{r}$

THUS $\quad y = \left(y_0 + \frac{M}{r}\right)e^{rt} - \frac{M}{r}$

- as $t \to \infty$ $\quad y \to \infty$ DUE TO e^{rt}

EXAMPLE 3.3

GIVEN: $RA\dot{h} + h = Rq_1$ or $\dot{h} + \frac{1}{RA} h = \frac{1}{A} q_1$

LIKE $\dot{y} + \frac{1}{\tau} y = M v$

TIME CONSTANT = RA = 2100 sec $h(t=0) = h_0 = 5ft$ $q_1 = 0.05\, u_s(t)$

SOLUTION LIKE $y = (y_0 - M\tau)e^{-t/\tau} + M\tau$ WHERE $M = \frac{1}{A} \cdot 0.05$

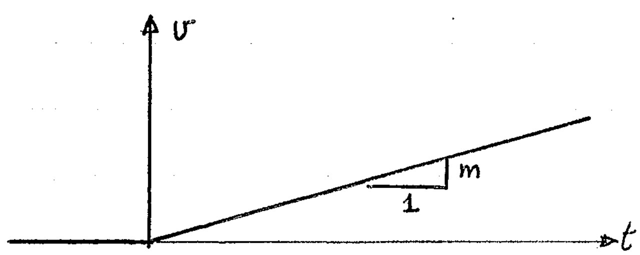
$\longrightarrow h(t) = \left(5 - \frac{0.05}{3} . 2100\right) e^{-t/2100} + \frac{0.05}{3}(2100)$

ANS $h(t) = -30\, e^{-t/2100} + 35$ ft.

- SUPPOSE HEIGHT OF TANK WAS 30 FT ? WILL TANK OVERFLOW
- WHAT IS EQUILIBRIUM HEIGHT ? 35 ft
- WHAT IS FREE RESPONSE ? $h_{FREE}(t) = h_0 e^{-t/\tau} = 5e^{-t/2100}$ ft $= h_H(t)$
- WHAT IS FORCED RESPONSE ? $h_{FORCED}(t) = \frac{0.05(2100)}{3}\left[1 - e^{-t/2100}\right]$ ft $= h_p(t)$
- WHAT IS TRANSIENT ? $h_{TRANS}(t) = -30\, e^{-t/2100}$ ft
- WHAT IS STEADY STATE ? $h_{SS}(t) = 35$ ft

- RESPONSE OF A FIRST ORDER SYSTEM TO A RAMP INPUT

$$\dot{y} = ry + Mv$$



$v(t) = \begin{cases} 0 & t \leq 0 \\ mt & t \geq 0 \end{cases}$ RAMP

$\dot{y} - ry = Mmt$ FOR $t > 0$

ASSUME

$y = y_H + y_P$ y_H SATISFIES $\dot{y} - ry = 0$ $y_H = Ae^{rt}$

y_P IS ANY PARTICULAR SOLUTION

LET $y_P = Ct + D$; $\dot{y}_P = C$

$\therefore$ $C - r(Ct + D) = Mmt$ $\Longrightarrow$ $C - rD = 0$ $-rC = Mm$

or $C = -\frac{Mm}{r}$ $D = \frac{C}{r} = -\frac{Mm}{r^2}$

- THUS $\quad y = Ae^{rt} - \frac{Mm}{r}\left(t + \frac{1}{r}\right)$ $\quad$ ONLY GOOD FOR $r \gtrless 0$

- TO FIND A : INITIAL CONDITION $\quad y(t=0) = y_0$

$$\Rightarrow \quad y_0 = A - \frac{Mm}{r^2} \quad \text{or} \quad A = y_0 + \frac{Mm}{r^2}$$

$$y = \underbrace{y_0 e^{rt}}_{\text{FREE}} + \underbrace{\frac{Mm}{r^2}\left(-1 + e^{rt}\right) - \frac{Mmt}{r}}_{\text{FORCED}}$$

FOR $r < 0 \quad \left(r = -\frac{1}{\tau}\right)$

$$y = y_0\, e^{-t/\tau} + Mm\tau^2\left(e^{-t/\tau} - 1\right) + Mm\tau t$$

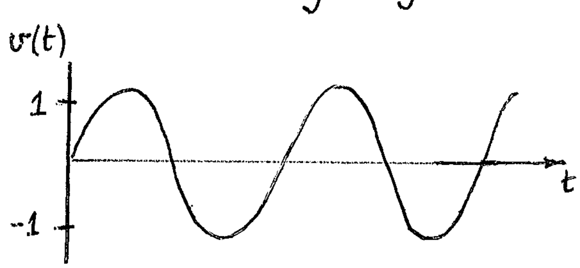
- LOOK AT THE FORCED RESPONSE. AS $t \to \infty$, THE FORCED RESPONSE TENDS TO $Mm\tau(t - \tau)$ FOR THE INPUT Mmt

- THE EQUILIBRIUM POINT OF $\dot{y} + \frac{1}{\tau}y = Mmt$ IS $y = Mmt\tau$. THUS THE FORCED RESPONSE IS THE SAME AS THE EQUILIBRIUM POINT OF SYSTEM EXCEPT A TIME LAG IS INTRODUCED. THAT IS, THE OUTPUT AT TIME t WAS THE INPUT AT TIME $t - \tau$ FOR VERY LARGE t. THIS PHENOMENON IS CALLED <u>STEADY STATE TIME LAG</u>

- NOTE THAT <u>FOR A UNIT STEP INPUT THERE IS NO STEADY STATE TIME LAG</u>

SINUSOIDAL RESPONSE TO A FIRST ORDER SYSTEM

$$\dot{y} = ry + Mv$$



$$v(t) = \begin{cases} 0 & t < 0 \\ \sin\omega t & t > 0 \end{cases}$$

AMPLITUDE OF $\sin\omega t$ IS 1

FREQUENCY IS GIVEN BY ω

- SUCH INPUTS OCCUR EXTENSIVELY IN NATURE (EARTH'S ROTATION & REVOLUTION PRODUCE SUCH INPUTS, RECIPROCATING ENGINES, ROTATING UNBALANCED MACHINES)

- SOLUTION: LET $y = y_H + y_P$ $\qquad$ y_H SOLVES $\dot{y} - ry = 0$

  y_P IS PARTICULAR SOLUTION

- LET

$$y_P = C\cos\omega t + D\sin\omega t$$

$$\dot{y}_P = -\omega C\sin\omega t + \omega D\cos\omega t$$

$$\dot{y} - ry = -\omega C\sin\omega t + \omega D\cos\omega t - r[C\cos\omega t + D\sin\omega t] = Mv = M\sin\omega t$$

$$\Rightarrow -\omega C - rD = M$$

$$\omega D - rC = 0$$

- TO SOLVE USE CRAMER'S RULE

$$\begin{bmatrix} -\omega & -r \\ -r & \omega \end{bmatrix} \begin{pmatrix} C \\ D \end{pmatrix} = \begin{pmatrix} M \\ 0 \end{pmatrix}$$

$$C = \frac{\begin{bmatrix} M & -r \\ 0 & \omega \end{bmatrix}}{\begin{bmatrix} -\omega & -r \\ -r & \omega \end{bmatrix}} = \frac{M\omega}{-\omega^2 - r^2} = -\frac{M\omega}{\omega^2 + r^2}$$

$$D = \frac{\begin{bmatrix} -\omega & M \\ -r & 0 \end{bmatrix}}{\begin{bmatrix} -\omega & -r \\ -r & \omega \end{bmatrix}} = \frac{rM}{-\omega^2 - r^2} = -\frac{rM}{\omega^2 + r^2}$$

$$\Rightarrow \quad y = y_P + y_H = \frac{-M}{\omega^2 + r^2}\left[\omega\cos\omega t + r\sin\omega t\right] + Ae^{rt}$$

- FOR THE STABLE CASE $r = -\frac{1}{\tau}$

$$y = \frac{M\tau}{\tau^2\omega^2 + 1}\left[\sin\omega t - \omega\tau\cos\omega t\right] + Ae^{-t/\tau}$$

- TO FIND A : IF $y(t=0)=y_o$

$$y_o = \frac{-M\tau(\omega\tau)}{\tau^2\omega^2+1} + A \quad \text{or} \quad A = y_o + \frac{M\omega\tau^2}{\tau^2\omega^2+1}$$

$$y = \underbrace{y_o e^{-t/\tau}}_{\text{FREE}} + \frac{M\tau}{\tau^2\omega^2+1}\underbrace{\left[\sin\omega t - \omega\tau\cos\omega t + \omega\tau e^{-t/\tau}\right]}_{\text{FORCED RESPONSE}}$$

- LOOK AT FORCED RESPONSE: AS $t \to \infty$ $\quad y \to \frac{M\tau}{\tau^2\omega^2+1}\left[\sin\omega t - \omega\tau\cos\omega t\right]$

 WHICH IS THE STEADY STATE OF THE SYSTEM

- EQUILIBRIUM PT OF SYSTEM $\dot{y} - ry = M\sin\omega t$ IS $y = -\frac{M}{r}\sin\omega t = M\tau\sin\omega t$

- Now IF $\sin\omega t - \omega\tau\cos\omega t$ IS WRITTEN AS $B\sin(\omega t+\phi)$

 THEN $B\cos\phi = 1 \qquad +B\sin\phi = -\omega\tau$

$$B = \sqrt{(B\cos\phi)^2 + (B\sin\phi)^2} = \sqrt{1+(\omega\tau)^2}$$

$$\phi = \tan^{-1}(-\omega\tau) \qquad \underline{\text{PHASE ANGLE}}$$

$\therefore$ STEADY STATE

$$y_{ss} = \frac{M\tau}{\tau^2\omega^2+1}\left[\sin\omega t - \omega\tau\cos\omega t\right] = \frac{M\tau}{\tau^2\omega^2+1}\cdot\sqrt{1+\omega^2\tau^2}\,\sin(\omega t+\phi)$$

$$= \frac{M\tau}{\sqrt{1+\tau^2\omega^2}}\sin(\omega t+\phi)$$

- IN COMPARISON TO THE EQUILIBRIUM PT (OR INPUT)
  - AMPLITUDE CHANGE BY $\frac{1}{\sqrt{1+\tau^2\omega^2}}$ (I.e. SMALLER)
  - ϕ WILL BE NEGATIVE. $\Longrightarrow \omega t+\phi < \omega t$ THUS THE OUTPUT WILL LAG THE INPUT
  - NOTE AS ω INCREASES, AMPLITUDE DECREASES & LAG INCREASES

FREE RESPONSE OF A SECOND ORDER SYSTEM

- EXAMPLE OF SECOND ORDER SYSTEM - FIRST TWO PROBLEMS IN YOUR PREVIOUS HW ASSIGNMENT. ALSO MASS-SPRING DASHPOT SYSTEM
- GENERAL FORM OF LINEAR SECOND ORDER O.D.E. W/ CONSTANT COEFFS

$$a\ddot{x} + b\dot{x} + cx = f$$

INPUT IS f OUTPUT IS x. THIS IS CALLED THE REDUCED FORM

- CAN ALSO BE WRITTEN AS

$$v = \dot{x}$$
$$a\dot{v} = f - bv - cx$$

} v & x ARE IN STATE VARIABLE FORM - HIGHEST ORDER OF DERIV. IS 1

- FOR NUMERICAL SOLUTION USE STATE VARIABLE FORM. REDUCED FORM ALLOWS FOR ANALYTICAL SOL'N FOR LOW-ORDER MODELS W/ SIMPLE INPUTS

- FOR FREE RESPONSE, SET $f = 0$. SOL'N IS $x(t) = Ae^{st}$

$$\dot{x} = sAe^{st} \quad \ddot{x} = s^2Ae^{st} \quad \text{or}$$
$$(as^2 + bs + c)\,Ae^{st} = 0$$

- CHARACTERISTIC EQN IS $as^2 + bs + c = 0$

$$s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- IF $b^2 - 4ac > 0$ 2 real unequal roots

$$s_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad s_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

SOL'N IS $x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

- SINCE 2^{nd} ORDER $\Rightarrow$ 2 UNKNOWNS A_1 & A_2 $\Rightarrow$ require 2 CONDITIONS TO FIND UNIQUE SOL'N. ASSUME $x(t=0) = x_0$, $\dot{x}(t=0) = \dot{x}_0$

- THIS IS LIKE DEFINING STATE VARIABLES AT $t=0$
- THUS

$$\left.\begin{aligned} x_0 &= A_1 + A_2 \\ \dot{x}_0 &= A_1 s_1 + A_2 s_2 \end{aligned}\right\} \quad A_1 = \frac{\dot{x}_0 - s_2 x_0}{s_1 - s_2} \qquad A_2 = \frac{s_1 x_0 - \dot{x}_0}{s_1 - s_2}$$

THUS ;

$$x(t) = \frac{1}{s_1 - s_2}\left[\dot{x}_0\left(e^{s_1 t} - e^{s_2 t}\right) + x_0\left(s_1 e^{s_2 t} - s_2 e^{s_1 t}\right)\right]$$

- IF $b^2 - 4ac = 0 \Rightarrow s_1, s_2 = \frac{-b}{2a} = s$ real, 2 repeated roots

SOLN' :

$$x(t) = (A_1 + A_2 t)e^{st}$$

- FOR SAME CONDITIONS

$$\left.\begin{aligned} x(t=0) &= x_0 = A_1 \\ \dot{x}(t=0) &= \dot{x}_0 = A_2 + A_1 s \end{aligned}\right\} \quad A_2 = \dot{x}_0 - x_0 s$$

$$x(t) = \left[x_0(1 - st) + \dot{x}_0 t\right]e^{st} \; ; \; s < 0 \Rightarrow \text{DECAYING EXPONENTIAL}$$

- IF $s < 0$ WE HAVE A DECAYING EXPONENTIAL
- IF $b^2 - 4ac < 0$ 2 complex conjugate roots

let $p = +\frac{b}{2a}$ $\quad q = \frac{\sqrt{4ac - b^2}}{2a}$

$$s_1 = -p + iq \qquad s_2 = -p - iq$$

SOLUTION :

$$x(t) = A_1 e^{(-p+iq)t} + A_2 e^{(-p-iq)t} = e^{-pt}\left[B_1 \cos qt + B_2 \sin qt\right]$$

- EULER FORMULAS $e^{i\theta} = \cos\theta + i\sin\theta$ ALLOWS THE ABOVE
- IF $p > 0$ THIS IS AN EXPONENTIALLY DECAYING SINUSOIDAL FN OF t
- FREQUENCY OF THE SINUSOIDAL PART IS q
- WE CAN WRITE AS BEFORE $B\sin(qt + \phi) = B_1 \cos qt + B_2 \sin qt$

w/ $B\cos\phi = B_2$ $\quad B\sin\phi = B_1$

THUS: $B = \sqrt{B_1^2 + B_2^2}$ $\quad \tan\phi = B_1/B_2$ $\quad$ PHASE SHIFT ϕ, AMPLITUDE B

SOLN :

$$x(t) = Be^{-pt}\sin(qt + \phi)$$

- IF $x(t=0) = x_0$, $\dot{x}(t=0) = \dot{x}_0$

$$x_0 = B\sin\phi$$

$$\dot{x}_o = -pB\sin\phi + Bq\cos\phi \implies \frac{1}{q}(\dot{x}_o + px_o) = B\cos\phi$$

$$\tan\phi = \frac{x_o q}{\dot{x}_o + px_o} \qquad B = \sqrt{x_o^2 + \frac{1}{q^2}(\dot{x}_o + px_o)^2}$$

- THE SOLUTION $x(t) = Be^{-pt}\sin(qt+\phi)$ IS BRACKETTED AT THE TOP & BOTTOM BY CURVES PROPORTIONAL TO e^{-pt}
- PG 125 IN PALM
- THE BRACKETTING CURVES HAVE A TIME CONSTANT $\tau = \frac{1}{p}$
- NOTE THAT FOR $t > 4\tau$ THE OSCILLATIONS HAVE ESSENTIALLY DIED OUT

- FOR THE FN $x(t) = Be^{-pt}\sin(qt+\phi)$

  $q\Delta t = 2\pi$ DEFINES THE <u>period of oscillation</u>

  $\Delta t = \frac{2\pi}{q}$ $\quad$ q IS THE FREQUENCY OF OSCILLATION IN RADIANS/SEC

** • CHARACTERISTIC EQUATION DETERMINES

  SYSTEM STABILITY

  TIME CONSTANT

  OSCILLATORY BEHAVIOR

  FREQUENCY OF OSCILLATION

- INITIAL CONDITIONS x_o, $\dot{x}_o$: DETERMINE AMPLITUDE & PHASE SHIFT

- LOOK AT $a\ddot{x} + b\dot{x} + cx = f$: TO DETERMINE STABILITY, LET $f = 0$

  LET $\dot{x}, \ddot{x} = 0$

  THUS $x = 0$ IS EQUILIBRIUM PT OF SYSTEM IF $c \neq 0$

- LOOK AT THE THREE CASES

(1) $\quad x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} \qquad s_1 = \dfrac{-b+\sqrt{b^2-4ac}}{2a} \qquad s_2 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$

(2) $\quad x(t) = (A_1 + A_2 t)e^{st} \qquad s = \dfrac{-b}{2a}$

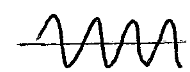
(3) $\quad x(t) = Be^{-pt}\sin(qt+\phi) \qquad p = \dfrac{b}{2a} \quad q = \dfrac{\sqrt{4ac-b^2}}{2a} \qquad \begin{array}{l} s_1 = -p+iq \\ s_2 = -p-iq \end{array}$

IF CASE (1) & (2) $\quad$ IF s_1, s_2 & s < 0 $\qquad x = 0$ IS EQUILIB PT

IN CASE (3) $\quad$ IF $p > 0$ (i.e. real part of s_1 & $s_2 < 0$)

OBSERVATIONS

** • THUS FOR STABILITY ROOTS OF CHAR EQN MUST HAVE NEGATIVE REAL PARTS

• FOR CASE (3) : IF $b=0$ $(p=0)$ WE HAVE NEUTRAL STABILITY 

(1) : IF $c = 0$ $(s_1 = 0) \Rightarrow$ WE HAVE NEUTRAL STABILITY ALSO IF ONE ROOT IS ZERO & THE SECOND IS NOT POSITIVE **

** • FOR ALL CASES IF REAL PART OF ANY ROOT IS POSITIVE THEN SYSTEM IS UNSTABLE

** • THE OBSERVATIONS ALSO HOLD FOR HIGHER ORDER SYSTEMS

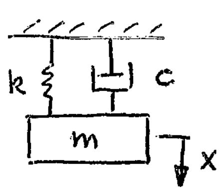
• FOR A SECOND ORDER SYSTEM $\Longrightarrow$ COEFFICIENT a IS NOT ZERO IN

$$a\ddot{x} + b\dot{x} + cx = f$$

• FOR ROOTS TO HAVE NEGATIVE REAL PARTS $\Rightarrow a, b, c \gtrless 0$

• IF b or c IS ZERO $\Longrightarrow$ SYSTEM IS NEUTRALLY STABLE

• LOOK AT



$m\ddot{x} + c\dot{x} + kx = 0 \qquad (ms^2 + cs + k = 0)$

WE CAN FIND THE CRITICAL DAMPING CONST, c_c, SO THAT $b^2 - 4ac = 0 \Rightarrow c^2 - 4mk = 0$

$$c_c = 2\sqrt{mk}$$

- CAN DEFINE DAMPING FACTOR $\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{mk}}$

IF $\zeta < 1$ SYSTEM IS UNDERDAMPED (COMPLEX ROOTS, OSCILLATORY)

$\zeta = 1$ SYSTEM IS CRITICALLY DAMPED (ONE REAL ROOT

$\zeta > 1$ SYSTEM IS OVERDAMPED (TWO DISTINCT REAL ROOTS)

IF NO DAMPING - THE FREQUENCY OF OSCILLATION FOR $a\ddot{x} + cx = 0$

$\omega_n = \sqrt{\frac{c}{a}}$ IN OUR CASE $\omega_n = \sqrt{\frac{k}{m}}$

$$\zeta = \frac{c}{2\sqrt{mk}} = \frac{c}{2\sqrt{m^2 \frac{k}{m}}} = \frac{c}{2\omega_n m} \Rightarrow \frac{c}{2m} = \zeta\omega_n = p$$

$$q = \frac{\sqrt{4ac-b^2}}{2a} = \sqrt{\frac{c}{a} - \left(\frac{b}{2a}\right)^2} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = \omega_n\sqrt{1-\zeta^2}$$

SUMMARY

$a\ddot{x} + b\dot{x} + cx = 0$

$\omega_n = \sqrt{\frac{c}{a}}$

$\zeta = \frac{b}{2\sqrt{ac}}$; $b_c = 2\sqrt{ac}$

$q = \omega_n\sqrt{1-\zeta^2} = \frac{\sqrt{4ac-b^2}}{2a}$

$p = \zeta\omega_n = \frac{b}{2a}$

$\tau = 1/p = \frac{2a}{b} = \frac{1}{\zeta\omega_n}$

$m\ddot{x} + c\dot{x} + kx = 0$

$\omega_n = \sqrt{\frac{k}{m}}$

$\zeta = \frac{c}{2\sqrt{mk}}$ $c_c = 2\sqrt{mk}$

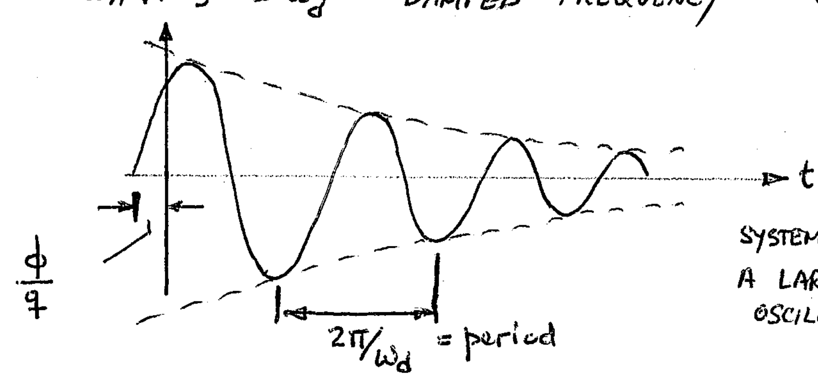
$q = \omega_n\sqrt{1-\zeta^2} = \frac{\sqrt{4mk-c^2}}{2m}$

$p = \zeta\omega_n = \frac{c}{2m}$

$\tau = \frac{1}{\zeta\omega_n}$ TIME CONST

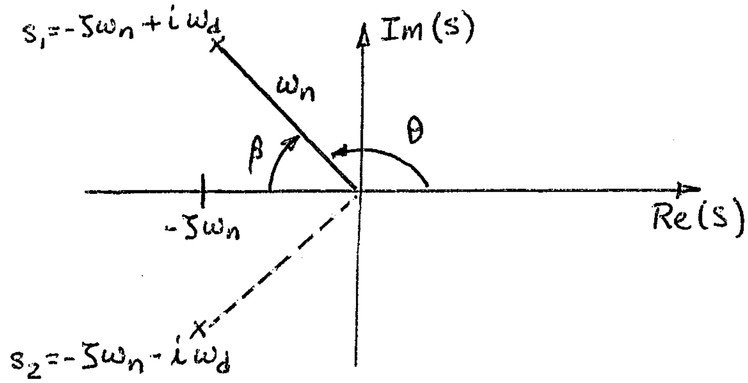
$$s_1, s_2 = -p \pm iq = -\zeta\omega_n \pm i\,\omega_n\sqrt{1-\zeta^2}$$

$\omega_n\sqrt{1-\zeta^2} = \omega_d$ DAMPED FREQUENCY $\omega_d < \omega_n$ FOR $\zeta < 1$



SYSTEM W/ DAMPING HAS A LARGER PERIOD OF OSCILLATION

- WE CAN GRAPHICALLY REPRESENT THE ROOTS



** NOTE: IN S PLANE IF $Re(s) > 0$ WE HAVE UNSTABLE SYSTEM

MAGNITUDE OF s_1 OR $s_2 = \omega_n$

$$s = re^{i\theta}$$

$$r = \sqrt{[Re(s)]^2 + [Im(s)]^2}$$

$$\theta = \tan^{-1}\frac{Im(s)}{Re(s)} = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{-\zeta} > \frac{\pi}{2}$$

$$\beta = \pi - \theta$$

NOTE: $\zeta = 0 \quad \theta = \frac{\pi}{2} \quad \beta = \pi/2 \quad \Rightarrow$ Imaginary axis

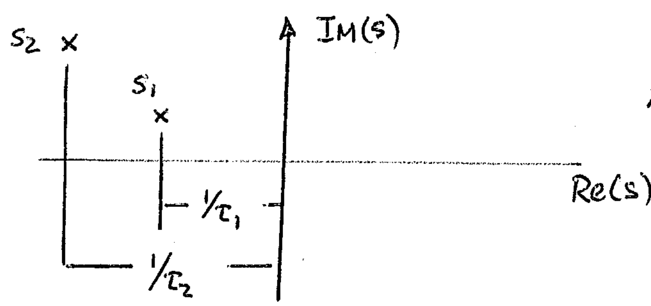
$\zeta = 1 \quad \theta = \pi \quad \beta = 0 \quad \Rightarrow$ Real axis — CRITICALLY DAMPED

$0 < \zeta < 1, \quad \frac{\pi}{2} < \theta < \pi, \quad 0 < \beta < \frac{\pi}{2}$ — UNDERDAMPED

- NOTE $\zeta\omega_n = \frac{1}{\tau}$ THIS IS DISTANCE FROM IMAGINARY AXIS TO THE MAGNITUDE OF THE REAL PART OF THE ROOT

- ROOTS LYING ALONG A CONSTANT "y" LINE $\Rightarrow$ THEY HAVE SAME ω_d
- ROOTS LYING ALONG A CONSTANT "x" LINE $\Rightarrow$ THEY HAVE SAME τ
- ROOTS LYING ALONG A CONSTANT θ LINE $\Rightarrow$ THEY HAVE SAME ζ

- ROOTS LYING FARTHER AWAY FROM IMAGINARY AXIS HAVE <u>SMALLER</u> TIME CONSTANTS THAN THOSE LYING CLOSER



$$1/\tau_1 < \frac{1}{\tau_2} \Rightarrow \tau_1 > \tau_2$$

- <u>DOMINANT ROOT</u> FOR A CHARACTERISTIC EQN IS THE ONE WITH THE LARGEST TIME CONSTANT. ITS EFFECT ON THE FREE RESPONSE OF A SYSTEM IS STILL EVIDENT WHEN THE EFFECT OF THE OTHER ROOTS HAVE DIED OUT.

CHOOSE $x_p(t) = D = const$; $\dot{x}_p = 0$; $\ddot{x}_p = 0$

$$0 + 0 + kD = 1 \quad \text{or} \quad D = 1/k = x_p(t)$$

- THUS

$$x(t) = x_h(t) + x_p(t) = \underbrace{x_h(t)}_{\text{TRANSIENT}} + \underbrace{\frac{1}{k}}_{\text{STEADY STATE}}$$

(EQUILIB)

- TO FIND THE COEFFICIENTS OF x_h (A_1, A_2 or B & ϕ): MUST BE GIVEN INITIAL CONDITIONS. MUST APPLY THEM TO TOTAL SOLUTION $x(t)$

- THUS IF SYSTEM IS UNDERDAMPED & $x(t=0) = x_o$ $\dot{x}(t=0) = \dot{x}_o$

$$x(t) = Be^{-pt}\sin(qt+\phi) + \frac{1}{k} = Be^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + \frac{1}{k}$$

$$x(t=0) = x_o = B\sin\phi + \frac{1}{k}$$

$$\dot{x}(t=0) = \dot{x}_o = -\zeta\omega_n B\sin\phi + B\omega_d\cos\phi$$

$$= -\zeta\omega_n\left(x_o - \frac{1}{k}\right) + B\omega_d\cos\phi \Rightarrow B\cos\phi = \frac{\dot{x}_o + \zeta\omega_n\left(x_o - \frac{1}{k}\right)}{\omega_d}$$

$$\cot\phi = \frac{\dot{x}_o + \zeta\omega_n\left(x_o - \frac{1}{k}\right)}{\omega_d\left(x_o - \frac{1}{k}\right)} \qquad B = \sqrt{\left(x_o - \frac{1}{k}\right)^2 + \frac{\left(\dot{x}_o + \zeta\omega_n\left(x_o - 1/k\right)\right)^2}{\omega_d^2}}$$

- IF $\dot{x}_o, x_o = 0$

$$\cot\phi = \frac{\zeta\omega_n}{\omega_d} \qquad B = \frac{1}{k\omega_d}\sqrt{\omega_d^2 + \zeta^2\omega_n^2} = \frac{\omega_n}{k\omega_d} = \frac{1}{k\sqrt{1-\zeta^2}}$$

$$\cot\phi = \frac{\zeta}{\sqrt{1-\zeta^2}} \quad \& \quad \tan\phi = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

- THUS $$x(t) = \frac{1}{k}\left[\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + 1\right]$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad \phi = \cot^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}} \quad \text{or} \quad \phi = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

ϕ IS IN 3rd QUADRAN

FOR UNDERDAMPED SYSTEM w/ $\dot{x}_o, x_o = 0$

IF m, k, c are > 0

- YOU SHOULD OBTAIN THE RESULTS FOR THE OVERDAMPED & CRITICALLY DAMPED CASE - SEE PG 134 TABLE 3.5

- NOTES ON CRITICALLY DAMPED, OVERDAMPED & UNDERDAMPED CASES
  - CRITICALLY DAMPED - STEADY STATE VALUE IS REACHED THE QUICKEST BUT WITHOUT OSCILLATION
  - OVERDAMPED - STEADY STATE VALUE IS REACHED WITHOUT OSCILLATION BUT TAKES LONGER
- UNDERDAMPED - SYSTEM OVERSHOOTS STEADY STATE VALUE & OSCILLATES ABOUT THE STEADY STATE VALUE. AS $\zeta \to 0$, TAKES LONGER TO SETTLE. $\zeta = 0$ OSCILLATIONS REMAIN.
- DESIGN INSTRUMENTS TO BE <u>CRITICALLY DAMPED</u>

- TRANSIENT RESPONSE TERMS - PG 137
  - MAXIMUM OVERSHOOT - MAXIMUM DEVIATION ABOVE STEADY STATE CONDITIONS

FOR THE STEP RESPONSE TO AN UNDERDAMPED SYSTEM

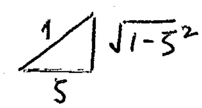
$$x(t) = \frac{1}{k}\left[\frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) + 1\right]$$

MAX OVERSHOOT OCCURS WHEN $\frac{dx}{dt} = 0$

$$\dot{x} = 0 = \frac{1}{k}\left[\frac{-\zeta\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \sin(\omega_d t+\phi) + \frac{\omega_d}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t} \cos(\omega_d t+\phi)\right]$$

$$0 = \frac{\omega_n e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}}\left[-\zeta \sin(\omega_d t+\phi) + \sqrt{1-\zeta^2}\cos(\omega_d t+\phi)\right]$$

$$\Rightarrow \tan(\omega_d t+\phi) = \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan\phi$$



THUS $\omega_d t = n\pi$ or $t = n\pi/\omega_d$.

TIME TO OVERSHOOT IS $t = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$ $(n=1)$

- WE CAN ESTIMATE THE RESPONSE TIME OF THE SYSTEM BY THE DOMINANT ROOT

- IF THE SEPARATION BETWEEN THE ROOTS IS LARGER THIS METHOD BECOMES BETTER AT ESTIMATING THE RESPONSE TIME OF THE SYSTEM

- FOR A 2^{nd} ORDER SYSTEM w/ TWO REAL ROOTS (OVERDAMPED CASE), ζ CAN INDICATE HOW MUCH SEPARATION EXISTS BETWEEN ROOTS

$$\zeta = \frac{n+1}{2\sqrt{n}} \qquad n \text{ is separation factor} \qquad \begin{aligned} s_1 &= -\text{const} \\ s_2 &= -n \cdot \text{const} \end{aligned}$$

- see pg 131 PALM

STEP RESPONSE OF SECOND ORDER SYSTEM

- REMEMBER THAT STEP RESPONSE MODELS ANY RAPID CHANGE IN INPUT FROM ONE LEVEL TO ANOTHER (LIKE A SWITCH)

- LET'S LOOK AT MASS-SPRING-DAMPER RESPONSE TO STEP INPUT

$$m\ddot{x} + c\dot{x} + kx = u_s(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$

- LET $x(t) = x_h(t) + x_p(t)$

$$x_h(t) \text{ SOLVES } m\ddot{x} + c\dot{x} + kx = 0 \Rightarrow x_h(t) = Ae^{st}$$

IF OVERDAMPED

$$x_h(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

UNDERDAMPED

$$x_h(t) = Be^{-pt}\sin(qt + \phi)$$

CRITICALLY DAMPED

$$x_h(t) = (A_1 + A_2 t)e^{st}$$

- $x_p(t)$ is any SOL'N THAT SATISFIES $m\ddot{x} + c\dot{x} + kx = 1$ FOR $t > 0$

** • THIS TIME IS ALSO KNOWN AS PEAK TIME $t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$

• since $\tan(\omega_d t+\phi) = \frac{\sqrt{1-\zeta^2}}{\zeta} \Longrightarrow \sin(\omega_d t+\phi) = \sqrt{1-\zeta^2}$

$$X = \frac{1}{K}\left[\frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\frac{\pi}{\sqrt{1-\zeta^2}}} \cdot \sqrt{1-\zeta^2} + 1\right]$$

** • MAX OVERSHOOT $= X - X_{ss} = X - \frac{1}{k} = \frac{1}{k}\, e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$

** • % MAX OVERSHOOT $= \frac{X - X_{ss}}{X_{ss}} \times 100\% = 100 \cdot e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$

DEFINED AS MAX OVER SHOOT COMPARED TO FINAL VALUE : $\frac{X - X_{ss}}{X_{ss}}$

** • DELAY TIME - TIME AT WHICH $X = 0,5\,X_{ss} = 0,5 \cdot \frac{1}{k}$

$$\Longrightarrow -0,5\sqrt{1-\zeta^2} = e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$$

$$\Longrightarrow t_d \cong \frac{1+0,7\zeta}{\omega_n} \qquad 0 \le \zeta \le 1$$

• RISE TIME - TIME WHICH $X = X_{ss}$ FOR THE FIRST TIME

$$\Longrightarrow \sin(\omega_d t+\phi) = 0 \quad \text{or} \quad \omega_d t+\phi = n\pi$$

THUS $X = X_s$ WHEN

$t = \frac{n\pi - \phi}{\omega_d}$; FOR FIRST TIME : $t_{r_{100}} = \frac{2\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$ **

• SETTLING TIME t_s : TIME FOR WHICH $X - X_s \le 0.02\, X_s$

$$\Longrightarrow \left|\frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\sin(\omega_d t+\phi)\right| \le 0.02$$

** FOR $\zeta < 0.7$ IF $\zeta\omega_n t = +4$ $t_s = \frac{4}{\zeta\omega_n}$ $\zeta\omega_n = \frac{1}{\tau}$

FOR $0.7 < \zeta \le 1$ $\quad t_s = -\frac{1}{\zeta\omega_n} \ln\left[\sqrt{1-\zeta^2}\,(0.02)\right]$

LESSON #8

- LOOK AT SINUSOIDAL INPUT TO 2^{nd} ORDER SYSTEM

$$m\ddot{x} + c\dot{x} + kx = f \qquad \text{WHERE} \quad f = dt$$

$x(t) = X_h + X_p$ WHERE X_h SOLVES $m\ddot{x} + c\dot{x} + kx = 0$

ALSO X_p IS ANY SOLUTION TO THE ODE

Since f IS A LINEAR FN $\Rightarrow$ CHOOSE $X_p = Ct + D$, $\dot{x}_p = C$, $\ddot{x}_p = 0$

$$m(0) + cC + k(Ct + D) = dt$$

$$\Rightarrow kC = d \quad \& \quad cC + kD = 0 \qquad C = d/k \quad \& \quad D = \frac{-c}{k}\left(\frac{d}{k}\right)$$

THUS

$$X_p = Ct + D = \frac{d}{k}\left(t - \frac{c}{k}\right)$$

- IF $\zeta < 1$ UNDERDAMPED CASE

$$x(t) = Be^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + \frac{d}{k}\left(t - \frac{c}{k}\right)$$

IF AT $t = 0$ $\quad x = x_o$ & $\dot{x} = \dot{x}_o$

$$x_o = B\sin\phi - \frac{dc}{k^2} \qquad \Rightarrow \qquad B\sin\phi = x_o + \frac{dc}{k^2}$$

$$\dot{x}_o = -\zeta\omega_n B\sin\phi + B\omega_d\cos\phi + \frac{d}{k}$$

$$\Rightarrow B\cos\phi = \left[\dot{x}_o - d/k + \zeta\omega_n\left(x_o + \frac{dc}{k^2}\right)\right]/\omega_d$$

$$\tan\phi = \frac{B\sin\phi}{B\cos\phi} \qquad \& \qquad B = \sqrt{(B\sin\phi)^2 + (B\cos\phi)^2}$$

FOR $x_o = \dot{x}_o = 0$ $\qquad B\sin\phi = \frac{dc}{k^2} \qquad B\cos\phi = \left[\left(\zeta\omega_n\frac{c}{k} - 1\right)d/k\right]/\omega_d$

$$\tan\phi = \frac{c/k \cdot \omega_d}{\zeta\omega_n \cdot c/k - 1} \qquad \text{WHERE} \quad c/k = \frac{2\zeta}{\omega_n} \qquad \Rightarrow \tan\phi = \frac{2\zeta\sqrt{1-\zeta^2}}{2\zeta^2 - 1}$$

and $B = \dfrac{d}{k\,\omega_d}$

THUS $\quad x(t) = Be^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + \frac{d}{k}\left(t - \frac{c}{k}\right)$

$$x(t) = \frac{d}{k}\left[\frac{e^{-\zeta\omega_n t}}{\omega_d}\sin(\omega_d t + \phi) + \left(t - \frac{2\zeta}{\omega_n}\right)\right]$$

THE OTHER CASES $\zeta = 1$ & $\zeta > 1$ ARE GIVEN IN TABLE 3.7 pg 147

- RETURN TO DOMINANT ROOT OF LAST LECTURE
- EXAMPLES

→ PROBLEM 3.17 FOR $f = 6t$

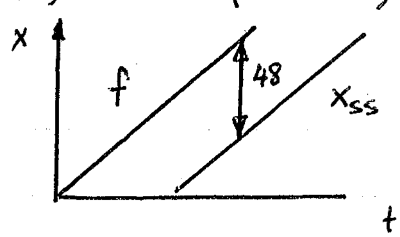
1) FIND STEADY STATE RESPONSE TO $\ddot{x} + 8\dot{x} + x = f$
2) WHAT IS DIFFERENCE BETWEEN $x(t)$ & $f(t)$ AT STEADY STATE
3) HOW LONG TO REACH STEADY STATE

1) IF $m\ddot{x} + c\dot{x} + kx = dt$ $\quad x_p$ (STEADY STATE) $= \frac{d}{K}\left(t - \frac{c}{K}\right)$

HERE $m = 1$ $c = 8$ $k = 1$ & $d = 6$

$$x_p \text{ or } x_{ss} = \frac{6}{1}\left(t - \frac{8}{1}\right)$$

2) $x(t) - f(t) = -6t + (6t - 48) = -48$



3) LOOK FOR DOMINANT ROOT

CHAR EQ IS $s^2 + 8s + 1 = 0 \Rightarrow s = \dfrac{-8 \pm \sqrt{8^2 - 4\cdot1\cdot1}}{2}$

$= -.127, \; -7.873$

$$x_h = A_1 e^{-.127t} + A_2 e^{-7.873t}$$

$\therefore \quad \tau_1 = \frac{1}{.127} \quad \& \quad \tau_2 = \frac{1}{7.873} \qquad \therefore \quad \tau_1 = 7.69 \text{ sec}$

DOMINANT ROOT

- TO REACH STEADY STATE $t \geq 4\tau_1 \Rightarrow x_h \leq 0.02\, x_h(t=0)$

$\therefore \quad t = 4(7.69) = 30.76$ sec TO REACH STEADY STATE

EXAMPLE 3.9 FIND $\xi, \omega_n, \tau, \omega_d$ FOR DOMINANT ROOT

1) $s = -2$, $s = -3 + i$ $s = -3 - i$

DOMINANT ROOT IS ONE CLOSEST TO IMAGINARY AXIS $\Rightarrow s = -2$

$\therefore \quad \tau = \frac{1}{2}$. SINCE THIS IS A REAL ROOT $\Rightarrow$ NO ξ, ω_n, ω_d FOR IT

2) $s = -3$, $s = -2 \pm 2i$, $s = -2 - 2i$

DOMINANT ROOT IS $s = -2 \pm 2i$ $\therefore \quad \tau = \frac{1}{|Re(s)|} = \frac{1}{2}$

$$s = -p \pm iq = -\xi\omega_n \pm i\omega_d \qquad \therefore \quad \omega_d = 2$$

also $|s| = \sqrt{Re(s)^2 + Im(s)^2} = \omega_n = \sqrt{(2)^2 + (2)^2} = 2\sqrt{2} = 2.828$

since $\omega_d = \omega_n\sqrt{1-\xi^2} \Rightarrow \sqrt{1 - \left(\frac{\omega_d}{\omega_n}\right)^2} = \xi = \sqrt{1 - \frac{1}{2}} = .707$

EXAMPLE 3.15 FOR $9\ddot{x} + c\dot{x} + 4x = 0$ FIND c SO THAT

1. MAX % OVERSHOOT IS SMALLER THAN 20%
2. 100% rise time is smaller than 3
3. PRIORITY IS TO OVERSHOOT (IE – WHAT DOES THIS IMPLY?)

2) SINCE THIS IS LIKE $m\ddot{x} + c\dot{x} + kx = 0$ & $\omega_n = \sqrt{\frac{k}{m}} \Rightarrow \omega_n = \frac{2}{3}$

SINCE $t_r \leq 3 \Rightarrow \omega_n t_r \leq 2 \Rightarrow$ LOOK AT FIG 3.14C $\Rightarrow \xi \leq 0.333$

1) FOR MAX OVERSHOOT $\leq 20\%$ $\Rightarrow$ $\zeta \geq .43$ (FIG 3.14a)

SINCE ζ CANNOT BE $\geq .43$ & LESS THAN .333
WE SATISFY OVERSHOOT & SET $\zeta = .43$ TO MINIMIZE RISE TIME

TO FIND C $\quad \zeta = \dfrac{c}{c_c} = \dfrac{c}{2\sqrt{mk}} = \dfrac{c}{2 \cdot 6}$ or $c = 12(.43) = 5.16$ units

- HIGHER ORDER (LINEAR) SYSTEMS RESPONSE
  - WE LOOK AT 2^{nd} ORDER SYSTEMS SINCE THEY REPRESENT "BASIS" FOR HIGHER ORDER SYSTEMS
  - ROOTS COULD BE REAL & DISTINCT, REAL & REPEATED, COMPLEX CONJ PAIRS LEADING TO THREE DISTINCT BEHAVIORS

  - HIGHER ORDER SYSTEMS ARE BASICALLY A COMBINATION OF THESE THREE TYPES.
  - CAN STILL USE $x = Ae^{st}$
  - CHARACTERISTIC EQN $a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_1 s + a_0 = 0$

    WHERE $a_1, a_2, \dots, a_n$ are <u>real</u>
  - SOLUTIONS OF FORM

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \dots + A_n e^{s_n t}$$ IF ALL ROOTS ARE REAL & DISTINCT

$$x(t) = (A_1 + A_2 t + A_3 t^2 + \dots + A_n t^{n-1}) e^{st}$$ IF n, real repeated roots

$$x(t) = B_1 e^{-p_1 t} \sin(q_1 t + \phi_1) + B_2 e^{-p_2 t} \sin(q_2 t + \phi_2) + \dots$$

FOR A SYSTEM W/ SETS OF COMPLEX CONJUGATE PAIRS

- TO FIND THE FORCED RESPONSE (OR X_p) USE METHODS WE DISCUSSED PREVIOUSLY

- IN CHARACTERISTIC EQN :
  - IF COMPLEX ROOTS ARE REPEATED ie $s_1 = s_3$, $s_2 = s_4$

$$x(t) = (B_1 + B_2 t)\, e^{-pt} \sin(qt + \phi)$$

- THIS CAN OCCUR FOR SYSTEMS WHICH ARE FOURTH ORDER OR HIGHER

- OFTEN IT IS NECESSARY TO OBTAIN A GENERAL IDEA OF SYSTEM RESPONSE
- TO DO THIS WE DO NOT NEED TO DETERMINE FULL SOL'N
- USE DOMINANT ROOT IDEA TO FIND OUT HOW LOOK BEFORE STEADY STATE IS REACHED

- LOOK AT EXAMPLE PG 153

$$y^{IV} + 24y''' + 225y'' + 900y' + 2500y = f$$

- FOR $y = Ae^{st}$
- CHAR EQ IS $s^4 + 24s^3 + 225s^2 + 900s + 2500 = 0$
- STEADY STATE RESPONSE (OR EQUILIB POINT) IS $2500y = f$

- ROOTS ARE $s = -10 \pm 5i \quad -2 \pm 4i$
- DOMINANT ROOT IS $-2 \pm 4i$ AND $\tau = \frac{1}{2}$

- NOTE THAT FOR $t \geq \frac{4}{10}$ RESPONSE IS GIVEN BY

$$B_2 e^{-2t} \sin(4t + \phi_2) + \frac{f}{2500} \quad \text{FOR } f = \text{STEP FN}$$

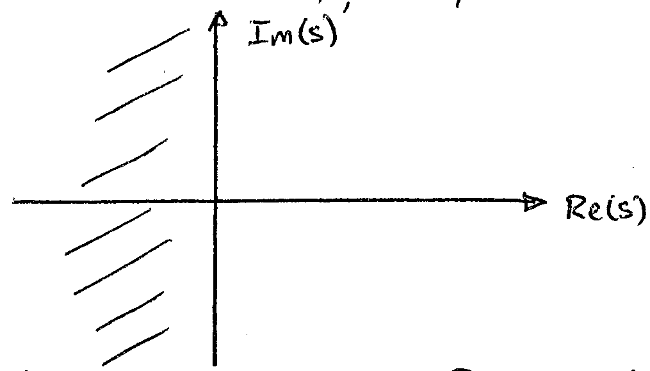
WHY ?

- $s = -2 \pm 4i = -\zeta\omega_n \pm i\omega_d \qquad \omega_d = 4 \qquad \omega_n = \sqrt{(2)^2 + 4^2} = 2\sqrt{5} = 4.47$

and $\zeta = \sqrt{1 - \left(\frac{\omega_d}{\omega_n}\right)^2} = 0.447$

- TO REACH $\sim 2\%$ OF STEADY STATE RESPONSE REQUIRES $t = 4\tau = 2$ sec

- AS MENTIONED PREVIOUSLY, A SYSTEM IS STABLE IF $Re(s) < 0$



- ROUTH HURWITZ IS A CRITERION THAT DEFINES # ROOTS IN RIGHT HALF PLANE
- STABILITY (ROUTH-HURWITZ CRITERION)
  - GIVEN IN APPENDIX F FOR A GENERAL SYSTEM
  - LOOK AT CHARACTERISTIC EQN $D(s) = 0$
- RULE 1 - NECESSARY CONDITION
  - IF ANY COEFFICIENTS OF THE EQUATION ARE ≤ 0 SYSTEM IS NOT STABLE (CAN BE NEUTRALLY STABLE OR UNSTABLE)
  - BETTER - ALL COEFFS MUST BE PRESENT & ALL OF SAME SIGN
  - HOWEVER CONVERSE IS NOT TRUE

EXAMPLE:

$D(s) = s^2 + 3 = 0$ NOT STABLE

$D(s) = s^3 + s^2 + 2s + 8 = 0$ MAY BE STABLE

$D(s) = s^3 + 4s^2 + s - 6 = 0$ UNSTABLE

LESSON #9

- FORM ROUTH ARRAY FOR $a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_2 s^2 + a_1 s + a_0 = 0$

$$
\begin{array}{c|ccccc}
s^n & a_n & a_{n-2} & a_{n-4} & a_{n-6} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & a_{n-7} & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & \cdots & \cdots & \cdots & \cdots & \\
s^2 & e_1 & e_2 & 0 & & \\
s & f_1 & 0 & & & \\
1 = s^0 & g_1 & 0 & & &
\end{array}
$$

where

$$b_1 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}$$

$$b_2 = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}$$

$$\vdots$$

$$b_k = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2k} \\ a_{n-1} & a_{n-(2k+1)} \end{vmatrix}$$

IF CHAR EQ IS EVEN POWER, LAST ELEMENT IN ROW FOR s^{n-1} is 0

$$c_1 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{vmatrix}$$

$$c_2 = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{vmatrix}$$

$$\vdots$$

$$c_k = -\frac{1}{b_1} \begin{vmatrix} a_{n-1} & a_{n-(2k+1)} \\ b_1 & b_{k+1} \end{vmatrix}$$

- IF THERE IS NO b_{k+1} ELEMENT PUT ZERO THERE
- CONTINUE THIS OPN UNTIL ALL ELEMENTS IN ARRAY ARE FOUND

RULE 2

- WHEN THE ARRAY IS COMPLETED THE NUMBER OF ROOTS IN RIGHT HALF PLANE IS EQUAL TO THE NUMBER OF SIGN CHANGES IN THE FIRST COLUMN OF THE ARRAY. IF NO SIGN CHANGES, ALL ROOTS IN LEFT HALF PLANE.

RULE 3 • IF RULE 1 + RULE 2 ARE SATISFIED, YOU HAVE NECESSARY & SUFFICIENT CONDITION FOR STABILITY

EXAMPLE : $r^3 + r^2 + 2r + 8 = 0$

- RULE 1 SATISFIED
- RULE 2

$$\begin{array}{c|cc} r^3 & 1 & 2 \\ r^2 & 1 & 8 \\ r & b_1 & b_2 \\ 1 & c_1 & \end{array}$$

$$b_1 = -\frac{1}{1}\begin{vmatrix} 1 & 2 \\ 1 & 8 \end{vmatrix} = -6$$

$$b_2 = 0$$

$$c_1 = -\frac{1}{-6}\begin{vmatrix} 1 & 8 \\ -6 & 0 \end{vmatrix} = +\frac{48}{6} = 8$$

2 SIGN CHANGE $\Rightarrow$ 2 ROOTS W/ POSITIVE REAL PARTS $\therefore$ UNSTABLE

EXAMPLE : $r^3 + 6r^2 + 11r + 6 = 0$

- RULE 1 SATISFIED

$$\begin{array}{c|cc} r^3 & 1 & 11 \\ r^2 & 6 & 6 \\ r & b_1 & b_2 \\ 1 & c_1 & \end{array}$$

$$b_1 = -\frac{1}{6}\begin{vmatrix} 1 & 11 \\ 6 & 6 \end{vmatrix} = \frac{+60}{6} = 10$$

$$b_2 = 0$$

$$c_1 = -\frac{1}{10}\begin{vmatrix} 6 & 6 \\ 10 & 0 \end{vmatrix} = \frac{-60}{-10} = 6$$

NO SIGN CHANGES $\Rightarrow$ SYSTEM STABLE

EXAMPLE : $r^3 + k_1 r^2 + r + k_2 = 0$

- RULE 1 SATISFIED IF $k_1 > 0$, $k_2 > 0$

$$\begin{array}{c|cc} r^3 & 1 & 1 \\ r^2 & k_1 & k_2 \\ r & b_1 & b_2 \\ 1 & c_1 & \end{array}$$

$$b_1 = -\frac{1}{k_1}\begin{vmatrix} 1 & 1 \\ k_1 & k_2 \end{vmatrix} = \frac{(k_2 - k_1)}{-k_1}$$

$$b_2 = 0$$

$$c_1 = \frac{-1}{-\frac{(k_2 - k_1)}{k_1}}\begin{vmatrix} k_1 & k_2 \\ b_1 & 0 \end{vmatrix} = \frac{-b_1 k_2 \cdot k}{(k_2 - k_1)}$$

$$= k_2$$

FOR STABILITY $k_2 - k_1 < 0 \Rightarrow k_1 > k_2$

SPECIAL CASES

RULE 4 IF A ZERO OCCURS IN THE FIRST COLUMN & REMAINING DERIVED ELEMENTS IN THE ROW ARE NOT ALL ZERO, REPLACE THE ZERO WITH A SMALL POSITVE NUMBER ε & CONTINUE TO FORM ARRAY. WHEN COMPLETED, LET $\varepsilon \to 0$

RULE 5 IF ENTRIES ABOVE & BELOW A ZERO IN FIRST COLUMN HAVE SAME SIGN, A PAIR OF PURE IMAGINARY ROOTS EXISTS. IF THE SIGNS ARE OPPOSITE, THERE IS ONE SIGN CHANGE AND AT LEAST ONE ROOT IN THE RH PLANE

EXAMPLE: $s^3 + 2s^2 + s + 2$ RULE 1 SATISFIED

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s & b_1 & b_2 \\ 1 & c_1 & \end{array}$$

$$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0 \longrightarrow \varepsilon$$

$$b_2 = 0$$

$$c_1 = -\frac{1}{\varepsilon}\begin{vmatrix} 2 & 2 \\ \varepsilon & 0 \end{vmatrix} = \frac{-2\varepsilon}{-\varepsilon} = 2$$

SINCE NO SIGN CHANGE & $b_1 = 0 \Rightarrow$ 2 IMAGINARY ROOTS

$\Rightarrow s^3 + 2s^2 + s + 2 = (s+a)(s^2+b)$ OR $b = 1$ $a = 2$ $\Rightarrow s = \pm i, -2$

- WE CAN DETERMINE THE DEGREE OF INSTABILITY - THE # OF ROOTS IN RIGHT HALF PLANE AND HOW CLOSE THESE ROOTS ARE TO THE IMAGINARY AXIS

RULE 6 IF ANY DERIVED ROW IS IDENTICALLY ZERO, THE SYSTEM IS NOT STABLE. THIS IMPLIES ROOTS ARE LOCATED SYMMETRICALLY ABOUT THE ORIGIN $(\pm r,\ \pm i\omega,\ a \pm ib$ & $-a \pm ib)$

- FORM THE AUXILIARY POLYNOMIAL CONTAINING THESE ROOTS BY USING THE ELEMENTS ABOVE THE ZERO ROW : $p(s)$
- TAKE $\frac{d}{ds}p(s)$. COEFFICIENTS NOW REPLACE ZERO ROW

- CONSTRUCT REST OF ARRAY

EXAMPLE

$$r^3 + 2r^2 + 4r + 8 = 0$$

RULE 1 SATISFIED

r^3	1	4
r^2	2	8
r	b_1	b_2
1	c_1	

$$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & 4 \\ 2 & 8 \end{vmatrix} = 0$$

$$b_2 = 0$$

- SINCE $b_1 = 0$ FORM $p(r) = 2r^2 + 8 = 0$

$$\frac{d}{dr}p = 4r$$

- REPLACE b_1 by 4 AND FIND C_1

$$c_1 = -\frac{1}{4}\begin{vmatrix} 2 & 8 \\ 4 & 0 \end{vmatrix} = 8$$

- THE AUXILIARY POLYNOMIAL IS A FACTOR OF CHAR. EQN

THUS $(2r^2+8)$ or r^2+4 IS A FACTOR OF r^3+2r^2+4r+8

$$(r^2+4)(r+a) = r^3+2r^2+4r+8 \implies a = 2$$

ROOTS ARE $r = -2$ $r = \pm 2i$

- NOTE SINCE NO SIGN CHANGE - ROOTS MUST LIE ON IMAGINARY AXIS

- RULE 7 ANY DERIVED ROW OF THE ARRAY MAY BE MULTIPLIED OR DIVIDED BY A NONZERO POSITIVE NUMBER WITHOUT CHANGING THE RESULT
  - CAN BE USED TO FACILITATE CALCULATIONS

EXAMPLE

$$r^5 + 2r^4 + 24r^3 + 48r^2 + 25r + 50 = 0$$

RULE 1 SATISFIED

r^5	1	24	25
r^4	2	48	50
r^3	b_1	b_2	b_3
r^2	c_1	c_2	
r	d_1	d_2	
1	e_1		

$$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & 24 \\ 2 & 48 \end{vmatrix} = 0$$

$$b_2 = -\frac{1}{2}\begin{vmatrix} 1 & 25 \\ 2 & 50 \end{vmatrix} = 0$$

$$b_3 = 0$$

- since b_1, b_2 are ZERO

$$p(r) = 2r^4 + 48r^2 + 50$$

$$p'(r) = 8r^3 + 96r$$

- REPLACE b_1, b_2 by 8 & 96 or 1 12

$$c_1 = -\frac{1}{1}\begin{vmatrix} 2 & 48 \\ 1 & 12 \end{vmatrix} = +24$$

$$c_2 = -\frac{1}{1}\begin{vmatrix} 2 & 50 \\ 1 & 0 \end{vmatrix} = +50$$

$$d_1 = \frac{-1}{24}\begin{vmatrix} 1 & 12 \\ 24 & 50 \end{vmatrix} = +9.917$$

$$d_2 = 0$$

$$e_1 = -\frac{1}{9.917}\begin{vmatrix} 24 & 50 \\ 9.917 & 0 \end{vmatrix} = 50$$

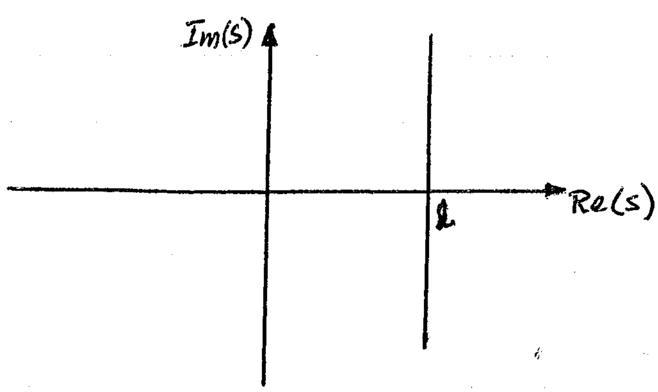
- BECAUSE OF THE ORIGINAL ROW, SYSTEM WAS UNSTABLE OR MARINALLY STABLE
- SINCE NO SIGN CHANGES $\Longrightarrow$ ROOTS ON IMAGINARY AXIS

** • THOSE ROOTS ARE ROOTS OF $p(r) = 2r^4 + 48r^2 + 50 = 0$

$$2(r^4 + 24r^2 + 25) = 0$$

$$2(r^2 + 1.0913)(r^2 + 22.909) = 0$$

- IN ORDER TO DETERMINE THE NUMBER OF ROOTS TO THE RIGHT OF A GIVEN VERTICAL LINE, REPLACE s by $\bar{s} + l$ IN THE CHAR EQUATION AND DO THE ROUTH-HURWITZ ARRAY



EXAMPLE 3.11 pg. 155

FOR $s^3+9s^2+26s+K=0$, FIND THE VALUE OF REQUIRED SO THAT THE DOMINANT TIME CONSTANT IS NO LARGER THAN $\frac{1}{2}$

$$\tau = \frac{1}{2} \Rightarrow Re(s) = \frac{-1}{\tau} = -2 \quad \text{or} \quad \ell = -2$$

REPLACE s by $\bar{s}-2$; s^2 by $(\bar{s}-2)^2 = \bar{s}^2-4\bar{s}+4$

s^3 by $(\bar{s}-2)^3 = \bar{s}^3-6\bar{s}^2+12\bar{s}-8$

$$s^3+9s^2+26s+K = (\bar{s}^3-6\bar{s}^2+12\bar{s}-8)+9(\bar{s}^2-4\bar{s}+4)+26(\bar{s}-2)+K =$$

$$\Rightarrow \bar{s}^3+3\bar{s}^2+2\bar{s}+K-24$$

$$\begin{array}{c|cc}
s^3 & 1 & 2 \\
s^2 & 3 & K-24 \\
s & b_1 & b_2 \\
1 & c_1 &
\end{array}$$

$$b_1 = -\frac{1}{3}\begin{vmatrix} 1 & 2 \\ 3 & K-24 \end{vmatrix} = \frac{K-24-6}{-3} = \frac{K-30}{-3}$$

$K-30<0$ or $K<30$

$b_2 = 0$

$$c_1 = \frac{-1}{(K-30)/-3}\begin{vmatrix} 3 & K-24 \\ \frac{K-30}{-3} & 0 \end{vmatrix}$$

$$= \frac{3}{K-30}\cdot\left[-\frac{(K-30)}{-3}(K-24)\right]$$

$$= K-24>0 \Rightarrow K>24$$

$\therefore 24<K<30$

STABILITY REVISITED

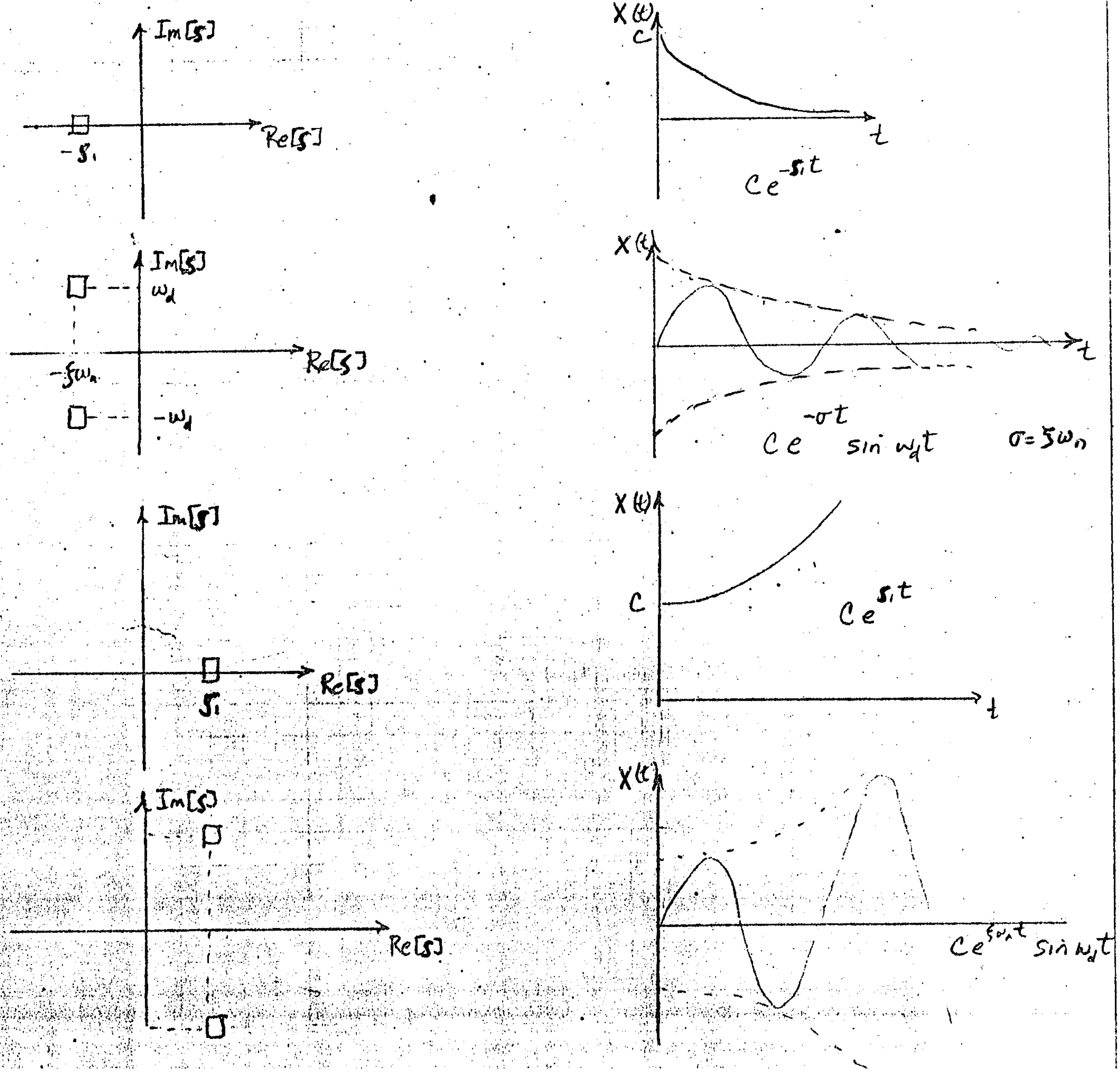
STABILITY IS A PROPERTY OF UNFORCED RESPONSE. IF AN UNFORCED SYSTEM IS UNSTABLE SO IS THE FORCED SYSTEM. (SUPERPOSITION)

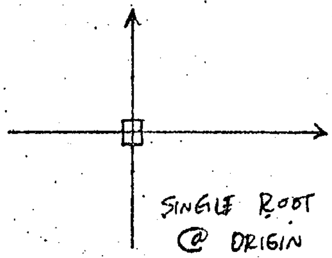
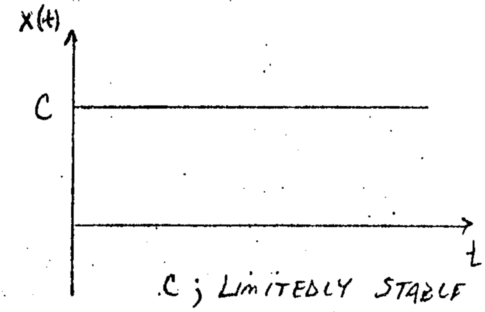
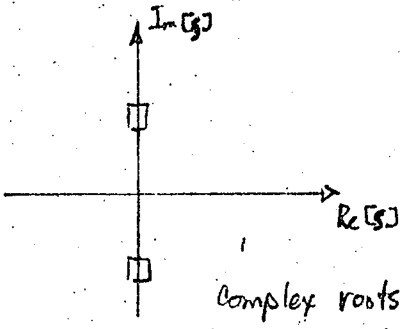
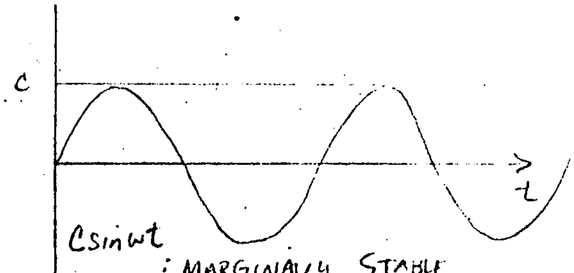
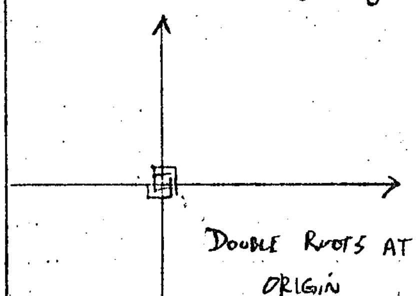
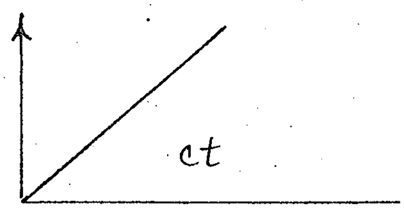
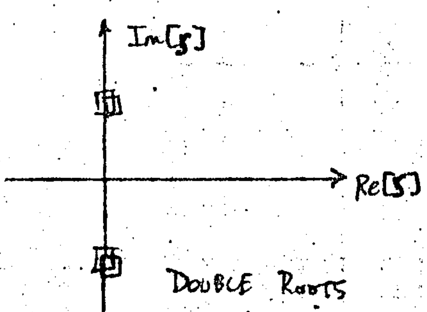
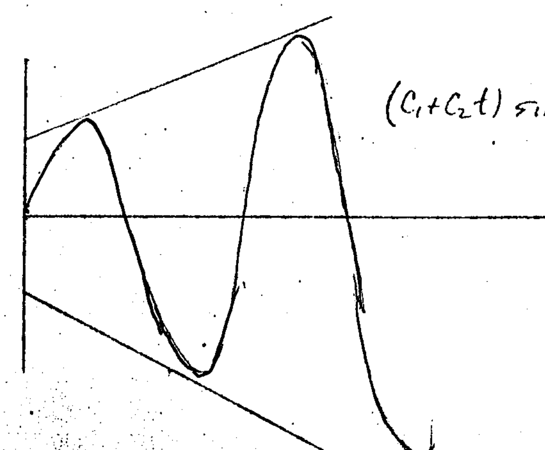
STABILITY IS DETERMINED BY THE ROOTS OF CHAR. EQN.

$$a_n s^n + a_{n-1} s^{n-1} + \dots + a_0 = 0$$

ROOTS MAY BE :
(1) REAL DISTINCT
(2) Complex Conjugates
(3) REPEATED

ROOTS | TRANSIENT MODE SHAPE



ROOTS	TRANSIENT MODE SHAPE
 SINGLE ROOT @ ORIGIN	 C ; LIMITEDLY STABLE
 Complex roots on Imaginary Axis	 $C\sin\omega t$: MARGINALLY STABLE
 DOUBLE ROOTS AT ORIGIN	 ct ; UNSTABLE
 DOUBLE ROOTS ON IMAGINARY AXIS	 $(C_1+C_2t)\sin\omega t$

A SYSTEM IS STABLE IF ALL ROOTS LIE IN LEFT-HALF-PLANE

A SYSTEM IS UNSTABLE IF ANY ROOT LIES IN RIGHT HALF-PLANE, OR MULTIPLE ROOT LIE @ ORIGIN OR IMAGINARY AXIS.

A SYSTEM IS LIMITEDLY STABLE IF A SINGLE ROOT LIES AT ORIGIN, AND ALL OTHERS IN LHP.

A SYSTEM IS MARGINALLY STABLE IF ONLY SINGLE PAIRS OF COMPLEX ROOTS LIE ON IMAGINARY AXIS AND ALL OTHER ROOTS ARE IN LHP.

Routh - Hurwitz Array

Summary

1. If no element in the first column is zero

    The number of roots of the characteristic equation with positive real parts is equal to the number of sign changes in the first column of the array.

2. Zeros in the first column, but no row of all zero elements

    If the sign of the element above the zero (ε) is the same sign as the element below it, it indicates that there are a pair of imaginary roots of the characteristic equation.

    If the sign of the element above the zero (ε) is opposite to that of the element below it, it indicates that there is one sign change (see 1.) above).

3. A row with all elements zero

    Indicates that there are roots of the characteristic equation of equal magnitude lying radially opposite in the s plane, i.e.,

    a) two real roots with equal magnitude, but opposite sign

    b) two conjugate imaginary roots

    c) two pair of complex conjugate roots symmetrically located about the origin of the s plane.

LINEARIZATION REVISITED

- THERE ARE MORE NONLINEAR MODELS THAN LINEAR MODELS
- PREDICTION OF THEIR BEHAVIOR IS MORE COMPLICATED; DEPENDENT ON MAGNITUDE OF INPUT & INITIAL CONDITIONS
- WE DESIRE LINEAR MODELS SINCE THEIR BEHAVIOR IS KNOWN (STABILITY DEPENDS ON CHARACTERISTIC EQN & RESPONSE IS DEPENDENT ON INPUT FORM)
- WE TRY TO LINEARIZE WHENEVER POSSIBLE ABOUT AN OPERATING POINT

LINEARIZATION METHODS p. 629–630

- SUPPOSE WE WANT TO LINEARIZE A FN $f(y,v)$ ABOUT $y=y_e$ & $v=v_e$

$$f(y,v) = f(y_e,v_e) + \left.\frac{\partial f}{\partial y}\right|_{y=y_e} \cdot (y-y_e) + \left.\frac{\partial f}{\partial v}\right|_{v=v_e} (v-v_e) + \text{HIGHER ORDER TERMS}$$

USING TAYLOR EXPANSION

- HIGHER ORDER TERMS ARE FNS OF $(y-y_e)^n$ & $(v-v_e)^m$
- LINEARIZATION THROWS OUT HIGHER ORDER TERMS IF $y-y_e \sim 0$, $v-v_e \sim 0$

** • LINEARIZATION IS USED TO INVESTIGATE LOCAL STABILITY

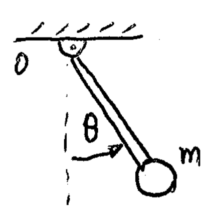
- LINEARIZED MODELS USED IN DESIGNING FEEDBACK CONTROL SYSTEMS

  WHY – IF CONTROLLER KEEPS SYSTEM NEAR OPERATING PT THIS IS SAME AS LINEARIZATION PROCESS, ESPECIALLY IF WE ARE LOCALLY STABLE. PERTURBATIONS WILL BE MINIMIZED

- HOW TO LINEARIZE : TWO METHODS

  1) FIRST LINEARIZE ALL NONLINEAR FNS & THEN DEVELOP DIFF EQS
  2) DEVELOP DIFF EQS THEN LINEARIZE

EXAMPLE



$$I_0 \ddot{\theta} = -mgL\sin\theta \qquad I_0 = mL^2$$

OPERATING PT IS $\theta = 0$

$\therefore \sin\theta \sim \theta$

SECOND METHOD

$$\text{if } \frac{d^n x}{dt^n} + a_{n-1}(x,y,t)\frac{d^{n-1}x}{dt^{n-1}} + a_{n-2}(x,y,t)\frac{d^{n-2}x}{dt^{n-2}} + \dots + a_0(x,y,t) = f(x,y,t)$$

REWRITE AS

$$x^{(n)} = f(x,y,t) - a_{n-1}x^{(n-1)} - a_{n-2}x^{(n-2)} - \dots - a_0(x,y,t)$$
$$= g(x,y,t,\ x^{(n-1)}, x^{(n-2)} \dots, x')$$

NOW TAKE

$$\left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_0\\ y=y_0\\ \vdots}}(x-x_0) + \left.\frac{\partial g}{\partial y}\right|_{y=y_0}(y-y_0) + \left.\frac{\partial g}{\partial t}\right|_{t=t_0}(t-t_0) + \left.\frac{\partial g}{\partial x^{(n-1)}}\right|_{x^{(n-1)}=x_0^{(n-1)}}(x^{(n-1)}-x_0^{(n-1)}) + \dots$$

EXAMPLE $\quad \dot{y} + 2y - 3x^2 = 0 \quad \Rightarrow \quad \dot{y} = (-2y + 3x^2) = g(x,y)$

$$\left.\frac{\partial g}{\partial x}\right|_{x=x_0} = 6x\Big|_{x=x_0} = 6x_0 \quad ; \quad \left.\frac{\partial g}{\partial y}\right|_{y=y_0} = -2$$

$$\therefore \quad \dot{y} = \left.\frac{\partial g}{\partial x}\right|_{x=x_0}(x-x_0) + \left.\frac{\partial g}{\partial y}\right|_{y=y_0}(y-y_0) = 6x_0(x-x_0) - 2(y-y_0)$$

Now since $\dot{(y-y_0)} = \dot{y}$ $\quad$ LET $\quad y-y_0 = \Upsilon \quad x-x_0 = \Xi$

$$\dot{\Upsilon} = 6x_0\,\Xi - 2\Upsilon$$

EXAMPLE : $\quad \ddot{x} + y\dot{x} + x^2 + xy + \dot{y}/z + y^2 z = 0$

$$\ddot{x} = -y\dot{x} - x^2 - xy - \dot{y}/z - y^2 z = g(x,y,z,\dot{x},\dot{y})$$

$$\left.\frac{\partial g}{\partial x}\right|_{x=x_0} = (-2x-y)_{\substack{x=x_0\\ y=y_0\\ \dot{x}_0\\ \vdots}} \quad ; \quad \left.\frac{\partial g}{\partial y}\right|_{y=y_0} = -\dot{x}_0 - x_0 - 2y_0 z_0$$

$$\frac{\partial g}{\partial z} = +\frac{\dot{y}_0}{z_0^2} - y_0^2 \qquad \frac{\partial g}{\partial \dot{x}} = -y_0 \qquad \frac{\partial g}{\partial \dot{y}} = -\frac{1}{z_0}$$

$$\ddot{\Xi} = -(2x_0 + y_0)\Xi - (\dot{x}_0 + x_0 + 2y_0 z_0)\Upsilon + \left(\frac{\dot{y}_0}{z_0^2} - y_0^2\right)Z - y_0\dot{\Xi} - \frac{1}{z_0}\dot{\Upsilon}$$

where $\Xi = x - x_0$, $\dot{\Xi} = \dot{x}$, $\ddot{\Xi} = \ddot{x}$, $\Upsilon = y - y_0$, $\dot{\Upsilon} = \dot{y}$, $Z = z - z_0$.

LAPLACE TRANSFORMS

- METHOD TO SOLVE THE INHOMOGENEOUS L.O.D.E & INITIAL CONDITIONS WITH RESORTING TO HOMOGENEOUS + PARTICULAR SOLNS
- METHOD LEADS TO AN ALGEBRAIC EQN THAT YIELDS OUTPUT
- THIS METHOD WILL ALLOW US TO PORTRAY THE SYSTEM STRUCTURE BY MEANS OF SIMPLIFIED BLOCK DIAGRAMS
- PARTICULARLY USEFUL
  - WHEN INPUTS ARE DISCONTINUOUS OR IMPULSIVE
  - WHEN INPUTS ARE PERIODIC BUT <u>NOT</u> SINUSOIDAL ONLY

- <u>DFN</u> IF $f(t)$ IS DEFINED FOR $t \geq 0$, AND IF

$$\int_0^\infty e^{-st} f(t)\,dt \quad \text{EXISTS,}$$

THEN

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

IS THEN LAPLACE TRANSFORM OF $f(t)$

$$F(s) = \mathcal{L}\{f(t)\}$$

ALSO

$$f(t) = \mathcal{L}^{-1}\{F(s)\} = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)\,e^{st}\,ds$$

- EXAMPLE: $f(t) = u_s(t) = \begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases}$

$$F(s) = \int_0^\infty e^{-st}\cdot u_s(t)\,dt = \left.\frac{e^{-st}}{-s}\right|_0^\infty = \frac{1}{s}$$

- NOTE THAT THIS TRANSFORM IS LINEAR

$$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$$

ALSO $\mathcal{L}\{\text{CONSTANT}\} = \dfrac{\text{CONSTANT}}{s}$

EXAMPLE: $f(t) = e^{at}, \quad t \geq 0 \qquad a = \text{constant}$

$$F(s) = \int_0^\infty e^{-st} e^{at}\,dt = \int_0^\infty e^{-(s-a)t}\,dt = -\frac{1}{s-a} e^{-(s-a)t}\Big|_0^\infty$$

$$= \frac{1}{s-a}$$

ALSO $\mathcal{L}^{-1}\left\{\dfrac{1}{s-a}\right\} = e^{at}$

EXAMPLE: SHOW THAT $\mathcal{L}\{t^{-1/2}\} = \sqrt{\pi/s}$

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_0^\infty e^{-st} t^{-1/2}\,dt = \frac{1}{\sqrt{s}} \int_0^\infty e^{-x} x^{-1/2}\,dx \qquad \text{USING } x = st$$

BUT $\displaystyle\int_0^\infty e^{-x} x^{-1/2}\,dx = \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

$$\therefore \quad \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}$$

LAPLACE TRANSFORMS OF DERIVATIVES & INTEGRALS

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_0^\infty e^{-st} \frac{df}{dt}\,dt \qquad \text{INTEGRATE BY PARTS}$$

LET $u = e^{-st} \quad dv = \dfrac{df}{dt}dt \quad \Rightarrow du = -s e^{-st}dt, \; v = f$

$$\int_0^\infty u\,dv = uv\Big|_0^\infty - \int_0^\infty v\,du \quad \Rightarrow \quad f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)\left[-s e^{-st} dt\right]$$

$$\therefore \mathcal{L}\left\{\frac{df}{dt}\right\} = -f(0) + s\int_0^\infty f(t) e^{-st} dt = sF(s) - f(0) = s\mathcal{L}\{f(t)\} - f(0)$$

NOTE: THIS IS TRUE IF $f(t)$ is bounded or grows slower than e^{+st} as $t \to \infty$

- MORE ON LAPLACE TRANSFORMS IN APPENDIX B.

- OTHER USEFUL FORMULAS

$$\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = s^2F(s) - sf(0) - \frac{d}{dt}f(0)$$

$$\mathcal{L}\left\{\frac{d^nf}{dt^n}\right\} = s^nF(s) - s^{n-1}f(0) - s^{n-2}\frac{d}{dt}f(0) - \dots - \frac{d^{n-1}}{dt^{n-1}}f(0)$$

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)d\tau$$

$$\mathcal{L}\{t\} = 1/s^2$$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

$$\mathcal{L}\{\sin\omega t\} = \frac{\omega}{s^2+\omega^2}$$

- ~~[illegible]~~ FINAL VALUE THEOREM - ONLY FOR STABLE SYSTEMS

$$\lim_{t\to\infty} f(t) = f_{ss} = \lim_{s\to 0} sF(s)$$

- INITIAL VALUE THEOREM - USEFUL IN INVERSE TRANSFORMS

$$\lim_{t\to 0^+} f(t) = f(0^+) = \lim_{s\to\infty} sF(s)$$

- APPLICATION TO SOLN'S OF DIFFERENTIAL EQS

$$\dot{y} + ry = f(t) \qquad f\text{-input} \qquad y\text{-output}$$

(1) TAKE TRANSFORM OF BOTH SIDES

$$\mathcal{L}\{\dot{y}\} + r\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$

$$sY(s) - y(0) \quad + rY(s) = F(s)$$

NOTE THIS IS AN ALGEBRAIC EQN

(2) SOLVE FOR $Y(s)$ – THE LAPLACE TRANSFORM OF OUTPUT

$$Y(s) = \frac{F(s) + y(0)}{s+r} = \frac{F(s)}{s+r} + \frac{y(0)}{s+r}$$

(3) TAKE $\mathcal{L}^{-1}$ OF EACH PARTIAL FRACTION

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) \quad , \quad \mathcal{L}^{-1}\left\{\frac{y(0)}{s+r}\right\} = y(0) \cdot e^{-rt}$$

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s+r}\right\} = \int_0^t e^{-r\tau} f(t-\tau)\, d\tau$$

p. 648 TABLE, FORMULA 9

$$\therefore \quad y(t) = \underbrace{y(0)\, e^{-rt}}_{\text{FREE RESPONSE}} + \underbrace{\int_0^t e^{-r\tau} f(t-\tau)\, d\tau}_{\text{FORCED RESPONSE}}$$

SUMMARY

- GIVEN $\ddot{y} + 4\dot{y} + 3y = 0$ WITH $y(0) = 3$, $\dot{y}(0) = 1$ (t INDEPENDENT VARIABLE)
- USE LAPLACE TRANSFORM TO FIND $Y(s)$

$$\left(s^2 Y - s \cdot 3 - 1\right) + 4\left(sY - 3\right) + 3Y = 0$$

$$Y\left(s^2 + 4s + 3\right) = 3s + 13$$

$$Y = \frac{3s+13}{s^2+4s+3}$$

- TAKE RHS AND WRITE AS PARTIAL FRACTION EXPANSION (SEE APPENDIX B.3)

$$\frac{3s+13}{s^2+4s+3} = \frac{A}{s+3} + \frac{B}{s+1}$$

& SOLVE FOR A & B $(A = -2 \;,\; B = 5)$

$$Y(s) = \frac{-2}{s+3} + \frac{5}{s+1}$$

- TAKE INVERSE LAPLACE TRANSFORM

$$y(t) = -2\mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = -2e^{-3t} + 5e^{-t}$$

- ASIDE – NEED TO KNOW PARTIAL FRACTION EXPANSION

$$\frac{s+5}{s^2+4s+3} = \frac{A}{s+3} + \frac{B}{s+1}$$ IF DENOM COMPLETELY FACTORABLE & FACTORS NOT SAME

$$\frac{s+5}{s^2+4s+4} = \frac{s+5}{(s+2)^2} = \frac{A}{(s+2)^2} + \frac{B}{(s+2)}$$ FACTORS SAME

$$\frac{s+5}{(s+1)(s^2+4s+5)} = \frac{A}{s+1} + \frac{Bs+C}{(s^2+4s+5)}$$ DENOM NOT FACTORABLE

LESSON # 11

TRANSFER FN

- TRANSFER FUNCTION OF A SYSTEM IS CONCERNED WITH FORCED RESPONSE OF SYSTEM UNDER ZERO INITIAL CONDITIONS

FOR THE EQN $\dot{y} + ry = f(t)$ WE FOUND

$$Y(s) = \frac{F(s)}{s+r} + \frac{y(0)}{s+r}$$

FOR THE TRANSFER FNS, TAKE $y(0) = 0$ THUS

$$\frac{Y(s)}{F(s)} = \frac{1}{s+r} = T(s)$$ TRANSFER FN

THUS THE SYSTEM TRANSFER FN $T(s)$ IS THE RATIO OF THE OUTPUT TO THE INPUT AFTER LAPLACE TRANSFORMATION

$$T(s) = \frac{\text{TRANSFORMED OUTPUT}}{\text{TRANSFORMED INPUT}}$$

NOTE THAT $T(s)$ HAS THE CHARACTERISTIC POLYNOMIAL AS ITS DENOMINATOR

- THUS WITH THE TRANSFER FN ONLY, WE CAN DETERMINE STABILITY & TIME CONSTANTS
- FOR 2^{nd} ORDER SYSTEM, THE TRANSFER FN GIVES US, DAMPING RATIO & DAMPED & UNDAMPED NATURAL FREQUENCIES ALSO.

- FOR A SYSTEM WITH MULTIPLE INPUTS, SET ALL INPUTS, EXCEPT THE ONE TO BE STUDIED, EQUAL TO ZERO, AND FIND THE TRANSFER FN AS BEFORE

- <u>IN GENERAL</u> (APPENDIX B.3)

  IF GIVEN A LINEAR ODE WITH CONST. COEFF

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_1 \frac{dx}{dt} + a_0 x = f(t)$$

THEN USE LAPLACE TRANSFORM TO GET

$$\underbrace{\left(a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0\right)}_{\text{CHARACTERISTIC POLYNOMIAL} \, = P(s)} X(s) = I(s) + F(s)$$

- $I(s)$ IS A FN TOTALLY DEPENDENT ON THE INITIAL CONDITIONS & THE COEFFS' $a_n, a_{n-1}, \dots, a_1, a_0$

- $F(s)$ IS A FN DEPENDENT OF THE FORCED RESPONSE $F(s) = \int_0^\infty e^{-st} f(t)\, dt$
- $X(s)$ IS THE OUTPUT $X(s) = \int_0^\infty e^{-st} x(t)\, dt$

- WE CAN WRITE

$$X(s) = \frac{I(s)}{P(s)} + \frac{F(s)}{P(s)}$$

AND

$$x(t) = \underbrace{\mathcal{L}^{-1}\left\{\frac{I(s)}{P(s)}\right\}}_{\text{DUE TO INITIAL CONDITIONS}} + \underbrace{\mathcal{L}^{-1}\left\{\frac{F(s)}{P(s)}\right\}}_{\text{DUE TO FORCED RESPONSE}}$$

THUS THE TRANSFER FN DUE TO FORCED RESPONSE w/ ZERO INITIAL CONDITIONS IS

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{P(s)}$$

MORE GENERALLY WE WILL HAVE $P(s)\,X(s) = W(s)\,F(s)$

AND $W(s)$ IS A FN THAT APPEAR IF INPUTS INCLUDE DERIVATIVES

$$\Rightarrow \quad T(s) = \frac{X(s)}{F(s)} = \frac{W(s)}{P(s)} = \frac{N(s)}{D(s)}$$

IF $P(s)$ & $W(s)$ HAVE SOME COMMON FACTORS, THEN $T(s) = \frac{N(s)}{D(s)}$

EXAMPLE Problem 4.2a pg 240

$$\ddot{x} + 8\dot{x} + 12x = 2 \quad w/\ x(0) = 0 \quad \dot{x}(0) = 1$$

$$\mathcal{L}\{\ddot{x} + 8\dot{x} + 12x\} = \mathcal{L}\{2\}$$

$$s^2 X - s\,x(0) - \dot{x}(0) + 8\left[sX - x(0)\right] + 12X = \frac{2}{s}$$

$$\{s^2 + 8s + 12\}\,X = 1 + \frac{2}{s}$$

$$P \qquad X = I + F$$

$$X = \frac{1}{s^2+8s+12} + \frac{2}{s(s^2+8s+12)} = \frac{1}{(s+2)(s+6)} + \frac{2}{s(s+2)(s+6)}$$

- FREE RESPONSE IS GIVEN BY $\frac{I(s)}{P(s)} = \frac{1}{(s+2)(s+6)}$

$$\frac{1}{(s+2)(s+6)} = \frac{A}{s+2} + \frac{B}{s+6}$$

$$A = \lim_{s \to -2} \frac{s+2}{(s+2)(s+6)} = (s+2)\frac{I(s)}{P(s)}$$

$$= +\frac{1}{4}$$

$$B = \lim_{s \to -6} (s+6)\frac{I(s)}{P(s)} = \lim_{s \to -6} \frac{1}{s+2}$$

$$= -\frac{1}{4}$$

THUS $x(t) = \mathcal{L}^{-1}\left\{\frac{1/4}{s+2}\right\} + \mathcal{L}^{-1}\left\{\frac{-1/4}{s+6}\right\}$

$$x(t) = \tfrac{1}{4}e^{-2t} - \tfrac{1}{4}e^{-6t}$$

FREE RESPONSE

- FORCED RESPONSE IS GIVEN BY $F(s)/P(s) = \frac{2}{s(s+2)(s+6)}$

$$\frac{2}{s(s+2)(s+6)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+6}$$

$$A = \lim_{s \to 0} \frac{s \cdot F(s)}{P(s)} = \frac{2}{12} = \frac{1}{6} \; ; \quad B = \lim_{s \to -2} \frac{(s+2)F(s)}{P(s)} = \frac{2}{-2 \cdot 4} = -\frac{1}{4}$$

$$C = \lim_{s \to -6} \frac{(s+6)F(s)}{P(s)} = \frac{2}{-6 \cdot -4} = +\frac{1}{12}$$

$$\therefore \quad \mathcal{L}^{-1}\left\{\frac{F(s)}{P(s)}\right\} = \mathcal{L}^{-1}\left\{\frac{1/6}{s} + \frac{-1/4}{s+2} + \frac{1/12}{s+6}\right\}$$

$$x(t) = \frac{1}{6} - \frac{1}{4}e^{-2t} + \frac{1}{12}e^{-6t}$$ FORCED RESPONSE

SOLUTION IS THE FORCED + FREE RESPONSE

- TO RECOVER THE DIFFERENTIAL EQN, GIVEN TRANSFER FN

$$T(s) = \frac{X(s)}{F(s)} = \frac{N(s)}{D(s)}$$

$$\therefore D(s)X(s) = N(s)F(s) \quad \text{\&} \quad \mathcal{L}^{-1}\{D(s)X(s)\} = \mathcal{L}^{-1}\{N(s)F(s)\}$$

FOR ZERO INITIAL CONDITIONS

NOTE THAT $s^n X(s) = \mathcal{L}\left\{\frac{d^n x}{dt^n}\right\}$

EXAMPLE GIVEN $y(t)$ INPUT ; $x(t)$ OUTPUT & $T(s) = \frac{X(s)}{Y(s)}$

$T(s) = \frac{s^2}{(2s+1)(5s+1)} = \frac{N(s)}{D(s)}$; FIND THE GOVERNIN EQN

$$\frac{X}{Y} = \frac{s^2}{10s^2+7s+1} \Rightarrow (10s^2+7s+1)X = s^2 Y$$

$$10\ddot{x} + 7\dot{x} + x = \ddot{y}$$

- WHEN THE NUMERATOR OF THE TRANSFER FN IS NOT CONSTANT

  WE CANNOT RELY ONLY ON THE CHARACTERISTIC POLYNOMIAL TO PREDICT

  THE SYSTEM RESPONSE ONLY

- MORE ON THIS TOPIC LATER

- MORE ON PARTIAL FRACTIONS & TRANSFER FNS (APP. B-3 pp 650)
- MOST TRANSFORMS OCCUR AS RATIO OF POLYNOMIALS
- GIVEN $T(s) = \dfrac{N(s)}{D(s)} = \dfrac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_2}{s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$

- FOR REAL SYSTEMS $m \leq n$

- SOLVE FOR ROOTS OF DENOMINATOR POLYNOMIAL

  ROOTS MAY BE : REAL & DISTINCT , REPEATED OR COMPLEX CONJUGATES

- IF ROOTS ARE REAL & DISTINCT

$$T(s) = \frac{N(s)}{(s+r_1)(s+r_2)\cdots(s+r_n)} = \frac{C_1}{s+r_1} + \frac{C_2}{s+r_2} + \dots + \frac{C_n}{s+r_n}$$

$$C_i = \lim_{s \to -r_i} (s+r_i)T(s) \quad \& \quad \mathcal{L}^{-1}\{T\} = \sum_{i=1}^{n} C_i e^{-r_i t}$$

- FOR REPEATED ROOTS : p repeated & $n-p$ real & distinct

$$T(s) = \frac{N(s)}{(s+r_1)^p (s+r_2)(s+r_3)\cdots(s+r_{n-p})}$$

$$= \frac{C_1}{(s+r_1)^p} + \frac{C_2}{(s+r_1)^{p-1}} + \dots + \frac{C_p}{(s+r_1)} + \frac{D_2}{s+r_2} + \frac{D_3}{s+r_3} + \dots \frac{D_{n-p}}{s+r_{n-p}}$$

$$C_i = \lim_{s \to -r_1} \left\{ \frac{1}{(i-1)!} \frac{d^{i-1}}{ds^{i-1}} \left[T(s)(s+r_1)^p \right] \right\} \quad i = 1, 2, \dots, p$$

$$D_j = \lim_{s \to -r_j} T(s)(s+r_j) \qquad j = 2, 3, \dots, n-p$$

$$\mathcal{L}^{-1}\{T(s)\} = \sum_{i=1}^{p} C_i \frac{t^{p-i}}{(p-i)!} e^{-r_1 t} + D_2 e^{-r_2 t} + D_3 e^{-r_3 t} + \dots + D_{n-p} e^{-r_{n-p} t}$$

EXAMPLE: $T(s) = \dfrac{s}{(s+1)(s+2)^2}$

$$T(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2} \qquad p=2$$

$$A = \lim_{s\to -1} (s+1)T(s) = \left.\frac{s}{(s+2)^2}\right|_{s=-1} = \frac{-1}{1^2} = -1$$

$$B = \lim_{s\to -2} \left\{ \frac{1}{(i-1)!} \frac{d^{i-1}}{ds^{i-1}} [T(s+2)^2] \right\} = \frac{1}{1!} \frac{d}{ds}\left(\frac{s}{s+1}\right) = 1 \cdot \left\{ \frac{-s}{(s+1)^2} + \frac{1}{s+1} \right\}_{s=-2}$$

$$= \frac{2}{(-1)^2} + \frac{1}{(-1)} = 1$$

$$C = \lim_{s\to -2} \left\{ \frac{1}{(1-1)!} \frac{d^0}{ds^0} [T(s)(s+2)^2] \right\} = \left.\frac{1}{0!} \frac{s}{s+1}\right|_{s=-2} = \frac{-2}{-1} = 2$$

$$\therefore \quad T(s) = \frac{-1}{s+1} + \frac{1}{s+2} + \frac{2}{(s+2)^2}$$

$$\mathcal{L}^{-1}\{T(s)\} = -1 \cdot e^{-t} + e^{-2t} + 2te^{-2t}$$

- WHAT IF ROOTS ARE COMPLEX CONJUGATES

LET $T(s) = \dfrac{N(s)}{D(s)} = \dfrac{P(s)}{Q(s)}$ WHERE $Q(s)$ IS $(s+p+iq)(s+p-iq)$

$$= (s+p)^2 + q^2$$

$P(s)$ CONTAINS THE REST OF THE TRANSFORM

$$T(s) = \frac{N(s)}{D(s)} = \frac{A}{s+r_1} + \frac{B}{s+r_2} + \dots + \frac{Cs+D}{s^2+as+b} + \dots$$

$$\frac{Cs+D}{s^2+as+b} = \frac{C[s+a/2] + D - C[a/2]}{(s+a/2)^2 + (b - a^2/4)}$$

HERE $p = \frac{a}{2}$; $q = \sqrt{b - \frac{a^2}{4}}$

$$\mathcal{L}^{-1}\left\{\frac{Cs+D}{s^2+as+b}\right\} = \frac{C}{\sqrt{b-\frac{a^2}{4}}}\, e^{-\frac{a}{2}t}\sin\left(\sqrt{b-\frac{a^2}{4}}\right)t + \left\{D - C\left[a/2\right]\right\}e^{-\frac{at}{2}}\cos\sqrt{\left(b-\frac{a^2}{4}\right)}t$$

IF $a^2 - 4b < 0$

- HERE $Q(s) = s^2+as+b$, & $T(s)Q(s) = P(s)$
- ALSO $\dfrac{N(s)}{D(s)} = \dfrac{P(s)}{Q(s)}$ $\therefore$ $\dfrac{N(s)}{D(s)} \cdot \dfrac{1}{P(s)} = \dfrac{1}{Q(s)}$

EXAMPLE 4.15 e p. 245

$$T(s) = \frac{4}{s(s^2+10s+100)} = \frac{N(s)}{D(s)}\ ; \quad P(s) = \frac{4}{s} \qquad Q(s) = s^2+10s+100$$

$$p = \frac{a}{2} = 5 \quad q = \sqrt{100-25} = 5\sqrt{3}$$

$$T(s) = \frac{A}{s} + \frac{Bs+D}{s^2+10s+100} \qquad A = \lim_{s\to 0} sT(s) = \frac{1}{25}$$

$$A(s^2+10s+100) + (Bs+D)s = 4$$

also $A+B=0$ & $10A+D=0$ $\quad\therefore\quad B = -1/25 \quad D = -2/5$

$$\therefore \mathcal{L}^{-1}\left\{\frac{Bs+D}{s^2+10s+100}\right\} = \frac{-1}{25\left[\sqrt{75}\right]} e^{-5t}\sin\sqrt{75}t + \left\{-2/5 + \frac{1}{25}(5)\right\} e^{-5t}\cos\sqrt{75}t$$

$$\mathcal{L}^{-1}\left\{\frac{A}{s}\right\} = 4$$

$$\& \quad \mathcal{L}^{-1}\{T(s)\} = 4 - \frac{1}{125\sqrt{3}} e^{-5t}\sin 5\sqrt{3}t - \frac{1}{5}e^{-5t}\cos 5\sqrt{3}t$$

LESSON #12

- BACK TO TRANSFER FNS & BLOCK DIAGRAMS
  - $T(s)$ GIVES RELATION BETWEEN INPUT & OUTPUT

$$T(s) = \frac{\text{OUTPUT}}{\text{INPUT}} = \frac{X(s)}{Y(s)} \quad \text{OR} \quad X(s) = T(s)Y(s)$$

- BLOCK DIAGRAMS GIVE GRAPHICAL CAUSE/EFFECT RELATIONSHIP

$$Y(s) \longrightarrow \boxed{T(s)} \longrightarrow X(s)$$

EXAMPLE $f = m\frac{d^2x}{dt^2} \Rightarrow F(s) = ms^2 X(s)$

REMEMBER $T(s)$ IS FOR ZERO INITIAL CONDITIONS

- IF x is input & f is output $\frac{F(s)}{X(s)} = ms^2 = T(s)$

$$X(s) \longrightarrow \boxed{ms^2} \longrightarrow F(s)$$

IF f is input & x is output $\frac{X(s)}{F(s)} = \frac{1}{ms^2} = T(s)$

$$F(s) \longrightarrow \boxed{\frac{1}{ms^2}} \longrightarrow X(s)$$

- LOOK AT p. 187 FIGURE 4.1. ARROWS INDICATE DIRECTION OF CAUSE/EFFECT
  - BLOCK DIAGRAMS DEAL WITH TRANSFORMS

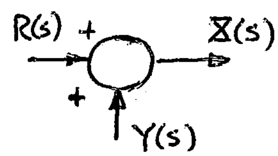
GAIN OR MULTIPLIER

$x(t) = K y(t)$ or $X(s) = K Y(s)$

$$Y(s) \longrightarrow \boxed{K} \longrightarrow X(s)$$

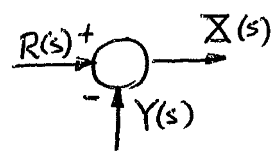
SUMMER REPRESENTS ADDITION

$x(t) = r(t) + y(t)$ or $X(s) = R(s) + Y(s)$



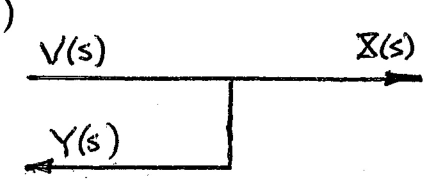
COMPARATOR REPRESENTS SUBTRACTION

$x(t) = r(t) - y(t)$ or $X(s) = R(s) - Y(s)$



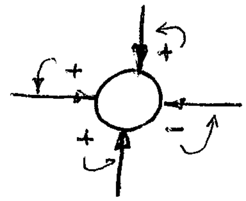
TAKE OFF POINT - USED TO CONNECT DIFFERENT PARTS OF DIAGRAM W/O CHANGING VALU

$y(t) = v(t) = x(t)$ or $Y(s) = V(s) = X(s)$



- NOTE THAT TRANSFER FN IS LIKE GAIN OR MULTIPLIER BLOCK DIAGRAM
- IN DRAWING BLOCK DIAGRAMS
  - FEEDBACK LOOPS ALLOW THE SAME VARIABLE TO BE BOTH INPUT & OUTPUT
  - FOR SUMMERS OR COMPARATORS, THE SIGN (+ OR −) SHOULD APPEAR

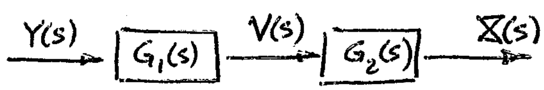
CLOCKWISE FROM THE VARIABLES ARROWHEAD



• BLOCK DIAGRAMS DISPLAY SYSTEM'S <u>STRUCTURE</u>. WE ADD THE FREE RESPONSE AT THE CONCLUSION OF ANALYSIS $\Longrightarrow$ ASSUME INITIAL CONDITIONS TO BE ZERO FOR BLOCK DIAGRAMS

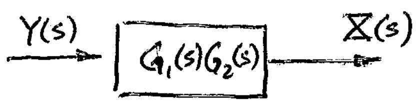
<u>BLOCK DIAGRAM ALGEBRA</u>

• <u>CASCADE</u>

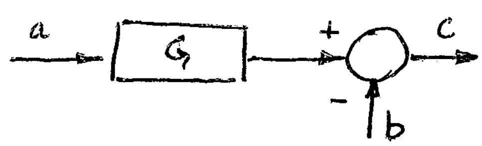


$$\frac{V(s)}{Y(s)} = G_1(s)\;;\quad \frac{X(s)}{V(s)} = G_2(s)$$

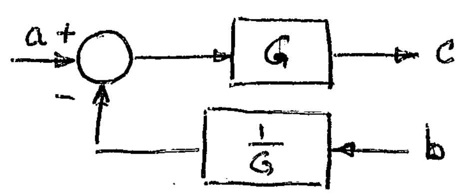
$$\frac{X(s)}{Y(s)} = G_1(s)G_2(s)$$



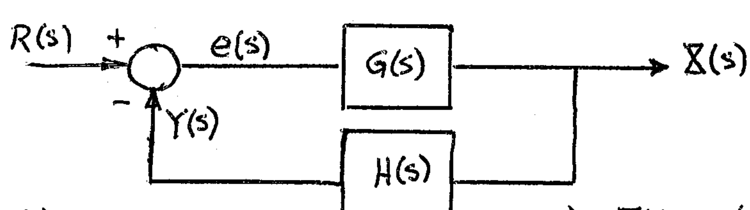
• <u>MOVING SUMMERS</u>



$$C = aG - b$$
$$= \left(a - b/G\right) G$$



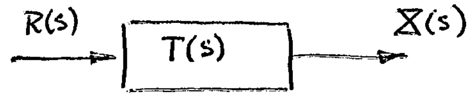
• <u>LOOP REDUCTION</u>



1) $R(s) - Y(s) = e(s)$

2) $Y(s) = H(s)X(s)$

3) $X(s) = e(s)G(s)$



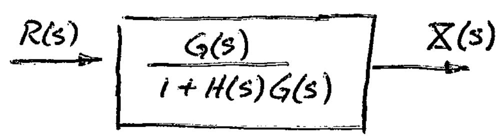
$$G(s) = \frac{X(s)}{e(s)}$$ IS THE FORWARD LOOP TRANSFER FN

$$H(s) = \frac{Y(s)}{X(s)}$$ IS THE FEEDBACK LOOP TRANSFER FN

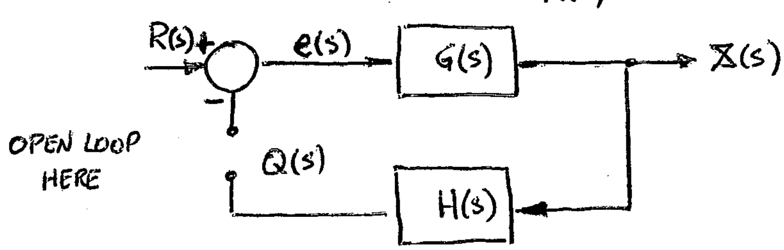
$$T(s) = \frac{X(s)}{R(s)}$$ FROM (1), (2) & (3) $$R(s) - H(s)X(s) = \frac{X(s)}{G(s)}$$

$$\therefore \quad R(s) = X(s)\left[\frac{H(s)G(s)+1}{G(s)}\right] \Rightarrow \underline{\underline{T(s) = \frac{G(s)}{1+H(s)G(s)}}}$$

T(s) IS THE <u>CLOSED LOOP TRANSFER FN</u>



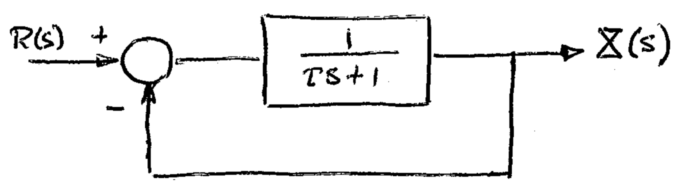
OPEN LOOP TRANSFER FN $\dfrac{Q(s)}{R(s)}$



1) $e(s) = R(s)$
2) $X(s) = e(s)G(s)$
3) $Q(s) = X(s)\,H(s)$

FROM (1), (2), (3) $\quad \dfrac{Q(s)}{X(s)} = H(s) \quad \dfrac{X(s)}{e(s)} = \dfrac{X(s)}{R(s)} = G(s) \;;\; \underline{\underline{\dfrac{Q(s)}{R(s)} = H(s)G(s)}}$

EXAMPLE



$$G(s) = \frac{1}{\tau s+1}$$

$$H(s) = 1$$

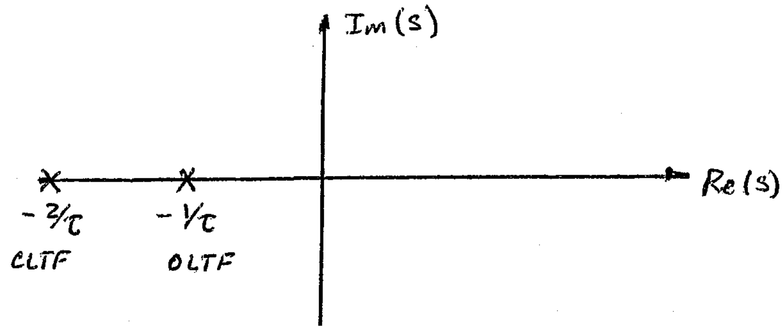
$$T(s) = \frac{G(s)}{1+H(s)G(s)} = \frac{\frac{1}{\tau s+1}}{1 + 1\cdot\frac{1}{\tau s+1}} = \frac{1}{\tau s+2}$$ CLOSED LOOP TRANSFER FN

$$\frac{Q(s)}{R(s)} = G(s)H(s) = \frac{1}{\tau s+1}\cdot 1 = \frac{1}{\tau s+1}$$ OPEN LOOP TRANSFER FN

- FOR THE CLOSED LOOP TRANSFER FN: $T(s) = \frac{1}{\tau} \cdot \frac{1}{s + 2/\tau}$
  AND THE ROOT OF THE CHARACTERISTIC polynomial IS $s = -2/\tau$

- FOR THE OPEN LOOP TRANSFER FN: $T(s) = \frac{1}{\tau} \cdot \frac{1}{s + 1/\tau}$
  THE ROOT OF THE CHARACTERISTIC POLYNOMIAL IS $s = -1/\tau$

- FROM OUR PREVIOUS WORK



- FOR A GIVEN INPUT, THE OPEN LOOP TRANSFER FN IMPLIES THAT THE CORRESPONDING DIFFERENTIAL EQN HAS A HIGHER TIME CONSTANT THAN THE CORRESPONDING DIFFERENTIAL EQN FOR THE CLOSED LOOP TRANSFER FN.

PROOF $$T(s) = \frac{X(s)}{Y(s)} = \frac{1}{\tau(s + 1/\tau)} \Rightarrow \dot{x} + \frac{1}{\tau} x = \frac{1}{\tau} y$$

Homogeneous eqn $\dot{x} + \frac{1}{\tau} x = 0$ has solution $x = A e^{-t/\tau}$

when $t = \tau$ x is 37% of A TIME CONST $= \tau$

$$T(s) = \frac{X(s)}{Y(s)} = \frac{1}{\tau} \frac{1}{(s + 2/\tau)} \Rightarrow \dot{x} + \frac{2}{\tau} x = \frac{1}{\tau} y$$

Homogeneous eqn has solution $x = A e^{-2t/\tau}$

when $t = \tau$ x is 14% of A TIME CONST $= \tau/2$

- THUS BY ADDING A CLOSED LOOP (FEEDBACK), THE SYSTEM'S STABILITY IS IMPROVED, CAUSING THE MOST DOMINANT ROOT OF THE SYSTEM TO MOVE FARTHER AWAY FROM THE IMAGINARY S AXIS. THUS FEEDBACK IMPROVES LINEARITY & INCREASES STABILITY THROUGH IMPROVED ROBUSTNESS (p19-27)

LESSON #13

FEEDBACK REDUCE PARAMETER SENSITIVITY of SYSTEM.

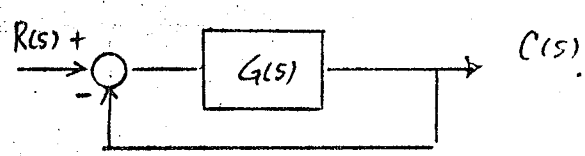
SUPPOSE, DUE TO PARAMETER VARIATION $G(s)$ IS CHANGED TO $G(s)+\Delta G(s)$

$|G(s)| \gg |\Delta G(s)|$ FOR ABOVE OPEN LOOP SYSTEM

$$C(s)+\Delta C(s) = [G(s)+\Delta G(s)]\, R(s)$$

$$\therefore \quad \Delta C(s) = \Delta G(s)\, R(s)$$

NOW CLOSE THE LOOP w/ UNITY FEED BACK



NOW $$C(s)+\Delta C(s) = \frac{G(s)+\Delta G(s)}{1+G(s)+\Delta G(s)}\, R(s)$$

$$\Delta C(s) \overset{\sim}{=} \frac{\Delta G(s)}{1+G(s)}\, R(s)$$ SHOW!

$\therefore$ SENSITIVITY TO PARAMETER VARIATIONS IS REDUCED BY A FACTOR OF $1/1+G(s)$

HINT: LET $1+G+\Delta G = (1+G)\left[1+\dfrac{\Delta G}{1+G}\right]$

NEXT $$\frac{1}{1+G+\Delta G} = \frac{1}{1+G}\left[\frac{1}{1+\frac{\Delta G}{1+G}}\right]$$

$$\cong \frac{1}{1+G}\left\{1-\frac{\Delta G}{1+G}+\dots\right\}$$

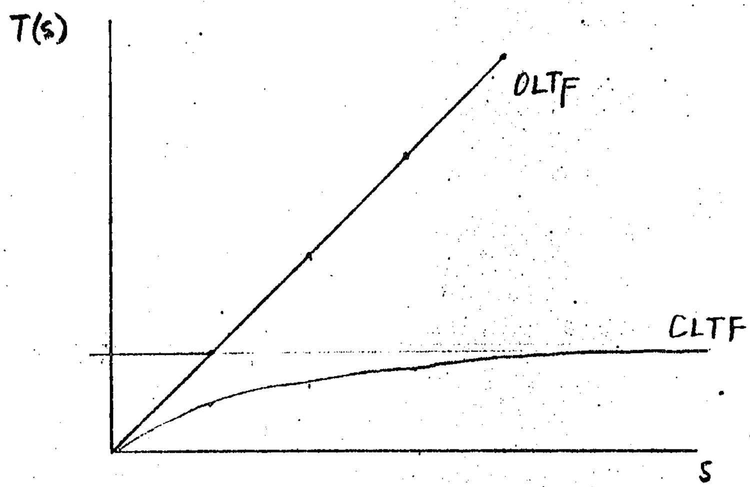
. FEED BACK IMPROVES LINEARITY

OPEN-LOOP X-FER FCTN $\quad G(s)$

CLOSED-LOOP X-FER FCTN $\quad \dfrac{G(s)}{1+G(s)}$

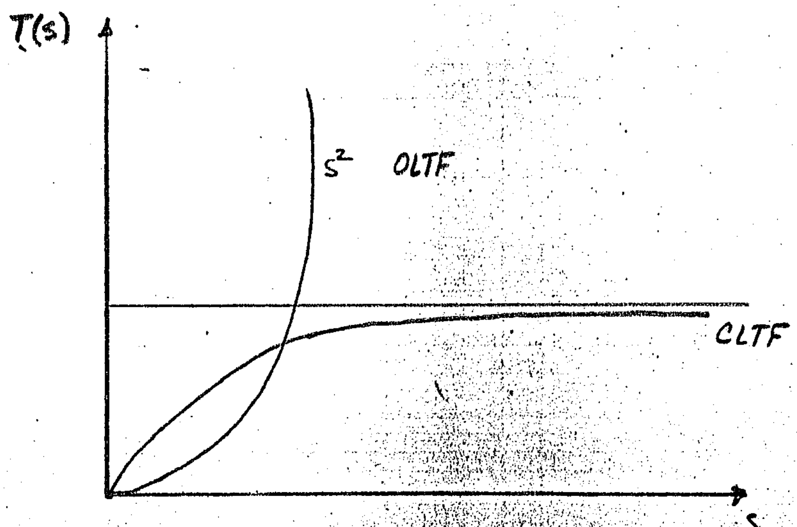
LET $G(s) = s \qquad \dfrac{s}{1+s}$

s	$\frac{s}{1+s}$
1	$\frac{1}{2}$
2	$\frac{2}{3}$
3	$\frac{3}{4}$
4	$\frac{4}{5}$
⋮	
∞	1



LET $G(s) = s^2 \qquad \dfrac{s^2}{1+s^2}$

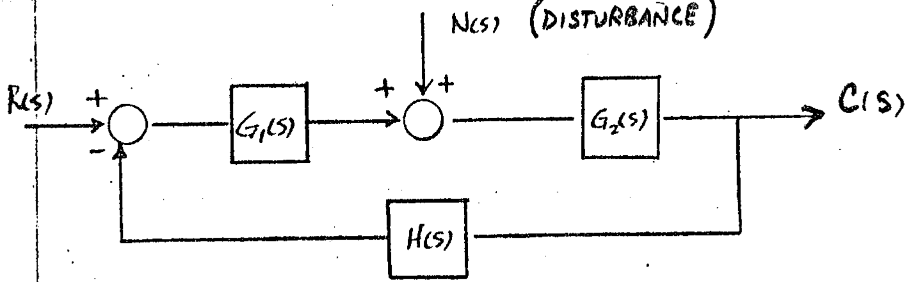
s^2	$\frac{s^2}{1+s^2}$
1	$\frac{1}{2}$
2	$4/5$
⋮	
∞	1



- LET y - output x - input $\therefore$ linear fn is $y = Kx \Rightarrow Y = KX \Rightarrow T(s) = \dfrac{Y}{X}$
  $\therefore T(s) = \text{const}$
- in first case for $s > 19$ K IS WITHIN 5% of a constant
- in second CASE for $s > 5$ K IS WITHIN 5% of a constant

• FEEDBACK ALLOWS DISTURBANCE REJECTION

LOOK AT SYSTEM SUBJECTED TO DISTURBANCE



NOTE: NOW NOT A SINGLE INPUT / SINGLE OUTPUT SYSTEM

LINEAR SYSTEM TREAT EACH SEPERATELY !

ASSUME $R(s) = 0$ AND FIND $T_N(s)$ TRANSFER FN DUE TO DISTURBANCE

$$T_N(s) = \frac{C_N(s)}{N(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

ASSUME $N(s) = 0$ AND FIND $T_R(s)$ TRANSFER FN DUE TO INPUT

$$T_R(s) = \frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)}$$

NOTE: SAME DENOMINATORS !

$$C(s) = C_N(s) + C_R(s) \quad \text{SINCE LINEAR}$$

$$C(s) = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)}\left[G_1(s)R(s) + N(s)\right]$$

IF $|G_1(s)H(s)| \gg 1$ AND $|G_1(s)G_2(s)H(s)| \gg 1$

$$T_N(s) = \frac{C_N(s)}{N(s)} = \frac{G_2(s)}{1 + G_1(s)G_2(s)H(s)} \to \frac{1}{G_1(s)H(s)} \longrightarrow 0$$

∴ N(s) IS REJECTED SINCE OUTPUT DUE TO $N(s) \to 0$

ON THE OTHER HAND AS $|G_1(s)G_2(s)H(s)| \gg 1$ & LOOK AT $T_R(s)$

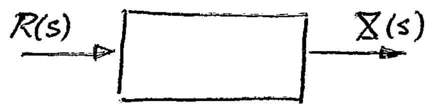
$$T_R(s) = \frac{C_R(s)}{R(s)} = \frac{G_1(s)G_2(s)}{1 + G_1(s)G_2(s)H(s)} \longrightarrow \frac{1}{H(s)}$$

FOR UNITY H(s)

$C_R(s) = R(s)$

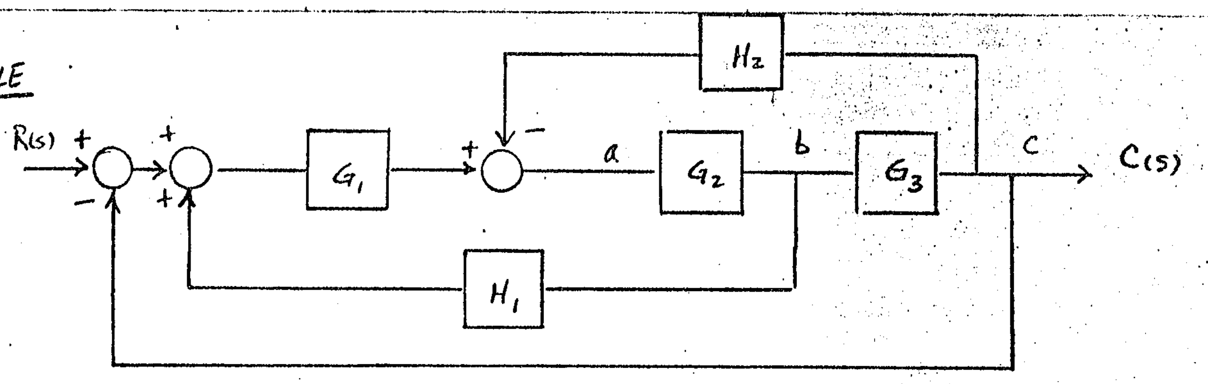
• CLOSED-LOOP SYSTEMS W/UNITY FEEDBACK EQUALIZE INPUT & OUTPUT

- REDUCTION OF BLOCK DIAGRAMS
  - WE WANT TO REDUCE DIAGRAMS TO THE POINT WHERE INPUT AND OUTPUT ARE RELATED SIMPLY BY



- WE CAN DO THIS USING RULES OF BLOCK DIAGRAM ALGEBRA
- RULES REQUIRES THAT ALGEBRAIC RELATIONS ARE MAINTAINED
- TWO DIAGRAMS ARE EQUIVALENT IF THE ALGEBRAIC RELATIONS BETWEEN INPUT & OUTPUT ARE THE SAME

EXAMPLE

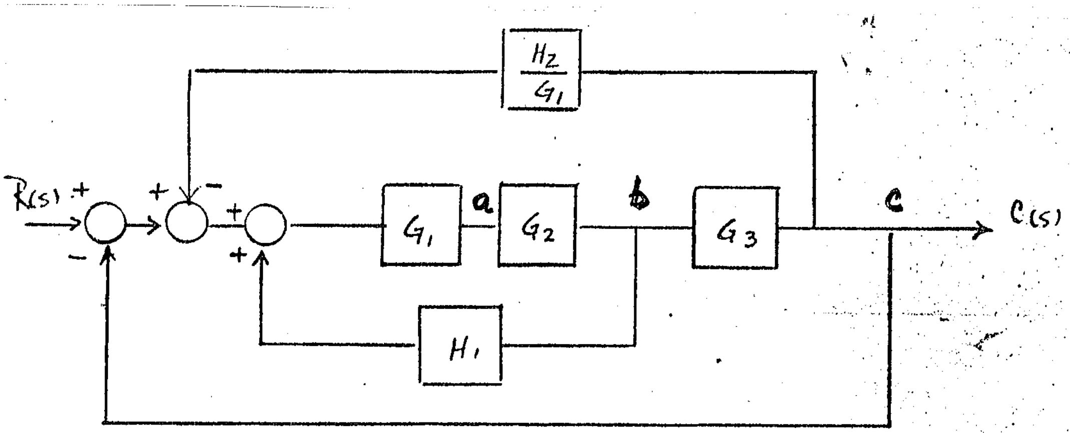


MOVE G_1-H_1 INSIDE H_2 FEEDBACK LOOP

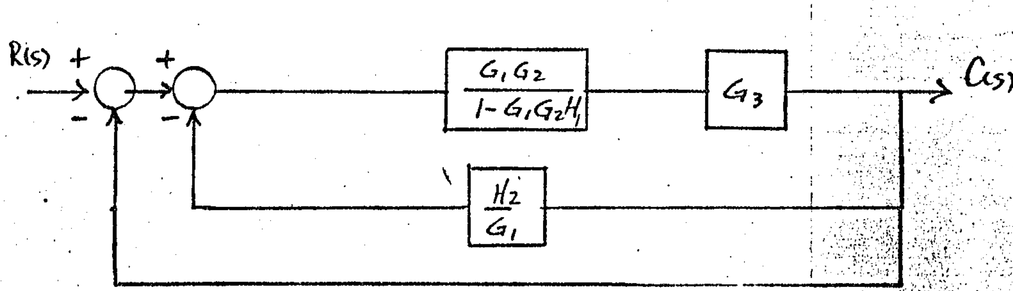
CHECK (a) SIGNAL

$$[(R-C)+H_1 b]G_1 - H_2 C = a$$

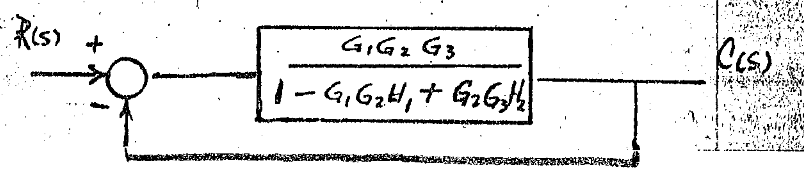
$$\left[(R-C)+H_1 b - \frac{H_2}{G_1}C\right]G_1 = a$$



REDUCE INNER-MOST LOOP : $\dfrac{G_1 G_2}{1 - G_1 G_2 H}$



REDUCE INNER-MOST LOOP $\dfrac{\dfrac{G_1 G_2 G_3}{1 - G_1 G_2 H_1}}{1 + \dfrac{G_1 G_2 G_3}{1 - G_1 G_2 H_1} \dfrac{H_2}{G_1}}$



LOOK AT DIAGRAMS ON PG 190 FOR RELOCATION OF TAKE OFF POINTS, SUMMERS ETC

DESIGN CRITERIA - CLASSICAL SENSE

- MUST ANSWER THE FOLLOWING QUESTIONS
  1. IS SYSTEM STABLE ... ROUTH HURWITZ
  2. DOES IT HAVE ACCEPTABLE ACCURACY ... STEADY STATE ERRORS
  3. IS TRANSIENT RESPONSE ACCEPTABLE ... PEAK OVERSHOOT SETTLING, DELAY, RISE, PEAK TIMES

- FOR ACCEPTABLE TRANSIENT RESPONSE: WANT OVERSHOOT, t_d, t_r, t_p, t_s TO BE SMALL

- ACCURACY : STEADY STATE ERROR IS DIFFERENCE BETWEEN DESIRED OUTPUT AND ACTUAL OUTPUT AFTER ALL TRANSIENTS HAVE DIED OUT.

EXAM - LESSON # 14

LESSON # 15

DESIGN CRITERIA - CLASSICAL SENSE

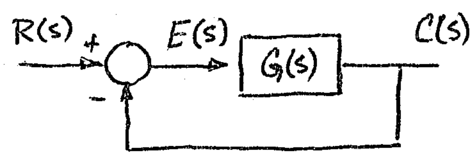
- MUST ANSWER THE FOLLOWING QUESTIONS
  1. IS SYSTEM STABLE ... ROUTH HURWITZ
  2. DOES IT HAVE ACCEPTABLE ACCURACY ... STEADY STATE ERRORS
  3. IS TRANSIENT RESPONSE ACCEPTABLE ... PEAK OVERSHOOT

     SETTLING, DELAY, RISE, PEAK TIME

- FOR ACCEPTABLE TRANSIENT RESPONSE: WANT OVERSHOOT, t_d, t_r, t_p, t_s TO BE SMALL

- ACCURACY : STEADY STATE ERROR IS DIFFERENCE BETWEEN DESIRED OUTPUT AND ACTUAL OUTPUT AFTER ALL TRANSIENTS HAVE DIED OUT.

STEADY STATE ERROR FOR STEP INPUT



$$E(s) = R(s) - C(s) = R(s) - G(s)E(s)$$

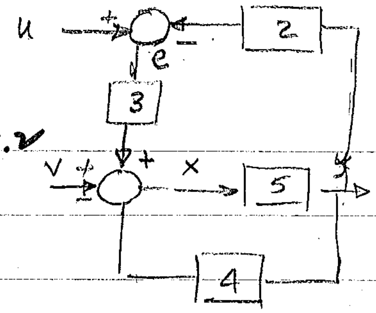
$$\text{or} \quad E(s) = \frac{R(s)}{1+G(s)} \qquad \text{FOR UNITY FEEDBACK}$$

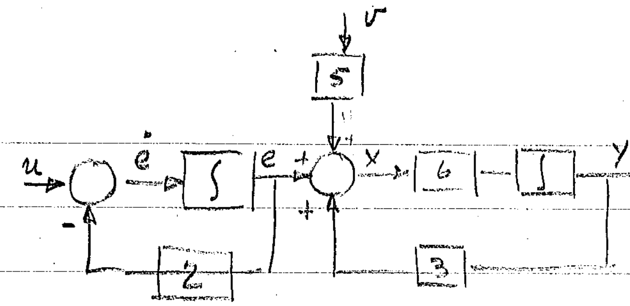
$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

IF $R(s)$ IS A STEP INPUT $R(s) = \frac{1}{s}$

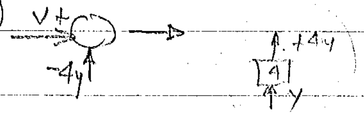
$$\therefore \; e_{ss} = \lim_{s\to 0} \frac{1}{1+G(0)}$$

pg 27
problem 1.2



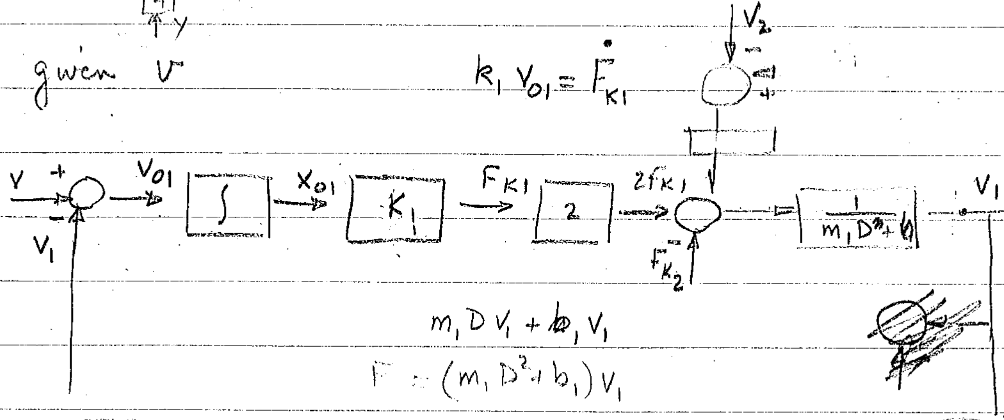


$X = V - 4y$



want V_1 given V

$k_1 V_{01} = \dot{F}_{K1}$



$m_1 D V_1 + b_1 V_1$

$F = (m_1 D^2 + b_1) V_1$

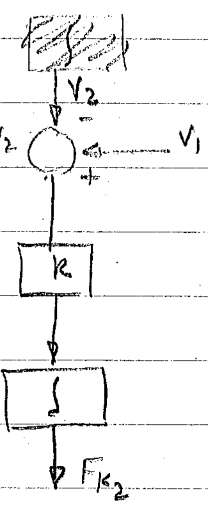
$\cancel{b_2 + m_2 V_2}$ $\quad (b_2 + m_2 D) V_2 = F_{K_2} =$

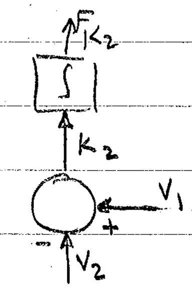
$\cancel{k_2}(X_1 - \cancel{X_2}) = \frac{\dot{F}_{K2}}{k_2} + V_2$

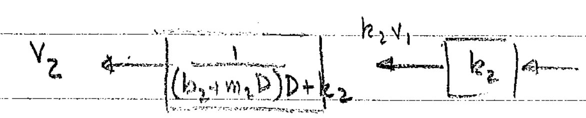
$k_2(x_1 - x_2) = F_{K2} = (b_2 + m_2 D) V_2$

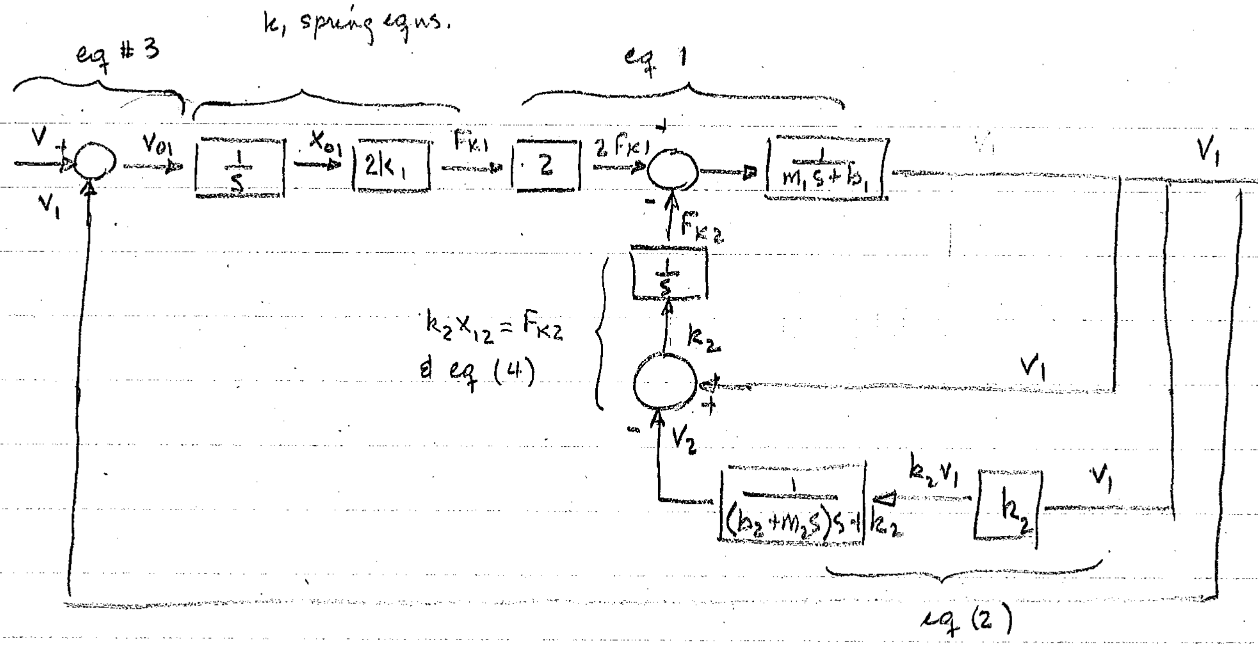
$k_2 (V_1 - V_2) = (b_2 + m_2 D) D V_2$

$k_2 V_1 = \left[(b_2 + m_2 D) D + k_2\right] V_2$









Vib Absorber

$$F_{m1} + F_{b1} = m_1 \frac{dv_1}{dt} + b_1 v_1 = (m_1 s + b_1) v_1$$

$$k_2 (x_1 - x_2) = F_{k2} \Rightarrow k_2 (v_1 - v_2) = \dot{F}_{k2}$$

$$\int \dot{F}_{k2}\, dt = F_{k2}$$

$$F_{k2} = k_2 (x_1 - x_2) = F_{b_2} + F_{m_2} = b_2 v_2 + m_2 \frac{dv_2}{dt}$$

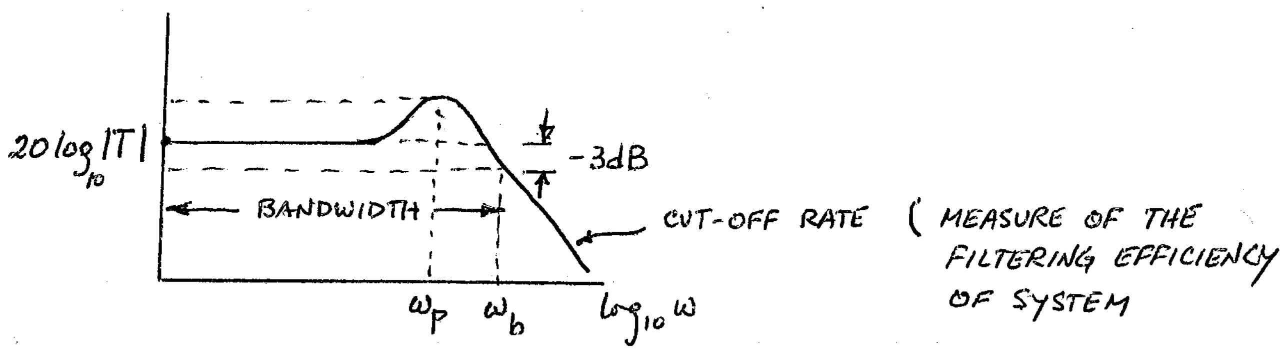
$$= (b_2 + m_2 s) v_2$$

$$k_2 (v_1 - v_2) = b_2 \frac{dv_2}{dt} + m_2 \frac{d^2 v_2}{dt^2} = (b_2 s + m_2 s^2) v_2$$

$$k_2 v_1 = \left[(b_2 s + m_2 s^2) + k_2 \right] v_2$$

IF V_1 is output V is input then V_{01}, V_2 are internal variable

- ACCURACY: PLOT AMPLITUDE OF OUTPUT TO AMPLITUDE OF INPUT VERSUS $\log_{10}$ OF FREQUENCY OF INPUT SINUSOID (SIMILAR TO MAGNITUDE OF $T(s)$)
- A SINUSOIDAL SIGNAL IS AN IMPORTANT TESTING INPUT AS THE FREQUENCY RESPONSE IS USEFUL TOOL FOR ANALYSIS & DESIGN OF CONTROL SYSTEMS



- $\log_{10}|T|$ - PRIMARILY DESCRIBES THE MAXIMUM OVERSHOOT; CAN ALSO RELATE t_s
- ω_p, ω_b - ARE BASICALLY DESCRIPTIVE OF t_d, t_r, t_p, t_s
- ω_b IS FREQUENCY 3dB BELOW LOW FREQUENCY ASYMPTOTE INTERSECTION WITH HIGH FREQUENCY ASYMPTOTE OR CUT OFF RATE
- ω_p IS THE FREQUENCY AT PEAK OF $\log_{10}|T|$ CURVE
- IN GENERAL IT IS HARDER TO WORK WITH $T(s)$ AND FINDING ITS ROOTS. IT IS EASIER TO WORK WITH $\log_{10}|T|$ (RELATED TO GAIN MARGIN) AND PHASE MARGIN (RELATED TO ϕ)
- $\log_{10}|T|$ AND ϕ ARE USED TO DESCRIBE THE STEADY STATE RESPONSE TO A SINUSOIDAL INPUT

FREQUENCY RESPONSE TRANSFER FN (STEADY STATE RESPONSE TO SINUSOIDAL INPUT)

- REMEMBER $\dot{y} + \frac{1}{\tau} y = M \sin \omega t = v(t)$ $\quad$ y - output, v - input
- WE FOUND FORCED RESPONSE

$$y = \frac{M\tau}{\sqrt{1+\omega^2\tau^2}} \sin(\omega t + \phi) + \frac{M\tau^2\omega}{(1+\omega^2\tau^2)} e^{-t/\tau}$$

- ALSO FOUND STEADY STATE SOL'N

$$y_{ss} = \frac{M\tau}{\sqrt{1+\omega^2\tau^2}} \sin(\omega t + \phi) \qquad \tan\phi = -\tau\omega$$
$$= B \sin(\omega t + \phi)$$

- TAKE LAPLACE TRANSFORM & FIND $T(s)$ OF $\dot{y} + \frac{1}{\tau} y = v(t)$

$$Y\left(s + \frac{1}{\tau}\right) = V(s) \quad \Rightarrow \quad T(s) = \frac{\tau}{s\tau + 1} = \frac{Y(s)}{V(s)}$$

- LAPLACE OF $v(t) = \frac{M\omega}{s^2 + \omega^2}$: ROOTS OF DENOM ARE $s = \pm j\omega$

- REPLACE s IN $T(s)$ by $j\omega$

$$T(j\omega) = \frac{\tau}{\tau j\omega + 1} = \frac{\tau}{\tau j\omega + 1} \cdot \frac{-\tau j\omega + 1}{-\tau j\omega + 1} = \frac{\tau[1 - \tau j\omega]}{1 + \tau^2\omega^2}$$

$$= \frac{\tau}{1+\tau^2\omega^2} - j\frac{\tau^2\omega}{1+\tau^2\omega^2} = A + j\hat{B}$$

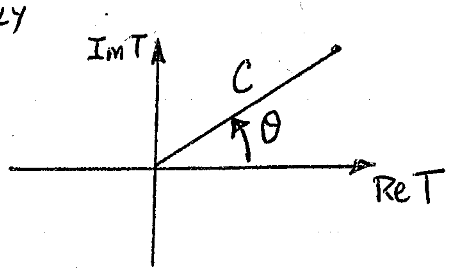
- ANY COMPLEX NUMBER $A + j\hat{B} = Ce^{j\theta}$ WHERE $C = \sqrt{A^2 + \hat{B}^2}$

$$\tan\theta = \hat{B}/A$$

- IN OUR CASE $C = \frac{\tau}{\sqrt{1+\tau^2\omega^2}}$ $\qquad$ $\tan\theta = -\tau\omega$

HERE $C \equiv |T(j\omega)|$ $\qquad$ $\phi = \theta \equiv \measuredangle T(j\omega)$

PHYSICALLY



- WHEN $j\omega$ REPLACES s IN $T(s)$ - FREQUENCY TRANSFER FN

- THE MAGNITUDE OF $T(j\omega)$ REPRESENTS $|y_{ss}| / |v|$

• THE ARGUMENT OF T, θ, IS THE SAME AS ϕ THE PHASE SHIFT OF THE OUTPUT RELATIVE TO THE INPUT.

THUS WE CAN USE TRANSFER FN TO CONSIDER FREQUENCY RESPONSE OF SYSTEM

$$M \sin \omega t \longrightarrow \boxed{T(s)} \longrightarrow B \sin(\omega t + \phi)$$

$$|T(j\omega)| = B/M \qquad \text{AMPLITUDE RATIO}$$

$$\measuredangle T(j\omega) = \phi = \tan^{-1}(\tau\omega) \qquad \text{PHASE SHIFT}$$

• NOTE THAT SINCE $|T(j\omega)| = \dfrac{\tau}{\sqrt{1+\tau^2\omega^2}}$ AS $\omega \to \infty$ $|T(j\omega)| \to 0$

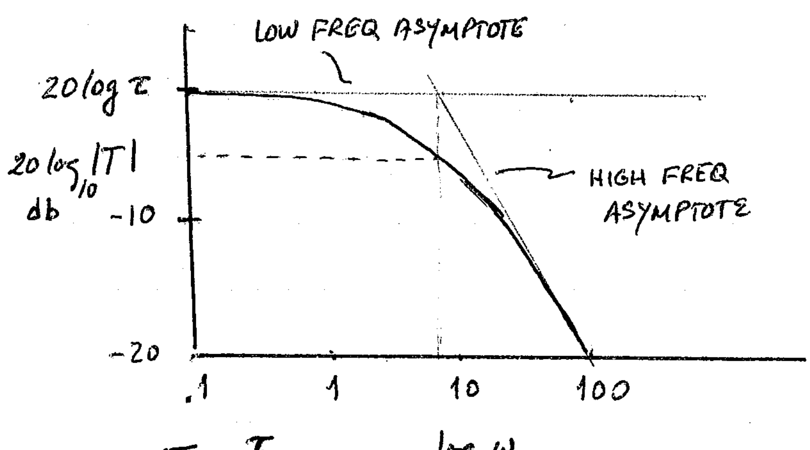
$$\measuredangle T(j\omega) \to -\pi/2$$

• AT HIGH FREQUENCY - SYSTEM'S INERTIA WILL NOT ALLOW OUTPUT TO FOLLOW INPUT

• TO PLOT FREQUENCY RESPONSE OR BODE DIAGRAMS

PLOT $20 \log_{10} |T(j\omega)|$ vs. $\log \omega$

UNITS: decibels (db)



PLOTTED ON SEMI LOG PAPER

PLOT FOR A 1^{ST} ORDER LAG

THUS FOR $T = \dfrac{\tau}{\tau j\omega + 1}$

$$20 \log_{10} |T| = 20 \log_{10} \tau - 20 \log_{10} (1+\tau^2\omega^2)^{1/2}$$

$$= 20 \log_{10} \tau - 10 \log_{10} (1+\tau^2\omega^2)$$

TO FIND LOW FREQ ASYMPTOTE LET $\omega \to 0$: HERE IT IS $20 \log \tau$

TO FIND HIGH FREQUENCY ASYMPTOTE LET $\omega \to \infty$

IN OUR CASE $10 \log_{10}(1+\omega^2\tau^2) \sim 20 \log_{10} \omega\tau \sim 20 \log_{10} \omega$

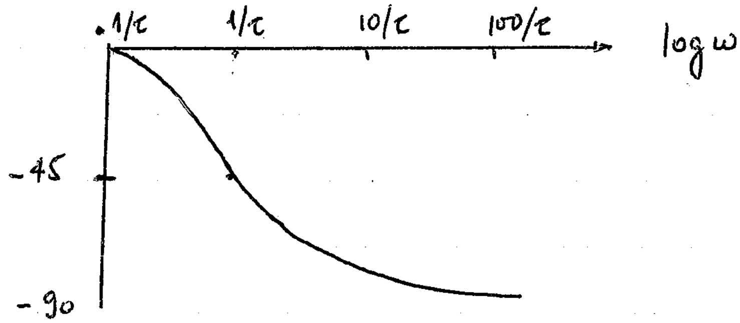
$\therefore \; 20 \log_{10} |T(j\omega)| \sim -20 \log_{10} \omega$

THIS IS STRAIGHT LINE WITH SLOPE $\dfrac{-20\,\text{db}}{\text{decade}}$

- DECADE IS 10 : 1 FREQUENCY RANGE

- WHERE LOW FREQ & HIGH FREQ ASYMPTOTE MEET IS KNOW AS BREAKPOINT FREQUENCY OR CORNER FREQUENCY $\omega_b = \frac{1}{\tau}$

- FOR THE PHASE SHIFT $\phi = \measuredangle T(j\omega) = \tan^{-1}(-\tau\omega)$
- AS ω GOES FROM 0 TO $+\infty$ ϕ GOES FROM 0 TO $-90°$



- NOTE THAT FOR $T(s) = \tau s + 1$. $|T(j\omega)| = \sqrt{1+\tau^2\omega^2}$

$\measuredangle T(j\omega) = \tan^{-1}(\tau\omega)$

- THIS IS MIRROR IMAGE OF $T(s) = \dfrac{1}{\tau s + 1}$

TRANSFER FN STANDARD FORMS

- $T(s)$ CAN BE WRITTEN IN THE FORM

$$T(s) = \frac{N(s)}{D(s)} = K \frac{N_1(s)\, N_2(s) \cdots}{D_1(s)\, D_2(s) \cdots}$$

AND THE FREQUENCY RESPONSE TRANSFER FN

$$T(j\omega) = K \frac{N_1(j\omega)\, N_2(j\omega) \cdots}{D_1(j\omega)\, D_2(j\omega) \cdots}$$

WHERE : $N_i(j\omega)$, $D_i(j\omega)$ ARE OF THE FORMS

$\tau s+1$	$\tau j\omega+1$	SIMPLE LEAD OR SIMPLE LAG
s^n	$(j\omega)^n$	N TH DERIVATIVE OR N TH INTEGRATION
K	K	CONSTANT GAIN TERM
$s^2+2\zeta\omega_n s+\omega_n^2$	$\left[1-\left(\frac{\omega}{\omega_n}\right)^2+2j\xi\left(\frac{\omega}{\omega_n}\right)\right]\omega_n^2$	SECOND ORDER TERM

EXAMPLES $\quad T(s) = \dfrac{A}{(\tau_1 s+1)(\tau_2 s+1)} \qquad T(j\omega) = \dfrac{A}{(\tau_1 j\omega+1)(\tau_2 j\omega+1)}$

$$= \frac{A(\tau s+1)}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad T(j\omega) = \frac{A(\tau j\omega+1)}{\omega_n^2\left[1-\left(\frac{\omega}{\omega_n}\right)^2+2j\xi\left(\frac{\omega}{\omega_n}\right)\right]}$$

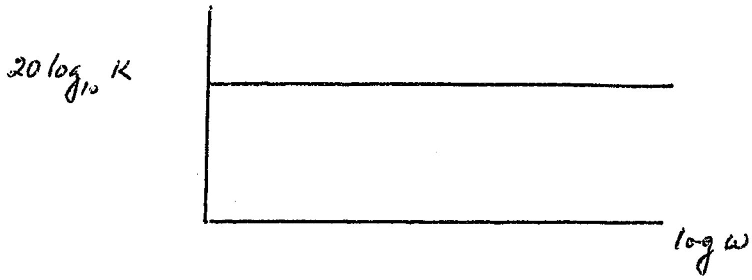
$$= \frac{A}{s(\tau s+1)} \qquad T(j\omega) = \frac{A}{(j\omega)(\tau j\omega+1)}$$

- GIVEN THE FORMS OF $T(s)$ OUR GOALS ARE TO OBTAIN THE PLOT OF $20\log|T(j\omega)|$ vs. $\log\omega$ & $\measuredangle T(j\omega)$ vs. $\log\omega$
- $20\log|T(j\omega)| = 20\left\{\sum \log|N_i(j\omega)| - \sum \log|D_i(j\omega)|\right\} + 20\log K$
- $\measuredangle T(j\omega) = \sum \phi_{N_i} - \sum \phi_{D_i}$

ϕ_{N_i} - IS PHASE SHIFT OF NUMERATOR TERM N_i
ϕ_{D_i} - IS PHASE SHIFT OF DENOMINATOR TERM D_i

LETS LOOK AT EACH TERM

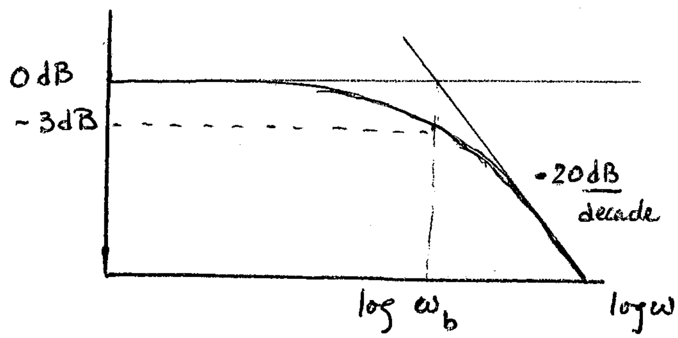
CONSTANT GAIN K (NOT A FN OF FREQUENCY



PHASE ANGLE $K = Ke^{j0}$ IF $K > 0$

$Ke^{j\pi}$ IF $K < 0$

SIMPLE LAG : $T(j\omega) = \dfrac{1}{\tau j\omega + 1}$



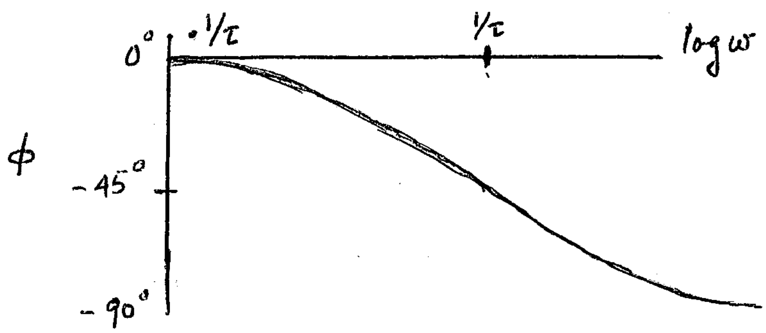
BREAK FREQUENCY $\omega_b = \frac{1}{\tau}$

FOR $\omega \ll \omega_b$: PHASE ANGLE is 0

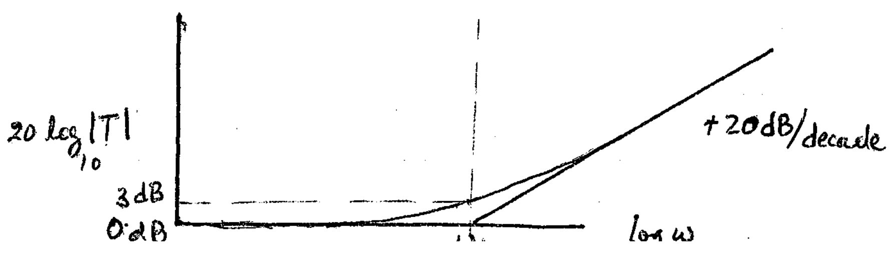
FOR $\omega \gg \omega_b$: PHASE ANGLE is -90

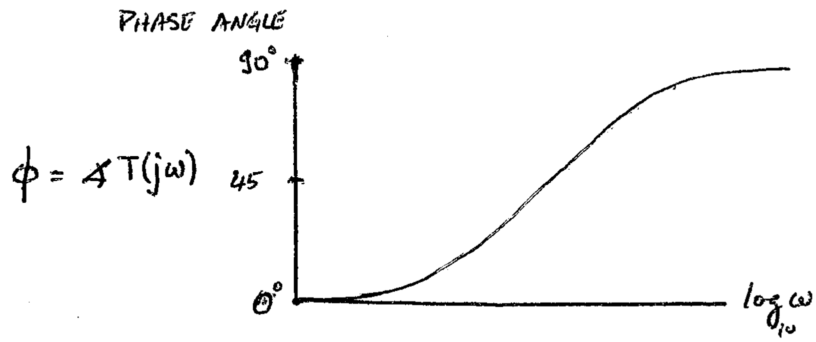
AT $\omega = \omega_b$ $\phi = \tan^{-1}(\omega\tau) = \tan^{-1}(-1)$

$= 45^0$



FIRST ORDER LEAD $T(j\omega) = \tau j\omega + 1$





INTEGRATION AND DIFFERENTIATION

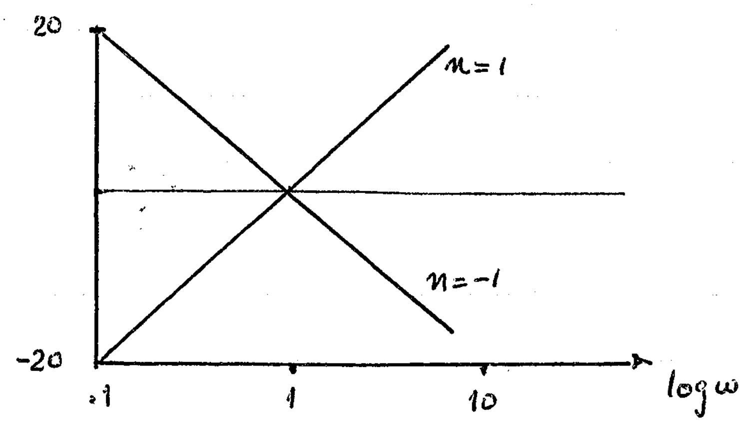
$$T(j\omega) = (j\omega)^n$$
$$= \omega^n \cdot e^{+\frac{\pi}{2} j n}$$

n - positive   DIFFERENTIATION

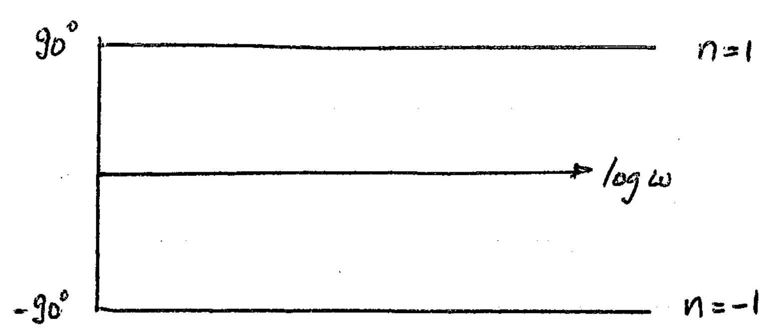
n - negative   INTEGRATION

$$20 \log |T(j\omega)| = 20 \log |j\omega|^n = 20n \log \omega$$

: THIS IS A STRAIGHT LINE WITH SLOPE $20n$ dB/decade.



PHASE ANGLE IS GIVEN BY $\frac{\pi}{2} \cdot n$ INDEPENDENT OF ω



SECOND ORDER TERM

$$T(j\omega) = \frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2 + j2\zeta\left(\frac{\omega}{\omega_n}\right)} \Rightarrow |T(j\omega)| = \frac{1}{\sqrt{\left(1-\left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\left(\frac{\omega}{\omega_n}\right)\right)^2}}$$

AREAS OF INTEREST

- FOR SMALL $\frac{\omega}{\omega_n}$ $\quad \log_{10}|T(j\omega)| \sim -\frac{1}{2}\log 1 = 0$ $\quad$ LOW FREQ ASYMPTOTE

- FOR ω/ω_n VERY LARGE $\quad \log_{10}|T(j\omega)| \sim -\frac{1}{2}\log_{10}\left(\frac{\omega}{\omega_n}\right)^4 = -2\log_{10}\left(\frac{\omega}{\omega_n}\right)$

$$m = 20\log_{10}|T(j\omega)| = -40\log_{10}|T(j\omega)| \Rightarrow$$ STRAIGHT LINE SLOPE = -40 dB/decade

HIGH FREQ ASYMPTOTE

- FOR $\frac{\omega}{\omega_n} = 1$ $\quad \log_{10}|T(j\omega)| = -\frac{1}{2}\log(2\zeta)^2 = \log\frac{1}{2\zeta}$

THUS $m = 20\log_{10}|T(j\omega)|$ is controlled by DAMPING RATIO ζ

- $1 = \omega/\omega_n$ IS WHERE HIGH & LOW FREQ ASYMPTOTES INTERSECT (CORNER OR BREAK FREQ)

- SINCE CONTROLLED BY ζ, YOU GET FAMILY OF CURVES (PALM PG 210)

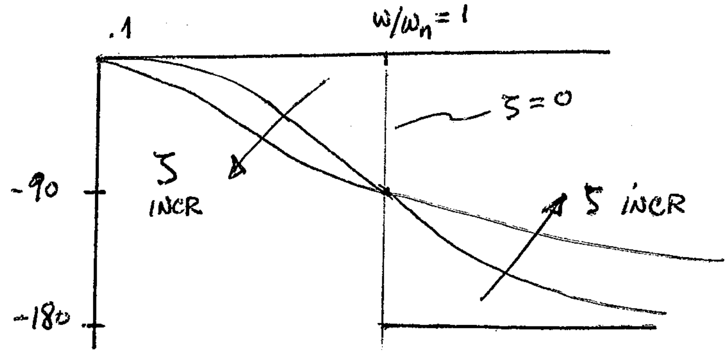
- LOOK AT PHASE ANGLE

$$\measuredangle T(j\omega) \Rightarrow \tan\phi = \frac{-2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}$$

FOR $\frac{\omega}{\omega_n} \ll 1$ $\quad \tan\phi \to 0$ FROM NEGATIVE SIDE $\quad \phi \to -0^\circ$

$\frac{\omega}{\omega_n} \gg 1$ $\quad \tan\phi \to 0$ FROM POSITIVE SIDE $\quad \phi \to -180^\circ$

$\omega/\omega_n = 1$ $\quad \tan\phi \to -\infty$ $\quad \phi \to -90^\circ$



SEE (PALM PAGE 210)

LOOK AT $|T(j\omega)|$ AGAIN

- NOTE $|T(j\omega)|$ ATTAINS A MAXIMUM AT $\dfrac{\omega}{\omega_n} = \sqrt{1-2\zeta^2}$ $\quad \zeta \le \dfrac{\sqrt{2}}{2}$

  AND ATTAINS A MAX AT $\dfrac{\omega}{\omega_n} = 0$ FOR $\zeta \geq \dfrac{\sqrt{2}}{2}$

- WHEN $\dfrac{\omega}{\omega_n} = \sqrt{1-2\zeta^2}$ $\quad |T(j\omega)| = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$ $\quad$ FOR $\zeta \le \sqrt{2}/2$

  $|T(j\omega)| = 1$ $\quad$ FOR $\zeta \geq \sqrt{2}/2$

- AT $\omega/\omega_n = 1$ $\quad |T(j\omega)| = \dfrac{1}{2\zeta}$ $\quad$ MAGNITUDE AT RESONANCE

- ω_n INDICATES THE BANDWIDTH FOR A SECOND ORDER SYSTEM

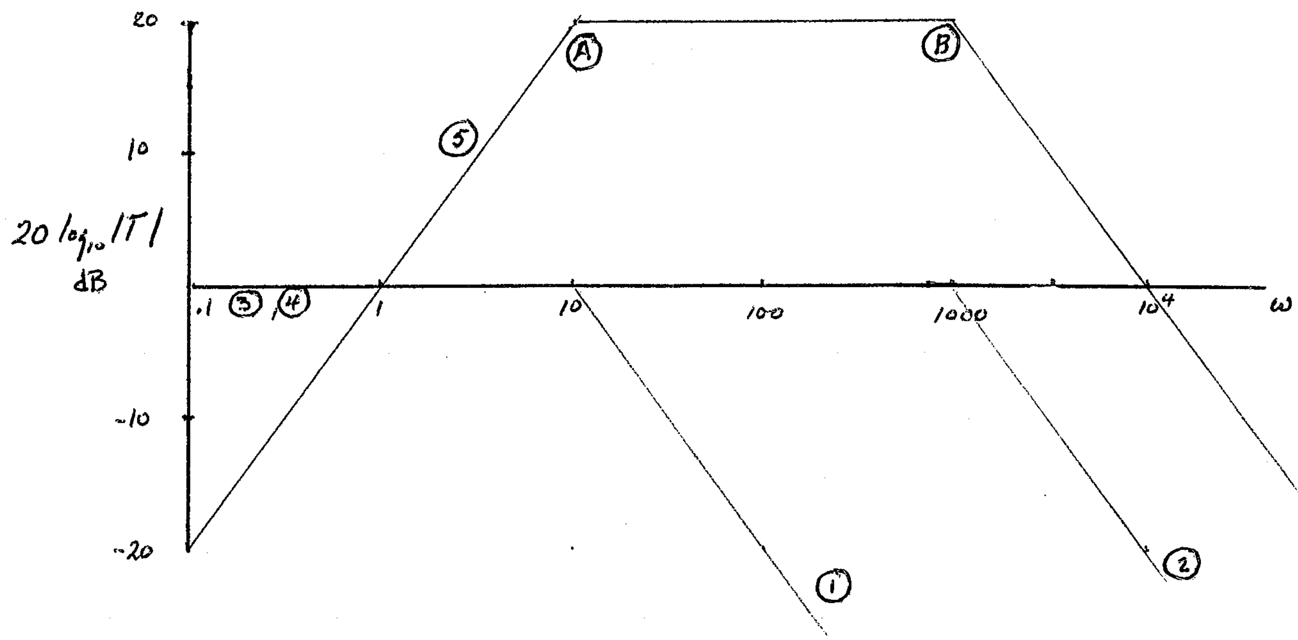
EXAMPLE OF COMPOSITE SYSTEM

$$T(s) = \frac{As}{(\tau_1 s+1)(\tau_2 s+1)} \qquad \begin{aligned}\tau_1 &= .1\\ \tau_2 &= .001\\ A &= 1\end{aligned}$$

- FORM $(T(j\omega))$: $\dfrac{Aj\omega}{(\tau_1 j\omega+1)(\tau_2 j\omega+1)}$

- $20\log_{10}|T(j\omega)| = 20\log_{10}|A| + 20\log_{10}|j\omega| - 20\log_{10}|\tau_1 j\omega+1| - 20\log_{10}|\tau_2 j\omega+1|$

$$|\tau_1 j\omega+1| = \sqrt{1+(\tau_1\omega)^2} \quad \& \quad |\tau_2 j\omega+1| = \sqrt{1+(\tau_2\omega)^2} \qquad |j\omega| = \omega$$



- NOTE (5) represents PLOT OF $20\log_{10}|j\omega|$

(3), (1) REPRESENT LOW & HIGH FREQ ASYMPTOTES OF $\frac{1}{\tau_1 j\omega+1}$

(4), (2) REPRESENT LOW & HIGH FREQ ASYMPTOTES OF $\frac{1}{\tau_2 j\omega+1}$

AT (A) SLOPE OF $20\log_{10}|j\omega|$ IS 20 dB/decade

OF $-20\log_{10}|\tau_1 j\omega+1|$ is -20 dB/decade

$-20\log_{10}|\tau_2 j\omega+1|$ is 0

$\Rightarrow$ $20\log_{10}|T(j\omega)|$ has slope of ZERO

AT (B) SAME AS (A) BUT NOW $-20\log_{10}|\tau_2 j\omega+1|$ HAS SLOPE OF -20 dB/DECADE

- CONNECT LINES TO DEFINE BODE PLOT

FOR THE PHASE ANGLE : $\measuredangle T(j\omega) = \measuredangle A + \measuredangle(j\omega) - \measuredangle(\tau_1 j\omega+1) - \measuredangle(\tau_2 j\omega+1)$

- REMEMBER $\measuredangle(j\omega) = 90^\circ$; $\measuredangle A = 0$; $\measuredangle(\tau_1 j\omega+1)$ varies from 0 TO -90

WHEN $\log_{10}\omega$ varies between $.1/\tau$ TO ∞ also at $\log_{10}\omega = \frac{1}{\tau}$, $\phi = -4$

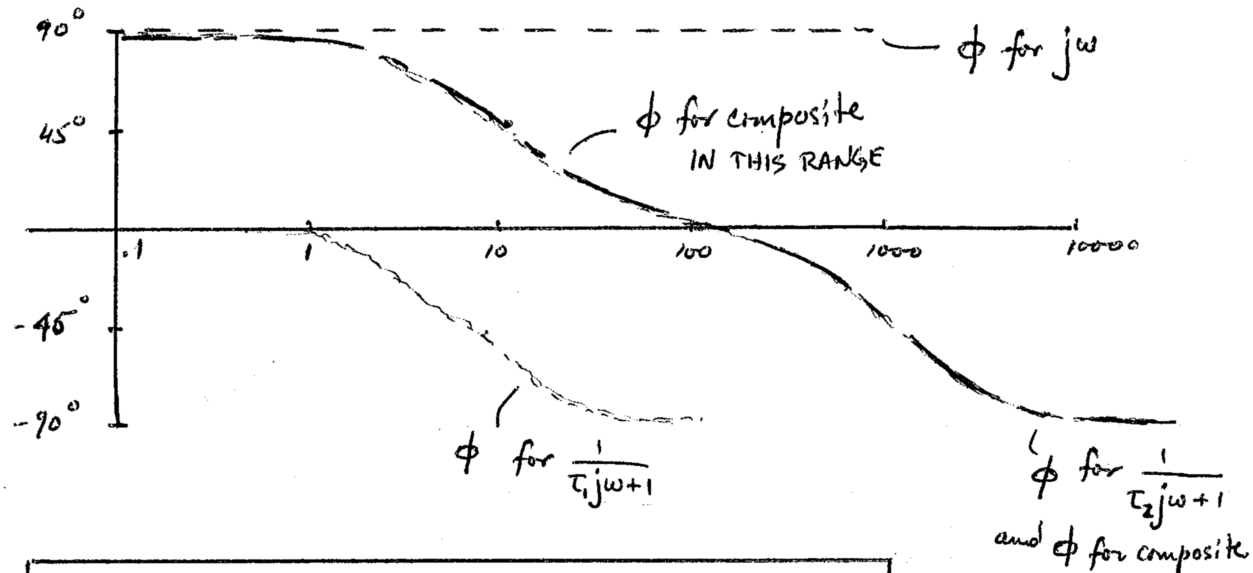
- THUS FOR $\tau_1 = .1$ @ $\frac{.1}{\tau_1} = 1 = \log_{10}\omega \Rightarrow 0^\circ = \phi$

$\tau_1 = .1$ @ $1.0/\tau_1 = 10 = \log_{10}\omega \Rightarrow 45^\circ = \phi$ and $\log_{10}\omega = \frac{10}{\tau_1} = 100 \Rightarrow \phi \rightarrow -90^\circ$

FOR $\frac{.1}{\tau_2} = \log_{10}\omega \Rightarrow \log_{10}\omega = 100 \qquad \angle(\tau_2 j\omega + 1) = 0^\circ$

@ $\frac{1.0}{\tau_2} = \log_{10}\omega = 1000 \qquad \angle = -45^\circ$

$\frac{10}{\tau_2} = 10000 \qquad \angle = -90^\circ$

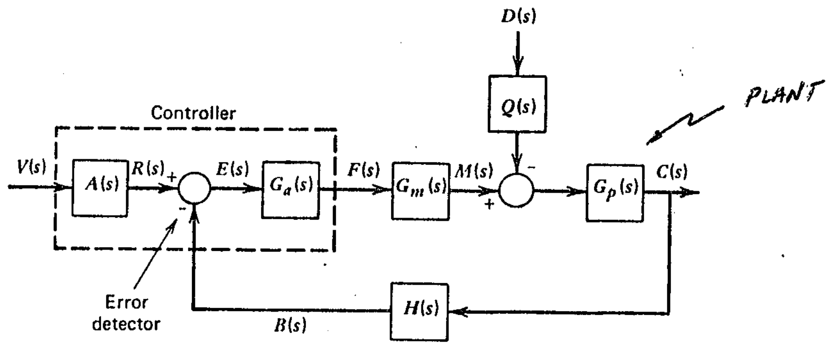


→ $\boxed{\text{HW. Do Prob. 4.15 b, c, f, h, 4.21, 4.22}}$

- FEEDBACK CONTROL - FINAL TOPIC TO BE DISCUSSED (MODELING, ANAL, CNT'L,
  - SYSTEM USED TO REGULATE / ADJUST FLOW OF ENERGY IN SOME MANNER
  - CLOSED LOOP / FEEDBACK CNT'L SYSTEM USES OUTPUT TO MODIFY SYSTEM ACTIONS TO ACHIEVE GOAL
  - OPEN LOOP SYSTEM MODIFIES FLOW ACCORDING TO SOME SET SCHEDULE
  - AUTO CONTROL SYSTEM - DOESN'T REQUIRE OUTSIDE INTERVENTION

- OPEN LOOP CONTROLLER - TIMER TO SWITCH LIGHTS ON/OFF
- CLOSED LOOP - THERMOSTAT COMMAND: DESIRED TEMP
  MEASURES ACTUAL TEMP & COMPARES TO COMMAND
  DISTURBANCES: OPENING DOOR CAUSE TRANSIENTS

- TO CONTROL EFFECTS OF TRANSIENTS, MUST DESIGN A DYNAMIC MODEL FOR SYSTEM TO BE CONTROLLED
- STRUCTURE OF CONTROL SYSTEM DEPENDS ON MODEL



Elements		Signals	
$A(s)$	Input elements	$B(s)$	Feedback signal
$G_a(s)$	Control logic elements	$C(s)$	Controlled variable or output
$G_m(s)$	Final control elements	$D(s)$	Disturbance input
$G_p(s)$	Plant elements	$E(s)$	Error or actuating signal
$H(s)$	Feedback elements	$F(s)$	Control signal
$Q(s)$	Disturbance elements	$M(s)$	Manipulated variable
		$R(s)$	Reference input
		$V(s)$	Command input

FIGURE 6.7 Terminology and basic structure of a feedback control system.

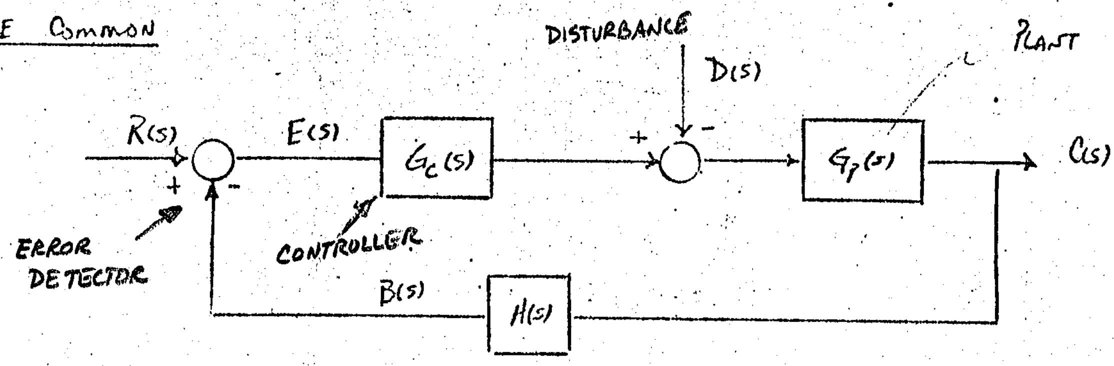
- CONTROLLER – LOGIC ELEMENT COMPARES $V(s)$ WITH $C(s)$; DECIDES WHAT TO DO
- CONTROL LOGIC ELEMENTS PRODUCE SIGNAL SENT TO FINAL CONTROL ELEMENT TO CONTROL PLANT (OBJECT TO BE CONTROLLED)
- LOGIC ELEMENTS ARE "BRAINS", FINAL CONTROL ELEMENTS ARE MUSCLE
- FINAL CONTROL ELEMENT PRODUCES SIGNAL TO BE ACTED UPON BY PLANT TO PRODUCE FINAL OUTPUT. THIS SIGNAL IS MANIPULATED VARIABLE $M(s)$

CONTROL SYSTEM CAN BE CLASSIFIED AS

- REGULATOR – CONTROLLED VARIABLE $C(s)$ IS TO BE KEPT CONSTANT
- FOLLOW UP SYSTEM – CONTROLLED VARIABLE IS KEPT NEAR COMMAND VALUE;
  EX. MACHINE TOOL SYSTEM FOR MAKING KEYS
- SERVOMECHANISM – $C(s)$ IS POSITION, VELOCITY OR ACCELERATION
- PROCESS CONTROL SYSTEM – THERMODYNAMIC PROCESS IS CONTROLLED VARIABLE
  TEMP, PRESSURE, FLOW RATE, ETC.

- OUR MAIN FUNCTION HERE IS TO DESIGN LOGIC ELEMENT

MORE COMMON



CONTROLLERS MAY ALSO BE CLASSIFIED BY

1. KIND OF POWER EMPLOYED TO IMPLEMENT THE CONTROL
   - PNEUMATIC
   - HYDRAULIC
   - ELECTRONIC

   $\Big\}$ MOTOR

2. CONTROL ACTION IT PRODUCES
   - TWO POSITION (ON/OFF)
   - PROPORTIONAL
   - INTEGRAL
   - PROPORTIONAL + INTEGRAL (PI)
   - PROPORTIONAL + DERIVATIVE (PD)
   - PID

TRANSDUCERS & ERROR DETECTORS

- NEED SENSORS TO PROVIDE MEASUREMENTS NECESSARY FOR FEEDBACK
- TRANSDUCER - DEVICE THAT CONVERTS ONE SIGNAL TO ANOTHER

  DISPL. TO VOLTAGE IN POTENTIOMETER
- LVDT (LINEAR VARIABLE DIFFERENTIAL TRANSFORMER) MEASURES DISPL. THROUGH CHANGES IN VOLTAGE OF SECONDARY WINDINGS ABOUT A MOVABLE MAGNETIC CORE
- STRAIN GAGES - MEASURES CHANGES IN WIRE RESISTANCE AS LENGTH CHANGES
- TACHOMETER - CHANGES IN VELOCITY MEASURED VIA CHANGES IN GENERATED VOLTAGE IN DC GENERATOR
- THERMOCOUPLES
- FLOW METERS $\Delta p \sim (\text{FLOW RATE})^2$ & Δp VIA BERNOLLI

  CAN DETERMINE DISPLACEMENT OF DIAPHRAGM

ERROR DETECTOR - DEVICE TO FIND DIFFERENCE BETWEEN TWO SIGNALS

- A BEAM ON A PIVOT USED TO COMPARE DISPL., FORCES, PRESSURES
- DETECTOR FOR VOLTAGE DIFF. DIFFERENTIAL AMPLIFIER

PG 329

ACTUATORS

- FINAL CONTROL ELEMENT THAT TAKE A LOW-LEVEL CONTROL SIGNAL TO PRODUCE A SIGNAL WITH ENOUGH POWER TO DRIVE PLANT
  "MUSLE" OF CONTROL SYSTEM
- FIGURE 2.19 ON PG 69 OF AN ARMATURE CONTROLLED D-C MOTOR HAS BLOCK DIAGRAM ON PG 331
- HYDRAULIC SERVOMOTOR P.79 & PNEUMATIC BELLOWS ON PG 87

CONTROL LAWS / CONTROL ACTIONS

- THESE ARE THE ALGORITHMS USED TO CONVERT THE ERROR SIGNAL INTO THE CONTROL SIGNAL - ACCOMPLISHED BY CONTROL LOGIC ELEMENTS

- ERROR SIGNAL CAN BE NONZERO IF CHANGE IN COMMAND OR DISTURBANCE
- WE CAN KEEP THE CONTROLLED VARIABLE NEAR ITS DESIRED VALUE

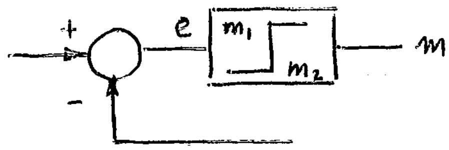
IF:
- MINIMIZE STEADY STATE ERROR (ACTUAL - DESIRED STEADYSTATE)
- MINIMIZE SETTLING TIME
- MINIMIZE MAX OVERSHOOT
- OTHER SPECIFICATIONS (BANDWIDTH, PARAMETER SENSITIVITY ETC)

- THESE ARE ACCOMPLISHED VIA THE FOLLOWING CONTROL LAWS

TWO-POSITION CONTROL ON/OFF CONTROL (THERMOSTAT)

- CONTROLLER OUTPUT DEPENDS ON MAGNITUDE OF ERROR SIGNAL

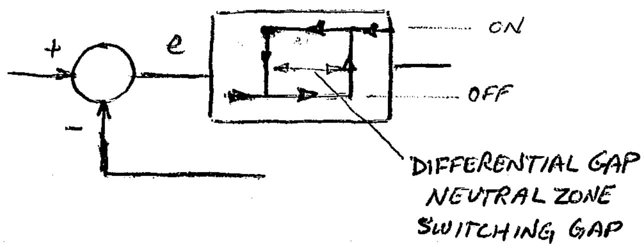
BANG-BANG. (REVERSIBLE MOTOR)



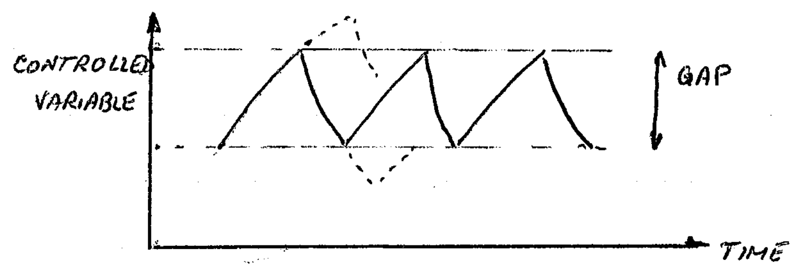
$m(t) = M_1$ for $e(t) > 0$

$m(t) = M_2$ for $e(t) < 0$

- REAL SYSTEM SENSOR & CONTROL VALVE DO NOT RESPOND INSTANTANEOUSLY



- ZONE EXISTS TO PREVENT FREQUENT ON-OFF SWITCHING
- UNDERSHOOT/OVERSHOOT OK IF TIME CONST. OF CNT'L ELEM << THAN TIME CONST. OF SYSTEM



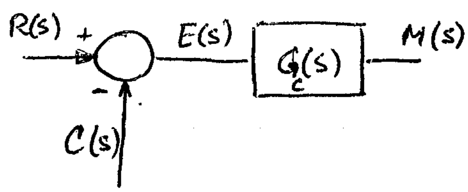
--- REAL LIFE

—— IDEAL (INSTANT RESPONSE

- USED IN SYSTEMS WHERE REQUIREMENTS NOT SEVERE. FOR FINER CONTROL:

LESSON # 18

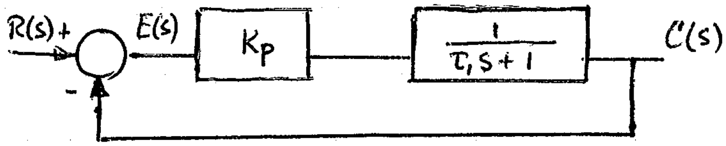
PROPORTIONAL CONTROL - SETS OUTPUT IN PROPORTION TO DEVIATION FROM DESIRED VALUE. K_p PROPORTIONAL GAIN



$G_c(s) = K$

$m(t) = K_p e(t)$ IN TIME DOMAIN

HOW DOES PROPORTIONAL CONTROL AFFECT A FIRST ORDER SYSTEM



$E(s) = R(s) - C(s)$

$K_p E(s)$ IS CONTROL SIGNAL

- CLOSED LOOP TRANSFER FN

$$\frac{C(s)}{R(s)} = \frac{K_p}{\tau s + (1 + K_p)}$$

- WHAT HAPPENS WHEN $R(s)$ IS A UNIT STEP INPUT $\mathcal{L}\{u_s(t)\} = \frac{1}{s}$ AT STEADY STATE CONDITIONS ?

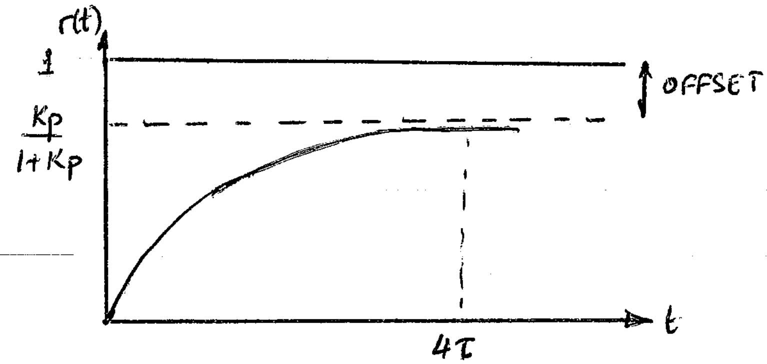
$$c_{ss}(t) = \lim_{t\to\infty} c(t) = \lim_{s\to 0} s\,C(s) = s\,\frac{K_p}{s\,[\tau s + (1+K_p)]} = \frac{K_p}{1+K_p}$$

- THUS FOR A STEP INPUT $u_s(t)=1$; THE OUTPUT $C(t)$ IS LESS THAN ONE ALWAYS EVEN IF NO DISTURBANCE

LOOK AT $e_{ss}(t) = r_{ss}(t) - c_{ss}(t) = 1 - \frac{K_p}{1+K_p} = \frac{1}{1+K_p}$ OFFSET ERROR

CAN BE MADE AS SMALL AS WE WANT IF $K_p >> 1$

- OUTPUT APPROACHES FINAL VALUE WITHOUT OSCILLATING ; TIME CONST $\sim \frac{1}{K_p}$
- OUTPUT ERROR IS INVERSELY PROPORTIONAL TO GAIN $e_{ss} \sim \frac{1}{K_p}$

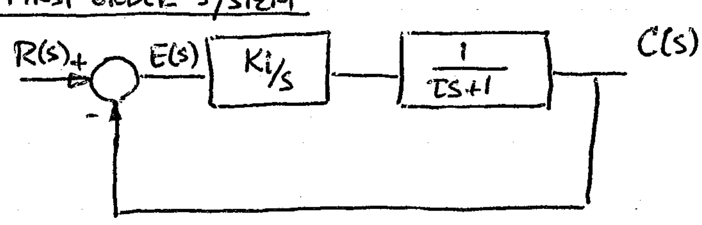


- NOTE w/ PROPORTIONAL CONSTANT ERROR EXISTS AT EQUIL **FOR STEP INPUT**

INTEGRAL CONTROL ALLOWS CONTROLLER TO PRODUCE INCREASING SIGNAL AS LONG AS ERROR IS NON ZERO

$$\frac{d\,m(t)}{dt} = k_i\, e(t) \quad \text{or} \quad M(s) = \frac{K_i}{s} E(s)$$

FOR FIRST ORDER SYSTEM



$$\frac{C(s)}{R(s)} = \frac{K_i}{s(\tau s+1)+K_i} = \frac{K_i}{\tau s^2 + s + K_i}$$

- FOR UNIT INPUT $\left(R(s)=\frac{1}{s}\right)$ THEN $c_{ss}(t) = \lim_{s\to 0} s\,C(s) = 1$

AND $e_{ss}(t) = r(t) - c_{ss}(t) = 0$: NO OFFSET UNDER UNIT INPUTS

- HOWEVER BECAUSE 2nd ORDER IT IS POSSIBLE FOR OSCILLATORY BEHAVIOR TO EXIST : $(\tau s^2 + s + K_i) = \tau\left(s^2 + \frac{1}{\tau}s + \frac{K_i}{\tau}\right) \Rightarrow \omega_n = \sqrt{\frac{K_i}{\tau}} \quad 2\zeta\omega_n = \frac{1}{\tau} \Rightarrow \zeta = \frac{1}{2\sqrt{\tau K_i}}$

- FOR NON OSCILLATORY BEHAVIOR $\zeta > 1 \Rightarrow K_i$ IS SMALL; HOWEVER RESPONSE WOULD BE SLUGGISH : THE TIME CONSTANT FOR SYSTEM WOULD BE LARGE
- FOR SLIGHT DAMPING K_i IS LARGE $\Rightarrow$ OSCILLATORY BEHAVIOR
- IMPROVED SS BEHAVIOR AT EXPENSE OF TRANSIENT PERFORMANCE

PROPORTIONAL + INTEGRAL CONTROL

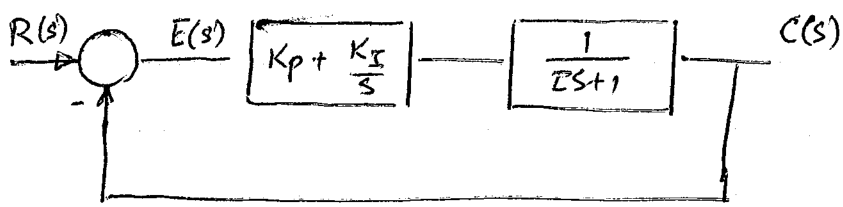
- FEATURES VARIABILITY TO SATISFY STEADY STATE & TRANSIENT SPECS

$$M(s) = K_P\left(1 + \frac{1}{T_i s}\right)E(s) = \left(K_P + \frac{K_I}{s}\right)E(s)$$

- INTEGRAL ACTION PROVIDES AN AUTOMATIC RESET OF THE CONTROLLER WHEN DISTURBED

T_i - RESET TIME $= K_P/K_I$

FIRST ORDER SYSTEM



$$\frac{C(s)}{R(s)} = \frac{K_P s + K_I}{\tau s^2 + (K_P+1)s + K_I}$$

$$\frac{K_P+1}{\tau} = 2\zeta\omega_n \qquad \omega_n = \sqrt{\frac{K_I}{\tau}} \qquad \therefore \quad \zeta = \frac{1+K_P}{2\sqrt{\tau K_I}}$$

- TIME CONST $= \dfrac{1}{\zeta\omega_n} = \dfrac{2\tau}{K_P+1}$ PICK K_P TO ACHIEVE TIME CONSTANT

- NOW PICK K_I TO CAUSE $\zeta > 1$

DISADVANTAGES - PRODUCES A CONTROL SIGNAL EVEN AFTER ERROR HAS VANISHED: $c_{ss} = 1$ $e_{ss} = 0$ FOR UNIT STEP INPUT

BUT $M_{ss} = \left(K_P + \frac{K_I}{s}\right)E_{ss} = K_I$

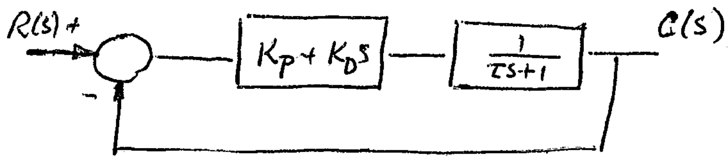
- INTRODUCES HIGHER OVERSHOOTS DUE TO NUMERATOR DYNAMICS

PROPORTIONAL + DERIVATIVE CONTROL

- METHOD BY WHICH CONTROLLER IS MADE AWARE ERROR $\rightarrow 0$
- DESIGN A CONTROLLER TO REACT TO DERIVATIVE OF ERROR
- USED TO DAMP OUT OSCILLATION

$$M(s) = K_P(1 + T_D s)E(s) = (K_P + K_D s)E(s)$$

LOOK AT DERIVATIVE CONTROL FOR A FIRST ORDER SYSTEM



$$\frac{C(s)}{R(s)} = \frac{K_P + K_D s}{s(\tau + K_D) + (1+K_P)}$$

- FOR UNIT INPUT, $c_{ss} = \lim_{s\to 0} s\,C(s) = s\,\dfrac{K_P + K_D s}{s(\tau+K_D)+(1+K_P)} \cdot \dfrac{1}{s} = \dfrac{K_P}{1+K_P}$

- $e_{ss} = r(t) - c_{ss} = 1 - \dfrac{K_P}{1+K_P} = \dfrac{1}{1+K_P}$; ERROR SAME AS PROPORTIONAL

- $m(t) = \lim_{s\to 0} s\,M(s) = \lim_{s\to 0} s\,(K_P + K_D s)\,E(s)$

$$= K_P\, e(t) + s\,K_D\, e(t) = \frac{K_P}{1+K_P} = c_{ss}$$

- PD control IMPROVES TRANSIENT RESPONSE BUT ALLOWS FOR A NON-ZERO DEVIATION IN THE ERROR
- HOWEVER IT IS DIFFICULT TO CONSTRUCT A DIFFERENTIATING DEVICE

PROPORTIONAL + INTEGRAL + DIFFERENTIAL CONTROL

- ALLOWS FOR MORE FLEXIBILITY - TRANSIENT RESPONSE IMPROVED; $e_{ss} = 0$

$$M(s) = \left(K_D s + K_P + \frac{K_I}{s}\right)E(s)$$

SUMMARY

NOTE - 90% OF CONTROLLERS ARE PI TYPE

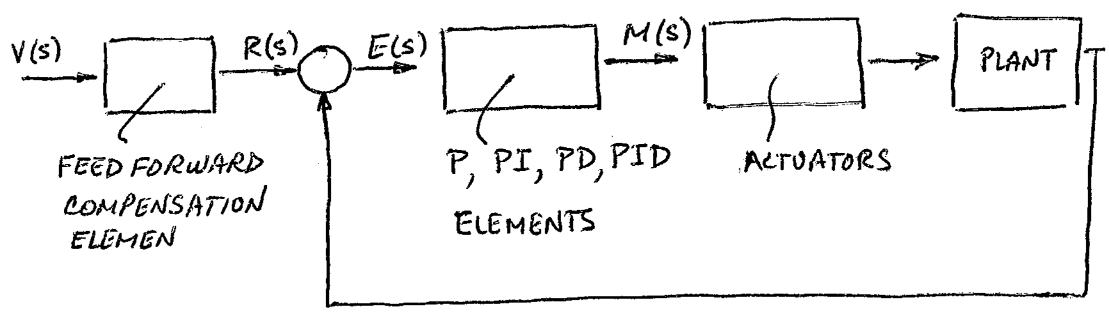
WHY? GOOD FOR HEAVILY DAMPED SYSTEM (FIRST ORDER W/ PI LEADS TO A 2nd ORDER SYSTEM W/ NUMERATOR DYNAMICS)

- LEADS TO LARGER OVERSHOOTS BUT FASTER RISE TIMES
- HAS ZERO STEADY STATE ERROR FOR STEP INPUTS

PROBLEMS - FOR SECOND ORDER SYSTEMS; REQUIRES HIGH K_P IF DAMPING IS LIGHT TO RETAIN STABILITY. SYSTEM UNSTABLE IF NO DAMPING

- PROPORTIONAL CONTROL W/ RAMP INPUTS PG. 342-344
  - VERY GOOD RESPONSE FOR FIRST ORDER SYSTEM
  - FOR SECOND ORDER SYSTEM – OFFSET ERROR CAN NEVER MADE ZERO
    SLIGHT DAMPING LEADS TO OSCILLATORY BEHAVIOR

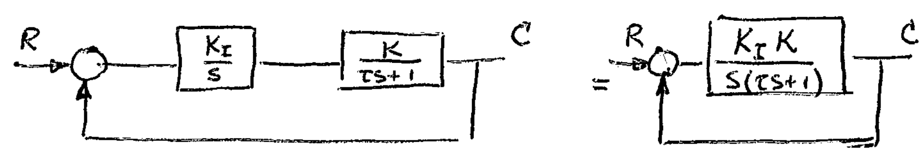
- FEED FORWARD COMPENSATION – MAY BE USED TO FOOL THE CONTROLLER TO PRODUCE ZERO STEADY STATE OFFSET ERROR FOR STEP INPUT AND TO REDUCE ERROR FOR RAMP INPUTS
  - THIS IS DONE BY PLACING AN INPUT ELEMENT TO CHANGE THE COMMAND INPUT PRIOR TO ARRIVAL AT THE ERROR DETECTOR



- NICE THING ABOUT <u>PID</u> – LIKE PI FOR SECOND ORDER SYSTEM REQUIRES HIGH $K_P K_D$ IF DAMPING IS LIGHT, BUT SINCE K_D IS PRESENT $\Rightarrow$ K_P NEED NOT BE HIGH

- WHICH LAW TO USE – DEPENDS ON SPECS AND EXPERIENCE

<u>EXAMPLE 6.25</u> PG 379



NOW:

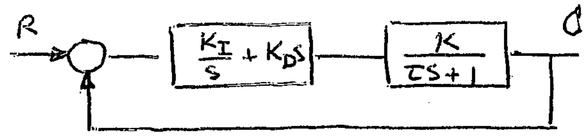
$$\frac{C(s)}{R(s)} = \frac{K_I K}{K_I K + s(\tau s+1)}$$

CHAR EQ IS $\tau s^2 + s + K_I K = 0$

$\Rightarrow \frac{K_I K}{\tau} = \omega_n^2$ AND $2\zeta\omega_n = \frac{1}{\tau}$

THUS $\zeta = \frac{1}{2}\sqrt{\frac{1}{\tau K_I K}}$ FOR OSCILLATORY BEHAVIOR $\zeta < 1 \Rightarrow \tau K_I K > \frac{1}{4}$

IF WE ADD DERIVATIVE ACTION (DERIVATIVE CONTROL CAN NEVER BE USED ALONE!)



$$\frac{C(s)}{R(s)} = \frac{\left(\frac{K_I + K_D s^2}{s}\right)\left(\frac{K}{\tau s+1}\right)}{1 + \left(\frac{K_I + K_D s^2}{s}\right)\left(\frac{K}{\tau s+1}\right)}$$

or $$\frac{C(s)}{R(s)} = \frac{K\left[K_I + K_D s^2\right]}{s(\tau s+1)+(K_I + K_D s^2)K} \Rightarrow \frac{KK_I + K_D K s^2}{(\tau + KK_D)s^2 + s + K_I K}$$

LOOK AT CHAR. EQN $(\tau + KK_D)s^2 + s + K_I K = 0$

$$\Rightarrow \omega_n^2 = \frac{K_I K}{\tau + KK_D} \quad \& \quad 2\zeta\omega_n = \frac{1}{\tau + KK_D} \quad \Rightarrow \quad \zeta = \frac{1}{2}\frac{1}{\sqrt{K_I K}} \cdot \frac{1}{\sqrt{\tau + K_D K}}$$

THUS IF $K_D > 0$ ζ WILL BE SMALLER MAKING SYSTEM MORE OSCILLATORY

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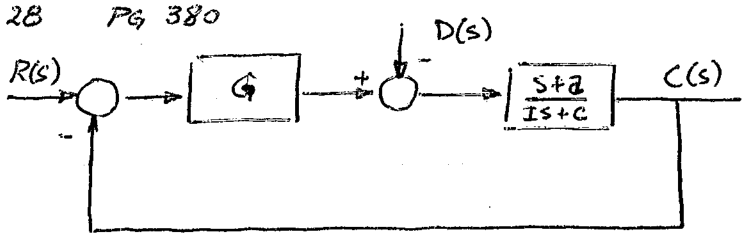
EXAMPLE 6.28 PG 380



GIVEN $I = 10$, $C = .1$ $a = 1$

a • FIND G SO THAT FOR $r(t)$ = UNIT STEP & $d(t) = 0$ e_{ss} IS MIN

DOMINANT TIME CONST $\leq .1$ SEC & $\zeta = 1$

$$T(s) = C(s)/R(s) = \frac{G(s)\,(s+a)}{(Is+C) + G(s)(s+a)}$$

SINCE WE MUST HAVE A ζ $\Rightarrow$ CHAR EQ MUST BE 2^{nd} ORDER

- P ACTION ONLY PRODUCES CHAR EQ 1^{ST} ORDER SINCE $G(s) = K_p$
- PI ACTION : $G(s) = K_p + \frac{K_I}{s}$ GIVES

$$T(s) = \frac{C(s)}{R(s)} = \frac{K_p s^2 + (K_p a + K_I)s + K_I a}{(K_p + I)s^2 + (K_p a + K_I + c)s + K_I a}$$

$$\omega_n^2 = \frac{K_I a}{K_p + I} \quad \text{and} \quad 2\zeta\omega_n = \frac{K_p a + K_I + c}{K_p + I} \quad ; \quad \text{ALSO} \quad \tau = .1 = \frac{2(K_p + I)}{K_p a + K_I + c}$$

AND $\zeta = 1 = \frac{1}{\tau}\sqrt{\frac{K_p + I}{K_I a}}$. SOLUTION OF THESE TWO LEADS TO $K_I = 12.22$

$K_p = -9.878$

FOR $r(t)$ = UNIT STEP $R(s) = \frac{1}{s}$

$$C(s) = T(s) \cdot \frac{1}{s} \quad \& \quad c_{ss}(t) = \lim_{s \to 0} s\,C(s) = \lim_{s \to 0} T(s=0)$$

$$c_{ss} = \frac{K_I a}{K_I a} = 1 \quad \& \quad e_{ss} = c_{ss} - r_{ss} = 1 - 1 = 0 \quad \checkmark$$

NOTE FOR PI $e_{ss} = 0$ FOR $r(t)$ = STEP INPUT IRRESPECTIVE OF NUMERATOR FOR FIRST ORDER SYSTEM & EVEN IF PLANT IS SECOND ORDER

IF WE USED I ACTION ONLY $G(s) = \frac{K_I}{s}$

$$\frac{C(s)}{R(s)} = \frac{K_I(s+a)}{s(Is+c) + K_I(s+a)} = \frac{K_I s + K_I a}{Is^2 + (c+K_I)s + K_I a}$$

IF $\zeta = 1$ $\quad \zeta\omega_n = \omega_n$ $\quad$ BUT $\tau = \frac{1}{\zeta\omega_n} = .1 \Rightarrow \zeta\omega_n = \omega_n = 10$

$$\Rightarrow \sqrt{\frac{K_I a}{I}} = 10 \quad \Rightarrow K_I = 1000$$

BUT $\frac{C + K_I}{I} = 2\zeta\omega_n$ $\quad \therefore \quad \zeta = \frac{C + K_I}{2\omega_n I} = 5$ $\quad \therefore$ CANNOT SATISFY REQ'TS SINCE ONLY HAVE K_I TO SATISFY 2 RQM'TS.

b) STEP RESPONSE $R(s) = \frac{1}{s}$

$$C(s) = \frac{1}{s}\left[1 - \frac{Is^2 + cs}{(K_p + I)s^2 + (K_p a + K_I + c)s + K_I a}\right]$$

$$= \frac{1}{s} - \frac{Is + c}{(K_p + I)s^2 + (K_p a + K_I + c)s + K_I a}$$

$$= \frac{1}{s} - \frac{10s + .1}{.122s^2 + 2.442s + 12.22}$$

$$\frac{1}{s} + \frac{A}{(s+10)^2} + \frac{B}{s+10} \qquad \begin{aligned} A &= 818.86 \\ B &= -81.967 \end{aligned}$$

$$c(t) = 1 + 818.86te^{-10t} - 81.967e^{-10t} = 1 + (At + B)e^{-\omega_n t}$$

overshoot (ie when $dC/dt = 0$) occurs when $t = \frac{1}{\omega} - \frac{B}{A} = .2$ sec

@ .2 sec $\quad$ overshoot $= \left[\dfrac{c(t=.2\,sec) - r(t)}{r(t)}\right] \times 100\% = \left[\dfrac{12.07 - 1}{1}\right] \times 100\% = 1207\%$

IF NO NUMERATOR DYNAMICS $\quad G(s) = \dfrac{1}{Is+C}$

$$T(s) = \frac{G(s)}{Is+C+G(s)} = \frac{K_p s + K_I}{Is^2 + (K_p+C)s + K_I}$$

ORDER OF NUMERATOR IS ONE LESS; OVERSHOOT STILL PRESENT BUT SMALLER. $\quad$ NOTE FOR $\zeta = 1$ $\quad 2\zeta\omega_n = 2\omega_n = \dfrac{K_p + C}{I}$ $\;\&\; \omega_n^2 = \dfrac{K_I}{I}$

SINCE $\dfrac{1}{\zeta\omega_n} = \tau = .1 \Rightarrow \omega_n = 10 \Rightarrow K_I = 1000 \;\&\; K_p = 20I - C = 199.9$

$$\therefore\; C(s) = T(s)\frac{1}{s} = \frac{1}{s}\,\frac{K_p s + K_I}{Is^2 + (K_p+C)s + K_I} = \frac{1}{s}\,\frac{199.9\,s + 1000}{10\,[s^2 + 20s + 100]}$$

$$= \frac{1}{10s}\,\frac{200(s+5)}{(s+10)^2} = \frac{A}{s} + \frac{B}{(s+10)^2} + \frac{C}{s+10}$$

$A = \lim\limits_{s\to 0} sC(s) = 1$ $\quad$ also $\quad B = +9.99 \quad C = -1$

$$c(t) = 1 + 9.99\,t\,e^{-10t} - 1\,e^{-10t} = 1 + (At + B)\,e^{-\omega_n t}$$

max overshoot occurs at $t = .\;$ sec $= \dfrac{1}{\omega_n} - \dfrac{B}{A}$

@ .2 sec $\quad$ overshoot $= \left[\dfrac{c(t=.2\,sec) - r(t)}{r(t)}\right] \times 100\% = \dfrac{1.133 - 1}{1} \times 100\% = 13.3\%$

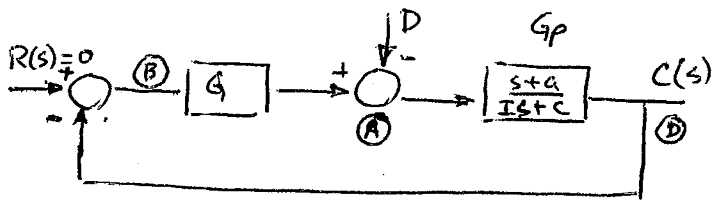
c) WHAT IS e_{ss} IF $r(t)$ is UNIT RAMP? $\quad R(s) = \dfrac{1}{s^2}$

$$\therefore\; C(s) = \frac{1}{s^2}T(s) \quad \text{and} \quad c_{ss} = \lim_{s\to 0} s\,C(s) = \lim_{s\to 0} \frac{1}{s}T(s)$$

$$\text{and } e_{ss} = \lim_{s\to 0} s\left[R(s) - C(s)\right] = s\left[\frac{1}{s^2} - \frac{1}{s^2}T(s)\right] = \frac{1}{s}\left[1 - T(s)\right]$$

$$= \frac{1}{s}\,\frac{Is^2 + cs}{(K_p + I)s^2 + (K_p a + K_I + C)s + K_I a} = \frac{C}{K_I a} = \frac{.1}{(12.22)\,1} = .008$$

d) FIND STEADY STATE DEVIATION IF $d(t) = u_s(t)$



AT (B) SIGNAL IS $-C(s)$; AT (A) SIGNAL IS $-GC-D$; AT (D) SIGNAL IS

$$(-GC-D)G_p = C \qquad \therefore \quad G_p D = C\left[GG_p + 1\right]$$

$$\therefore \quad \frac{C(s)}{D(s)} = \frac{-G_p}{GG_p+1} = \frac{-s(s+a)}{s(Is+c)+(K_p s+K_I)(s+a)} = \frac{-s^2-as}{(I+K_p)s^2+(K_p a+K_I+c)s+K_I a}$$

IF $\quad D(s) = \frac{1}{s} \quad$ ($d(t)$ = UNIT STEP)

$$C(s) = \frac{-(s+a)}{(I+K_p)s^2+(K_p a+K_I+c)s+K_I a}$$

$$c_{ss} = \lim_{s\to 0} sC(s) = 0 \qquad \text{STEADY STATE DEVIATION}$$

THUS A UNIT STEP DISTURBANCE PRODUCES <u>NO</u> STEADY STATE OUTPUT

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DUE MARCH 29 : 6.21, 6.23, 6.26

SUGGEST YOU READ UP ON P, PI, PD, I, PID CONTROL LAWS

→ SUGGEST YOU READ § 6.8 – 6.11 TO SEE HOW THESE CONTROL LAWS ARE IMPLEMENTED ELECTRONICALLY, PNEUMATICALLY, HYDRAULICALLY

- <u>A WORD ABOUT NUMERATOR DYNAMICS & DOMINANT ROOT APPROX.</u>

CLTF $\Rightarrow \quad T(s) = \dfrac{N(s)}{D(s)}$

- ROOTS OF $D(s)$ (i.e. VALUES OF s FOR WHICH $D(s)=0$) ARE CALLED CLOSED LOOP ZEROS OF THE SYSTEM

- <u>IN GENERAL</u>

- CLOSED LOOP ZEROS TEND TO MAKE SYSTEM MORE RESPONSIVE

- ROOTS OF $D(s)$ ARE MORE IMPORTANT WHEN CLOSER TO THE IMAGINARY AXIS (DOMINANT ROOT EFFECT)
- FURTHER FROM IMAGINARY AXIS, THEY MAY BE NEGLECTED IN AN APPROXIMATION
- WE HAVE SEEN THAT NUMERATOR DYNAMICS CAN AFFECT ADVERSELY THE SOLUTION AND HENCE THE RESPONSE OF THE SYSTEM
- NUMERATOR DYNAMICS WILL APPEAR FOR CONTROLLER USING PD, PI OR PID ACTION, EVEN WHEN THE PLANT HAS NONE OF ITS OWN
- NUMERATOR DYNAMICS CAN EXIST WHEN INPUT INVOLVES DERIVATIVES

EXAMPLE

- $m\ddot{x} + c\dot{x} + kx = P_o(t) + A\dot{P}_o(t) + B\ddot{P}_o(t)$ $\quad P_o(t)$ input ; $x(t)$ output
- VIBRATION ABSORBER PROBLEM
- SERIES RLC CIRCUIT P. 40

LAPLACE TRANSFORM OF THE ABOVE IS $(ms^2+cs+k)X = P_o(1+As+Bs^2)$

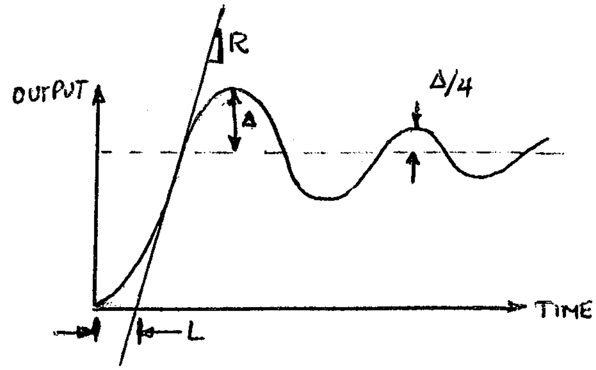
THUS $\quad T(s) = \dfrac{X}{P_o} = \dfrac{1+As+Bs^2}{ms^2+cs+k}$

- EVEN IF $ms^2+cs+k=0$ HAS DECAYING SOLUTIONS (STABLE SYSTEM) NUMERATOR DYNAMICS CAN CAUSE SYSTEM INSTABILITY OR OSCILLATORY BEHAVIOR

LESSON # 19

ZIEGLER - NICHOLS RULES

- IN SOME CASES IT IS DIFFICULT TO OBTAIN TRANSFER FN. MODELS
- EMPIRICAL FORMULAS HAVE BEEN USED DEVELOPED BY (Z-N).
- TWO SEPARATE METHODS
  - ONE METHOD REQUIRES OPEN LOOP STEP RESPONSE (PROCESS REACTION)
  - SECOND METHOD USES EXPERIMENTAL RESULTS AFTER CONTROLLER IS INSTALLED (ULTIMATE CYCLE)
- RULES USE THE QUARTER - DECAY CRITERION – 2^{ND} OVERSHOOT $= \frac{1}{4}$ 1^{ST} O.S.



- PROCESS REACTION METHOD
  - ← DETERMINE OPEN LOOP RESPONSE
  - ← FIND LINE TANGENT TO STEEPEST PART OF CURVE – R
  - ← DETERMINE WHERE LINE CROSSES TIME AXIS – L

- FOR P CONTROL $K_p = 1/RL$

  PI CONTROL $K_p = 0.9/RL$ $\quad K_I = .273/R$

  PID $K_p = 1.2/RL$ $\quad K_I = .6/R$ $\quad K_D = 2.4/R$

- NOTE: THIS METHOD <u>CANNOT</u> BE USED FOR 1^{ST} & 2^{ND} ORDER SYSTEMS ONLY FOR 3^{RD} OR HIGHER W/ SUFFICIENT DAMPING.

- <u>ULTIMATE CYCLE METHOD</u>.

- RESTRICTIONS – TURN OFF ALL EXCEPT PROPORTIONAL CONTROL & INCREASE GAIN UNTIL SUSTAINED OSCILLATIONS ARE ATTAINED (SYSTEM IS ON MARGIN OF STABILITY – TWO COMPLEX CONJUGATE ROOTS ON IMAGINARY AXIS & REST IN LEFT HALF OF S-PLANE) THIS VALUE OF GAIN IS CALLED K_{pu}. THE PERIOD OF THESE OSCILLATIONS IS CALLED T_u

- FOR P CONTROL $K_p = 0.5\,K_{pu}$

  PI CONTROL $K_p = 0.45\,K_{pu}$ $\quad K_I = .542\,K_{pu}/T_u$

  PID $K_p = 0.6\,K_{pu}$ $\quad K_I = 1.2\,K_{pu}/T_u$ $\quad K_D = 4.8\,K_{pu}/T_u$

EXAMPLE: FOR A 3^{RD} ORDER SYSTEM $a_3 s^3 + a_2 s^2 + a_1 s + a_0 + K_p = 0$ CHAR EQN

BY RESTRICTIONS $(s+s_3)(s^2+\omega^2) = 0$ CHAR EQN **ALSO a_3, a_2, a_1 & $a_0 + K > 0$ ROUTH CRIT**

$$\Rightarrow \quad (a_0 + K_p)/a_3 = s_3\omega^2 \quad \text{AND} \quad a_1/a_3 = \omega^2 \quad \text{AND} \quad a_2/a_3 = s_3$$

$$\therefore \quad \underline{\underline{\omega}} = \pm\underline{\underline{\sqrt{a_1/a_3}}} \qquad \underline{\underline{K_p}} = a_3 s_3 \omega^2 - a_0 = a_3 \cdot \frac{a_2}{a_3}\cdot\frac{a_1}{a_3} - a_0 = \underline{\underline{\frac{a_2 a_1}{a_3} - a_0}}$$

AND $\underline{T_u = \frac{2\pi}{\omega}}$

- FOR THIRD AND FOURTH ORDER SYSTEM, IT IS EASY TO FIND K_{Pu} ANALYTICALLY
- FOR HIGHER ORDERS MUCH HARDER - WE USE ROOT LOCUS FOR THAT TO FIND ROOTS ON IMAGINARY AXIS, HENCE K_{Pu} & T_u
- MORE ON ROOT LOCUS LATER

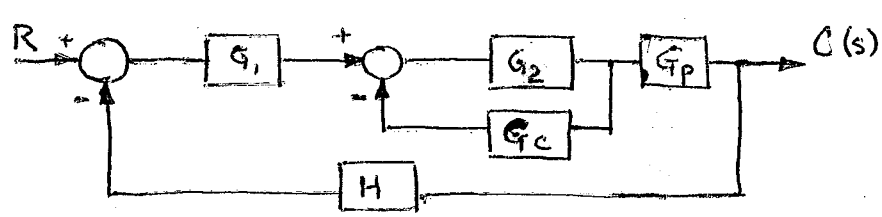
NONLINEARITIES AND PERFORMANCE

- NONLINEARITIES MAY HAVE SIGNIFICANT EFFECT ON SYSTEM PERFORMANCE
- START-UP OF CONTROLLER IF I.C.'S DIFFERENT THAN CONDITIONS FOR LINEARIZATION
- EACH PHYSICAL ELEMENT THAT PROVIDES CONTROL DOESN'T PROVIDE INFINITE ENERGY BUT HAS A MAXIMUM LEVEL - SATURATION NONLINEARITY
- WHEN CONTROLLER REQUIRES ELEMENT TO DO SOMETHING IT CAN'T THE ELEMENT IS OVERDRIVEN
- CONTROLLER W/ INTEGRAL ACTION ACTING ON OVERDRIVEN ELEMENT WILL NOT RESPOND IMMEDIATELY BUT SENDS A LARGE SIGNAL LATER
- THIS CAN RESULT IN LARGE OVERSHOOTS
- SELECT GAINS SO THAT THIS DOESN'T HAPPEN : EXAMPLE P. 408-409

COMPENSATION

- SOMETIMES IMPOSSIBLE TO SATISFY ALL PERFORMANCE SPECS USING PID
- SOMETIMES MOST CAN BE SATISFIED USING P, I, PI, PD OR PID
- TO SATISFY THE REST USE COMPENSATOR
- 3 TYPES: SERIES, FEEDBACK, FEED FORWARD DISTURBANCE
- PHYSICAL DEVICES FOR COMPENSATORS ARE LIKE THOSE ACTUATORS §6.8-6.11

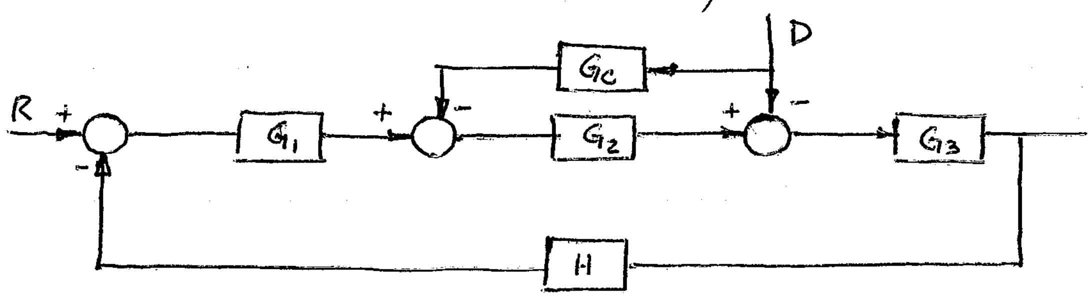
FEEDBACK OR PARALLEL OR CASCADE



- USED TO ACHIEVE BETTER CONTROL OVER VARIABLES IN FORWARD PATH OF SYSTEM
- USED WHEN PLANT CANNOT BE APPROXIMATED BY 2ND OR 1ST ORDER SYSTEM
- DISADV. – MORE HARDWARE INVOLVED

FEED FORWARD DISTURBANCE

- NORMALLY DISTURBANCE IS ACCOUNTED FOR THROUGH OUTPUT OF PLANT BEFORE CONTROLLER CAN ACT
- BY MEASURING DISTURBANCE AND USING IT TO AUGMENT SIGNAL FROM CONTROLLER TO FINAL CONTROL ELEMENT, THE CONTROLLER RESPONSE IS BETTER

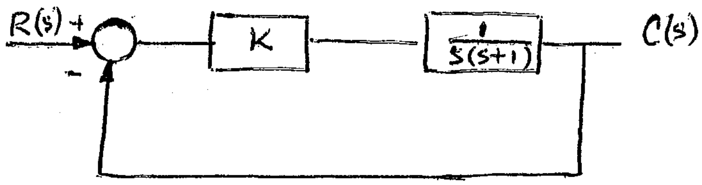


- THIS COMPENSATOR IMPROVES SPEED OF RESPONSE TO A DISTURBANCE
- DISADV: IT IS AN OPENLOOP TECHNIQUE (CONTAINS NO SELF CORRECTION)
- SELF CORRECTION CAN BE OBTAINED BY USING PI CONTROL FOR G_1

DO 7.1, 7.7 AND TURN IN MARCH 29

- REMEMBER FROM ZIEGLER-NICHOLS RULES DETERMINED K_{Pu} FROM WHICH K_P, K_I & K_D CAN BE OBTAINED
- THIS IS SIMPLE FOR 3rd ORDER OR 4th ORDER SYSTEM; DIFFICULT FOR 5th ORDER & HIGHER
- WE DEVELOP A GRAPHICAL METHOD FOR REPRESENTING SYSTEM BEHAVIOR – ROOT LOCUS PLOT
- MORE DETAILED THAN ROUTH-HURWITZ (R-H GIVES # OF ROOTS IN R-H PLANE)
- ROOT LOCUS PLOT - SKETCHES ROOT LOCATIONS OF CHARACTERISTIC EQN IN TERMS OF SOME PARAMETER, MOST OFTEN PROPORTIONAL GAIN.

EXAMPLE



$$T(s) = \frac{C(s)}{R(s)} = \frac{K}{s(s+1)+K}$$

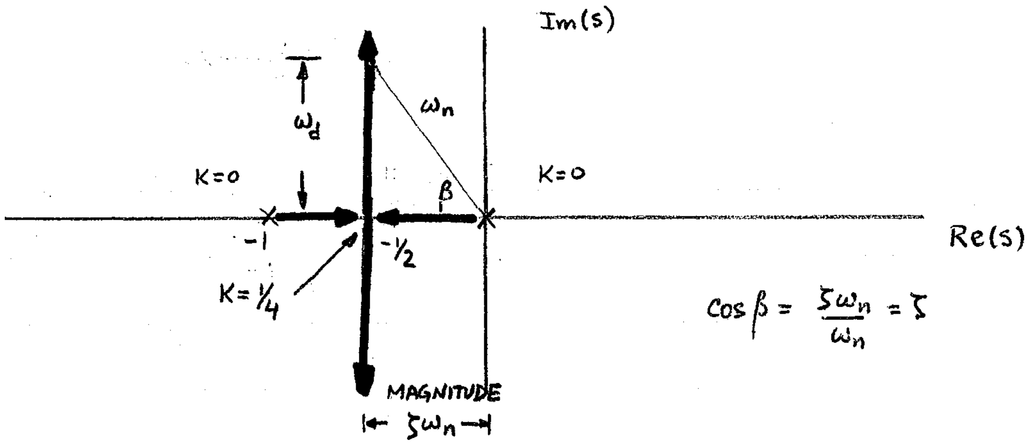
- $T(s) = \dfrac{\text{Numerator of } T}{\text{Denominator of } T} = \dfrac{N(s)}{D(s)}$; Denominator of $T(s)$ = CHARACTERISTIC EQN

- $D(s) = s^2 + s + K$ ROUTH REQUIRES $K > 0$ FOR STABILITY

- ROOTS $s = -\frac{1}{2} \pm \frac{1}{2}(1-4K)^{1/2}$

  $0 < K < 1/4$ REAL
  $K > 1/4$ COMPLEX
  $K = 1/4$ REPEATED

ROOT LOCUS SKETCH



$$\cos\beta = \frac{\zeta\omega_n}{\omega_n} = \zeta$$

LESSON # 20

- WE CAN WRITE $D(s) = s^2 + s + K = 0$ AS $1 + \dfrac{K}{s(s+1)} = 0$
- K PARAMETER TO BE STUDIED

$$D(s) = 1 + K\,\frac{Z(s)}{P(s)} = 0$$

OFTEN $\dfrac{Z(s)}{P(s)}$ IS SAME AS $G(s)H(s)$

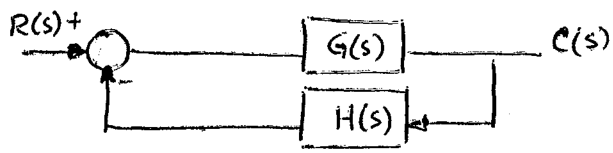
LOOK AT $$K\frac{Z(s)}{P(s)} = \frac{K(s+z_1)(s+z_2)\cdots(s+z_q)}{s^n(s+p_1)(s+p_2)\cdots(s+p_r)} = \frac{K}{s^n}\,\frac{\prod(s+z_i)}{\prod(s+p_j)}$$

$s = -z_1, -z_2, \ldots$ ARE CALLED THE ZEROES OF $\frac{KZ}{P}$; GIVEN BY O

$s = -p_1, -p_2, \ldots$ ARE CALLED THE POLES OF KZ/P ; GIVEN BY X

NOTE THAT ZEROES MAKE $\dfrac{KZ(s)}{P(s)} = 0$; POLES MAKE $\dfrac{KZ(s)}{P(s)} = \infty$

• WILL APPLY ROOT LOCUS TO FEEDBACK SYSTEMS (TO CHARACTERISTIC EQN)



$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

SUPPOSE $G(s) = \frac{N_G(s)}{D_G(s)}$ AND $H(s) = \frac{N_H(s)}{D_H(s)}$

• $$T(s) = \frac{N_G(s)\,D_H(s)}{D_G(s)\,D_H(s) + N_G(s)N_H(s)}$$ CLTF

• CHARACTERISTIC EQN IS.

$$D_G(s)\,D_H(s) + N_G(s)\,N_H(s) = 0$$

• WE CAN WRITE

$$F(s) = 1 + \frac{N_G(s)\,N_H(s)}{D_G(s)\,D_H(s)} = 0 \;;\; \text{WITH } H(s)=1 \Rightarrow 1 + G(s) = 0$$

• THE OPEN LOOP TRANSFER FN IS GIVEN BY

$$G(s)\,H(s) = \frac{N_G(s)\,N_H(s)}{D_G(s)\,D_H(s)}$$ OLTF

• NOTE THAT THE POLES OF CHAR EQ ($D_G D_H$) ARE POLES OF OLTF
ZEROES OF CHAR EQ ($N_G N_H$) ARE ZEROES OF OLTF

• CLOSED LOOP TRANSFER FN POLES ARE THE ROOTS OF CHAR EQ ($N_G N_H + D_G D_H$)
ZEROES ARE ROOTS OF $N_G D_G$

THUS $$F(s) = \frac{\text{CLOSED LOOP CHARACTERTIC POLY}}{\text{OPEN LOOP CHARACTERISTIC POLY.}}$$

• SINCE

$$D(s) = 1 + K\frac{Z(s)}{P(s)} = 0 \Rightarrow K\frac{Z(s)}{P(s)} = -1 = 1e^{j\phi}$$

WHERE $\phi = \pm(2k+1)180^\circ \quad k = 0, 1, 2, \ldots$

AND $\left|\frac{KZ(s)}{P(s)}\right| = 1$ & $\measuredangle \frac{KZ(s)}{P(s)} = \pm(2k+1)180^\circ$

- MAGNITUDE RQM'T

SINCE $\left|\frac{K Z(s)}{P(s)}\right| = \left|\frac{K(s+z_1)(s+z_2)\cdots(s+z_q)}{s^n(s+p_1)(s+p_2)\cdots(s+p_r)}\right| = 1$

$$\Rightarrow \frac{K}{|s^n|} \frac{|s+z_1||s+z_2|\cdots|s+z_q|}{|s+p_1||s+p_2|\cdots|s+p_r|} = 1$$

- PHASE RQM'T

$$\measuredangle K\frac{Z(s)}{P(s)} = \measuredangle K + \measuredangle(s+z_1) + \measuredangle(s+z_2) + \dots + \measuredangle(s+z_q) - \measuredangle s^n - \measuredangle(s+p_1) - \measuredangle(s+p_2) - \dots - \measuredangle(s+p_r) = \pm(2k+1)\,180°$$

- CAN BE OBTAINED FROM THE FACT THAT $(s+z_i) = |s+z_i|\,e^{j\phi_{z_i}}$

WHERE $\phi_{z_i} = \measuredangle(s+z_i)$; ALSO $K = |K|e^{j0}$; $s^n = |s|^n e^{jn\phi_0}$

THUS THE PHASE RQM'T CAN BE WRITTEN

$$\measuredangle K\frac{Z(s)}{P(s)} = 0 + \phi_{z_1} + \phi_{z_2} + \dots + \phi_{z_q} - \phi_{p_1} - \phi_{p_2} - \dots - \phi_{p_r} - n\phi_0 = \pm(2k+1)\cdot 180°$$

$$= \sum \phi_{z_i} - \sum \phi_{p_j} = \pm(2k+1)\,180°$$ DOESN'T DEPEND ON K

- WE CAN USE THESE RQM'TS TO PLOT ROOT LOCUS

PLOTTING GUIDES

(1) ROOT LOCUS IS SYMMETRIC ABOUT THE REAL AXIS WHY?

(2) THE NUMBER OF LOCI (OR BRANCHES) EQUALS THE NUMBER OF POLES OR ZEROES, WHICHEVER IS LARGER

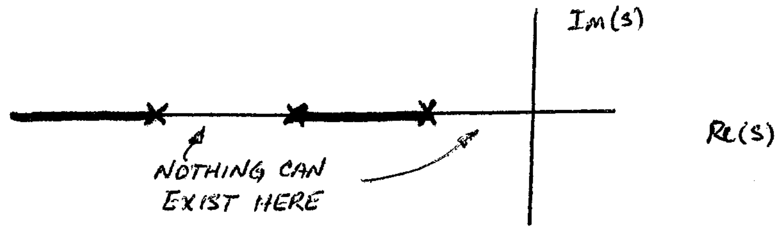
(3) EACH BRANCH OF THE ROOT LOCUS STARTS AT A POLE (i.e. WHEN $K=0$) AND ENDS AT A ZERO ($K=\infty$)

$D(s) = P(s) + K Z(s) = 0$; $K=0 \Rightarrow P(s)=0$ BRANCHES START AT ROOTS OF $P(s)$ – poles of $D(s)$

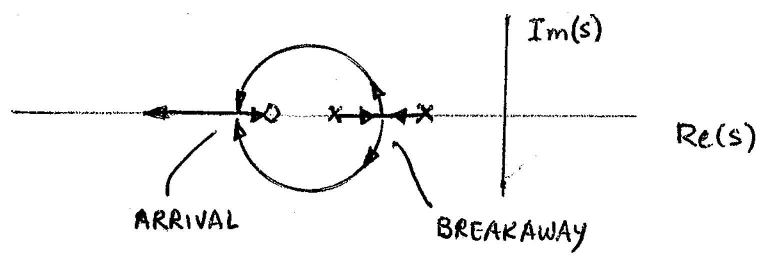
$\frac{D(s)}{K} = \frac{P(s)}{K} + Z(s) = 0$; $K \doteq \infty \Rightarrow Z(s) = 0$ BRANCHES END AT ROOTS OF $Z(s)$ – ZEROES OF $D(s)$

- IF NO. OF POLES > NO. OF ZEROES EXCESS POLES TERMINATE AT ZEROES THAT ARE FOUND AT INFINITY
- IF NO OF POLES < NO OF ZEROES EXCESS ZEROES AT INFINITY IS WHERE BRANCHES START

(4) ALONG THE REAL AXIS THE LOCUS INCLUDES ALL POINTS TO THE LEFT OF AN ODD NUMBER OF <u>REAL</u> POLES & ZEROES (PG 448-451



(5) A BREAK AWAY POINT OCCURS WHEN K IS AT A RELATIVE MAXIMUM WRT S ; AN ARRIVAL POINT OCCURS WHEN K IS AT A RELATIVE MINIMUM WRT S



LOOK FOR $dK/ds = 0$

$\dfrac{d^2K}{ds^2} > 0$ ARRIVAL $\qquad \dfrac{d^2K}{ds^2} < 0$ BREAKAWAY

- <u>EXAMPLE</u> $D(s) = s^2 + s + K = 0 \qquad \dfrac{dK}{ds} + 2s + 1 = \dfrac{dD}{ds} = 0$

FOR $\dfrac{dK}{ds} = 0 \Rightarrow s = -\tfrac{1}{2}$; $\dfrac{d^2K}{ds^2} + 2 = \dfrac{d^2D}{ds^2} = 0 \Rightarrow \dfrac{d^2K}{ds^2} < 0$

BREAKAWAY

- WE CAN WRITE $s^2 + s + K = 0 \Rightarrow 1 + \dfrac{K}{s(s+1)} = 0$

SINCE 2 POLES AT $s=0, s=-1 \Rightarrow$ 2 LOCI

FOR LARGE K

(6) IF n_p (NO. OF POLES) $>$ n_z (NO. OF ZEROES), AS $K \to \infty$, THERE WILL BE $(n_p - n_z)$ BRANCHES. LOCUS WILL BECOME ASYMPTOTIC TO STRAIGHT LINES INTERSECTING THE REAL AXIS AT ANGLES GIVEN BY

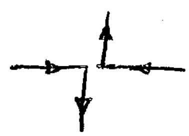
$$\Theta_k = \pm \frac{(2k+1)}{n_p - n_z} 180^\circ \qquad k = 0, 1, 2, \ldots$$

IF $n_z > n_p$, THEN AS $k \to 0$ $(n_z - n_p)$ BRANCHES BEHAVE AS ABOVE

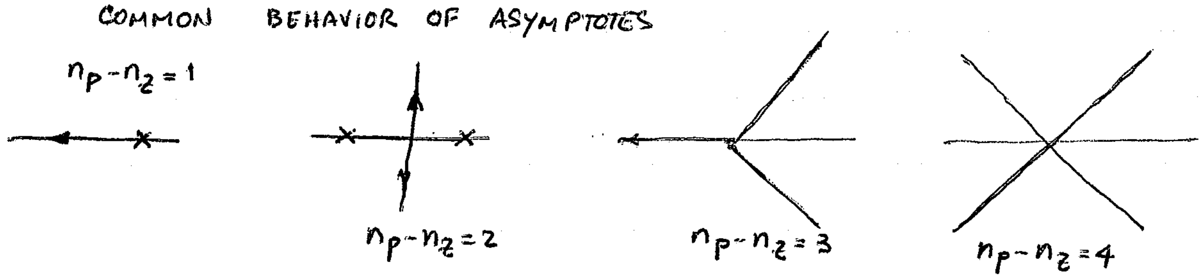
EXAMPLE $\quad D(s) = s^2 + s + K = 0 \quad$ or $\quad 1 + \frac{K}{s^2+s} = 0$

THERE ARE 2 POLES; NO ZEROES

$$\Theta = \pm \frac{(2k+1)180^\circ}{2-0} \; ; \quad \text{for } k=0 \quad \Theta_0 = \pm 90^\circ$$



COMMON BEHAVIOR OF ASYMPTOTES



(7) THE ASYMPTOTES OF (6) INTERSECT THE REAL AXIS AT A POINT (CALLED THE CENTROID OF THE ROOTS)

$$CG = \frac{\sum(+p_i) - \sum(+z_i)}{n_p - n_z}$$

ALGEBRAIC SUMS OF THE VALUES OF POLES & ZEROES

EXAMPLE: $+p_1 = 0 \quad +p_2 = -1 \qquad \sum(+p_i) = -1 \qquad n_p - n_z = 2$

$$CG = -\tfrac{1}{2}$$

(8) POINTS WHERE ROOT LOCUS CROSSES IMAGINARY AXIS: $s = j\omega$

ω IS THE CROSS OVER FREQUENCY

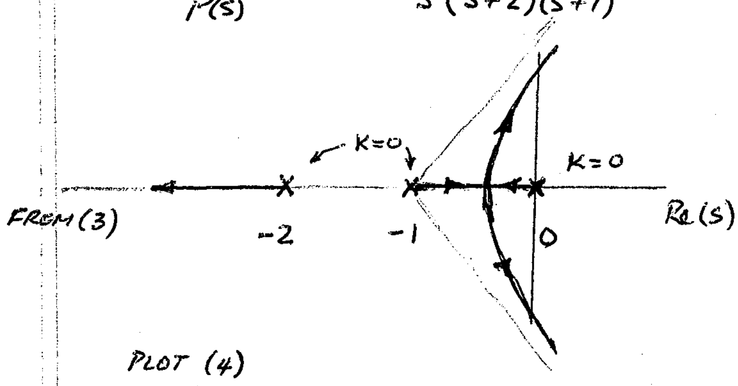
NOTE: IF $\omega \neq 0$; THE VALUE OF K THAT CAUSES CROSS OVER IS K_{pu} IN ZIEGLER-NICHOLS RULE

THIS VALUE CAN BE OBTAINED BY ROUTH-HURWITZ

EXAMPLE

$D(s)$: $s^3 + 3s^2 + 2s + K = 0 \quad \Rightarrow \quad 1 + \frac{K}{s(s^2+3s+2)} = 0$

$1 + K\frac{Z(s)}{P(s)} = 1 + \frac{K}{s(s+2)(s+1)} = 0$



PLOT (4)

- 3 POLES AT $0, -1, -2$
- 3 BRANCHES GUIDE 2
- BREAK AWAY

$3s^2 + 6s + 2 + \frac{dK}{ds} = 0$

$\frac{dK}{ds} = 0 \Rightarrow 3s^2 + 6s + 2 = 0$

or $s = \frac{-6 \pm \sqrt{36 - 4(3)(2)}}{6}$

$= -1 \pm \frac{\sqrt{3}}{3}$

$= -.423, -1.58$

BREAKAWAY PT AT $-.423$

- $CG = \frac{\sum \text{POLES}}{n_p} = -\frac{3}{3} = -1$

- ASYMPTOTES FOR LARGE K

$\theta = \pm\frac{(2k+1)}{n_p - n_z} 180^\circ$; $\pm 60^\circ$ & 180° ($k=0$, $k=1$)

TO FIND CROSS OVER: put $s = j\omega$ INTO $D(s)$

$-j\omega^3 - 3\omega^2 + 2j\omega + K = 0 \quad \Rightarrow \quad K = 3\omega^2$

$2\omega - \omega^3 = 0 \quad \omega = 0 \quad \omega = \pm\sqrt{2}$

$\Rightarrow K = 0, \underline{K = 6}$ K_{pu}

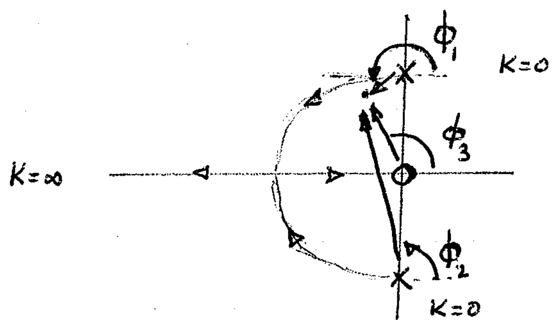
- IN MOST CASES THE FIRST EIGHT RULES ARE ENOUGH TO PLOT LOCUS; IF POLES ARE COMPLEX YOU NEED THE FOLLOWING

(9) ANGLE OF DEPARTURE (ARRIVAL) FROM A COMPLEX POLE (OR ZERO) MAY BE FOUND USING THE ANGLE CRITERION AND A POINT ON THE PATH ARBITRARILY CLOSE TO THE POLE (ZERO)

LET $D(s) = s^2 + Ks + 1 = 0 \quad \Rightarrow \quad 1 + \frac{Ks}{s^2+1} = 0$

THIS HAS A ZERO at $s=0$

AND TWO POLES AT $s = \pm j$



• $K = -\frac{(s^2+1)}{s} = -\frac{(s+j)(s-j)}{s}$

• ANGLE CRITERION SAYS

$\measuredangle K = \measuredangle(-1) + \measuredangle(s+j) + \measuredangle(s-j) - \measuredangle s$

$0 = \pm(2k+1)180^\circ + \phi_1 + \phi_2 - \phi_3$

• ASSUME s IS SO CLOSE THAT ϕ_2, ϕ_3 are UNCHANGED

$\therefore \quad \phi_1 = \phi_3 - \phi_2 \pm (2k+1)180^\circ$

$\phi_1 = 90^\circ - 90^\circ \pm 180^\circ \qquad \therefore \quad \phi_1 = \pm 180^\circ$ for $k=0$

$= \pm 540^\circ = \pm 180^\circ$ $k=1$

ANGLE OF DEPARTURE IS 180°

(10) FOR A POLYNOMIAL

$$s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_1 s + a_0 = 0$$

THE SUM OF THE ROOTS ARE

$$r_1 + r_2 + r_3 + \dots + r_n = -a_{n-1}$$

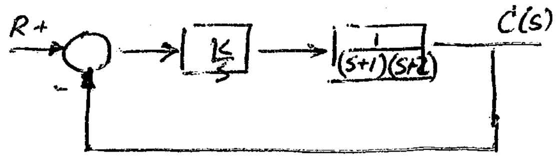
(11) TO DETERMINE THE VALUE OF K AT ANY s

FROM $1 + \frac{K\,Z(s)}{P(s)} = 0 \quad \Rightarrow \quad K = -\frac{P(z)}{Z(s)}$

LESSON #21

USING THE ROOT LOCUS TO COMPARE PERFORMANCE OF TWO SYSTEMS

EXAMPLE : PLANT $\dfrac{1}{(s+1)(s+2)}$ w/ INTEGRAL GAIN K/s



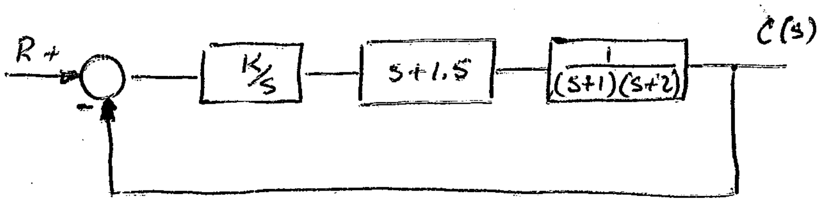
$$T(s) = \frac{C(s)}{R(s)} = \frac{K}{s(s+1)(s+2)+K}$$

Denominator is $s(s+1)(s+2)+K = s^3+3s^2+2s+K$

or $1 + \dfrac{K\,Z(s)}{P(s)} = 1 + \dfrac{K}{s(s+1)(s+2)} = 0$

WE SOLVED THIS AND SHOWED THAT FOR $K=6$ WE GET MARGINAL STABILITY, THUS WE WERE RESTRICTED IN K

SUPPOSE WE PUT SERIES COMPENSATOR BETWEEN CONTROLLER & PLANT



$$T(s) = \frac{C(s)}{R(s)} = \frac{K(s+1.5)}{s(s+1)(s+2)+K(s+1.5)}$$

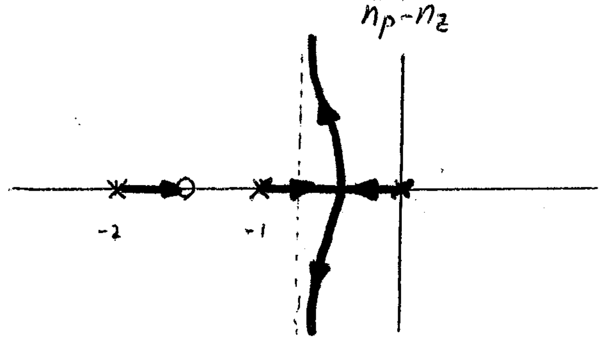
DENOM IS $D(s) = s(s+1)(s+2) + K(s+1.5) = 0$

or $1 + K\dfrac{Z(s)}{P(s)} = 1 + K\dfrac{(s+1.5)}{s(s+1)(s+2)} = 0$ THUS 3 POLES, 1 ZERO

$\Rightarrow$ 3 BRANCHES

$$CG = \frac{\Sigma \text{POLES} - \Sigma \text{ZEROES}}{n_p - n_z} = \frac{0+(-1)+(-2) - (-1.5)}{3-1} = \frac{-3+1.5}{2} = -.75$$

ASYMPTOTES $\theta = \pm \dfrac{(2k+1)}{n_p-n_z}180^\circ = \pm\dfrac{(2k+1)}{3-1}180^\circ$; for $k=0$ $\theta = \pm 90^\circ$



BREAK AWAY PT

$$\frac{dD}{ds} = 0 = 3s^2+6s+2+\cancel{\frac{dK}{ds}}(s+1.5)+K$$

$$0 = (3s^2+6s+2) - \frac{s(s+1)(s+2)}{s+1.5}$$

THUS $(s+1.5)(3s^2+6s+2) - (s^3+3s^2+2s) = 0$ or $2s^3+7.5s^2+9s+3=0$

- SOLUTION TO FIND S : USE NEWTON-RAPHSON TECHNIQUE

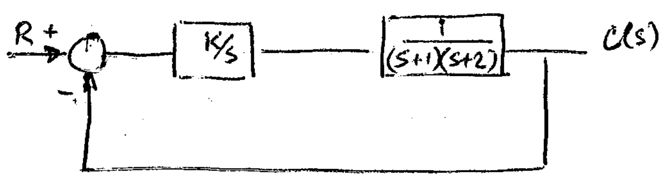
$$s_{n+1} = s_n - \frac{f'(s_n)}{f(s_n)} \qquad f'(s) = 6s^2+15s+9 = 3(3s^2+5s+3)$$

- SINCE WE EXPECT BREAKAWAY BETWEEN $-1 \leq s \leq 0$ CHOOSE $s_0 = .5$

  NEWTON RAPHSON GIVES $s_1 = -.542$

- NOTE THAT COMPENSATOR MAKES SYSTEM STABLE FOR ALL $K>0$
- SHIFT LOCUS PLOT AWAY FROM RIGHT HALF PLANE
- WE CAN NOW PICK K FOR INTEGRAL CONTROLLER FOR GOOD RESPONSE CHARACTERISTICS

LET'S LOOK AT THE RESPONSE OF OLD SYSTEM



$$T(s) = \frac{C(s)}{R(s)} = \frac{K}{s(s+1)(s+2)+K}$$

FOR DENOMINATOR $D(s) = s^3+3s^2+2s+K$ CHAR EQN

- LOOK AT ROOT LOCUS - WHEN PLOT BREAKS AWAY

$$D(s) = 0 = (s+s_1)(s+s_2)(s+\bar{s}_2) \qquad \bar{s}_2 \text{ IS CONJUGATE OF } s_2$$

- WHEN s_2 & $\bar{s}_2$ ARE COMPLEX $\Rightarrow \zeta < 1$
- AT BREAK POINT: $s_2 = \bar{s}_2$ & $\zeta = 1$ SYSTEM IS CRITICALLY DAMPED

  $\Rightarrow$ CRITICAL DAMPING IS AT $s = -.423$

$$D(s) \Rightarrow 1 + \frac{KZ(s)}{P(s)} = 1 + \frac{K}{s(s+1)(s+2)} = 0$$

; PUT $s = -.423$ AND SOLVE FOR $K = .385$

- ALSO SINCE $s^3+3s^2+2s+K=0$ w/ $K=.385$

$$(s+s_1)(s+.423)(s+.423) = 0 \qquad \text{AND} \quad s_1 = 2.154$$

- (REMEMBER $s_1+s_2+s_3 = -3$)
- $$T(s) = \frac{C_1}{s+2.154} + \frac{C_2}{(s+.423)^2} + \frac{C_3}{(s+.423)}$$

- TIME CONSTANTS ARE $\frac{1}{.423}$ & $\frac{1}{2.154}$

- THUS DOMINANT $\tau = \frac{1}{.423} = 2.36$

- LET'S LOOK AT WHAT HAPPENS FOR THE NEW SYSTEM

  SINCE BREAKAWAY REPRESENTS $\zeta = 1$ THEN COMPUTE K FOR $\zeta = 1$ $(s = -.542)$

$$1 + \frac{K(s+1.5)}{s(s+1)(s+2)} = 0 \Rightarrow K = .378$$

- SINCE K HERE IS LESS THAN THAT OF ORIGINAL SYSTEM $\Rightarrow$ FOR $K = .385$

  NEW SYSTEM WILL HAVE COMPLEX CONJUGATE ROOTS. WHY?

- IF COMPLEX CONJUGATE ROOTS $\zeta < 1$ WHY?
- WHAT ABOUT THIRD ROOT?
- FOR NEW SYSTEM WHAT IS DOMINANT ROOT?

EXAMPLE (8.2e) PG 479

$$D(s) = s(s^2 + 2s + 2) + K = 0 \Rightarrow 1 + \frac{KZ(s)}{P(s)} = 1 + \frac{K}{s(s^2+2s+2)}$$

- THIS HAS HOW ANY POLES & HOW MANY ZEROES? NO. OF BRANCHES?
- WHERE ARE THEY LOCATED?
- WHAT ARE THE ASYMPTOTES & WHERE IS THE CG?
- WHAT ARE THE ANGLES OF DEPARTURE
- WHAT IS THE SKETCH OF THE LOCUS
- DOES THE LOCUS CROSS THE IMAGINARY AXIS? WHERE?

  USE ROUTH ARRAY TO FIND THIS

s^3	1	2
s^2	2	K
s	a	
1	b	

$$a = -\frac{1}{2}\begin{vmatrix} 1 & 2 \\ 2 & K \end{vmatrix} = -\frac{1}{2}(K-4)$$

FOR MARGINALLY STABLE $K = 4 \Rightarrow a = 0$

USE ε RULE TO FIND B

$$b = -\frac{1}{\varepsilon}\begin{vmatrix} 2 & K \\ \varepsilon & 0 \end{vmatrix} = \frac{-\varepsilon K}{-\varepsilon} = K = 4 > 0$$

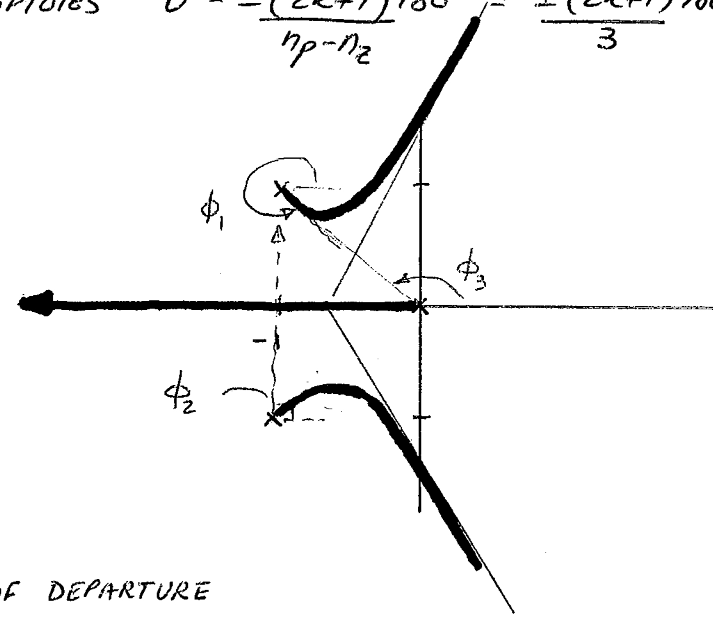
$\therefore$ SINCE NO SIGN CHANGES, NO ROOTS IN RHP

SOLN: 3 POLES, NO ZEROES $\Rightarrow$ 3 BRANCHES

LOCATED AT $S=0$ & $S = \frac{-2 \pm \sqrt{4-4(2)}}{2} = -1 \pm j$

- $CG = \frac{\sum \text{ROOT OF POLES} - \sum \text{ROOTS OF ZEROES}}{n_p - n_z} = \frac{0 + (-1+j) + (-1-j)}{3-0} = -\frac{2}{3}$

- ASYMPTOTES $\theta = \frac{\pm(2k+1)180^\circ}{n_p - n_z} = \frac{\pm(2k+1)180^\circ}{3} = \pm 60^\circ (k=0), \pm 180^\circ (k=1)$



- ANGLE OF DEPARTURE

$$\phi_1 + \phi_2 + \phi_3 = 180^\circ$$

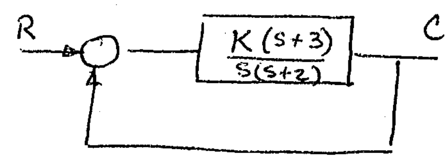
$$\phi_1 + 90^\circ + 135^\circ = 180^\circ \qquad \phi_1 = -45^\circ$$

USING ROUTH ARRAY $\Rightarrow$ $2S^2 + K = 0$ or $2S^2 + 4 = 0$ $\quad S = \pm\sqrt{2}j$

CROSS OVER

$K = 4$ IS K_{pu}

EXAMPLE 8.4



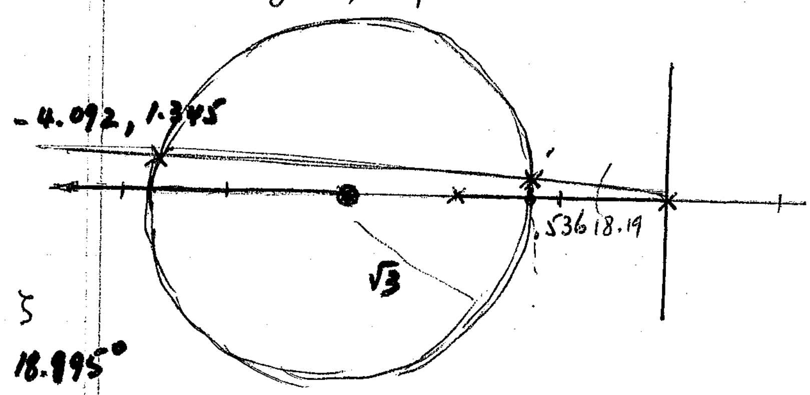
$$T(s) = \frac{C(s)}{R(s)} = \frac{K(s+3)}{s(s+2)+K(s+3)}$$

Denominator $s(s+2) + K(s+3) = 0$

$$1 + \frac{K(s+3)}{s(s+2)} = 0$$

one zero, 2 poles : 2 branches

$$CG = \frac{\Sigma \text{poles} - \Sigma \text{zeros}}{n_p - n_z} = \frac{(0-2)-(-3)}{2-1} = \frac{+1}{1} = 1$$



$$\theta = \pm\frac{(2k+1)\,180^\circ}{n_p - n_z} = \pm\frac{(2k+1)\,180^\circ}{2-1} = \pm 180^\circ$$

$\cos\beta = \zeta$

$\beta = 18.995^\circ$

Break away pts $\quad \frac{dD}{ds} = 0 = (s+2) + s + \cancel{\frac{dK}{ds}}^{\,0}(s+3) + K = 0$

$$= 2s+2 - \frac{s(s+2)}{s+3} = 0 \Rightarrow 2(s+1)(s+3) - s(s+2) = 0$$

$2(s^2+4s+3)$

or $2s^2+8s+6-s^2-2s = \cancel{2}s^2+6s+6=0$

$$s = \frac{-6 \pm \sqrt{36-4(6)}}{2} = -3 \pm \cancel{2}\sqrt{3}$$

$= \cancel{.464, 6.46}$

$-4.732 \quad -1.26$

$s+3 = \pm\sqrt{3}$

$(x+3)^2 + y^2 = 3$

$y = \cancel{\sqrt{3}}$

$-.3287x$

$(x+3)^2 + .108x^2 = 3$

$\cancel{x^2+6x+9}$

$1.108x^2 + 6x + \cancel{9} = \cancel{3}$

$$\frac{-6 \pm \sqrt{36-4(6)(1.108)}}{2.216}$$

$s^2 + (2.647)s + 1.941$

$-1.3235 \pm .435j$

$s^2 + (2+K)s + 3K \qquad s^2 + 8.183s + 18.549 \qquad -2.728 \pm 1.3448$

$2+K = 2\zeta\omega_n \qquad \omega_n^2 = 3K$

$2+K = 1.9\,\omega_n$

$(2+K)^2 = (1.9)^2 \cdot 3K$

$4 + 4K + K^2 = 10.83K \qquad K^2 - 6.83K + 4 = 0$

$$K = \frac{6.83 \pm \sqrt{(6.83)^2 - 4\cdot 4}}{2}$$

$K = .647 \qquad \underline{6.183}$

$\therefore \quad \tau = \frac{1}{\zeta\omega_n}$ for smallest time const. $\Rightarrow \zeta\omega_n$ largest

$\therefore \underline{K = 6.183}$

$$\frac{d^2D}{ds^2} = 2 + \frac{d^2K}{ds^2}(s+3) + 2\cancelto{0}{\frac{dK}{ds}} = 0 \qquad \frac{d^2K}{ds^2} = \frac{-2}{s+3}$$

for $s = -1.268 \qquad \frac{d^2K}{ds^2} = \frac{-2}{1.732} < 0$ breakaway

$s = -4.732 \qquad \frac{d^2K}{ds^2} = \frac{-2}{-1.732} > 0$ breakin

---

PLEASE READ SECTIONS 8.5 – 8.7

- **NEXT EXAM 12 April** FREQ. RESPONSE PLOTS CH 4
  P, PI, PID CONTROLS, ZIEGLER-NICHOLS CH 6, CH 7
  ROOT LOCUS CH. 8

EX: PROB. 8.1a PG. 479

$$s^2 + 4s + 3b = 0 \qquad b \geq 0$$

LET $3b = -(s^2+4s)$

DIV BY (s^2+4s) $\quad \dfrac{3b}{s^2+4s} = -1 \quad$ or $\quad 1 + \dfrac{3b}{s^2+4s} = 0$

LET $K = 3b \geq 0 \;\Rightarrow\; 1 + \dfrac{K}{s(s+4)} = 0$

HOW MANY POLES, ZEROES ?

PROB. 8.1 c

$$s^2 + (2+b)s + 2 + 2b = 0 \qquad b \leq 0$$

ISOLATE b TERMS $\quad (s^2+2s+2) + b(s+2) = 0$

$$\Rightarrow \quad b(s+2) = -(s^2+2s+2)$$

DIVIDE BY $-(RHS)$

$$\frac{b(s+2)}{s^2+2s+2} = -1 \quad \text{or} \quad 1 + \frac{b(s+2)}{s^2+2s+2} = 0$$

LET $K = b \leq 0$ THIS WILL LEAD TO <u>COMPLIMENTARY ROOT LOCUS</u>

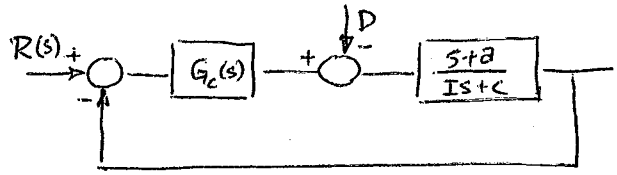
<u>READ SECTION 8.5</u> – <u>YOU ARE RESPONSIBLE FOR COMPLIMENTARY ROOT LOCUS</u>

LOOK AT PG 461 & 470

- DIFFERENCES IN ANGLE CRITERION & MAGNITUDE CRITERION
- GUIDE 3a, 4a BCK, 5a
- LOCI FOR ASYMPTOTES
- USE ANGLE & MAGNITUDE CRITERION FOR COMPLIMENTARY ROOT LOCUS, TO FIND TAKE OFF ANGLES FOR

LESSON # 22

EX: 8.10 PG 480 Problem 6.28



w/ $a = 1$ $c = .1$ $I = 10$

WE FOUND $G_c(s) = -9.878 + \frac{12.22}{s}$

FOR $\tau = 0.1$ $\zeta = 1$

- WHAT IF c VARIES $.05 \le c \le 0.15$ PART a)
- WHAT IF a VARIES $0.5 \le a \le 1.5$ PART b)

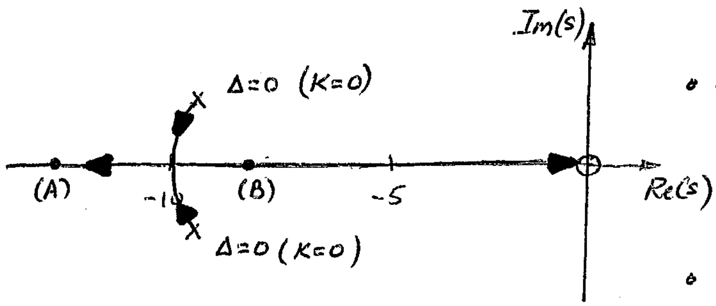
$$G_c = K_p + K_I/s = -9.878 + 12.22/s$$

WE FOUND $T(s) = \dfrac{C(s)}{R(s)} = \dfrac{K_p s^2 + (K_p a + K_I)s + K_I a}{(K_p + I)s^2 + (K_p a + K_I + c)s + K_I a}$

a) c VARIES a IS <u>FIXED</u> AT 1, LET $\Delta = c - .05$ $0 \le \Delta \le .1$

DENOM: $(.122 s^2 + (2.392 + \Delta)s + 12.22)$ or $1 + \dfrac{\Delta s}{.122[s^2 + 19.61s + 100]} = 0$

poles at $-9.803 \pm 1.974i$; zeroes at $s = 0$ let $K = \dfrac{\Delta}{.122}$



- since 2 poles and one zero $\Rightarrow$ 2 BRANCHES
- $D(s) = .122[s^2 + 19.61s + Ks + 100] = 0$

or $2s + 19.61 + \cancel{\frac{dK}{ds}}\,s + K = 0$

$\therefore$ $2s + 19.61 - \dfrac{(s^2 + 19.61s + 100)}{s} = 0$ or

$s^2 - 100 = 0 \Rightarrow s = \pm 10$ $s = -10$ break in pt

ALSO $s = -10$ is DOUBLE ROOT $\Rightarrow s^2 + 20s + 10 = 0$ or $K = .39$

$\Delta = K(.122)$ $\therefore$ $\Delta = .0476$ at $s = -10$

NOTE: FOR $\Delta = .1$ $K = .8197$ AND $D(s) = .122[s^2 + 20.43s + 100] = 0$

$s = -8.13$ & $s = -12.3$ (A & B ON PLOT)

THUS FOR VARIATION OF C SYSTEM IS STABLE

b) c IS FIXED AT 0.1 & a VARIES ; LET $\Delta = a - 0.5$ $\quad 0 \le \Delta \le 1$

NOTE THAT THIS WILL AFFECT BOTH NUMERATOR & DENOMINATOR

SINCE

$$T(s) = \frac{C(s)}{R(s)} = \frac{K_P s^2 + (K_P a + K_I s) + K_I a}{(K_P + I) s^2 + (K_P a + K_I + C) s + K_I a}$$

$$D(s) = \text{DENOM} = .122 s^2 + (7.38 - 9.878\Delta) s + 12.22\Delta + 6.11 = 0$$

or

$$1 + \frac{\Delta(-9.878 s + 12.22)}{.122 (s^2 + 60.49 s + 50)} = 0$$

or if $K = -80.967\Delta$ THIS GIVES RISE TO COMPLIMENTARY ROOT LOCUS

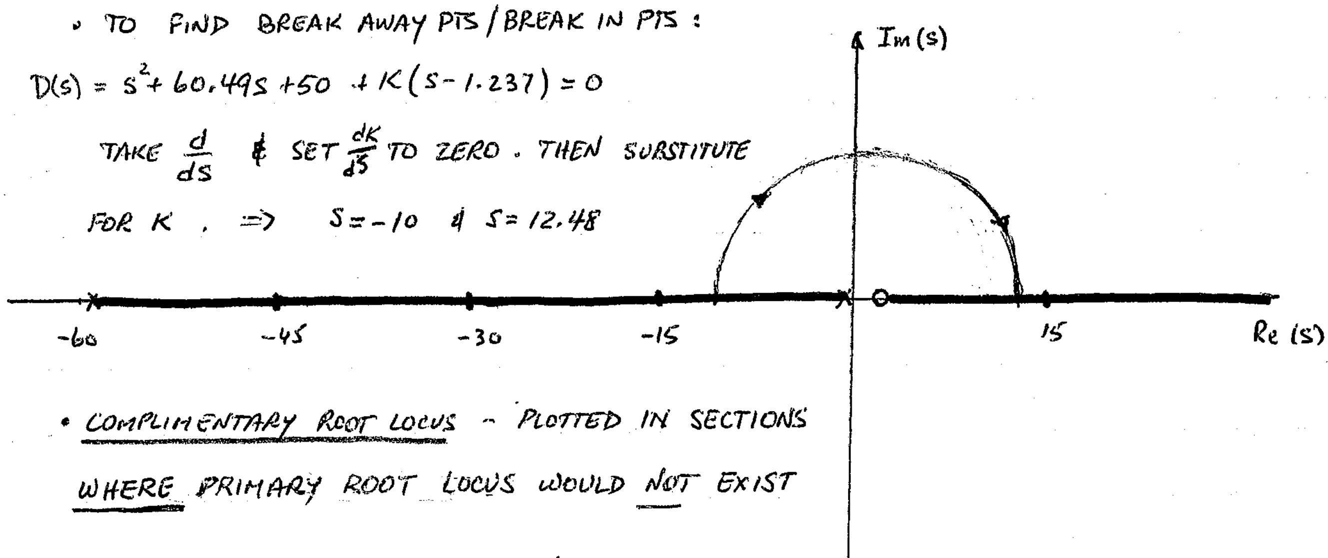
$$1 + \frac{K(s - 1.237)}{s^2 + 60.49 s + 50} = 0$$

- POLES AT $s = -.838$ & -59.65 ZERO AT $s = 1.237$
- TO FIND BREAK AWAY PTS / BREAK IN PTS :

$D(s) = s^2 + 60.49 s + 50 + K(s - 1.237) = 0$

TAKE $\frac{d}{ds}$ & SET $\frac{dK}{ds}$ TO ZERO . THEN SUBSTITUTE

FOR K , $\Rightarrow$ $s = -10$ & $s = 12.48$



- COMPLIMENTARY ROOT LOCUS - PLOTTED IN SECTIONS

WHERE PRIMARY ROOT LOCUS WOULD NOT EXIST

PHASE & MAGNITUDE RQM'TS

- ONLY OTHER CHANGES ARE $\Sigma\phi_{z_i} - \Sigma\phi_{p_j} = \pm n\,360°$
- AND $\dfrac{K|Z(s)|}{|P(s)|} = -1$

- TO FIND CROSS OVER PTS $\quad s^2 + (60.49 + K)s + (50 - 1.237K) = 0$

s^2	1	50 − 1.237 K
s	60.49+K	0
1	d	

$$d = \frac{-1}{60.49+K}\begin{vmatrix} 1 & 50-1.237K \\ 60.49+K & 0 \end{vmatrix} = 0$$

$\Rightarrow (60.49 + K)(50 - 1.237K) = 0$ or

$\Delta = .748, -.5 \Leftarrow K = -60.49, 40.42 \longleftarrow 3024.5 - 24.826K - 1.237K^2 = 0$

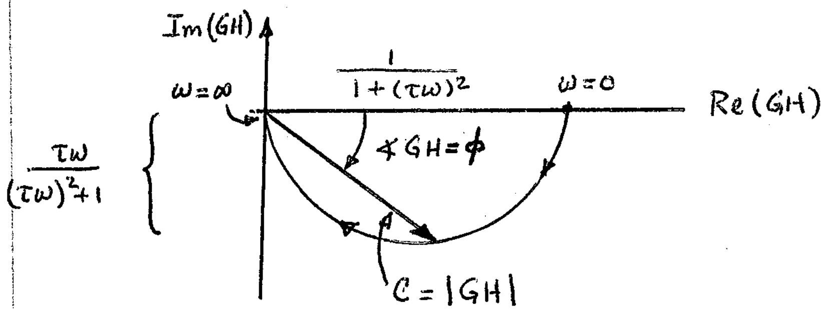
NYQUIST DIAGRAMS CH. 9

- FREQUENCY RESPONSE PLOTS OF O.L.T.F. CONTAIN MUCH INFO ABOUT BEHAVIOR OF CLOSED LOOP SYSTEM
- FOR GAIN ADJUSTMENTS WHERE SERIES COMPENSATOR IS REQUIRED, THESE PLOTS CAN HELP SELECT COMPENSATOR PARAMETERS
- SERIES LEAD & LAG COMPENSATORS USED WIDELY IN INDUSTRY.
- WHY USE THIS METHOD? - EASIER TO GET DATA FOR OPEN LOOP SYSTEM
- EASIER FOR HIGHER ORDER SYSTEMS - DON'T NEED CHARACTERISTIC ROOTS
- COMPENSATORS EASIER TO DESIGN WITH THIS METHOD

$\boxed{\text{• REREAD SECTION 4.5 ON FREQUENCY RESPONSE}}$

- RECALL OLTF IS $G(s)H(s)$
- NYQUIST DIAGRAM IS A POLAR PLOT OF $|G(s)H(s)|$ AND $\measuredangle G(s)H(s)$ AS ω GOES FROM ZERO TO ∞

EXAMPLE: $G(s)H(s) = \frac{1}{\tau s+1} \Rightarrow G(j\omega)H(j\omega) = \frac{1}{\tau j\omega+1} = \frac{1}{\tau^2\omega^2+1} - j\frac{\tau\omega}{\tau^2\omega^2+1}$

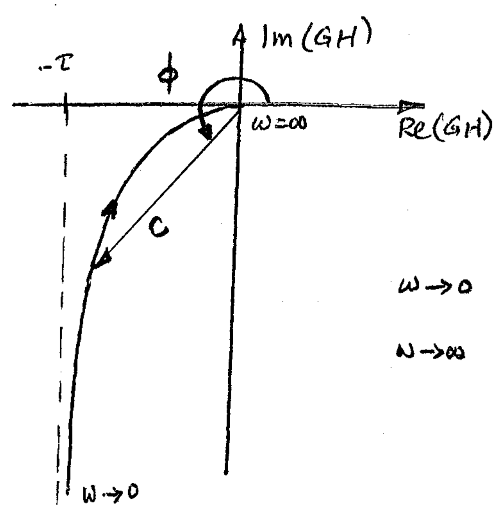


$$= Ce^{j\phi}$$

AND $\phi = \tan^{-1}(-\tau\omega)$

$$C = \frac{1}{\sqrt{1+(\tau\omega)^2}}$$

- EXAMPLE: $G(s)H(s) = \frac{1}{s(\tau s+1)} \Rightarrow G(j\omega)H(j\omega) = \frac{1}{j\omega(\tau j\omega+1)}$



$$GH = \frac{-\tau}{1+(\tau\omega)^2} - j\frac{1}{\omega[1+(\tau\omega)^2]}$$

FOR $\omega \to 0$ $GH \to -\tau - j\infty$

FOR $\omega \to \infty$ $GH \to -0 - j0$

$\omega \to 0$ LOW FREQ ASYMPTOTE

$\omega \to \infty$ HIGH FREQ ASYMPTOTE

$GH = Ce^{j\phi}$ $C = \frac{1}{\omega\sqrt{1+\tau^2\omega^2}}$

$\phi = \tan^{-1}\left(\frac{-1}{-\tau\omega}\right)$

NYQUIST STABILITY CRITERION — BASED ON CONFORMAL MAPPING

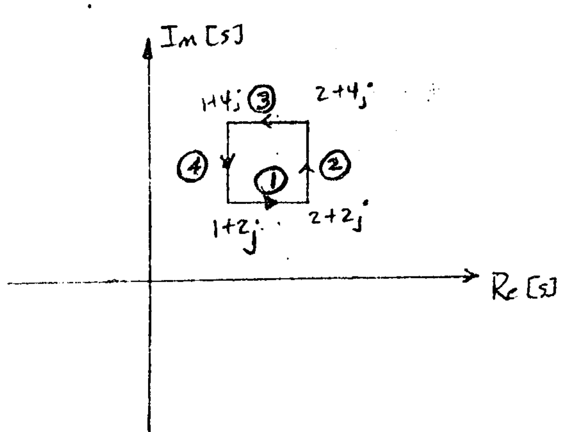
RELATES OPEN-LOOP FREQ. RESPONSE $GH(j\omega)$ TO THE NO. OF ZEROS AND POLES OF $1+GH(s)$ THAT LIE IN THE RHP.

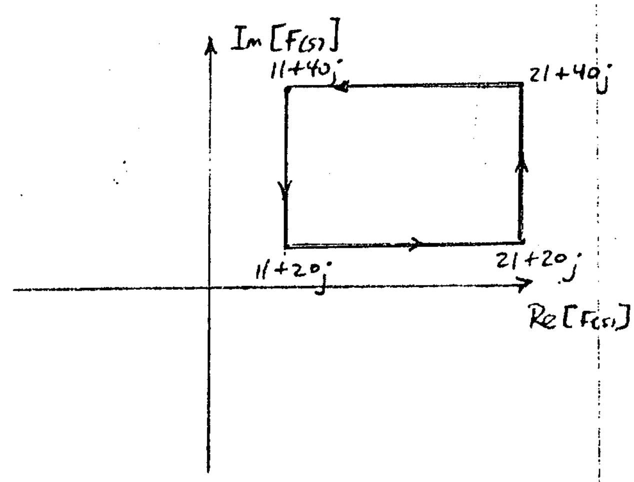
THE ABSOLUTE STABILITY OF A SYSTEM (CLOSED LOOP) CAN BE DETERMINED GRAPHICALLY FROM THE OPEN-LOOP FREQ. RESPONSE CURVES.

∴ NO NEED TO DETERMINE CLOSED-LOOP POLES. IE CHARACTERISTIC EQN OF CLOSED LOOP TRANSFER FN

MAPPING PROBLEM — GIVEN AN S-PLANE CONTOUR, WHAT IS THE CORRESPONDING PATH IN ANOTHER PLANE, THE $F(s)$ PLANE FOR EXAMPLE.

LET $F(s) = 1 + G(s)H(s)$ CHAR. EQN





$G(s)H(s) = 10s$

$F(s) = 1 + 10s$

ALONG ① $\quad s = \alpha + 2j \qquad 1 \le \alpha \le 2$

$F(s) = 1 + 10(\alpha + 2j)$

$F(s) = 1 + 10\alpha + 20j = (1+10\alpha) + 20j$

ALONG ② $\quad s = 2 + \beta j \qquad 2 \le \beta \le 4$

$F(s) = 1 + 10(2 + \beta j)$

$F(s) = 21 + 10\beta j$

ALONG ③ $\quad s = \gamma + 4j \qquad 1 \le \gamma \le 2$

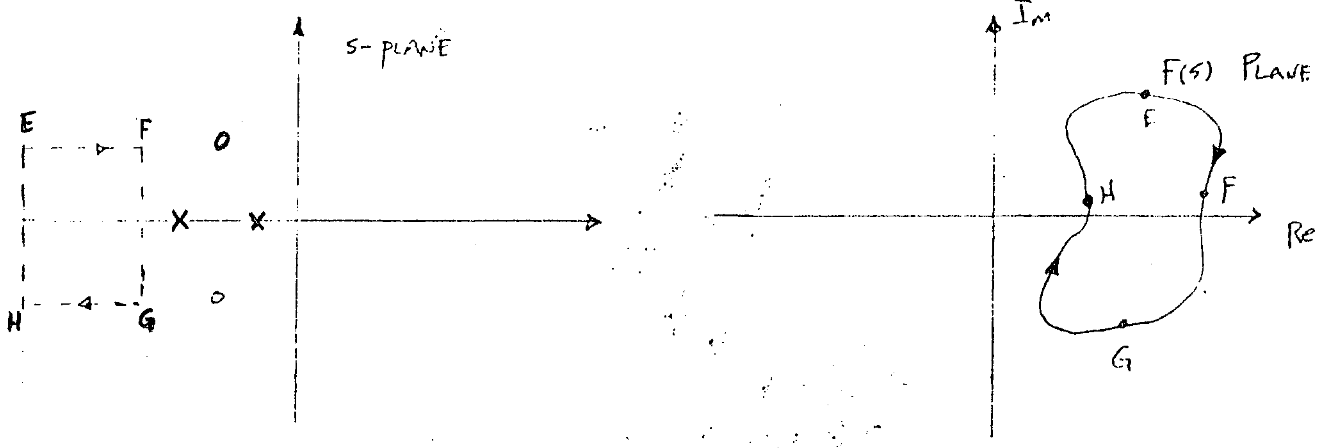
$F(s) = 1 + (10)(\gamma + 4j)$

$= 1 + 10\gamma + 40j$

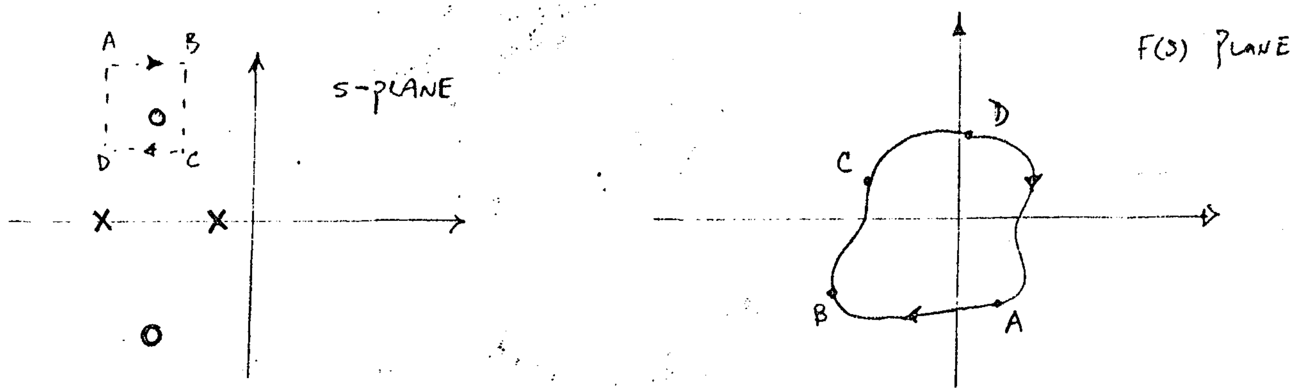
ALONG ④ $\quad s = 1 + \delta j \qquad 2 \le \delta \le 4$

$F(s) = 1 + 10(1 + \delta j) = 11 + 10\delta j$

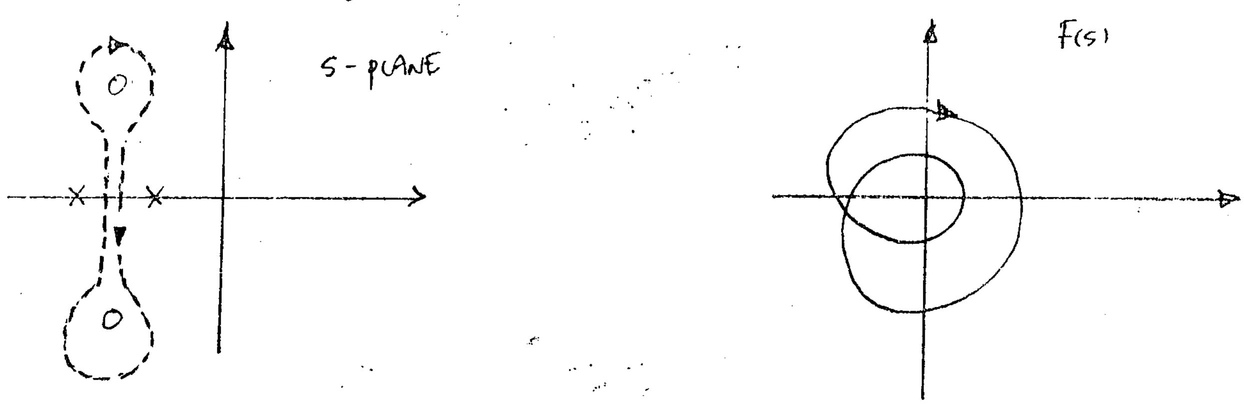
IF THE CONTOUR IN THE S-PLANE, DOES NOT ENCIRCLE A ZERO, THE CONTOUR IN THE F(S) PLANE DOESN'T ENCIRCLE THE ORIGIN.



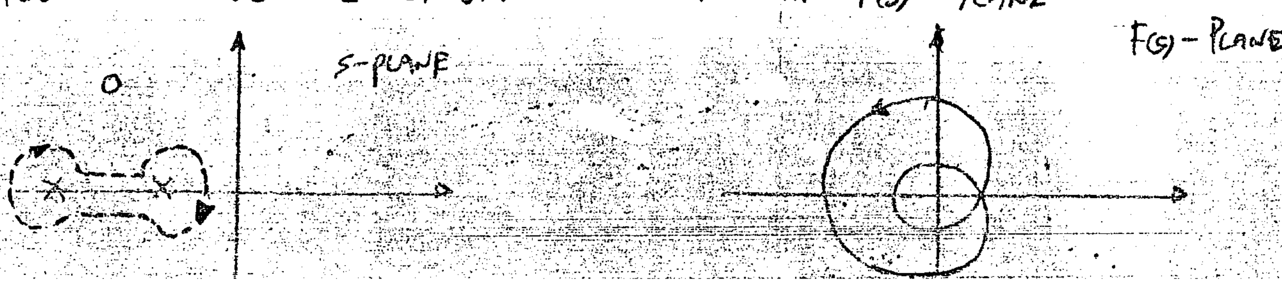
IF YOU ENCIRCLE A ZERO C.W. IN THE S-PLANE, YOU ENCIRCLE THE ORIGIN C.W. IN THE F(S) PLANE.



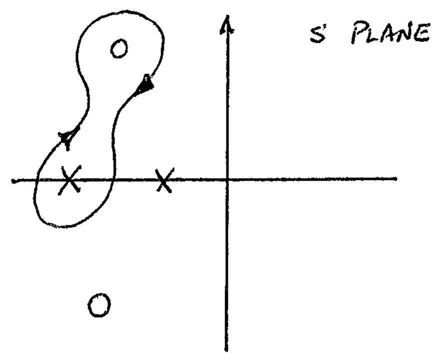
ENCIRCLE 2 ZEROES

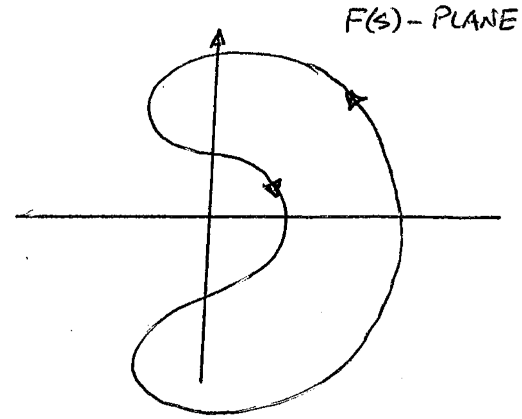


IF YOU ENCIRCLE A POLE CW IN THE S-PLANE W/ A CONTOUR, YOU ENCIRCLE THE ORIGIN CCW IN THE F(S) - PLANE



IF WE ENCIRCLE ONE POLE AND ONE ZERO





$$F(s) = 1 + G(s)\,H(s) \qquad G(s) = \frac{N_G(s)}{D_G(s)} \qquad H(s) = \frac{N_H(s)}{D_H(s)}$$

$$F(s) = \frac{D_G(s)\,D_H(s) + N_G(s)\,N_H(s)}{D_G(s)\;D_H(s)} \quad \& \quad G(s)H(s) = \frac{N_G(s)\;N_H(s)}{D_G(s)\;D_H(s)} = OLTF$$

$$T(s) = CLTF = \frac{G}{1+GH} = \frac{N_G(s)\,D_H(s)}{D_G(s)\,D_H(s) + N_G(s)\,N_H(s)}$$

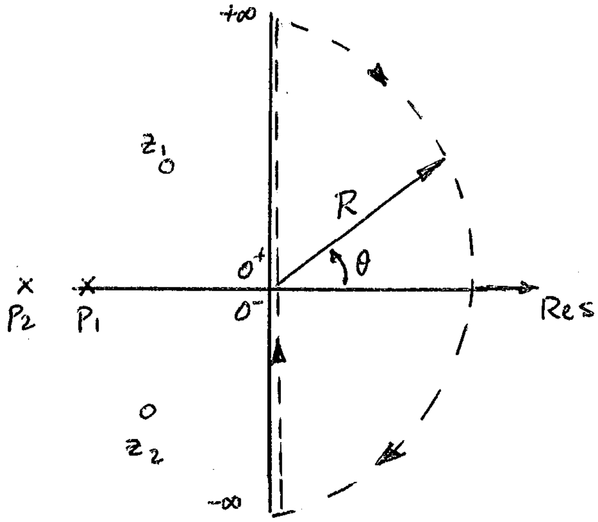
$$\Rightarrow F(s) = \frac{\text{POLES OF CLTF}}{\text{POLES OF OLTF}} = \frac{\text{ZEROES OF } F(s)}{\text{POLES OF } F(s)}$$

$$= \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

NOTE : THE POLES OF THE CLOSED LOOP TRANSFER FN CAUSE $F(s) = 0$

$\Rightarrow$ THEY MAP TO THE ORIGIN OF F(S) PLANE

- REMEMBER - A STABLE SYSTEM HAS ALL THE ROOTS OF THE CHARACTERISTIC EQN (I.E. THE DENOM OF TRANSFER FN = POLES OF CLTF) IN LEFT HALF PLANE
- THUS IF WE TOOK A CONTOUR IN THE R.H. S-PLANE AND FOUND ANY ENCIRCLEMENTS OF THE ORIGIN THEN SYSTEM IS UNSTABLE
- CONTOUR WE USE IS D- CONTOUR



DEFINE :

Z = NO. OF ZEROES OF $F(s)$ IN RH S-PLANE

N = NO. OF CW ENCIRCLEMENTS OF ORIGIN IN $F(s)$ - PLANE

P = NO. OF POLES OF $G(s)H(s)$ IN RH S-PLANE

- REMEMBER Z REPRESENTS ROOTS OF CHAR. EQ ; WE USE CW ENCIRCLEMENTS BECAUSE THEY REPRESENT ZEROES OF $F(s)$ (IE ROOTS OF CHAR EQ = POLES OF CLTF) OR EXCESS NO. OF ZEROES ; REMEMBER P ALSO REPRESENTS THE DENOM OF $F(s)$ (IE THE POLES OF O.L.T.F.)
- NYQUIST STATES

$$Z = N + P$$

PROCEDURE : (1) MAP $F(s)$ TO THE $F(s)$ - PLANE USING D - CONTOUR
(2) THE MAPPING WILL GIVE YOU N
(3) P IS KNOWN FROM $G(s)H(s)$

FOR A STABLE SYSTEM (CLOSED LOOP SYSTEM)

$$Z = 0 \quad \text{AND} \quad N = -P$$

- WITH $F(s) = 1 + G(s)H(s)$. WE PLOT $F(s)$ AS s FOLLOWS THE D-CONTOUR AND COUNT THE ENCIRCLEMENTS OF THE ORIGIN.
- HOWEVER SUPPOSE WE PLOT $\tilde{F}(s) = G(s)H(s)$; THEN WE COUNT THE ENCIRCLEMENTS OF $\tilde{F}(s) = -1 + 0j$

PROOF $\rightarrow$ $F(s) - 1 = \tilde{F}(s)$; WHEN $F(s) = 0 \Rightarrow \tilde{F}(s) = -1$

NYQUIST STABILITY CRITERION

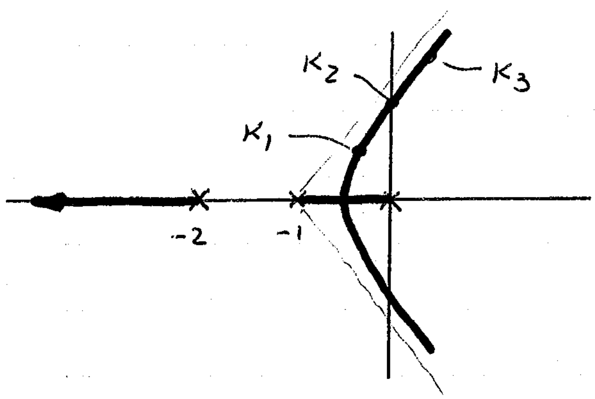
FOR THE CLOSED LOOP SYSTEM $T(s) = \dfrac{G(s)}{1 + G(s)H(s)}$, IF THE OPEN LOOP TRANSFER FN. $G(s)H(s)$ HAS K POLES IN THE RIGHT-HALF S-PLANE

AND $\lim_{s \to \infty} G(s)H(s) = \text{constant}$, THEN FOR STABILITY THE $G(j\omega)H(j\omega)$ LOCUS, AS $-\infty \le \omega \le \infty$ MUST ENCIRCLE THE $-1+0j$ POINT K TIMES IN THE CCW DIRECTION

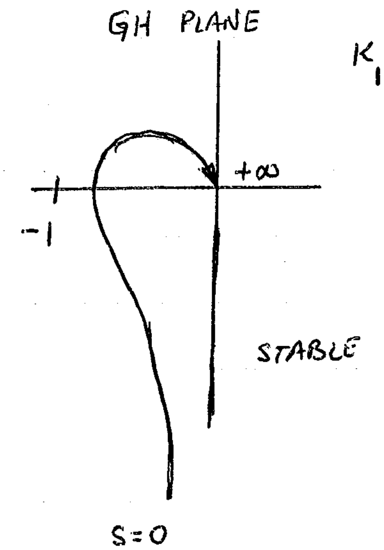
PROOF - STABILITY $Z = 0 = N + P = N + K \Rightarrow N = -K$

THE (−) MEANS CCW SINCE N IS NO OF CW ENCIRCLEMENTS

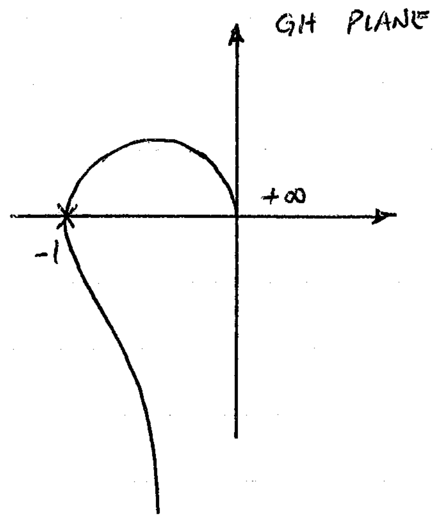
ROOT LOCUS AND CORRESPONDING NYQUIST



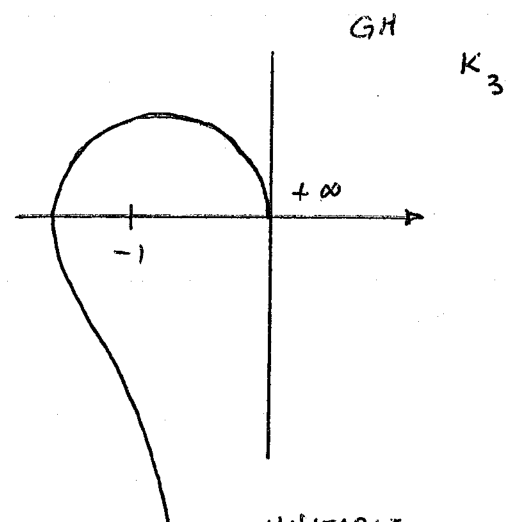
$$GH = \frac{K}{s(s+2)(s+1)}$$



STABLE

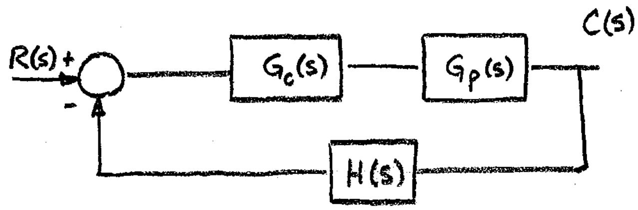


MARGINALLY STABLE



UNSTABLE

- NOW LET US APPLY THIS RESULT TO SEVERAL PROBLEMS
- RECAP $Z = N + P$
    - Z IS THE NO. OF ZEROES OF $F(s)$ $\{$ZEROES OF $F(s)$ ARE ROOTS OF THE CHARACTERISTIC EQN = DENOM OF $T(s) = \frac{C(s)}{R(s)}\}$
    - N IS NO. OF CW ENCIRCLEMENTS OF $-1+0j$ IN GH PLANE
    - P IS NO OF POLES OF GH INSIDE THE S-PLANE D CONTOUR

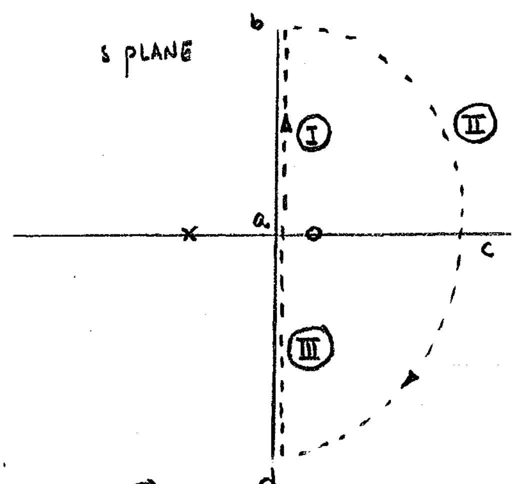


EXAMPLE: LET $G_c(s) = K$

$$G_p(s) = \frac{s-1}{s+2}$$

$$H(s) = 1$$

- $$T(s) = \frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)}$$
- $$GH = G_c G_p H = K\frac{s-1}{s+2}$$
- NOTE GH HAS POLE AT $s=-2$, ZERO AT $s=$
- REMEMBER $F(s) = 1 + GH = 1 + \frac{K(s-1)}{s+2}$
- SINCE GH HAS NO POLE IN RH S-PLANE $\Rightarrow P = 0$
- NOW PLOT $G(s)H(s)$ FOR s IN D-CONTOUR $\Rightarrow N$



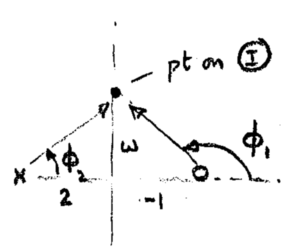
	a	b
ALONG I $s = j\omega$	$0^+ \le \omega$	$\le R$

$$GH = \frac{K(j\omega-1)}{j\omega+2} = \frac{K\sqrt{\omega^2+1}}{\sqrt{\omega^2+4}}e^{j\phi}$$

$$|GH| = \frac{K\sqrt{\omega^2+1}}{\sqrt{\omega^2+4}}$$

$$\phi = \angle GH = 180^\circ - \tan^{-1}\left(\frac{\omega}{1}\right) - \tan^{-1}\left(\frac{\omega}{2}\right)$$

$$= \phi_1 - \phi_2$$



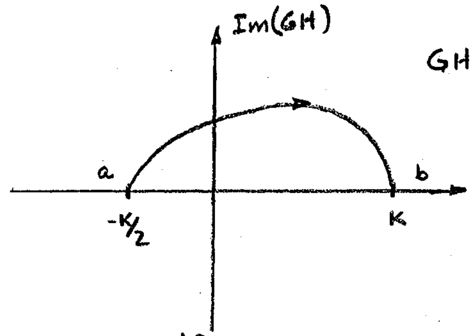
NOTE $\angle K = 0$ SINCE $K = Ke^{j0}$

as $\omega \to 0^+$ $|GH| \to \frac{K}{2}$ $\angle GH = 180^\circ$

$\omega \to \infty$ $|GH| \to K$ $\angle GH = 180^\circ - 90^\circ - 90^\circ = 0^\circ$

- FOR $\omega = 0^+$ $\quad GH = |GH| e^{j\phi} = \frac{K}{2} e^{j180^\circ} = -\frac{K}{2}$

  $\omega = \infty$ $\quad GH = |GH| e^{j\phi} = K e^{j0} = K$



ALONG (II) $\quad s = Re^{j\theta} \qquad R \to \infty \qquad -90^\circ \le \theta \le 90^\circ$

$$GH = \frac{K(Re^{j\theta} - 1)}{(Re^{j\theta} + 2)} \qquad \text{AS } R \to \infty \qquad GH \to \frac{K Re^{j\theta}}{Re^{j\theta}} \to K$$

$\Rightarrow \quad |GH| = K \;\; \& \;\; \angle GH = 0^\circ$ $\qquad$ THAT IS b, c, d ALL MAP TO K

ALONG (III) $\quad s = j\omega \qquad -\infty \le \omega \le 0^-$ $\qquad$ THIS PATH IS MIRROR IMAGE OF (I)

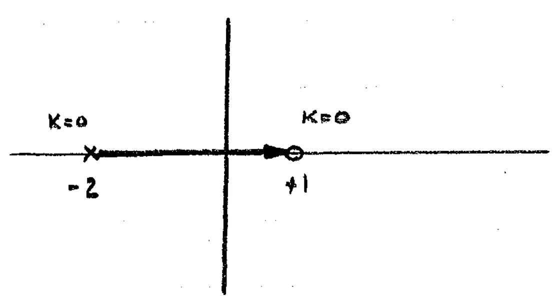
- REMEMBER FOR STABILITY PATH SHOULD <u>NOT</u> ENCIRCLE -1 $\Rightarrow -\frac{K}{2} > -1$ or $K < 2$

  MARGINAL STABILITY IF PASSES THROUGH -1 $\qquad -\frac{K}{2} = -1$ or $K = 2$

  UNSTABLE IF PASSES TO LEFT OF -1 $\Rightarrow -\frac{K}{2} < -1$ or $K > 2$

- THUS DEPENDING ON K $\qquad Z = 0 \;\; (K \le 2) \qquad Z = 1 \;\; (K > 2)$

- <u>COMPARE WITH ROOT LOCUS</u> : $\quad F(s) = 1 + GH = 1 + K\frac{Z(s)}{P(s)} = 1 + K\frac{(s-1)}{s+2} = 0$



- CROSSES AT $s = 0$ $\quad (K=2)$

  MARGINAL STABILITY

- USE ROUTH CRITERIA $\quad 1 + K\frac{(s-1)}{s+2} = 0 \qquad (s+2) + K(s-1) = 0 \quad s(1+K) + (2-K) = 0$

  $\Rightarrow \quad 1 + K > 0 \qquad 2 - K > 0$

  $-1 \le K < 2$

- USE $K < 2$ IN INEQUALITY FOR STABLE SYSTEM
- NOTE THAT THIS GIVES REGION ON REAL AXIS OF GH PLANE WHEN *NYQUIST PLOT* PASSES THROUGH -1

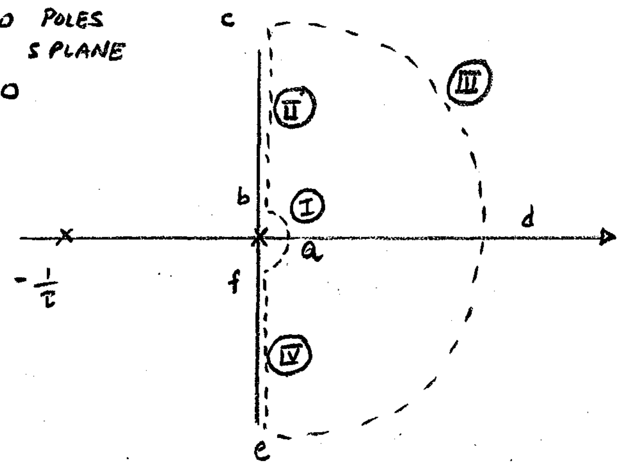
ANOTHER EXAMPLE

- GIVEN $GH = \dfrac{K}{s(\tau s+1)}$

DUE TO THE s IN DENOMINATOR MUST MODIFY D CONTOUR

NOTE NO POLES IN R-H S PLANE

$\therefore P = 0$



ON III $s = Re^{j\theta}$ AS $R \to \infty$

ALONG: I $s = \epsilon e^{j\theta}$ AS $\epsilon \to 0$

$-90^\circ \le \theta \le 90^\circ$

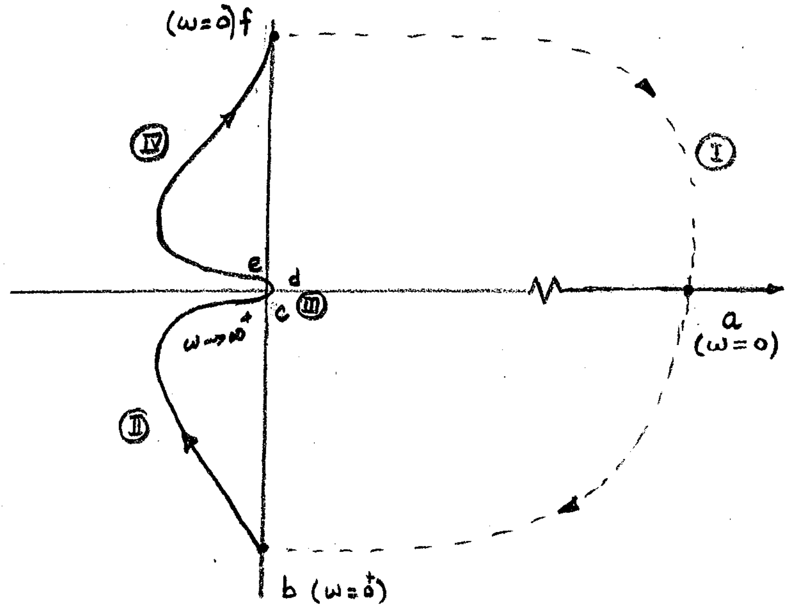
$$GH = \frac{K}{\epsilon e^{j\theta}(\tau \epsilon e^{j\theta} + 1)} \cong \frac{K}{\epsilon e^{j\theta}} = \frac{K}{\epsilon} e^{-j\theta}$$

$|GH|$ is $\frac{K}{\epsilon}$ AND $\measuredangle GH$ is $-\theta$

AT f AS $\epsilon \to 0$ $\quad |GH| \to \infty \quad \measuredangle GH = -(-90^\circ) = 90^\circ$

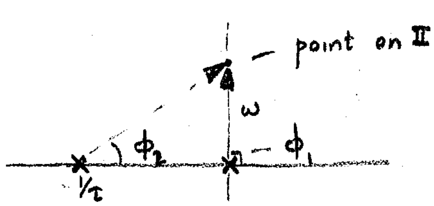
a AS $\epsilon \to 0$ $\quad |GH| \to \infty \quad \measuredangle GH = -(0^\circ) = 0^\circ$

b AS $\epsilon \to 0$ $\quad |GH| \to \infty \quad \measuredangle GH = -(90^\circ) = -90^\circ$



ALONG II $\quad s = j\omega \quad \epsilon \le \omega \le R$

$$GH = \frac{K}{j\omega(\tau j\omega + 1)}$$



$$GH = \frac{K}{\omega\sqrt{\tau^2\omega^2+1}} e^{j\phi}$$

$$\phi = -90^\circ - \tan^{-1}(\tau\omega)$$

$$= \phi_1 - \phi_2$$

THUS $|GH| = \dfrac{K}{\omega\sqrt{\tau^2\omega^2+1}}$ & $\measuredangle GH = -90^\circ - \tan^{-1}(\tau\omega)$

AS $\omega \to \epsilon$ $|GH| \to \dfrac{K}{\epsilon} \to \infty$ & $\measuredangle GH \to -90^\circ$

$\omega \to +\infty$ $|GH| \to \dfrac{K}{\omega(\tau\omega)} \to 0$ & $\measuredangle GH \to -90^\circ - 90^\circ = -180^\circ$

- ALONG III $s = Re^{j\theta}$ θ goes FROM $+90^\circ \to -90^\circ$; $GH = \dfrac{K}{Re^{j\theta}(\tau Re^{j\theta}+1)} \cong \dfrac{K}{\tau R^2 e^{2j\theta}}$

$|GH| = \dfrac{K}{\tau R^2}$ AND $\measuredangle GH = -2\theta \Rightarrow -180^\circ \le \measuredangle GH \le 180^\circ$

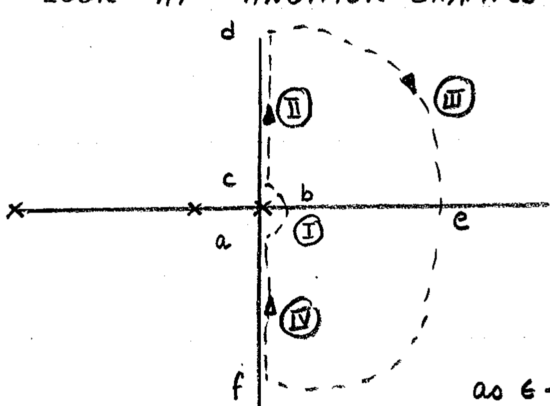
AS $R \to \infty$ $|GH| \to 0$

- NOTE THAT SINCE -1 IS NOT ENCIRCLED $N = 0$ & $Z = 0$
- NOTE THAT IV IS THE MIRROR IMAGE OF II

- WHEN GH HAS NO POLES OR ZEROES IN RIGHT-HALF S-PLANE ARE CALLED MINIMUM PHASE FUNCTIONS IF $K > 0$. OTHERWISE IT IS A NON-MINIMUM PHASE FN.

- FOR MINIMUM PHASE FUNCTIONS - YOU NEED ONLY PLOT $s = j\omega$ FOR $0 \le \omega \le \infty$ USING EITHER THE MODIFIED CONTOUR OR THE REGULAR CONTOUR DEPENDING ON WHETHER GH INVOLVES POWERS OF S

LOOK AT ANOTHER EXAMPLE $GH = \dfrac{K}{s(s+2)(s+20)}$ WHICH CONTOUR? WHY?

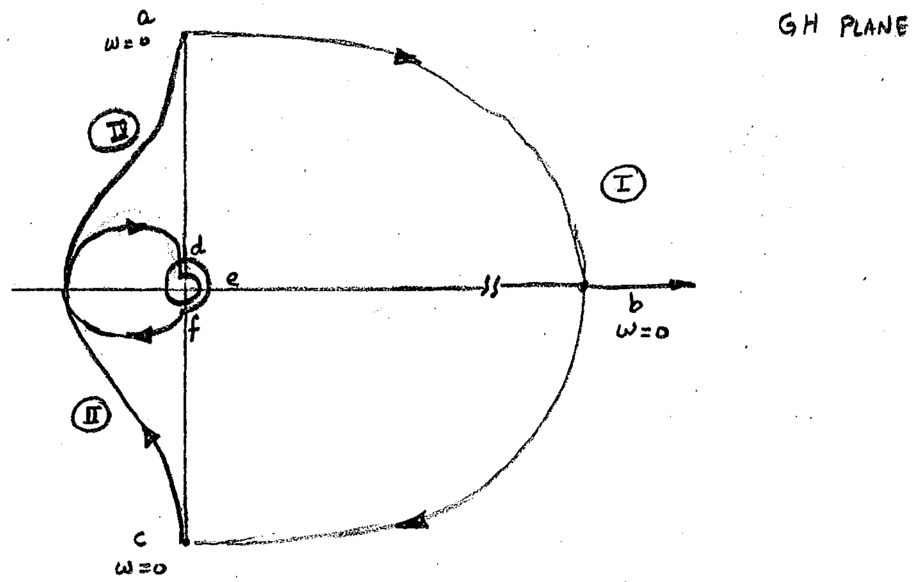


ALONG I $s = \epsilon e^{j\theta}$ $-90^\circ \le \theta \le 90^\circ$

$$\frac{K}{\epsilon e^{j\theta}(\epsilon e^{j\theta}+2)(\epsilon e^{j\theta}+20)} \cong \frac{K}{40\epsilon e^{j\theta}} = \frac{K}{40\epsilon} e^{-j\theta}$$

$|GH| = \dfrac{K}{40\epsilon}$ $\measuredangle GH = -\theta$

as $\epsilon \to 0$ $|GH| \to \infty$ for all values of θ



ALONG (II) $\quad s = j\omega \qquad \epsilon \le \omega \le R \qquad \epsilon \to 0 \quad R \to \infty$

$$GH = \frac{K}{j\omega(j\omega+2)(j\omega+20)} = \frac{K}{\omega\sqrt{\omega^2+4}\,\sqrt{\omega^2+400}}\, e^{j\phi}$$

$$\phi = -90^\circ - \tan^{-1}(\omega/4) - \tan^{-1}(\omega/20)$$

for $\omega = \epsilon \quad \dfrac{K}{\epsilon \cdot 2 \cdot 40} \to \infty$ as $\epsilon \to 0$; $\quad \phi \to -90^\circ$

for $\omega = R \quad \dfrac{K}{R \cdot R \cdot R} \to 0$ as $R \to \infty \qquad \phi \to -270^\circ$

- ALONG (III) $\quad s = Re^{j\theta} \qquad \theta$ goes from $90^\circ \to -90^\circ$

$$GH = \frac{K}{Re^{j\theta}(Re^{j\theta}+2)(Re^{j\theta}+20)} \cong \frac{K}{R^3 e^{3j\theta}} = \frac{K}{R^3} e^{-3j\theta}$$

- AS $\quad \underline{\theta} \to \;$ 90 $\;$ 60 $\;$ 30 $\;$ 0 $\;$ -30 $\;$ -60 $\;$ -90
 THEN $\;\measuredangle GH\;$ -270 $\;$ -180 $\;$ -90 $\;$ 0 $\;$ 90 $\;$ 180 $\;$ 270 $\quad \Big\} \; |GH| = \dfrac{K}{R^3} \to 0$ as $R \to \infty$

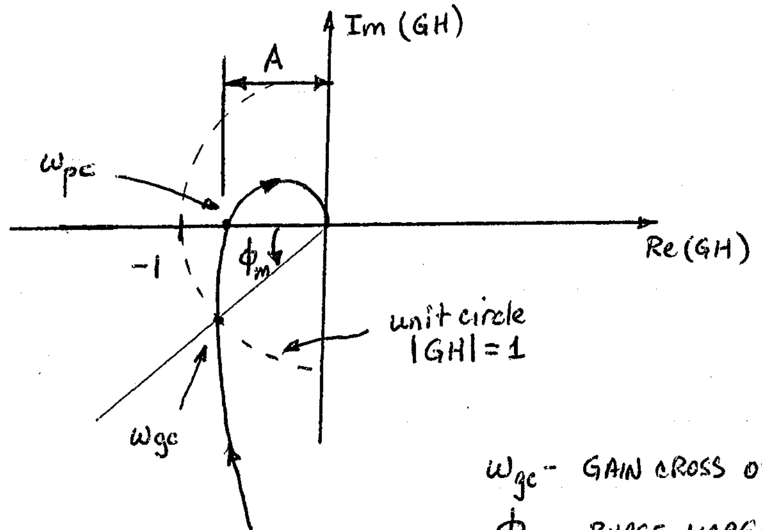
ALONG (IV) THIS IS MIRROR IMAGE OF (II)

- NOTE CROSSING THE NEGATIVE REAL AXIS IN GH-PLANE IS EQUIVALENT TO THE ROOT LOCUS PLOT CROSSING THE IMAGINARY AXIS AND DENOTES A CONDITIONALLY STABLE SYSTEM.

- FOR MINIMUM PHASE - IF, AS YOU TRAVEL ALONG CURVE IN GH PLANE IN DIRECTION OF INCR. ω, YOU CAN SEE -1 PT TO YOUR RIGHT SYSTEM IS UNSTABLE

PHASE AND GAIN MARGINS - RELATIVE STABILITY

- NYQUIST THEOREM PROVIDES A CONVENIENT MEASURE OF THE RELATIVE STABILITY OF THE SYSTEM.
- DRAW A CIRCLE OF RADIUS 1 ABOUT ORIGIN OF GH - PLANE



ω_{gc} - GAIN CROSS OVER FREQ WHERE $|GH| = 1$

ϕ_m - PHASE MARGIN, MEASURED POSITIVE DOWNWARD FROM NEGATIVE REAL AXIS

ω_{pc} - PHASE CROSS OVER FREQ WHERE $\measuredangle GH = 180^\circ$

A - IS $|GH|$ WHEN $\measuredangle GH = -180^\circ$

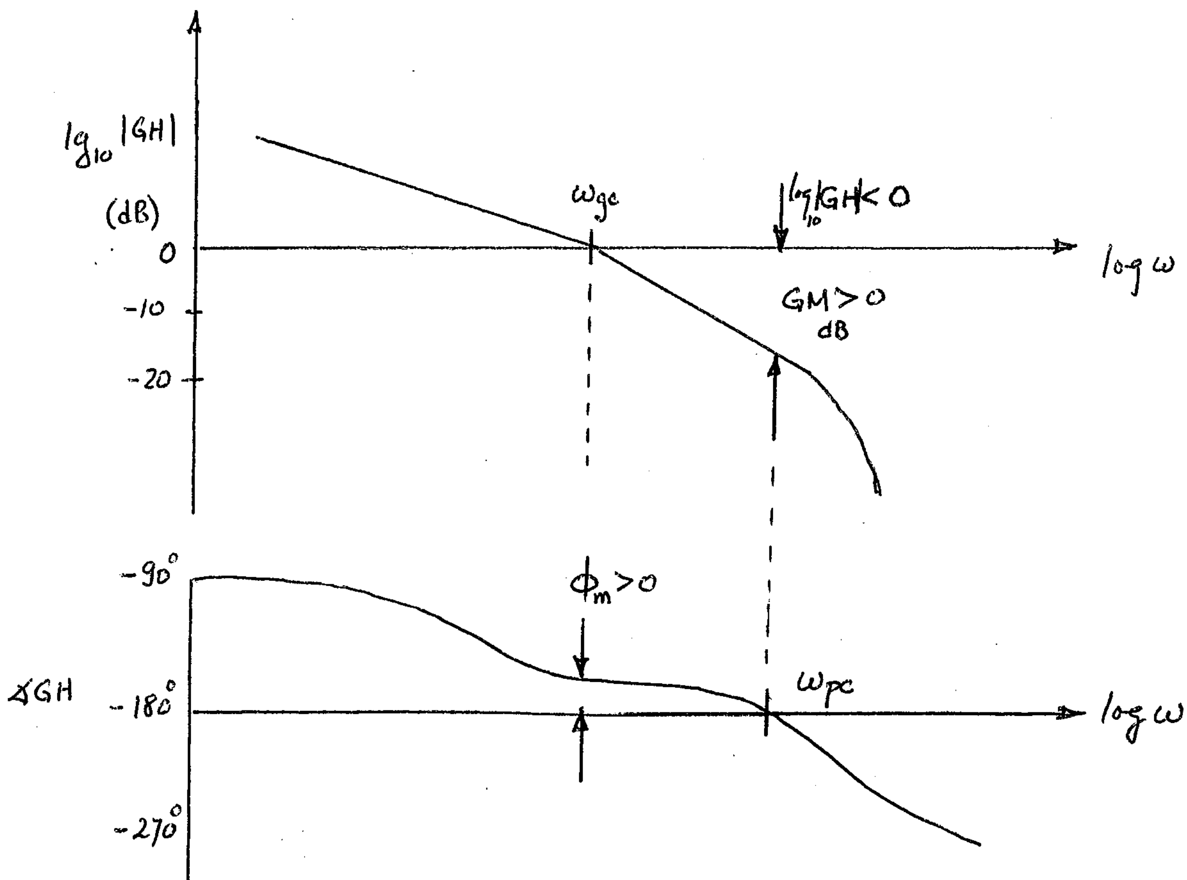
GM - GAIN MARGIN $= 1/A$

- GAIN MARGIN IS THE RATIO OF THE GAIN REQUIRED TO FORCE THE NYQUIST PLOT THROUGH THE $-1+0j$ POINT (K_c) TO THE DESIGN GAIN (K)

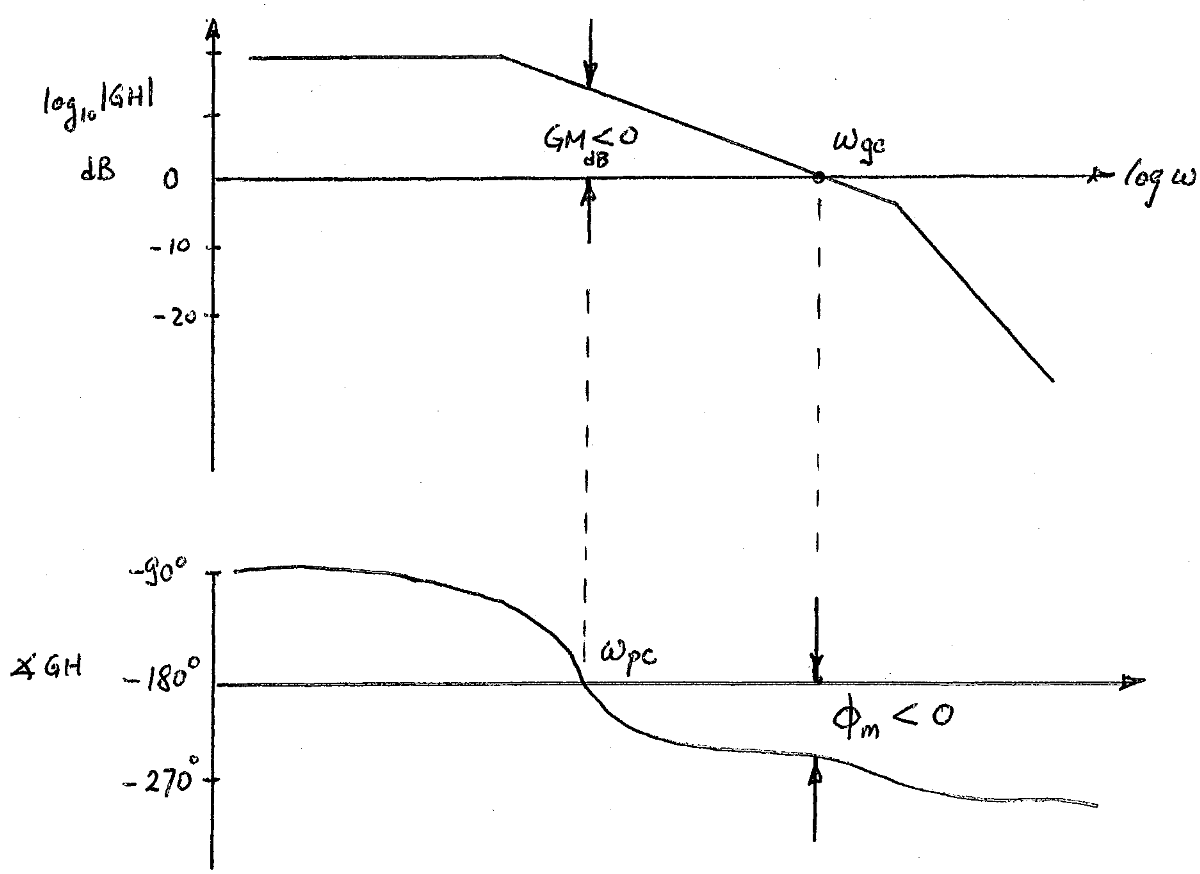
$$GM = K_c/K = \frac{1}{A} = \frac{1}{|GH|}\bigg|_{\omega=\omega_{pc}}$$

- $\boxed{\text{FOR A STABLE SYSTEM} \quad GM > 0 \quad \text{AND} \quad \phi_M > 0}$

- IT IS MORE COMMON TO LOOK AT ϕ_m AND GM THROUGH BODE PLOTS. THE PLOTS ARE OF $20\log_{10}|GH|$ in dB VS. $\log\omega$; $\log_{10}\omega$ VS. ϕ.
- RECALL $\log_{10} 1 = 0$; SO THAT WHEN $|GH| = 1 \Rightarrow 0dB$
- SINCE $GM = 1/A \Rightarrow \log_{10} GM = -\log_{10}|GH|$ in dB. FOR STABLE NYQUIST $A < 1 \Rightarrow \log_{10} A < 0 \Rightarrow GM_{dB} > 0$ $\boxed{GM_{dB} = 20\log_{10} GM} = -20\log_{10}|GH|$



STABLE SYSTEM



UNSTABLE SYSTEM

HELP SESSION ON BODE PLOTS

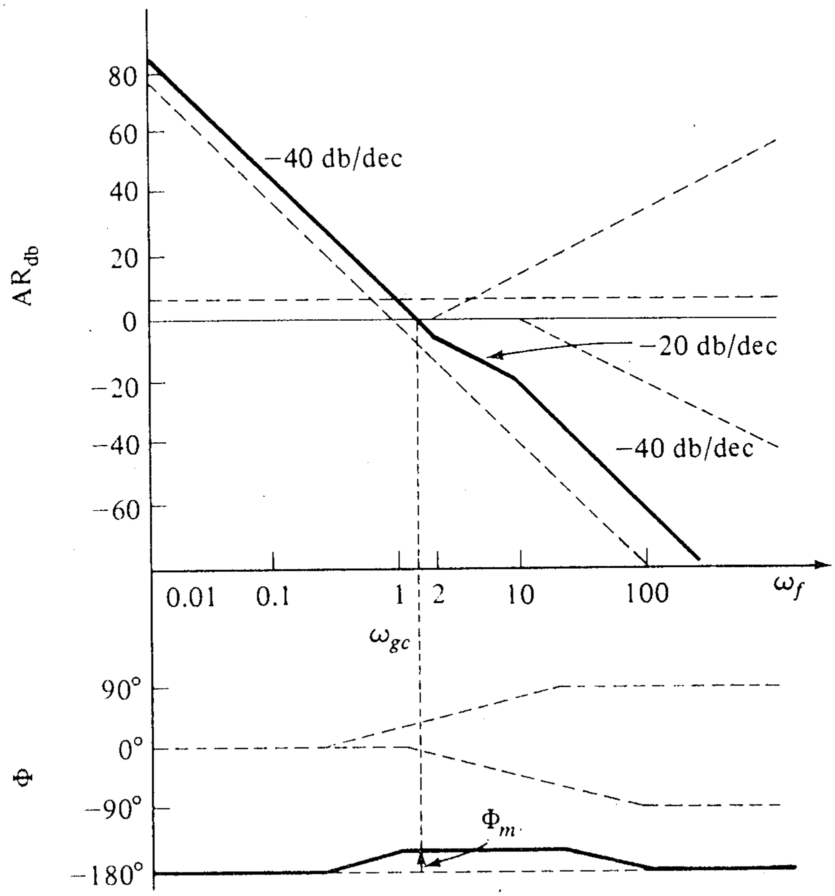


Figure 8-2-6. Bode plots of $\dfrac{K_1(j\omega/2 + 1)}{5(j\omega)^2(j\omega/10 + 1)}$ for $K^1 = 10$.

The asymptotic Bode plots are a reasonable approximation of the exact plots except in the vicinity of the corner frequencies of lightly damped second-order terms. Fortunately, exact values in these regions usually are not needed for the analysis of well-behaved systems. However, when these regions cannot be avoided and must be considered, computer plots should be used or corrections can be made to the asymptotic plots based on a knowledge of the appropriate damping ratios.

8-3 BODE PLOTS AND STABILITY

The Nyquist stability criterion for a minimum phase system as developed in Sec. 7-5 states that a system is stable if the phase margin Φ_m is positive; furthermore, the larger the phase margin, the more stable the system. The approximate phase margin can be easily obtained from the Bode plots of the $KZ(j\omega)/P(j\omega)$ function.* Bode plots, therefore, can be used to obtain information about both the absolute and relative stability of a system.

* The open-loop frequency transfer function $GH(j\omega)$ is customarily used.

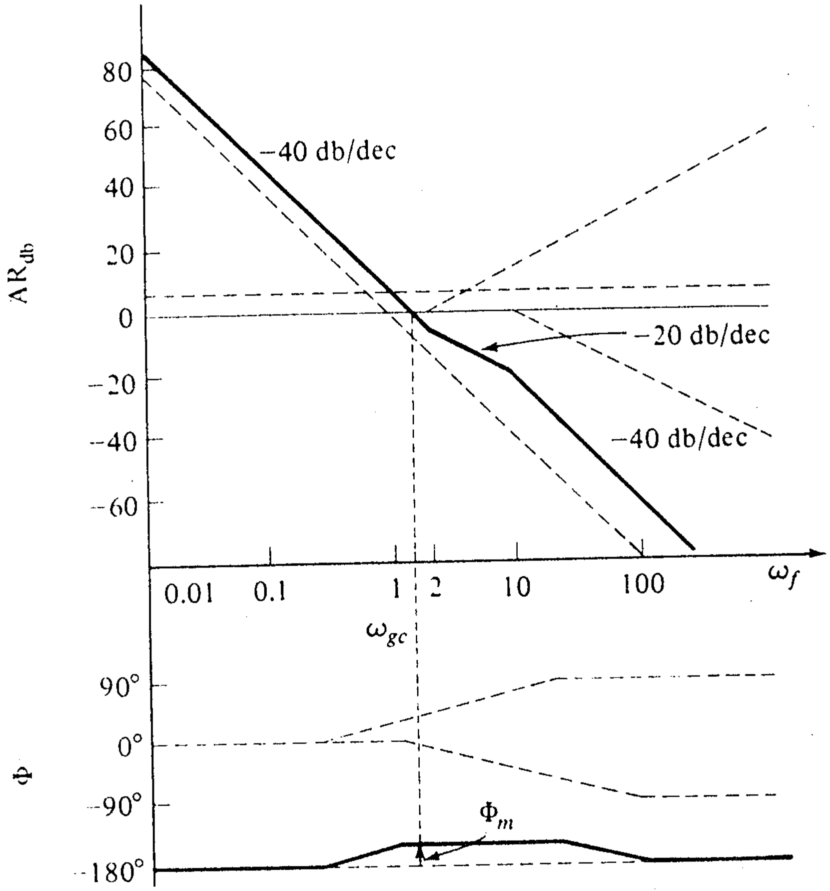


Figure 8-2-6. Bode plots of

$$GH = \frac{K_1 (s+2)}{s^2 (s+10)} \qquad K_1 = 10$$

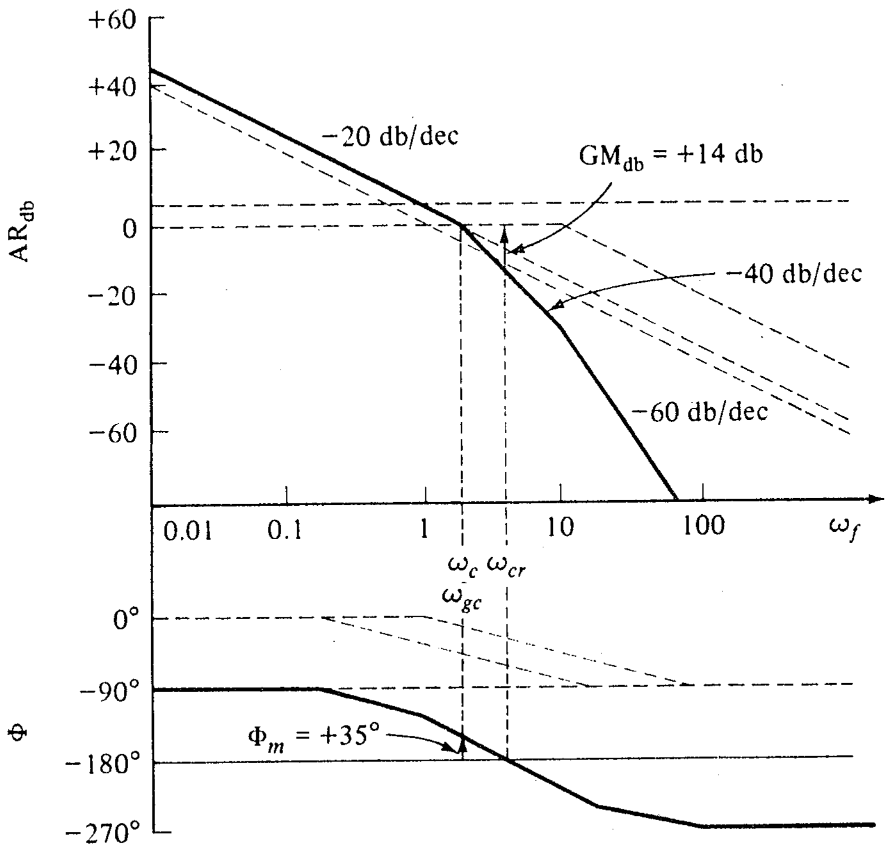


Figure 8–2–5. Bode plots of

$$\frac{K_1}{20} \frac{1}{s(s/2+1)(s/10+1)} \qquad K_1 = 40$$

or $$\frac{K_1}{s(s+2)(s+10)}$$

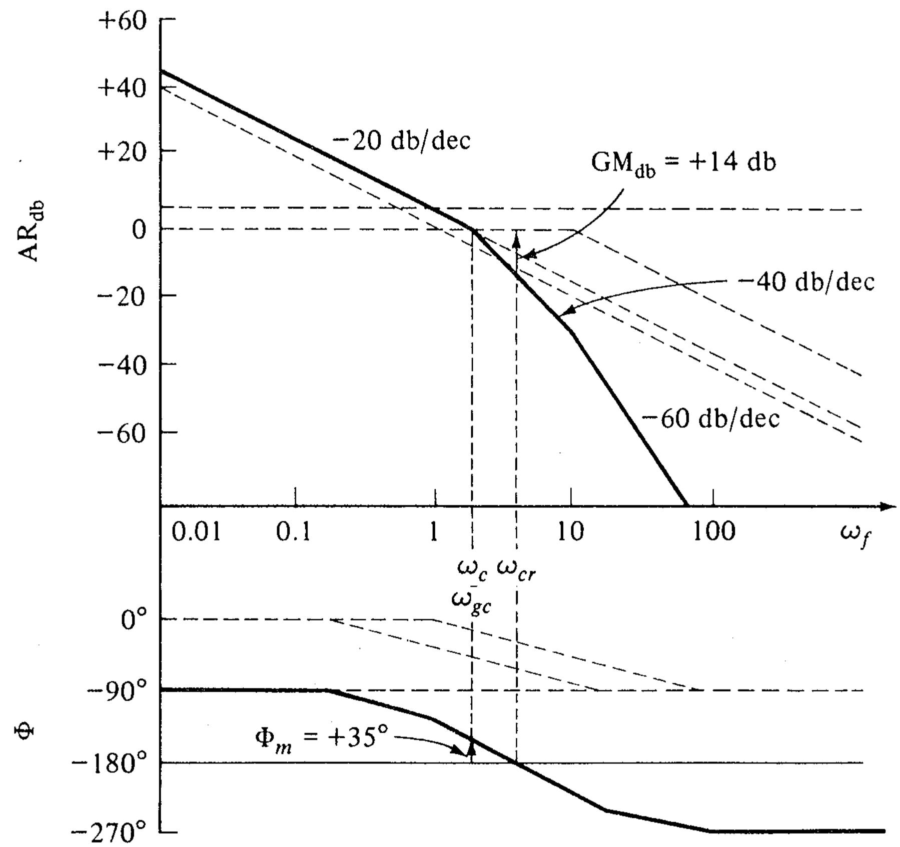


Figure 8–2–5. Bode plots of

$$\frac{K_1}{20}\frac{1}{j\omega(j\omega/2 + 1)(j\omega/10 + 1)} \qquad \text{for } K_1 = 40.$$

phase plots produces the total phase angle associated with the function. Notice that increasing or decreasing the value of K moves the overall AR_{db} curve up or down with respect to the 0-db line but does not affect the phase angle plot.

For the next example, let

$$K\frac{Z(s)}{P(s)} = \frac{K_1(s + 2)}{s^2(s + 10)} \tag{8–2–19}$$

which in the frequency domain is

$$K\frac{Z(j\omega)}{P(j\omega)} = \frac{K_1}{5}\frac{j\omega/2 + 1}{(j\omega)^2(j\omega/10 + 1)} \tag{8–2–20}$$

If K_1 is set equal to 10, K is 2 and K_{db} is +6 db. The corner frequency of the real zero is 2, and that of the real pole is 10. Each AR_{db} curve is drawn in Fig. 8–2–6 and added to produce the overall AR_{db} plot. The phase angle of the zero passes through +45° at $\omega_c = 2$, reaching +90° at $\omega = 20$. This and the other individual phase angles are drawn and then added to obtain the overall phase angle plot. The overall phase angle starts at −180° at low frequencies because of the double pole at the origin. The phase lead of the zero is felt first, since it has a lower corner frequency than the pole. The phase lag of the pole eventually counteracts the effect of the zero, and the phase angle returns to −180° at higher frequencies.

COMPUTER RESULTS FOR EXAMPLE 2.8a

A	B	C	D	E
1.00000000 5	1.00139755 5	1.0013976	0.50000000 4	8.00000000 7
2.00000000 5	1.96892082 4	1.9689208	1.00000008 4	3.99999862 6
3.00000000 5	3.31832477 7	3.3183233	2.00000007 4	2.00000552 6
4.00000000 5	3.50505891 7	3.5050604	4.00000005 4	0.99999079 5
5.00000000 4		5.5731849 +0.2641298i	7.99999999 2	0.50000485 2
6.00000014 4		5.5731849 −0.2641298i		
6.99999993 2	7.05992816 40	7.0599281		

EXERCISES

2.6-1. Using Algorithm 2.10 and a desk calculator, find the real root of

$$x^3 + 2x - 1 = 0$$

correct to seven significant figures. Determine the remaining roots from the reduced polynomial, using the quadratic formula. How accurate are these solutions?

2.6-2. Write a computer program for finding a real root of a polynomial of any degree, based on Algorithm 2.10.

2.6-3. Using the program developed in Exercise 2.6-2, find the real positive roots of the following polynomials:

$(a)\ x^5 - 3x^3 + x^2 - 1 = 0 \qquad (b)\ x^3 + 3x - 1 = 0$

2.6-4. The polynomial $x^4 + 2.8x^3 - 0.38x^2 - 6.3x - 4.2$ has four real zeros. Find them, using Algorithm 2.10.

2.6-5. The polynomial

$$p(x) = x^8 - 170x^6 + 7{,}392x^4 - 39{,}712x^2 + 51{,}200$$

has the zeros $\pm 10, \pm 8, \pm 2, \pm\sqrt{2}$. Find these roots on a computer in ascending order of magnitude, choosing initial approximations within 10 percent of the exact solutions. Then change the coefficient of x^2 to $-39{,}710$, and solve the problem once again. Observe the change in the solutions.

2.7 COMPLEX ROOTS AND MULLER'S METHOD

The methods discussed up to this point allow us to find an isolated zero of a function once an approximation to that zero is known. These methods are not very satisfactory when all the zeros of a function are required or when good initial approximations are not available. For polynomial functions there are methods which yield an approximation to all the zeros simultaneously, after which the iterative methods of this chapter can be applied to obtain more accurate solutions. Among such methods may be mentioned the quotient-difference algorithm [2] and the method of Graeffe [5].

A method of recent vintage, expounded by Muller [6], has been used on computers with remarkable success. This method may be used to find any prescribed number of zeros, real or complex, of an arbitrary function. The method is iterative, converges almost quadratically in the vicinity of a root, does not require the evaluation of the derivative of the function, and obtains both real and complex roots even when these roots are not simple.

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MULLER'S METHOD FOR HW PROBLEMS 8-5, 8-

Moreover, the method is global in the sense that the user need not supply an initial approximation. In this section we describe briefly how the method is derived, omitting any discussion of convergence, and we discuss its use in finding both real and complex roots. We will especially emphasize the problem of finding complex zeros of polynomials with real coefficients since this problem is of great concern in many branches of engineering.

Muller's method is an extension of the secant method. To recall, in the secant method we determine, from the approximations x_i, x_{i-1} to a root of $f(x) = 0$, the next approximation x_{i+1} as the zero of the linear polynomial $p(x)$ which goes through the two points $\{x_i, f(x_i)\}$ and $\{x_{i-1}, f(x_{i-1})\}$. In Muller's method, the next approximation, x_{i+1}, is found as a zero of the *parabola* which goes through the three points $\{x_i, f(x_i)\}$, $\{x_{i-1}, f(x_{i-1})\}$, and $\{x_{i-2}, f(x_{i-2})\}$.

Let x_i, x_{i-1}, x_{i-2} be three distinct approximations to a root of $f(x) = 0$. We use the abbreviations

$$f_i = f(x_i) \qquad f_{i-1} = f(x_{i-1}) \qquad f_{i-2} = f(x_{i-2})$$

In Chap. 4, we show that, with

$$f[x_i, x_{i-1}] = \frac{f_i - f_{i-1}}{x_i - x_{i-1}}$$
$$f[x_i, x_{i-1}, x_{i-2}] = \frac{f[x_i, x_{i-1}] - f[x_{i-1}, x_{i-2}]}{x_i - x_{i-2}} \tag{2.45}$$

0

the function

$$p(x) = f_i + f[x_i, x_{i-1}](x - x_i) + f[x_i, x_{i-1}, x_{i-2}](x - x_i)(x - x_{i-1})$$

is the unique parabola which goes through the three points $\{x_i, f_i\}$, $\{x_{i-1}, f_{i-1}\}$, and $\{x_{i-2}, f_{i-2}\}$. In the more customary way of writing down polynomials, we get that

$$p(x) = a_0 + a_1 x + a_2 x^2$$

with

$$\begin{aligned} a_2 &= f[x_i, x_{i-1}, x_{i-2}] \\ a_1 &= f[x_i, x_{i-1}] - (x_i + x_{i-1})a_2 \\ a_0 &= f_i - x_i(f[x_i, x_{i-1}] - x_{i-1}a_2) \end{aligned} \tag{2.46}$$

Once the numbers a_0, a_1, a_2 have been calculated from (2.45) and (2.46), the roots of $p(x)$ can be determined from the quadratic formula, written

$$x = \frac{2a_0}{-a_1 \pm (a_1^2 - 4a_0a_2)^{1/2}} \tag{2.47}$$

The sign before the radical in (2.47) is selected so that the denominator is largest in magnitude, and the corresponding root is taken as the next approximation, x_{i+1}. The reason for writing the quadratic formula in this form, as explained in Sec. 1.4, is to obtain maximum accuracy in the formula. The process is then repeated, using x_{i-1}, x_i, x_{i+1} as the three basic approximations. If the roots obtained from (2.47) are real, the situation is pictured graphically in Fig. 2.7. Note, however, that even if the root being sought is real, we may encounter complex approximations because the solutions given by (2.47) may be complex. However, in such cases the complex component will normally be so small in magnitude that it can be neglected. In fact, in the subroutine described later, any complex components encountered when seeking a real root are suppressed.

For reasons of accuracy and convenience, Muller proposed a somewhat different but equivalent sequence of steps for obtaining the next approximation, x_{i+1}. This sequence of steps is described in Algorithm 2.11

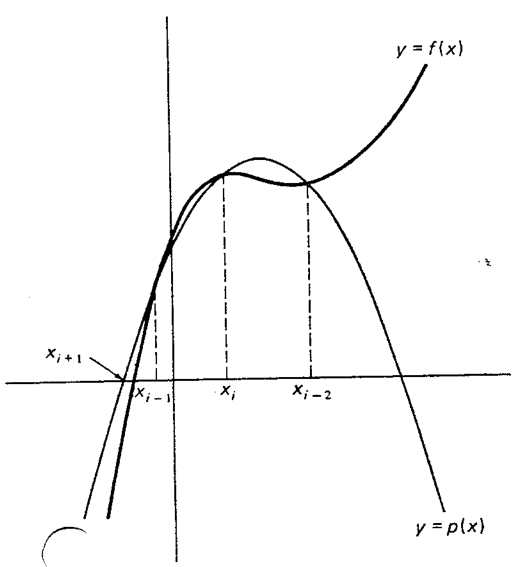


Fig. 2.7

Algorithm 2.11 Muller's method

1. Let x_i, x_{i-1}, x_{i-2} be three approximations to a zero ξ of $f(x)$. Compute f_i, f_{i-1}, f_{i-2}.

2. Compute

$$h_i = x_i - x_{i-1}$$
$$\lambda_i = \frac{h_i}{x_{i-1} - x_{i-2}}$$

3. Compute

$$g_i = (1 + 2\lambda_i)(f_i - f_{i-1}) - \lambda_i^2(f_{i-1} - f_{i-2})$$

4. Compute

$$\lambda_{i+1} = \frac{-2f_i(1 + \lambda_i)}{g_i \pm [g_i^2 - 4f_i(1 + \lambda_i)\lambda_i(f_i - f_{i-1} - \lambda_i(f_{i-1} - f_{i-2}))]^{1/2}}$$

choosing the sign so that the denominator will always have the largest magnitude.

5. Compute

$$x_{i+1} = x_i + h_i\lambda_{i+1}$$
$$h_{i+1} = h_i\lambda_{i+1}$$

6. Compute

$f_{i+1} = f(x_{i+1})$ and increase i by 1

7. Repeat steps 3 to 6 until either of the following criteria is satisfied for prescribed $\varepsilon_1, \varepsilon_2$:

(a) $\dfrac{|x_i - x_{i-1}|}{|x_i|} < \varepsilon_1$

(b) $|f(x_i)| < \varepsilon_2$

A complete subroutine based on this algorithm is given below. The arguments in this subroutine have the following meanings:

KN ≡ number of roots previously computed, normally zero.
N ≡ number of roots desired.
RTS ≡ an array of initial estimates of all desired roots; set to zero if no better estimates are available.
MAXIT ≡ maximum number of iterations per root allowed.
EP1 ≡ relative error tolerance on x_i.
EP2 ≡ error tolerance on $f(x_i)$.
FN ≡ FN(Z,FZ) is a subroutine which, for given Z, returns FZ = f(Z).

FNREAL ≡ a logical variable; if TRUE, the subroutine forces all approximations to all the roots to be real. This makes it possible to use this routine even if $f(x)$ is defined only for real x.

```
      SUBROUTINE MULLER(KN,N,RTS,MAXIT,EP1,EP2,FN,FNREAL)
      COMPLEX RTS(1)
      LOGICAL FNREAL
      COMPLEX RT,H,DELFPR,FRTDEF,LAMBDA,DELF,DFPRLM,NUM,
     *DEN,G,SQR,FRT,FRTPRV
C
C     INITIALIZATION
      EPS1=AMAX1(EP1,1.E-12)
      EPS2=AMAX1(EP2,1.E-20)
      IBEG=KN+1
      IEND=KN+N
C
      DO 100 I = IBEG,IEND
      KOUNT=0
COMPUTE FIRST THREE ESTIMATES FOR ROOT AS
C        RTS(I)+.5, RTS(I) -.5, RTS(I) .
    1 H=.5
      RT=RTS(I)+H
      ASSIGN 10 TO NN
                                                 GO TO 70
   10 DELFPR = FRTDEF
      RT=RTS(I)-H
      ASSIGN 20 TO NN
                                                 GO TO 70
   20 FRTPRV=FRTDEF
      DELFPR=FRTPRV-DELFPR
      RT=RTS(I)
      ASSIGN 30 TO NN
                                                 GO TO 70
   30 ASSIGN 80 TO NN
      LAMBDA = -.5
COMPUTE NEXT ESTIMATE FOR ROOT
   40 DELF = FRTDEF - FRTPRV
      DFPRLM = DELFPR*LAMBDA
      NUM = -FRTDEF*(1. + LAMBDA)*2.
      G = (1. + LAMBDA*2.)*DELF - LAMBDA*DFPRLM
      SQR = G*G + 2.*NUM*LAMBDA*(DELF - DFPRLM)
      IF (FNREAL .AND. REAL(SQR) .LT. 0.) SQR = 0.
      SQR = CSQRT(SQR)
      DEN = G + SQR
      IF (REAL(G)*REAL(SQR) + AIMAG(G)*AIMAG(SQR) .LT. 0.)
     *DEN = G - SQR
      IF (CABS(DEN) .EQ. 0.) DEN = 1.
      LAMBDA = NUM/DEN
```



```
      FRTPRV = FRTDEF
      DELFPR = DELF
      H = H*LAMBDA
      RT = RT + H
      IF (KOUNT .GT. MAXIT)                      GO TO 100
C
   70 KOUNT = KOUNT + 1
      CALL FN(RT,FRT)
      FRTDEF = FRT
      IF (I .LT. 2)                              GO TO 75
      DO 71 J = 2,I
      DEN = RT - RTS(J - 1)
      IF (CABS(DEN) .LT. EPS2)                   GO TO 79
   71 FRTDEF = FRTDEF/DEN
   75                                            GO TO NN,(10,20,30,80)
   79 RTS(I) = RT + .001
                                                 GO TO 1
CHECK FOR CONVERGENCE
   80 IF (CABS(H) .LT. EPS1*CABS(RT))            GO TO 100
      IF (AMAX1(CABS(FRT),CABS(FRTDEF)) .LT. EPS2)  GO TO 100
C
CHECK FOR DIVERGENCE
      IF (CABS(FRTDEF) .LT. 10.*CABS(FRTPRV))  GO TO 40
      H = H/2.
      LAMBDA = LAMBDA/2.
      RT = RT - H
                                                 GO TO 70
  100 RTS(I) = RT
                                                 RETURN
      END
```

Muller's method, like the other algorithms described in this chapter, finds one zero at a time. To find more than one zero it uses a procedure known as *deflation*. If, for example, one zero ξ_1 has already been found, the routine calculates the next zero by working with the function

$$f_1(x) = \frac{f(x)}{x - \xi_1} \tag{2.48}$$

If r zeros $\xi_1, \ldots, \xi_r$ have already been found, the next root is obtained by working with the deflated function

$$f_r(x) = \frac{f(x)}{(x - \xi_1)(x - \xi_2) \cdots (x - \xi_r)} \tag{2.49}$$

If no estimates are given, the routine always looks for zeros in order of increasing magnitude since this will usually minimize round-off-error growth. Also, all zeros found using deflated functions are tested for accuracy by substituting into the original function $f(x)$. In applying the Muller subroutine, the user can specify the number of zeros desired. Some functions, for example, may have an infinite number of zeros, of which only the first few may be of interest.

Example 2.9 Bessel's function $J_0(x)$ is given by the infinite series

$$J_0(x) = 1 - \frac{x^2}{2^2 \cdot 1 \cdot 1} + \frac{x^4}{2^4 \cdot 2! \cdot 2!} - \frac{x^6}{2^6 \cdot 3! \cdot 3!} + \cdots$$

It is known that $J_0(x)$ has an infinite number of real zeros. Find the first three positive zeros, using Algorithm 2.11. The machine results given below were obtained on an IBM 7094 using a standard library subroutine for $J_0(x)$ based on the series given above. The values of $J_0(x)$ were computed to maximum accuracy.

The iterations were all started with the approximations $x_0 = -1$, $x_1 = 1$, $x_2 = 0$ and were continued until either of the following error criteria was satisfied:

(a) $\dfrac{|x_{i+1} - x_i|}{|x_{i+1}|} < 10^{-6}$

(b) $|J_0(x_i)| < 10^{-20}$

The converged values are correct to at least six significant figures. Note that the zeros are obtained in ascending order of magnitude.

All the following examples were run on a CDC 6500 computer using Algorithm 2.11. The error criteria for these examples were $\varepsilon_1 = \varepsilon_2 = 10^{-8}$, and all used the same starting values (0.5, −0.5, 0.0) followed by deflation. Although the results are printed to 8 significant figures, one should recall that on a CDC 6500 the floating-point word length is 14 decimal digits. The output consists of the real and imaginary (if applicable) parts of the

COMPUTER RESULTS FOR EXAMPLE 2.9

ROOT 1	ROOT 2	ROOT 3
−0.09999999E 01	−0.09999999E 01	−0.09999999E 01
0.09999999E 01	0.09999999E 01	0.09999999E 01
0.	0.	0.
0.20637107E 01	0.36557332E 01	0.47983123E 01
0.23167706E 01	0.44416171E 01	0.59396663E 01
0.23970029E 01	0.50863190E 01	0.70758440E 01
0.24047983E 01	0.55024961E 01	0.88981197E 01
0.24048255E 01	0.55202182E 01	0.92976399E 01
0.24048255E 01	0.55200780E 01	0.86854592E 01
	0.55200780E 01	0.86529856E 01
		0.86537299E 01
		0.86537278E 01

converged approximations to the roots, and the real and imaginary parts of the value of the function at those roots.

Example 2.10 Find all the zeros of the polynomial $p(x) = x^3 + x - 3$.

ROOT REAL PART	ROOT IMAGINARY PART	F(ROOT) REAL PART	F(ROOT) IMAGINARY PART
1.2134117E+00	0.	−4.2632564E−14	0.
−6.0670583E−01	1.4506122E+00	2.8421709E−14	4.2632564E−14
−6.0670583E−01	−1.4506122E+10	2.8421709E−14	−2.6290081E−13

Compare these results with those obtained in Example 2.7, where we computed the solutions on a desk calculator. Note that since $p(x)$ has real coefficients, the complex roots occur in complex-conjugate pairs. Note too that no estimates of the complex roots had to be provided. Newton's method, although it can be used to find complex roots, must be supplied with a good estimate of that root, an estimate that is normally very difficult to obtain. Observe that the error in F(ROOT) is considerably smaller than 10^{-8} as required by the error criterion. In fact, in the last iteration, the error must have been reduced from something like 10^{-7} to 10^{-14}, indicating that the method converges almost quadratically.

Example 2.11 Find all the zeros of the polynomial $p(x) = x^3 - x - 1$ considered in Example 2.2.

Note that since this example was calculated in complex arithmetic, the approximation to the one real root $x = 1.3247180$ has a small imaginary component.

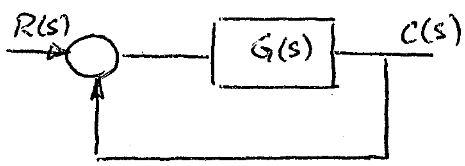
ROOT REAL PART	ROOT IMAGINARY PART	F(ROOT) REAL PART	F(ROOT) IMAGINARY PART
−6.6235898E−01	−5.6227951E−01	0.	3.5527137E−15
−6.6235898E−01	5.6227951E−01	0.	1.4210855E−14
1.3247180E+00	−1.3134765E−14	1.4921397E−13	−5.6014953E−14

Example 2.12 Find all the zeros of the polynomial $f(x) = x^3 - 3x^2 + 4$. This polynomial has a simple zero at $x = -1$ and a double zero at $x = 2$. Note the extremely good accuracy obtained by this method in both modes at a double zero. Most other methods, including Newton's, would have some difficulty producing accurate results at multiple zeros.

REAL MODE

ROOT REAL PART	ROOT IMAGINARY PART	F(ROOT) REAL PART	F(ROOT) IMAGINARY PART
−1.0000000E+00	0.	2.8421709E−14	0.
2.0000000E+00	0.	0.	0.
2.0000000E+00	0.	2.8421709E−14	0.

- GENERALLY DESIRE $GM > 6\,dB$ THIS CORRESPONDS TO RESPONSE OBTAINED USING ZIEGLER-NICHOLS SETTING FOR ULTIMATE CYCLE METHOD
- $30^\circ \le \phi_m \le 60^\circ$
- SINCE PROPORTIONAL CONTROL ONLY AFFECTS THE MAGNITUDE OF GH AND NOT THE PHASE ANGLE, IT SHOULD BE SELECTED LAST AFTER THE PHASE MARGIN CRITERIA HAS BEEN SET.
- IF PID CONTROL DESIGN D THEN I THEN P.
- EXAMPLE TO DETERMINE PHASE MARGIN AND GAIN CROSS OVER FREQ.



$$G(s) = \frac{\omega_n^2}{s(s+2\zeta\omega_n)} \qquad H(s) = 1$$

$$T(s) = \frac{G}{1+GH} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

TO FIND PHASE MARGIN : $s = j\omega_{gc}$ $\qquad |GH| = 1$ AT $\omega = \omega_{gc}$

$$\Rightarrow \left| \frac{\omega_n^2}{j\omega_{gc}(j\omega_{gc} + 2\zeta\omega_n)} \right| = 1 \quad \Rightarrow \text{IF WE LET } \Omega = \frac{\omega_{gc}}{\omega_n}$$

$$\left| \frac{1}{j\Omega\,(j\Omega + 2\zeta)} \right| = 1 \quad \text{or} \quad \frac{1}{\Omega\,(\Omega^2 + 4\zeta^2)^{1/2}} = 1$$

$$\Omega^2(\Omega^2 + 4\zeta^2) = 1 \quad \text{or} \quad \Omega^4 + 4\zeta^2\Omega^2 - 1 = 0$$

$$\Omega^2 = \frac{-4\zeta^2 \pm \sqrt{16\zeta^4 - 4(1)(-1)}}{2} \quad \Rightarrow \quad \Omega = \sqrt{-2\zeta^2 + \sqrt{4\zeta^4 + 1}}$$

- THIS GIVES ω_{gc} AS A FN OF ζ

$$\angle GH = \angle \frac{1}{j\Omega\,(j\Omega + 2\zeta)} = \angle \frac{1}{2\zeta} \cdot \frac{1}{j\omega} \cdot \frac{1}{\left(j\frac{\Omega}{2\zeta} + 1\right)}$$

$$\angle GH = \angle \frac{1}{2\zeta} + \angle \frac{1}{j\omega} + \angle \frac{1}{j\frac{\Omega}{2\zeta} + 1} = 0 - 90^\circ - \tan^{-1}\left(\frac{\Omega}{2\zeta}\right)$$

Now $\tan^{-1}\left(\frac{a}{b}\right) = 90^\circ - \tan^{-1}(b/a) \Rightarrow -\tan^{-1}\left(\frac{\Omega}{2\zeta}\right) = -90^\circ + \tan^{-1}\left(\frac{2\zeta}{\Omega}\right)$

and $\measuredangle GH = -180^\circ + \tan^{-1}\left(\frac{2\zeta}{\Omega}\right)$; SINCE WE MEASURE ϕ_m FROM -180° LINE

$$\phi_m = \tan^{-1}\left(\frac{2\zeta}{\Omega}\right)$$

- BY MEANS OF BODE DIAGRAM $\quad GH = \dfrac{1}{j\Omega(j\Omega+2\zeta)} = \dfrac{1}{2\zeta}\cdot\dfrac{1}{j\Omega}\cdot\dfrac{1}{\left(j\left(\frac{\Omega}{2\zeta}\right)+1\right)}$

$$20\log_{10}|GH| = -20\log_{10}\Omega - 20\log_{10}2\zeta - 10\log_{10}\left(\left(\frac{\Omega}{2\zeta}\right)^2+1\right)$$

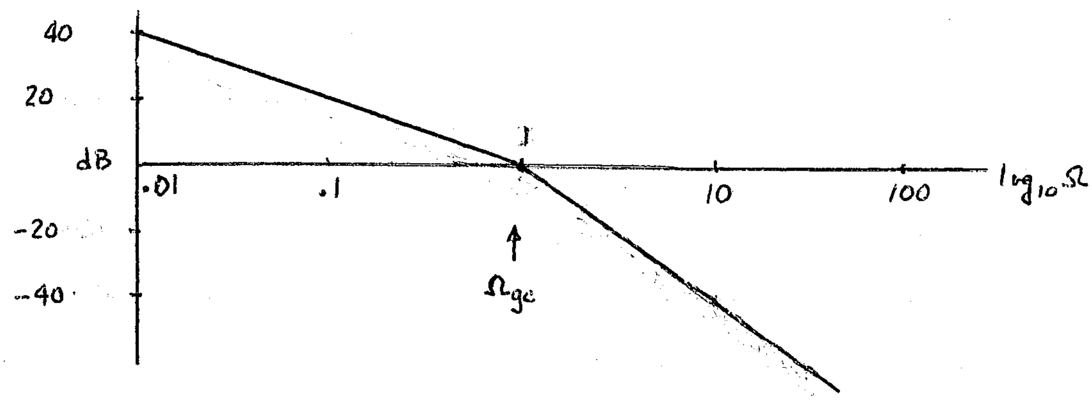
FOR $\underline{\underline{\zeta = 1/2}}$ AND PLOT VERSUS $\log_{10}\Omega$

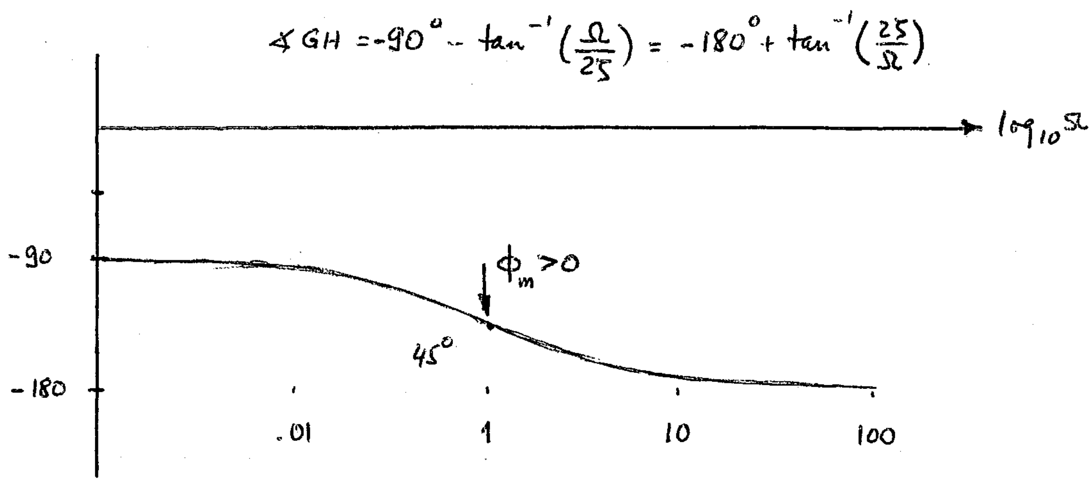
① $-20\log_{10}2\zeta = 0$; ② $-20\log_{10}\Omega$ HAS SLOPE -20 dB/dec

③ FOR LAST TERM BREAK FREQ IS WHEN $\frac{\Omega}{2\zeta} = 1$ OR $\Omega = 1$

④ LAST TERM HAS SLOPE OF -20 dB/dec FOR $\Omega \gg \Omega_{gc}$

PLOT FOR





NOTE THIS WILL GIVE AN ∞ GM SINCE ω_{pc} IS AT ∞

ACTUAL $\Omega_{gc} = .956 \quad \phi_m = 46.3^\circ$

- WHAT IF ϕ_m AND GAIN MARGIN OF A SYSTEM ARE NOT SATISFACTORY
- OFTEN CONTROLLER AND PLANT TOGETHER REQUIRE COMPENSATION

SYSTEM COMPENSATION – MODIFY SYSTEM RESPONSE TO OBTAIN

(1) STABILITY

(2) ACCEPTABLE STEADY STATE ERROR

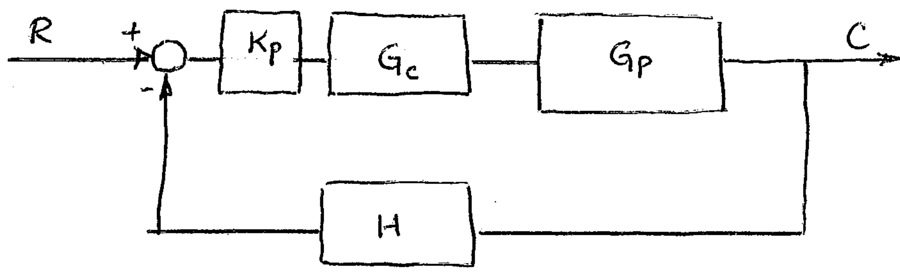
(3) ACCEPTABLE TRANSIENT RESPONSE

(4) MINIMIZE DISTURBANCE EFFECTS

SIMPLEST METHOD IS BY LEAD OR LAG COMPENSATION

LEAD COMPENSATION

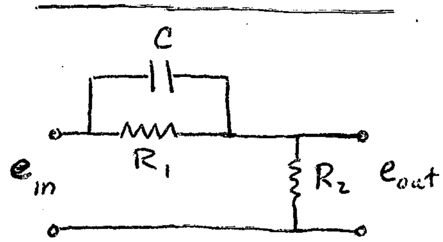
ASSUME



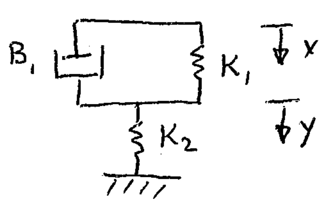
$K_p G_c G_p = G$

GH - open loop transfer fn

WITHOUT COMPENSATION : $G_c = 1$



$$G_c = \frac{E_{out}(s)}{E_{in}(s)} = \frac{\tau_2}{\tau_1} \frac{1+\tau_1 s}{1+\tau_2 s} \qquad \tau_1 > \tau_2$$



$$G_c = \frac{Y(s)}{X(s)} = \frac{\tau_2}{\tau_1} \frac{1+\tau_1 s}{1+\tau_2 s}$$

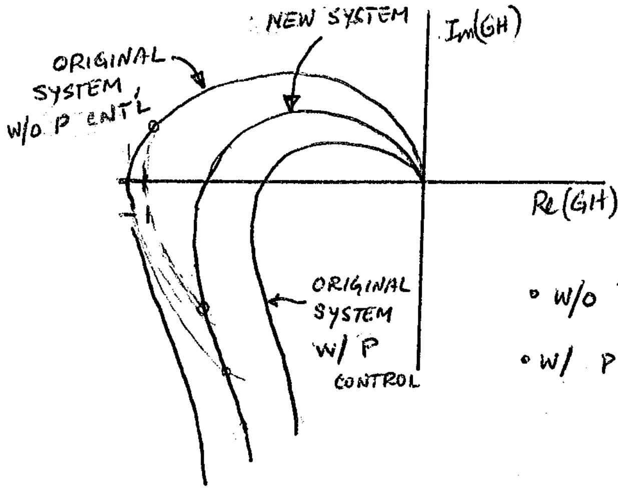
ELECTRICAL SYSTEM $\tau_1 = R_1 C_1$ $\quad \tau_2 = \dfrac{R_2}{R_1+R_2}\tau_1$

MECHANICAL SYSTEM $\tau_1 = B_1/K_1$ $\quad \tau_2 = \dfrac{K_1}{K_1+K_2}\tau_1$

LOOK AT CONTRIBUTION OF G_c AT STEADY STATE $(s \to 0)$

- $G_c \to \tau_2/\tau_1 < 1 \Rightarrow$ $|GH|$ AT STEADY STATE FOR SYSTEM w/ COMPENSATION < WITHOUT COMPENSATION

- THUS TO GET SAME GAIN AS ORIGINAL SYSTEM - CAN INCREASE PROPORTIONAL GAIN OF ORIGINAL SYSTEM K_p IF $K_p > 1$



- ADVANTAGES
  - INCREASES STABILITY
  - RESHAPES HIGH FREQUENCY PORTION OF CURVE
- W/O P CONTROL, INCREASES PHASE CROSSOVER FREQ ω_{pc}
- W/ P CONTROL, CAN BE USED TO INCREASE GAIN K_p
  - INCREASES SPEED OF RESPONSE
  - ADDS TO PHASE MARGIN

- NORMALLY PICK τ_1 & τ_2 SO THAT LEAD COMPENSATOR PUTS POLE & ZERO AWAY FROM ORIGIN $\Rightarrow$ ROOT OF DOMINANT TIME CONSTANT IS SHIFTED TO LEFT AWAY FROM ORIGIN $\Rightarrow$ INCREASE SPEED OF RESPONSE (CAN SEE THIS IN ROOT LOCUS PLOT)

- HOW TO DO THIS: PUT POLE & ZERO OF COMPENSATOR TO LEFT OF ALL POLES & ZEROES WITH POLE OF COMPENSATOR CLOSER TO ORIGIN. NOMINALLY CHOOSE $\frac{\tau_1}{\tau_2} = 10$
- LOOK AT FIG 9.9 PG 494 — FOR PHASE SHIFT / LOG MAGNITUDE
- HOW MUCH PHASE MARGIN IS ADDED TO SYSTEM:

GIVEN: $\tau_1/\tau_2 = a$ NEED TO DETERMINE $\Delta\phi_m$

$$\sin\Delta\phi_m = \frac{a-1}{a+1}$$

- IF YOU KNOW HOW MUCH PHASE MUST BE ADDED WHAT MUST BE τ_1/τ_2:

GIVEN: $\Delta\phi_m$ FIND $\tau_1/\tau_2 = a$

$$a = \frac{1+\sin\Delta\phi_m}{1-\sin\Delta\phi_m}$$

THIS OCCURS AT A FREQUENCY $\omega_m = \frac{1}{\sqrt{\tau_1\tau_2}} = \frac{1}{\tau_2\sqrt{a}}$

Lead Compensators

The effects of the compensator in terms of time domain specifications (characteristic roots) can be shown with the root locus plot. Consider the second-order plant with the real distinct roots $s = -\alpha, -\beta$. The root locus for this system with proportional control is shown in Figure 9.12*a*. The smallest dominant time constant obtainable is τ_1, marked in the figure. With lead compensation, the root locus becomes that shown in Figure 9.12*b*. The pole and zero introduced by the compensator reshape the locus so that a smaller dominant time constant can be obtained. This is done by choosing the proportional gain high enough to place the roots close to the asymptotes.

Designing a lead compensator with the root locus is done as follows. We assume that the specifications do not include required values for the static error coefficients. If they do, the design should be done with the Bode plot. The same is true if phase and gain margins are specified.

1. From the time domain specifications (time constant, damping ratio, etc.), the required locations of the dominant closed-loop poles are determined.
2. From the root locus plot of the uncompensated system, determine whether or not the desired closed-loop poles can be obtained by adjusting the open-loop gain. If not, determine the net angle associated with the desired closed-loop pole by drawing vectors to this pole from the open-loop poles and zeros. The difference between this angle and $-180°$ is the *angle deficiency.*
3. Locate the pole and zero of the compensator so that they will contribute the angle required to eliminate the deficiency. A method for doing this is presented in Example 9.1.

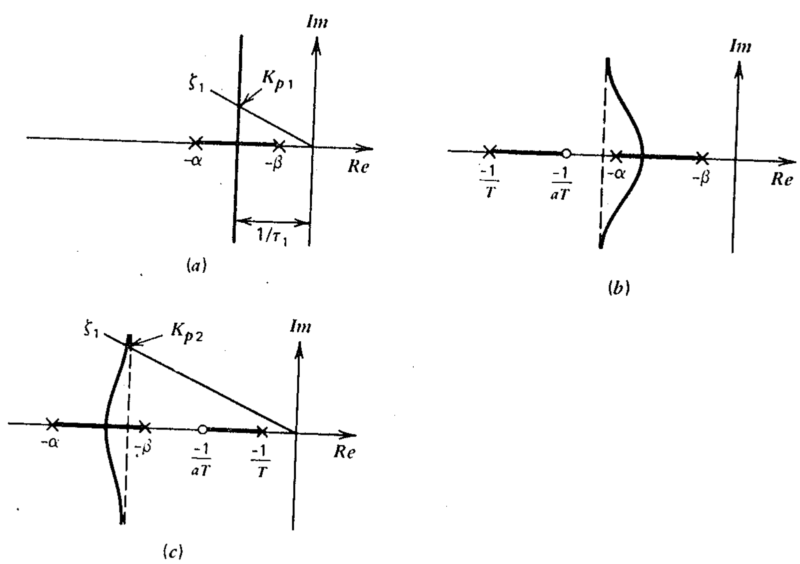


FIGURE 9.12 Effects of series lead and lag compensators. (*a*) Uncompensated system's root locus. (*b*) Root locus with lead compensation. (*c*) Root locus with lag compensation.

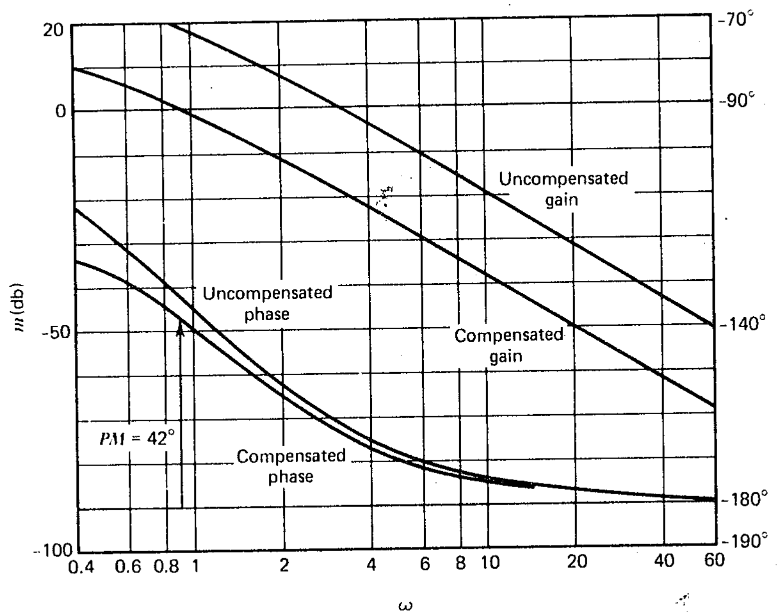


FIGURE 9.17 Bode plots for the uncompensated and compensated systems in Example 9.4.

Since the transient response is satisfactory, a lag compensator is indicated. Set $K = 10$ to achieve the desired value of C_v, and construct the Bode plots for the uncompensated open-loop system. These are the same as those shown in Figure 9.16 and repeated in Figure 9.17. Keeping in mind the lag introduced by the compensator, we attempt to achieve a phase margin of $40° + 5° = 45°$. The uncompensated system would have a phase margin of 45° if the gain crossover occurs at $\omega = 1$. Thus, $\omega_g' = 1$. The gain at this frequency is 18 db, so $m' = 18$. From (9.6-5),

$$a = 10^{-18/20} = 0.126$$

For step 3, we choose T to place $\omega = 1/aT$ one decade below $\omega = 1$. Thus, $1/aT = 0.1$, and $T = 79.4$. The lag compensator that results is

$$G_c(s) = \frac{10s + 1}{79.4s + 1}$$

The open-loop transfer function of the compensated system is

$$G_c(s)G(s)H(s) = \frac{K(10s + 1)}{s(s + 1)(79.4s + 1)}$$

With $K = 10$, $C_v = 10$, and the Bode plots are given in Figure 9.17. The phase margin is 42°, and the gain margin is infinite, so the specifications have been met.

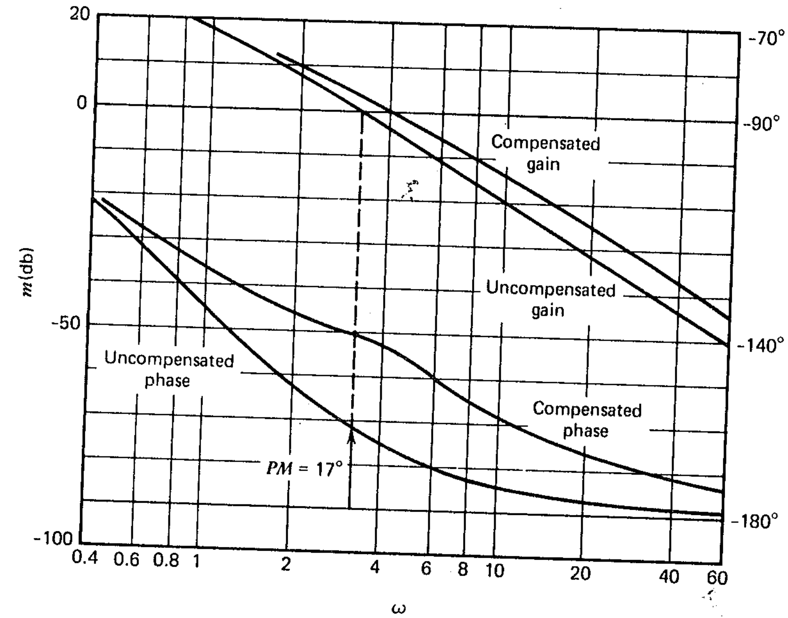


FIGURE 9.16 Bode plots for the uncompensated and compensated systems in Example 9.3.

The Bode plot of the open-loop uncompensated system is shown in Figure 9.16. The phase margin is 17°, and the gain margin is infinite, because the phase curve never falls below $-180°$. Thus, $40° - 17° = 23°$ must be added to the phase curve in order to meet the specifications, so we choose to try a lead compensator with $\phi_m = 23°$. From (8.9-19), we have

$$a = \frac{1 + \sin 23°}{1 - \sin 23°} = 2.283$$

From step 3, we compute

$$-20 \log \sqrt{a} = -3.585 \text{ db}$$

The open-loop gain of the uncompensated system is -3.585 db at approximately $\omega = 4$ rad/sec. Take this frequency to be ω_m, and solve for T from (9.6-1).

$$T = \frac{1}{4\sqrt{2.283}} = 0.1655$$

From (9.4-1), these values of a and T give a compensator with the transfer function

$$G_c(s) = \frac{1}{2.283}\,\frac{0.3778s + 1}{0.1655s + 1}$$

The open-loop transfer function of the compensated system is

$$G_c(s)G(s)H(s) = \frac{K(0.3778s + 1)}{2.283s(s + 1)(0.1655s + 1)} \qquad \textbf{(9.6-4)}$$

HOW TO DESIGN FOR A PROPORTIONAL - LEAD COMPENSATOR CONTROLLER :

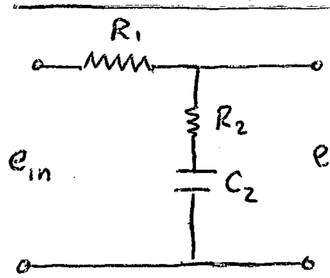
1. SET K_p TO MEET UNCOMPENSATED SYSTEM'S STEADY STATE ERROR
2. PLOT BODE PLOTS AND DETERMINE ϕ_m FOR UNCOMPENSATED SYSTEM
3. DETERMINE $\Delta\phi_m$ REQUIRED TO ACCOMPLISH SPECS.
4. USE $a = \dfrac{1+\sin\Delta\phi_m}{1-\sin\Delta\phi_m}$ TO DETERMINE a
5. FIND ω FOR UNCOMPENSATED SYSTEM SO THAT $20\log_{10}|GH| = -20\log\sqrt{a}$

   LET THIS $\omega = \omega_m = \dfrac{1}{\tau_2\sqrt{a}}$ $\Rightarrow$ GIVES τ_2, THIS $\omega = \omega_{gc}$ OF COMPENSATED SYSTEM
6. PLOT BODE PLOTS FOR COMPENSATED SYSTEM TO CHECK SPECS.

   IF NOT MET, CHECK WHAT ϕ_m FOR NEW SYSTEM IS AND WHAT $\Delta\phi_m'$ IS REQUIRED TO MEET SPECS
7. LET $(\Delta\phi_m)_{new} = (\Delta\phi_m)_{old} \pm \Delta\phi_m'$
8. GO TO STEP 4

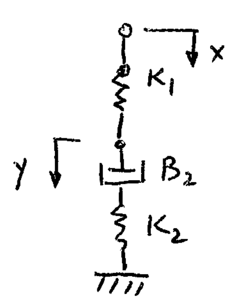
- IN STEP 7 $\pm$ IS DEPENDENT ON EXPERIENCE
- NOTE THIS IS A TRIAL & ERROR METHOD

<u>LAG COMPENSATION</u>



$$G_c = \frac{E_{OUT}(s)}{E_{IN}(s)} = \frac{1+\tau_2 s}{1+\tau_1 s} \qquad \tau_1 > \tau_2$$

$$\tau_2 = R_2 C_2 \qquad \tau_1 = (R_1+R_2)C_2 = R_1 C_2 + \tau_2$$



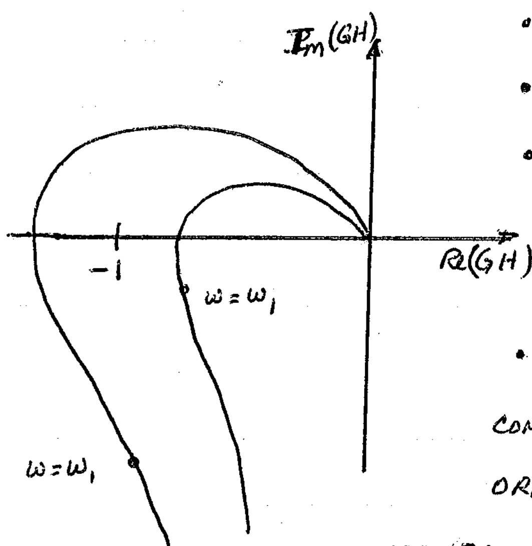
$$G_c = \frac{Y(s)}{X(s)} = \frac{1+\tau_2 s}{1+\tau_1 s}$$

$$\tau_2 = B_2/K_2 \qquad \tau_1 = \frac{K_1+K_2}{K_1}\tau_2$$

- LOOK AT CONTRIBUTION OF G_c AT STEADY STATE $(s \to 0)$
  - $G_c \to 1 \Rightarrow |GH|$ AT STEADY STATE IS SAME FOR BOTH COMPENSATED AND UNCOMPENSATED SYSTEM
- AS $\omega \to \infty$ IN NYQUIST PLOT $|G_c| \to \frac{\tau_2}{\tau_1} < 1 \Rightarrow |GH|_{new\ system}$ LESS THAN THAT OF OLD SYSTEM - SYSTEM ATTENUATED
- ATTENUATION AT HIGH FREQ IS AN ADVANTAGE OF SYSTEM

IMPLICATIONS

- STEADY STATE ERROR IS UNCHANGED (SINCE $G_c \to 1$ AS $s \to 0$)
- SUPPOSE e_{ss} IS LARGE $\Rightarrow$ INCREASE K_p TO DECREASE e_{ss}
- $\Rightarrow$ FOR FIRST ORDER SYSTEM YOU CHANGE TIME CONST $\tau \sim \frac{1}{K_p}$
- $\Rightarrow$ FOR 2nd ORDER SYSTEM YOU AFFECT ζ, ω_n WHY?
- SUPPOSE ζ, ω_n ARE OK - USE LAG COMPENSATOR TO INCREASE GAIN WITHOUT SUBSTANTIAL CHANGE IN PEAK OF $20 \log_{10} |T|$ CURVE FOR CLOSED LOOP SYSTEM



- CAN INCREASE STABILITY
- LAG COMPENSATION - SUBTRACTS PHASE MARGIN
- DECREASES PHASE CROSSOVER FREQ W/O P CNT'L

- NORMALLY PICK τ_1 & τ_2 SO THAT LAG COMPENSATOR PUTS POLE & ZERO CLOSER TO ORIGIN $\Rightarrow$ ROOT OF DOMINANT TIME CONSTANT IS BASICALLY LEFT UNCHANGED $\Rightarrow$ THUS RESPONSE OF SYSTEM IS LEFT UNCHANGED (VISUALIZED VIA ROOT-LOCUS)

- HOW TO DO THIS: PUT POLE & ZERO OF COMPENSATOR TO RIGHT OF ALL POLES & ZEROES WITH POLE OF COMPENSATOR AWAY FROM ORIGIN. CHOOSE $\tau_1/\tau_2 > 4 \Rightarrow$ PHASE MARGIN CHANGE WILL BE $> -5°$

- LOOK AT FIG 9.11, PG 494 FOR LAG COMPENSATOR
- $(\Delta\phi_m)_{LAG} = -(\Delta\phi_m)_{LEAD}$
- THUS

$$\sin(-\Delta\phi_m) = \frac{a-1}{a+1} \qquad a = \tau_1/\tau_2$$

- HOW TO DESIGN A PROPORTIONAL - LAG COMPENSATOR CONTROLLER
- THIS SYSTEM USES THE FACT THAT PHASE CURVE IS UNCHANGED NEAR ω_{gc}

(1) SET OPEN LOOP GAIN K_P OF UNCOMPENSATED SYSTEM TO MEET STEADY STATE ERROR REQ'TS

(2) CONSTRUCT BODE PLOT FOR UNCOMPENSATED SYSTEM. DETERMINE FREQ ω' TO ACHIEVE DESIRED PHASE MARGIN + 5° (5° IS FOR $\tau_1/\tau_2 = 10$. ADDITION OF 5° IS REQUIRED SINCE LAG COMP <u>DECREASED</u> P.M.)

(3) FOR THIS ω' FIND $20\log|GH| = m$

(4) THEN $\tau_2/\tau_1 = 10^{-m/20}$

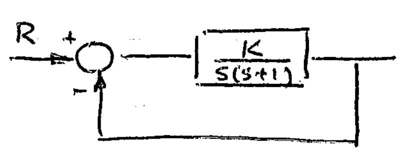
(5) CHOOSE $\frac{1}{\tau_2} = \frac{1}{10}\omega'$ $\Rightarrow$ FROM (4) τ_1 IS NOW FOUND

(6) CONSTRUCT BODE PLOT OF COMPENSATED SYSTEM. CHECK IF SPECS ARE MET. IF NOT THEN

(7) CHOOSE τ_1 SO THAT $\omega' >> \frac{1}{\tau_2}$ AND GO BACK TO (5)

e.g. CHOOSE $\frac{1}{\tau_2} = \frac{1}{15}\omega'$; FIND τ_1 AND PLOT BODE

<u>EXAMPLES</u> - LEAD COMPENSATOR



$$GH = \frac{K}{s(s+1)}$$

$$T = \frac{K}{s^2+s+K} \quad ; \quad \text{FOR STEP INPUT } R(s) = \frac{1}{s}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[R(s) - C(s)\right] = sR(s)\left[1 - T\right]$$

$$= 0$$

- BUT THE STATIC VELOCITY ERROR IS $\lim_{s\to 0} \frac{sR(s)}{1+GH} = e_{v_{ss}}$

FOR RAMP INPUT $R(s) = \frac{1}{s^2}$

$$e_{v_{ss}} = \frac{1}{K}$$

- TO MINIMIZE $e_{v_{ss}}$ TAKE $K = 10$ FOR EXAMPLE
- BUT $K = 10$ $\Rightarrow$ CHAR EQ = $s^2 + s + 10$ $\quad \omega_n = \sqrt{10}$ $\quad 2\zeta\omega_n = 1$ $\quad \tau = 2$

  AND $\zeta = \dfrac{1}{2\sqrt{10}} = .158$ VERY OSCILLATORY

- LOOK AT EXAMPLE 9.3 PG 502 WANT $GM \geq 6dB$ $\quad \phi_m \geq 40°$
- AT 0 dB $\phi_m = PM = 17°$ THIS IS LESS THAN $40°$

  AT $\phi = -180°$ $\quad \omega_{gm} = GM = \infty$ THIS IS $> 6\,dB$ OK
- $\Delta\phi_m = 40° - 17° = 23°$

- $a = \tau_1/\tau_2 = \dfrac{1 + \sin 23°}{1 - \sin 23°} = 2.283$

- $-20 \log \sqrt{2.283} = -3.585\,dB$ $\Rightarrow$ $\omega_m = 4dB = \dfrac{1}{\tau_2 \sqrt{a}}$

  $\Rightarrow \tau_2 = .1655$ $\qquad G_c = \dfrac{\tau_2}{\tau_1} \dfrac{1 + \tau_1 s}{1 + \tau_2 s} = \dfrac{1}{2.283} \dfrac{1 + .3778s}{1 + .1655s}$

  AND $\quad GH = \dfrac{K}{2.283} \dfrac{1 + .3778s}{1 + .1655s} \dfrac{1}{s(s+1)}$

  TO ACHIEVE SAME GAIN $\Rightarrow \dfrac{K}{2.283} = 10$ or $K = 22.83$

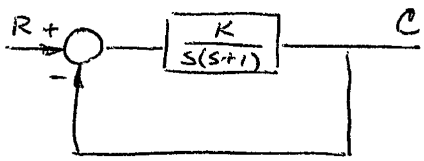
- PLOT NEW SYSTEM BODE PLOT

  NOTE THAT AT $\omega = \omega_m = 4dB$ $\quad \phi_m = 37°$

- CHOOSE $\phi_m = 23° + 3° = 26°$ $\Rightarrow a = 2.476$ $\Rightarrow \omega_m = 2.5$

  $\Rightarrow \tau_2 = .254$ & $\tau_1 = .629$ & $K = 24.76$

EXAMPLE OF <u>LAG COMPENSATOR</u>



$$GH = \frac{K}{s(s+1)} \qquad T = \frac{G}{1+GH} = \frac{K}{s^2+s+K}$$

FOR $K=1$ $\quad \omega_n = 1 \quad 2\zeta\omega_n = 1 \quad \zeta = .5 \quad \tau = 2$

AND $T = \dfrac{1}{(s+.5+j.866)(s+.5-.866j)}$

- SUPPOSE AGAIN WE WANT STATIC VELOCITY ERROR $= \lim_{s\to 0} \dfrac{sR(s)}{1+GH}$ FOR RAMP INPUT $\left(\frac{1}{s^2}\right)$ TO BE 0.1.

  $e_{v_{ss}} = \dfrac{1}{K} \Rightarrow K = 10 \Rightarrow \zeta, \omega_n, \tau$ changes

- SUPPOSE $\zeta = .5 \quad \tau = 2 \quad \omega_n = 1$ ARE ACCEPTABLE (TRANSIENT RESPONSE OK)

- LOOK AT EXAMPLE 9.4 pg 505 WANT $GM \geq 6\,dB \quad \phi_m \geq 40^\circ$

(1) SET $K = 10$ TO SATISFY $e_{v_{ss}}$

(2) PLOT BODE FOR GH UNCOMPENSATED

(3) LOOK FOR ω' THAT PRODUCES $\phi_m = 40^\circ + 5^\circ = 45^\circ$

(4) $\omega' = 1 \Rightarrow 20 \log |GH|_{\omega'=1} = 18\,dB = m$

(5) $\tau_2/\tau_1 = 10^{-18/20} = .126$

(6) CHOOSE $\dfrac{1}{\tau_2} = \dfrac{1}{10}\omega' = .1 \Rightarrow \tau_2 = 10 \qquad \tau_1 = \dfrac{\tau_2}{.126} = 79.37$

(7) $$G_c = \frac{1+\tau_2 s}{1+\tau_1 s} = \frac{1+10s}{1+79.37s}$$

(8) $$(GH)_{compensated} = \frac{K}{s(s+1)} G_c = \frac{10}{s(s+1)} \frac{1+10s}{1+79.37s}$$

(9) PLOT $(GH)_{compensated}$ & CHECK ϕ_m, GM

$$\left.\begin{array}{l} \omega_{gc} = 0.9 \Rightarrow \phi_m = 42^\circ \\ \omega_{pc} = \infty \Rightarrow GM = \infty \end{array}\right\} \text{MET SPECS.}$$

LAST ITEM

- LEAD COMPENSATOR $\omega \sim 0$ G_c ATTENUATES SIGNAL AMPLITUDE
  $\omega \sim \infty$ $G_c = 1$ SIGNAL AMPLITUDE UNCHANGED
- ACTS LIKE A HIGH PASS FILTER FOR $\omega > \frac{1}{\tau_2}$ (PASSES SIGNALS UNALTERED FOR $\omega >> \frac{1}{\tau_2}$)

- LAG COMPENSATOR $\omega \sim 0$ $G_c = 1$ SIGNAL AMPLITUDE UNCHANGED
  $\omega \sim \infty$ $G_c = \tau_2/\tau_1$ SIGNAL ATTENUATED
- ACTS LIKE A LOW PASS FILTER FOR $\omega < \frac{1}{\tau_1}$ (PASSES SIGNALS UNALTERED FOR $\omega << \frac{1}{\tau_1}$)

- COMPARE FIGURES P. 494 & FIGURES ON P. 204

EML 4312
Automatic Control Systems

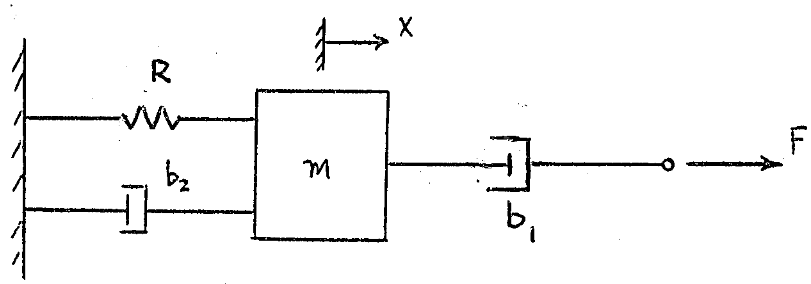
Homework Set #1
Spring 1988

Due: 28 Jan 1988

Determine the modelling differential equations for the following systems in terms of the input and output specified. State elemental, compatibility (path), and continuity (vertex) equations used to determine your answers.

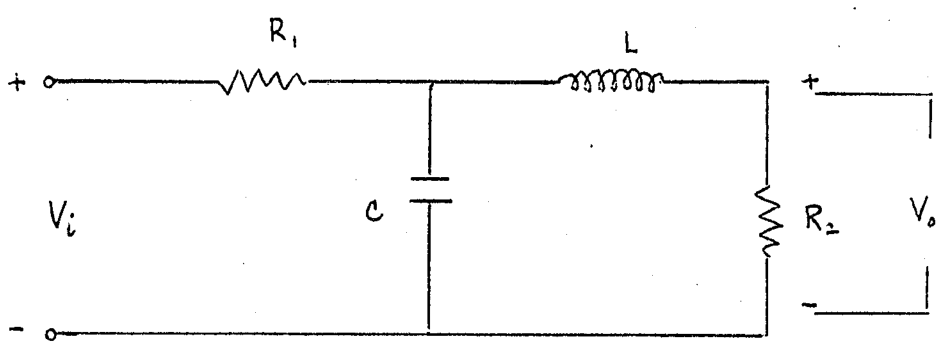
Problem 1-1

Input: F Output: x



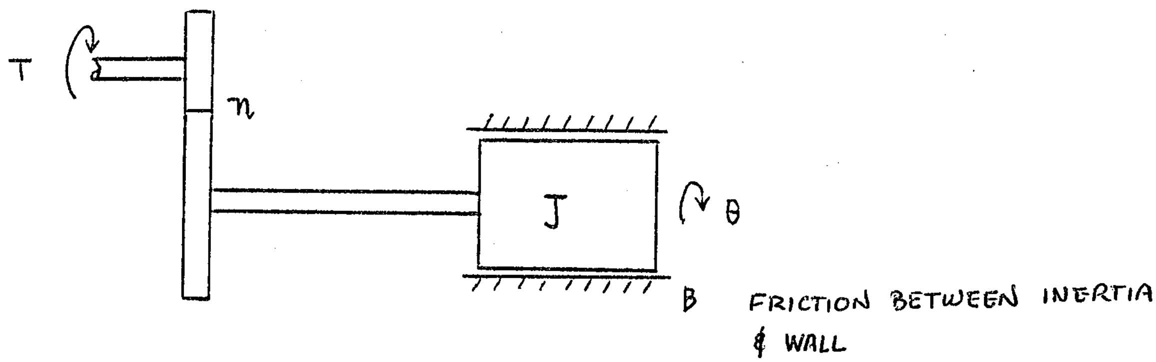
Problem 1-2

Input: V_i Output: V_o



Problem 1-3

Input: T Output: θ



Problem 1-4

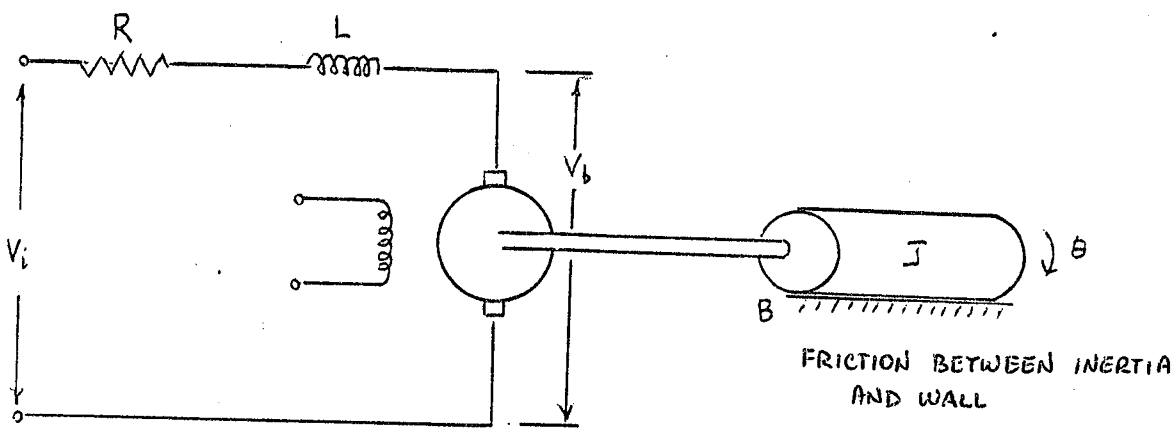
An armature controlled DC motor, with:

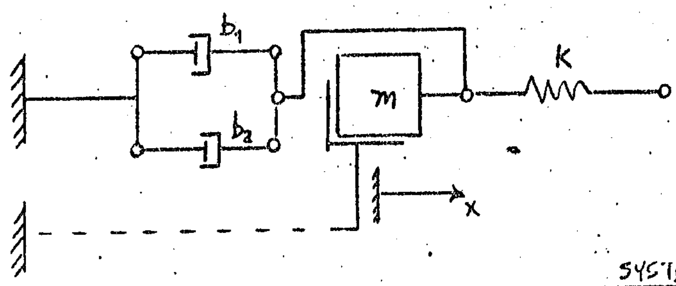
Motor torque = $T = K_m i$

Back EMF voltage = $V_b = K_b \dot{\theta}$

where K_m and K_b are constants.

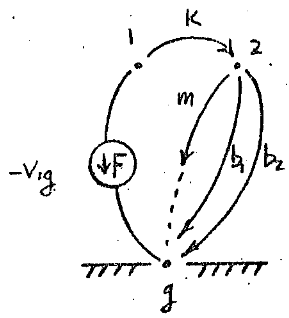
INPUT: V_i OUTPUT: θ





SYSTEM MODEL (NOTE: MASS IS IN PARALLEL W/ PARALLEL DAMPERS)

LINEAR GRAPH



ELEMENTAL EQNS:

$$V_{12} = \frac{1}{K}\frac{dF}{dt} \qquad F_m = m\frac{dV_2}{dt}$$

$$F_{b_1} = b_1 V_2 \qquad F_{b_2} = b_2 V_2$$

CONTINUITY EQNS: node 1 $\quad -F + F_K = 0$

node 2 $\quad F_K - F_m - F_{b_1} - F_{b_2} = 0$

COMPATIBILITY EQNS:

$$-V_{1g} + V_{12} + V_{2g} = 0$$

$$V_{g2} + V_{2g} = 0 \qquad \text{(OBVIOUSLY!)}$$

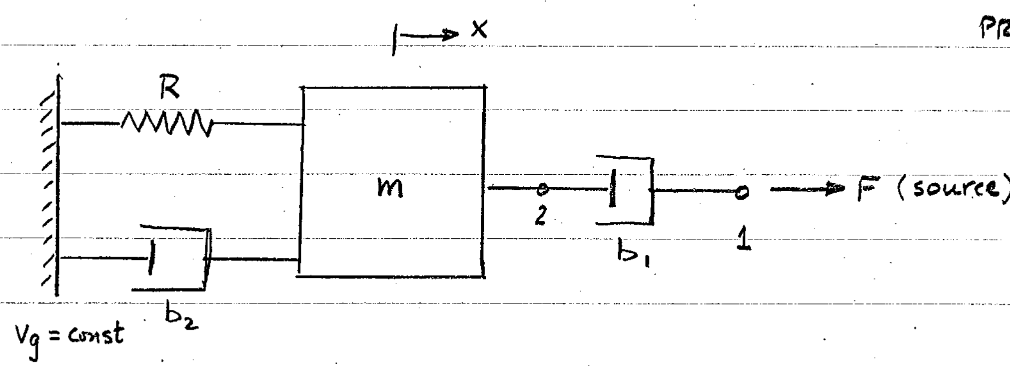
COMBINING THE TWO NODAL EQNS GIVES

$$F_m + F_{b_1} + F_{b_2} = F$$

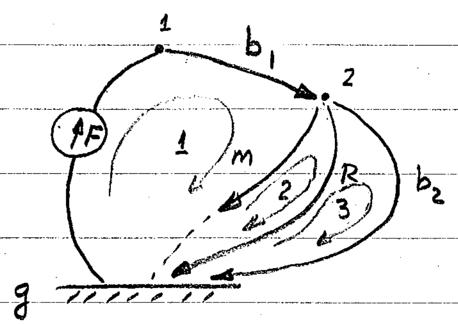
$$F_m = m\frac{dV_2}{dt} = m\frac{d^2x_2}{dt^2} \quad ; \quad F_{b_1} = b_1\frac{dx_2}{dt}$$

$$m\frac{d^2}{dt^2}x_2 + b_1\frac{dx_2}{dt} + b_2\frac{dx_2}{dt} = F$$

$$\boxed{m\frac{d^2x}{dt^2} + (b_1+b_2)\frac{dx_2}{dt} = F}$$



LINEAR GRAPH



ELEMENTAL EQNS $b-s = 5-1=4$

$F_{b_1} = b_1 V_{12}$ (1)

$F_{b_2} = b_2 V_{2g}$ (2)

$F_m = m \dfrac{dV_{2g}}{dt}$ (3)

$F_R = R\, x_{2g}$ (4)

CONTINUITY $n-2 = 3-1=2$

$F = F_{b_1}$ (5)

$F_{b_1} = F_m + F_R + F_{b_2}$ (6)

COMPATABILITY $b-n+1 = 5-3+1 = 3$

$-V_{1g} + V_{12} + V_{2g}$ (due to mass) $= 0$ (7)

$-V_{2g} + V_{2g} = 0$ twice

Put (5) into (6) and (2-4) into 6 to get

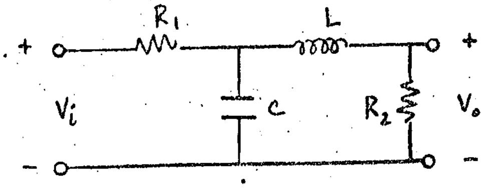
$$F = F_m + F_R + F_{b_2} = m\, \dot{V}_{2g} + R\, x_{2g} + b_2 V_{2g}$$

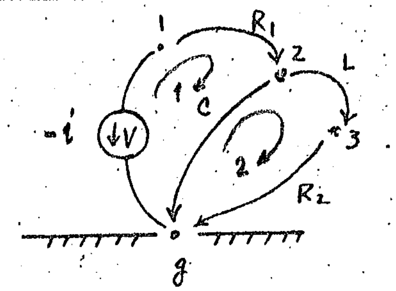
ALSO $V_{2g} = \dot{x}_{2g}$ and $x_{2g} = x$ gives

$$F = m\ddot{x} + Rx + b_2\dot{x}$$

Knowing F determines V_{12}. Knowing x determines V_{2g}. From (7) $V_{1g} = V_{12} + V_{2g}$

KEY – H.W. SET#1 | EML 4312 | PROBLEM 1-2





ELEMENTAL EQNS: $(b-s = 5-1 = 4)$

$$i_{R_1} = \frac{1}{R_1} V_{12} \qquad i_{R2} = \frac{1}{R_2} V_3$$

$$V_{23} = L\frac{di_L}{dt} \qquad i_c = C\frac{dV_2}{dt}$$

CONTINUITY EQNS: $n-1 = 4-1 = 3$

node 2 $\quad i_{R_1} = i_c + i_L \quad (1)$

node 1 $\quad i = i_{R_1} \quad (1')$

node 3 $\quad i_L = i_{R2} \quad (2)$

COMPATIBILITY EQNS: $b-n+1 = 5-4+1 = 2$

$$-V_{1g} + V_{12} + V_{2g} = 0 \qquad (3)$$

$$-V_{2g} + V_{23} + V_{3g} = 0 \qquad (4)$$

$$V_i = V_{1g} \qquad V_o = V_{3g}$$

FROM (3) $\quad V_i = V_{12} + V_{2g} \quad (5)$

FROM (1) $\quad V_{12} = R_1 C\frac{dV_{2g}}{dt} + \frac{R_1}{L}\int V_{23}\,dt + (V_{23})_o \quad (6)$

SUBSTITUTE (6) INTO (5)

$$V_i = R_1 C\frac{dV_{2g}}{dt} + \frac{R_1}{L}\int V_{23}\,dt + (V_{23})_o + V_2 \qquad (7)$$

FROM (4) & (7)

$$V_i = R_1 C\frac{d}{dt}(V_{23} + V_{3g}) + \frac{R_1}{L}\int V_{23}\,dt + (V_{23})_o + (V_{23} + V_{3g}) \qquad (8)$$

FROM (2) $\quad \frac{V_{3g}}{R_2} - \frac{1}{L}\int V_{23}\,dt - (V_{23})_o = 0 \quad (9)$

DIFFERENTIATE (9):

$$\frac{1}{R_2}\frac{dV_{3g}}{dt} - \frac{1}{L}V_{23} = 0 \qquad (10)$$

$$V_{23} = \frac{L}{R_2}\frac{dV_{3g}}{dt} \qquad (11)$$

NOW SUBSTITUTE (11) INTO (8)

$$V_i = R_1 C \frac{d}{dt}\left[\frac{L}{R_2}\frac{d}{dt}V_{3g} + V_{3g}\right] + \frac{R_1}{L}\cdot\frac{L}{R_2}\int_0^t \frac{d}{dt}V_{3g}\,dt + (V_{3g})_0 + \frac{L}{R_2}\frac{dV_{3g}}{dt} + V_{3g}$$

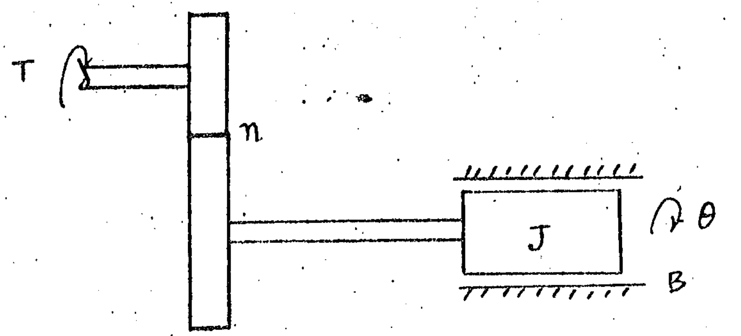
$$V_i = \frac{R_1 C L}{R_2}\frac{d^2}{dt^2}V_{3g} + R_1 C\frac{dV_{3g}}{dt} + \frac{R_1}{R_2}V_{3g} + \frac{L}{R_2}\frac{d}{dt}V_{3g} + V_{3g} + \left(\frac{L}{R_2}\frac{dV_{3g}}{dt}\right)_0$$

$$V_i = \frac{R_1 C L}{R_2}\frac{d^2}{dt^2}V_{3g} + \left(R_1 C + \frac{L}{R_2}\right)\frac{dV_{3g}}{dt} + \left(1 + \frac{R_1}{R_2}\right)V_{3g} + \cancel{\left(\frac{L}{R_2}\frac{dV_{3g}}{dt}\right)_0}$$

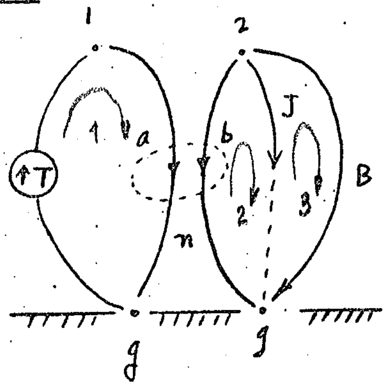
ASSUME ZERO

WITH $V_{3g} = V_0$

$$\boxed{\frac{R_1 C L}{R_2}\ddot{V}_0 + \left(R_1 C + \frac{L}{R_2}\right)\dot{V}_0 + \left(1 + \frac{R_1}{R_2}\right)V_0 = V_i}$$



LINEAR GRAPH



ELEMENTAL EQNS: $b-s = 5-1 = 4$

$$T_b = -\frac{1}{n} T_a \quad (1) \qquad T_J = J\frac{d\omega_2}{dt} \quad (2) \qquad T_B = B\omega_2 \quad (3) \qquad \omega_b = n\omega_a \quad (4)$$

CONTINUITY EQNS: $n-p = 4-2 = 2$

NODE 1 $\quad T = T_a \quad (5)$

NODE 2 $\quad -T_b = T_J + T_B \quad (6)$

COMPATIBILITY EQNS: $b-n+p = 5-4+2 = 3$

$$\frac{d\theta_2}{dt} = \omega_2$$

$$-\omega_1 + \omega_a = 0 \implies \omega_1 = \omega_a \quad (7)$$

$$-\omega_b + \omega_2 = 0 \implies \omega_2 = \omega_b \quad (8)$$

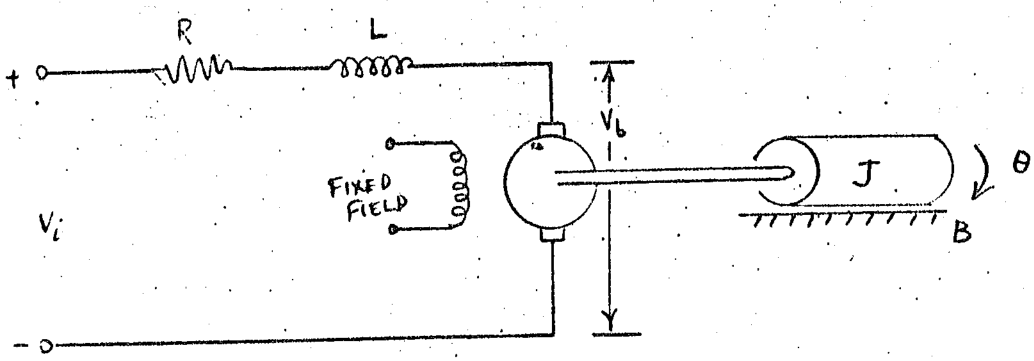
$$-\omega_2 + \omega_2 = 0$$

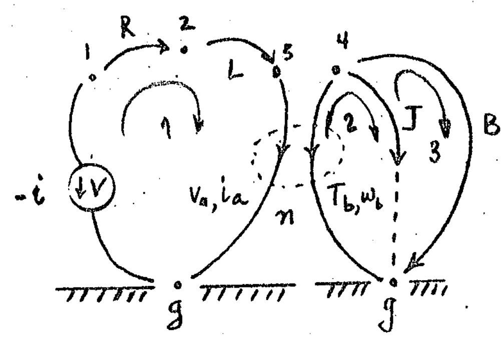
$$-T_b = J\frac{d\omega_2}{dt} + B\omega_2 \quad \text{FROM (6), (2 \& 3)}$$

BUT $T_b = -\frac{1}{n} T_a$ AND $T = T_a$ FROM (1) & (5)

THUS $\quad \frac{1}{n} T = J\frac{d^2}{dt^2}\theta_2 + B\frac{d\theta_2}{dt}$.

$$\boxed{J\ddot{\theta} + B\dot{\theta} = \frac{1}{n} T}$$





TRANSFORMATION OF CURRENT THRU THE FIXED FIELD INTO TORQUE IS MODELLED AS A TRANSFORMING TRANSDUCER. THE THRU VARIABLE CURRENT TRANDUCED TO THE THRU VARIABLE TORQUE.

ELEMENTAL EQNS: $b-s = 7-1=6$

$$i_R = \frac{1}{R} V_{12} \quad (1) \qquad T_J = J\frac{d\omega_4}{dt} \quad (5)$$

$$V_{23} = L\frac{di_L}{dt} \quad (2) \qquad T_B = B\omega_4 \quad (6)$$

$$T_b = -\frac{1}{n} i_a \quad (3) \qquad \left(\text{NOTE: } -\frac{1}{n} = K_m\right)$$

$$\omega_b = n V_a \quad (4) \qquad (\text{NOTE: } V_a \text{ is } V_b;\ V_b = \frac{1}{n}\omega_b$$

$$V_b = K_b \omega_4 \quad \therefore K_b = \frac{1}{n} = -K_m$$

CONTINUITY EQNS: $n-p = 6-2=4$

node 1 $\quad -i + i_R = 0 \quad (7)$

node 2 $\quad i_R = i_L \quad (8)$ $\}\ i_R = i_a$

node 3 $\quad i_L = i_a \quad (9)$

node 4 $\quad T_b = T_J + T_B \quad (10)$

COMPATIBILITY EQNS: $b-n+p = 7-6+2=3$

$$V_{1g} = V_{12} + V_{23} + V_{3g} \quad (11) \qquad (\text{NOTE: } V_{3g} = V_b)$$

$\omega_4 = \omega_4$ TWICE (OBVIOUSLY!)

$V_{1g} = V_i$

FROM (11)

$$V_i = V_{12} + V_{23} + V_b \quad (12)$$

FROM (12), (8, 9, 1), (2, 9) (4)

$$V_i = R i_a + L\frac{di_a}{dt} + K_b\omega_4 \quad (A)$$

FROM (3) $\quad i_a = \dfrac{1}{k_m} T_b$

SUBSTITUTE FOR i_a IN (A)

$$V_i = \frac{R}{k_m} T_b + \frac{L}{k_m} \frac{dT_b}{dt} + k_b \omega_4 \qquad (B)$$

FROM NODE (4) $\quad T_b = T_J + T_B$

FROM (5, 10, 6) $\quad T_b = J\dfrac{d\omega_4}{dt} + B\omega_4 \qquad (C)$

$$\omega_4 = \frac{d\theta_4}{dt} \qquad T_b = J\frac{d^2}{dt^2}\theta_4 + B\frac{d}{dt}\theta_4 \qquad (D)$$

$$V_i = \frac{R}{k_m}\left[J\ddot{\theta}_4 + B\dot{\theta}_4\right] + \frac{L}{k_m}\frac{d}{dt}\left[J\ddot{\theta}_4 + B\dot{\theta}_4\right] + k_b\dot{\theta}_4 \qquad \text{FROM (B \& D)}$$

$$V_i = \frac{RJ}{k_m}\ddot{\theta}_4 + \frac{RB}{k_m}\dot{\theta}_4 + \frac{LJ}{k_m}\dddot{\theta}_4 + \frac{LB}{k_m}\ddot{\theta}_4 + k_b\dot{\theta}_4$$

$$\boxed{\frac{LJ}{k_m}\dddot{\theta}_4 + \frac{RJ + LB}{k_m}\ddot{\theta}_4 + \left[\frac{RB}{k_m} + k_b\right]\dot{\theta}_4 = V_i}$$

EML 4312
Automatic Control Systems

Homework Set #2
Summer 1986

Due: 2 June 1986

Problem 2-1

Classify the systems described by the following differential equations:

$\dot{y} = \frac{dy}{dt}$, $\ddot{y} = \frac{d^2y}{dt^2}$ etc.

(a) $\ddot{y} + t^2\dot{y} - 6y = u(t)$

(b) $\dddot{y} + \ddot{y}y + 4 = 0$

(c) $\frac{\partial^2 y}{\partial t^2} = a\frac{\partial^2 y}{\partial x^2}$

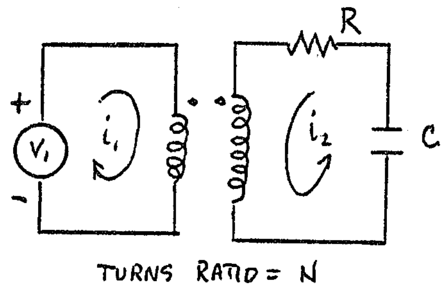
(d) $\ddot{y} + a\dot{y} + by^2 = u(t)$

(e) $\dot{y} + ay = u(t)$ if $t < t_1$
$\dot{y} + by = u(t)$ if $t \geq t_1$

(f) $\dot{y} + a \max(0, y) = 0$

Problem 2-2

a) Draw the linear graph for the ideal transformer circuit shown, noting that the transformer is a four terminal element.
b) Show that an ideal transformer has zero instantaneous power flow into it.



Problem 2-3 Problem 2.13 in Palm.

Problem 2-4 Problem 2.20 in Palm.

EML 4312
Automatic Control Systems

Homework Set #2
Spring 1988

Due: 2 Feb 1988

Problem 2-1

Classify the systems described by the following differential equations:

$\dot{y} = \dfrac{dy}{dt}, \quad \ddot{y} = \dfrac{d^2y}{dt^2} \quad$ etc.

(a) $\ddot{y} + t^2\dot{y} - 6y = u(t)$

(b) $\dddot{y} + |\ddot{y}|y + 4 = 0$

(c) $\dfrac{\partial^2 y}{\partial t^2} = a\dfrac{\partial^2 y}{\partial x^2}$

(d) $\ddot{y} + a\dot{y} + by^2 = u(t)$

(e) $\dot{y} + ay = u(t) \quad$ if $\quad t < t_1$
$\dot{y} + by = u(t) \quad$ if $\quad t \geq t_1$

(f) $\dot{y} + a\,\max(0,y) = 0$

Problem 2-2

a) Draw the linear graph for the ideal transformer circuit shown, noting that the transformer is a four terminal element.
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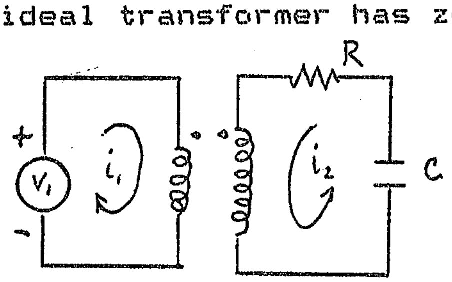


TURNS RATIO = N

Problem 2-3 Problem 2.13 in Palm.

Problem 2-4 Problem 2.20 in Palm.

EML 4312
Automatic Control Systems

Homework Set #3
Spring 1988

Due: 4 Feb 1988

Problem 3-1

Indicate which of the following first order systems are stable.

(a) $a\dfrac{dx}{dt} - bx = y$ y - input , x - output

(b) $m\dfrac{dv}{dt} + bv = F$ F - input, V - output

(c) $-a\dfrac{dx}{dt} - bx = y$ y - input, x - output

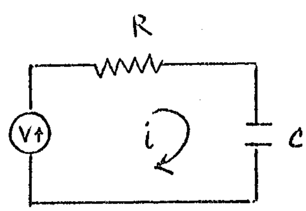
(d) $-a\dfrac{dx}{dt} + bx = y$ y - input, x - output

(e) $a\dfrac{dx}{dt} + bx = -y$ y - input, x - output

(f) $a\dfrac{dx}{dt} - bx = y$ x - input, y - output

Problem 3-2

Write the system differential equation for the series RC circuit with input v and output i. Sketch the response of the system to a step change in voltage. What is the time constant (τ) for this system?



Problem 3-3 Problem 3.5 in Palm.

Problem 3-4 Problem 3.18 in Palm.

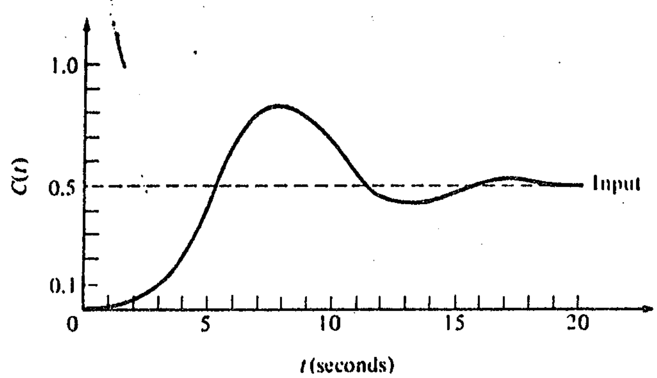
EML 4312
Automatic Control Systems

Homework Set #4
Summer 1986

Due: 18 June 1986

Problem 4-1

Shown below is the time response of a particular system to a step input. Find the delay, rise, peak, and settling times, as well as the peak overshoot as a percent of the input and the number of oscillations.



Problem 4-2

In judging acceptable system performance the following three criteria are generally used:

(1) Is the system stable?
(2) Does the system have acceptable accuracy?
(3) Is the transient response acceptable?

How are each of these criteria specified? How can a system be tested to show that these criteria are satisfied?

Problem 4-3 Problem 3.6 in Palm.

Problem 4-4 Problem 3.10 in Palm.

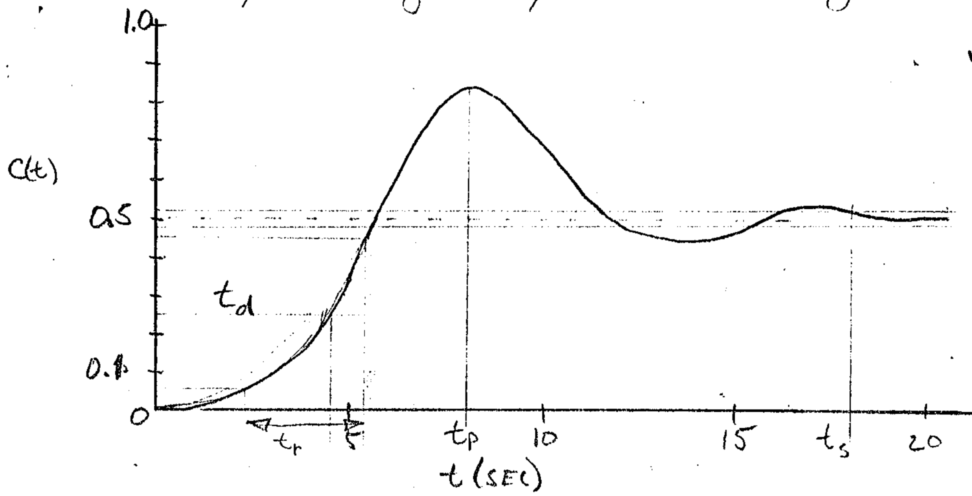
Problem 4-5 Problem 3.13 in Palm.

Problem 4-6 Problem 3.14 in Palm.

Problem 4-7 Problem 3.21 in Palm.

-18-86 | STEVE BOGGS | EML 4312 | PB 4-1

Shown below is the time response of a particular system to a step input. Find the delay, rise, peak and settling time as well as the peak overshoot as a percent of the input and the # of oscillations.



X_{MAX} = MAX. OVERSHOOT = MAX. DEVIATION OF THE OUTPUT X ABOVE X_{SS}

$$X_{MAX} = \frac{X_{MAX} - X_{SS}}{X_{SS}}(100) = \frac{0.84 - 0.5}{0.5}(100) = \boxed{68\% = X_{MAX}}$$ ✓

t_d = time required to reach 50% of its final value

$\boxed{t_d = 4.3 \text{ sec}}$ ✓

t_r = time required for output to rise from 10% to 90% of its final value.

$\boxed{t_r = 5.1 - 2.3 = 2.8 \text{ sec}}$ ✓

t_p = the time at which the max. overshoot occurs.

$\boxed{t_p = 8.0 \text{ sec}}$ ✓

t_s = time required for oscillations to stay with-in 2%

$\boxed{t_s = 18 \text{ sec}}$ ✓

In judging acceptable system performance the following three criteria are generally used:

1.) Is the system stable?
2.) Does the system have acceptable accuracy?
3.) Is the transient response acceptable?

How are each of these criteria specified? How can a system be tested to show that these criteria are satisfied?

1.) SPECIFIED
A system is stable if with zero input and arbitrary initial conditions, the resulting system response goes to zero, and if with a bounded input the resulting system output is bounded.

TESTED
1ST ORDER — lim of free response goes to 0 as $t \to \infty$
lim of final response goes to bounded output as $t \to \infty$

2ND & 3RD ORDER — ROUTH-HURWITZ CRITERION
OR ROUTH ARRAY

2.) SPECIFIED
Is the system response very close to the desired steady-state value? The steady state error must be within acceptable limits

TESTING.
difference between steady state value and desired value is steady state error
✓ calculate max overshoot and percent max overshoot to tell if response goes too far.

3.) SPECIFIED
How long before actual response is close enough to steady state to be acceptable?

TESTING. calculating the peak, delay, rise and settling times

-8-86 | STEVE BOGGS | EML 4312 | PB 4-3 (3.6 PALM)

Find the free response and the step response of the following models. If the solution is oscillating, it should be expressed in the form of a sine function with a phase shift.

a) $\ddot{x} + 4\dot{x} + 8x = 2 \qquad x(0) = 0 \quad \dot{x}(0) = 1$

$$x = \frac{-4 \pm \sqrt{16-32}}{2} = -2+2i, \; -2-2i$$

$$x(t) = Be^{-at}\sin(bt+\phi) \qquad B = +2\sqrt{A_R^2 + A_I^2} \qquad A_R = \frac{x(0)}{2} = 0$$

$$B = 1/2 \qquad A_I = -\frac{\dot{x}(0) + a x(0)}{2b}$$

$$B\sin\phi = 2A_R \qquad A_I = \frac{-(1 + 2(0))}{4} = -1/4$$

$$\phi = 0 \text{ RAD}$$

$$\boxed{\text{FREE} \qquad x(t) = \tfrac{1}{2} e^{-2t}\sin(2t + 0)} \quad \checkmark$$

assume $\quad x(t) = Be^{-at}\sin(bt+\phi) + d/k$

$$\dot{x}(t) = -aBe^{-at}\sin(bt+\phi) + bBe^{-at}\cos(bt+\phi) \qquad \tan\phi = \frac{b}{a} \qquad \phi \text{ is } 3^{rd}$$

$$\phi = \tan^{-1} 1 = .785$$

$$x(0) = B\sin\phi + d/k \qquad B\sin\phi = x(0)^{\,0} - d/k$$

$$\dot{x}(0) = -aB\sin\phi + bB\cos\phi \qquad B\cos\phi = \frac{\dot{x}(0) + aB\sin\phi}{b} = \frac{\dot{x}(0) + a(x(0)^{\,0} - d/k)}{b}$$

$$B^2(\sin^2\phi + \cos^2\phi) = d^2/k^2 + \frac{1 - d/k\, a + \frac{a^2d^2}{k^2}}{b^2}$$

$$B^2 = \frac{d^2}{k^2}\left[1 + \frac{\frac{k^2}{d^2} - \frac{k}{d}a + a^2}{b^2}\right]$$

$$B = \frac{d}{k}\frac{\sqrt{b^2+a^2}}{b} - \sqrt{\frac{1 - \frac{d}{k}a}{b^2}}$$

$$B = \frac{d}{k}\left[\frac{\sqrt{b^2+a^2}}{b} - \sqrt{\frac{k^2 - kda}{d^2b^2}}\right]$$

$$x(t) = \frac{d}{k}\left[\left(\frac{\sqrt{b^2+a^2}}{b} - \sqrt{\frac{k^2-kda}{d^2b^2}}\right)e^{-2t}\sin(2t+\phi) + 1\right]$$

$$x(t) = \frac{1}{4}\left[\left(\frac{\sqrt{4+4}}{2} - \sqrt{\frac{64 - 8(2)(2)}{4(4)}}\right)e^{-2t}\sin(2t + 3.927) + 1\right]$$

$$= \frac{1}{4}\left[(\sqrt{2} - 2)e^{-2t}\sin(2t + 3.927) + 1\right]$$

STEPPED $\quad \boxed{x(t) = \frac{1}{4}\left[-\sqrt{2}\,e^{-2t}\sin(2t + 3.927) + 1\right]}$ ✗

$x(0) = 0 \;\; ? \qquad \dot{x}(0) = 1 \;\; ?$

b. $\ddot{x} + 8\dot{x} + 12x = 2 \qquad x(0) = 0 \qquad \dot{x}(0) = 1$

$$x = \frac{-8 \pm \sqrt{64-48}}{2} = -4 \pm 2$$

$$x = -6, -2$$

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$A_1 = \frac{\dot{x}(0) - s_2 x(0)}{s_1 - s_2} = -\frac{1}{4}$$

$$A_2 = \frac{s_1(x(0)) - \dot{x}(0)}{s_1 - s_2} = x(0) - A_1 = \frac{1}{4}$$

$$\boxed{\text{FREE} \qquad x(t) = -\frac{1}{4}e^{-6t} + \frac{1}{4}e^{-2t}} \checkmark$$

ASSUME

$$x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t} + \frac{d}{k}$$

$$\dot{x}(t) = s_1 A_1 e^{s_1 t} + s_2 A_2 e^{s_2 t}$$

$$\ddot{x}(t) = s_1^2 A_1 e^{s_1 t} + s_2^2 A_2 e^{s_2 t}$$

$$A_1 = x(0) - A_2 - \frac{d}{k} \qquad A_1 = \frac{\dot{x}(0) - s_2 A_2}{s_1}$$

$$x(0) - A_2 - \frac{d}{k} = \frac{\dot{x}(0) - s_2 A_2}{s_1}$$

$$s_2 A_2 - s_1 A_2 = \dot{x}(0) + s_1 \frac{d}{k} - \cancel{s_1 (x) 0}$$

$$A_2 = \frac{1 + s_1 \frac{d}{k}}{s_2 - s_1} = \frac{1}{s_2 - s_1} + \frac{d}{k}\frac{s_1}{s_2 - s_1}$$

$$A_1 = \cancel{x(0)}^{0} - \left(\frac{1}{s_2 - s_1} - \frac{d}{k}\frac{s_1}{s_2 - s_1}\right) + \frac{d}{k}\left(\frac{s_2 - s_1}{s_2 - s_1}\right)$$

$$A_1 = -\frac{1}{s_2 - s_1} - \frac{d}{k}\frac{s_2}{s_2 - s_1}$$

$$x(t) = \frac{d}{k}\left[\left(-\frac{k}{d(s_2 - s_1)} - \frac{s_2}{s_2 - s_1}\right)e^{s_1 t} + \left(\frac{k}{d(s_2 - s_1)} + \frac{s_1}{s_2 - s_1}\right)e^{s_2 t} + 1\right]$$

$$x(t) = \frac{2}{12}\left[\left(-\frac{12}{2(4)} - \frac{-2}{4}\right)e^{-6t} + \left(\frac{12}{2(4)} + \frac{-6}{4}\right)e^{-2t} + 1\right]$$

$$x(t) = \left(-\frac{1}{4} + \frac{1}{12}\right)e^{-6t} + \left(\frac{1}{4} - \frac{1}{4}\right)e^{-2t} + \frac{1}{6}$$

$$\text{STEP} \qquad \boxed{x(t) = -\frac{1}{6}e^{-6t} + \frac{1}{6}} \checkmark$$

c) $\ddot{x} + 4\dot{x} + 4x = 2 \qquad x(0) = 0 \qquad \dot{x}(0) = 1$

$$x = \frac{-4 \pm \sqrt{16-16}}{2} = -2, -2$$

$$x(t) = A_1 e^{-2t} + tA_2 e^{-2t}$$

$A_1 = x(0) = 0$

$A_2 = \dot{x}(0) - s_1(x(0)) = 1$

$$\boxed{\text{FREE} \quad x(t) = t e^{-2t}}$$ ✓

assume

$$x(t) = A_1 e^{s_1 t} + tA_2 e^{s_1 t} + \frac{d}{k}$$

$$\dot{x}(t) = sA_1 e^{st} + A_2 e^{st} + sA_2 t e^{st}$$

$$A_1 = \cancel{x(0)} - \cancel{t A_2} - \frac{d}{k} \qquad -\frac{d}{k} = A_1$$

$$A_2 = \frac{\dot{x}(t) - sA_1}{1 + st} = \frac{1 - 2\frac{d}{k}}{1} \qquad 1 - 2\frac{d}{k} = A_2 = -\frac{d}{k}\left[-\frac{k}{d} + 2\right]$$

$$x(t) = -\frac{d}{k} e^{st} + t\,\frac{-d}{k}\left(\frac{-k}{d} + 2\right) e^{st} + \frac{d}{k}$$

$$x(t) = \frac{d}{k}\left(-e^{st} - t\left(\frac{-k}{d} + 2\right) e^{st} + 1\right)$$

$$x(t) = \frac{1}{2}\left[-e^{-2t} + (2t - 2t)e^{-2t} + 1\right]$$

$$\boxed{\text{STEP} \quad x(t) = \frac{1}{2}\left[1 - e^{-2t}\right]}$$ ✓

Find $\zeta, \omega_n, \omega_d$, and τ for the models in Prob 3.6

a) $\ddot{x} + 4\dot{x} + 8x = 2 \quad x(0)=0 \quad \dot{x}(0)=1$

$s^2+4s+8=0$

$s^2+2\zeta\omega_n s+\omega_n^2=0$

$4=2\zeta\omega_n \qquad 8=\omega_n^2$

$\frac{2}{\omega_n}=\frac{2}{2\sqrt{2}}=\zeta$

$\omega_d=\sqrt{1-\zeta^2}\,\omega_n$

$=\sqrt{1-\frac{1}{2}}\,2\sqrt{2}=2$

$\tau=\frac{1}{\zeta\omega_n}=\frac{1}{\frac{\sqrt{2}}{2}(2\sqrt{2})}$

$=\frac{1}{2}$

$$\boxed{\begin{aligned}&\omega_n=\sqrt{8}=2\sqrt{2}\\&\zeta=\frac{\sqrt{2}}{2}\\&\omega_d=2\\&\tau=\frac{1}{2}\end{aligned}}$$

✓ ✓

b) $\ddot{x} + 8\dot{x} + 12x = 2$

$8=2\zeta\omega_n \quad 12=\omega_n^2$

$\zeta=\frac{8}{2(2\sqrt{3})}=\frac{2}{\sqrt{3}}=\frac{2\sqrt{3}}{3}$

$\omega_d=\omega_n\sqrt{1-\zeta^2}$

$=2\sqrt{3}\sqrt{1-\frac{4}{9}(3)}$

$2\sqrt{3}\sqrt{\frac{-3}{9}}=2i$

$\tau=\frac{1}{\zeta\omega_n}=\frac{1}{\frac{2\sqrt{3}}{3}(2\sqrt{3})}$

$=\frac{1}{\frac{4}{3}(3)}=\frac{1}{4}$

$$\boxed{\begin{aligned}&\cancel{\omega_n}=\sqrt{12}=2\sqrt{3}\\&\zeta=\frac{2\sqrt{3}}{3}\\&\omega_d\cancel{=2i}\\&\tau=\cancel{\tfrac{1}{4}}\ \frac{1}{2}\end{aligned}}$$

do not apply to real roots

dominant root

c) $\ddot{x} + 4\dot{x} + 4x = 2$

$4=2\zeta\omega_n \quad 4=\omega_n^2$

$\frac{4}{2(2)}=\zeta=1$

$\omega_d=\omega_n\sqrt{1-\zeta^2}$

$=2\sqrt{1-1}=0$

$\tau=\frac{1}{\zeta\omega_n}=\frac{1}{1(2)}=\frac{1}{2}$

$$\boxed{\begin{aligned}&\cancel{\omega_n=2}\\&\zeta=1\\&\cancel{\omega_d=0}\\&\tau=\frac{1}{2}\end{aligned}}$$

✓ ✓

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For the model given in Pb 3.6a, compute the max. percent overshoot, the actual max. overshoot, the peak time, the 100% rise time, the delay time, and the settling time.

$$\text{max percent overshoot} = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}} =$$

$$\text{max percent overshoot} = 100\, e^{-\pi \frac{\sqrt{2}}{2}/\sqrt{1-1/2}}$$

$$= 100\, e^{-\pi\left(\frac{\sqrt{2}}{2}\right)/\sqrt{1/2}} = 100\, e^{-\pi}$$

$$\boxed{\text{max percent overshoot} = 100\, e^{-\pi} \approx 4.3214\,\%}$$ ✓

$$\text{max overshoot} = \frac{1}{k} e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

$$= \frac{1}{8} e^{-\pi \frac{\sqrt{2}}{2}/\sqrt{1/2}} = \frac{1}{8} e^{-\pi}$$

$x_{max} - x_{ss} =$

$$\boxed{\text{max overshoot} = \frac{1}{8} e^{-\pi} \approx 0.0054\,X}$$ .01

$$\text{peak time} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$= \frac{\pi}{2\sqrt{2}\left(\frac{\sqrt{2}}{2}\right)}$$

$$\boxed{\text{peak time} = \frac{\pi}{2}}$$ ✓

$$\text{100\% rise time} = t_{r_{100}} = \frac{2\pi - \phi}{\omega_n\sqrt{1-\zeta^2}}$$

$$= \frac{2\pi - 3.927}{2\sqrt{2}\sqrt{1-\left(\frac{\sqrt{2}}{2}\right)^2}}$$

$$= \frac{2.3562}{2}$$

$$\boxed{t_{r/100} = 1.1781}$$

$$\text{delay time} = t_d \simeq \frac{1 + 0.7\zeta}{\omega_n} \qquad 0 \le \zeta \le 1$$

$$t_d = \frac{1 + .7\left(\frac{\sqrt{2}}{2}\right)}{2\sqrt{2}}$$

$$\boxed{t_d = 0.5286}$$ ✓

$$\text{settling time} = t_s = 4\tau = \frac{4}{\zeta\omega_n}$$

$$t_s = \frac{4}{\frac{\sqrt{2}}{2}(2\sqrt{2})} = \frac{4}{2}$$

$$\boxed{t_{s_{2\%}} = 2}$$ ✓

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a certain system is described by the model

$$\ddot{x} + c\dot{x} + 4x = f$$

Set the value of the damping constant c so that both of the following specifications are satisfied. Give priority to the overshoot specification. If both cannot be satisfied state reason.

1.) Max % overshoot as small as possible & no greater than 20%

2.) 100% rise time as small as possible & no greater than 3.

$c = 2\zeta\omega_n \qquad \omega_n^2 = 4 \quad \therefore \omega_n = 2$

$c = 4\zeta$

max % overshoot $= 100e^{-\pi\zeta/\sqrt{1-\zeta^2}} \leq 20$

100% rise $t_r|_{100} = \dfrac{2\pi - \phi}{\omega_n\sqrt{1-\zeta^2}} \qquad \tan\phi = \dfrac{\sqrt{1-\zeta^2}}{\zeta}$

$$100\,e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 20$$

$$-\pi\zeta/\sqrt{1-\zeta^2} = -1.6094$$

$$\zeta/\sqrt{1-\zeta^2} = 0.5123$$

$$\zeta^2 = 0.2625(1-\zeta^2)$$

$$1.2625\zeta^2 = .2625$$

$$\zeta^2 = .2079$$

$$\zeta = 0.4559 \longrightarrow \phi = \tan^{-1}\frac{\sqrt{1-.2079}}{.4559} = 1.0979 \text{ rad}$$

$$\phi = 4.2389 \text{ rad}$$

$$t_r|_{100} = \frac{2\pi - 4.2389}{2\sqrt{1-\zeta^2}} = 1.1484$$

CHECK OTHER EXTREMES
(OVER)

$$100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 1$$

$$\pi\zeta/\sqrt{1-\zeta^2} = 4.6052$$

$$\zeta^2 = 2.1488(1-\zeta^2)$$

$$\zeta^2 = 0.6824$$

$$\zeta' = 0.8261 \qquad \phi = \tan^{-1}\frac{\sqrt{1-.6824}}{.8261} = .5987$$

$$t_r = \frac{2\pi - 3.7403}{2\sqrt{1-.6824}} = 2.2561$$

with max overshoot of 1%
and $t_r|_{100} = 2.2561$

$$c = 4\zeta = 4(.8261)$$

$$c = 3.3044$$

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Use the Routh-Hurwitz criterion to determine the allowable range of K so that the systems with the following characteristic eqn. will be stable.

a.) $s^3 + as^2 + Ks + b = 0$ (a, b known const)

$K > \frac{b}{a}$ & $K > 0$

$\boxed{0 < \frac{b}{a} < K}$

b.) $s^3 + as^2 + bs + K = 0$

$ab > K \quad K > 0$

$\boxed{0 < K < ab}$

c.) $s^3 + 3s^2 + 3s + 1 + K = 0$

$9 > (1+K)$

$\boxed{0 < K < 8}$

EML 4312
Automatic Control Systems

Homework Set #5
Spring 1988

Due: 25 Feb 1988

Problem 5-1

Convert the following complex numbers to polar coordinate form (i.e. $Z = A\, e^{j\theta}$, where θ is in degrees).

(a) $Z = 5 - 3j$

(b) $Z = 5 / (2 + 3j)$

(c) $Z = 2\,(1 + 3.5j) / (1 + 0.5j)\,(1 + 1.9j)$

(d) $Z = (jw + 2)\,(jw + 3) / ((jw)^2 + 2jw + 1)\,(jw + 4)$, $w = 2$

Problem 5-2

Find the Inverse Laplace Transform, f(t), for the following Laplace Transforms:

(a) $F(s) = 6 / (s+1)\,(s+2)\,(s+4)$

10 (b) $F(s) = (s^2 + 8s + 15) / (s+2)\,(s+3)\,(s+5)$

(c) $F(s) = (3s + 1) / s\,(s+1)\,(s+1)$

Problem 5-3

Find the Laplace Transforms X(s) for the function x(t) governed by the following system differential equations.

(a) $\ddot{x} + 2\dot{x} + 4x = 5\,u(t)$ $\quad x(0) = 2,\ \dot{x}(0) = -3$

(b) $\dddot{x} + 2\dot{x} - x = 3t - t^2$ $\quad x(0) = 0,\ \dot{x}(0) = -2,\ \ddot{x}(0) = 1$

EML 4312
Automatic Control Systems

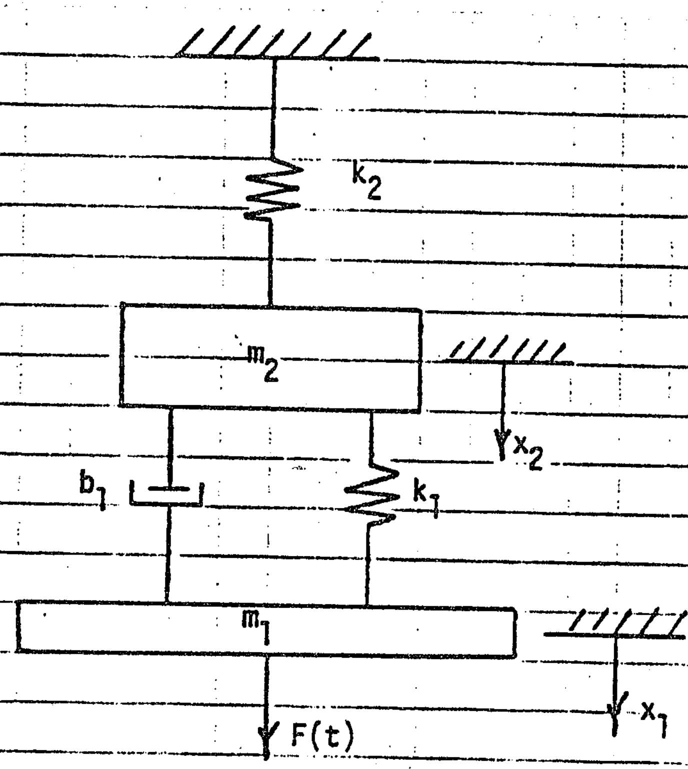
Homework Set #6
Spring 1988

Due: 15 March 1988

Problem 6-1

Draw the block diagram for the mechanical system shown below. Assume zero initial conditions.

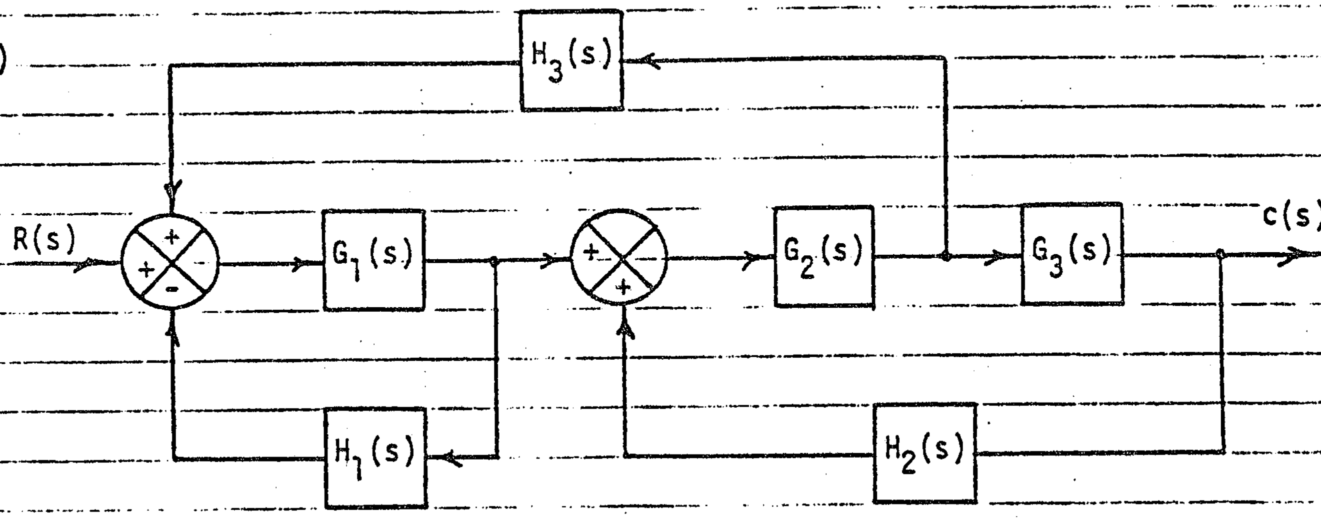
Input : F(t)
Output: $x_2(t)$



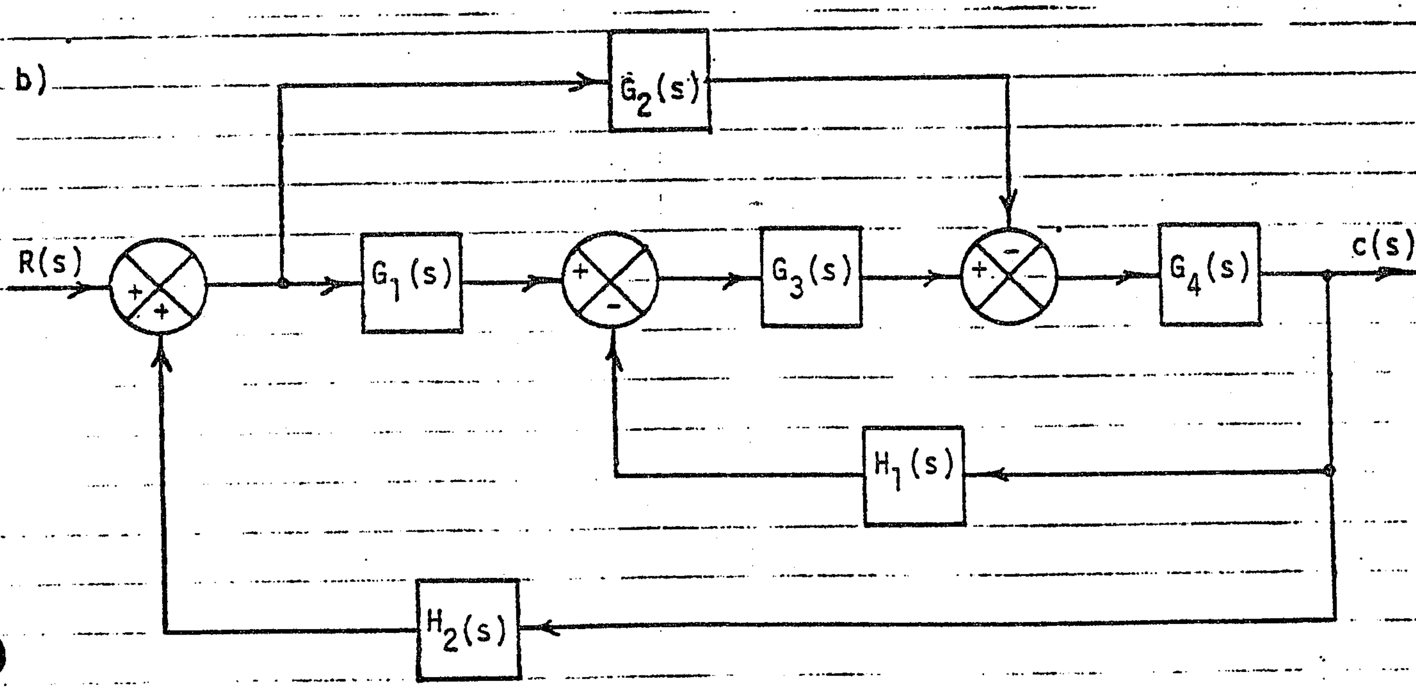
Problem 6-2

Reduce the block diagrams below to obtain the transfer function C(s)/R(s) for each system. Clear all fractions in the numerator and denominator of your final answer.

a)

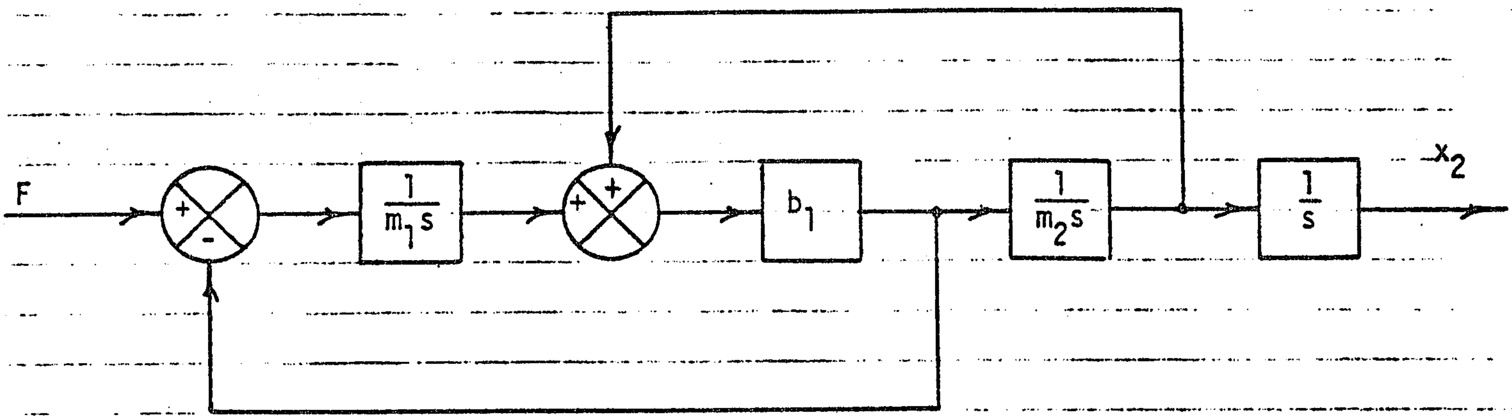


b)



Problem 6-3

Find the differential equation relating the output $x_2(t)$, to the input, $F(t)$, for the system represented by the following block diagram.

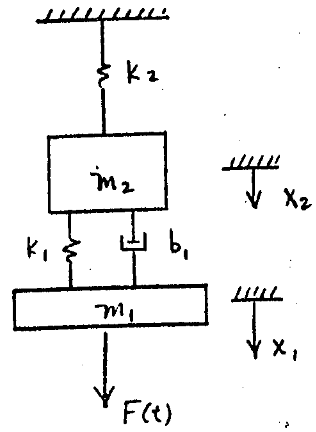


Problem 6-4

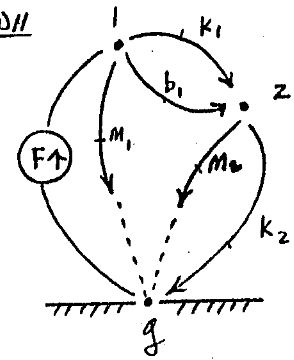
For what values of K are the following closed-loop systems stable?

(a) $$\frac{G}{1 + GH} = \frac{1}{s^5 + s^4 + 12s^3 - 3s^2 + (5 - K)s + 2 + K}$$

(b) $$\frac{G}{1 + GH} = \frac{1}{s^3 + 8s^2 + 7s + K}$$



LINEAR GRAPH



COMPATIBILITY

$$F_{m_1} = -F_{b_1} - F_{k_1} + F(t)$$

$$F_{m_2} = -F_{k_2} + F_{b_1} + F_{k_1}$$

THERE IS NO SET SOLN.

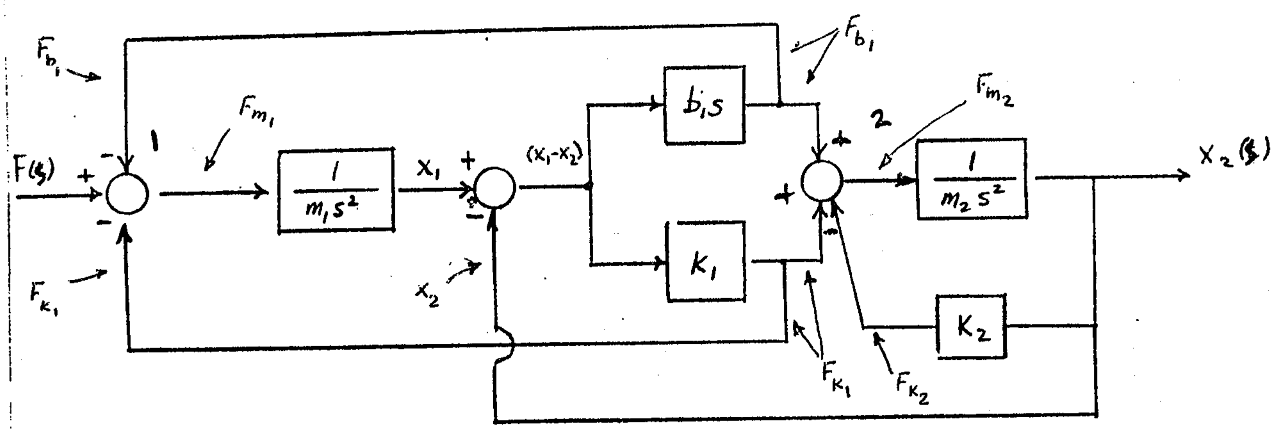
$$F_{m_1} = m_1\, dv_1/dt \Rightarrow F_{m_1} = m_1 s V_1 = m_1 s^2 X_1$$

$$F_{k_1} = k_1 (X_1 - X_2)$$

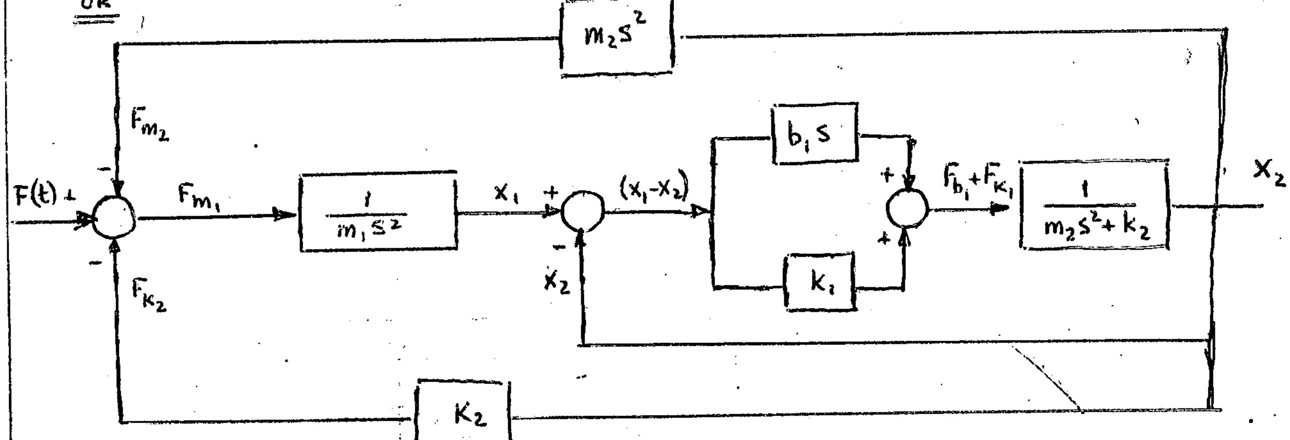
$$F_{b_1} = b_1 (\dot{X}_1 - \dot{X}_2) \Rightarrow F_{b_1} = b_1 s (X_1 - X_2)$$

$$F_{k_2} = K_2 (X_2)$$

$$F_{m_2} = m_2 s V_2 = m_2 s^2 X_2$$



OR

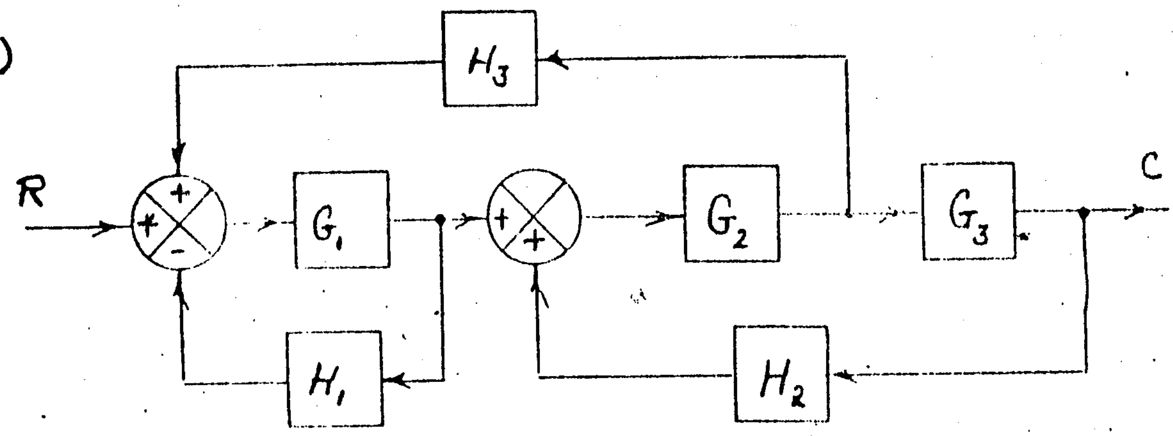


GIVEN: THE BLOCK DIAGRAMS SHOWN

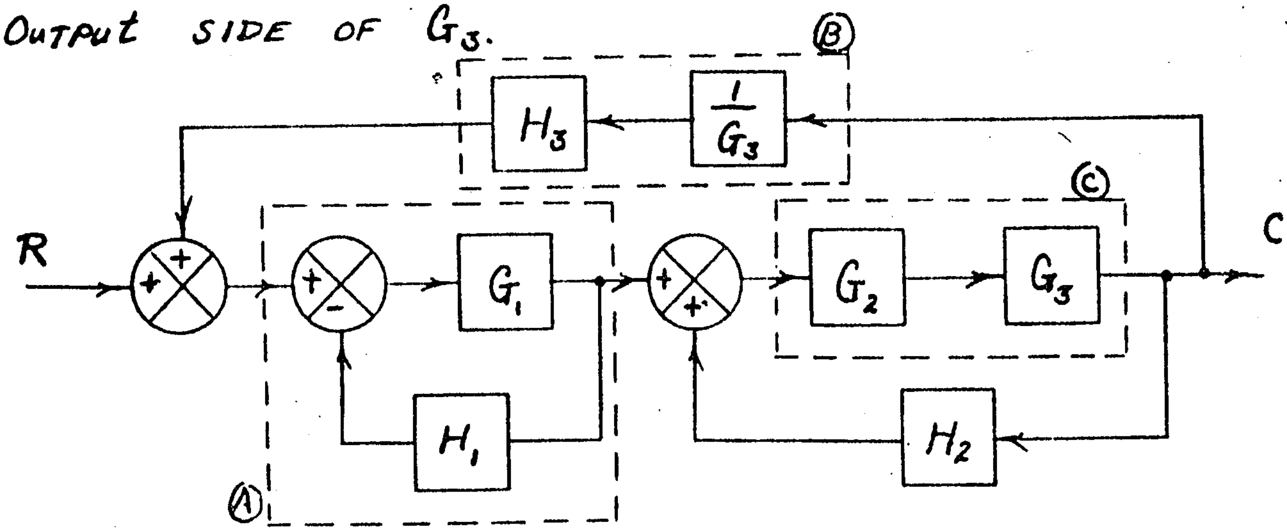
FIND: REDUCE THE BLOCK DIAGRAMS TO OBTAIN A TRANSFER FUNCTION $\frac{C(s)}{R(s)}$. CLEAR ALL FRACTIONS.

SOLUTION

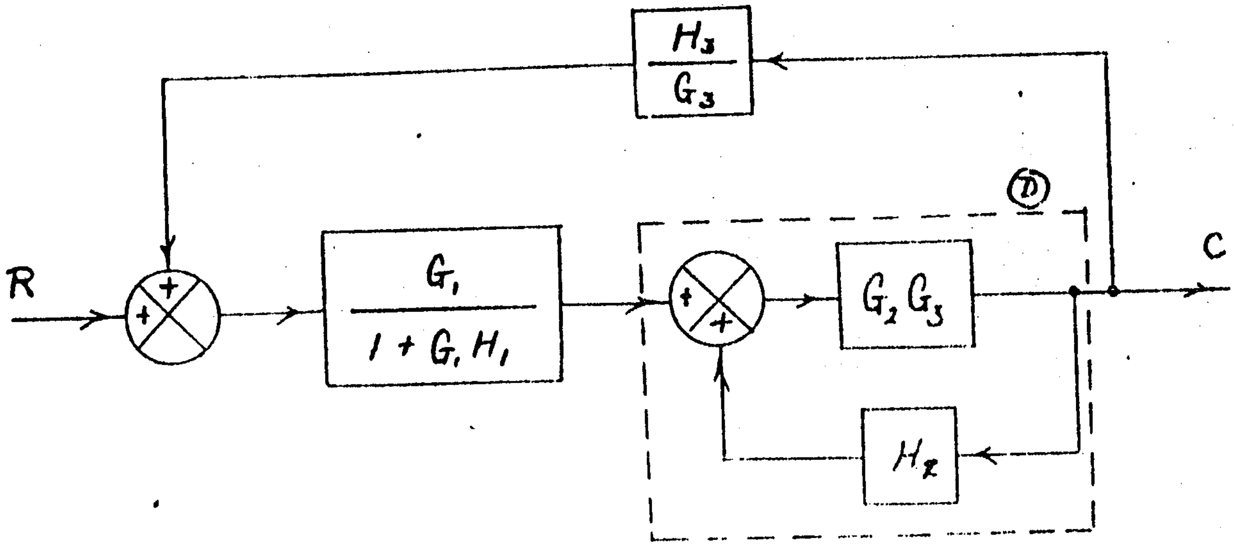
a)



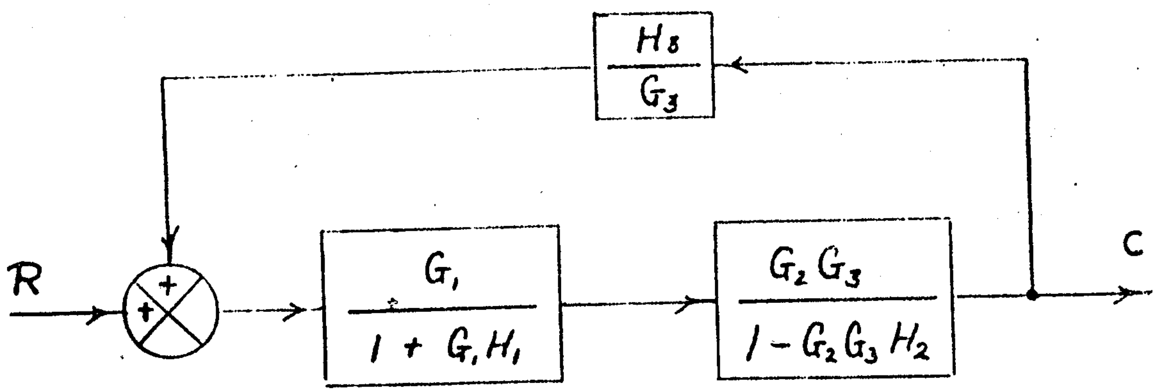
BREAK THE FIRST SUMMING JUNCTION INTO TWO PARTS AND MOVE THE PICKOFF POINT FROM AHEAD OF G_3 TO THE OUTPUT SIDE OF G_3.



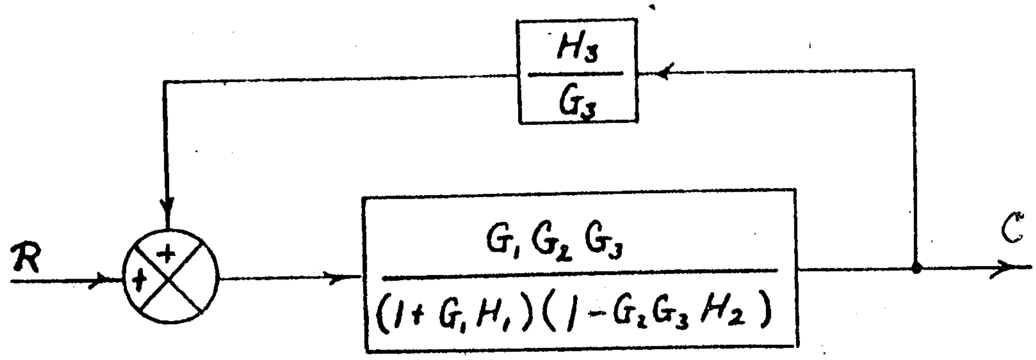
REDUCE FEEDBACK LOOP LABELED Ⓐ. COMBINE BOXES IN Ⓑ & Ⓒ RESPECTIVELY.



REDUCE FEEDBACK LOOP D



COMBINE THE TWO BLOCKS IN THE FORWARD PATH



REDUCE THE FEEDBACK LOOP.

$$R \longrightarrow \boxed{\dfrac{\dfrac{G_1 G_2 G_3}{(1+G_1H_1)(1-G_2G_3H_2)}}{1 - \dfrac{G_1 G_2 \cancel{G_3} H_3}{(1+G_1H_1)(1-G_2G_3H_2)\cancel{G_3}}}} \longrightarrow C$$

$$\frac{C(s)}{R(s)} = \frac{\dfrac{G_1 G_2 G_3}{(1+G_1H_1)(1-G_2G_3H_2)}}{1 - \dfrac{G_1 G_2 H_3}{(1+G_1H_1)(1-G_2G_3H_2)}}$$

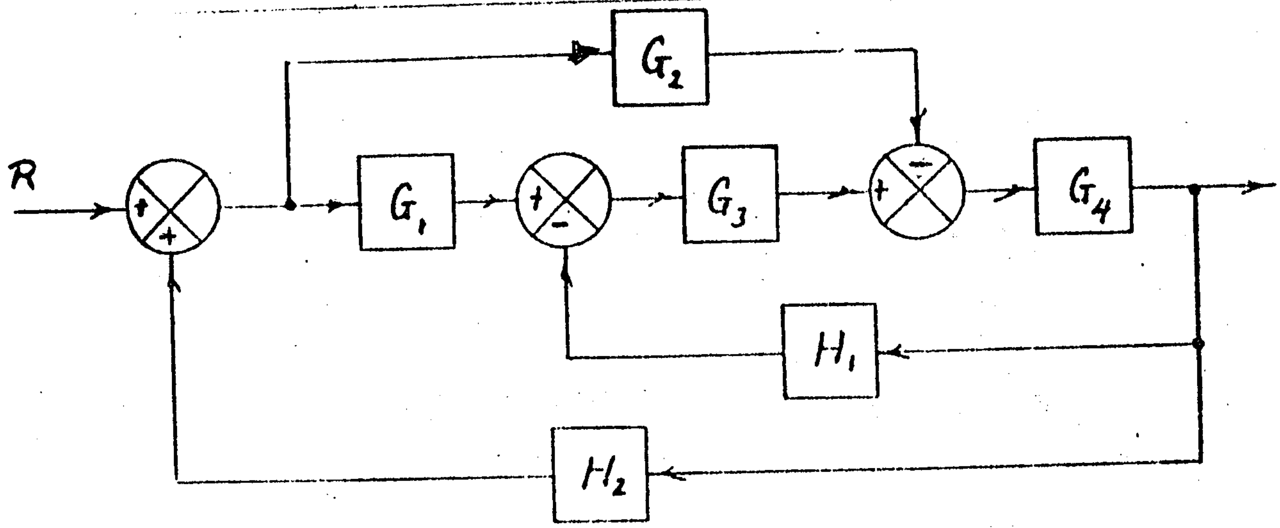
MULTIPLYING NUMERATOR & DENOMINATOR BY $(1+G_1H_1)(1-G_2G_3H_2)$ yields.

$$\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3}{(1+G_1H_1)(1-G_2G_3H_2) - G_1G_2H_3}$$

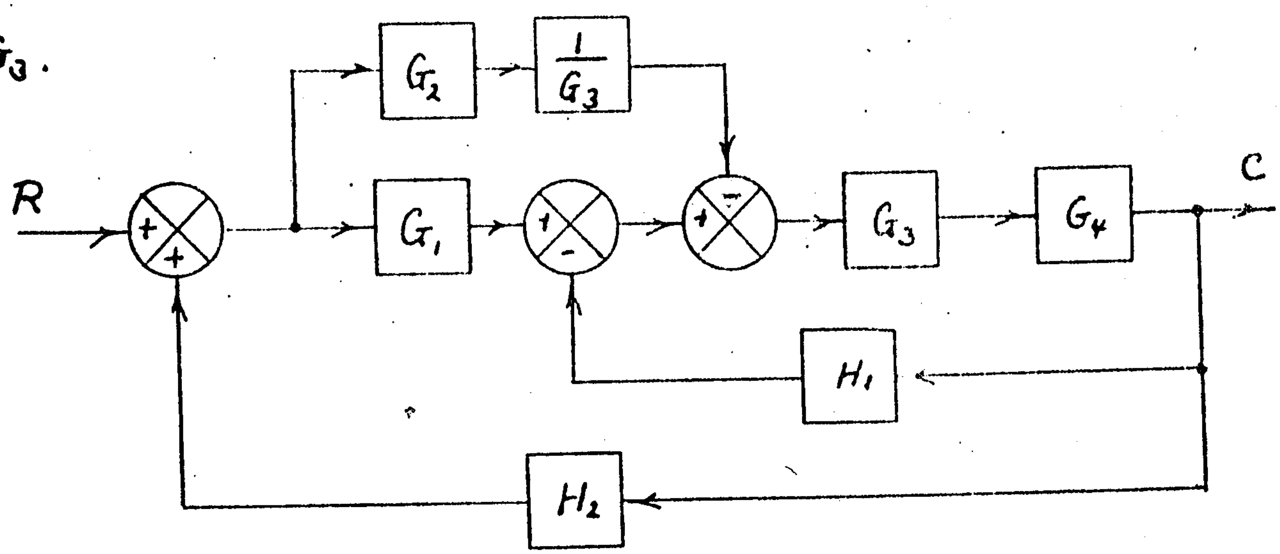
EXPANDING THE DENOMINATOR.

$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3}{-G_1G_2G_3H_1H_2 - G_2G_3H_2 - G_1G_2H_3 + G_1H_1 + 1}}$$

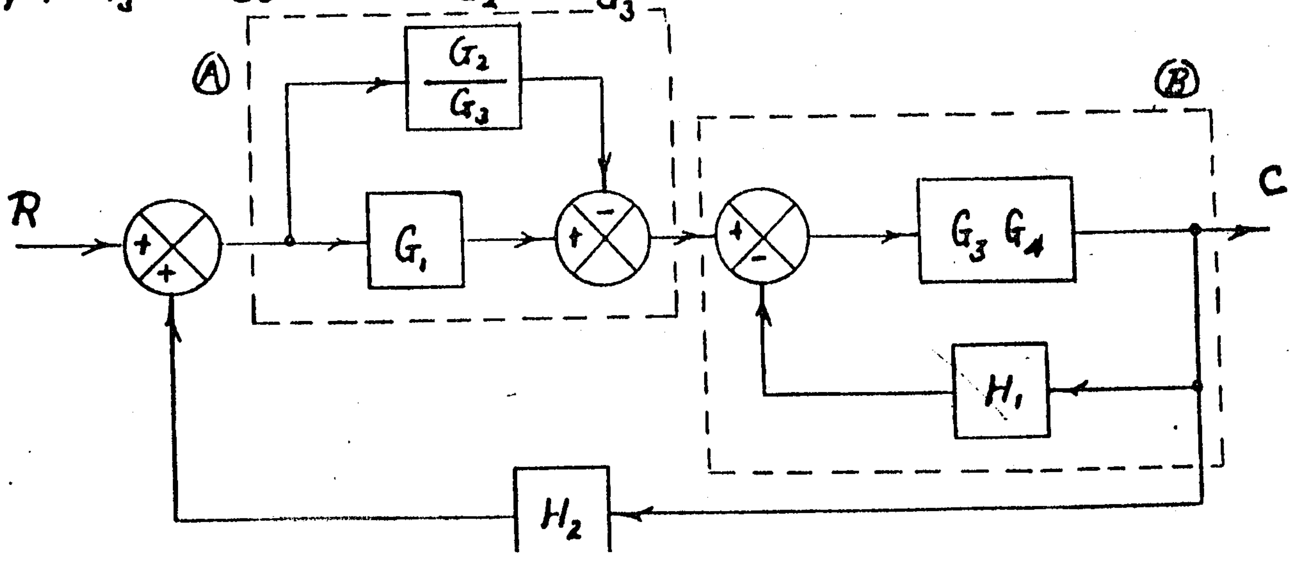
b)



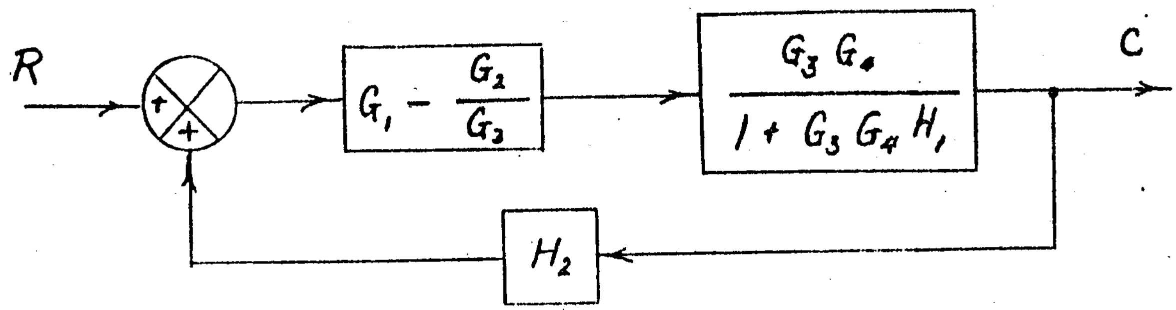
MOVE SUMMING JUNCTION FROM BEHIND G_3 TO AHEAD OF G_3.



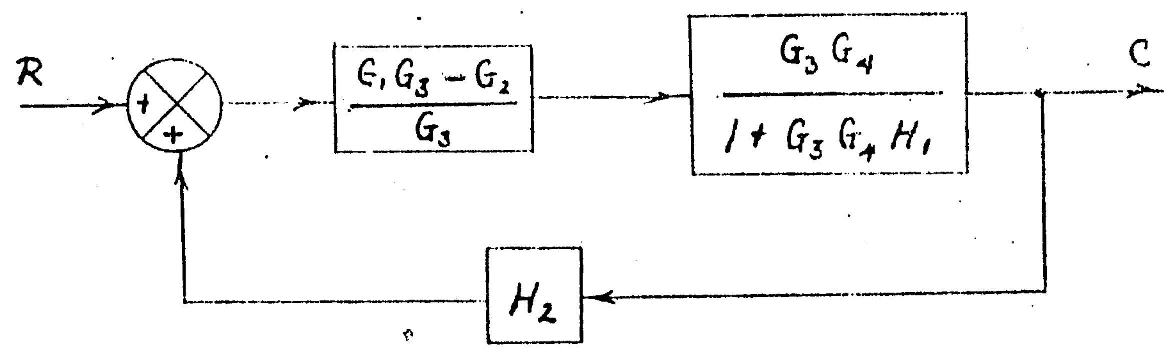
SWITCH THE ORDER OF THE SUMMING JUNCTIONS BETWEEN G_1 & G_3. Combine G_2 & $\frac{1}{G_3}$ and G_3 & G_4.



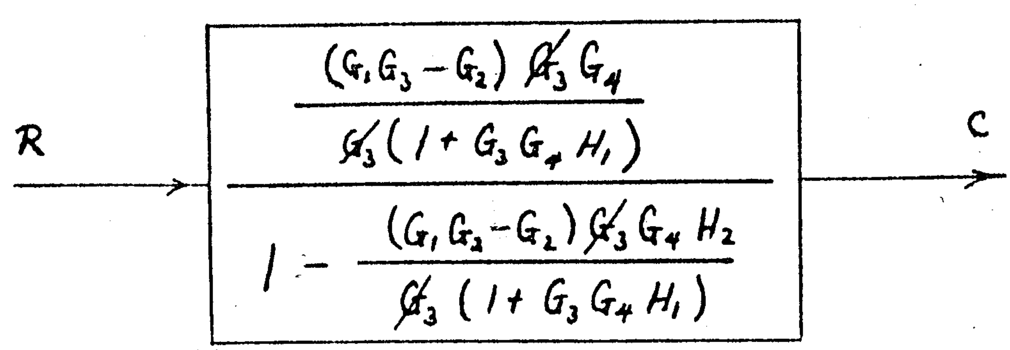
REDUCE FEEDFORWARD LOOP (A) AND FEEDBACK LOOP (B)



PUT THE TERMS IN THE FIRST BOX OF THE FORWARD LOOP OVER A COMMON DENOMINATOR.



COMBINE THE TWO BLOCKS OF THE FORWARD PATH AND REDUCE THE FEEDBACK LOOP.



$$\frac{C(s)}{R(s)} = \frac{\dfrac{(G_1G_3 - G_2)G_4}{(1 + G_3G_4H_1)}}{1 - \dfrac{(G_1G_3 - G_2)G_4H_2}{(1 + G_3G_4H_1)}}$$

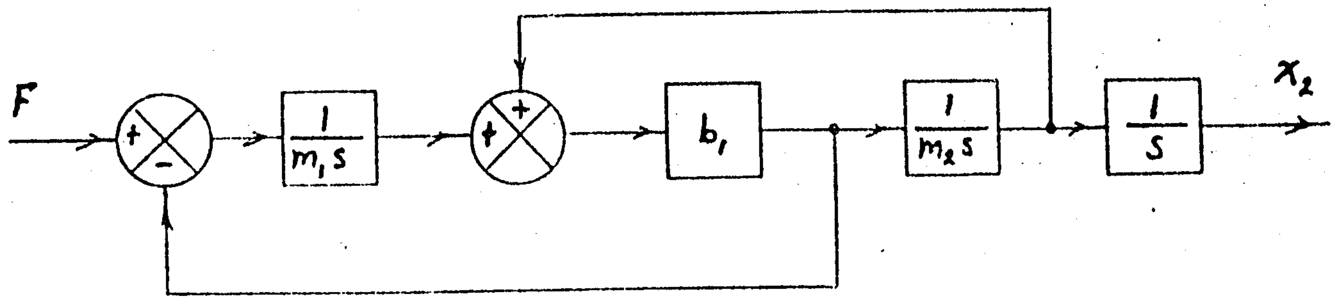
MULTIPLY BOTH NUMERATOR AND DENOMINATOR BY $(1 + G_3G_4H_1)$

$$\frac{C(s)}{R(s)} = \frac{(G_1G_3 - G_2)G_4}{(1 + G_3G_4H_1) - (G_1G_3 - G_2)G_4H_2}$$

EXPANDING

$$\boxed{\frac{C(s)}{R(s)} = \frac{G_1G_3G_4 - G_2G_4}{-G_1G_3G_4H_2 + G_2G_4H_2 + G_3G_4H_1 + 1}}$$

GIVEN: The SYSTEM BLOCK DIAGRAM

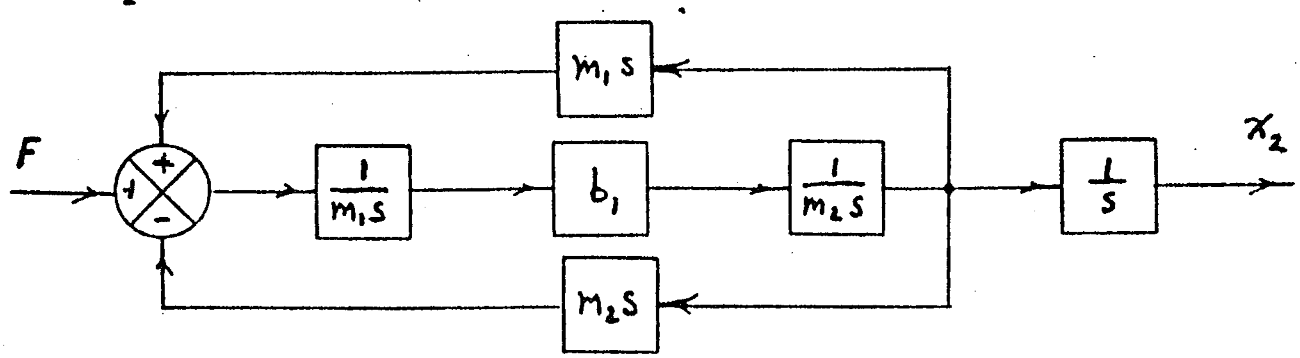


FIND: THE DIFFERENTIAL EQUATION RELATING $x_2(t)$ AND $F(t)$.

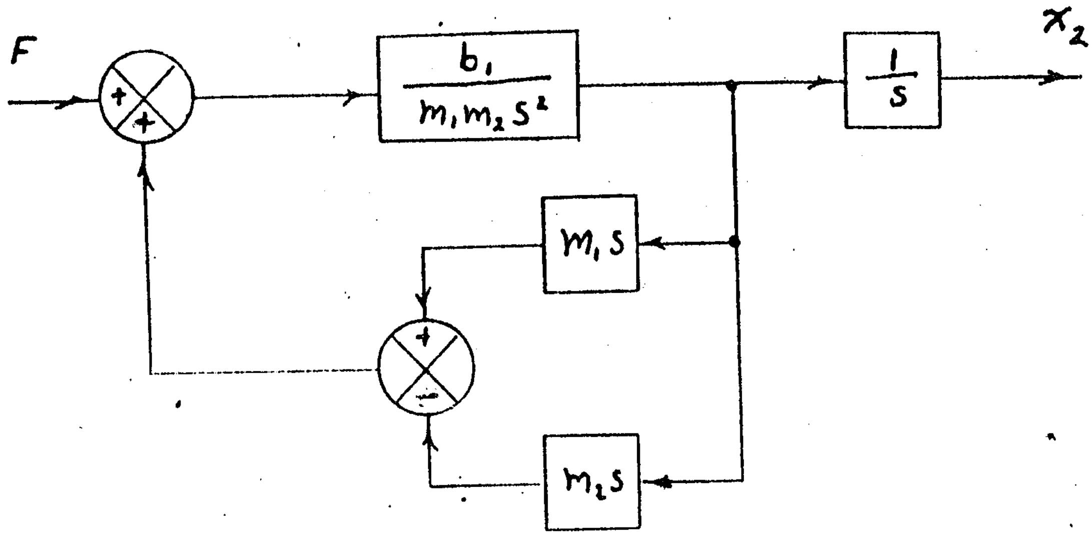
SOLUTION

FIRST, REDUCE THE BLOCK DIAGRAM TO FIND THE TRANSFER FUNCTION $\frac{X_2(s)}{F(s)}$.

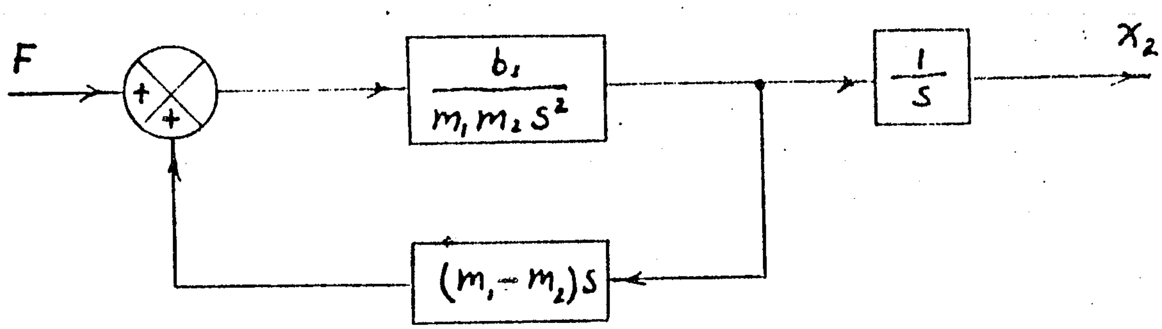
MOVE SUMMING JUNCTION FROM BEHIND $\frac{1}{m_1 s}$ AHEAD OF $\frac{1}{m_1 s}$. ALSO MOVE THE PICKOFF POINT FROM AHEAD OF $\frac{1}{m_2 s}$ TO BEHIND $\frac{1}{m_2 s}$.



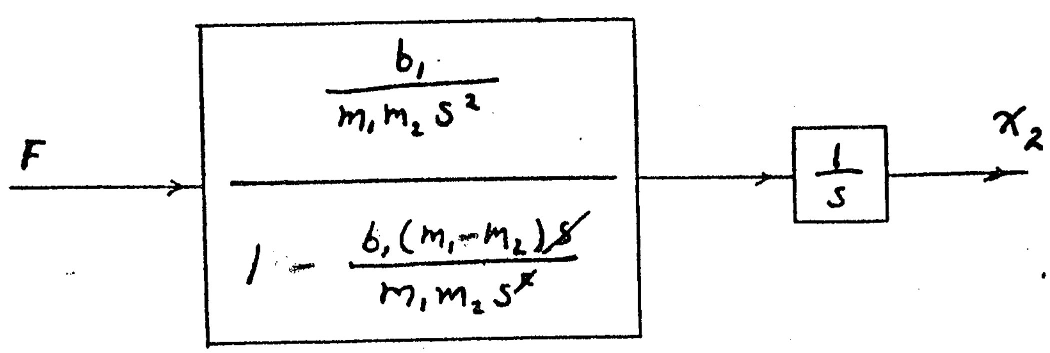
COMBINE THE PARALLEL FEEDBACK PATHES. COMBINE BLOCKS IN FORWARD PATH.



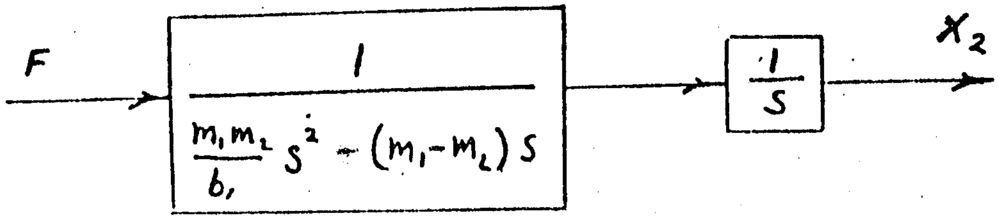
REDUCE PARALLEL FEEDBACK PATHES.



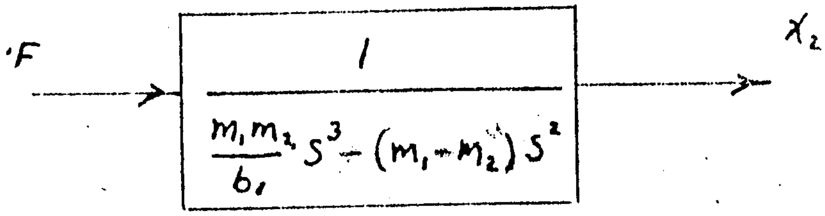
REDUCE FEEDBACK LOOP



MULTIPLYING NUMERATOR AND DENOMINATOR BY $\frac{m_1 m_2 s^2}{b_1}$



COMBINING THE TWO BOXES ABOVE



∴ The TRANSFER FUNCTION IS:

$$\frac{X_2(s)}{F(s)} = \frac{1}{\frac{m_1 m_2}{b_1} s^3 - (m_1 - m_2) s^2}$$

TO FIND THE DIFFERENTIAL EQUATION, CROSS MULTIPLY TO GET

$$\left[\frac{m_1 m_2}{b_1} s^3 - (m_1 - m_2) s^2 \right] X_2(s) = F(s)$$

TAKING THE INVERSE LAPLACE TRANSFORM, ASSUMING ZERO INITIAL CONDITIONS

$$\boxed{\frac{m_1 m_2}{b_1} \dddot{X}_2(t) - (m_1 - m_2) \ddot{X}(t) = F(t)}$$

6-4

$$\frac{G}{1+GH} = \frac{1}{s^5+s^4+12s^3-3s^2+(5-K)s+(2+K)}$$

Since the denominator is the characteristic equation for the closed-loop system we can use Routh Hurwitz. Rule #1 is not satisfied due to the s^2 term and the system is unstable for all values of K

$$\frac{G}{1+GH} = \frac{1}{s^3+8s^2+7s+K}$$

Again using Routh Hurwitz, Rule #1 requires $K>0$

$$\begin{array}{c|cc} s^3 & 1 & 7 \\ s^2 & 8 & K \\ s & b_1 & 0 \\ 1 & c_1 & \end{array}$$

$$b_1 = -\frac{1}{8}\begin{vmatrix} 1 & 7 \\ 8 & K \end{vmatrix} = -\frac{1}{8}(K-56) > 0 \quad \therefore \quad K<56$$

$$c_1 = \frac{-1}{-\frac{(K-56)}{8}}\begin{vmatrix} 8 & K \\ -\frac{1}{8}(K-56) & 0 \end{vmatrix} = \frac{8}{K-56}\cdot\left[-\left(-\frac{1}{8}(K-56)\right)K\right]$$

$$= K > 0$$

$$\therefore \quad \underline{0<K<56}$$

HW #7

4.15 b $\qquad T(s) = \dfrac{1+.01s}{1+.03s}$

$$20 \log_{10} |T| = 10 \log_{10}(1+10^{-4}\omega^2) - 10 \log_{10}(1+9\times10^{-4}\omega^2)$$

- the first term has a break freq at $\omega = \frac{1}{.01} = 100$. It's high freq asymptote has a slope of -20 dB/decade
- the 2nd freq has a break freq at $\omega = \frac{1}{.03} = 33.3$. It's high freq asymptote has a slope of $+20$ dB/decade
- both have low freq asymptotes with slopes of 0 dB/decade

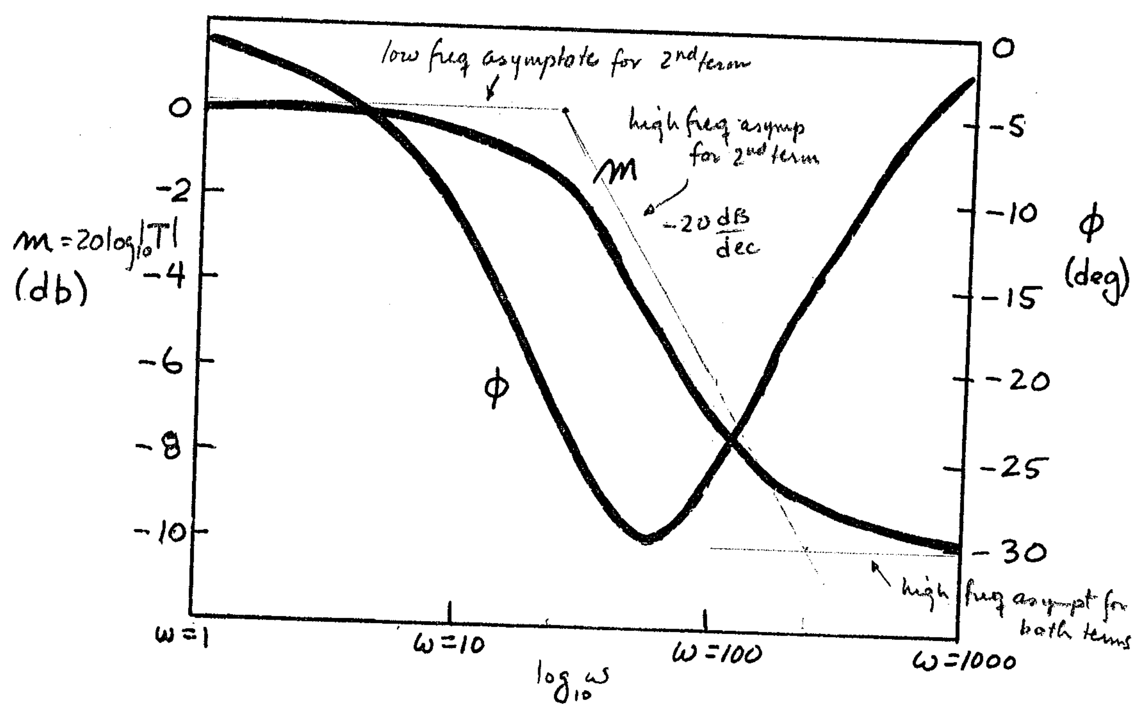
$\therefore$ from $\omega < 1$ to $\omega = 33.3$ slope $= 0$

33.3 to 100 slope $= -20$ dB/decade (2nd term dominat)

100 to ∞ slope $= 0$ (slopes $-20+20=0$)

For ϕ just plot $\tan^{-1}(.01\omega) - \tan^{-1}(.03\omega)$; for large ω $\phi \to 90° - 90° = 0$; for small ω $\phi \to 0 - 0 = 0$

b)



4.15c $\quad T(s) = \dfrac{5}{(10s+1)(4s+1)}$

$m = 20\log_{10}|T| = 20\log_{10}5 - 10\log_{10}(1+100\omega^2) - 10\log_{10}(1+16\omega^2)$

for ω small $\quad m = 20\log_{10}5 = 13.98$

break freq for 2nd term is $\omega = \frac{1}{4} = .25$ and causes a change in slope of -20 dB/decade

break freq for 1st term is $\omega = \frac{1}{10} = .1$ and causes a change in slope of -20 dB/decade

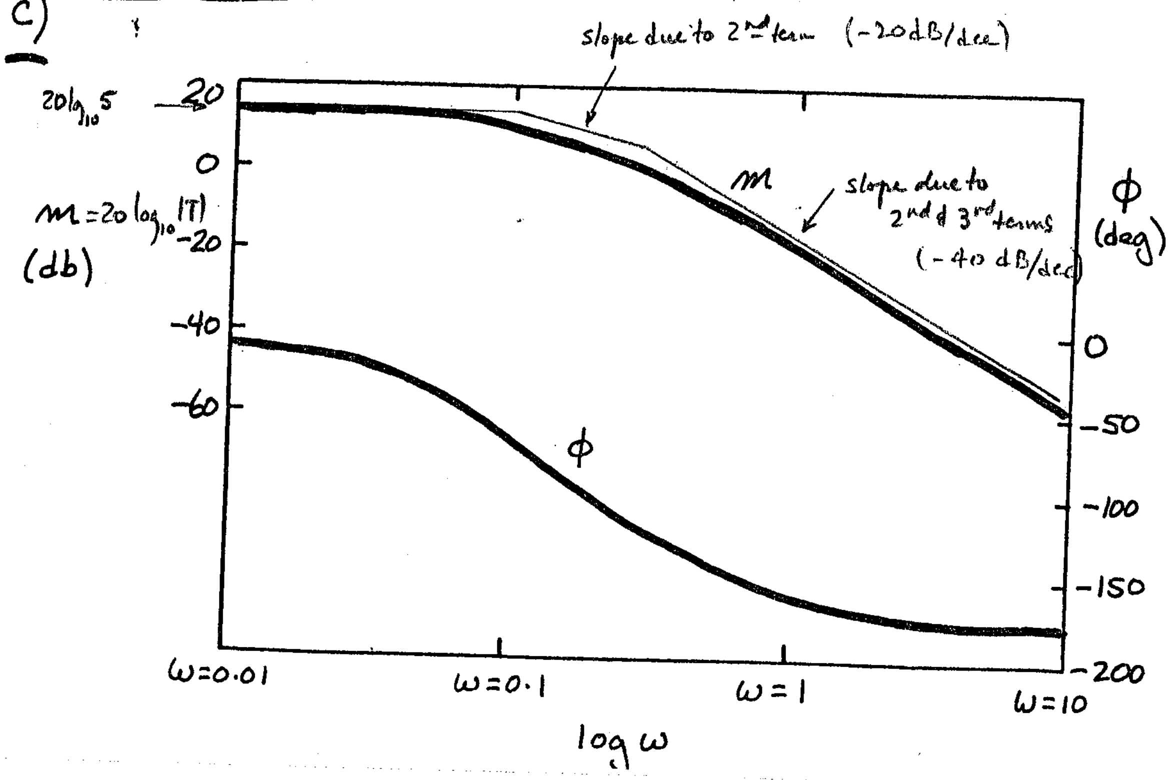
$\therefore$ between $\omega < .1$ and $\omega = .1$ $\quad m = 13.98$ $\quad$ slope $= 0$

$\omega > .1$ but $\omega < .25$ $\quad$ slope $= -20\,\frac{dB}{dec}$ (2nd term)

$\omega > .25$ $\quad$ slope $= -40\,\frac{dB}{dec}$ (2nd & 3rd term dominate)

for $\phi = \tan^{-1}(-4\omega) + \tan^{-1}(-10\omega)$ since the const is positive it has a $\phi = 0°$; for very large ω, $\phi = -90° - 90° = -180°$ each term contributes $-90°$ at very large ϕ. Plot the rest.

c) 4.15 continued



$$T(s) = \frac{10}{s^2(s+1)}$$

4.15f $\quad 20\log_{10}|T| = m = 20\log_{10}10 - 40\log_{10}\omega - 10\log_{10}(1+\omega^2)$

for small ω the constant term has a value of 20, the 2^{nd} term also has a value of 40 when $\omega = .1$; the 3^{rd} term doesn't contribute

$\therefore$ at $\omega = .1 \quad m = 60$

The 2^{nd} term has a slope of -40 dB/decade for all values of ω

the break freq for the last term is $\omega = \frac{1}{1} = 1$ and causes a change in slope of -20 dB/dec.

$\therefore$ for $\omega < 1$ the curve has a slope dominated by the 2^{nd} term ($-40\frac{dB}{dec}$

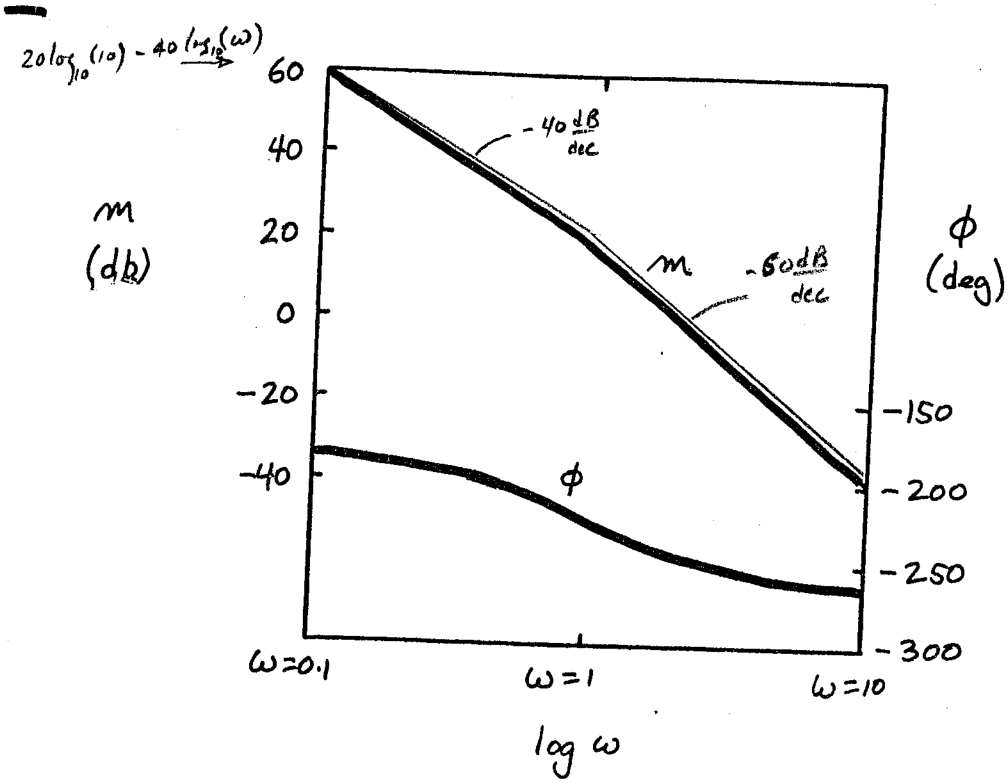
at $\omega = 1$ the curve has a slope dominated by 2^{nd} & 3^{rd} terms ($-60\frac{dB}{dec}$

for ϕ the const produces no phase change so

$\phi = -90^\circ \cdot 2 - \tan^{-1}(\omega)$ the first term is due to $\frac{1}{s^2}$

for small ω $\phi = -180^\circ$; as $\omega \to \infty$ $\phi \to -270^\circ$

f)



4.15h $\qquad T(s) = \dfrac{s^2}{(2s+1)(5s+1)}$

$20\log_{10}|T| = m = 40\log_{10}\omega - 10\log_{10}(1+4\omega^2) - 10\log_{10}(1+25\omega^2)$

the first term comes from s^2. for small ω the first term dominates and has a slope of +40 dB/dec at $\omega = .01$ $\quad m = -80$

the 3rd term has a break freq at $\omega = \frac{1}{5} = .2$ and changes the slope by −20 dB/dec from that pt on. The 2nd term has a break freq at $\omega = \frac{1}{2} = .5$ and changes the slope by another −20 dB/dec from that pt on

Thus for $\omega = .01$ to $\omega = .2$ $\quad$ m starts at −80 w/ slope of $+40\frac{dB}{dec}$

for $\omega > .2$ but less than .5 $\quad$ m slope is (+40−20 = 20 dB/dec

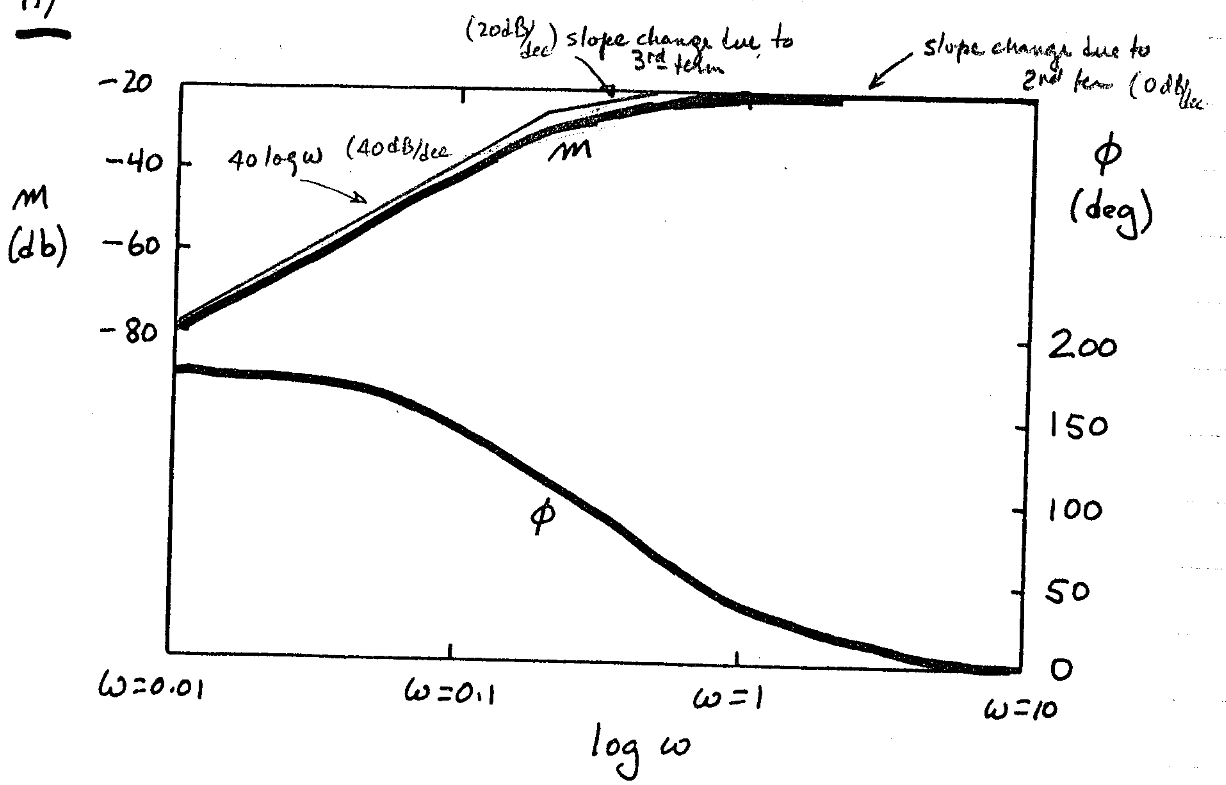
for $\omega > .5$ $\quad$ m slope is (+40−20−20) = 0 dB/dec

For $\phi = 90°\cdot 2 - \tan^{-1}(2\omega) - \tan^{-1}(5\omega)$; the first term is from s^2

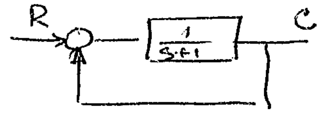
for small ω $\phi = 180°$ and decreases to zero as $\omega \to \infty$

each $\tan^{-1}$ term $\to -90°$ as $\omega \to \infty$. A decade above each break frequency is where ϕ decreases by 90°. ie at $\omega = 2$ ϕ goes from 180° to 90° and at $\omega = 5$ ϕ has almost dropped to zero

h)



Use final value theorem to find e_{ss} for



4.21 $E(s) = R(s) - C(s) = R(s) - \frac{1}{s+1}E(s)$

$$\therefore \quad \frac{s+2}{s+1}E(s) = R(s) \Rightarrow E(s) = \frac{s+1}{s+2}R(s)$$

unit step a) $R(s) = \frac{1}{s} \Rightarrow E(s) = \frac{s+1}{s+2}\frac{1}{s}$, $e_{ss} = \lim_{s\to 0} sE(s) = \frac{1}{2}$ ANS.(a)

unit ramp b) $R(s) = \frac{1}{s^2} \Rightarrow E(s) = \frac{s+1}{s+2}\frac{1}{s^2}$

$$e_{ss} = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0}\frac{s+1}{s+2}\frac{1}{s} \to \infty \quad \text{ANSWER (b)}$$

4.22 Let the input be v and the output be y, so that:

Find unit step response for
a) $T(s) = \frac{Ks}{s+d}$
b) $T(s) = \frac{K(s+c)}{s+d}$
c) $T(s) = \frac{1}{s+d}$.

for b)
① $c=0$
② $c>d$
③ $c<d$

$$V(s) = \frac{1}{s}, \quad Y(s) = T(s)V(s) = T(s)\frac{1}{s}$$

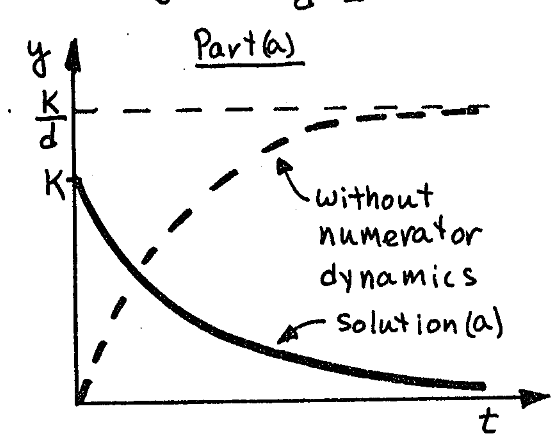
Assume that $y(0)=0$.

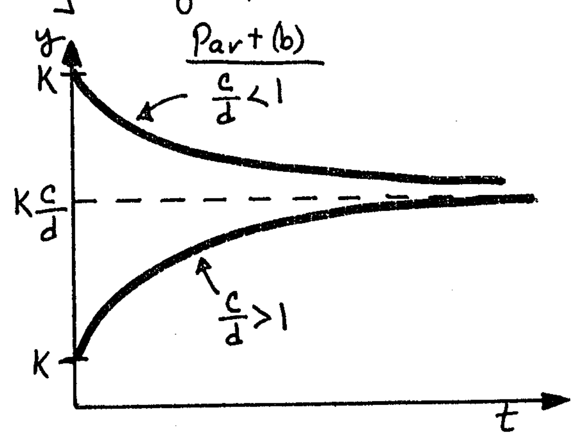
a) $Y(s) = K\frac{s}{s+d}\frac{1}{s} = \frac{K}{s+d} \Rightarrow y(t) = Ke^{-dt}$

Note that as $t\to 0+$, $y(0+) = K$. Thus y jumps instantaneously from $y(0-)=0$ to $y(0+)=K$.

b) $Y(s) = K\frac{s+c}{s+d}\frac{1}{s} = \frac{K}{d}\left(\frac{c}{s} + \frac{d-c}{s+d}\right)$

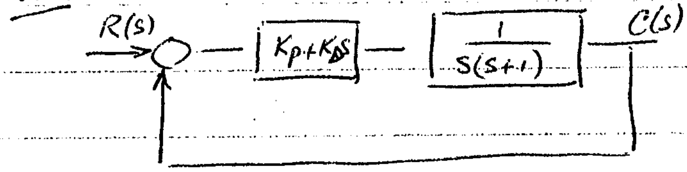
$$\Rightarrow y(t) = \frac{K}{d}\left[c + (d-c)e^{-dt}\right] \Rightarrow y(0+) = K$$





c) Also for a) without numerator dynamics $T(s) = \frac{K}{s+d}$ & $Y(s) = T(s)V(s) = \frac{K}{s+d}\cdot\frac{1}{s}$
for step input $\therefore$ $Y(s) = \frac{K}{d}\left[\frac{1}{s} - \frac{1}{s+d}\right] \Rightarrow \frac{K}{d}\left[1 - e^{-dt}\right] = y(t)$

6.21



(1) Find transfer fn

$$T(s) = \frac{C(s)}{R(s)} = \frac{\frac{K_P + K_D s}{s(s+1)}}{1 + \frac{K_P + K_D s}{s(s+1)}}$$

$$\Rightarrow T(s) = \frac{K_P + K_D s}{s(s+1) + K_P + K_D s} = \frac{K_P + K_D s}{s^2 + (1+K_D)s + K_P}$$; NOTE DENOM IS CHARACTER. EQN

Denominator is of the form $s^2 + 2\zeta\omega_n s + \omega_n^2 \Rightarrow 2\zeta\omega_n = 1 + K_D$, $\omega_n^2 = K_P$

from data $\tau = 1 = \frac{1}{\zeta\omega_n} = \frac{2}{1+K_D} \Rightarrow \underline{K_D = 1}$

also since $\zeta = .9$ and $\tau = 1 \Rightarrow \omega_n = 1.111$ & $\omega_n^2 = \underline{1.235 = K_P}$

finally $e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s \to 0} s[R(s) - C(s)] = \lim_{s\to 0} s\left[\frac{1}{s} - \frac{1}{s}\frac{K_P + K_D s}{s^2 + (1+K_D)s + K_P}\right]$

for unit input $R(s) = \frac{1}{s}$

$\underline{e_{ss}} = 1 - \frac{K_P}{K_P} = \underline{0}$ Thus (a) is automatically satisfied for all values of K_P & K_D (ie for any PD controller) for a step input

10p.

6.23 From no. (3) this implies that $\zeta = 1$ and from step no. (4)

$4\tau = 4/5$ or $\tau = 1/5 = \frac{1}{\zeta\omega_n}$; note that $\zeta = 1$ implies two dominant roots, why?

To satisfy this the above implies that we must have a transfer fn which is 2nd order in denominator (of the form $s^2 + 2\zeta\omega_n s + \omega_n^2$) and must have 2 degrees of freedom to satisfy (3) & (4) ; SO from problem 6.21 we note that we should choose PI (P, I only have one degree of freedom)

Thus $G(s) = K_P + \frac{K_I}{s}$ and

$$T(s) = \frac{C(s)}{R(s)} = \frac{\frac{K_P s + K_I}{s} \cdot \frac{5}{s+5}}{1 + \frac{K_P s + K_I}{s} \cdot \frac{5}{s+5}} = \frac{5K_P s + 5K_I}{s(s+5) + 5K_P s + 5K_I}$$

The denominator (characteristic eqn) is $s^2 + (5+5K_p)s + 5K_I = s^2 + 2\zeta\omega_n s + \omega_n^2$

$\Rightarrow \quad 2\zeta\omega_n = 5(1+K_p) = \frac{2}{\tau} = 2.5 \quad \Rightarrow \quad \underline{K_p = 1}$

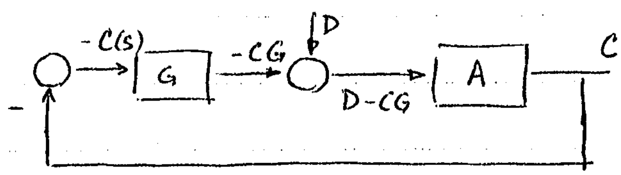
$\Rightarrow \quad 5K_I = \omega_n^2 \quad ; \quad$ since $2\zeta\omega_n = 2.5$ & $\zeta = 1 \Rightarrow \omega_n = 5 \Rightarrow \underline{K_I = 5}$

For (1) $\quad e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[R(s) - C(s)\right] = \lim_{s\to 0} s\left[\frac{1}{s} - \frac{1}{s}\,\frac{5(K_p s + K_I)}{s^2 + 5s(1+K_p) + 5K_I}\right]$

$= 1 - \frac{5K_I}{5K_I} = 0 \quad$ note that $e_{ss} = 0$ for all K_I & K_p for a step input

For (2) we need to find $c_{ss} = \lim_{s\to 0} s\,C(s)$ for a step disturbance

$\Rightarrow$ find Transfer Function $\frac{C(s)}{D(s)}$ when $R(s) = 0$



$\therefore \quad C = A(D - CG)$ or

$\therefore \quad C\left[1 + AG\right] = AD$

or $\quad \frac{C}{D} = \frac{A}{1+AG} \quad$ where $A = \frac{5}{s+5}$

$G = K_p + \frac{K_I}{s}$

$\therefore \quad \frac{C}{D} = \frac{\frac{5}{s+5}}{1 + \frac{K_p s + K_I}{s}\cdot\frac{5}{s+5}} = \frac{5s}{s(s+5) + 5(K_p s + K_I)}$

and $\lim_{s\to 0} s\,C(s) = \lim_{s\to 0} s\cdot\frac{1}{s}\,\frac{5s}{s^2 + 5s(1+K_p) + 5K_I} = 0$

thus PI <u>always</u> satisfies the no steady state deviation irrespective of K_p & K_I

20 p.

<u>6.26</u>

<u>(a)</u> Since the plant $G_p(s) = \frac{1}{Is + C}$ is first order, we have shown in class that for proportional control

$$T(s) = \frac{C(s)}{R(s)} = \frac{K_p}{Is + (c + K_p)} \quad ; \quad \text{if } R(s) = \frac{1}{s^2} \Rightarrow C(s) = \frac{K_p}{Is^3 + (c+K_p)s^2}$$

The steady state error for this case is $e_{ss} = \lim_{s\to 0} s\left(R(s) - C(s)\right) = \lim_{s\to 0}\left[\frac{1}{s} - \frac{K_p}{Is^2 + (c+K_p)s}\right]$
$= \infty$

Thus proportional control won't work. Try integral control $G = \frac{K_I}{s}$

$$T(s) = \frac{C(s)}{R(s)} = \frac{\frac{K_I}{s}\frac{1}{Is+c}}{1+\frac{K_I}{s}\frac{1}{Is+c}} = \frac{K_I}{Is^2+cs+K_I}$$

now $$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left[R(s)-C(s)\right] = \lim_{s\to 0} s\left[\frac{1}{s^2} - \frac{1}{s^2}\frac{K_I}{Is^2+cs+K_I}\right]$$

$$= \lim_{s\to 0} \frac{s}{s^2}\left[\frac{Is^2+cs}{Is^2+cs+K_I}\right] = \frac{c}{K_I}$$ for unit ramp $R(s)=\frac{1}{s^2}$

for $c = 0.1$ and for $e_{ss} = 0.01 \implies \underline{K_I = 10}$

To find the damping ratio and time constant, look at denominator of $T(s)$ = characteristic eqn. $Is^2+cs+K_I = s^2+2\zeta\omega_n+\omega_n^2$

$$\implies 2\zeta\omega_n = \frac{c}{I} \quad \text{and} \quad \frac{K_I}{I} = \omega_n^2 \implies \underline{\omega_n = 1} \text{ since } I = 10$$

and $2\zeta\omega_n = 2\zeta = \frac{0.1}{10} = .01 \implies \underline{\zeta = 0.005}$ $\quad \tau = \frac{1}{\zeta\omega_n} = \frac{1}{.005} = 200$

(b) To meet (1) and (2) you need two degrees of freedom, so try PI action

$$T(s) = \frac{C(s)}{R(s)} = \frac{\left(K_p+\frac{K_I}{s}\right)\cdot\frac{1}{Is+c}}{1+\left(K_p+\frac{K_I}{s}\right)\frac{1}{Is+c}}$$ where $G = K_p + K_I/s$

$$\implies T(s) = \frac{K_p s + K_I}{(Is+c)s + K_p s + K_I} = \frac{K_p s + K_I}{Is^2+(c+K_p)s+K_I}$$

Now $$E(s) = R(s) - C(s) = R(s)\left[1 - T(s)\right] = R(s)\frac{Is^2+Cs}{Is^2+(c+K_p)s+K_I}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\cdot\frac{1}{s^2}\frac{Is^2+cs}{Is^2+(c+K_p)s+K_I} = \frac{c}{K_I}$$

- as in part (a) for $c = 0.1 \implies \underline{K_I = 10}$ for $e_{ss} = 0.01$
- for $\zeta = 1$ look at denom of $T(s)$ = characteristic eqn

$$\implies Is^2+(c+K_p)s+K_I = s^2+2\zeta\omega_n s+\omega_n^2 \implies 2\zeta\omega_n = \frac{c+K_p}{I}\ ; \ \omega_n^2 = \frac{K_I}{I}$$

as before for $K_I = 10$, $I = 10$ $\underline{\omega_n = 1}$

also $2\zeta\omega_n = 2\zeta = \frac{c+K_p}{I} = \frac{0.1+K_p}{10} \implies$ for $\zeta = 1$ $\underline{K_p = 19.9}$

(c) since $T(s) = \frac{C(s)}{R(s)} \Rightarrow c(t) = \mathcal{L}^{-1}\{T(s)R(s)\}$ where $R(s) = \frac{1}{s}$

$$= \mathcal{L}^{-1}\{C(s)\}$$

unit step. Thus $C(s) = \frac{1}{s}\,\frac{K_p s + K_I}{Is^2 + (c+K_p)s + K_I} = \frac{1}{Is}\,\frac{K_p s + K_I}{(s^2 + 2\zeta\omega_n + \omega_n^2)}$

$$= \frac{1}{Is}\,\frac{K_p s + K_I}{s^2+2s+1} = \frac{1}{Is}\,\frac{K_p s + K_I}{(s+1)^2}$$

By partial fraction

$$C(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

$\underline{A} = \lim_{s\to 0} s\,C(s) = \frac{K_I}{I} = \underline{1}$

$\underline{B} = \lim_{s\to -1} (s+1)^2 C(s) = \frac{K_I - K_p}{-I} = \underline{.99}$

also $A(s+1)^2 + B(s+1)s + Cs = \frac{K_p s + K_I}{I} \Rightarrow \underline{B = -A = -1}$

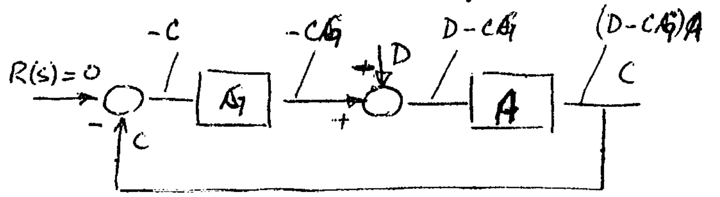
$$\therefore \underline{c(t) = 1 + .99\,t e^{-t} - e^{-t}}$$

This will produce an overshoot when $\frac{dc(t)}{dt} = 0$ or at $\underline{t = 2\text{ sec}}$

The overshoot will be $100\left[\frac{1 - c(t)}{1}\right]\Bigg|_{t=2\text{sec}} = 100\left[.99(2)e^{-2} - e^{-2}\right] = \underline{13.26\%}$

- To reduce overshoot accept a larger time constant
- The overshoot exists because of numerator dynamics $(K_p s + K_I)$

(d)



$(D - CG)A = C \Rightarrow$

$$\frac{C}{D} = \frac{-A}{1+AG}$$

$G = K_p + K_I/s$

$$T_d(s) = \frac{C}{D} = \frac{\frac{-1}{Is+c}}{1 + \left(K_p + \frac{K_I}{s}\right)\left(\frac{1}{Is+c}\right)} = \frac{-s}{s(Is+c) + (K_p s + K_I)}$$

we do this to find the deviation (ie c_{ss}) for a unit step disturbance. Hence we need the relation between C & D

now $c_{ss} = \lim_{s\to 0} s\,C(s) = \lim_{s\to 0} s \cdot T_d(s)\cdot \overset{D(s)}{D(s)} = \lim_{s\to 0} s\cdot\frac{1}{s}\,T_d(s) = \lim_{s\to 0} T_d(0)$

$$= 0$$

NNp.

Thus the unit step disturbance produces no deviation

For the unit ramp disturbance $D(s) = \frac{1}{s^2}$

$$\therefore \quad c_{ss} = \lim_{s\to 0} s\,C(s) = \lim_{s\to 0} s\,T_d(s)\,D(s) = \lim_{s\to 0} \frac{1}{s^2}\cdot s \cdot \frac{-s}{s(Is+c)+K_P s+K_I}$$

$$= \frac{-1}{K_I} = -0.1$$

(e) To satisfy all three must choose PID

Thus $G = K_P + \frac{K_I}{s} + K_D s = \frac{K_P s + K_I + K_D s^2}{s}$

and

$$T(s) = \frac{C(s)}{R(s)} = \frac{\frac{K_I + K_P s + K_D s^2}{s}\cdot\frac{1}{Is+c}}{1+\frac{K_I + K_P s + K_D s^2}{s}\cdot\frac{1}{Is+c}} = \frac{K_I + K_P s + K_D s^2}{s(Is+c)+K_I+K_P s+K_D s^2}$$

$$= \frac{K_I + K_P s + K_D s^2}{(I+K_D)s^2+(c+K_P)s+K_I}$$

To meet the first spec

$$e_{ss} = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s\left[R(s)-C(s)\right] = sR(s)\left[1-T(s)\right]$$

$$\xi_s = \lim_{s\to 0} s\cdot\frac{1}{s^2}\left[\frac{s(Is+c)}{s(Is+c)+K_I+K_P s+K_D s^2}\right] = \frac{c}{K_I} \quad \text{as before } \underline{K_I = 10}$$

To meet 2^{nd} & 3^{rd} specs, look at denominator

$$(I+K_D)s^2+(c+K_P)s+K_I = s^2+2\zeta\omega_n s+\omega_n^2$$

where $2\zeta\omega_n = \frac{c+K_P}{I+K_D}$ & $\frac{K_I}{I+K_D} = \omega_n^2$

we are told that $\zeta = 1$ and $\tau = 0.1 = \frac{1}{\zeta\omega_n}$ $\Rightarrow$ $\underline{\omega_n = 10}$

This implies that $\underline{K_D = -9.9}$ and $\underline{K_P = 1.9}$

20p.

To find the response $c(t) = \mathcal{L}^{-1}(C(s)) = \mathcal{L}^{-1}(T(s)R(s))$ for $R(s) = \frac{1}{s}$

Then $$C(s) = \frac{1}{s}\,\frac{K_I + K_P s + K_D s^2}{(I+K_D)s^2 + (c+K_P)s + K_I} = \frac{1}{(I+K_D)s}\,\frac{K_I + K_P s + K_D s^2}{(s^2+20s+100)}$$

$$= \frac{1}{I+K_D}\cdot\frac{1}{s}\,\frac{K_I + K_P s + K_D s^2}{(s+10)^2}$$

Use partial fractions to find

$$C(s) = \frac{A}{s} + \frac{B}{s+10} + \frac{C}{(s+10)^2}$$

$$\underline{A} = \lim_{s\to 0} s\,C(s) = \frac{K_I}{(I+K_D)\cdot 100} = \underline{1}$$

$$\underline{C} = \lim_{s\to -10} (s+10)^2 C(s) = \underline{999}$$

also $$\frac{A(s+10)^2 + Bs(s+10) + Cs}{s(s+10)^2} = \frac{K_I + K_P s + K_D s^2}{(I+K_D)s(s+10)^2} \Rightarrow A+B = \frac{K_D}{I+K_D} \text{ or } \underline{B = -100}$$

thus

$$\underline{c(t) = 1 + 999te^{-10t} - 100e^{-10t}}$$

Max overshoot occurs at $\frac{d\,c(t)}{dt} = 0$ or at $\underline{t = 0.2 \text{ sec}}$

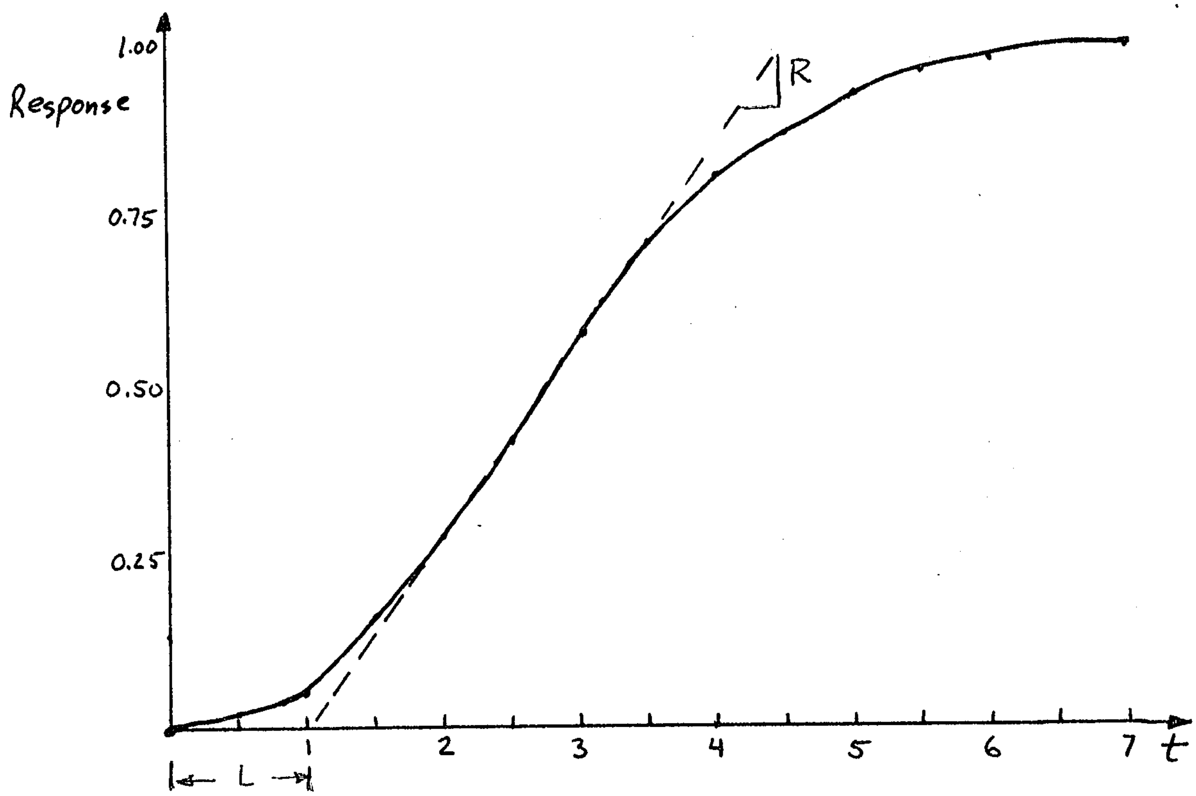
Thus $\underline{\text{Max overshoot for a unit step}} = \frac{1 - c(t)}{1}\Big|_{t=.2} \times 100 = \underline{1452\%}$

<u>overshoot</u> is strictly due to <u>numerator dynamics</u>

To reduce overshoot either take $\zeta > 1$ or use a larger time constant

CHAPTER SEVEN: Control System Design: Modeling Considerations and Alternative Control Structures

7.1 a) Plotting the response data (see below), and comparing with Figure 7.1, we estimate $R = 0.29$ and $L = 1$. Since this line is estimated "by eye", some variations in R and L can be expected.



From Table 7.1:

<u>P-Control</u> $\quad K_p = \dfrac{1}{RL} = 3.45$

<u>PI-Control</u> $\quad K_p = \dfrac{0.9}{RL} = 3.105, \quad K_I = \dfrac{K_p}{T_I} = \dfrac{3.105}{3.3L} = 0.941$

<u>PID-Control</u> $\quad K_p = \dfrac{1.2}{RL} = 4.14 \quad K_I = \dfrac{K_p}{T_I} = \dfrac{4.14}{2L} = 2.07$

$$K_D = K_p T_D = 0.5L(4.14) = 2.07$$

10p.

7.1 Continued

b) Using P-control, the closed-loop transfer function is : $\frac{G_p G_c}{1+G_p G_c}$
where $G_c = K_p$

$$\therefore \quad T(s) = \frac{0.5\,K_p}{s^3 + 1.5s^2 + 1.5s + 0.5 + 0.5K_p}$$

Let $s = i\omega$. Then from the characteristic equation: (see Exaple 7.2 p. 395)

$$i(-\omega^3 + 1.5\omega) + (-1.5\omega^2 + 0.5\omega + 0.5K_p) = 0$$

$$\Rightarrow \omega_u = 1.225, \quad K_{pu} = 3.5, \quad P_u = \frac{2\pi}{\omega_u} = 5.13 \text{ minutes}$$

From Table 7.1, pg 395

P-control: $K_p = 0.5\,K_{pu} = 1.75$

PI-Control: $K_p = 0.45\,K_{pu} = 1.575$

$$K_I = \frac{K_p}{T_I} = \frac{1.575}{0.83P_u} = 0.370$$

PID-Control: $K_p = 0.6\,K_{pu} = 2.1$

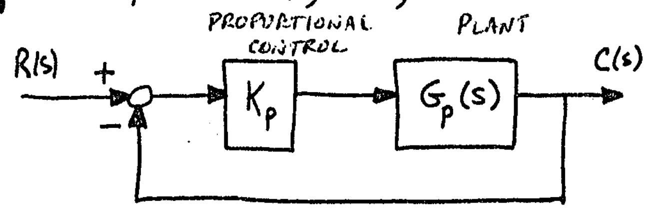
$$K_I = \frac{K_p}{T_I} = \frac{2.1}{0.5P_u} = 0.819$$

$$K_D = K_p T_D = 2.1\,(0.125\,P_u) = 1.347$$

---

c) The gains from the process-reaction method are about $\frac{1}{2}$ of those from the ultimate-cycle method.

$\boxed{7.7}$ For parts (a), (b), (c):



$$T(s) = \frac{C(s)}{R(s)} = \frac{K_p G_p(s)}{1 + K_p G_p(s)}$$

a) For $\tau_p = 0$, $G_p(s) = 1/s$ and

$$T(s) = \frac{K_p}{s + K_p} \Rightarrow \tau = \frac{1}{K_p} \quad \text{since } T(s) = \frac{1}{\frac{1}{K_p}s + 1} = \frac{1}{\tau s + 1}$$

For $\tau = 1$, $K_p = 1$ ANSWER (a)

---

b) For $\tau_p = 0$ and $\tau = 0.1$, $K_p = \frac{1}{0.1} = 10$ ANSWER (b)

---

c) For $\tau_p = 0.06$,

$$T(s) = \frac{K_p}{0.06s^2 + s + K_p}$$

now look at denom (char eqn) and find its roots

(a) For $K_p = 1$, $s = -15.6, -1.07$ $\Rightarrow \tau = 0.935$ (*), which is a non-oscillatory response with τ close to the desired value of $\tau = 1$. (a 6.5% error)

(b) For $K_p = 10$, $s = -8.33 \pm 9.86i$, which is an oscillatory response (not desired), with $\tau = 0.12$. (*) This is an error of $0.12 - 0.1 = 0.02$, or 20%.

10p.

(*) dominant time constant is found by looking for root closest to imaginary axis in (c) for first case $\tau = \frac{1}{1.07} = .935$ for second case $\tau = \frac{1}{8.33} = 0.12$

EML 4312
Automatic Control Systems

Homework Set # 8
Spring 1988

Due: 5 April 1988

Problem 8-1

Determine by inspection, system stability for each of the following closed-loop transfer functions.

10

(a) $$\frac{G}{1 + GH} = \frac{1}{(s + 2)(s + 5)}$$

(b) $$\frac{G}{1 + GH} = \frac{1}{s(s + 2)}$$

(c) $$\frac{G}{1 + GH} = \frac{5(s + 2)}{(s^2 + 2s + 1)(s + 1)}$$

(d) $$\frac{G}{1 + GH} = \frac{1}{(s + 2)(s - 1)}$$

(e) $$\frac{G}{1 + GH} = \frac{1}{(s^2 + 4)(s - 2)}$$

Problem 8-2

Apply the Routh-Hurwitz test to the following closed-loop transfer functions and comment on stability. (Just apply the necessary conditions)

8

2 pts each

(a) $$\frac{G}{1 + GH} = \frac{s - 2}{4s^3 + 9s^2 + 3s + 15}$$

(b) $$\frac{G}{1 + GH} = \frac{10}{s^3 + 4s^2 + 2s}$$

(c) $$\frac{G}{1 + GH} = \frac{18}{-s^5 - 2s^4 - 15s^3 - 12s^2 - 3s - 1}$$

(d) $$\frac{G}{1 + GH} = \frac{15}{s^3 - 2s^2 + 3s + 4}$$

Problem 8-3

Apply the Routh-Hurwitz criterion to the following closed-loop transfer functions and determine system stability. If the system is unstable, determine the number of roots in the right half-plane. (Use the Routh Array)

20 pts

(a) $$\frac{G}{1 + GH} = \frac{1}{s^5 + 3s^4 + 5s^3 + 5s^2}$$

5 pts each

(b) $$\frac{G}{1 + GH} = \frac{10(s + 2)}{s^4 + 4s^3 + 6s^2 + 4s + 10}$$

(c) $$\frac{G}{1 + GH} = \frac{15s}{s^5 + 3s^4 + 3s^3 + 9s^2 + 5s + 8}$$

(d) $$\frac{G}{1 + GH} = \frac{1}{s^4 + 5s^3 + 11s^2 + 25s + 30}$$

Problem 7-4

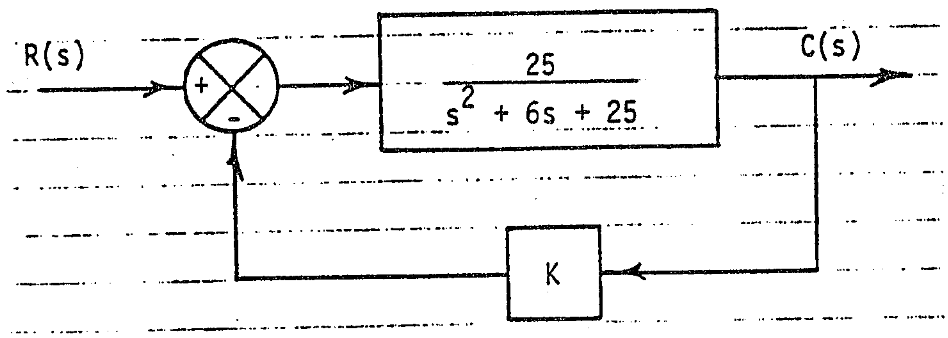
For what values of K are the following closed-loop systems stable?

(a) $$\frac{G}{1 + GH} = \frac{1}{s^5 - s^4 + 2s^3 + 3s^2 + (5 - K)s + 2 + K}$$

(b) $$\frac{G}{1 + GH} = \frac{1}{s^3 + 8s^2 + 17s + K}$$

Problem 8-4

Given the following feedback control system,



$K = 2$

$r(t) = Au(t)$, step input

calculate the following parameters for the closed-loop system.

12 (a) natural frequency, ω_n. 2 pts. each

(b) damping ratio, ζ.

(c) 2% settling time, t_s,

(d) maximum output value, $c_{max}(t)$,

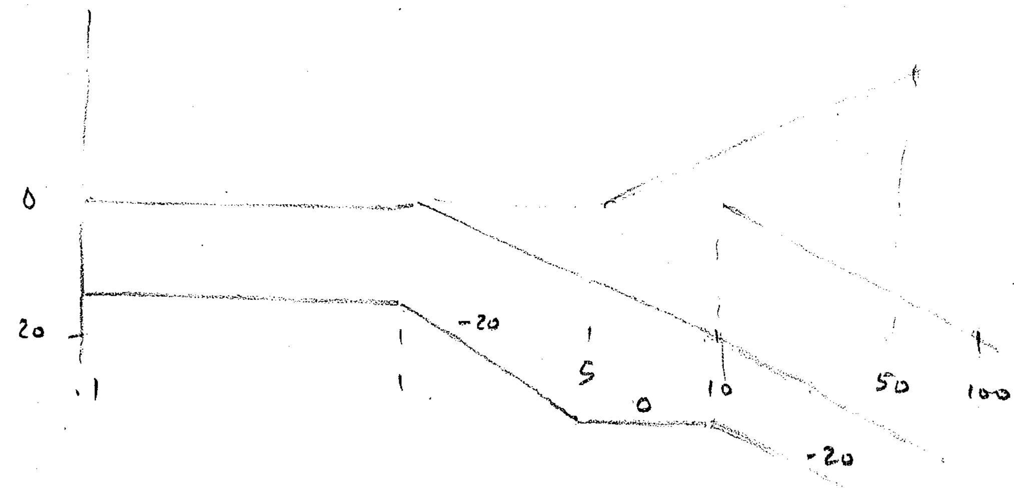
(e) steady state output, c_{ss},

(f) steady state error, E_{ss}.

$$T = \frac{5 \cdot 5\left(\frac{s}{5}+1\right)}{10 \cdot 5\,(s+1)\left(\frac{s}{10}+1\right)} = \frac{1}{2}$$

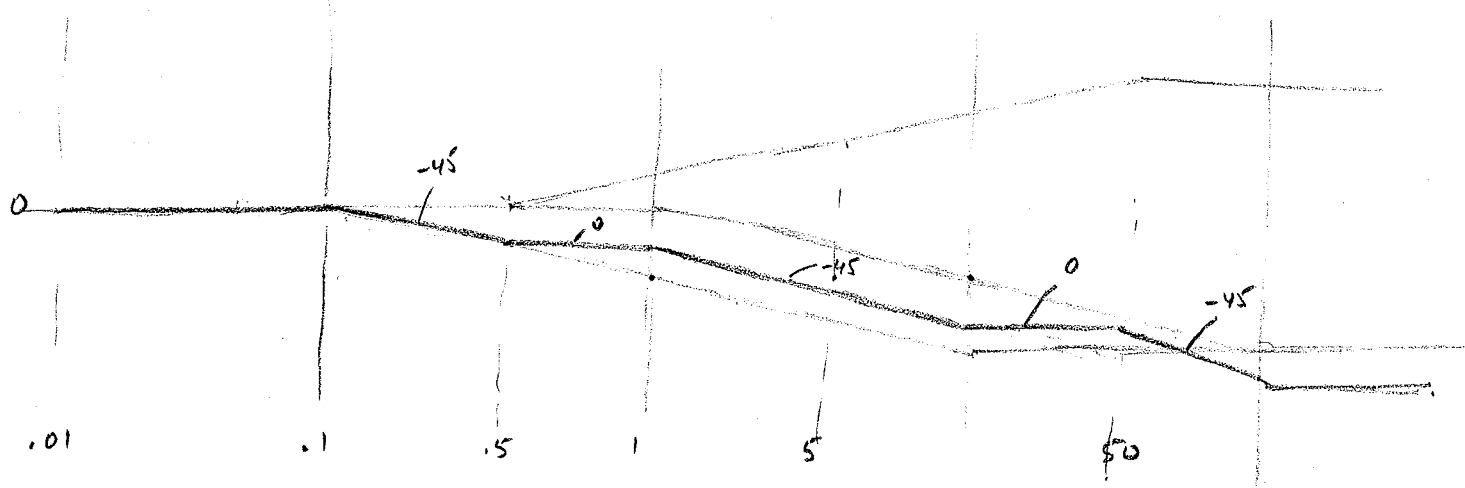
$\tau = 1/5 \quad \tau = 1 \quad \tau = 1/10$

$\omega_b = 5 \quad \omega_b = 1 \quad \omega_b = 10$



$$-20\log 2 + 10\log\left(\left(\frac{\omega}{5}\right)^2+1\right) - 10\log\left(\omega^2+1\right) - 10\log\left(\left(\frac{\omega}{10}\right)^2+1\right)$$

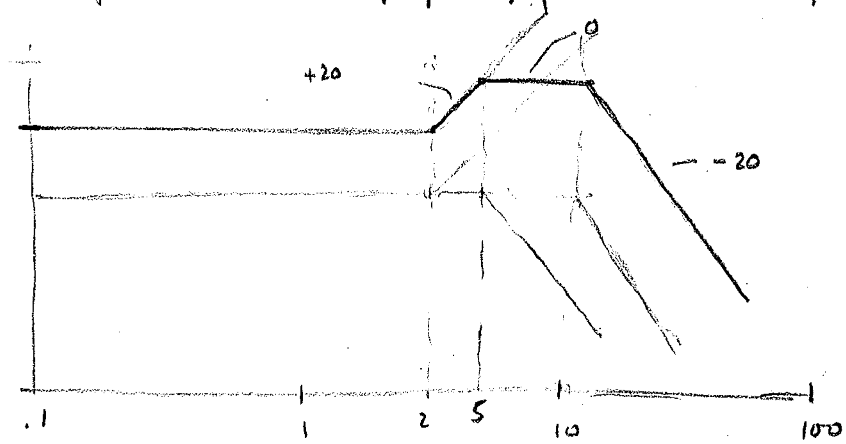
$$\phi = \tan^{-1}\left(\frac{\omega}{5}\right) - \tan^{-1}(\omega) - \tan^{-1}\left(\frac{\omega}{10}\right)$$



$$T = \frac{250 \cdot 2\left(\frac{1}{2}s+1\right)}{5\left(\frac{1}{5}s+1\right)\cdot 10\left(\frac{1}{10}s+1\right)} = \frac{10\,(\tau_1 s+1)}{(\tau_2 s+1)(\tau_3 s+1)} \qquad \tau_1 = 1/2 \quad \tau_2 = 1/5 \quad \tau_3 = \frac{1}{10}$$

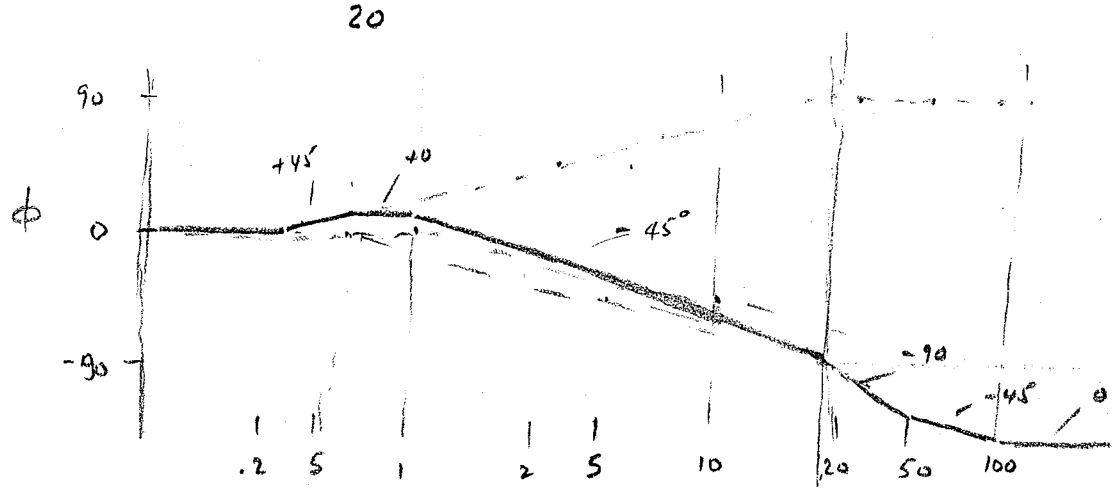
break frequencies at $\omega = 2$ $\quad \omega = 5$ $\quad \omega = 10$

each one of factors has low freq asymptote = 0 high freq asymptote −20



$$20\log_{10}|T| = 20\log_{10}|10| + 10\log_{10}(\tau_1^2\omega^2+1) - 10\log_{10}(\tau_2^2\omega^2+1) - 10\log_{10}(\tau_3^2 \ldots$$

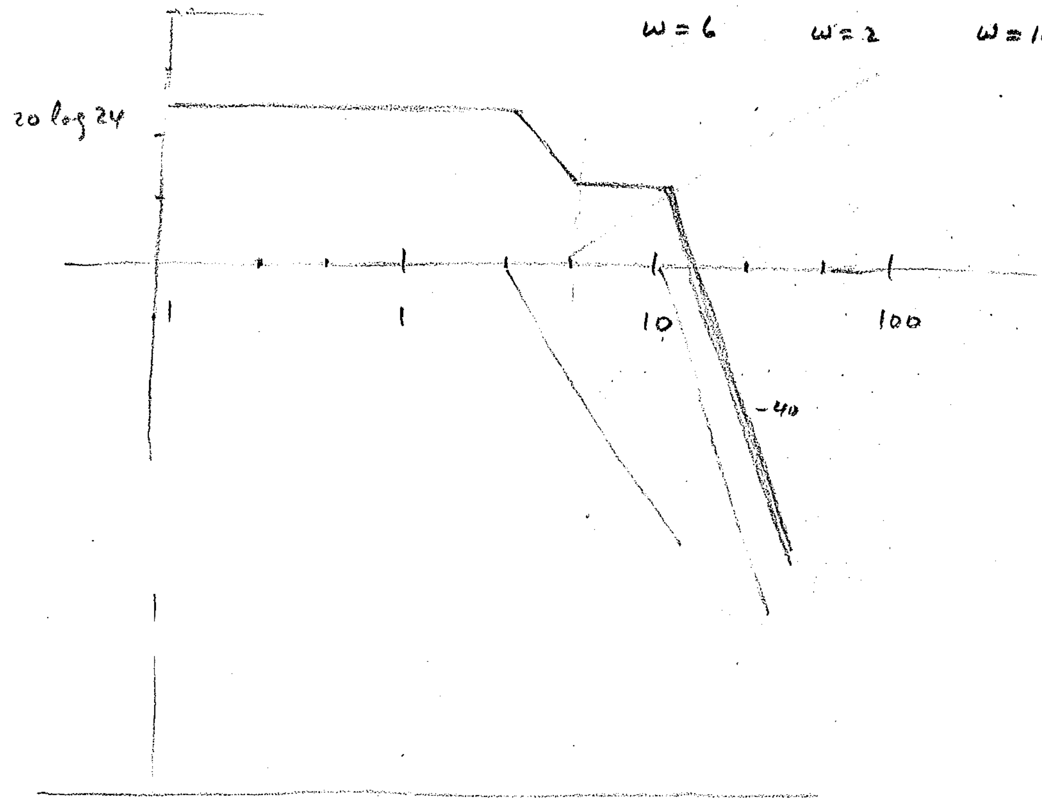
20



$$\frac{800\,(s+6)}{(s+2)\,(s^2+10s+100)} = \frac{800\cdot 6\,(s/6+1)}{2\,(s/2+1)\cdot 100\,\left((s/10)^2+(s/10)+1\right)} = \frac{24\,(\quad)}{(\quad)(\quad)}$$

$\tau = 1/6$ $\tau = 1/2$

$\omega = 6$ $\omega = 2$ $\omega = 10$



DETERMINE BY INSPECTION, SYS. STABILITY FOR EACH OF THE FOLLOWING CLOSED-LOOP TRANSFER FUNCTIONS.

a.) $\dfrac{G}{1+GH} = \dfrac{1}{(s+2)(s+5)}$

ROOTS @ $s = -2, -5$

SYS IS STABLE.

b.) $\dfrac{G}{1+GH} = \dfrac{1}{s(s+2)}$

ROOTS @ $s = 0, -2$

SYS IS ~~MARGINALLY~~ limitedly STABLE

c.) $\dfrac{G}{1+GH} = \dfrac{5(s+2)}{(s^2+2s+1)(s+1)}$

ROOTS @ $s = -1,$

SYS IS STABLE

d.) $\dfrac{G}{1+GH} = \dfrac{1}{(s-2)(s+1)}$

ROOTS @ $s = 2, -1$

SYS IS UNSTABLE

e.) $\dfrac{G}{1+GH} = \dfrac{1}{(s^2+4)(s+2)}$

ROOTS @ $s = \pm 2i, -2$

SYS IS MARGINALLY STABLE

APPLY THE ROUTH-HURWITZ TEST to the following closed-loop transfer functions AND COMMENT ON STABILITY. (JUST APPLY the NECESSARY CONDITIONS)

a.) $\frac{G}{1+GH} = \frac{s-2}{4s^3+9s^2+3s+15}$ $b_2 > 0$ $b_1 > 0$ $b_0 > 0$

SYS MAY BE STABLE BY R-H. CRITERION

ROUTH ARRAY SHOWS SYS TO BE UNSTABLE

b.) $\frac{G}{1+GH} = \frac{10}{s^3+4s^2+2}$ $b_1 \not> 0$

SYS IS UNSTABLE

S-TERM MISSING

c.) $\frac{G}{1+GH} = \frac{18}{-s^5-2s^4-15s^3-12s^2-3s-1}$

$= \frac{-18}{s^5+2s^4+15s^3+12s^2+3s+1}$

SYSTEM MAY BE STABLE

d.) $\frac{G}{1+GH} = \frac{15}{s^3-2s^2+3s+4}$ $b_2 \not> 0$

SYS IS UNSTABLE

PROBLEM 8.3a

$$\frac{G}{1+GH} = \frac{1}{s^5+3s^4+5s^3+5s^2}$$

s^5	1	5	0
s^4	3	5	0
s^3	a_1	a_2	0
s^2	b_1	b_2	
s	c_1	0	
1	d_1		

$$a_1 = \frac{-1}{3}\begin{vmatrix}1 & 5\\ 3 & 5\end{vmatrix} = \frac{10}{3}$$

$$a_2 = -\frac{1}{3}\begin{vmatrix}1 & 0\\ 3 & 0\end{vmatrix} = 0$$

$$b_1 = -\frac{1}{10/3}\begin{vmatrix}3 & 5\\ 10/3 & 0\end{vmatrix} = 5$$

$$b_2 = -1/10/3\begin{vmatrix}3 & 0\\ \frac{10}{3} & 0\end{vmatrix} = 0$$

$$c_1 = -1/5\begin{vmatrix}10/3 & 0\\ 5 & 0\end{vmatrix} = 0 \Rightarrow \varepsilon$$

$$d_1 = -\frac{1}{\varepsilon}\begin{vmatrix}5 & 0\\ \varepsilon & 0\end{vmatrix} = 0$$

- SINCE $c_1 = 0 \Rightarrow$ FROM ROW ABOVE IT ($b_1s^2+b_2 = 5s^2 = 0$) THERE ARE TWO ROOTS AT ORIGIN

NO SIGN CHANGES - SYSTEM HAS 2 ROOTS AT ORIGIN, SYSTEM <u>UNSTABLE</u>

---

b.) $$\frac{G}{1+GH} = \frac{10(s+2)}{s^4+4s^3+6s^2+4s+10}$$

s^4	1	6	10
s^3	4	4	0
s^2	5	10	0
s	-4	0	0
1	10		

$$c_1 = \frac{-1}{4}\begin{vmatrix}1 & 6\\ 4 & 4\end{vmatrix} = 5$$

$$c_2 = \frac{-1}{4}\begin{vmatrix}1 & 10\\ 4 & 0\end{vmatrix} = 10$$

$$c_3 = \frac{-1}{4}\begin{vmatrix}1 & 0\\ 4 & 0\end{vmatrix} = 0$$

$$d_1 = \frac{-1}{5}\begin{vmatrix}4 & 4\\ 5 & 10\end{vmatrix} = -4$$

$$d_2 = \frac{-1}{5}\begin{vmatrix}4 & 0\\ 5 & 0\end{vmatrix} = 0$$

$$e_1 = \frac{1}{4}\begin{vmatrix}5 & 10\\ -4 & 0\end{vmatrix} = 10$$

$$\boxed{\begin{array}{c}\text{2 SIGN CHANGES}\\ \text{SYS. IS UNSTABLE}\\ \text{2 ROOTS IN RIGHT}\\ \text{HALF PLANE}\end{array}}$$

c.) $\frac{G}{1+GH} = \frac{15s}{s^5+3s^4+3s^3+9s^2+5s+8}$

$$\begin{array}{c|ccc}
s^5 & 1 & 3 & 5 \\
s^4 & 3 & 9 & 8 \\
s^3 & 0 & 7/3 & 0 \\
s^2 & 9-2.33/\epsilon & 8 & 0 \\
s & \frac{21-5.44/\epsilon}{9-7/3\epsilon} & 0 & 0 \\
1 & 8 & &
\end{array}$$

$c_1 = \frac{-1}{3}\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0$

$c_2 = \frac{-1}{3}\begin{vmatrix} 1 & 5 \\ 3 & 8 \end{vmatrix} = \frac{7}{3}$

$d_1 = \frac{-1}{\epsilon}\begin{vmatrix} 1 & 9 \\ \epsilon & 7/3 \end{vmatrix} = \frac{7/3 - 9\epsilon}{-\epsilon} = 9 - 2.33/\epsilon \Rightarrow -$

$d_2 = \frac{-1}{\epsilon}\begin{vmatrix} 1 & 8 \\ \epsilon & 0 \end{vmatrix} = 8$

$e_1 = \frac{-1}{9-7/3\epsilon}\begin{vmatrix} 0 & 7/3 \\ 9-7/3\epsilon & 8 \end{vmatrix} = \frac{21-5.44/\epsilon}{9-7/3\epsilon} \Rightarrow +$

$e_2 = \frac{-1}{9-7/3\epsilon}\begin{vmatrix} 0 & 0 \\ 9-7/3\epsilon & 0 \end{vmatrix} = 0$

$f_1 = \frac{-1}{\frac{21-5.44/\epsilon}{9-7/3\epsilon}}\begin{vmatrix} 9-2.33/\epsilon & 8 \\ \frac{21-5.44/\epsilon}{9-7/3\epsilon} & 0 \end{vmatrix} = 8$

2 SIGN CHANGES SYS UNSTABLE 2 ROOTS IN RIGHT HALF PLANE

d.) $\frac{G}{1+GH} = \frac{1}{s^4+5s^3+11s^2+25s+30}$ ✓

$$\begin{array}{c|ccc}
s^4 & 1 & 11 & 30 \\
s^3 & 5 & 25 & 0 \\
s^2 & 6 & 30 & 0 \\
s & 0 & 0 & 0 \\
1 & 30 & &
\end{array}$$

$c_1 = \frac{-1}{5}\begin{vmatrix} 1 & 11 \\ 5 & 25 \end{vmatrix} = 6$

$c_2 = \frac{-1}{5}\begin{vmatrix} 1 & 30 \\ 5 & 0 \end{vmatrix} = 30$

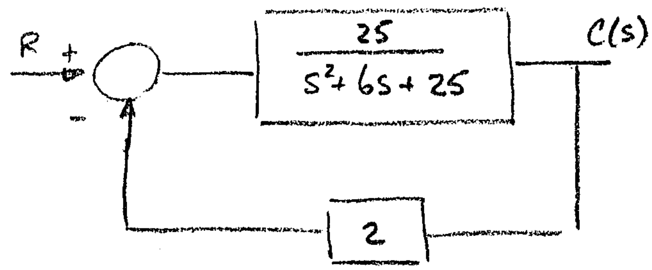
$d_1 = \frac{-1}{6}\begin{vmatrix} 5 & 25 \\ 6 & 30 \end{vmatrix} = 0$

$d_2 = \frac{-1}{6}\begin{vmatrix} 5 & 0 \\ 6 & 0 \end{vmatrix} = 0$

$e_1 = \frac{-1}{\epsilon}\begin{vmatrix} 6 & 30 \\ \epsilon & 0 \end{vmatrix} = 30$

NO SIGN CHANGES 1 ROOT ON ZERO AXIS SYS. IS MARGINALLY STABLE.

8-4



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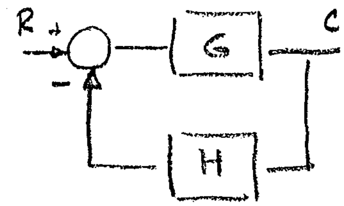

$$T(s) = \frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{25}{s^2+6s+75}$$

$s^2+6s+75$ is characteristic eqn. $\quad s^2+6s+75 = s^2+2\zeta\omega_n s+\omega_n^2$

(a) $\Rightarrow$ $\underline{\omega_n} = \sqrt{75} = 5\sqrt{3} = \underline{8.66}$

(b) $2\zeta\omega_n = 6 \Rightarrow \underline{\zeta} = \frac{6}{2\omega_n} = \frac{\sqrt{3}}{5} = \underline{.3464}$

(c) To find the 2% settling time $= 4\tau$ we need the dominant root's time constant $\tau = \frac{1}{\zeta\omega_n} \Rightarrow \frac{1}{3}$ $\quad\therefore\quad$ $\underline{t_s} = 4\tau = \underline{4/3 \text{ seconds}}$

(d) To find the max. value $C_{max}(t)$: use Fig 3.14 (a) TO FIND maximum percent overshoot $\frac{C_{max}-C_{ss}}{C_{ss}} \times 100\%$ for $\zeta = .3464 \Rightarrow \sim 30\%$

METHOD A

To complete the problem we need to find C_{ss} for a step input $r(t) = Au(t)$ or $R(s) = \frac{A}{s}$

now $\quad C_{ss} = \lim_{s\to 0} s\,C(s) = \lim_{s\to 0} s\,T(s)R(s) = \lim_{s\to 0} A\,T(s) = A\,T(0) = \frac{A}{3}$

Thus $\quad \frac{C_{max}-C_{ss}}{C_{ss}} = .3 \quad$ or $\quad C_{max} = 1.3\,C_{ss} = \frac{1.3}{3}A = \underline{.433A}$

METHOD 2

or write $\quad T(s) = \frac{1}{\frac{s^2}{25}+\frac{6}{25}s+3} = \frac{1}{ms^2+cs+k} \quad$ Thus $k = 3 \quad m = \frac{1}{25} \quad c = \frac{6}{25}$

roots are $s = -3 \pm 8.124j$. Thus from Table 3.5 pg 134

$$c(t) = \frac{A}{3}\left[\frac{8.66}{8.124}\, e^{-(.3464)(8.66)t}\sin(8.124t+\phi)+1\right]$$

$$\phi = \tan^{-1}\left(\frac{1-\zeta^2}{\zeta}\right) = 248.52^\circ$$

maximum occurs at first overshoot and $t = t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = .3867\text{ sec}$ $\quad$ eqn 3.5-13 pg 138

This value of t into $c(t)$ gives $1.311\frac{A}{3} = \underline{.437A}$

(e) see (d) for details $e_{ss} = \lim_{s\to 0} s\,C(s) = \frac{A}{3}$

(f) $\underline{e_{ss}} = \lim_{s\to 0} s\,E(s) = \lim_{s\to 0} s\left[R(s) - C(s)\right] = \lim_{s\to 0} s\left[R(s) - T(s)R(s)\right]$

$$= \lim_{s\to 0} s\,R(s)\left[1 - T(s)\right] = A \lim_{s\to 0}\left[1 - T(s)\right]$$

$$= A\left[1 - \frac{1}{3}\right] = \underline{\frac{2}{3}A}$$

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DETERMINE BY INSPECTION, SYS. STABILITY FOR EACH OF THE FOLLOWING CLOSED-LOOP TRANSFER FUNCTIONS.

a.) $\frac{G}{1+GH} = \frac{1}{(s+2)(s+5)}$ roots @ $s=-2,-5$

$\boxed{\text{SYS IS STABLE.}}$

b.) $\frac{G}{1+GH} = \frac{1}{s(s+2)}$ roots @ $s=0,-2$

SYS IS ~~MARGINALLY~~ limitedly STABLE

c.) $\frac{G}{1+GH} = \frac{5(s+2)}{(s^2+2s+1)(s+1)}$ roots @ $s=-1$, ✓

$\boxed{\text{SYS IS STABLE}}$

d.) $\frac{G}{1+GH} = \frac{1}{(s-2)(s+1)}$ roots @ $s=2,-1$

$\boxed{\text{SYS IS UNSTABLE}}$

e.) $\frac{G}{1+GH} = \frac{1}{(s^2+4)(s+2)}$ roots @ $s=\pm 2i,-2$

$\boxed{\text{SYS IS MARGINALLY STABLE.}}$

APPLY THE ROUTH-HURWITZ TEST to the following closed-loop transfer FUNCTIONS AND COMMENT ON STABILITY. (JUST APPLY the NECESSARY CONDITIONS)

a.) $\frac{G}{1+GH} = \frac{s-2}{4s^3+9s^2+3s+15}$

$b_2 > 0 \quad b_1 > 0 \quad b_0 > 0$

$\boxed{\text{SYS MAY BE STABLE BY R-H. CRITERION}}$ ✓

ROUTH ARRAY SHOWS SYS TO BE UNSTABLE

b.) $\frac{G}{1+GH} = \frac{10}{s^3+4s^2+2}$

$b_1 \not> 0$

$\boxed{\text{SYS IS UNSTABLE}}$ ✓

S-TERM MISSING

c.) $\frac{G}{1+GH} = \frac{18}{-s^5-2s^4-15s^3-12s^2-3s-1}$

$= \frac{-18}{s^5+2s^4+15s^3+12s^2+3s+1}$ ✓

$\boxed{\text{SYSTEM MAY BE STABLE}}$

d.) $\frac{G}{1+GH} = \frac{15}{s^3-2s^2+3s+4}$ ✓

$b_2 \not> 0$

$\boxed{\text{SYS IS UNSTABLE}}$

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APPLY THE ROUTH-HURWITZ CRITERION TO THE FOLLOWING CLOSED-LOOP TRANSFER FUNCTIONS AND DETERMINE SYS. STABILITY. IF THE SYS. IS UNSTABLE, DETERMINE THE NUMBER OF ROOTS IN THE RIGHT HALF-PLANE. (USE THE ROUTH-ARRAY)

a.) $\frac{G}{1+GH} = \frac{1}{s^5+3s^4+5s^3+5s^2+3s+1}$

$$\begin{array}{c|ccc} s^5 & 1 & 5 & 3 \\ s^4 & 3 & 5 & 1 \\ s^3 & 10/3 & 8/3 & 0 \\ s^2 & 2.6 & 1 & 0 \\ s & 1.385 & 0 & 0 \\ 1 & 1 & & \end{array}$$

$c_1 = -\frac{1}{3}\begin{vmatrix} 1 & 5 \\ 3 & 5 \end{vmatrix} = \frac{10}{3}$

$c_2 = -\frac{1}{3}\begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} = \frac{8}{3}$

$c_3 = -\frac{1}{3}\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0$

$d_1 = -\frac{1}{10/3}\begin{vmatrix} 3 & 5 \\ 10/3 & 8/3 \end{vmatrix} = -.3\left(8 - \frac{50}{3}\right) = 2.6$

$d_2 = -\frac{1}{10/3}\begin{vmatrix} 3 & 1 \\ 10/3 & 0 \end{vmatrix} = 1$

$d_3 = -\frac{1}{10/3}\begin{vmatrix} 3 & 0 \\ \frac{10}{3} & 0 \end{vmatrix} = 0$

$e_1 = -\frac{1}{2.6}\begin{vmatrix} 10/3 & 8/3 \\ 2.6 & 1 \end{vmatrix} = 1.385$

$f_1 = -\frac{1}{1.385}\begin{vmatrix} 2.6 & 1 \\ 1.385 & 0 \end{vmatrix} = 1$

$e_2 = -\frac{1}{2.6}\begin{vmatrix} 10/3 & 0 \\ 2.6 & 0 \end{vmatrix} = 0$

NO SIGN CHANGES ∴ SYS IS STABLE ✓

b.) $\frac{G}{1+GH} = \frac{10(s+2)}{s^4+4s^3+6s^2+4s+10}$

$$\begin{array}{c|ccc} s^4 & 1 & 6 & 10 \\ s^3 & 4 & 4 & 0 \\ s^2 & 5 & 10 & 0 \\ s & -4 & 0 & 0 \\ 1 & 10 & & \end{array}$$

$c_1 = -\frac{1}{4}\begin{vmatrix} 1 & 6 \\ 4 & 4 \end{vmatrix} = 5$

$c_2 = -\frac{1}{4}\begin{vmatrix} 1 & 10 \\ 4 & 0 \end{vmatrix} = 10$

$c_3 = -\frac{1}{4}\begin{vmatrix} 1 & 0 \\ 4 & 0 \end{vmatrix} = 0$

$d_1 = -\frac{1}{5}\begin{vmatrix} 4 & 4 \\ 5 & 10 \end{vmatrix} = -4$

$d_2 = -\frac{1}{5}\begin{vmatrix} 4 & 0 \\ 5 & 0 \end{vmatrix} = 0$

$e_1 = \frac{1}{4}\begin{vmatrix} 5 & 10 \\ -4 & 0 \end{vmatrix} = 10$

2 SIGN CHANGES

SYS. IS UNSTABLE
2 ROOTS IN RIGHT HALF PLANE

✓

c.) $$\frac{G}{1+GH} = \frac{15s}{s^5+3s^4+3s^3+9s^2+5s+8}$$

s^5	1	3	5
s^4	3	9	8
s^3	0	$7/3$	0
s^2	$9-2.33/\epsilon$	8	0
s	$\frac{21-5.44/\epsilon}{9-7/3\epsilon}$	0	0
1	8		

$$c_1 = \frac{-1}{3}\begin{vmatrix}1 & 3\\ 3 & 9\end{vmatrix} = 0$$

$$c_2 = \frac{-1}{3}\begin{vmatrix}1 & 5\\ 3 & 8\end{vmatrix} = \frac{7}{3}$$

$$d_1 = \frac{-1}{\varepsilon}\begin{vmatrix}1 & 9\\ \varepsilon & 7/3\end{vmatrix} = \frac{7/3 - 9\varepsilon}{-\varepsilon} = 9 - 2.33/\varepsilon \Rightarrow -$$

$$d_2 = \frac{-1}{\varepsilon}\begin{vmatrix}1 & 8\\ \varepsilon & 0\end{vmatrix} = 8$$

$$e_1 = \frac{-1}{9-7/3\epsilon}\begin{vmatrix}0 & 7/3\\ 9-7/3\epsilon & 8\end{vmatrix} = \frac{21-5.44/\epsilon}{9-7/3\epsilon} \Rightarrow +$$

$$e_2 = \frac{-1}{9-7/3\epsilon}\begin{vmatrix}0 & 0\\ 9-7/3\epsilon & 0\end{vmatrix} = 0$$

$$f_1 = \frac{-1}{\frac{21-5.44/\epsilon}{9-7/3\epsilon}}\begin{vmatrix}9-2.33/\epsilon & 8\\ \frac{21-5.44/\epsilon}{9-7/3\epsilon} & 0\end{vmatrix} = 8$$

2 SIGN CHANGES

SYS UNSTABLE
2 ROOTS IN RIGHT HALF PLANE

d.) $$\frac{G}{1+GH} = \frac{1}{s^4+5s^3+11s^2+25s+30}$$

s^4	1	11	30
s^3	5	25	0
s^2	6	30	0
s	0	0	0
1	30		

$$c_1 = \frac{-1}{5}\begin{vmatrix}1 & 11\\ 5 & 25\end{vmatrix} = 6$$

$$c_2 = \frac{-1}{5}\begin{vmatrix}1 & 30\\ 5 & 0\end{vmatrix} = 30$$

$$d_1 = \frac{-1}{6}\begin{vmatrix}5 & 25\\ 6 & 30\end{vmatrix} = 0$$

$$d_2 = \frac{-1}{6}\begin{vmatrix}5 & 0\\ 6 & 0\end{vmatrix} = 0$$

$$e_1 = \frac{-1}{\varepsilon}\begin{vmatrix}6 & 30\\ \varepsilon & 0\end{vmatrix} = 30$$

NO SIGN CHANGES
1 ROOT ON ZERO AXIS

SYS. IS MARGINALLY STABLE.

PROBLEM 8.3a

$$\frac{G}{1+GH} = \frac{1}{s^5 + 3s^4 + 5s^3 + 5s^2}$$

$$\begin{array}{c|ccc}
s^5 & 1 & 5 & 0 \\
s^4 & 3 & 5 & 0 \\
s^3 & a_1 & a_2 & 0 \\
s^2 & b_1 & b_2 & \\
s & c_1 & 0 & \\
1 & d_1 & &
\end{array}$$

$$a_1 = \frac{-1}{3}\begin{vmatrix} 1 & 5 \\ 3 & 5 \end{vmatrix} = \frac{10}{3}$$

$$a_2 = -\frac{1}{3}\begin{vmatrix} 1 & 0 \\ 3 & 0 \end{vmatrix} = 0$$

$$b_1 = -\frac{1}{10/3}\begin{vmatrix} 3 & 5 \\ 10/3 & 0 \end{vmatrix} = 5$$

$$b_2 = -1/10/3\begin{vmatrix} 3 & 0 \\ \frac{10}{3} & 0 \end{vmatrix} = 0$$

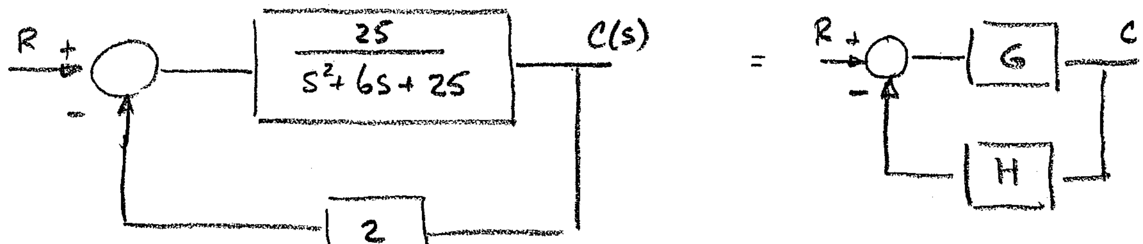
$$c_1 = -1/5\begin{vmatrix} 10/3 & 0 \\ 5 & 0 \end{vmatrix} = 0 \Rightarrow \varepsilon$$

$$d_1 = -\frac{1}{\varepsilon}\begin{vmatrix} 5 & 0 \\ \varepsilon & 0 \end{vmatrix} = 0$$

- SINCE $c_1 = 0 \Rightarrow$ FROM ROW ABOVE IT ($b_1 s^2 + b_2 = 5s^2 = 0$) THERE ARE TWO ROOTS AT ORIGIN

NO SIGN CHANGES - SYSTEM HAS 2 ROOTS AT ORIGIN, SYSTEM <u>UNSTABLE</u>

8-4



$$T(s) = \frac{C(s)}{R(s)} = \frac{G}{1+GH} = \frac{25}{s^2+6s+75}$$

$s^2+6s+75$ is characteristic eqn. $\quad s^2+6s+75 = s^2+2\zeta\omega_n s+\omega_n^2$

(a) $\Rightarrow$ $\underline{\omega_n} = \sqrt{75} = 5\sqrt{3} = \underline{8.66}$

(b) $2\zeta\omega_n = 6 \Rightarrow \underline{\zeta} = \frac{6}{2\omega_n} = \frac{\sqrt{3}}{5} = \underline{.3464}$

(c) To find the 2% settling time $= 4\tau$ we need the dominant root's time constant $\quad \tau = \frac{1}{\zeta\omega_n} \Rightarrow \frac{1}{3} \quad \therefore \quad \underline{t_s} = 4\tau = \underline{4/3 \text{ seconds}}$

(d) To find the max. value $C_{max}(t)$: use Fig 3.14 (a) TO FIND maximum percent overshoot $\frac{C_{max}-C_{ss}}{C_{ss}} \times 100\%$ for $\zeta = .3464 \Rightarrow \sim 30\%$

To complete the problem we need to find C_{ss} for a step input $r(t) = Au(t)$ or $R(s) = \frac{A}{s}$

now $\quad C_{ss} = \lim_{s\to 0} sC(s) = \lim_{s\to 0} sT(s)R(s) = \lim_{s\to 0} AT(s) = AT(0) = \frac{A}{3}$

Thus $\quad \frac{C_{max}-C_{ss}}{C_{ss}} = .3 \quad$ or $\quad C_{max} = 1.3C_{ss} = \frac{1.3}{3}A = \underline{.433A}$

$\underline{\underline{or}}$ write $\quad T(s) = \frac{1}{\frac{s^2}{25}+\frac{6}{25}s+3} = \frac{1}{ms^2+cs+k} \quad$ Thus $k=3 \quad m=\frac{1}{25} \quad c=\frac{6}{25}$

roots are $s = -3 \pm 8.124j$. Thus from Table 3.5 pg 134

$$c(t) = \frac{A}{3}\left[\frac{8.66}{8.124}e^{-(.3464)(8.66)t}\sin(8.124t+\phi)+1\right]$$

$$\phi = \tan^{-1}\left(\frac{1-\zeta^2}{\zeta}\right) = 248.52^\circ$$

maximum occurs at first overshoot and $t = t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = .3867 \text{ sec}$ $\quad$ eqn 3.5-13 pg 138

This value of t into $c(t)$ gives $1.311\frac{A}{3} = .437A$

(e) see (d) for details $c_{ss} = \lim_{s \to 0} s\,C(s) = \frac{A}{3}$

(f) $\underline{e_{ss}} = \lim_{s \to 0} s\,E(s) = \lim_{s \to 0} s\left[R(s) - C(s)\right] = \lim_{s \to 0} s\left[R(s) - T(s)R(s)\right]$

$$= \lim_{s \to 0} s\,R(s)\left[1 - T(s)\right] = A \lim_{s \to 0}\left[1 - T(s)\right]$$

$$= A\left[1 - \frac{1}{3}\right] = \underline{\frac{2}{3}A}$$

FOR WHAT VALUES OF K ARE THE FOLLOWING CLOSED-LOOP SYS. STABLE?

a) $\frac{G}{1+GH} = \frac{1}{s^5 - s^4 + 2s^3 + 3s^2 + (5-K)s + (2+K)}$

$$\begin{array}{c|ccc} s^5 & 1 & 2 & (5-K) \\ s^4 & -1 & 3 & (2+K) \\ s^3 & 5 & 7 & 0 \\ s^2 & \frac{22}{5} & 2+K & 0 \\ s^1 & 4.73-1.14K & 0 & 0 \\ 1 & 2+K & & \end{array}$$

$c_1 = \frac{-1}{-1}\begin{vmatrix} 1 & 2 \\ -1 & 3 \end{vmatrix} = 5$

$c_2 = \frac{-1}{-1}\begin{vmatrix} 1 & (5-K) \\ -1 & (2+K) \end{vmatrix} = (2+K)+(5-K) = 7$

$d_1 = -\frac{1}{5}\begin{vmatrix} -1 & 3 \\ 5 & 7 \end{vmatrix} = \frac{22}{5}$

$d_2 = -\frac{1}{5}\begin{vmatrix} -1 & 2+K \\ 5 & 0 \end{vmatrix} = \frac{--5(2+K)}{5} = 2+K$

$e_1 = \frac{-1}{22/5}\begin{vmatrix} 5 & 7 \\ \frac{22}{5} & 2+K \end{vmatrix} = \frac{(10+5K) - \frac{154}{5}}{-\frac{22}{5}}$

$4.73 - 1.14K$

$e_2 = \frac{-1}{4.4}\begin{vmatrix} 5 & 0 \\ 4.4 & 0 \end{vmatrix} = 0$

$f_1 = \frac{-1}{4.73-1.14K}\begin{vmatrix} 4.4 & 2+K \\ 4.73-1.14K & 0 \end{vmatrix} = 2+K$

$\boxed{\text{DESPITE K SYS. ALREADY HAS 2 SIGN CHANGES SYS IS UNSTABLE}}$ ✓

b) $\frac{G}{1+GH} = \frac{1}{s^3 + 8s^2 + 17s + K}$

$$\begin{array}{c|cc} s^3 & 1 & 17 \\ s^2 & 8 & K \\ s & 17-\frac{K}{8} & 0 \\ 1 & K & \end{array}$$

$b_1 = \frac{-1}{8}\begin{vmatrix} 1 & 17 \\ 8 & K \end{vmatrix} = 17 - \frac{K}{8}$

$c_1 = \frac{-1}{17-\frac{K}{8}}\begin{vmatrix} 8 & K \\ 17-\frac{K}{8} & 0 \end{vmatrix} = K$

$K > 0$

$17 - \frac{K}{8} = 0$

$17(8) = K$

$136 = K$

$\boxed{136 \geq K > 0}$ ✓

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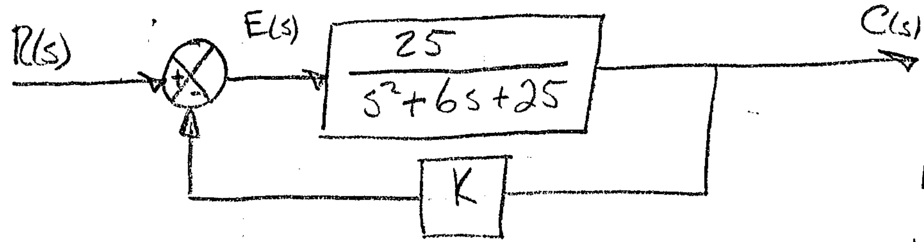
GIVEN THE FOLLOWING FEEDBACK CONTROL SYS.
CALC. THE FOLLOWING PARAMETERS FOR THE CLOSED LOOP SYS.



$K = 3$

$r(t) = A u(t) =$ step input

$R(s) = \frac{A}{s}$

$$T_F(s) = \frac{\frac{25}{s^2+6s+25}}{1+\frac{75}{s^2+6s+25}} = \frac{25}{s^2+6s+100}$$

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$\boxed{\omega_n = \sqrt{100} = 10}$$

$$2\zeta(10) = 6$$
$$\zeta = \frac{6}{20} = 3/10 = \zeta$$

$$\tau = \frac{1}{\zeta\omega_n} \quad \boxed{2\% = 4\tau = \frac{4}{\frac{3}{10}(10)} = \frac{4}{3} \text{ sec.}}$$

$$C_{SS} = \lim_{s\to 0} s\,C(s) = \lim_{s\to 0} s\left(\frac{25}{s^2+6s+100}\right)\frac{A}{s} = \lim_{s\to 0}\frac{25A}{s^2+6s+100} = \frac{25A}{100} \quad \boxed{\frac{A}{4} = C_{ss}}$$

$$\frac{C_{MAX} - C_{SS}}{C_{SS}}(100) = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}} = 100\, e^{-\pi .3/\sqrt{1-.3^2}}$$

$$C_{MAX} - C_{SS} = .3723\left(\frac{A}{4}\right)$$

$$\boxed{C_{MAX} = \frac{A}{4} + .09\cancel{3}1A}$$

$$E_{SS} = \lim_{s\to 0} s\,E(s) \qquad E(s) = R(s) - k\,C(s) \qquad \frac{C(s)}{R(s)} = \frac{25}{s^2+6s+100}$$

$$\lim_{s\to 0} s\left(\frac{A}{s} - \frac{75}{s^2+6s+100}\right) = A - \frac{75s}{s^2+6s+100} = A \qquad C(s) = \frac{25}{s^2+6s+100}R(s)$$

$$\boxed{E_{SS} = A} \quad .75A$$

EML 4312
Automatic Control Systems

Homework Set #9
Spring 1988

Due: 12 April 1988

Problem 8-1

Sketch the "Root Locus" for the systems represented by the following open-loop transfer functions. Calculate and label all pertinent information (i.e. break-away points, angle of asymptotes, number of branches, angle of departure/arrival at complex poles/zeros, etc.).

(a) $GH(s) = \dfrac{K(s + 9)}{(s + 1)(s + 5)}$

(b) $GH(s) = \dfrac{K(s - 9)}{(s + 1)(s + 5)(s + 7)(s + 10)}$

(c) $GH(s) = \dfrac{K}{(s^2 + 2s + 2)(s + 10)}$

(d) $GH(s) = \dfrac{K(s + 10)}{(s^2 + 2s + 2)}$

NOTE: For part (b) only, do not calculate break-away or arrival points.

REMEMBER WHAT WAS SAID ABOUT GH & ITS RELATION TO $K\dfrac{Z(s)}{P(s)}$

PROB. 8-2 DO 8.5 IN PALM. LET $\Delta = I-20$ IN YOUR CALCULATIONS

PROB. 8-3 DO 8.12 IN PALM. LET $\Delta = \zeta - .4$ IN YOUR CALCULATIONS IN (b)

$\Delta = p - .5$ IN YOUR CALCULATIONS IN (c)

Problem 8-2

For each of the transfer functions given below, construct the approximate Bode plots (i.e. asymptotic magnitude (db) and phase angle vs. frequency). In addition, calculate the exact magnitude (db) and phase angle at frequencies of w=1 rad/sec and w=10 rad/sec. Compare the exact results with those obtained using the approximate Bode plots.

(a) $GH(s) = \dfrac{250(s + 2)}{(s + 5)(s + 10)}$

(b) $GH(s) = \dfrac{5(s + 5)}{s(s + 1)(s + 10)}$

(c) $GH(s) = \dfrac{800(s + 6)}{(s + 2)(s^2 + 10s + 100)}$

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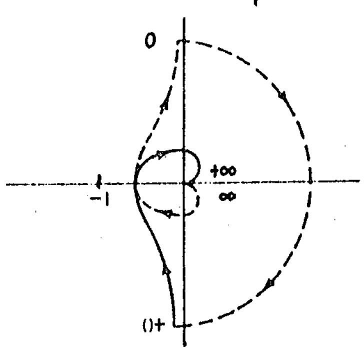
Automatic Control Systems

Homework Set #10
Spring 1988

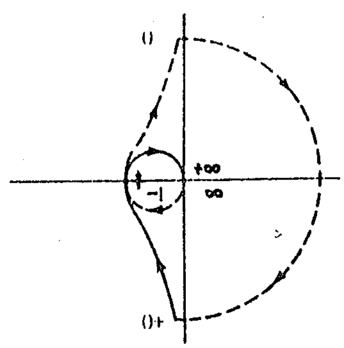
Due: 19 April 1988

Problem 10-1

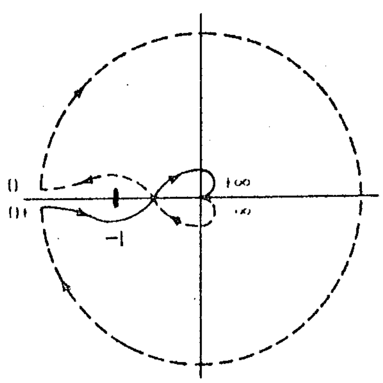
a) For each of the Nyquist diagrams sketched below determine the number of encirclements of the −1 point. With the number of poles in the right half-plane as shown with the figure, describe the stability of each system.
b) Determine which are minimum phase fns. assuming there are no zeros in RH plane.



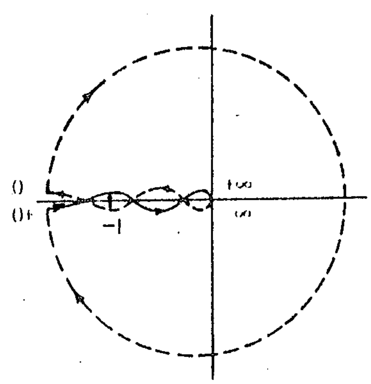
(a) $N_p = 0$



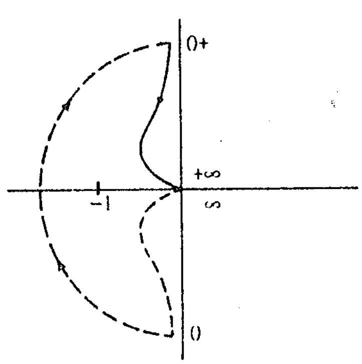
(b) $N_p = 0$



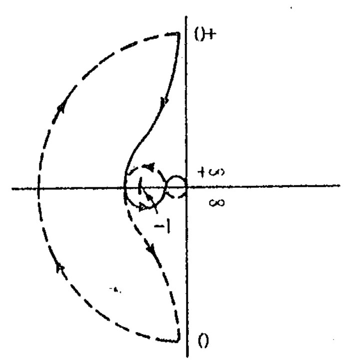
(c) $N_p = 0$



(d) $N_p = 0$



(e) $N_p = 1$



(f) $N_p = 1$

Problem 10-2

The open-loop transfer functions for several feedback control systems are given below. For each system, describe the open-loop stability, sketch the Nyquist diagram for positive values of K, and describe the closed-loop system stability.

(a) $GH(s) = \frac{K(s - 4)}{s(s + 6)}$

(b) $GH(s) = \frac{K(s + 3)}{(s + 10)(s + 5)(s + 2)}$

(c) $GH(s) = \frac{K(s + 4)}{s(s + 6)(s - 2)}$

(d) $GH(s) = \frac{K}{(s^2 + 4)(s + 2)}$

(e) $GH(s) = \frac{K}{s^2(s + 2)(s + 5)}$

EML 4312
Automatic Control Systems

Homework Set #10
Summer 1986

Due: 11 August 1986

Problem 10-1

Given the following transfer function

$$GH(s) = \frac{K(s + 12)}{(s + 1)(s + 2)(s + 6)}$$

(a) Construct the approximate Bode plots for a gain constant K = 20.

(b) Determine the gain margin and phase margin from the Bode plots.

(c) Describe the system stability.

(d) Find the new phase margin and gain margin if the gain constant is increased to K = 40.

e) Discuss the relative stability of the system for K = 40 vs K = 20.

f) For what value of gain K does the system become unstable? Confirm your results using the Routh-Hurwitz Criteria.