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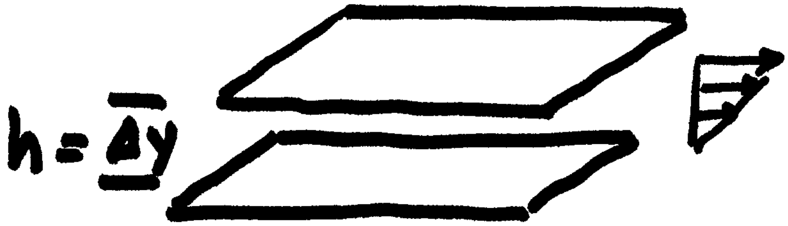
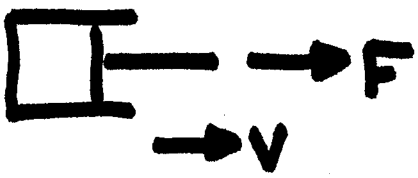


$$\tau = \mu \frac{du}{dy}$$

if TOP $u = V$
BOTTOM $u = 0$
$\Delta u = V$

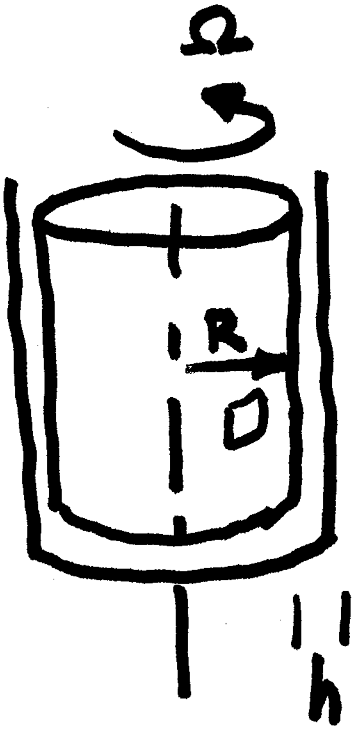
$$\tau = \frac{\mu V}{h}$$

$$F_{\text{TOP PLATE}} = \tau A = \frac{\mu V}{h} A$$



$F = cV$ (EQUIV. DAMPER)

$$c_{eq} = \frac{\mu A}{h}$$



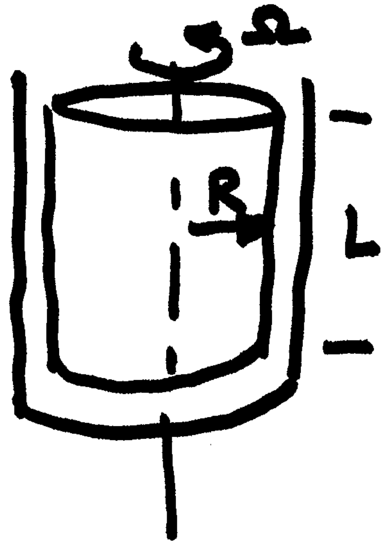
$$\tau = \frac{\mu du}{dy}$$

NEGLECT BOTTOM

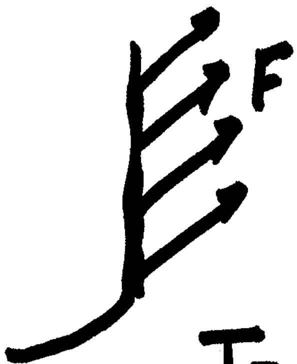
$$\text{Torque} = c_{eq}\Omega$$

$$du = \Delta u = V = \Omega R$$

$$\tau = \mu \frac{du}{dy} = \frac{\mu V}{h} = \frac{\mu \Omega R}{h}$$



$\text{Torque} = FR$



$F = \tau \cdot 2\pi R L$

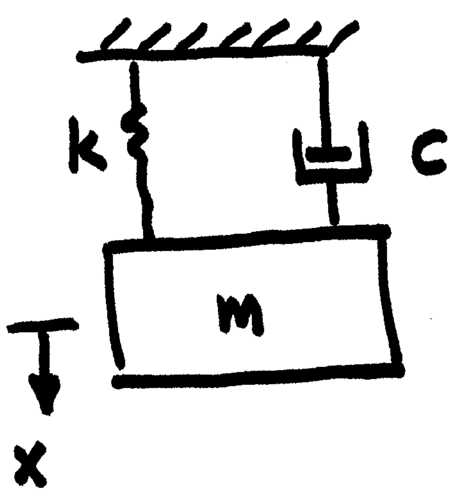
$\text{Torque} = FR = \tau \cdot 2\pi R^2 L$

$$T = \mu \frac{\Omega R}{h} \cdot 2\pi R^2 L$$

$$T = \frac{2\mu\pi R^3 L}{h}\,\Omega$$

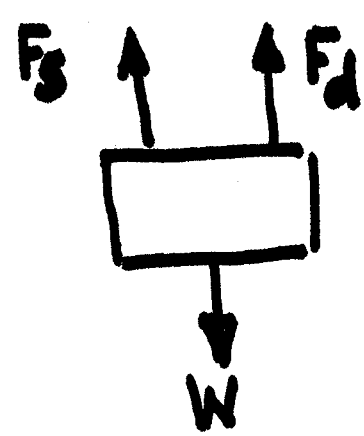
$$T = c_{eq}\Omega$$

$$c_{eq} = \frac{2\pi\mu R^3 L}{h}$$



$$\delta_{st} = \frac{W}{k}$$

Free Body Diag



$$+\!\downarrow \Sigma F = m\ddot{x} = W - F_s - F_d$$

$$m\ddot{x} = W - k(x+\delta_{st}) - c(\dot{x}+\dot{\delta}_{st})$$

$$= \underline{\underline{W}} - k(x+\underline{\underline{\delta_{st}}}) - c\dot{x}$$

$$= -kx - c\dot{x}$$

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \leftarrow \quad x(t) = \mathscr{C}e^{pt}$$

$$(mp^2 + cp + k)\,\mathscr{C}e^{pt} = 0$$

$$p = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

$c^2 - 4km = 0 \quad \Longrightarrow$ critically damped

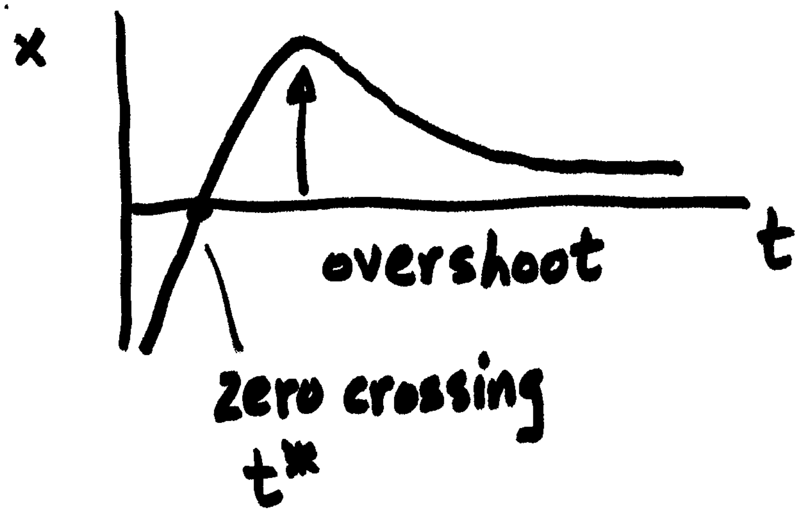
$$c_c = 2\sqrt{km} = 2m\sqrt{\frac{k}{m}} = 2m\omega_n$$

$\frac{c}{c_c} = \zeta$ (ZETA) DAMPING RATIO

$$p = -\frac{c}{2m} = -\frac{c}{c_c}\cdot\frac{2m\omega_n}{2m} = -\zeta\omega_n \qquad \zeta = 1$$

$$p = -\omega_n$$

$$x(t) = (\mathscr{C}_1 + t\mathscr{C}_2)e^{pt} = (\mathscr{C}_1 + \mathscr{C}_2 t)e^{-\omega_n t}$$



zero crossing

$$\mathcal{C}_1 + \mathcal{C}_2 t^* = 0$$

$$t^* = -\mathcal{C}_1 / \mathcal{C}_2$$

$$\left.\begin{array}{l} x(t=0) = x_0 \\ \dot{x}(t=0) = v_0 \end{array}\right\} \rightarrow x = (\mathcal{C}_1 + \mathcal{C}_2 t)e^{-\omega_n t}$$

$$\mathcal{C}_1 = x_0$$

$$\mathcal{C}_2 - \mathcal{C}_1 \omega_n = v_0$$

$$\mathcal{C}_2 = v_0 + x_0 \omega_n$$

$$c^2 - 4km > 0 \qquad p = \frac{-c \pm \sqrt{c^2 - 4km}}{2m}$$

$$p_1 = \frac{-c + \sqrt{}}{2m} \qquad p_2 = \frac{-c - \sqrt{}}{2m}$$

$$\left(\frac{c^2 - 4km}{c_c^2}\right) c_c^2 = (\zeta^2 - 1) \cdot 4m^2 \omega_n^2 \qquad \frac{\sqrt{c^2 - 4km}}{2m}$$

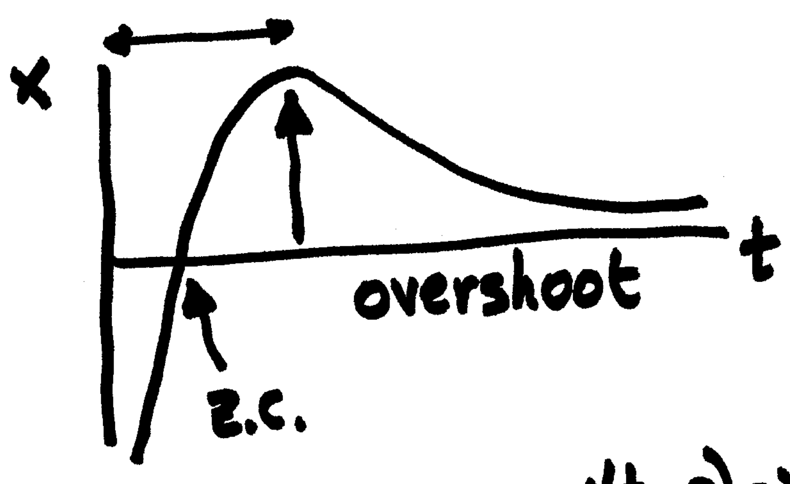
$$\frac{\sqrt{c^2-4km}}{2m} = \sqrt{\zeta^2-1}\,\omega_n$$

$$p_1 = \frac{-c+\sqrt{\quad}}{2m} = \left(-\zeta+\sqrt{\zeta^2-1}\right)\omega_n$$

$$p_2 = \frac{-c-\sqrt{\quad}}{2m} = \left(-\zeta-\sqrt{\zeta^2-1}\right)\omega_n$$

overdamped case $c > c_c$ $\zeta > 1$

$$x = \mathcal{C}_1 e^{p_1 t} + \mathcal{C}_2 e^{p_2 t}$$



overshoot

$$\frac{dx}{dt} = 0 \Rightarrow t_{os}$$

$$x(t=0) = x_0 = \mathcal{C}_1 + \mathcal{C}_2$$

$$\dot{x}(t=0) = v_0 = \mathcal{C}_1 p_1 + \mathcal{C}_2 p_2$$

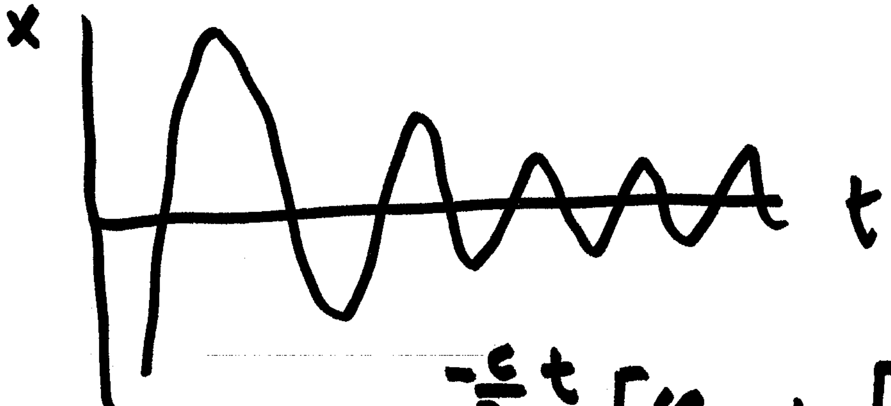
$$x(t^*) = \mathcal{C}_1 e^{p_1 t^*} + \mathcal{C}_2 e^{p_2 t^*} = 0$$

$$\mathcal{C}_1/\mathcal{C}_2 = e^{(p_2-p_1)t^*}$$

$$\frac{\ln\left(\mathcal{C}_1/\mathcal{C}_2\right)}{p_2 - p_1} = t^* \quad \text{z.c. time}$$

---

underdamped case $\quad c < c_c \quad \zeta < 1$



$$m\ddot{x} + c\dot{x} + kx = 0$$

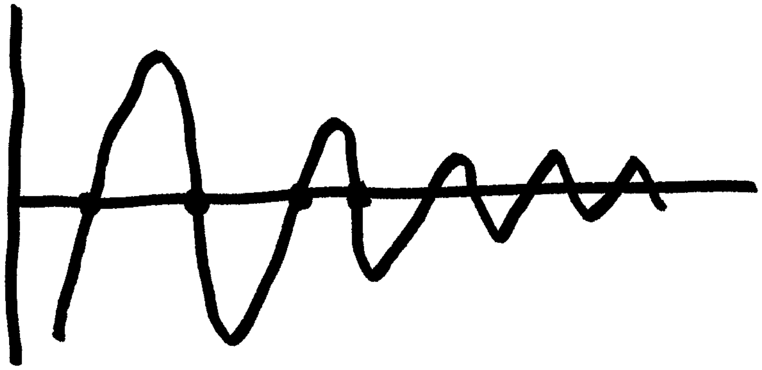
$$x(t) = e^{-\frac{c}{2m}t}\left[\mathcal{C}_1 \sin\frac{\sqrt{4km - c^2}}{2m}t + \mathcal{C}_2 \cos\frac{\sqrt{4km - c^2}}{2m}t\right]$$

$$\frac{c}{2m} = \zeta\omega_n \qquad \frac{\sqrt{4km - c^2}}{2m} = \sqrt{1 - \zeta^2}\,\omega_n$$

$$x(t) = e^{-\zeta\omega_n t}\left[\mathcal{C}_1 \sin\sqrt{1-\zeta^2}\,\omega_n t + \mathcal{C}_2 \cos\sqrt{1-\zeta^2}\,\omega_n t\right]$$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$

$$x(t) = e^{-\zeta\omega_n t}\left[A\sin(\omega_d t + \phi)\right]$$



$\omega_d t + \phi = n\pi$ (zero crossings)

$t = \dfrac{n\pi - \phi}{\omega_d}$ $\quad n=1 \Rightarrow 1^{st}$ zero cross

$$\dot{x}(t) = 0 = Ae^{-\zeta\omega_n t}\left[-\zeta\omega_n \sin(\omega_d t + \phi) + \omega_d \cos(\omega_d t + \phi)\right]$$

$\tan(\omega_d t^* + \phi) = \dfrac{\omega_d}{\zeta\omega_n} = \dfrac{\sqrt{1-\zeta^2}}{\zeta}$

S	A
T	C

1, $\sqrt{1-\zeta^2}$, ζ

$\sin(\omega_d t + \phi) = \sqrt{1-\zeta^2}$

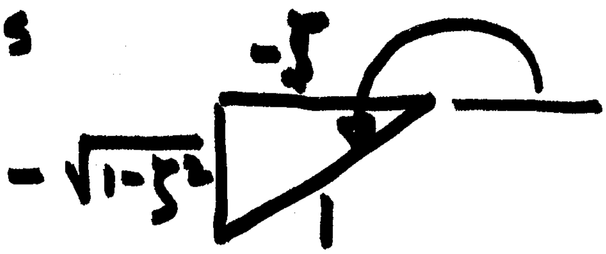
max's occur when

$$x = e^{-\zeta\omega_n t^*} A \sin(\omega_d t + \phi)$$

$$= e^{-\zeta\omega_n t^*} A \sqrt{1-\zeta^2} \qquad \sin(\omega_d t^* + \phi) = \sqrt{1-\zeta^2}$$

$$\tan(\omega_d t^* + \phi) = \frac{\sqrt{1-\zeta^2}}{\zeta}$$

FOR MINS



$$\sin(\omega_d t^* + \phi) = -\sqrt{1-\zeta^2}$$

$$x(t) = -A e^{-\zeta\omega_n t^*} \sqrt{1-\zeta^2}$$

U.D. $\zeta<1$

C.D. $\zeta=1$

O.D. cases $\zeta>1$

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C.D. $x = e^{-\omega_n t}(C_1 + C_2 t)$ $\zeta = 1$

O.D. $x = C_1 e^{p_1 t} + C_2 e^{+p_2 t}$ $\zeta > 1$

U.D. $x = A e^{-\zeta \omega_n t}[\sin(\omega_d t + \phi)]$ $\zeta < 1$

$p_1 = (-\zeta + \sqrt{\zeta^2 - 1})\omega_n$

$p_2 = (-\zeta - \sqrt{\zeta^2 - 1})\omega_n$

$\omega_d = \sqrt{1 - \zeta^2}\,\omega_n$ $\omega_d < \omega_n$

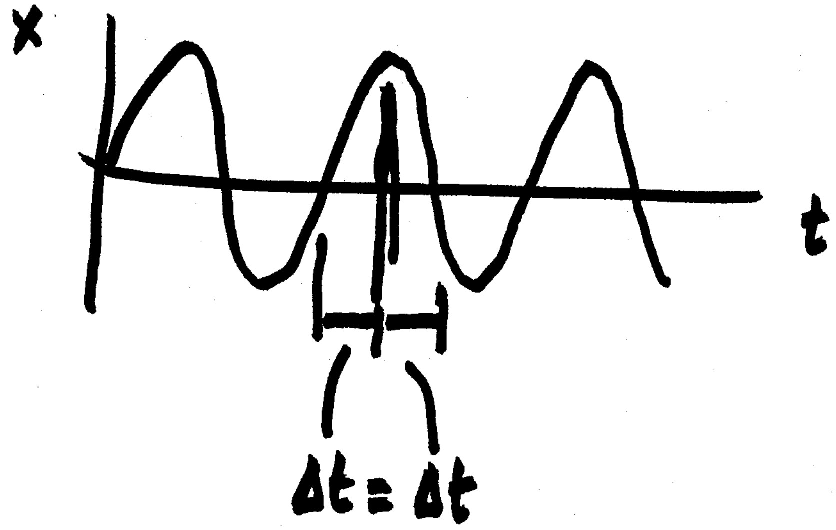
UNDAMPED SYSTEM $x = A\sin(\omega_n t + \phi)$

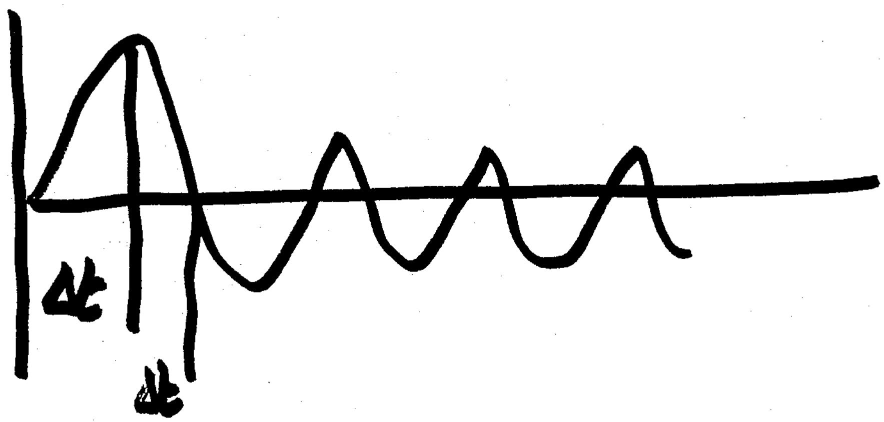
1) $\omega_d < \omega_n$

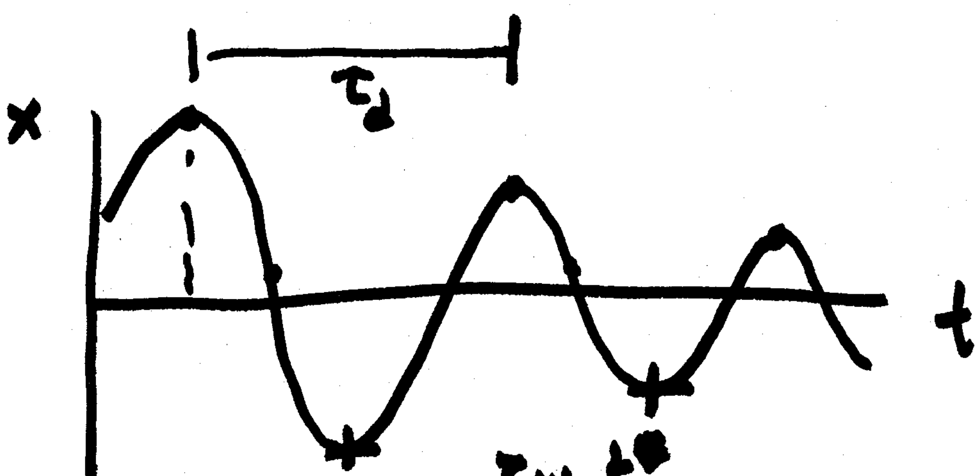
2) AMPLITUDE ↓ AS A FN. OF TIME

3) $\omega \tau = 2\pi \Rightarrow \tau_d > \tau_n = \dfrac{2\pi}{\omega_n}$

$$\frac{2\pi}{\omega_n \sqrt{1-\zeta^2}}$$







at max $\quad x_j(t^*) = Ae^{-\zeta\omega_n t^*} \sqrt{1-\zeta^2}$

$x_{j+1}(t^*) = Ae^{-\zeta\omega_n (t+\tau_d)} \sqrt{1-\zeta^2}$

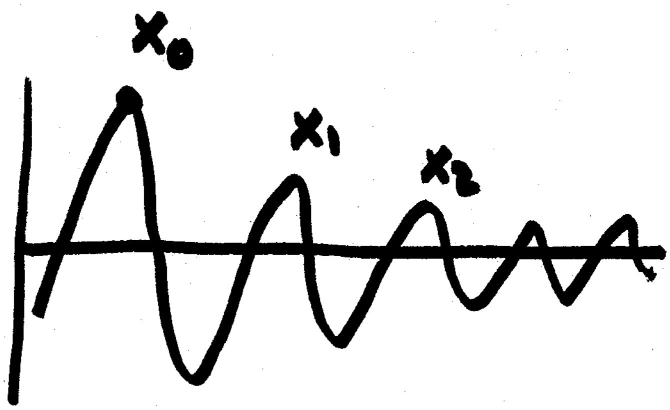
$$\frac{x_j}{x_{j+1}} = e^{+\zeta \omega_n \tau_d}$$

$$\omega_d \tau_d = 2\pi$$

$$\delta = \ln\left(\frac{x_j}{x_{j+1}}\right) = \zeta \omega_n \tau_d = \zeta \omega_n \frac{2\pi}{\omega_d}$$

LOG. DECREMENT

$$= \zeta \omega_n \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$



what is x_n

$$\delta = \ln \frac{x_0}{x_1} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\frac{x_0}{x_n}\,? = \frac{x_0}{x_1}\cdot\frac{x_1}{x_2}\cdot\frac{x_2}{x_3}\cdot \ldots \cdot \frac{x_{n-1}}{x_n} = \left(\frac{x_0}{x_1}\right)^n$$

$$\ln\left(\frac{x_0}{x_n}\right) = \text{[illegible]}\ n \ln\left(\frac{x_0}{x_1}\right) = n\delta$$

$$c_c = 2\sqrt{mk} \qquad k_{eq} = 2\,(40\,N/mm) = 80000\,\frac{N}{m}$$

$$m = 1\ kg$$

$$c_c = 2\sqrt{1.8\times10^4} = 565.69\ N\text{-}s/m \ \text{ or } \ .566\,\frac{N\text{-}s}{mm}$$

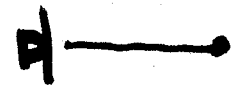
$$\frac{c}{c_c} = \frac{.2}{.566} = .354 = \zeta < 1 \qquad \text{UNDERDAMPED}$$

$$x = \underline{\underline{A}}\,e^{-\zeta\omega_n t}\ \sin(\omega_d t + \underline{\underline{\phi}})$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{80000}{1}} = 282.84\ rad/s$$

$$x(t=0) = 0$$

$$\dot{x}(t=0) = 80\,m/s$$



$$x = A \cdot 1 \sin\phi = 0 \qquad \sin\phi = 0$$

$$\dot{x} = A\left[-\zeta\omega_n \sin(\omega_d t + \phi) + \omega_d \cos(\omega_d t + \phi)\right]e^{-\zeta\omega_n t}$$

$$\dot{x}(t=0) = \underbrace{-\zeta\omega_n A\sin\phi}_{=0} + \omega_d A \underbrace{\cos\phi}_{1} = 80 \qquad \underline{\underline{\omega_d A = 80}}$$

$$\omega_n\sqrt{1-\zeta^2}\,A = 80 \implies A = .302\ m$$

$$\ln\left(\frac{X_0}{X_n}\right) = n \cdot \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

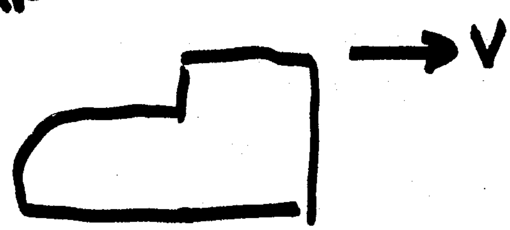
How much time to reach X_n

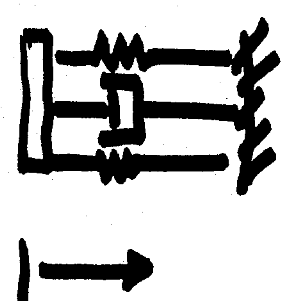
$$\delta = \ln\left(\frac{X_j}{X_{j+1}}\right) = \zeta\omega_n\tau_d$$

$$n\tau_d = \Delta t = \frac{1}{\zeta\omega_n} n\delta = \frac{1}{\zeta\omega_n}\ln\left(\frac{X_0}{X_n}\right)$$

PG 183 FIG 2.92 PROBL 2.82 PG 182

SIMILAR TO





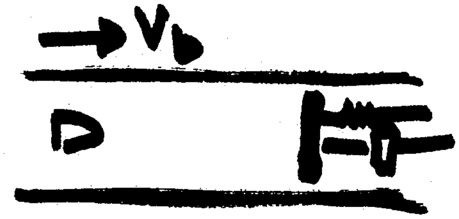
X_{max}, what is t for X_{max}

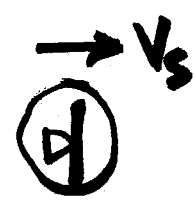
$V = 80$ m/s

$K = 40$ N/mm

$C = .2$ N-s/mm

$m = 1$ kg





$$m_b V_b = (m_b + m_s) V_s \qquad m_s \ll m_b$$

$$V_s = V_b$$

$\dot{x}(t)=0 \Rightarrow \tan(\omega_d t+\phi)=\dfrac{\sqrt{1-\zeta^2}}{\zeta}$

veloc must be zero at max. disp. $\Rightarrow \sin(\omega_d t+\phi)=\sqrt{1-\zeta^2}$

$$\frac{\sin^{-1}\left(\sqrt{1-\zeta^2}\right)-\phi}{\omega_n\sqrt{1-\zeta^2}}=t$$

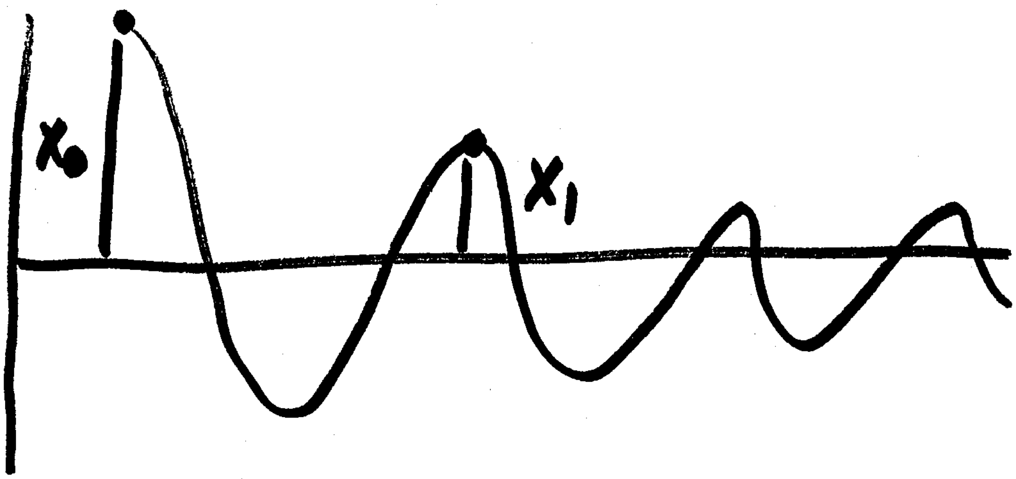
$\phi=0 \quad \underline{\zeta=.354} \quad \underline{\omega_n=282.84\ rad/s}$

$t=.00457\ sec$

$\omega_d=\omega_n\sqrt{1-\zeta^2}=264.57\ rad/s$, $A=.302m$

$x=Ae^{-\zeta\omega_n t}\sin(\omega_d t+\phi)$

$=.179m$



origi $\frac{x_0}{x_1} = 12$ if $\zeta_{new} = 2\zeta_{orig}$ WHAT is x_1

$$\ln\left(\frac{x_0}{x_1}\right) = \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \quad ; \quad \zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

$\delta = 2.485$ → $\zeta_{orig} = .368$

$$\zeta_{new} = 2\zeta_{orig} = .736$$

$$\delta_{new} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 6.822 \qquad \delta = \ln\left(\frac{x_0}{x_1}\right)$$

$$x_1 = x_0 e^{-\delta} = x_0/917.5 \Rightarrow \frac{x_0}{x_1} = 917.5$$

IF WE LOOK AT ENERGY DISSIPATED BY DAMPER IN ONE CYCLE

$$\Delta W = \int_0^{\tau_d} c\dot{x}\, d(\dot{x}) = \frac{c\dot{x}^2}{2}\Big|_0^{\tau}$$

$$x = A\sin(\omega_d t + \phi) \nearrow$$

$$\Delta W = c\omega_d \pi A^2$$

$$W = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}mA^2\omega_d^2$$

$$\frac{\Delta W}{W} = \frac{2c\pi}{m\omega_d} = \qquad\qquad \frac{c}{2m} = \zeta\omega_n$$

$$= \frac{4\pi\zeta\omega_n}{\omega_n\sqrt{1-\zeta^2}} = \frac{4\pi\zeta}{\sqrt{1-\zeta^2}}$$

FOR SMALL ζ $\quad \frac{\Delta W}{W} \cong 4\pi\zeta$ SPECIFIC DAMPING CAPACITY

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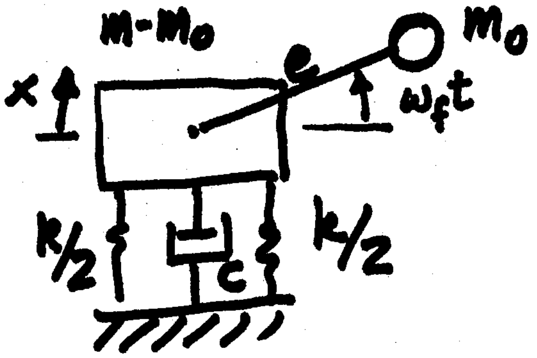
ROTATING UNBALANCE

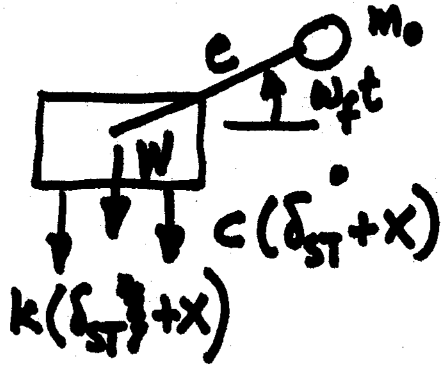


TOTAL SYSTEM MASS = m_o

$$\delta_{st} = \frac{mg}{k}$$

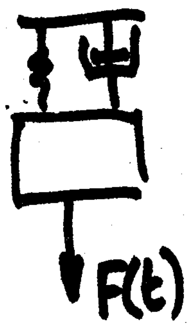
X: IS THE DISPL MEASURED FROM EQUIL OF ~~MAIN~~ TOTAL MASS

X: DEFINE FORCED MOTION OF $m - m_o$



$$(m-m_o)\frac{d^2x}{dt^2} + m_o\frac{d^2}{dt^2}(x + e\sin\omega_f t)$$

$$= -k(\delta_{st} + x) - c(\dot{\delta}_{st} + \dot{x}) - W$$



$$m\ddot{x} + c\dot{x} + kx = \underbrace{m_o e\omega_f^2}_{P_o}\sin\omega_f t$$

$$m\ddot{x} + c\dot{x} + kx = P_o \sin\omega_f t$$

$$x_{ss} = x_p = \frac{P_o}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}\sin(\omega_f t - \psi)$$

$$\tan\psi = \frac{c\omega_f}{k - m\omega_f^2} = \frac{2\zeta r}{1 - r^2} \qquad \zeta = \frac{c}{c_c} \qquad r = \frac{\omega_f}{\omega_n}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

FOR ROTATING UNBALANCE

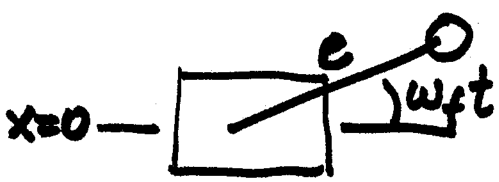
$$x_p = \frac{m_0 e \omega_f^2}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \sin(\omega_f t - \psi)$$

$$\tan\psi = \frac{c\omega_f}{k - m\omega_f^2}$$

TOTAL DISPL = $x_{TRANSIENT} + x_p$

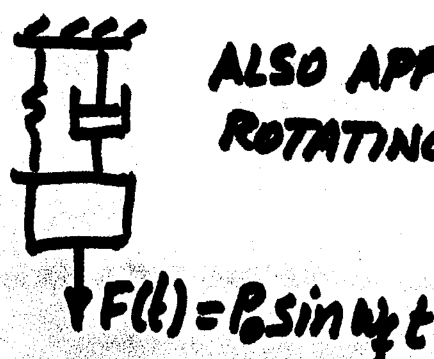
OVERDAMPED C.D. UNDERDAMPED

FREE VIBS FOR A DAMPED SYSTEM



WHEN MAIN MASS PASSES THROUGH $x_p = 0 \Rightarrow \psi = \omega_f t$

- THE SAME GRAPHS THAT DEFINE HOW δ VARIES WITH m & k FOR



ALSO APPLIES FOR ROTATING UNBALANCE

- VARIATION OF δ DUE TO VARIATIONS IN ω_f IS NOT THE SAME

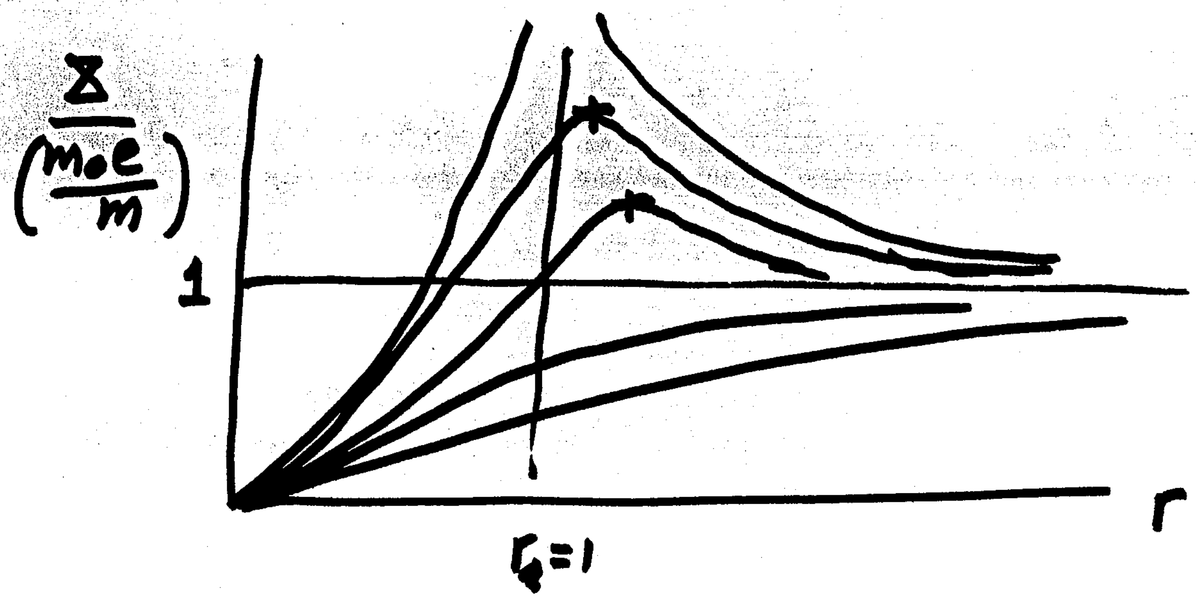
$$x_p = \frac{m_o e \omega_f^2}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \sin(\omega_f t - \psi)$$

$$= \frac{m_o e \omega_f^2 / k \cdot \frac{m}{m}}{\sqrt{\left(1 - \frac{m}{k}\omega_f^2\right)^2 + \left(\frac{c\omega_f}{k}\right)^2}}$$

$$= \underbrace{\left\{ \frac{m_o e}{m} \bullet \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \right\}}_{X_{RU}} \sin(\omega_f t - \psi)$$

$$\frac{X_{RU}}{(m_o e/m)} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\frac{X}{X_o = P_o/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$



- FOR VERY LARGE ω_f $(r \to \infty)$ ALL CURVES TEND TO $\dfrac{X_{RU}}{(m_o e/m)} = 1$

- $\dfrac{d}{dr}\left(\dfrac{X_{RU}}{m_o e/m}\right) = 0 \Rightarrow$ max. OCCURS

  WHEN $r = \dfrac{1}{\sqrt{1-2\zeta^2}} > 1$

  ONLY TRUE UNTIL $\zeta = \dfrac{1}{\sqrt{2}}$

  r HAS LOCAL MAX WHEN $\zeta \leq \dfrac{1}{\sqrt{2}}$

  FOR $\zeta > \dfrac{1}{\sqrt{2}}$ NO <u>LOCAL</u> MAX

$$X_{RU_{max}} = \frac{m_o e}{m} \cdot \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

THIS IS FOR LOCAL MAXES $\zeta \leq \dfrac{1}{\sqrt{2}}$

$$X_{RU}\Big|_{r=1} = \frac{m_o e}{m} \cdot \frac{1}{2\zeta}$$

$$X_{RU_{max}} = X_{RU}\Big|_{r=1} \cdot \frac{1}{\sqrt{1-\zeta^2}}$$

\# 3.43 on PG 256/257

$m_o = 0.5$ kg

$e = 0.15$ m

$$k_{eq} = \frac{3EI}{L^3}$$

m_T = MASS OF TAIL = [illegible] = 240 kg

TO FIND THE MASS $(m - m_o)$

MASS OF ROTOR BLADES + DRIVE SYSTEM =

$m_1 = 20$ kg

MUST ADD THE EFFECT MASS DUE TO TAIL AS IT ACTS AS EFFECTIVE SPRING

$$m = m_1 + 0.25\, m_T = 20 + 0.25(240) = 80 \text{ kg}$$

$$m - m_o = 79.5 \text{ kg}$$

1) $k_{eq} = \dfrac{3EI}{\ell^3} = \dfrac{3(2.5\times10^6)}{4^3} = 1.172\times10^5$ N/m

2) $\omega_n = \sqrt{\dfrac{k_{eq}}{m}} = 38.28$ rad/s

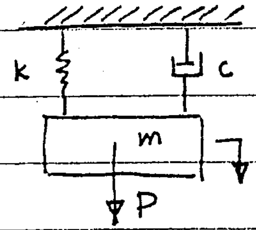
$$\omega_f = 2\pi f_f = 2\pi\,(1500\ \text{rpm})\,\frac{1\,\text{min}}{\text{sec}}\cdot\frac{1}{2\pi} =$$

$$r = \frac{\omega_f}{\omega_n} = 4.1 \qquad \zeta = 0.15$$

$$\frac{m_0 e}{m} = \frac{(0.5)(0.15)}{80}\left(\frac{r^2}{\sqrt{(1-r^2)^2(2\zeta r)^2}}\cdot\frac{m_0 e}{m}\right) = .00397\ \text{m}$$

SAMPLE HANDOUT

FORCED MOTION OF M-K-C SYSTEM DUE TO P(t)



①

DE: $m\ddot{x} + c\dot{x} + kx = P_0 \sin\omega_f t = P(t)$

STEADY STATE SOLUTION: $x_p = \dfrac{P_0/k}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t - \psi)$

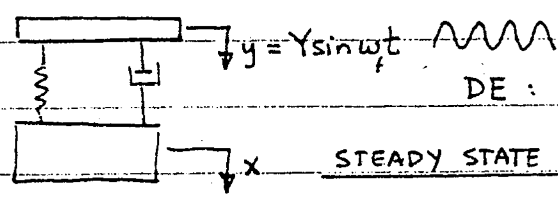
or $x_p = X \sin(\omega_f t - \psi)$ with $\psi = \tan^{-1}\left(\dfrac{2\zeta r}{1-r^2}\right)$

TRANSMITTED FORCE TO SUPPORT:

$F_T \sin(\omega_f t - \beta - \psi) = F_{TRAN} = kx + c\dot{x} = \dfrac{P_0 \sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t - \beta - \psi)$

$\beta = \tan^{-1}(-2\zeta r)$

FORCED MOTION OF M-K-C SYSTEM DUE TO OSCILLATING SUPPORT



②

DE: $m\ddot{x} + c\dot{x} + kx = ky + c\dot{y} = Yk\sqrt{1+(2\zeta r)^2} \sin(\omega_f t - \beta$

STEADY STATE SOLUTION:

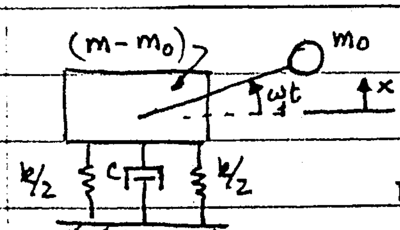
$x_p = \dfrac{Y\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t - \beta - \psi) = X_{OS} \sin(\omega_f t - \beta - \psi$

TRANSMITTED FORCE TO OSCILLATING BASE:

$F_{TRAN} = k(x-y) + c(\dot{x}-\dot{y}) = \dfrac{Yk\, r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t - \beta - \psi) = F_{T_{OS}} \sin(\omega_f t - \beta - \psi$

RELATIVE DISPLACEMENT (motion of mass relative to base): $z = (x-y)$

$z_p = \dfrac{Y\, r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t - \beta - \psi) = Z \sin(\omega_f t - \beta - \psi$



FORCED MOTION OF M-K-C SYSTEM DUE TO THE ECCENTRIC MASS m_0

③

DE: $m\ddot{x} + c\dot{x} + kx = m_0 e \omega^2 \sin\omega_f t$

STEADY STATE SOLUTION:

$x_p = X_{RU} \sin(\omega_f t - \psi) = \dfrac{m_0 e}{m} \dfrac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t - \psi)$

TRANSMITTED FORCE TO THE SUPPORT:

$$F_{TRANS} = kx + c\dot{x} = F_{T_{RU}} \sin(\omega_f t - \beta - \psi) = \frac{m_0 e}{m} k \frac{r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t - \beta - \psi)$$

$$F_T = k X \sqrt{1+(2\zeta r)^2}$$

NOTE THAT $MF = \frac{X}{X_0} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ from ①

$$TR = \left(\frac{F_T}{P_0}\right)_① = \left(\frac{X_{OS}}{Y}\right)_② = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$\left(\frac{F_{T_{OS}}}{Yk}\right)_② = \left(\frac{F_{T_{RU}}}{\frac{m_0 e}{m} k}\right)_③ = \frac{r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$\left(\frac{Z}{Y}\right)_② = \left(\frac{X_{RU}}{\frac{m_0 e}{m}}\right)_③ = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

Even though we looked at 3 different phenomena, the resulting DE's displacements, transmitted forces are similar in form.

OS - oscillating support

RU - rotating unbalance

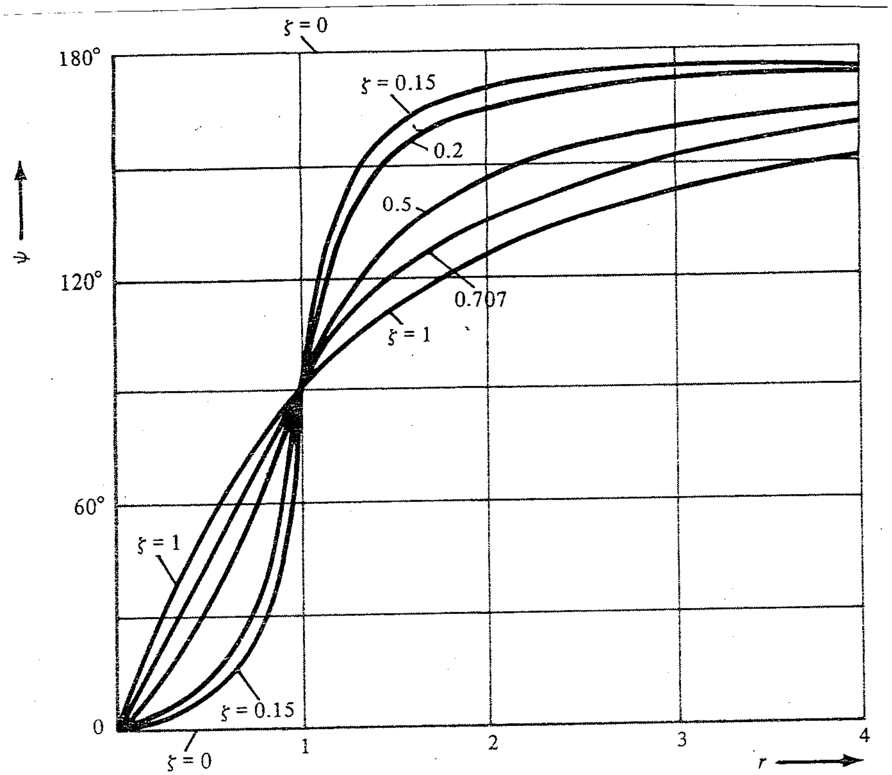


Figure 4-16

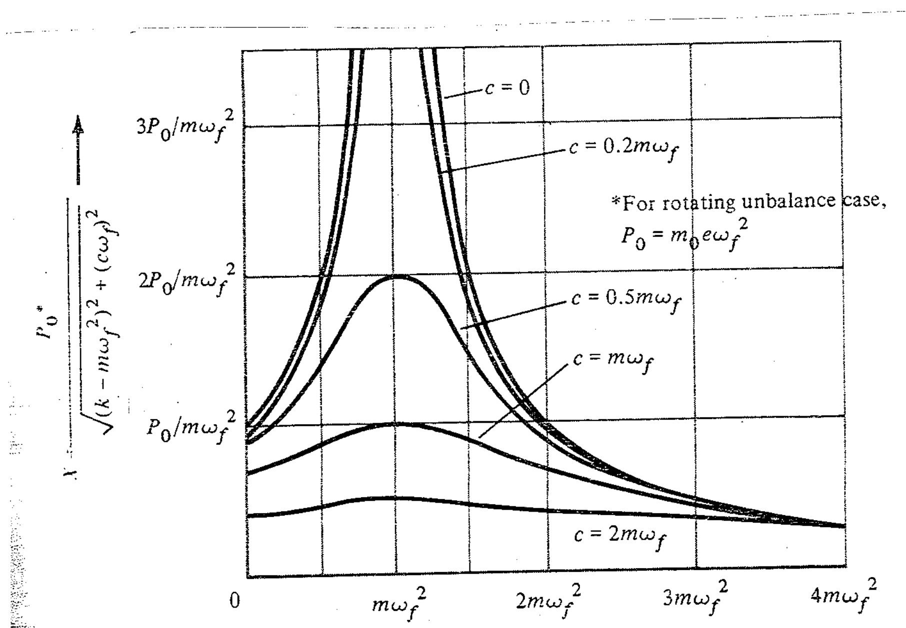


Figure 4-17

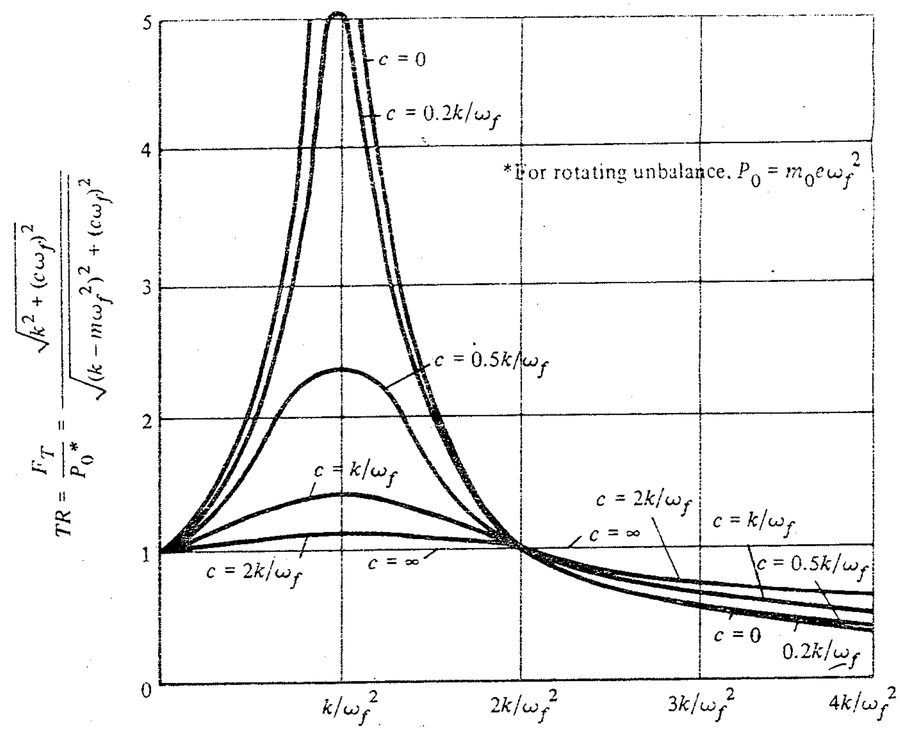


Figure 4-24

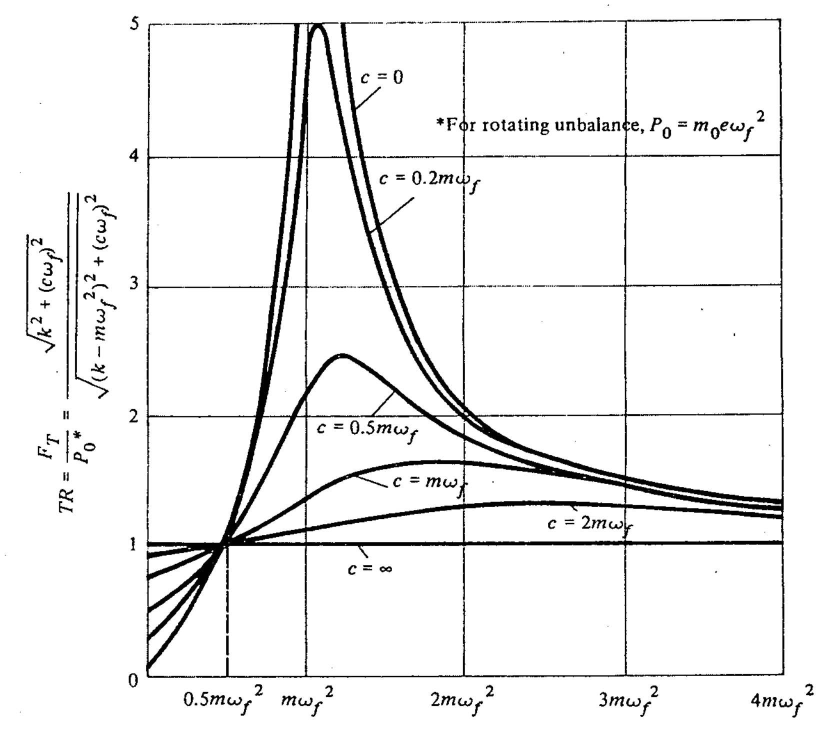


Figure 4-23

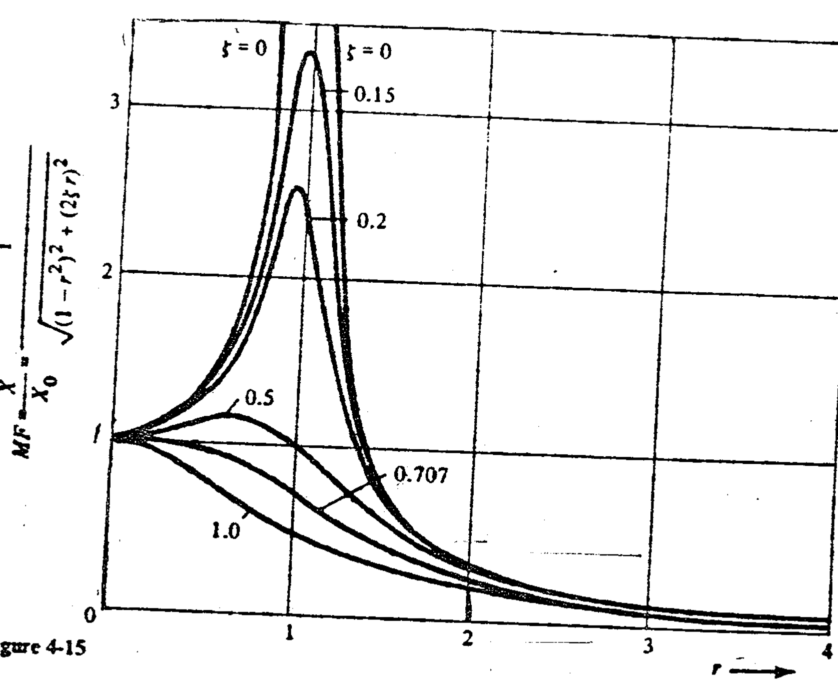


Figure 4-15

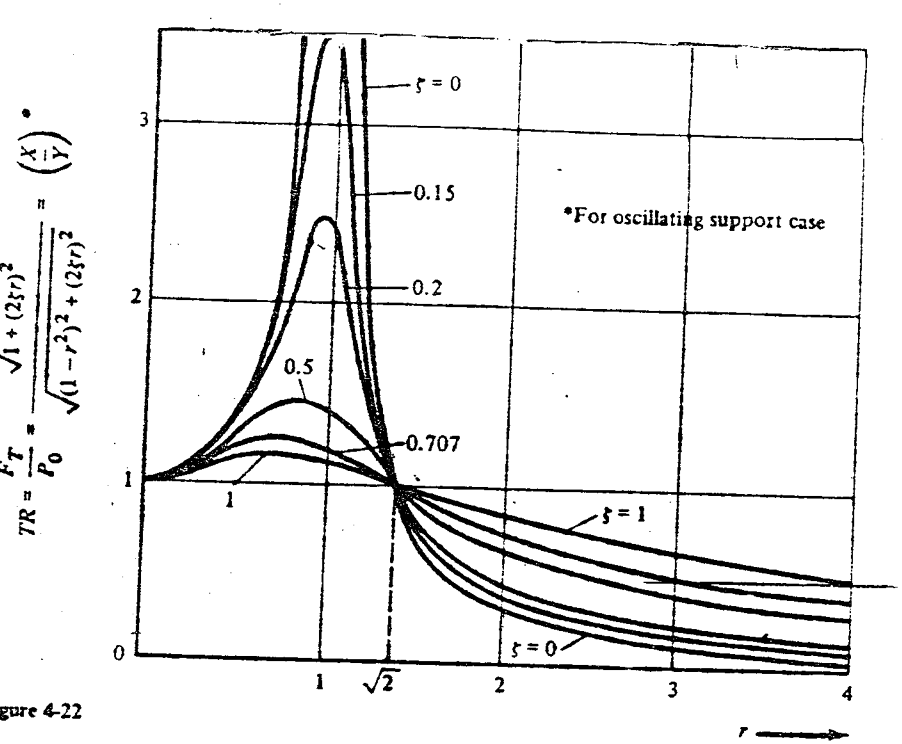


Figure 4-22

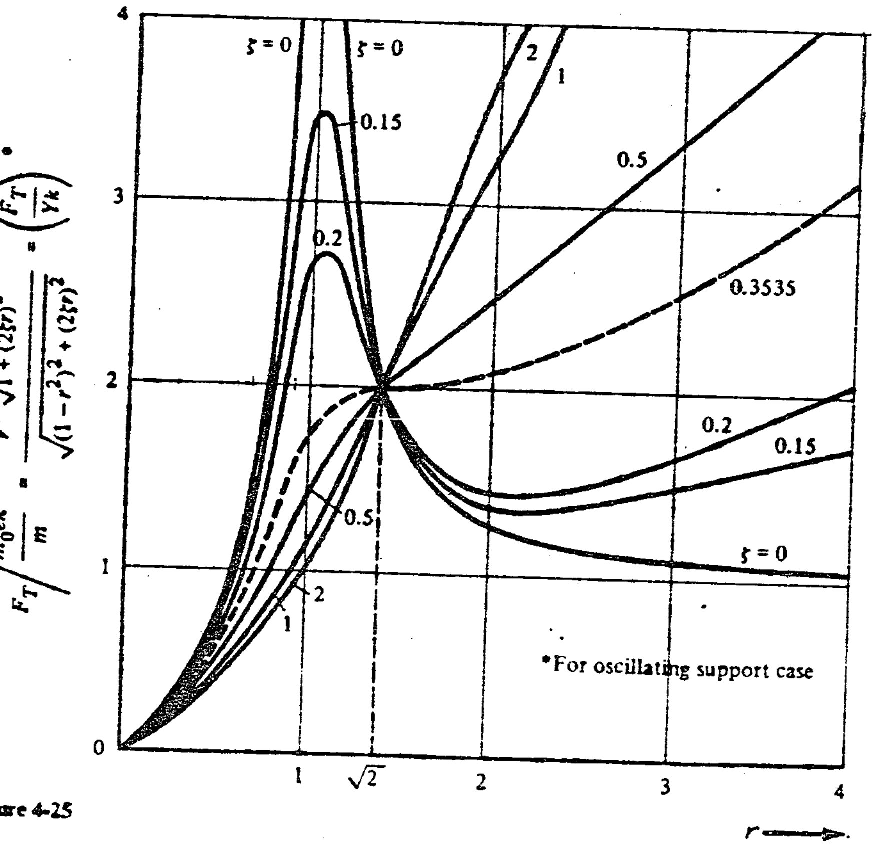


Figure 4-25

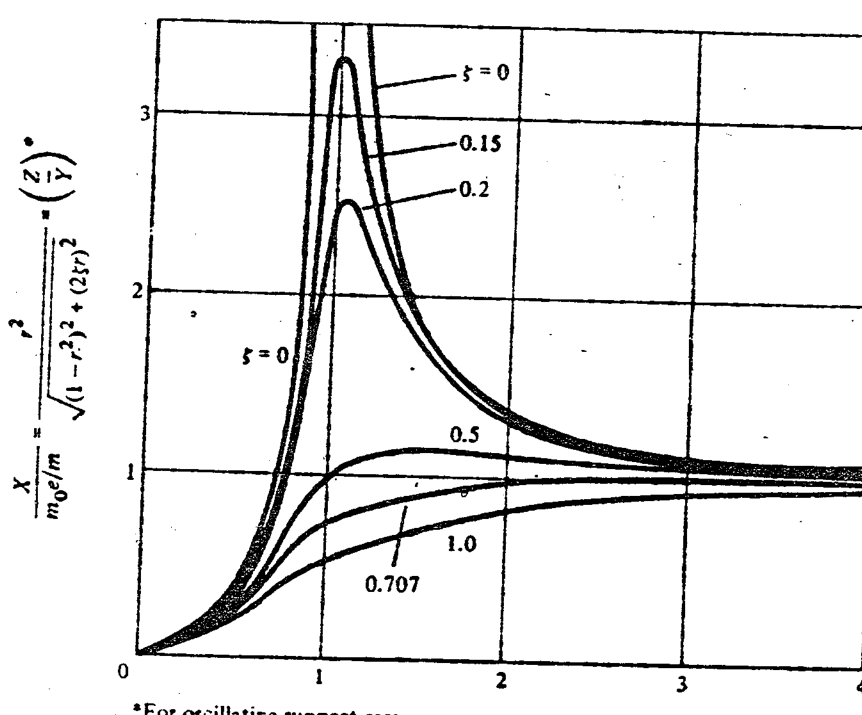


Figure 4-20

Problem 3.43 Erratum

- The main mass for this problem is the blades drive system + the rotor blades LESS the eccentric mass, i.e., $20\,kg - 0.5\,kg = 19.5\,kg = (m - m_o)$
- The eccentric mass $m_o = 0.5\,kg$, $e = 0.15\,m$
- $\zeta_{TAIL} = 0.15$

The tail section ONLY acts to support the drive system and blades and thus provides the elasticity (spring equivalent) and the damping (viscous equivalent) for the mass (main + eccentric). Tail information is used to get k

$$k_{eq} = \frac{3EI}{l^3} = \frac{3\,(2.5\times10^6\,N\text{-}m^2)}{(4)^3\,m^3} = 1.172 \times 10^5\,\frac{N}{m}$$

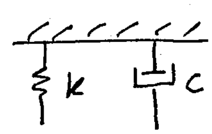
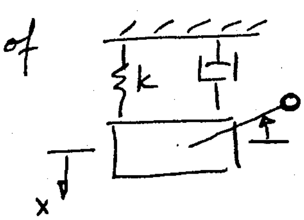
- the tail has weight and so it acts like a spring having weight. This is important for $\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}}$

$$m_{eq} = m_{\substack{drivesys\\+blades}} + 0.25\,m_{tail} = 20 + 0.25(240) = 80\,kg$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{1.172\times10^5}{80}} = 38.27\,rad/sec$$

$$\omega_f = 1500\,\frac{rev}{min} \times \frac{2\pi\,rad}{rev} \times \frac{1\,min}{60\,sec} = 50\pi\,rad/sec = 157.08\,rad/sec$$

$$r = \frac{\omega_f}{\omega_n} = \frac{157.08}{38.27} = 4.1$$

- This work so far is like finding  part of 
- THE MASS OF TAIL DOES NOT PLAY A PART IN x_p
- Now to find the actual forced response, we need to use

$$x_p = \frac{m_o e}{m} \cdot \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \psi)$$

where $m = m_{drivesy+blade} = 20\,kg$, $m_o = 0.5\,kg$, $r = 4.1$, $e = 0.15\,m$, $\zeta = 0.15$ $\Big\} \rightarrow X_{RU} = 0.003975\,m$

when $\omega_f \to \infty \Rightarrow r \to \infty$ $\quad X_{RU} = .00375\,m = \frac{m_o e}{m}$

$$\psi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = -4.45°$$

WHEN MAIN MASS IS AT X=0, ECCENTRICITY WILL BE AT AN ANGLE OF 4.45° BELOW HORIZONTAL

Accelerometers

as

Vibration Measuring Devices

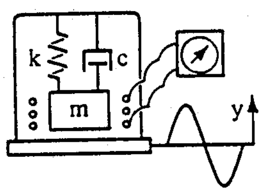
The "How and Why"

of their Design

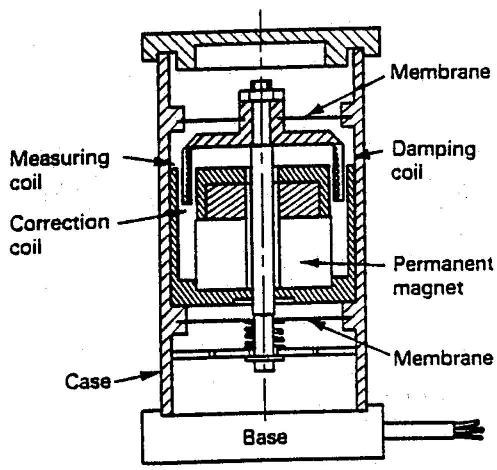
Cesar Levy

Mechanical Vibrations Lecture

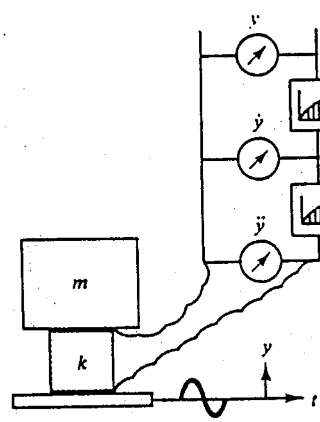
EML 4220


k
c
m
y

(g)


Membrane
Measuring coil
Damping coil
Correction coil
Permanent magnet
Membrane
Case
Base

(h)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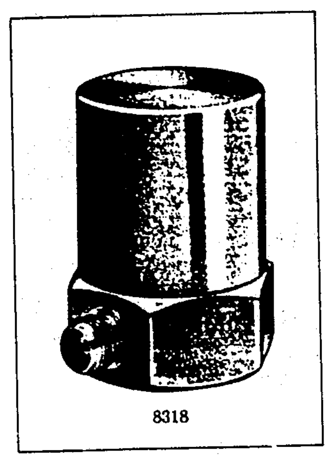

8318

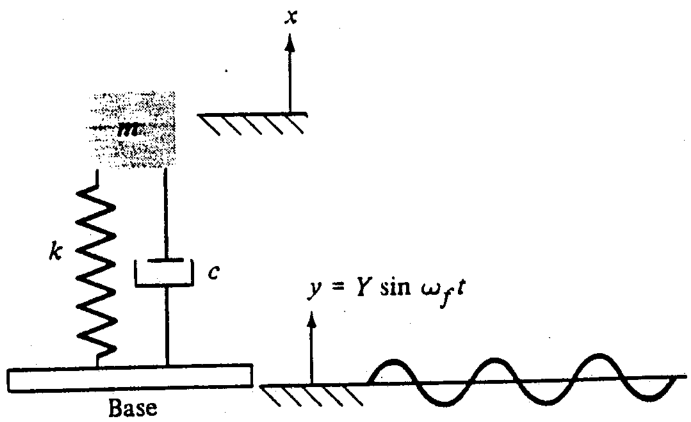
(j)

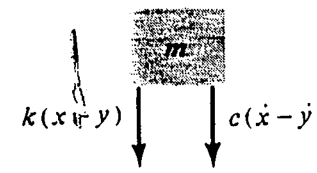
WHY USE ACCELEROMETERS

- *GIVES ACCELERATION OF BODY ON WHICH IT IS ATTACHED*
- *SMALL SIZE*

HOW DO THEY WORK

- *COMPARES MOTION OF SEISMIC MASS RELATIVE TO BASE AND RELATES IT TO THE MOTION OF THE BASE*





(b)

$$m\ddot{x} = -k(x-y) - c(\dot{x}-\dot{y})$$

GOVERNING EQUATION ON RELATIVE DISPL. z = x-y

$$m\ddot{z} + c\dot{z} + kz = m\omega_f^2 Y \sin\omega_f t$$

HAS STEADY STATE SOLUTION

$$z = \frac{m\omega_f^2 Y/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \phi)$$

$$r = \frac{\omega_f}{\omega_n} \qquad \zeta = \frac{c}{2m\omega_n} \qquad \omega_n = \sqrt{\frac{k}{m}}$$

MAXIMUM AMPLITUDE-RELATIVE MOTION

$$Z = Y \frac{(\omega_f^2/\omega_n^2)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

WE WANT TO REPRESENT ACCELERATION OF THE BASE

$$\ddot{y} = -\omega_f^2 Y \sin \omega_f t$$

HOW IS THIS DONE ?

$$-z\omega_n^2 = \frac{-\omega_f^2 Y \sin(\omega_f t - \phi)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

WHAT DOES THIS MEAN ?

$$\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \approx 1$$

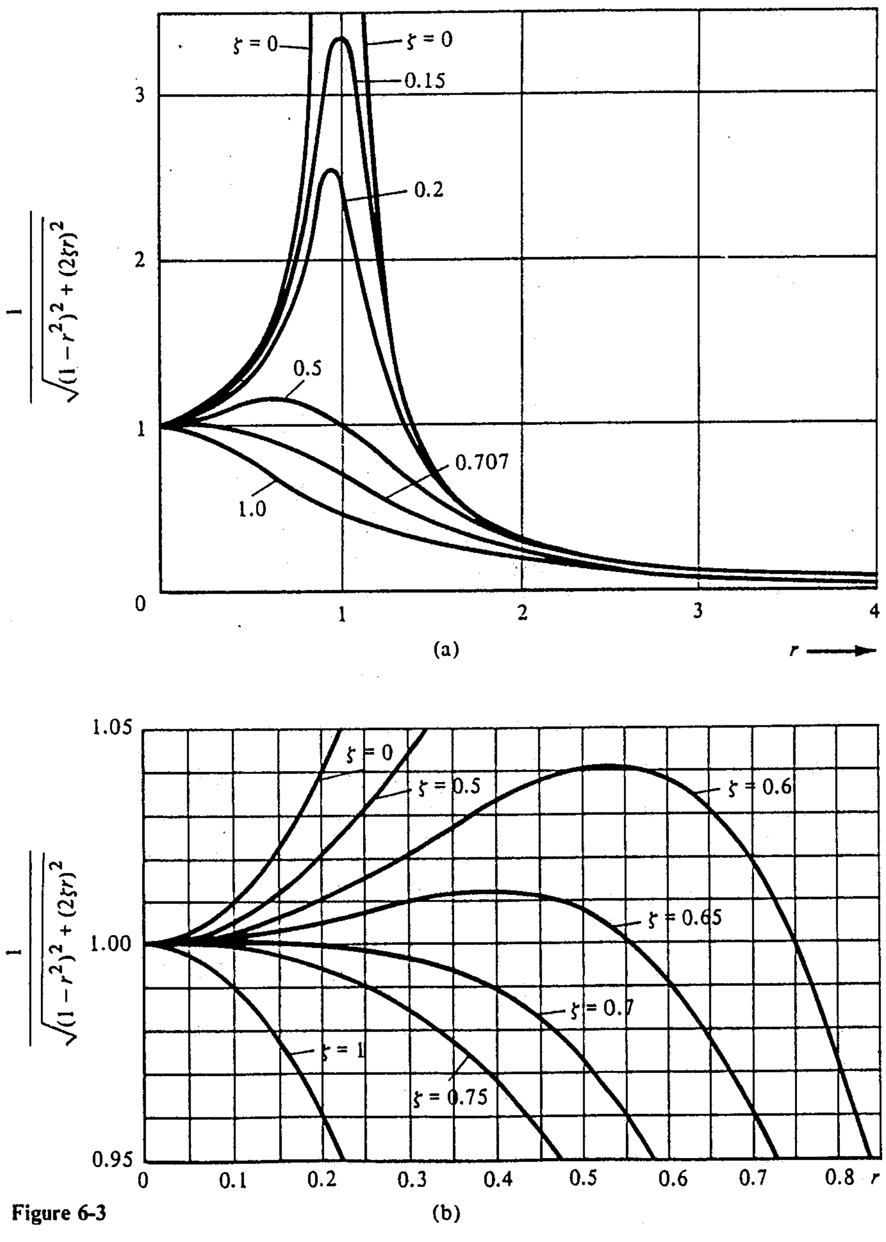


Figure 6-3

==> *r* **MUST BE SMALL**

EITHER *k* **MUST BE LARGE OR**

m **MUST BE SMALL**

for $\zeta > 1$ $\quad x = C_1 e^{-s_1 t} + C_2 e^{-s_2 t}$

$$s_1 = \zeta\omega_n + \omega_n\sqrt{\zeta^2-1} \qquad s_2 = \zeta\omega_n - \omega_n\sqrt{\zeta^2-1}$$

if $\left.\begin{array}{l} x = x_0 \text{ @ } t=0 \\ \dot{x} = \dot{x}_0 \text{ @ } t=0 \end{array}\right\} \Longrightarrow \quad C_1 = \dfrac{-x_0 s_2 - \dot{x}_0}{s_1 - s_2} \qquad C_2 = \dfrac{\dot{x}_0 + s_1 x_0}{s_1 - s_2}$

if $\quad m\ddot{x} + c\dot{x} + kx = P(t)$

then $\quad x(t) = \dfrac{1}{2m\omega_n\sqrt{\zeta^2-1}} \displaystyle\int_0^t P(\tau)\left\{ e^{-s_2(t-\tau)} - e^{-s_1(t-\tau)} \right\} d\tau$ for general load

for $\zeta = 1$ $\quad x = (C_1 + C_2 t)e^{-\omega_n t}$

if $\left.\begin{array}{l} x = x_0 \text{ @ } t=0 \\ \dot{x} = \dot{x}_0 \text{ @ } t=0 \end{array}\right\} \Longrightarrow \quad \begin{array}{l} x_0 = C_1 \\ \dot{x}_0 + x_0\omega_n = C_2 \end{array}$

if $\quad m\ddot{x} + c\dot{x} + kx = P(t)$

$$x(t) = \frac{1}{m}\int_0^t P(\tau)\,(t-\tau)\,e^{-\omega_n(t-\tau)}\,d\tau \quad \text{for general load}$$

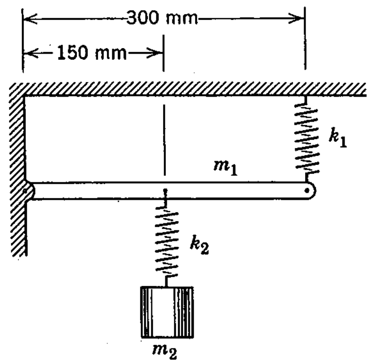
for $\zeta < 1$ $\quad x = \bar{C}_1 e^{-\zeta\omega_n t}\sin(\omega_d t + \varphi) = e^{-\zeta\omega_n t}\left[C_1 \sin\omega_d t + C_2\cos\omega_d t \right]$

if $\left.\begin{array}{l} x = x_0 \text{ @ } t=0 \\ \dot{x} = \dot{x}_0 \text{ @ } t=0 \end{array}\right\} \Longrightarrow \quad \begin{array}{l} C_2 = x_0 \\ C_1 = \dfrac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \end{array}$

if $\quad m\ddot{x} + c\dot{x} + kx = P(t)$

$$x(t) = \frac{1}{m\omega_d}\int_0^t P(\tau)\,e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)\,d\tau$$

FOR accelerometer replace $x(t)$ by $z(t)$ & $P(\tau)$ by $-m\ddot{y}(\tau)$



PROBLEM 9.59 Using Rayleigh's principle, determine the lowest natural frequency of the system shown.

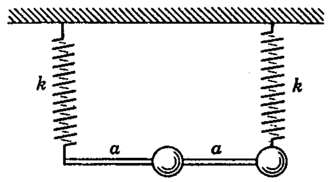
$k_1 = 10$ N/mm

$k_2 = 20$ N/mm

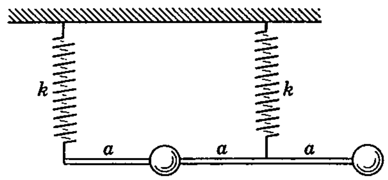
$m_1 = 5$ kg

$m_2 = 3$ kg

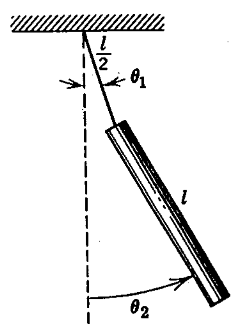
Answer: $f_n = 9.02$ Hz



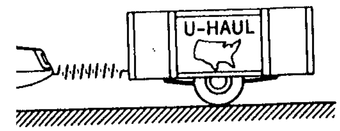
PROBLEM 9.60 Using Rayleigh's principle, determine the two natural frequencies and mode shapes of the two-mass system of Problem 9.16.



PROBLEM 9.61 Using Rayleigh's principle, determine the two natural frequencies and mode shapes of the two-mass system of Problem 9.17.



PROBLEM 9.62 For the slender rod suspended as a pendulum, Problem 9.18, use Rayleigh's principle to determine the two natural frequencies and mode shapes of small vibrations.



Determine the two natural fre-
:ame and pendulum. The pen-
e in the same plane in which

$$)_2^2 = \frac{m_1+m_2}{m_1}\left(\frac{g}{l}\right)$$

Two identical solid circular cyl-
ı a mass m, are connected by a
h a modulus k. Determine the
y(es) of small oscillations, if the
splaced from their equilibrium
ll without slipping on the hori-

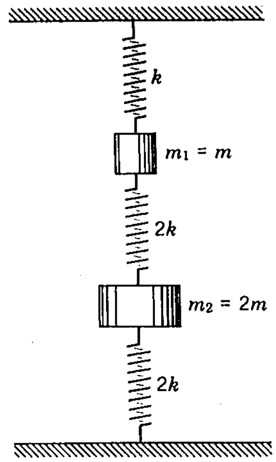
$$\omega_2^2 = \frac{4}{3}\frac{k}{m}$$

A platform supports a circular
elastically suspended from the
; with a modulus of k. Each has
mine the natural frequencies and
the system, if the cylinder rolls
;.

$$\sqrt{\frac{3k}{4m}};\ 3x_2 = x_1$$

The seismic mass m is mounted
tween two springs, each with a
The frame has identical mass of
the natural frequencies of free
e is no friction.

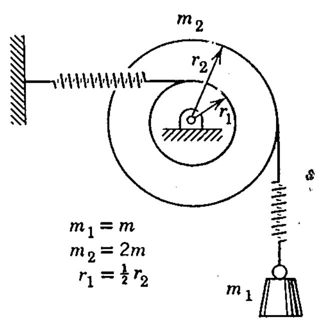
$$);\ \omega_2^2 = \frac{4k}{m}$$



PROBLEM 9.9 Determine the two natural frequencies and mode shapes of the two-mass systems shown.

Answer: $\omega_1^2 = \frac{k}{m}$; $\chi^{(1)} = +1$

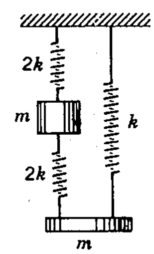
$$\omega_2^2 = 4\frac{k}{m};\ \chi^{(2)} = -\tfrac{1}{2}$$



PROBLEM 9.10 Determine the natural frequencies for the system given here. The pulley can be considered as a solid circular cylinder.

Answer: $\omega_1^2 = 0.117\frac{k}{m}$;

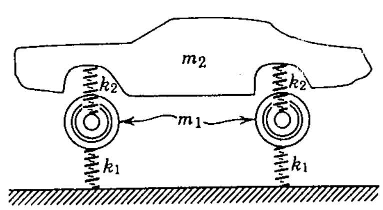
$$\omega_2^2 = 2.133\frac{k}{m}$$



PROBLEM 9.11 Determine the natural frequencies and mode shapes for the two-mass system. Both masses move only vertically. Do not consider rotation of the lower mass.

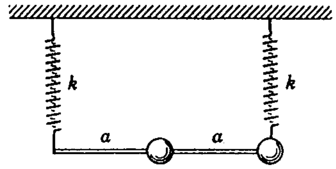
Answer: $\omega_1^2 = 1.439\frac{k}{m}$; $\chi_1 = 1.281$

$$\omega_2^2 = 5.562\frac{k}{m};\ \chi_2 = -0.781$$



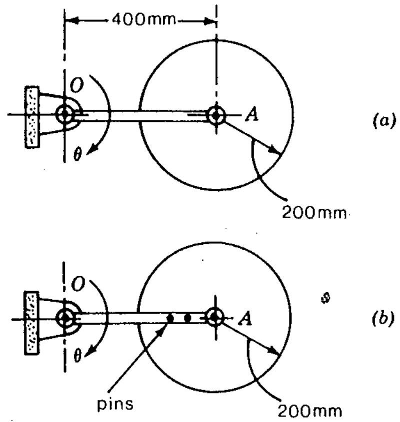
PROBLEM 9.12 A large automobile manufacturer analyzed the problem of the automobile by taking an entire automobile apart. By weighing each section, the following values of equivalent masses were found.

m_1	axle mass	180 kg
m_2	body mass	670 kg
k_2	springs	45.5 N/mm
k_1	tires	538 N/mm



PROBLEM 1.53 Two identical springs support a rigid rod and two identical masses. Choose coordinates that will describe the motion of each mass. Determine an expression for the kinetic energy of the system in terms of your coordinates.

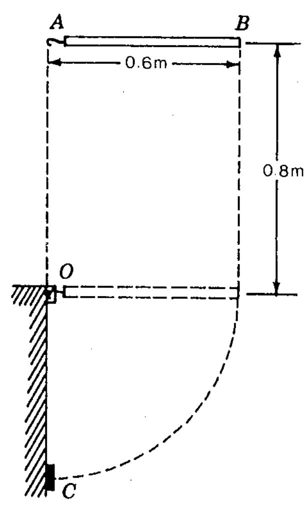
Answer: $T = m\dot{x}^2 + \frac{1}{4}ma^2\dot{\theta}^2$



PROBLEM 1.54 The uniform circular disk of 200-mm radius has a mass of 30 kg and is mounted on the rotary bar OA in two ways. In case (a), the disk is not pinned and rotates freely at A. Since there is no friction, the disk has no rigid body rotation. In case (b), the disk is pinned to the bar and has rigid body rotation.

If the systems are released from rest in the positions shown, determine the angular velocity of the bar OA as it passes through its vertical position, in each case.

Answer: (a) $\omega_{OA} = 7$ rad/s
(b) $\omega_{OA} = 6.6$ rad/s



PROBLEM 1.55 The uniform rod AB has a mass of 2 kg and is released from rest from the horizontal position shown. As it falls, the end A becomes hooked at pin O. End B remains free. Determine the speed at which end B strikes the stop at C.

Answer: $v_B = 7.28$ m/s

which is quadratic in $(\omega/\omega_a)^2$. Using the values of $\omega_a^2 = 6500/18.29$, $\omega_p^2 = 2600/73.16$, and $\mu = 0.25$, this simplifies to

$$10\left(\frac{\omega}{\omega_a}\right)^4 - 14.5\left(\frac{\omega}{\omega_a}\right)^2 + 2 = 0$$

Solving for ω/ω_a yields

$$\left(\frac{\omega}{\omega_a}\right)^2 = 0.1544, 1.2956 \quad \text{or} \quad \frac{\omega}{\omega_a} = 0.3929, 1.1382$$

Hence the three roots satisfying $|Xk/F_0| = 1$ are 0.3929, 1.1180, and 1.1382. Following the example of Figure 5.12 indicates that the driving frequency may vary between $0.3929\omega_a$ and $1.1180\omega_a$, or since $\omega_a = 18.857$,

$$7.4089 < \omega < 21.0821 \text{ (rad/s)}$$

before the response of the primary mass is amplified or the system is in danger of experiencing resonance.

□

The preceding discussion and examples illustrate the concept of *performance robustness*; that is, the examples illustrate how the design holds up as the parameter values (k, k_a, etc.) drift from the values used in the original design. Example 5.3.2 illustrates that the mass ratio greatly affects the robustness of absorber designs. This is stated in the caption of Figure 5.13; up to a certain point, increasing μ increases the robustness of the absorber design. The effects of damping on absorber design are examined in the next section.

5.4 DAMPING IN VIBRATION ABSORPTION

As mentioned in Section 5.3, damping is often present in devices and has the potential for destroying the ability of a vibration absorber to protect the primary system fully by driving X to zero. In addition, damping is sometimes added to vibration absorbers to prevent resonance or to improve the effective bandwidth of operation of a vibration absorber. Also, a damper by itself is often used as a vibration absorber by dissipating the energy supplied by an applied force. Such devices are called *vibration dampers* rather than absorbers.

First consider the effect of modeling damping in the standard vibration absorber problem. A vibration absorber with damping in both the primary and absorber system is illustrated in Figure 5.15. This system is dynamically equal to the system of Figure 4.14 of Section 4.5. The equations of motion are given in matrix form by equation (4.126) as

$$\begin{bmatrix} m & 0 \\ 0 & m_a \end{bmatrix}\begin{bmatrix} \ddot{x}(t) \\ \ddot{x}_a(t) \end{bmatrix} + \begin{bmatrix} c + c_a & -c_a \\ -c_a & c_a \end{bmatrix}\begin{bmatrix} \dot{x}(t) \\ \dot{x}_a(t) \end{bmatrix} + \begin{bmatrix} k + k_a & -k_a \\ -k_a & k_a \end{bmatrix}\begin{bmatrix} x(t) \\ x_a(t) \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}\sin\omega t \qquad (5.27)$$

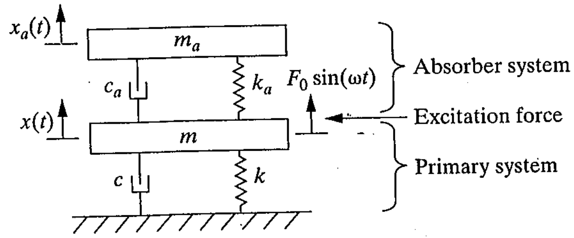


Figure 5.15 Schematic of a vibration absorber with damping in both the primary and absorber system.

Note, as was mentioned in Section 4.5, that these equations cannot necessarily be solved by using the modal analysis technique of Chapter 4 because the equations do not decouple $(KM^{-1}C \neq CM^{-1}K)$. The steady-state solution can be calculated, however, by using a combination of the exponential approach discussed in Section 2.3 and the matrix inverse used in previous sections for the undamped case.

To this end, let $F_0 \sin \omega t$ be represented in exponential form by $F_0 e^{j\omega t}$ in equation (5.27) and assume that the steady-state solution is of the form

$$\mathbf{x}(t) = \mathbf{X}e^{j\omega t} = \begin{bmatrix} X \\ X_a \end{bmatrix} e^{j\omega t} \tag{5.28}$$

where X is the amplitude of vibration of the primary mass and X_a is the amplitude of vibration of the absorber mass. Substitution into equation (5.27) yields

$$\begin{bmatrix} (k + k_a - m\omega^2) + (c + c_a)\omega j & -k_a - c_a\omega j \\ -k_a - c_a\omega j & (k_a - m_a\omega^2) + c_a\omega j \end{bmatrix} \begin{bmatrix} X \\ X_a \end{bmatrix} e^{j\omega t} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix} e^{j\omega t} \tag{5.29}$$

Note that the coefficient matrix of the vector $\mathbf{X}$ has complex elements. Dividing equation (5.29) by the nonzero scalar $e^{j\omega t}$ yields a complex matrix equation in the amplitudes X and X_a. Calculating the matrix inverse using the formula of Example 4.1.4, reviewed in Window 5.3, and multiplying equation (5.29) by the inverse from the right yields

$$\begin{bmatrix} X \\ X_a \end{bmatrix} = \frac{\begin{bmatrix} (k_a - m_a\omega^2) + c_a\omega j & k_a + c_a\omega j \\ k_a + c_a\omega j & k + k_a - m\omega^2 + (c + c_a)\omega j \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix}}{\det(K - \omega^2 M + \omega j C)} \tag{5.30}$$

Here the determinant in the denominator is given by (recall Example 4.1.4)

$$\det(K - \omega^2 M + \omega j C) = mm_a\omega^4 - (c_a c + m_a(k + k_a) + k_a m)\omega^2 + k_a k + [(kc_a + ck_a)\omega - (c_a(m + m_a) + cm_a)\omega^3]j \tag{5.31}$$

and the system coefficient matrices M, C, and K are given by

$$M = \begin{bmatrix} m & 0 \\ 0 & m_a \end{bmatrix} \quad C = \begin{bmatrix} c + c_a & -c_a \\ -c_a & c_a \end{bmatrix} \quad K = \begin{bmatrix} k + k_a & -k_a \\ -k_a & k_a \end{bmatrix}$$

Simplifying the matrix vector product yields

$$X = \frac{[(k_a - m_a\omega^2) + c_a\omega j]F_0}{\det(K - \omega^2 M + \omega jC)} \tag{5.32}$$

$$X_a = \frac{(k_a + c_a\omega j)F_0}{\det(K - \omega^2 M + \omega jC)} \tag{5.33}$$

which expresses the magnitude of the response of the primary mass and absorber mass, respectively. Note that these values are now complex numbers and are multiplied by the complex value $e^{j\omega t}$ to get the time responses.

Equations (5.32) and (5.33) are the two-degree-of-freedom version of the frequency response function given for a single-degree-of-freedom system in equation (2.42). The complex nature of these values reflect a magnitude and phase. The magnitude is calculated following the rules of complex numbers and is best done with a symbolic computer code, or after substitution of numerical values for the various physical constants. It is important to note from equation (5.32) that unlike the tuned undamped absorber, the response of the primary system cannot be exactly zero even if the tuning condition is satisfied. Hence the presence of damping ruins the ability of the absorber system to exactly cancel the motion of the primary system.

Equations (5.32) and (5.33) can be analyzed for several specific cases. First, consider the case for which the internal damping of the primary system is neglected $(c = 0)$. If the primary system is made of metal, the internal damping is likely to be very low and it is reasonable to neglect it in many circumstances. In this case the determinant of equation (5.31) reduces to the complex number

$$\det(K - \omega^2 M + \omega Cj)$$
$$= [(-m\omega^2 + k)(-m_a\omega^2 + k_a) - m_a k_a \omega^2] + [(k - (m + m_a)\omega^2)c_a\omega]j \tag{5.34}$$

The maximum deflection of the primary mass is given by equation (5.32) with the determinant in the denominator evaluated as given in equation (5.34). This is the ratio of two complex numbers and hence is a complex number representing the phase and the amplitude of the response of the primary mass. Using complex arithmetic (see Window 5.4) the amplitude of the motion of the primary mass can be written as the real number

$$\frac{X^2}{F_0^2} = \frac{(k_a - m_a\omega^2)^2 + \omega^2 c_a^2}{[(k - m\omega^2)(k_a - m_a\omega^2) - m_a k_a \omega^2]^2 + [k - (m + m_a)\omega^2]^2 c_a^2\omega^2} \tag{5.35}$$

It is instructive to examine this amplitude in terms of the dimensionless ratios introduced in Section 5.3 for the undamped vibration absorber. The amplitude X is written in terms of the static deflection $\Delta = F_0/k$ of the primary system. In addition, consider the mixed "damping ratio" defined by

$$\zeta = \frac{c_a}{2m_a\omega_p} \tag{5.36}$$

Window 5.4
Reminder of Complex Arithmetic

The response magnitude given by equation (5.32) can be written as the ratio of two complex numbers:

$$\frac{X}{F_0} = \frac{A_1 + B_1 j}{A_2 + B_2 j}$$

where A_1, A_2, B_1, and B_2 are real numbers and $j = \sqrt{-1}$. Multiplying this by the conjugate of the denominator divided by itself yields

$$\frac{X}{F_0} = \frac{(A_1 + B_1 j)(A_2 - B_2 j)}{(A_2 + B_2 j)(A_2 - B_2 j)} = \frac{(A_1 A_2 + B_1 B_2)}{A_2^2 + B_2^2} + \frac{B_1 A_2 - A_1 B_2}{A_2^2 + B_2^2} j$$

which indicates how X/F_0 is written as a single complex number of the form $X/F_0 = a + bj$. This is interpreted, as indicated, that the response magnitude has two components: one in phase with the applied force and one out of phase. The magnitude of X/F_0 is the length of the preceding complex number (i.e., $|X/F_0| = \sqrt{a^2 + b^2}$. This yields

$$\left|\frac{X}{F_0}\right| = \sqrt{\frac{A_1^2 + B_1^2}{A_2^2 + B_2^2}}$$

which corresponds to the expression given in equation (5.35). (Also see Appendix A.)

where $\omega_p = \sqrt{k/m}$ is the original natural frequency of the primary system without the absorber attached. Using the standard frequency ratio $r = \omega/\omega_p$, the ratio of natural frequencies $\beta = \omega_a/\omega_p$, (where $\omega_a = \sqrt{k_a/m_a}$), and the mass ratio $\mu = m_a/m$, equation (5.35) can be rewritten as

$$\frac{X}{\Delta} = \frac{Xk}{F_0} = \sqrt{\frac{(2\zeta r)^2 + (r^2 - \beta^2)^2}{(2\zeta r)^2(r^2 - 1 + \mu r^2)^2 + [\mu r^2\beta^2 - (r^2 - 1)(r^2 - \beta^2)]^2}} \tag{5.37}$$

which expresses the dimensionless amplitude of the primary system. Note from examining equation (5.37) that the amplitude of the primary system response is determined by four physical parameter values:

- μ the ratio of the absorber mass to the primary mass
- β the ratio of the decoupled natural frequencies
- r the ratio of the driving frequency to the primary natural frequency
- ζ the ratio of the absorber damping and $2m_a\omega_p$

These four numbers can be considered as design variables and are chosen to give the smallest possible value of the primary mass's response, X, for a given application.

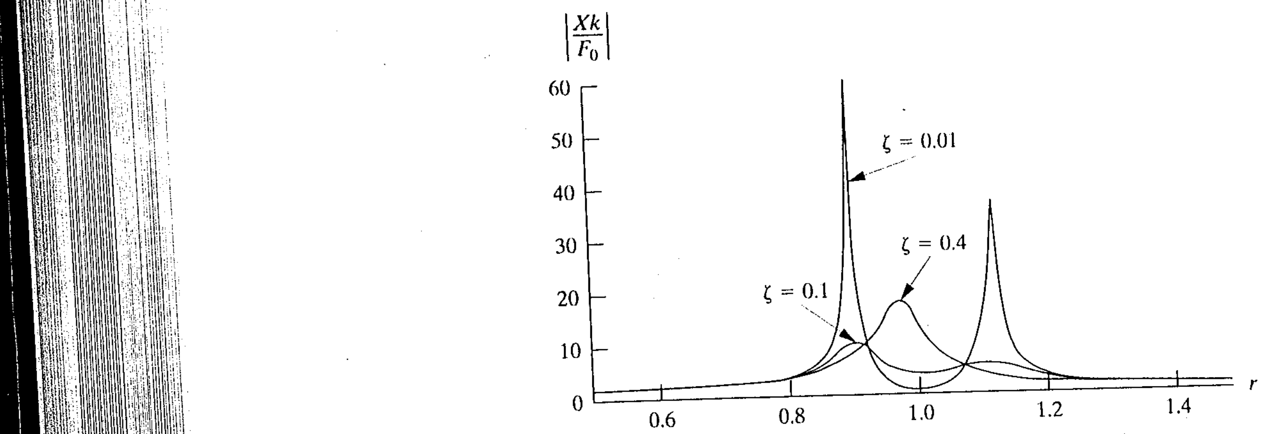


Figure 5.16 Normalized amplitude of vibration of the primary mass as a function of the frequency ratio for several values of the damping in the absorber system for the case of negligible damping in the primary system [i.e., a plot of equation (5.37)].

Figure 5.16 illustrates how the damping value, as reflected in ζ, affects the response for a fixed value of μ, β, and r.

As mentioned at the beginning of this section, damping is often added to the absorber to improve the bandwidth of operation. This effect is illustrated in Figure 5.16. Recall that if there is no damping in the absorber ($\zeta = 0$), the magnitude of the response of the primary mass as a function of the frequency ratio r is as illustrated in Figure 5.12 (i.e., zero at $r = 1$ but infinite at $r = 0.781$ and $r = 1.281$). Thus the completely undamped absorber has poor bandwidth (i.e., if r changes by a small amount, the amplitude grows). In fact, as noted in Section 5.3, the bandwidth, or useful range of operation of that undamped absorber, is $0.897 \le r \le 1.103$. For these values of r, $|Xk/F_0| \le 1$. However, if damping is added to the absorber ($\zeta \neq 0$), Figure 5.16 results, and the bandwidth, or useful range of operation, is extended. The price for this increased operating region is that $|Xk/F|$ is never zero in the damped case (see Figure 5.16).

Examination of Figure 5.16 shows that as ζ is varied, the amplification of $|Xk/F_0|$ over the range of r can be reduced. The design question now becomes: For what values of the mass ratio μ, the absorber damping ratio ζ, and the frequency ratio β is the magnitude $|Xk/F_0|$ smallest over the region $0 \le r \le 2$? Just increasing the damping with μ and β fixed does not necessarily yield the lowest amplitude. Note from Figure 5.16 that $\zeta = 0.1$ produces a smaller amplification over a larger region of r that does the higher ratio, $\zeta = 0.4$. Figures 5.17 and 5.18 yield some hint of how the various parameters affect the magnitude by providing plots of $|Xk/F_0|$ for various combinations of ζ, μ, and β.

A solution of the best choice of μ and ζ is discussed again in Section 5.5. Note from Figure 5.18 that $\mu = 0.25$, $\beta = 0.8$, and $\zeta = 0.27$ yield a minimum value of $|Xk/F_0|$ over a large range of values of r. However, amplification of the response X

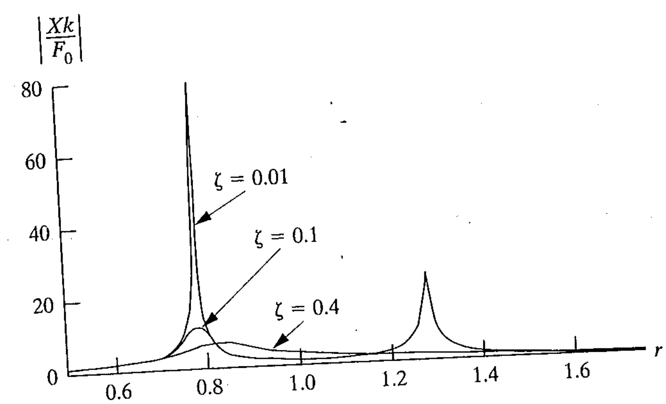


Figure 5.17 Repeat of the plot of Figure 5.16 with $\mu = 0.25$ and $\beta = 1$ for several values of ζ. Note that in this case, $\zeta = 0.4$ yields a lower magnitude than does $\zeta = 0.1$.

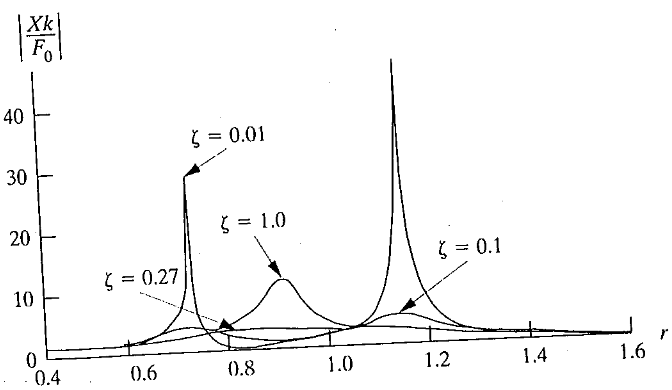


Figure 5.18 Repeat of the plots of Figure 5.16 with $\mu = 0.25$, $\beta = 0.8$ for several values of ζ. In this case $\zeta = 0.27$ yields the lowest amplification over the largest bandwidth.

still occurs (i.e., $|Xk/F_0| > 1$ for values of $r < \sqrt{2}$), but no order-of-magnitude increases in $|X|$ occurs as in the case of the undamped absorber.

Next consider the case of an appended absorber mass connected to an undamped primary mass only by a dashpot, an arrangement illustrated in Figure 5.19. Systems of this form arise in the design of vibration reduction devices for rotating systems such as engines, where the operating speed (and hence the driving frequency) varies over a wide range. In such cases a viscous damper is added to the end of the crankshaft (or other rotating device) as indicated in Figure 5.20. The shaft spins through an angle θ_1 with torsional stiffness k and inertia J_1. The damping inertia J_2 spins through an angle θ_2 in a viscous film providing a damping force $c_a(\dot{\theta}_1 - \dot{\theta}_2)$. If an external harmonic torque is applied of the form $M_0 e^{\omega t j}$, the equation of motion of this system becomes

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} c_a & -c_a \\ -c_a & c_a \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} M_0 \\ 0 \end{bmatrix} e^{\omega t j} \quad (5.38)$$

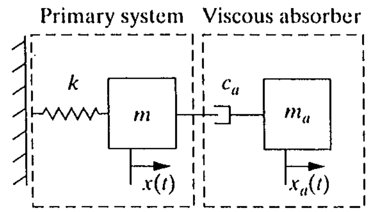


Figure 5.19 Damper–mass system added to a primary mass (with no damping) to form a viscous vibration absorber.

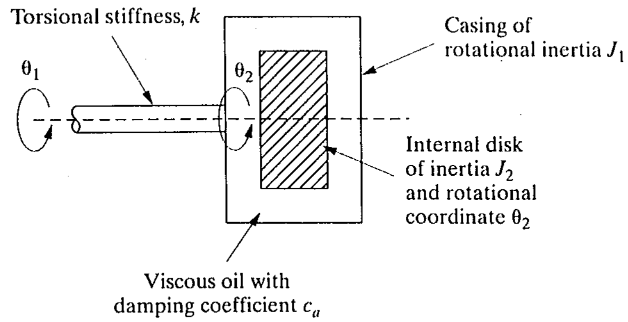


Figure 5.20 Viscous damper and mass added to a rotating shaft for broadband vibration absorption. Often called a *Houdaille damper.*

This is a rotational equivalent to the translational model given in Figure 5.19. It is easy to calculate the undamped natural frequencies of this two-degree-of-freedom system. They are

$$\omega_p = \sqrt{\frac{k}{J_1}} \text{ and } \omega_a = 0$$

The solution of this set of equations is given by equations (5.32) and (5.33) with m and m_a replaced by J_1 and J_2, respectively, $c = 0$, $k_a = 0$, and F_0 replaced by M_0. Equation (5.32) is given in nondimensional form as equation (5.37). Hence letting $\beta = \omega_a/\omega_p = 0$ in equation (5.37) yields that amplitude of vibration of the primary inertia J_1 [i.e., the amplitude of $\theta_1(t)$] is described by

$$\frac{Xk}{M_0} = \sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2(r^2 + \mu r^2 - 1)^2 + (r^2 - 1)^2 r^2}} \tag{5.39}$$

where $\zeta = c/(2J_2\omega_p)$, $r = \omega/\omega_p$, and $\mu = J_2/J_1$. Figure 5.21 illustrates several plots of Xk/M_0 for various values of ζ for a fixed μ as a function of r. Note again that the highest damping does not correspond to the largest-amplitude reduction.

The various absorber designs discussed previously, excluding the undamped case, result in a number of possible "good" choices for the various design parameters. When faced with a number of good choices, it is natural to ask which is the best choice. Looking for the best possible choice among a number of acceptable or good choices can be made systematic by using methods of optimization introduced in the next section.

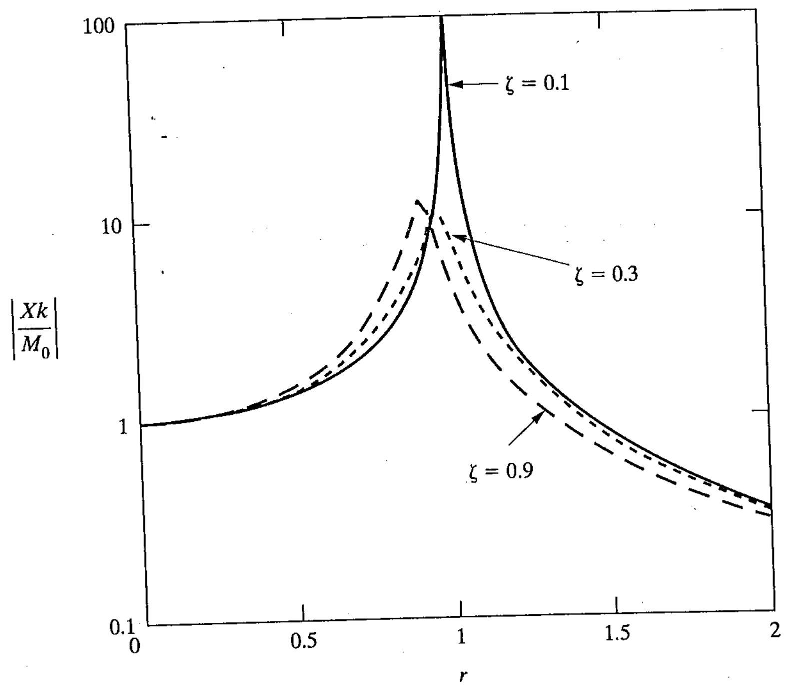


Figure 5.21 Amplitude curves for a system with a viscous absorber, a plot of equation (5.39), for the case $\mu = 0.25$ and for three different values of ζ.

5.5 OPTIMIZATION

In the design of vibration systems the best selection of system parameters is often sought. In the case of the undamped vibration absorber of Section 5.3 the best selection for values of mass and stiffness of the absorber system is obvious from examining the expression for the amplitude of vibration of the primary system. In this case the amplitude could be driven to zero by tuning the absorber mass and stiffness to the driving frequency. In the other cases, especially when damping is included, the choice of parameters to produce the best response is not obvious. In such cases optimization methods can often be used to help select the best performance. Optimization techniques often produce results that are not obvious. An example is in the case of the undamped primary system or the damped absorber system discussed in the preceding section. In this case Figures 5.16 to 5.18 indicate that the best selection of parameters does not correspond to the highest value of the damping in the system as intuition might dictate. These figures essentially represent an optimization by trial and error. In this section a more systematic approach to optimization is suggested by taking advantage of calculus.

Recall from elementary calculus that minimums and maximums of particular functions can be obtained by examining certain derivatives. Namely, if the first derivative vanishes and the second derivative of the function is positive, the function has obtained a minimum value. This section presents a few examples where optimization procedures are used to obtain the best possible vibration reduction for various isolator and absorber systems. A major task of optimization is first deciding what quantity should be minimized to best describe the problem under study. The next question of interest is to decide which variables to allow to vary during the optimization. Optimization methods have developed over the years that allow the parameters during the optimization to satisfy constraints, for example. This approach is often used in design for vibration suppression.

Recall from calculus that a function $f(x)$ experiences a maximum (or minimum) at value of $x = x_m$ given by the solution of

$$f'(x_m) = \frac{d}{dx}[f(x_m)] = 0 \tag{5.40}$$

If this value of x causes the second derivative, $f''(x_m)$, to be less than zero, the value of $f(x)$ at $x = x_m$ is the maximum value that $f(x)$ takes on in the region near $x = x_m$. Similarly, if $f''(x_m)$ is greater than zero, the value of $f(x_m)$ is the smallest or minimum value that $f(x)$ obtains in the interval near x_m. Note that if $f''(x) = 0$, at $x = x_m$, the value $f(x_m)$ is neither a minimum or maximum for $f(x)$. The points where $f'(x)$ vanish are called *critical points*.

These simple rules were used in Section 2.2, Example 2.2.3, for computing the value (r_{peak}) where the maximum value of normalized magnitude of the steady-state response of a harmonically driven single-degree-of-freedom system occurs. The second derivative test was not checked because several plots of the function clearly indicated that the curve contains a global maximum value rather than a minimum. In both absorber and isolator design, plots of the magnitude of the response can be used to avoid having to calculate the second derivative (second derivatives are often unpleasant to calculate).

If the function f to be minimized (or maximized) is a function of two variables [i.e., $f = f(x, y)$], the preceding derivative tests become slightly more complicated and involve examining the various partial derivatives of the function $f(x, y)$. In this case the critical points are determined from the equations

$$f_x(x, y) = \frac{\partial f(x, y)}{\partial x} = 0$$

$$f_y(x, y) = \frac{\partial f(x, y)}{\partial y} = 0 \tag{5.41}$$

Whether or not these critical points (x, y) are a maximum of the value $f(x, y)$ or a minimum depend on the following:

1. If $f_{xx}(x, y) > 0$ and $f_{xx}(x, y)f_{yy}(x, y) > f_{xy}^2(x, y)$, then $f(x, y)$ has a relative minimum value at x, y.

2. If $f_{xx}(x, y) < 0$ and $f_{xx}(x, y)f_{yy}(x, y) > f_{xy}^2(x, y)$, then $f(x, y)$ has a relative maximum value at x, y.
3. If $f_{xy}^2(x, y) > f_{xx}(x, y)f_{yy}(x, y)$, then $f(x, y)$ is neither a maximum nor a minimum value, the point x, y is a *saddle* point.
4. If $f_{xy}^2(x, y) = f_{xx}(x, y)f_{yy}(x, y)$, the test fails and the point x, y could be any or none of the above.

Plots of $f(x, y)$ can also be used to determine whether or not a given critical point is a maximum, minimum, saddle point or neither. These rules can be used to help solve vibration design problems in some circumstances. As an example of using these optimization formulations for designing a vibration suppression system, recall the damped absorber system of Section 5.4. In this case the magnitude of the primary mass normalized with respect to the input force (moment) magnitude is given in equation (5.39) to be

$$\frac{Xk}{M_0} = \sqrt{\frac{4\zeta^2 + r^2}{4\zeta^2(r^2 + \mu r^2 - 1)^2 + (r^2 - 1)^2 r^2}} = f(r, \zeta) \tag{5.42}$$

which is considered to be a function of the mixed damping ratio ζ and the frequency ratio r for a fixed mass ratio μ.

In Section 5.4, values of $f(r)$ are plotted versus r for several values of ζ in an attempt to find the value of ζ that yields the smallest maximum value of $f(r, \zeta)$. This is illustrated in Figure 5.21. Figure 5.22 illustrates the magnitude as a function of both ζ and r. From the figure it can be concluded that the derivative $\partial f/\partial r = 0$ yields the maximum value of the magnitude for each fixed ζ.

Looking along the ζ axis, the partial derivative $\partial f/\partial\zeta = 0$ yields the minimum value of $f(r, \zeta)$ for each fixed value of r. The best design, corresponding to the smallest of the largest amplitudes, is thus illustrated in Figure 5.22. This point corresponds

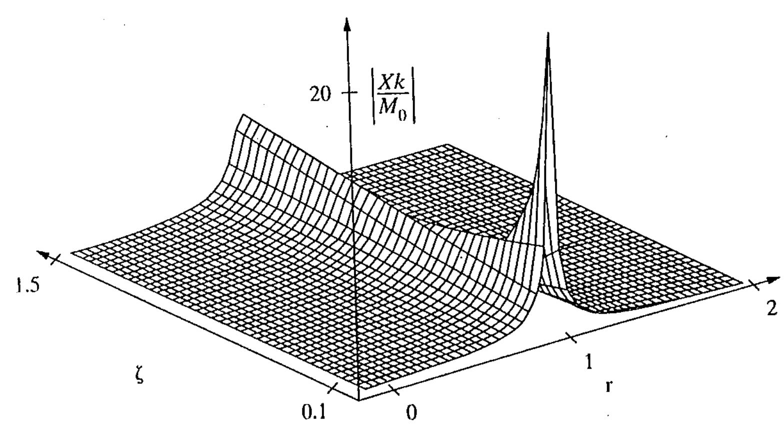


Figure 5.22 Plot of the normalized magnitude of the primary system versus both ζ and r [i.e., a two-dimensional plot of equation (5.42) for $\mu = 0.25$]. This illustrates that the most desirable response is obtained at the *saddle point*.

to a saddle point and can be calculated by evaluating the appropriate first partial derivatives.

First consider $\partial(Xk/M_0)/\partial\zeta$. From equation (5.42), the function to be differentiated is of the form

$$f = \frac{A^{1/2}}{B^{1/2}} \tag{5.43}$$

where $A = 4\zeta^2 + r^2$ and $B = 4\zeta^2(r^2 + \mu r^2 - 1)^2 + (r^2 - 1)^2 r^2$. Differentiating and equating the resulting derivatives to zero yields

$$\frac{\partial f}{\partial \zeta} = \frac{1}{2}\frac{A^{-1/2}dA}{B^{1/2}} - \frac{1}{2}A^{1/2}\frac{dB}{B^{3/2}} = 0 \tag{5.44}$$

Solving this yields the form $[B\,dA - A\,dB]/2B^{3/2} = 0$ or

$$B\,dA = A\,dB \tag{5.45}$$

where A and B are as defined previously and

$$dA = 8\zeta \qquad \text{and} \qquad dB = 8\zeta(r^2 + \mu r^2 - 1)^2 \tag{5.46}$$

Substitution of these values of A, dA, B, and dB into equation (5.45) yields

$$(1 - r^2)^2 = (1 - r^2 - \mu r^2)^2 \tag{5.47}$$

For $\mu \neq 0$, $r > 0$, this has the solution

$$r = \sqrt{\frac{2}{2 + \mu}} \tag{5.48}$$

Similarly, differentiating equation (5.42) with respect to r and substituting the value for r obtained previously yields

$$\zeta_{op} = \frac{1}{\sqrt{2(\mu + 1)(\mu + 2)}} \tag{5.49}$$

Equation (5.49) reveals the value of ζ that yields the smallest amplitude at the point of largest amplitude (resonance) for the response of the primary mass. The maximum value of the displacement for the optimal damping is given by

$$\left(\frac{Xk}{M_0}\right)_{max} = 1 + \frac{2}{\mu} \tag{5.50}$$

which is obtained by substitution of (5.48) and (5.49) into equation (5.42). This last expression suggests that μ should be as large as possible. However, the practical consideration that the absorber mass should be smaller than the primary mass requires $\mu \leq 1$. The value $\mu = 0.25$ is fairly common.

The second derivative conditions for the function f to have a saddle point (condition 3 in the preceding list) are too cumbersome to calculate. However, the plot of Figure 5.22 clearly illustrates that these conditions are satisfied. Furthermore, the plot indicates that f as a function of ζ is convex and f as a function of r is concave so that the saddle point condition is also the solution of minimizing the maximum value $f(r, \zeta)$, called the *min-max problem* in applied mathematics and optimization.

Example 5.5.1

A viscous damper–mass absorber is added to the shaft of an engine. The mass moment of inertia of the shaft system is 1.5 kg · m²/rad and has a torsional stiffness of 6×10^3N · m/rad. The nominal running speed of the engine is 2000 rpm. Calculate the values of the added damper and mass moment of inertia such that the primary system has a magnification (Xk/M_0) of less than 5 for all speeds and is as small as possible at the running speed.

Solution Since $\omega_p = \sqrt{k/J}$, the natural frequency of the engine system is

$$\omega_p = \sqrt{\frac{6.0 \times 10^3\text{N} \cdot \text{m/rad}}{1.5 \text{ kg} \cdot \text{m}^2/\text{rad}}} = 63.24 \text{ rad/s}$$

The running speed of the engine is 2000 rpm or 209.4 rad/s, which is assumed to be the driving frequency (actually, it is a function of the number of cylinders). Hence the frequency ratio is

$$r = \frac{\omega}{\omega_p} = \frac{209.4}{63.24} = 3.31$$

so that the running speed is well away from the maximum amplification as illustrated in Figures 5.21 and 5.22 and the absorber is not needed to protect the shaft at its running speed. However, the engine spends some time getting to the running speed and often runs at lower speeds. The peak response occurs at

$$r_{\text{peak}} = \frac{\omega}{\omega_p} = \sqrt{\frac{2}{2 + \mu}}$$

as given by equation (5.48) and has a value of

$$\left(\frac{Xk}{M_0}\right)_{\max} = 1 + \frac{2}{\mu}$$

as given by equation (5.50). The magnification is restricted to be 5, so that

$$1 + \frac{2}{\mu} \leq 5, \quad \text{or} \quad \mu \geq 0.5$$

Thus $\mu = 0.5$ is chosen for the design. Since the mass of the primary system is $J_1 = 1.5$ kg · m²/rad and $\mu = J_2/J_1$, the mass of the absorber is

$$J_2 = \mu J_1 = \frac{1}{2}(1.5)\text{kg} \cdot \text{m}^2/\text{rad} = 0.75 \text{ kg} \cdot \text{m}^2 \cdot \text{rad}$$

The damping value required for equation (5.50) to hold is given by equation (5.49) or

$$\zeta_{\text{op}} = \frac{1}{\sqrt{2(\mu + 1)(\mu + 2)}} = \frac{1}{\sqrt{2(1.5)(2.5)}} = 0.3651$$

Recall from Section 5.4 [just following equation (5.39)] that $\zeta = c/(2J_2\omega_p)$, so that the optimal damping constant becomes

$$c_{op} = 2\zeta_{op}J_2\omega_p = 2(0.3651)(0.75)(63.24) = 34.638 \text{ N}\cdot\text{m}\cdot\text{s/rad}$$

The two values of J_2 and c given here form an optimal solution to the problem of designing a viscous damper–mass absorber system so that the maximum deflection of the primary shaft is satisfied $|Xk/M_0| < 5$. This solution is optimal in terms of a choice of ζ, which corresponds to the saddle point of Figure 5.22 and yields a minimum value of all maximum amplifications.

□

Optimization methods can also be useful in the design of certain types of vibration isolation systems. For example, consider the model of a machine mounted on an elastic damper and spring system as illustrated in Figure 5.23. The equations of motion of the system of Figure 5.23 are

$$m\ddot{x}_1 + c(\dot{x}_1 - \dot{x}_2) + k_1x_1 = F_0\cos\omega t$$
$$c(\dot{x}_1 - \dot{x}_2) = k_2x_2 \qquad (5.51)$$

Because no mass term appears in the second equation, the system given by equation (5.51) is of third order. Equation (5.51) can be solved by assuming periodic motions of the form

$$x_1(t) = X_1e^{j\omega t}, \quad \text{and} \quad x_2(t) = X_2e^{j\omega t} \qquad (5.52)$$

and considering the exponential representation of the harmonic driving force. Substitution of equation (5.52) into (5.51) yields

$$(k_1 - m\omega^2 + jc\omega)X_1 - jc\omega X_2 = F_0$$
$$jc\omega X_1 - (k_2 + jc\omega)X_2 = 0 \qquad (5.53)$$

Solving for the amplitudes X_1 and X_2 yields

$$X_1 = \frac{F_0(k_2 + jc\omega)}{k_2(k_1 - m\omega^2) + jc\omega(k_1 + k_2 - m\omega^2)} \qquad (5.54)$$

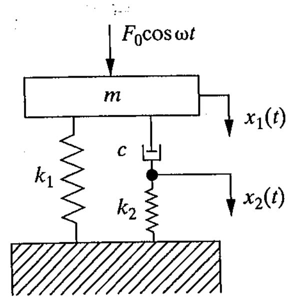


Figure 5.23 Model of a machine mounted on an elastic foundation through an elastic damper to provide vibration isolation.

and

$$X_2 = \frac{c\omega F_0 j}{k_2(k_1 - m\omega^2) + c\omega(k_1 + k_2 - m\omega^2)j} \tag{5.55}$$

These two amplitude expressions can be simplified further by substituting the nondimensional quantities $r = \omega/\sqrt{k_1/m}$, $\gamma = k_2/k_1$, and $\zeta = c/(2\sqrt{k_1 m})$. The force transmitted to the base is the vector sum of the two forces k_1x_1 and k_2x_2. Using complex arithmetic and a vector sum (recall Section 2.3) the force transmitted can be written as

$$\text{T.R.} = \frac{F_T}{F_0} = \frac{\sqrt{1 + 4(1 + \gamma)^2\zeta^2 r^2}}{\sqrt{(1 - r^2)^2 + 4\zeta^2 r^2(1 + \gamma - r^2\gamma)^2}} \tag{5.56}$$

which describes the transmissibility ratio for the system of Figure 5.23.

The force transmissibility ratio can be optimized by viewing the ratio F_T/F_0 as a function of r and ζ. Figure 5.24 yields a plot of F_T/F_0 versus r for $\gamma = 0.333$ and for several values of ζ. This illustrates that the value of the damping ratio greatly affects the transmissibility at resonance. A three-dimensional plot of F_T/F_0 versus r and ζ is given in Figure 5.25, which illustrates that the saddle point value of ζ and r yields the best design for the minimum transmissibility of the maximum force transmitted.

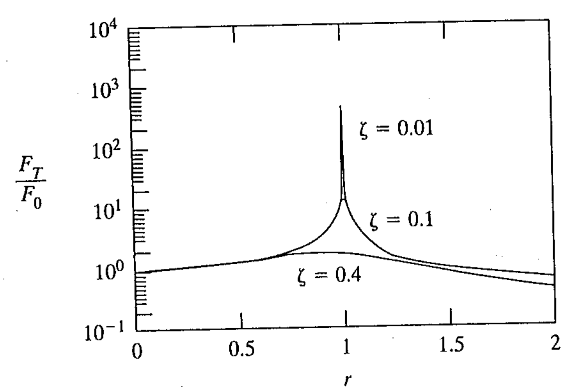


Figure 5.24 Plot of equation (5.56) illustrating the effect of damping on the magnification of force transmitted to ground.

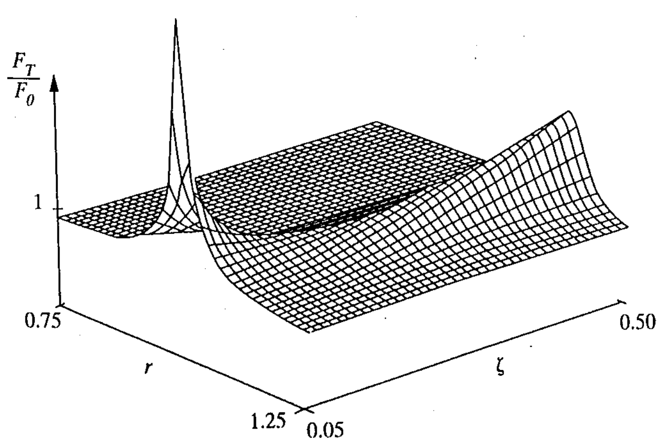


Figure 5.25 Plot of equation (5.56) illustrating F_T/F_0 versus ζ versus r. The plot shows the point where damping minimizes the maximum transmissibility.

The saddle point illustrated in Figure 5.25 can be found from the derivative of T.R. as given in equation (5.56). These partial derivatives are

$$\frac{\partial(\text{T.R.})}{\partial\zeta} = 0 \quad \text{yields} \quad r_{\max} = \frac{\sqrt{2(1+\gamma)}}{\sqrt{1+2\gamma}} \tag{5.57}$$

and

$$\frac{\partial(\text{T.R.})}{\partial r} = 0 \quad \text{yields} \quad \zeta_{\text{op}} = \frac{\sqrt{2(1+2\gamma)/\gamma}}{4(1+\gamma)} \tag{5.58}$$

These values of r correspond to an optimal design of this type of isolation device. At the saddle point, the value of T.R. becomes

$$(\text{T.R.})_{\max} = 1 + 2\gamma \tag{5.59}$$

which results from substitution of equations (5.57) and (5.58) into equation (5.56). This illustrates that as long as $\gamma < 1$, T.R. < 3 and the isolation system will not cause much difficulty at resonance.

Example 5.5.2

An isolation system is to be designed for a machine modeled by the system of Figure 5.23 (i.e., an elasticity coupled viscous damper). The mass of the machine is $m = 100$ kg and the stiffness $k_1 = 400$ N/m. The driving frequency is 10 rad/s at nominal operating conditions. Design this system (i.e., choose k_2 and c) such that the maximum transmissibility ratio at any speed is 2 (i.e., design the system for "start up" or "run through"). What is the T.R. at the normal operating condition of a driving frequency of 10 rad/s?

Solution For $m = 100$ kg and $k_1 = 400$ N/m, $\omega_n = \sqrt{400/100} = 2$ rad/s, so that the normal operating condition is well away from resonance (i.e., $r = \omega/\omega_n = 10/2 = 5$ at running conditions). Equation (5.59) yields that the maximum value for T.R. is

$$(\text{T.R.})_{\max} = 1 + 2\gamma \leq 2$$

so that $\gamma = 0.5$ and $k_2 = (0.5)(k_1) = (0.5)(400\text{ N/m}) = 200$ N/m. With $\gamma = 0.5$, the optimal choice of damping ratio is given by equation (5.58) to be

$$\zeta_{\text{op}} = \frac{\sqrt{2(1+2\gamma)/\gamma}}{4(1+\gamma)} = 0.4714$$

Hence the optimal choice of damping coefficient is

$$c_{\text{op}} = 2\zeta_{\text{op}}\omega_n m = 2(0.4714)(2)(100) = 188.56 \text{ kg/s}$$

The T.R. value at nominal operating frequency of $\omega = 10$ rad/s is given by equation (5.56) to be ($r = 10/2 = 5$)

$$\text{T.R.} = \frac{\sqrt{1 + 4(1+0.5)^2(0.4714)^2(5)^2}}{\sqrt{(1-5^2)^2 + 4(0.4714)^2(5)^2[1 + 0.5 - 5^2(0.5)]^2}} = 0.12$$

Hence the design $k_2 = 200$ N/m and $c = 188.56$ kg/s will protect the surroundings by a T.R. of 0.12 (i.e., only 12% of the applied force is transmitted to ground) and limits the force transmitted near resonance to a factor of 2.

□

5.6 VISCOELASTIC DAMPING TREATMENTS

A common and very effective way to reduce transient and steady-state vibration is to increase the amount of damping in the system so there is greater energy dissipation. This is especially useful in aerospace structures applications, where the added mass of an absorber system may not be practical. While a rigorous derivation of the equations of vibration for structures with damping treatments is beyond the scope of this book, formulas are presented that provide a sample of design calculations for using damping treatments.

A damping treatment consists of adding a layer of viscoelastic material, such as rubber, to an existing structure. The combined system often has a higher damping level and thus reduces unwanted vibration. This procedure is described by using the *complex stiffness* notation. The concept of complex stiffness results from considering the harmonic response of a damped system of the form

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{j\omega t} \tag{5.60}$$

Recall from Section 2.3 that the solution to equation (5.60) can be calculated by assuming the form of the solution to be $x(t) = Xe^{j\omega t}$, where X is a constant and $j = \sqrt{-1}$. Substitution of the assumed form into equation (5.60) and dividing by the nonzero function $e^{j\omega t}$ yields

$$[-m\omega^2 + (k + j\omega c)]X = F_0 \tag{5.61}$$

This can be written as

$$\left[-m\omega^2 + k\left(1 + \frac{\omega c}{k} j\right)\right]X = F_0 \tag{5.62}$$

or

$$[-m\omega^2 + k^*]X = F_0 \tag{5.63}$$

where $k^* = k(1 + \bar{\eta} j)$. Here $\bar{\eta} = \omega c/k$ is called the *loss factor* and k^* is called the *complex stiffness*. This illustrates that in steady state, the viscous damping in a system can be represented as an "undamped" system with a complex-valued stiffness. The imaginary part of the stiffness, $\bar{\eta}$, corresponds to the energy dissipation in the system. Since the loss factor has the form

$$\bar{\eta} = \frac{c}{k}\omega \tag{5.64}$$

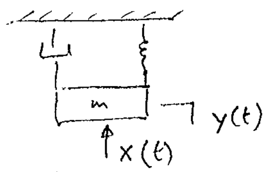
the loss factor depends on the driving frequency and hence is said to be frequency dependent. Hence the value of the energy dissipation term depends on the value of the driving frequency of the external force exciting the structure.

Brüel & Kjær

Modal Analysis - Theory and Application

m, c, k time invariant

force- line lower to Channel A $x(t)$
response signal channel B $y(t)$
thin wax on accelerometer good to several KHz



F

$$m\ddot{y} + c\dot{y} + ky = x$$

$$(-m\omega^2 + cj\omega + k)Y = X$$

$$\frac{Y}{X} = \frac{1}{-m\omega^2 + cj\omega + k}$$

independent of input & output

ω = freq in rad/s
f = hertz $\quad \omega = 2\pi f$

This session will be devoted to modal testing techniques, and the use of Bruel & Kjaer modal analysis instruments and software.

LINEAR SYSTEMS

$f_n = 412$

$$Q = \frac{f_n}{f_2 - f_1}$$

f_2 -, f_1 - } ½ power pts at 3 dB

$$Q = \frac{412}{412.5 - 411} = 275$$

$$\zeta = \frac{1}{2Q} = \frac{1}{550} \doteq .002$$

$$= \frac{c}{2\sqrt{km}} \quad \text{for } \zeta \leq .1$$

$$y(t) = \int_0^t h(t-\tau)\,x(\tau)\,d\tau$$

convolution

impulse response

Day 1

- Introduction to modal analysis
- Frequency response functions
- Using a dual-channel FFT analyzer
- Recognizing "good" and "bad" modal data
- Structural excitation techniques
- The software
- Structural modeling and data acquisition

Day 2

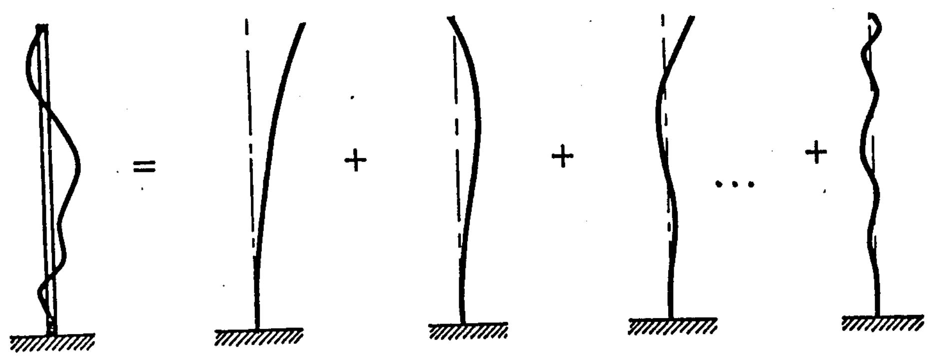
- Mode shapes
- Software organization, commands, and tables
- Verifying the modal model
- Structural Dynamic Modification (SDM)
- Forced Response Simulation (FRS)

Modal Behaviour

- The Dynamic Response of a structure is the sum of a discrete set of independent predictable motions

- These motions are called
  * Normal Modes

- A Mode is described by
  * Natural Frequency & Damping
  * Mode Shape



$$X(y,t) = a_1(t)\{\emptyset\}_1 + a_2(t)\{\emptyset\}_2 + a_3(t)\{\emptyset\}_3 \;\ldots\; + a_n(t)\{\emptyset\}_n$$

- Modal Analysis is the process of determining these modal parameters

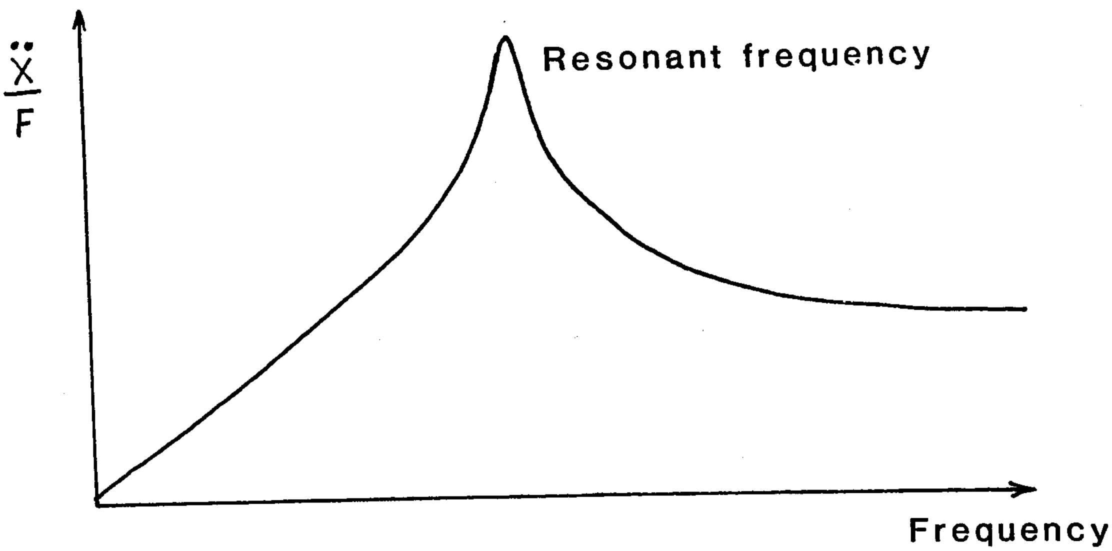
MODAL PARAMETERS

EIGENVALUES ----- RESONANT FREQUENCIES
DAMPING COEFFICIENTS

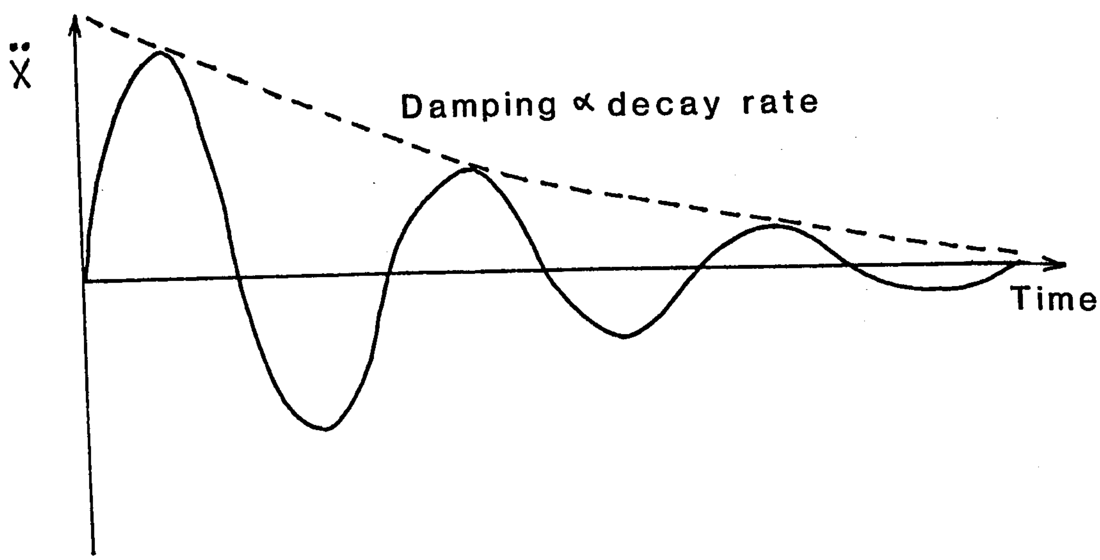
EIGENVECTORS --- MODE SHAPES

RESIDUES ------- SCALING FACTORS
(Engineering units)

EIGENVALUE

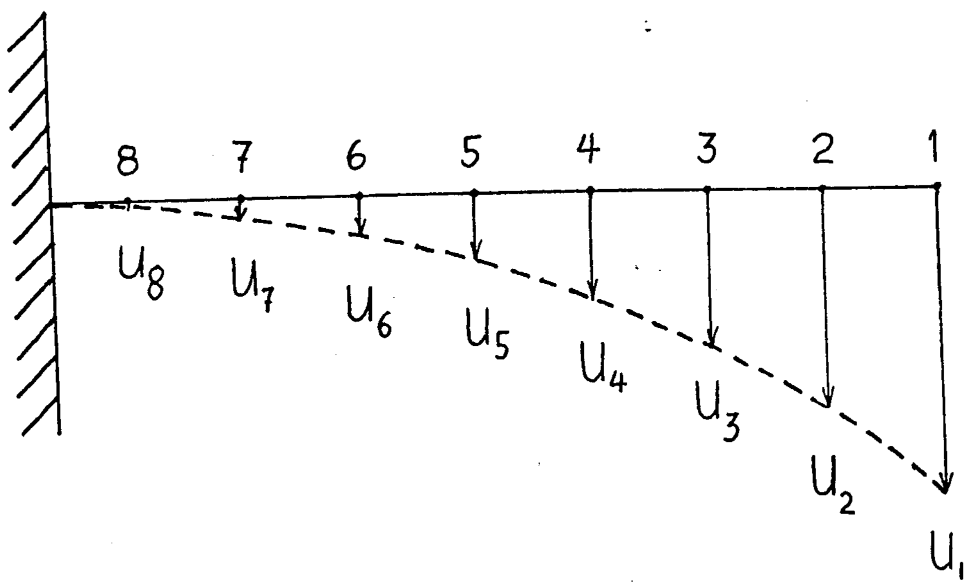


$\ddot{X}$ / F
Resonant frequency
Frequency



$\ddot{X}$
Damping ∝ decay rate
Time

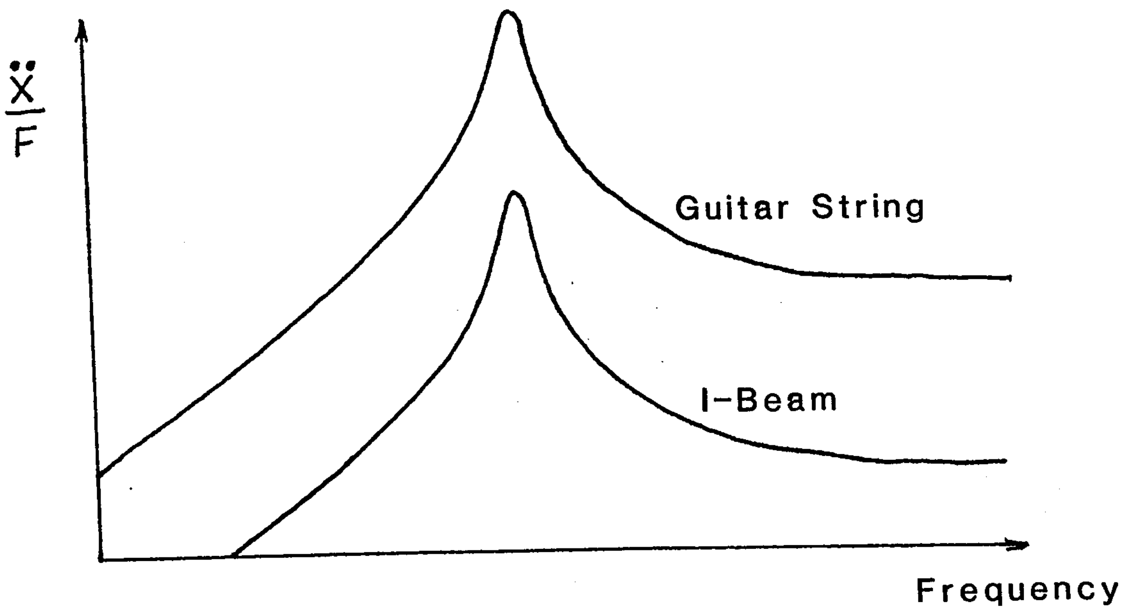
EIGENVECTOR



Underformed ————————

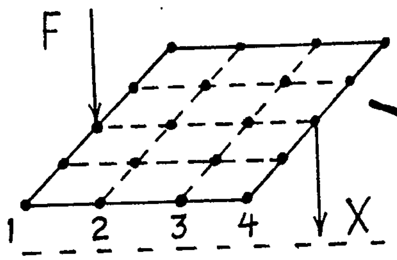
deformed - - - - - - - - -

RESIDUE



Residue indicates the strength of a mode

REVIEW: MODAL MODEL

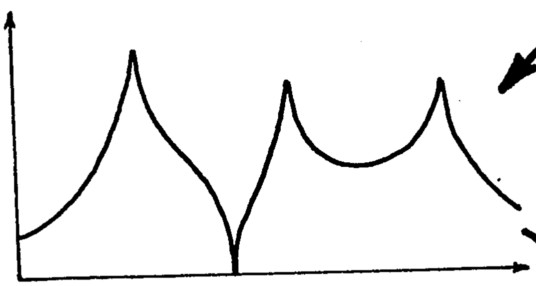


1) SPATIAL MODEL: DOF'S

2) EQUATIONS OF MOTION:

FRF matrix

$$\begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix} = \begin{bmatrix} H_{1,1} & H_{1,2} & \cdots & H_{1,n} \\ H_{2,1} & & & \vdots \\ \vdots & & & \\ H_{n,1} & & \cdots & H_{n,n} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix}$$



3) *Measurements*

FRF's (one row or column)

Curve fit

4) MODAL MODEL:

Each element of the matrix is written in terms of the modal parameters.

Freq.	Damp.	Mode shape
ω_1	ζ_1	$\{u\}_1$
ω_2	ζ_2	$\{u\}_2$
$\vdots$	$\vdots$	$\vdots$
ω_r	ζ_r	$\{u\}_r$

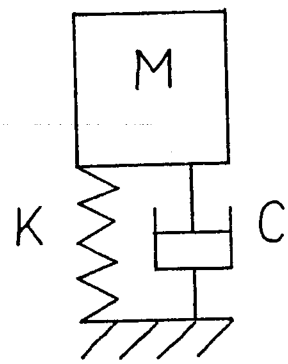
REVIEW: MODAL MODEL

SUMMARY:

THE MODAL MODEL GIVES THE RESPONSE AT ONE OR MORE DOF's DUE TO A FORCE EXCITATION AT ONE OR MORE DOF's.

THE PROBLEM SIZE HAS BEEN REDUCED FROM THE NUMBER OF SPATIAL DOF's TO THE NUMBER OF MODES.

FREQUENCY TERMS



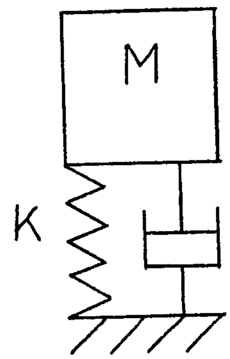
UNDAMPED NATURAL FREQUENCY:

$$\omega_n = \sqrt{\frac{K}{M}}$$

DAMPED NATURAL FREQUENCY:

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

DAMPING TERMS



DAMPING:

C (force/velocity or mass/time)

CRITICAL DAMPING:

C_c (no oscillations)

$C_c = 2\sqrt{KM}$

DAMPING RATIO:

ζ (Zeta)

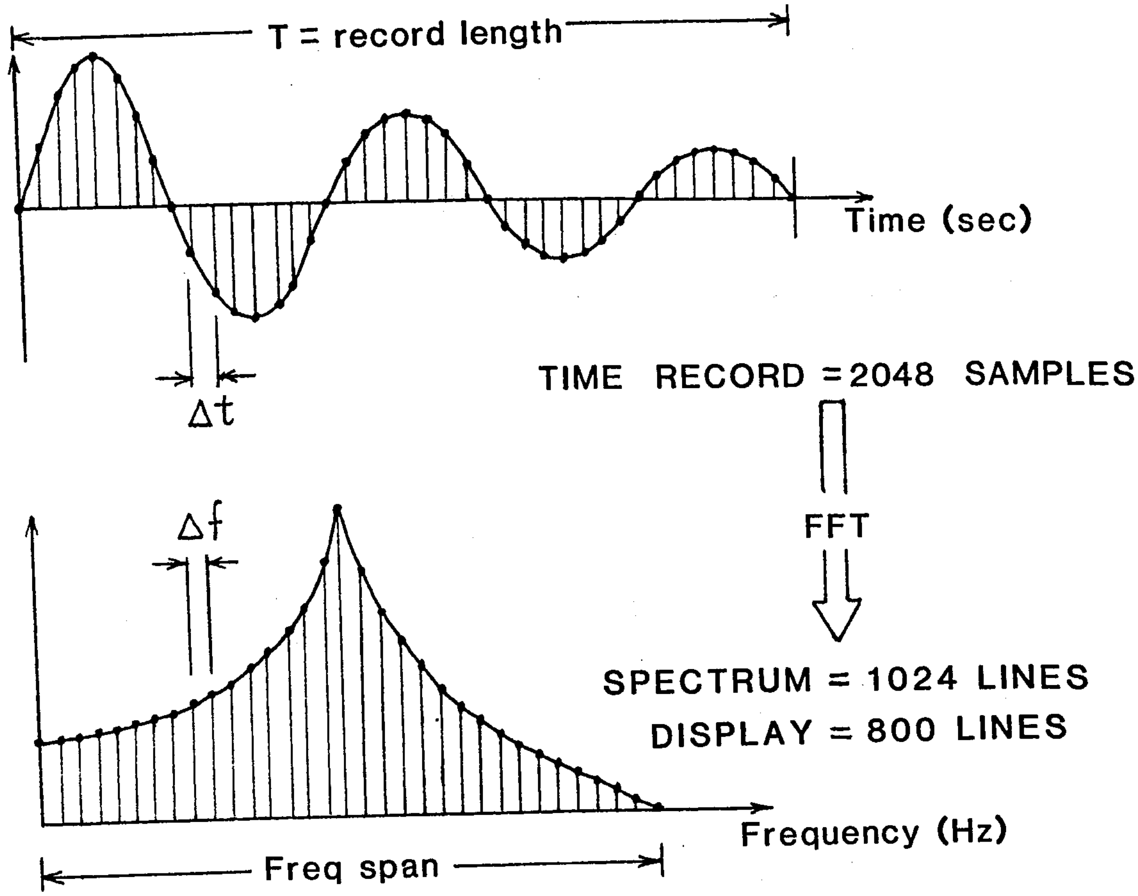
$\zeta = C/C_c$

DECAY RATE:

σ (1/time, Hz)

$\sigma = \zeta\,\omega_n = \zeta\sqrt{K/M}$

FFT FUNDAMENTALS



FFT PARAMETERS:

Δf = FREQ SPAN / 800

$T = 1 / \Delta f$

$\Delta t = T / 2048$

FFT FUNDAMENTALS

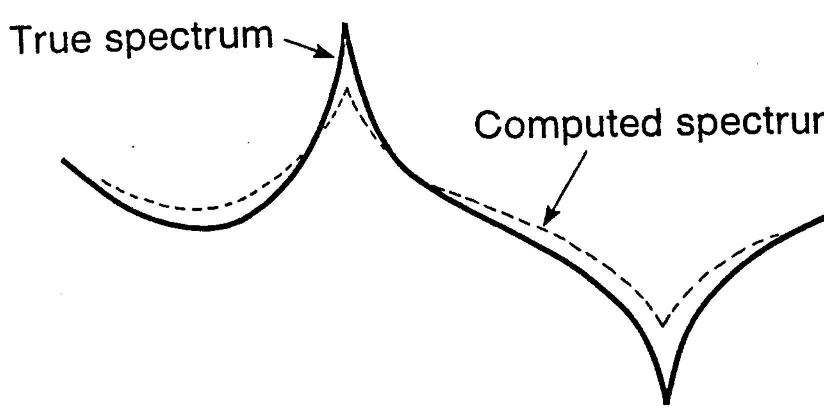
CONSEQUENCES OF USING A TIME LIMITED SIGNAL IN AN FFT ANALYZER.

1. FREQUENCY RESOLUTION:

$$\Delta f = 1 / T$$

2. RESOLUTION BIAS ERROR (LEAKAGE):

   - PEAKS CAN BE MEASURED TOO LOW.
   - VALLEYS CAN BE MEASURED TOO HIGH.



FREQUENCY RESOLUTION

GOOD FREQUENCY RESOLUTION IS IMPORTANT FOR ACCURACY WHEN MEASURING FREQUENCY RESPONSE FUNCTIONS OF MECHANICAL STRUCTURES.

TYPICALLY, MECHANICAL STRUCTURES EXHIBIT VERY SHARP RESONANCE PEAKS.

IF FREQUENCY RESOLUTION IS INADEQUATE, THESE PEAKS WILL BE MEASURED TOO LOW (RESOLUTION BIAS ERROR).

INCREASE FREQUENCY RESOLUTION BY:

- ZOOM ANALYSIS
- USE A LARGE FFT TRANSFORM SIZE (2048 time points = 800 lines)

AVERAGING: DUAL CHANNEL FFT

INPUT AUTOSPECTRUM:
(For modal analysis input is usually force)

$$G_{AA} = \frac{1}{n}\sum(S_A^* S_A)$$

OUTPUT AUTOSPECTRUM:
(Output is usually response)

$$G_{BB} = \frac{1}{n}\sum(S_B^* S_B)$$

gives amplitude of power spectrum

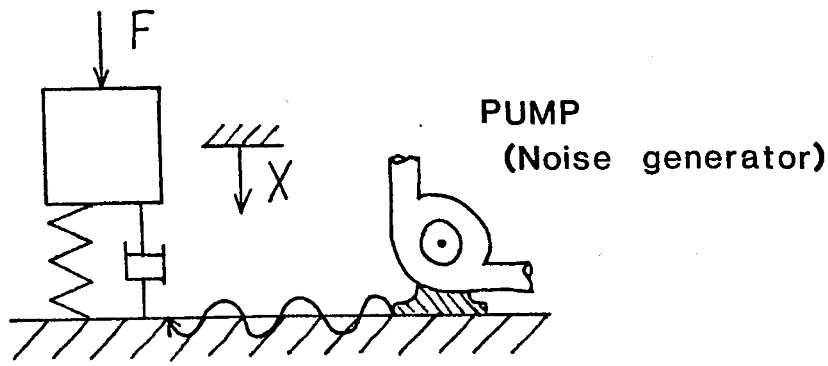
CROSS–SPECTRUM:

$$G_{AB} = \frac{1}{n}\sum(S_A^* S_B)$$

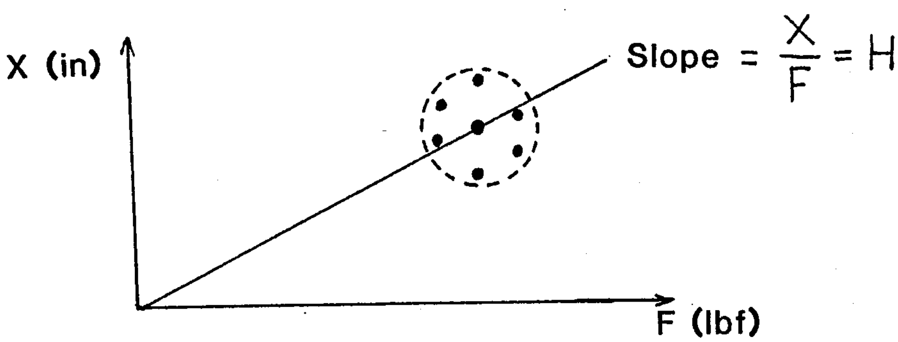
Gives phase

n = averaging number or ensemble size
S_A = instantaneous spectrum of channel A
S_B = instantaneous spectrum of channel B
* = complex congugate

AVERAGING RANDOM NOISE



- CONSIDER EXCITATION (FORCE) AT A SINGLE FREQUENCY.



- DUAL CHANNEL FFT AVERAGES RANDOM SCATTER FOR EACH FREQUENCY LINE IN THE FREQUENCY RESPONSE FUNCTION ESTIMATE.

CALCULATING FREQUENCY RESPONSE

$H_1 = G_{AB} / G_{AA}$

AVERAGES OUT UNCORRELATED RANDOM NOISE MIXED IN THE OUTPUT (RESPONSE) CHANNEL. H_1 IS THE TRADITIONAL METHOD.

$H_2 = G_{BB} / G_{BA}$

AVERAGES OUT UNCORRELATED RANDOM NOISE MIXED IN THE INPUT (FORCE) CHANNEL. CAN BE BETTER SUITED THAN H_1 WHEN USING SHAKER EXCITATION.

COHERENCE FUNCTION

$$\gamma_{AB}^2 = \frac{|G_{AB}|^2}{G_{AA} \cdot G_{BB}}$$

RELATES HOW MUCH OF THE MEASURED OUTPUT SIGNAL IS LINEARLY RELATED TO THE MEASURED INPUT SIGNAL.

CAN BE USED TO CHECK THE QUALITY OF THE FREQUENCY RESPONSE FUNCTION.

$$0 < \gamma_{AB}^2 < 1$$

G_{AB} , G_{AA} AND G_{BB} ARE AVERAGED OVER MANY RECORDS.

FOR ONE RECORD ONLY (NO AVERAGING):

$$\gamma^2(f) = 1$$

The Coherence Function

• **Definition**

$$\gamma^2(f) = \frac{|G_{AB}(f)|^2}{G_{AA}(f) \cdot G_{BB}(f)}$$

function of frequency

$$0 \leq \gamma^2(f) \leq 1$$

Expresses degree of

Linear

relationship between A(f) and B(f)

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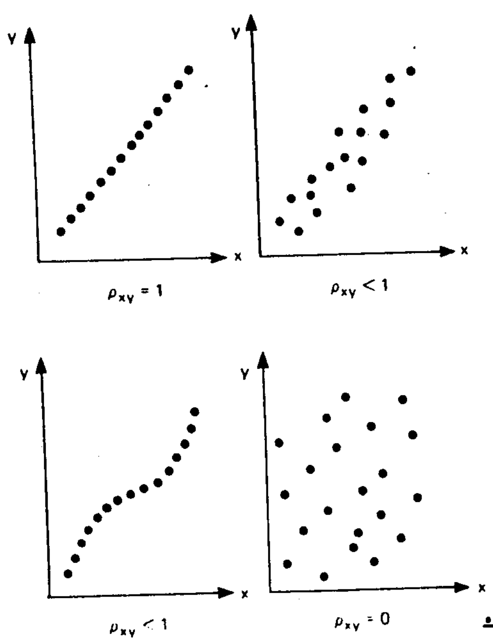
The Coherence Function

• **Correlation Coefficient**

$$\rho_{xy} = \frac{\sigma_{xy}}{\sigma_x \cdot \sigma_y}$$



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COHERENCE

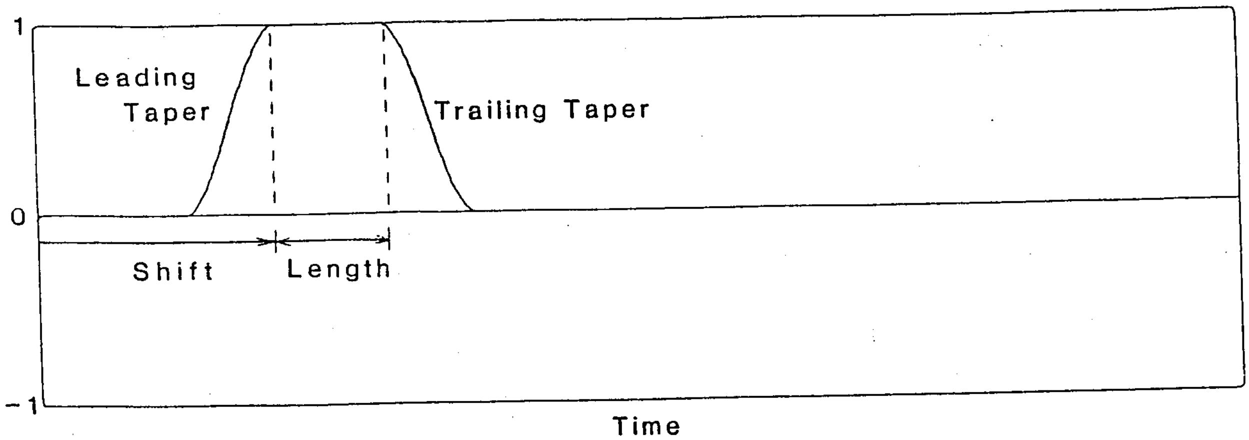
REASONS FOR LOW COHERENCE:

- NON–LINEAR SYSTEM
- NOISE IN MEASURED OUTPUT SIGNAL
- NOISE IN MEASURED INPUT SIGNAL
- OTHER INPUTS NOT CORRELATED WITH MEASURED INPUT SIGNAL
- LEAKAGE (RESOLUTION BIAS ERROR)
- NO COMPENSATION FOR PROPAGATION TIME

Setting Up the Analyzer

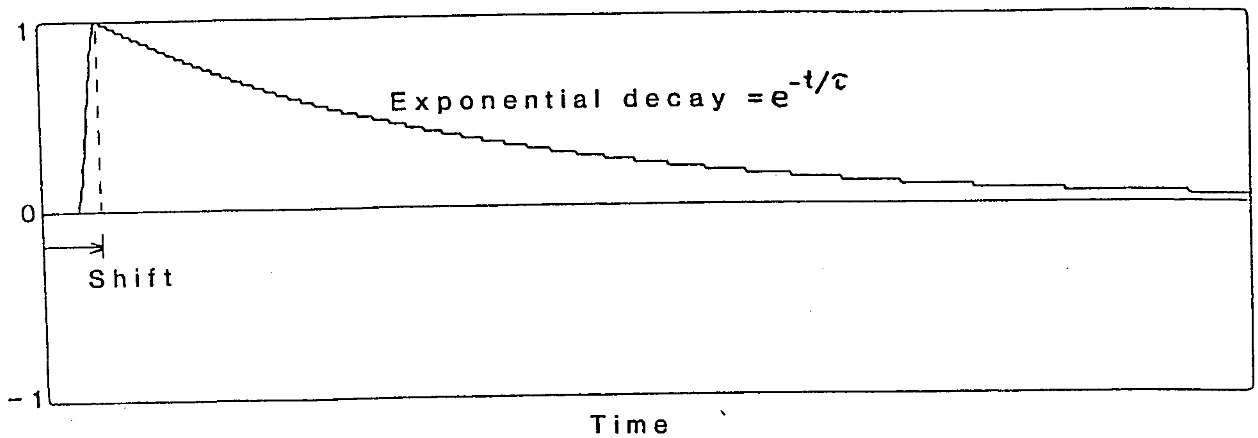
- Frequency Range
- Trigger
- Windows
- Attenuators
- Display

FORCE WINDOW



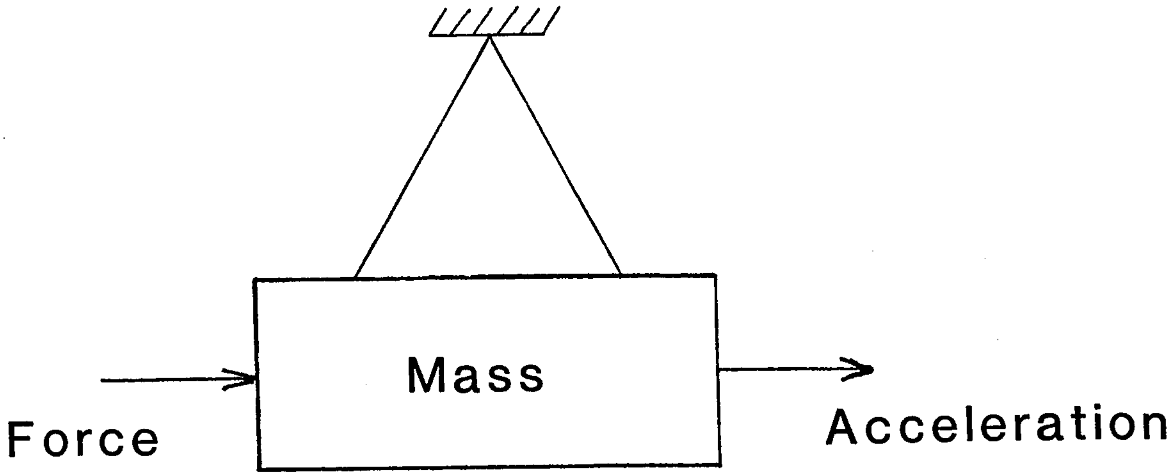
PURPOSE: ELIMINATE NOISE

EXPONENTIAL WINDOW



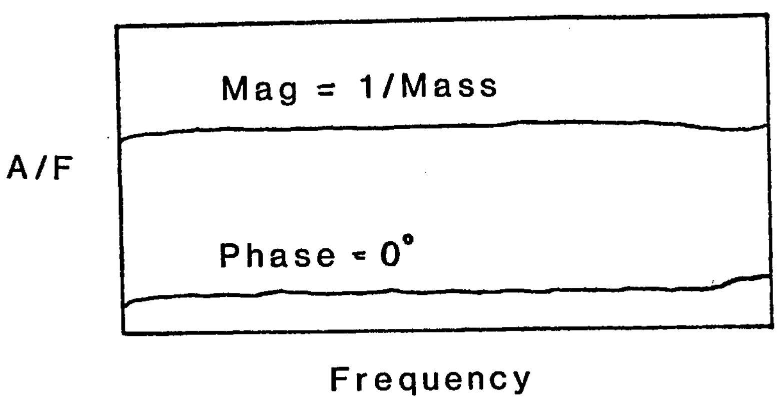
PURPOSE: ELIMINATE TRUNCATION ERROR (LEAKAGE)

Free Mass Calibration



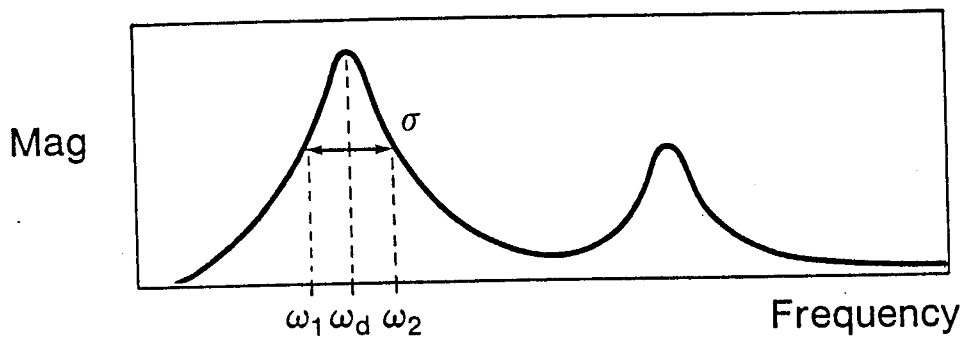
$$H(f)=A/F=1/Mass$$

Calibrates the entire measuring chain



MODAL PROPERTY CALCULATION

1. From the Magnitude of the Frequency Response Function



The damped natural frequency ω_d is the frequency of maximum magnitude

The damping σ_r can be found from the half power points ω_1, ω_2

$$Q = \frac{\omega_d}{\omega_2 - \omega_1}$$ (= Quality factor)

$$\zeta_r = \frac{1}{2Q}$$ (= damping ratio, damping coefficient, or percentage of critical damping)

$$\sigma_r = \zeta_r \cdot \omega_d$$ (= decay rate)

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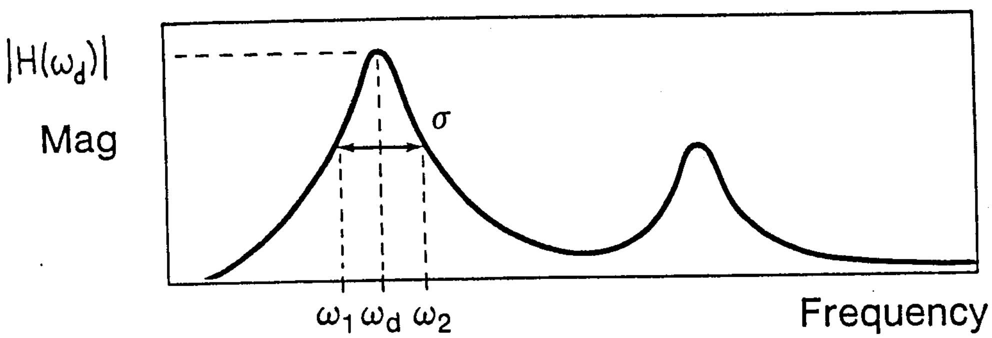
MODAL PROPERTY CALCULATION

1. From the Magnitude of the Frequency Response Function



The damped natural frequency ω_d is the frequency of maximum magnitude

The residue ,R_r , can be found from the magnitude of the frequency response function at ω_d.

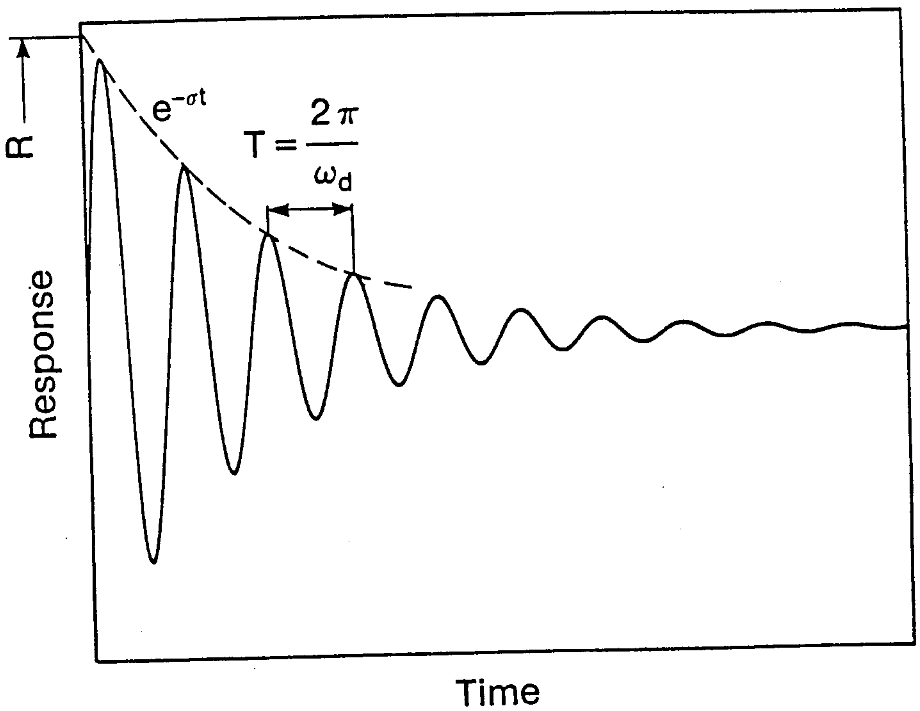
$$R_r \approx 2\,|H(\omega_d)|\,\sigma_r$$

σ_r = damping

NOTE:
The residue is indicative of the "strength" of a mode. For example, an I–beam and a guitar string may have a mode with identical natural frequency and damping, but the force necessary to excite the modes is very different.

MODAL PROPERTY CALCULATION

2. From the Impulse Response



τ = time constant of the decay

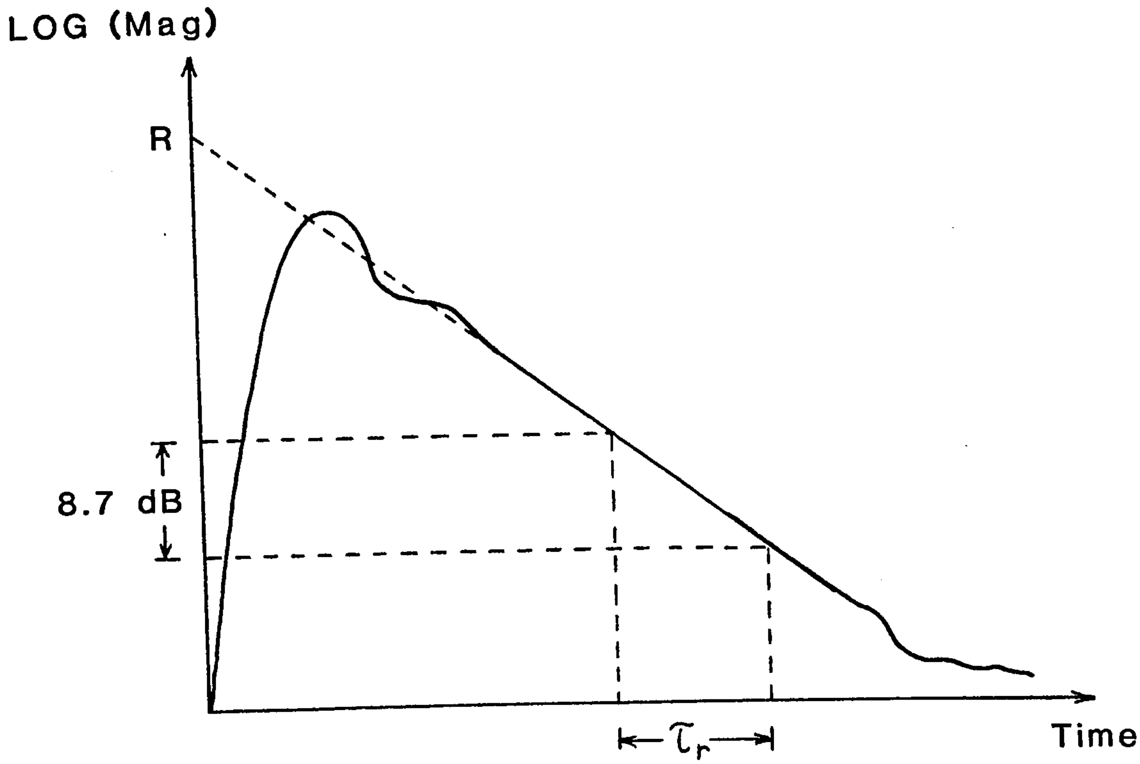
$$Q = \pi f_d \tau$$

$$\zeta_r = \frac{1}{2Q}$$

$$\sigma_r = \zeta_r \cdot \omega_d$$

R = Residue

IMPULSE RESPONSE FUNCTION



R = Residue

$$\sigma_r = 1/\tau_r$$

$$\zeta_r = 1/(2\pi f_r \tau_r)$$

$$Q_r = \pi f_r \tau_r$$

MOD55

Modal Analysis

Step–by–Step Experimental Procedure

1 Setup

- Decide test point and directions (DOF's)
- Mount Structure. ("Free" or Fixed)
- Choose, Adjust Excitation
- Setup Analyzer/Transducers
- Calibrate Measuring Chain
- Make Trial Measurements

2 Measurements

- Measure one row or one column in FRF matrix

3 Parameter Estimation

- Estimate – Natural Frequencies
  – Dampings
  – Residues

4 Draw Mode Shapes

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Multi Degree of Freedom Curve Fitting Procedure

In almost all modal analyses there will be one or more well defined, lightly coupled, and lightly damped mode(s). The procedure presented in this appendix is based on this assumption.

Procedure:

1. Document the frequency of as many peaks as possible.
2. Use an SDOF curve-fitting algorithm wherever possible and process the associated mode shapes, which are from here on referred to as "known modes".
3. Include one or more of these "known modes" in the MDOF frequency band.
4. Select the number of poles (modes) expected in this frequency range and at least two computational modes (one for inertia correction above the selected MDOF frequency range and one for flexibility residuals below the selected MDOF frequency range.
5. The algorithm will yield a frequency and damping table each time the MDOF, etc command is executed.
6. Compare the table of contents with the information gathered in Step 1.
7. The process now becomes iterative. By changing the number of poles (modes) and computational modes in the MDOF command, many FRD (Frequency and damping) lists are generated.
8. It is a good practice to synthesize the FRF used for the MDOF and to compare it with experimental results. Keep a copy of the synthesis and the frequency and damping table (FRD) for each iteration. The "best" MDOF, is when the poles in the MDOF generated FRD match the information gained in Step 1. There may also be other peaks that were not obvious in Step 1 that could be actual modes.

9. When the best number of poles (modes) and computational modes have been selected editing sometimes becomes necessary. Modes with excessively high damping or that are out of order with the "known" modes can be discarded prior to processing the mode shapes.

10. Proceed with generating the mode shapes.

11. Compare MDOF modes with common modes that were acquired with a SDOF fit.

12. If all information lines up, the other modes in that necessitated an MDOF fit are most likely accurate.

13. The section in the seminar that defines the Mode Indicator Function is helpful and is easily implemented in an SMS autosequence program.

A. For further software command information see the curve-fitting section of the Modal 3.0 manuals.