Mechanical Vibrations

Harmonically Excited Systems

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Figures and content adapted from

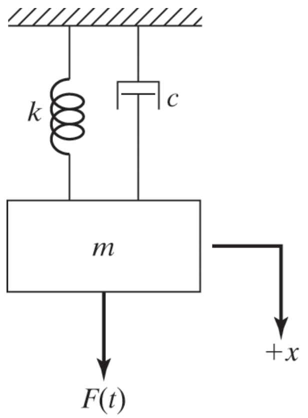
Textbook: Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition.

Chapter 3: Harmonically Excited Systems

Learning Objectives

- Find the responses of undamped and viscously damped single-degree-of-freedom systems subjected to different types of harmonic force, including base excitation and rotating unbalance.
- Distinguish between transient, steady-state, and total solutions.
- Understand the variations of magnification factor and phase angles with the frequency of excitation and the phenomena of resonance and beats.
- Study the responses of a damped system to a simple harmonic force, harmonic motion of the base and under a rotating unbalance, and the force transmitted to the base in each case.
- Identify self-excited problems and investigate their stability aspects.

Force Vibration



A mechanical or structural system is said to undergo forced vibration whenever external energy is supplied to the system during vibration.

The applied force or displacement excitation may be harmonic, nonharmonic but periodic, nonperiodic, or random in nature.

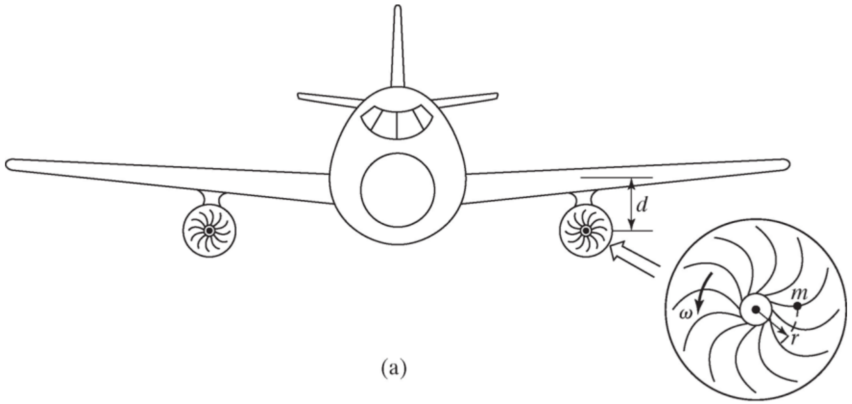
The nonperiodic excitation may have a long or short duration. The response of a dynamic system to suddenly applied nonperiodic excitations is called *transient response.*

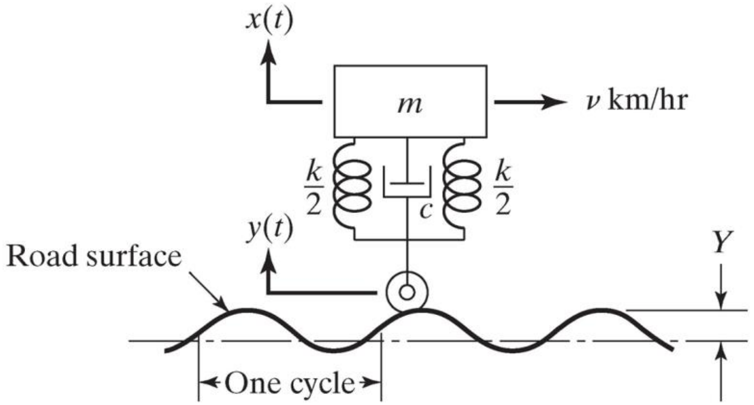
A harmonic, nonharmonic but periodic excitation will produce a steady-state response as long as the excitation is applied.

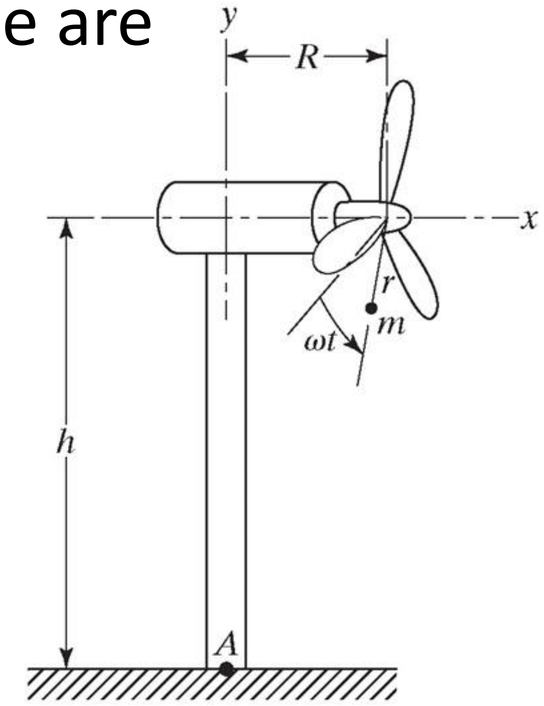
Harmonic excitation

The *response* of a system to a *harmonic excitation* is also *harmonic, and* with same frequency of excitation.

The vibration produced by an unbalanced rotating machine, the oscillations of a bridge or a tall tower due to a steady wind, and the vertical motion of an automobile on a sinusoidal road surface are examples of harmonically excited vibration.

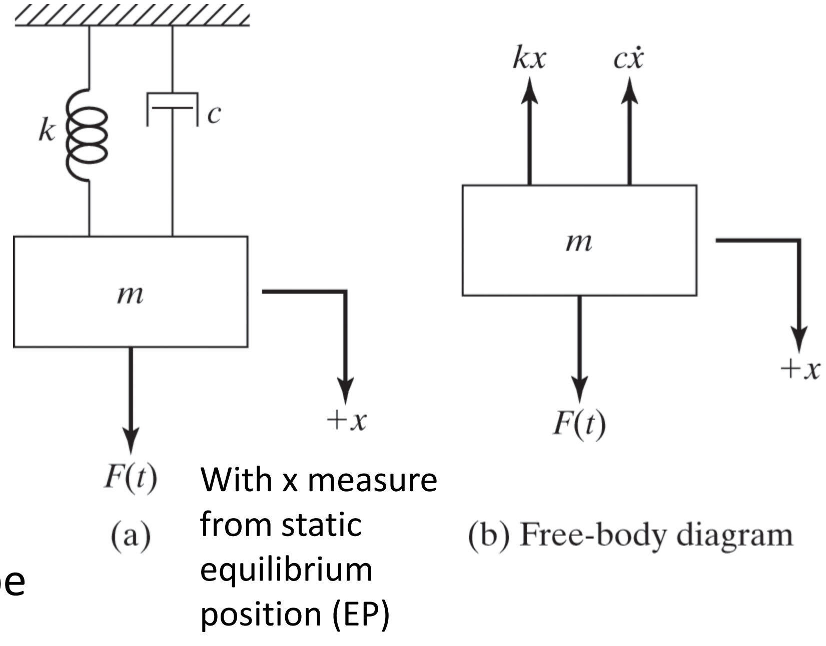






Equation of motion

$$m\ddot{x} + c\dot{x} + kx = F(t)$$



Since this equation is nonhomogeneous, its general **solution** x(t) is given by the sum of the homogeneous solution, $x_h(t)$, and the particular solution, $x_{\mathrm{p}}(t)$

$$x(t) = x_h(t) + x_p(t)$$

The **particular solution** will have the same form as the external function and can be calculated following :

$$x_p(t) = AF(t) + BF'(t) + CF''(t) + \cdots + CF^n(t)$$

The **homogeneous solution** represent the free vibration of the system, that dies out with time under each of the three possible conditions of damping (underdamping, critical damping, and overdamping) and under all possible initial conditions.

$$x_h(t) = \begin{cases} \zeta = 0, x(t) = X_0 \cos(\omega_{\,n} t - \phi) \\ \zeta < 1, x(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_{\,d} t - \phi) \\ \zeta = 1, x(t) = \mathrm{C}_1 e^{-\omega_n t} + \mathrm{C}_2 t e^{-\omega_n t} \\ \zeta > 1, x(t) = \mathrm{C}_1 e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)t} + \mathrm{C}_2 e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)t} \end{cases}$$

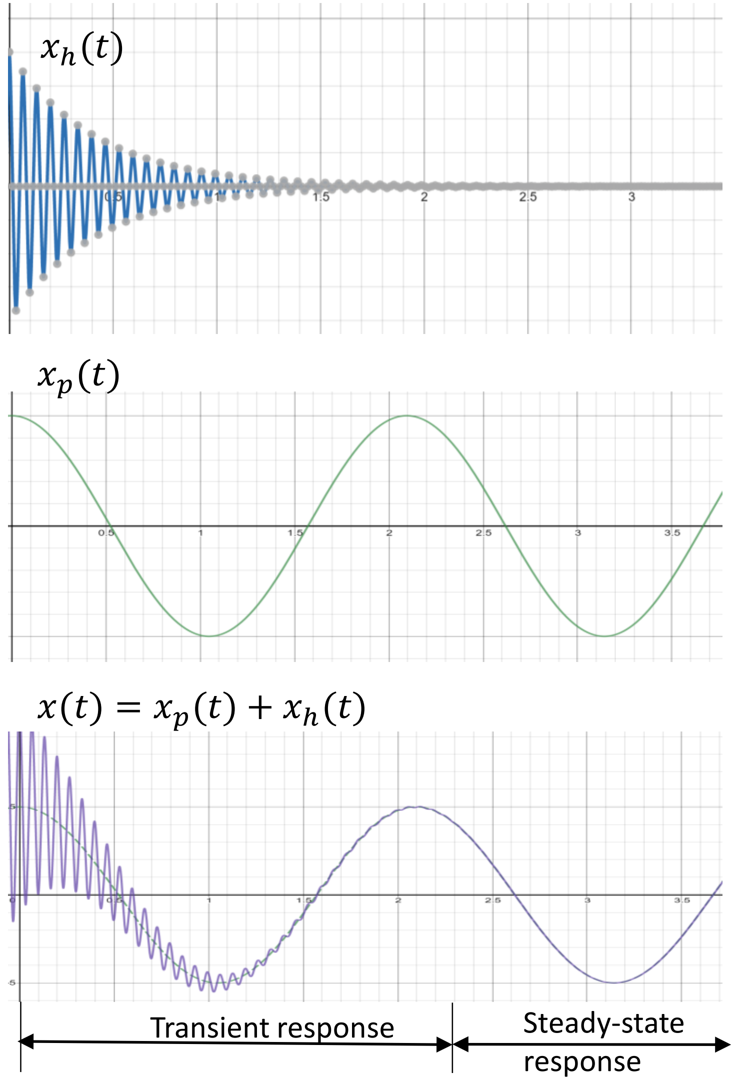
Transient and steady-state solution

If damping is present, It can be seen that $x_h(t)$ dies out and x(t) becomes $x_p(t)$ after some time.

The part of the motion that dies out due to damping (the free-vibration part) is called **transient.**

The rate at which the transient motion decays depends on the values of the system parameters k, c, and m.

To do the analysis of harmonic motion we ignore the transient motion and derive only the particular solution, which represents the **steady-state response**, under harmonic forcing functions.



Solution to a constant force

The **particular solution** will have the same form as the function and can be calculated following :

$$\boxed{F(t) = F_o} \qquad \boxed{m\ddot{x} + c\dot{x} + kx = F(t)}$$

$$x_p(t) = AF(t) + BF'(t) + CF''(t) + \cdots + CF^n(t)$$

$$\left.x_p(t) = AF_0, \quad \dot{x}_p(t) = \ddot{x}_p\,(t) = 0\right\} \quad m0 + c0 + kAF_o = F_o$$

$$\boxed{A = \frac{1}{k}} \qquad \boxed{x_p(t) = \frac{F_0}{k}}$$

The **total solution** will be

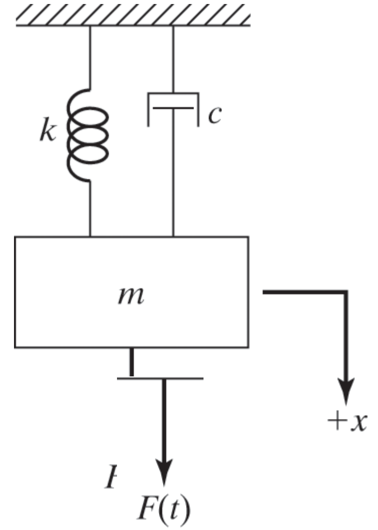
$$x(t) = x_p(t) + x_h(t) = \frac{F_0}{k} + x_h(t)$$

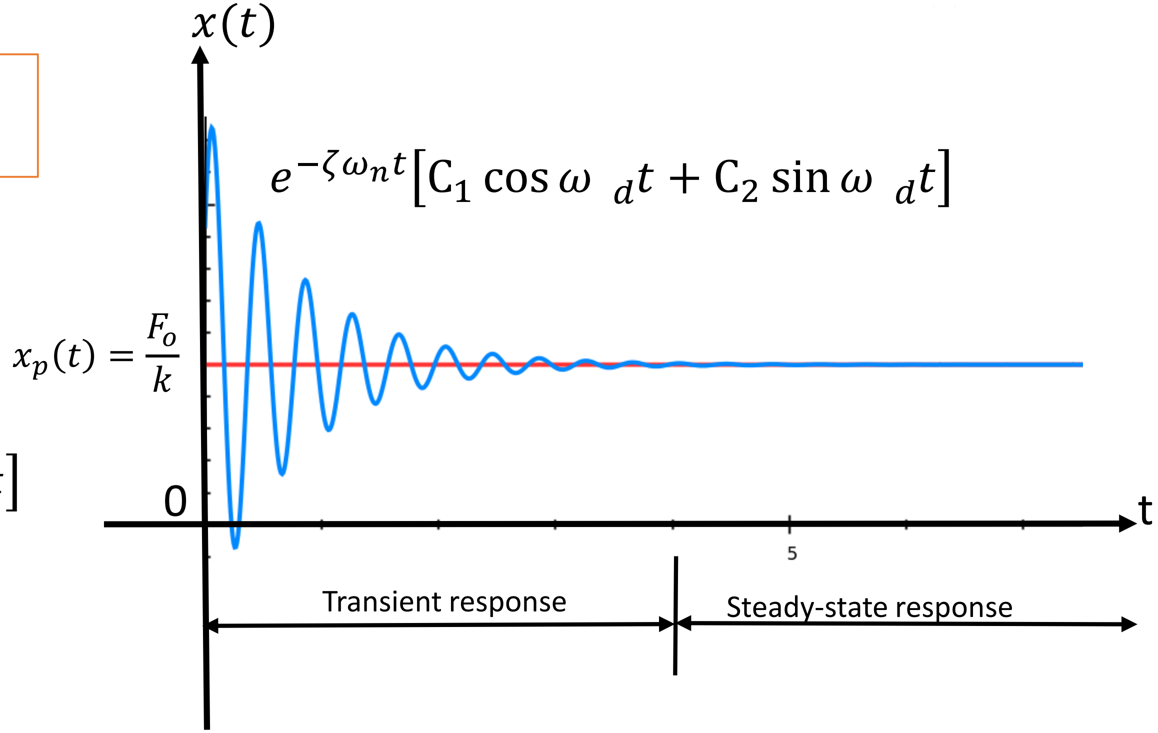
For a **undamped system**

$$x(t) = x_p(t) + x_h(t) = \frac{F_0}{k} + e^{-\zeta\omega_n t}[\mathrm{C}_1 \cos\omega_d t + \mathrm{C}_2 \sin\omega_d t]$$

with

$$\left.\begin{aligned} x(o) &= x_0 \\ \dot{x}(o) &= v_0 \end{aligned}\right\} \quad \begin{aligned} \mathrm{C}_1 &= x_0 - \frac{F_0}{k} \\ \mathrm{C}_2 &= \frac{v_0 + \zeta\omega_n\left(x_0 - \frac{F_0}{k}\right)}{\omega_d} \end{aligned}$$





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Harmonically Excited Undamped Systems

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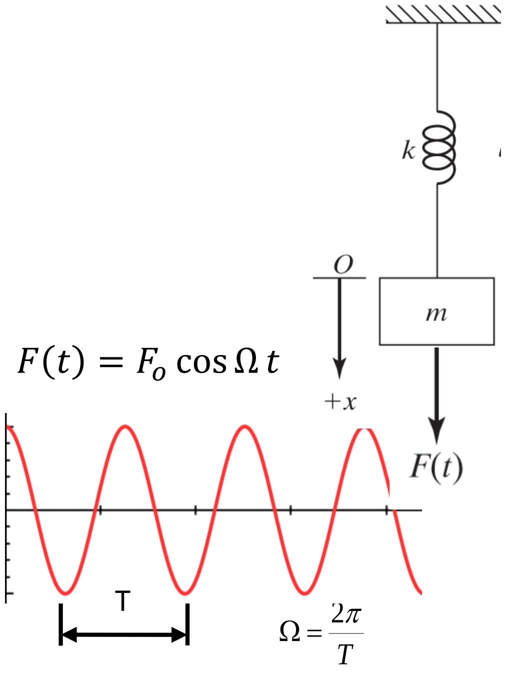
Chapter 3: Harmonically Excited Systems

Solution to a harmonic force (Mass- spring System)

Equation of motion: $m\ddot{x} + kx = F(t)$



The **total solution** is : $x(t) = x_p(t) + x_h(t)$

Where: $x_h(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$ $\quad x_p(t) = A \cos \Omega t + B \sin \Omega t$

The constants C_1 and C_2 on the **homogeneous solution** depend on the **initial conditions**, and have to be calculated for the whole equation.

To find A and B, derivate the **particular solution** and substitute in the equation of motion

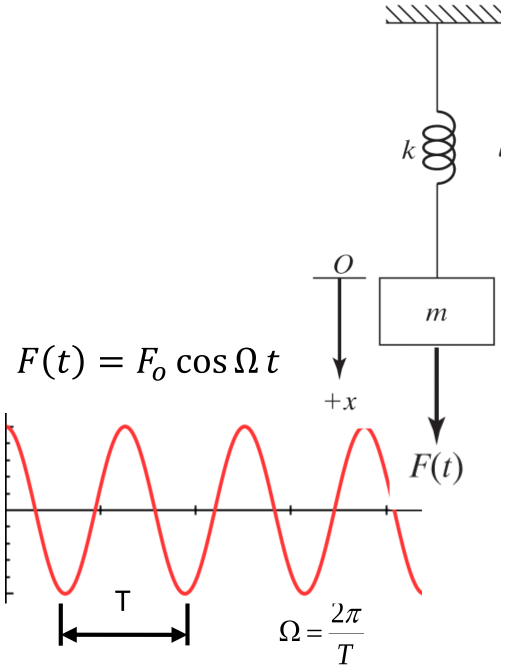
$$\dot{x}_p(t) = -A\Omega \sin \Omega t + B\Omega \cos \Omega t$$
$$\ddot{x}_p(t) = -A\Omega^2 \cos \Omega t - B\Omega^2 \sin \Omega t$$

Before we find the final solution lets introduce the following concepts:

$r = \frac{\Omega}{\omega_n}$ Frequency Ratio

$\delta_{st} = \frac{Fo}{k}$ Static deflection

Solution to a harmonic force (Mass –Spring System)



Equation of motion: $m\ddot{x} + kx = F(t)$

The **particular solution** is : $x_p(t) = A \cos \Omega t + B \sin \Omega t$

and the derivatives:

$$\dot{x}_p(t) = -A\Omega \sin \Omega + B\Omega \cos \Omega t$$

$$\ddot{x}_p(t) = -A\Omega^2 \cos \Omega t - B\Omega^2\, sin\, \Omega t$$

$$m\left(-A\Omega^2 \cos \Omega t - B\Omega^2 \sin \Omega t\right) + k(A \cos \Omega t + B \sin \Omega t) = F_0\, cos\, \Omega t$$

$$\left(-Am\Omega^2 + Ak\right) \cos \Omega t + \left(-Bm\Omega^2 + kB\right) \sin \Omega t = Fo\, cos\, \Omega t$$

$$\left(-Am\Omega^2 + Ak\right) = Fo$$

$$\left(-Bm\Omega^2 + kB\right) = 0$$

$$A = F_o \frac{1}{k - m\Omega^2}$$

$$B = 0$$

In term of natural frequency, frequency ratio *r* and static deflection

$$A = \frac{Fo}{k} \frac{1}{1 - \left(\frac{\Omega}{\omega_n}\right)^2} = \delta_{st} \frac{1}{1 - r^2}$$

The particular solution:

$$\boxed{x_p(t) = \delta_{st} \frac{1}{1 - r^2} \cos \Omega t}$$

$$\boxed{\boldsymbol{M} = \frac{X}{\delta_{st}} = \frac{1}{1 - r^2}}$$

Magnification factor, or amplitud ratio

Solution to a harmonic force

The **total solution** is : $x(t) = x_h(t) + x_p(t)$

Then: $$x(t) = C_1\cos(\omega_n t) + C_2\sin(\omega_n t) + \delta_{st}\frac{1}{1-r^2}\cos\Omega\, t$$

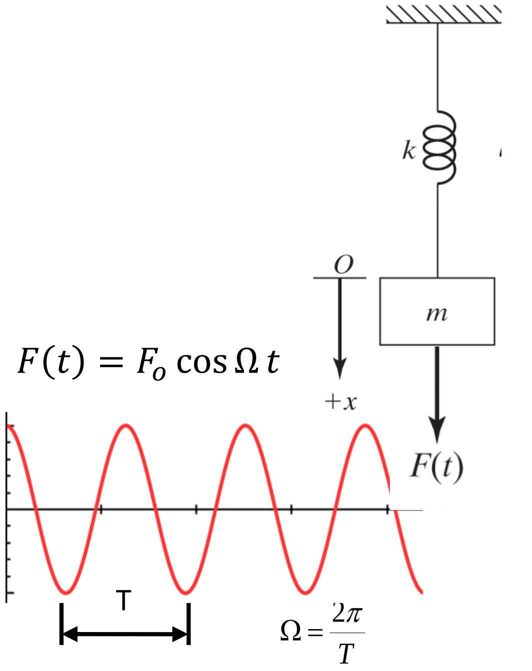
The constants C_1 and C_2 on the **homogeneous solution** depend on the **initial conditions**, and have to be calculated for the whole equation.

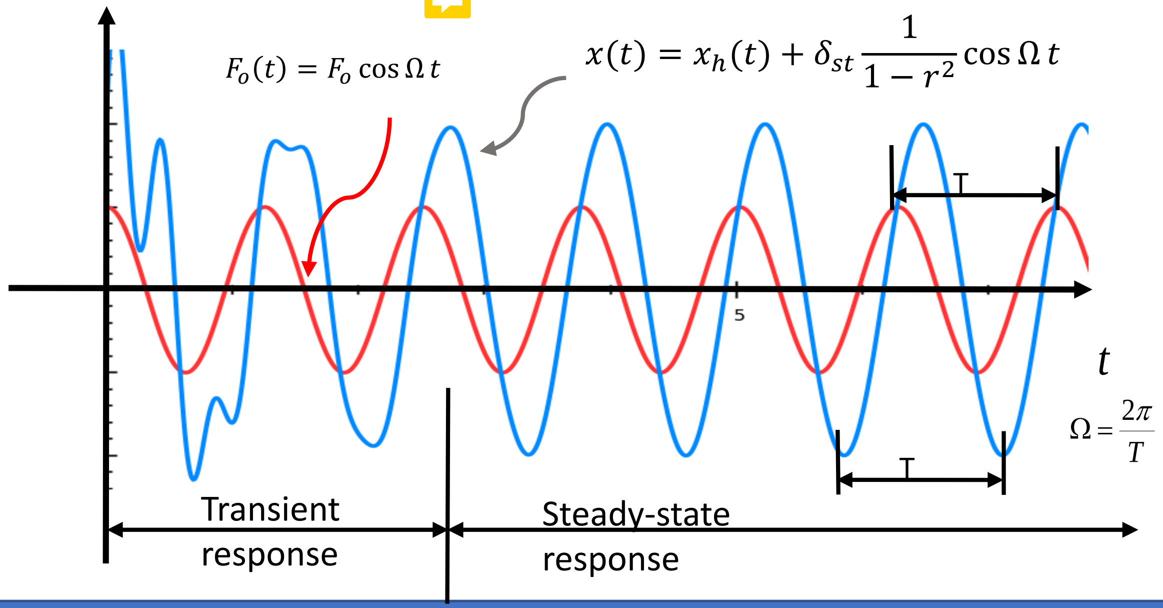
When applying initial conditions $\boldsymbol{x_0}$ and $\boldsymbol{v_0}$

$$C_1 = x_o - \frac{F_o}{k - m\Omega^2} = x_o - \delta_{st}\frac{1}{1-r^2} \qquad C_2 = \frac{v_o}{\omega_n}$$

The **total solution** is :

$$\boxed{x(t) = \left(x_o - \delta_{st}\frac{1}{1-r^2}\right)\cos(\omega_n t) + \frac{v_o}{\omega_n}\sin(\omega_n t) + \delta_{st}\frac{1}{1-r^2}\cos\Omega\, t}$$





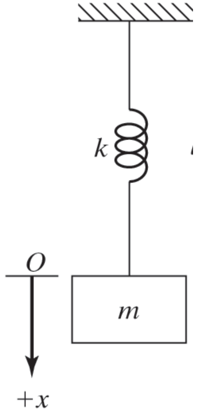
Magnification factor (M)

Also called amplification factor or amplitude ratio

$$M = \frac{X}{\delta_{st}} = \frac{1}{1 - r^2}$$

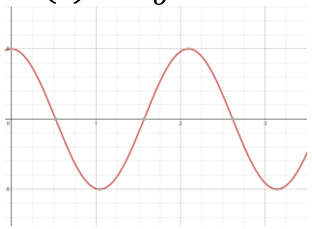
$$F(t) = F_o \cos \Omega\, t$$

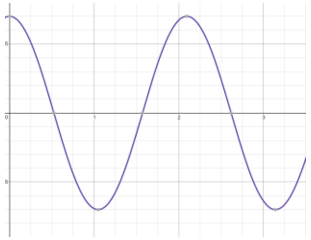
$$x_p(t) = \delta_{st}\ \boldsymbol{M} \cos \Omega\, t$$



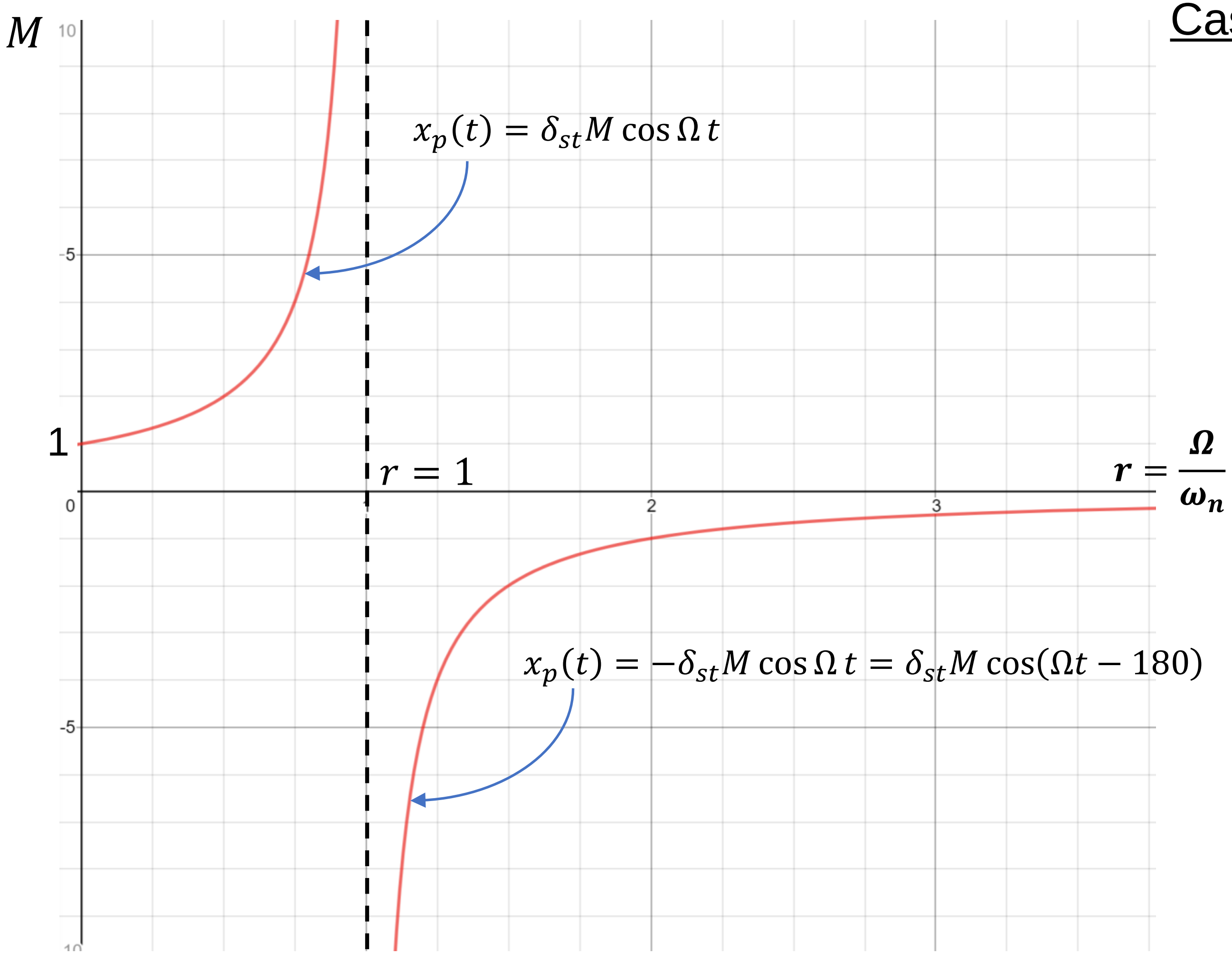
Case 1: $r < 1$

$F(t) = F_o \cos \Omega\, t$



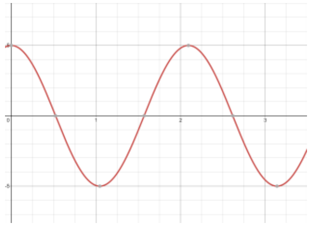


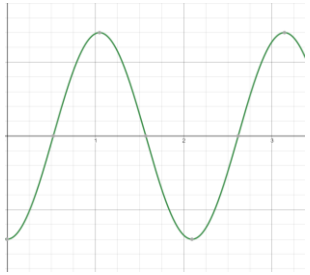
Response in phase with excitation



Case 2: $r > 1$

$F(t) = F_o \cos \Omega\, t$





Response in out of phase with excitation by 180°

Magnification factor (M)

Also called amplification factor or amplitude ratio

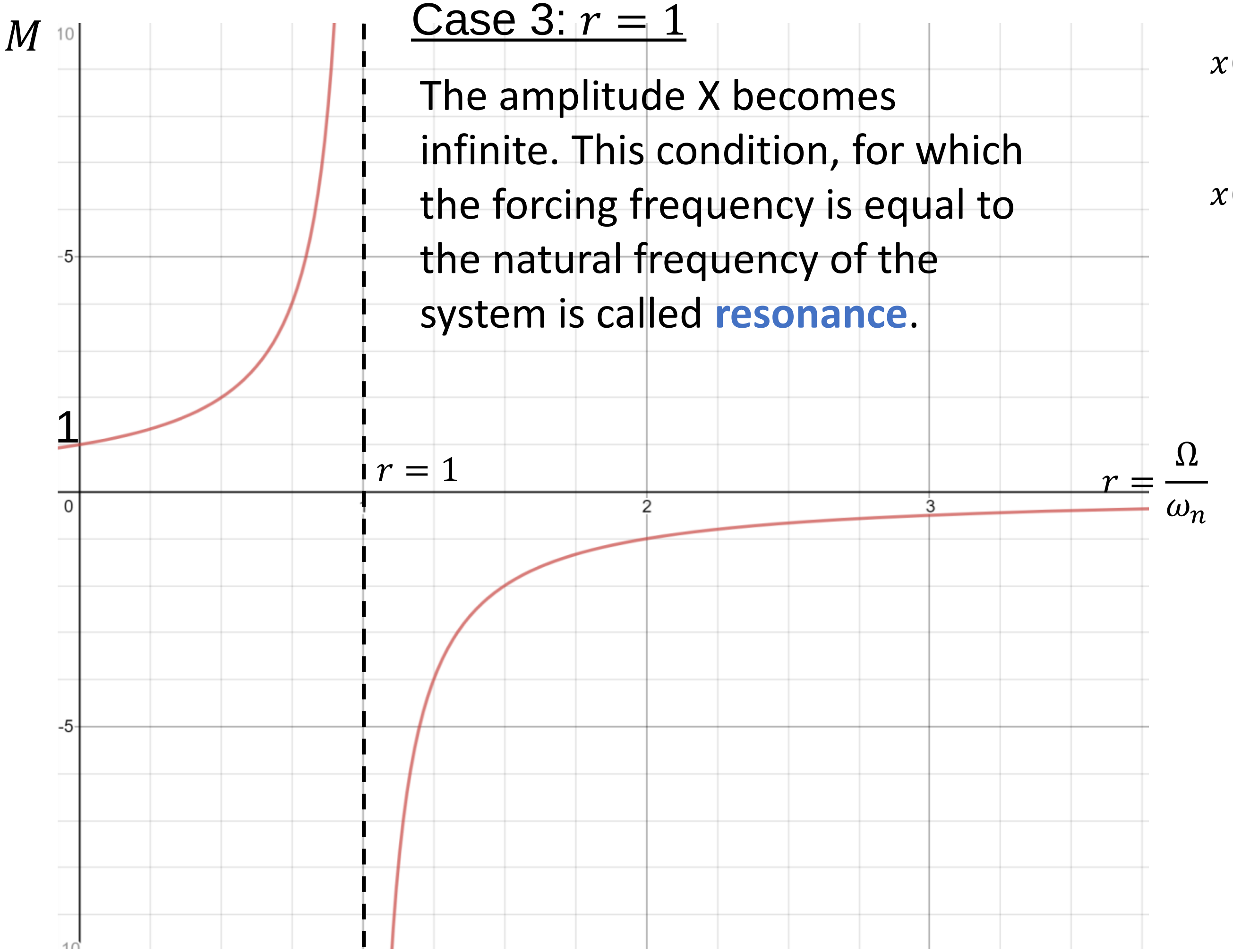
$$M = \frac{X}{\delta_{st}} = \frac{1}{1 - r^2}$$

$$F(t) = F_o \cos \Omega t$$

$$x_p(t) = \delta_{st}\ \boldsymbol{M} \cos \Omega t$$

<u>Case 3: $r = 1$</u>

The amplitude X becomes infinite. This condition, for which the forcing frequency is equal to the natural frequency of the system is called **resonance**.

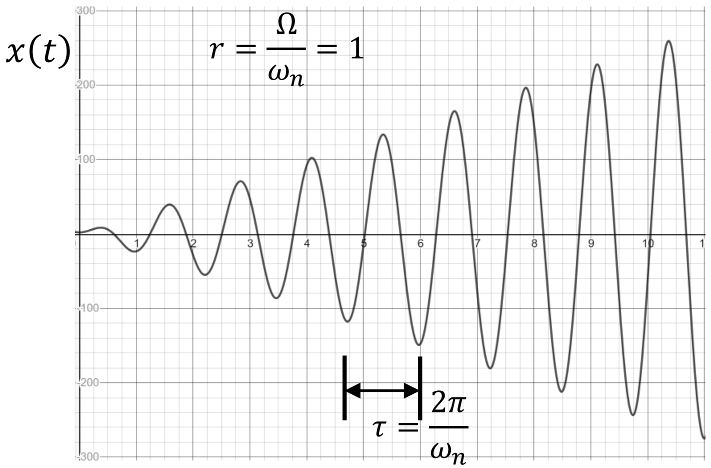


$$x(t) = \left(x_o - \delta_{st}\frac{1}{1 - r^2}\right)\cos(\omega_n t) + \frac{v_o}{\omega_n}\cos(\omega_n t) + \delta_{st}\frac{1}{1 - r^2}\cos \Omega t$$

$$x(t) = (x_o)\cos(\omega_n t) + \frac{v_o}{\omega_n}\cos(\omega_n t) + \delta_{st}\frac{\cos \Omega t - \cos(\omega_n t)}{1 - r^2}$$

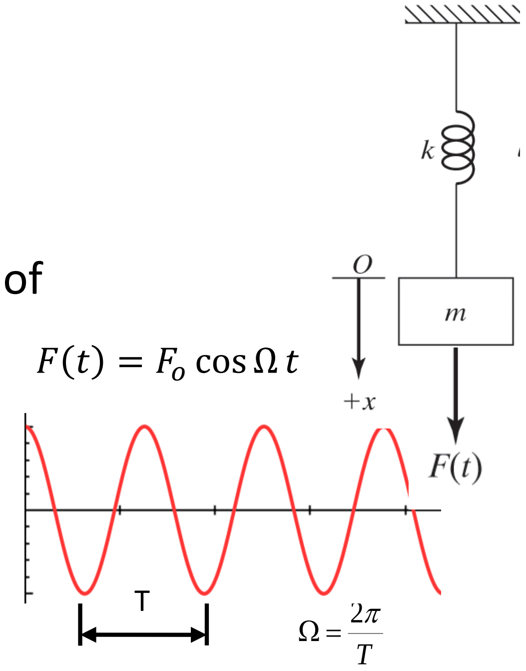
Since the last term of this equation takes an indefinite form for $\Omega = \omega_n$,, we apply L'Hospital's rule to evaluate the limit of this term, and the response of the system at resonance becomes

$$\boxed{x(t) = (x_o)\cos(\omega_n t) + \frac{v_o}{\omega_n}\cos(\omega_n t) + \frac{\delta_{st}\omega_n t}{2}\sin\omega_n t}$$



The transient response increases linearly with time

Beating Phenomenon



Case 4: $\boldsymbol{\Omega - \omega_n = \varepsilon}$

Occurs when the forcing frequency is close to, but not exactly equal to, the natural frequency of the system. In this kind of vibration, the amplitude builds up and then diminishes in a regular pattern

The **total solution** is :

$$x(t) = \left(\cancel{x_o} - \frac{F_o/k}{1-r^2}\right)\cos(\omega_n t) + \cancel{\frac{v_o}{\omega_n}}\cos(\omega_n t) + \frac{F_o/k}{1-r^2}\cos\Omega\, t$$

If we have $x_0 = v_0 = 0$

$$x(t) = \left(\frac{F_o/k}{1-r^2}\right)(\cos\Omega\, t - \cos(\omega_n t))$$

- Using a trigonometry identity, the response becomes:

$$x(t) = \left(\frac{F_o/m}{{\omega_n}^2 - \Omega^2}\right)\left(-2\sin\left(\frac{\Omega - \omega_n}{2}t\right)\sin\left(\frac{\Omega + \omega_n}{2}t\right)\right)$$

- With: $\dfrac{F_o/k}{1-r^2} = \dfrac{F_o/m}{{\omega_n}^2 - \Omega^2}$

$$\Omega + \omega_n \approx 2\Omega$$

$$\Omega - \omega_n = \varepsilon$$

$${\omega_n}^2 - \Omega^2 = (\omega_n - \Omega)(\omega_n + \Omega) = -2\varepsilon\Omega$$

$$\boxed{x(t) = \left(\frac{F_o/m}{2\varepsilon\Omega}\right)\left(2\sin\left(\left(\frac{\varepsilon}{2}\right)t\right)\sin(\Omega t)\right)}$$

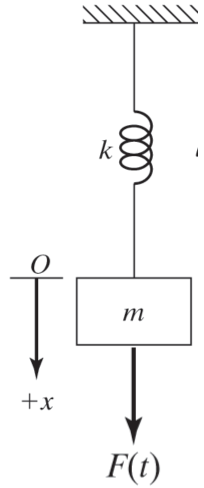
Beating Phenomenon

Case 4: $\Omega - \omega_n = \varepsilon$

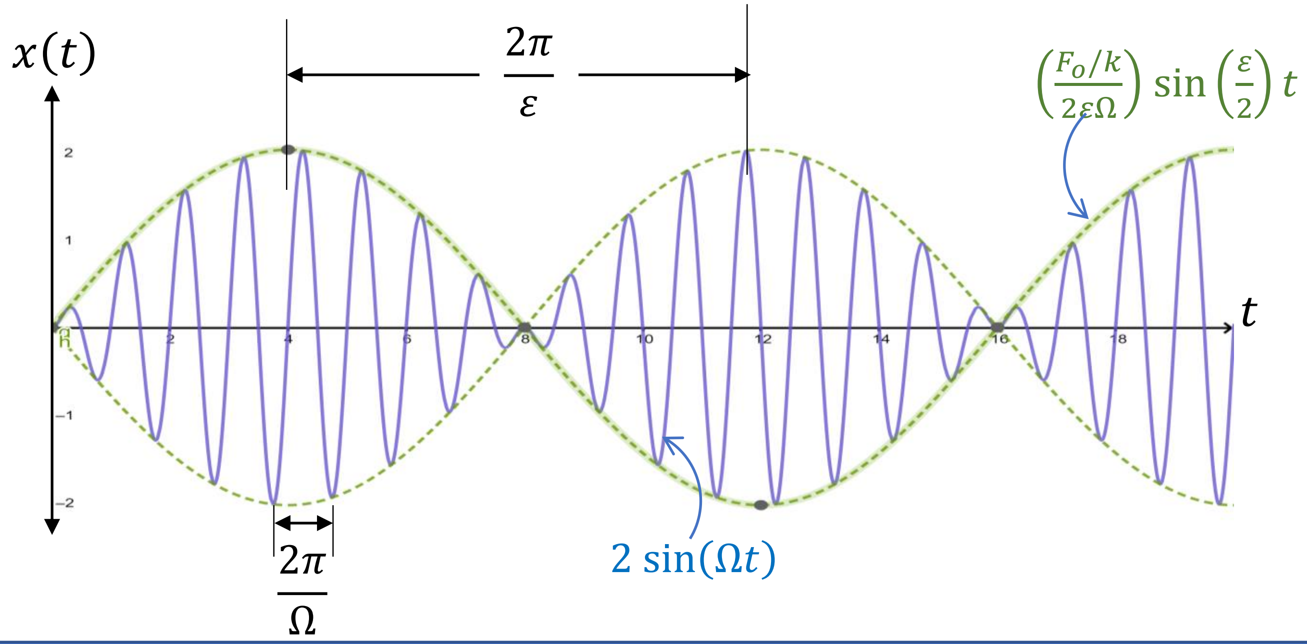
- When the forcing frequency Ω be slightly less (ε) than the natural frequency, the response becomes:

$$x(t) = \left(\frac{F_o/m}{2\varepsilon\Omega}\right)\left(2\sin\left(\left(\frac{\varepsilon}{2}\right)t\right)\sin(\Omega t)\right)$$

- The frequency of beating is equal to $\varepsilon = \Omega - \omega_n$.
- It represent the time when the response magnitude atains a minimun or maximum.



$F(t) = F_o \cos \Omega t$



Formula Sheet for Force vibration Transient response Undamped systems $\zeta = 0$

Natural Frequency

$$\omega_n = \sqrt{\frac{k_e}{m_e}}$$

Frequency ratio

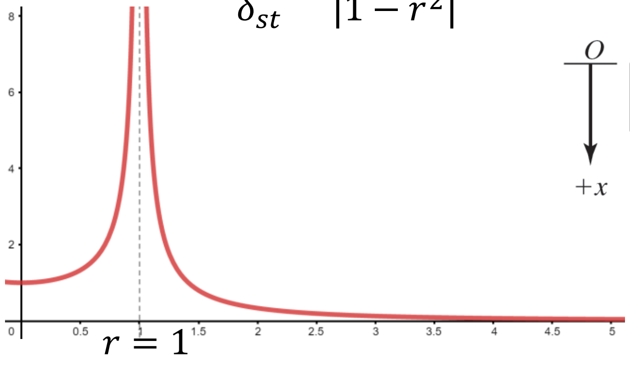
$$r = \frac{\Omega}{\omega_n}$$

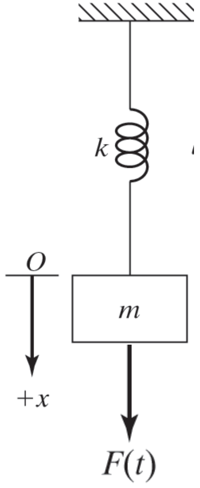
Static deflection

$$\delta_{st} = \frac{Fo}{k}$$

Magnification factor

$$\boldsymbol{M} = \frac{X}{\delta_{st}} = \frac{1}{|1 - r^2|}$$





Governing equation $\quad m_e\ddot{x} + k_e x = F(t)$

$$\ddot{x} + \omega_n^2 x = F(t)/m$$

Force : $\quad F(t) = F_o \cos \Omega\, t$

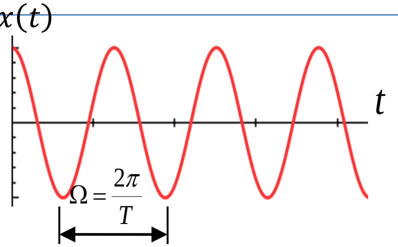
Response : $\quad x(t) = x_h(t) + x_p(t)$

$r \neq 1$

$$x(t) = \left(x_o - \delta_{st}\frac{1}{1-r^2}\right)\cos(\omega_n t) + \frac{v_o}{\omega_n}\sin(\omega_n t) + \delta_{st}\frac{1}{1-r^2}\cos(\Omega\, t - \varphi)$$

$r < 1 \quad \varphi = 0°$ $\qquad$ $r > 1 \quad \varphi = 180°$

For no initial conditions the system vibrates at forcing frequency

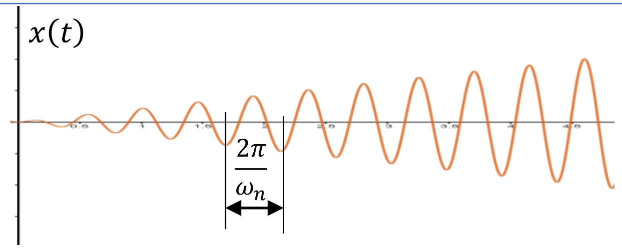


$r = 1$

$$x(t) = (x_o)\cos(\omega_n t) + \frac{v_o}{\omega_n}\cos(\omega_n t) + \frac{\delta_{st}\omega_n t}{2}\sin\omega_n t$$

$\varphi = 90°$

Resonance



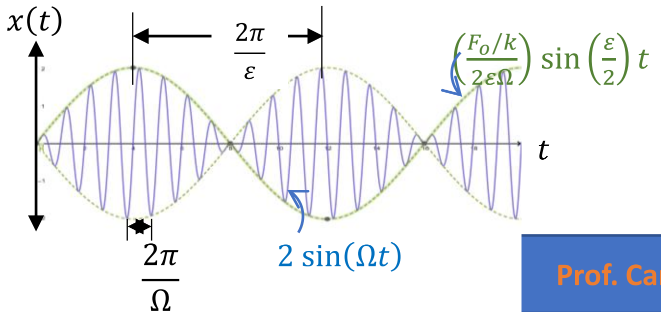
For no initial conditions the system vibrates following :

$r \approx 1$

$$x(t) = \left(\frac{F_o/m}{2\varepsilon\Omega}\right)\left(2\sin\left(\left(\frac{\varepsilon}{2}\right)t\right)\sin(\Omega t)\right)$$

Beating

$$\varepsilon = \Omega - \omega_n$$



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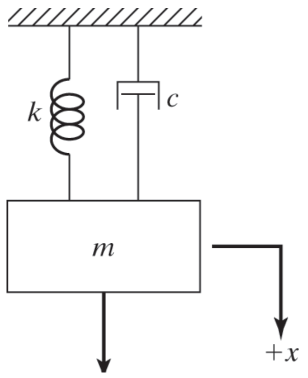
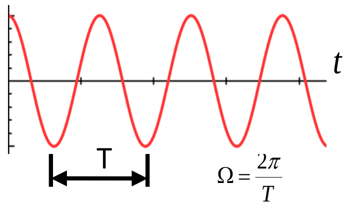
CASES TO STUDY:

Case 1: Harmonic force



$$F(t) = F_o \cos \Omega\, t$$

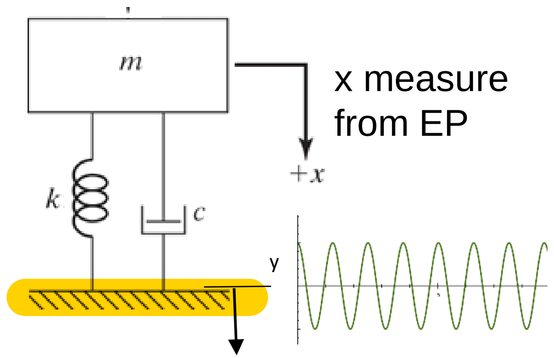


Absolute response:
$x(t)$

Force Transmitted to the base:

$$\boldsymbol{F_T(t) = c\dot{x} + kx}$$

Case 2: Moving base



$$y(t) = Y_o \sin \Omega\, t$$

Absolute response:
$\boldsymbol{x(t)}$

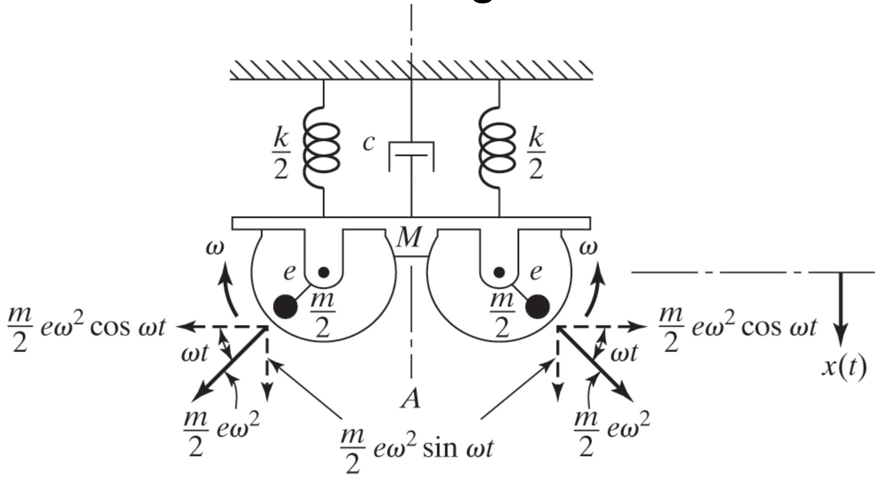
Relative Response:

$$\mathbf{z}(\boldsymbol{t}) = (\boldsymbol{x} - \boldsymbol{y})$$

Force Transmitted to the base:

$$\boldsymbol{F_T(t) = c\dot{x} + kx}$$

Case 3: Rotating unbalance



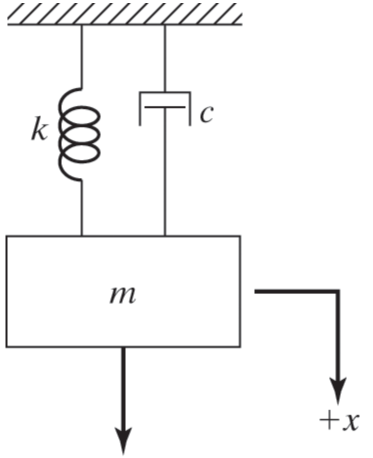
$$F(t) = me\omega^2 \sin(\omega t)$$

Absolute response:
$\boldsymbol{x(t)}$

Force Transmitted to the base:

$$\boldsymbol{F_T(t) = c\dot{x} + kx}$$

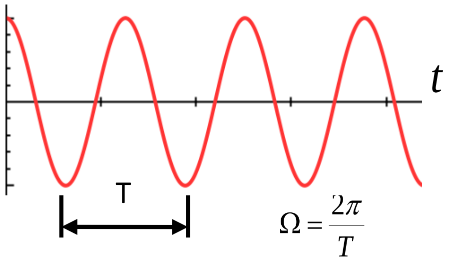
Solution to a harmonic force (Mass-Damper–Spring System)



Equation of motion: $m\ddot{x} + c\dot{x} + kx = F(t)$

The **total solution** is : $x(t) = x_p(t) + x_h(t)$

$F(t) = F_o \cos \Omega\, t$



Where: $x_p(t) = A \cos \Omega\, t + B \sin \Omega\, t$

To find A and B, derivate the **particular solution** and substitute in the equation of motion

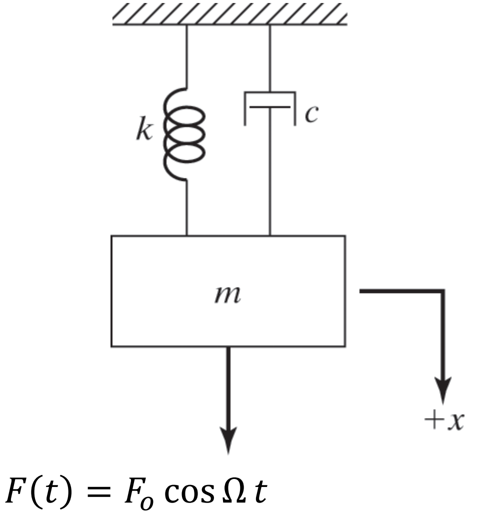
$$\dot{x}_p(t) = -A\Omega \sin \Omega\, t + B\Omega \cos \Omega\, t$$
$$\ddot{x}_p(t) = -A\Omega^2 \cos \Omega\, t - B\Omega^2\, sin\, \Omega\, t$$

$$x_h(t) = \begin{cases} \zeta < 1, x(t) = C_1 e^{-\zeta\omega_n t} \cos(\omega_{\,d} t) + C_2 e^{-\zeta\omega_n t}\, sin(\omega_{\,d} t) \\ \zeta = 1, x(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t} \\ \zeta > 1, x(t) = C_1 e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)t} + C_2 e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)t} \end{cases}$$

The **homogeneous solution** represent the free vibration of the system, that dies out with time under each of the three possible conditions of damping (underdamping, critical damping, and overdamping) and under all possible initial conditions.

The constants C_1 and C_2 depend on the **initial conditions**, and have to be calculated for the whole equation.

Solution to a harmonic force (Mass-Damper–Spring System)



The **particular solution**

$$x_p(t) = A\cos\Omega t + B\sin\Omega t$$

$$\dot{x}_p(t) = -A\Omega\sin\Omega t + B\Omega\cos\Omega t$$

$$\ddot{x}_p(t) = -A\Omega^2\cos\Omega t - B\Omega^2 \sin\Omega t$$

substitute in the equation of motion $m\ddot{x} + c\dot{x} + kx = F(t)$

$$m\left(-A\Omega^2\cos\Omega t - B\Omega^2\sin\Omega t\right) + c(-A\Omega\sin\Omega t + B\Omega\cos\Omega t) + k(A\cos\Omega t + B\sin\Omega t) = F_0\cos\Omega t$$

$$\left(-Am\Omega^2 + Bc\Omega + Ak\right)\cos\Omega t + \left(-Bm\Omega^2 - Ac\Omega + kB\right)\sin\Omega t = Fo\cos\Omega t$$

$$\left(-Am\Omega^2 + Bc\Omega + Ak\right) = Fo$$
$$\left(-Bm\Omega^2 - Ac\Omega + kB\right) = 0$$

$$A\left(k - m\Omega^2\right) + Bc\Omega = Fo$$
$$B\left(k - m\Omega^2\right) = Ac\Omega$$

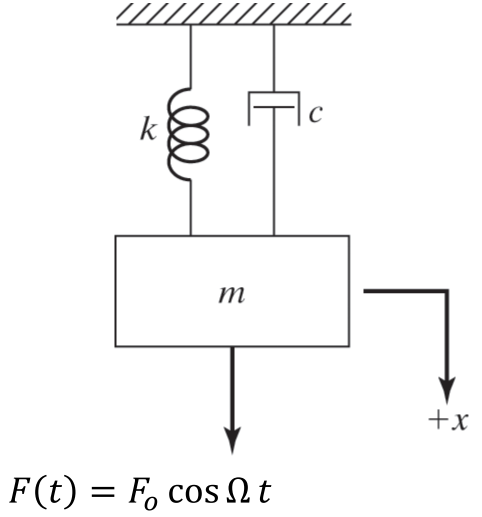
$$A = \frac{Fo\left(k - m\Omega^2\right)}{(k - m\Omega^2)^2 + (c\Omega)^2} \qquad B = \frac{Fo(c\Omega)}{(k - m\Omega^2)^2 + (c\Omega)^2}$$

In terms of

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{c}{2\sqrt{km}} \qquad r = \frac{\Omega}{\omega_n}$$

$$\boxed{A = \frac{Fo}{k}\frac{\left(1 - r^2\right)}{(1 - r^2)^2 + (2\zeta r)^2}} \qquad \boxed{B = \frac{Fo}{k}\frac{2\zeta r}{(1 - r^2)^2 + (2\zeta r)^2}}$$

Solution to a harmonic force (Mass-Damper–Spring System)



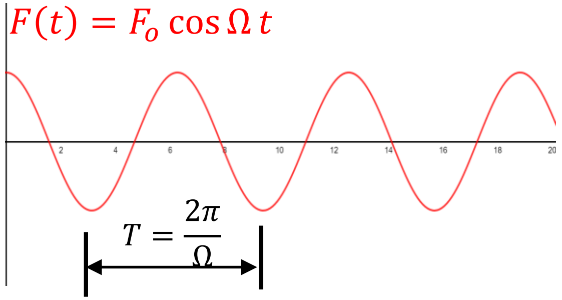
The **particular solution** $x_p(t) = A\cos\Omega\, t + B\sin\Omega\, t = X_P\cos(\Omega t - \varphi)$

$$A = \frac{Fo}{k}\frac{(1-r^2)}{(1-r^2)^2+(2\zeta r)^2}$$

$$B = \frac{Fo}{k}\frac{2\zeta r}{(1-r^2)^2+(2\zeta r)^2}$$

$$X_P = \sqrt{A^2+B^2} = \sqrt{\left(\frac{Fo}{k}\right)^2\frac{(1-r^2)^2+(2\zeta r)^2}{[(1-r^2)^2+(2\zeta r)^2]^2}}$$

$$\varphi = \tan^{-1}\frac{B}{A} = \tan^{-1}\frac{2\zeta r}{1-r^2}$$



The amplitude of the solution

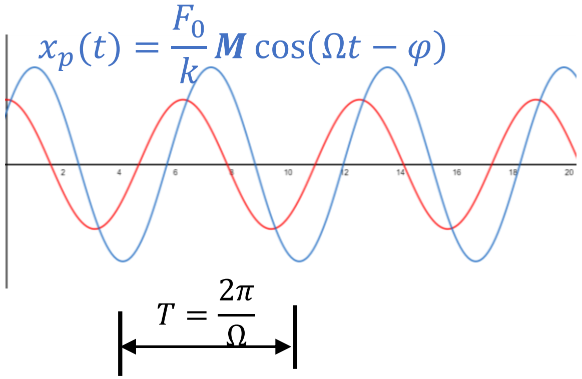
$$\boxed{X_P = \frac{Fo}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}}$$

$$\boxed{\varphi = \tan^{1}\frac{2\zeta r}{1-r^2}}$$

Magnification factor

$$\boxed{\boldsymbol{M} = \frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}}$$

$$\boxed{x_p(t) = \frac{F_0}{k}\boldsymbol{M}\cos(\Omega t - \varphi)}$$



Figures and content adapted from Textbook:
Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition
Prof. Carmen Muller-Karger, PhD

Magnification factor (*M*)

(Also called amplification factor or amplitude ratio)

$$M = \frac{X}{\delta_{st}}$$

Magnification factor For M-D-S system with harmonic force

- **Maximum** take place at

$$\frac{\partial M}{\partial r} = \frac{-\left[2(1-r^2)(-2r) + 2(4\zeta^2)r\right]}{\left[(1-r^2)^2 + (2\zeta r)^2\right]^{3/2}} = 0$$

$$4r\left[-1 + r^2 + 2\zeta^2\right] = 0$$

- The **critical frequency** will be a little bit less than the natural frequency:

$$r_{crit} = \sqrt{1-2\zeta^2}$$

$$r = 1 \quad M = \frac{1}{2\zeta}$$

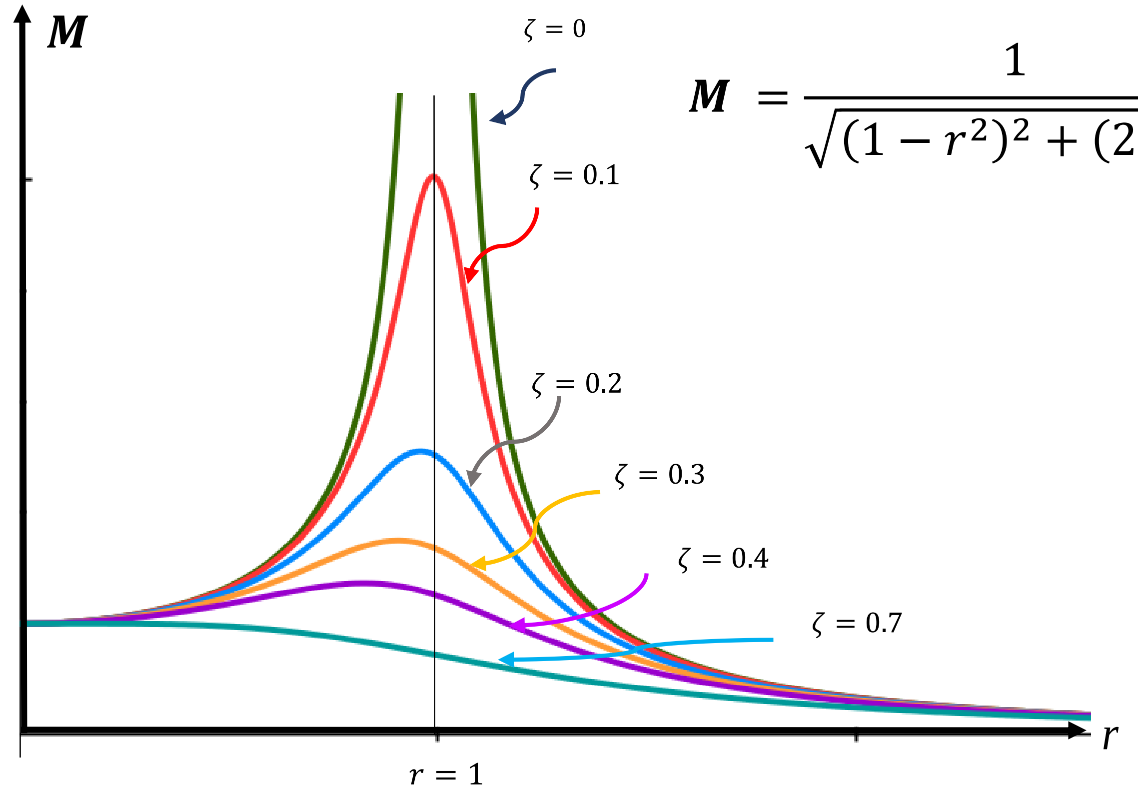
$$r_{crit} = \frac{\Omega_{crit}}{\omega_n}$$

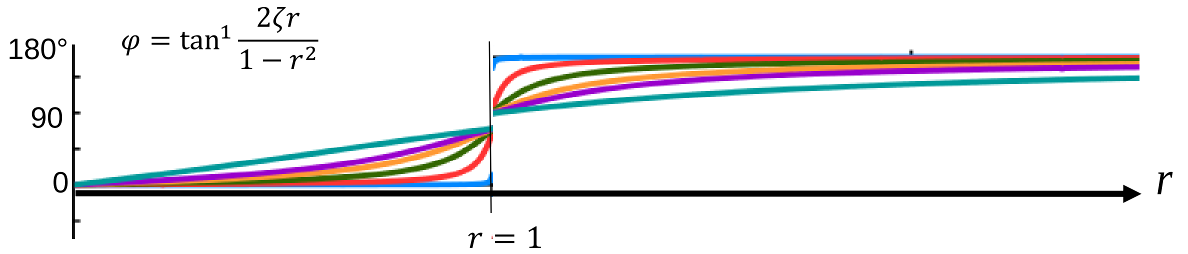
$$M_{crit} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

maximum value of M

Phase angle is 90 for r=1, regarless the value of damping ratio.

$$M = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$





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Prof. Carmen Muller-Karger, PhD

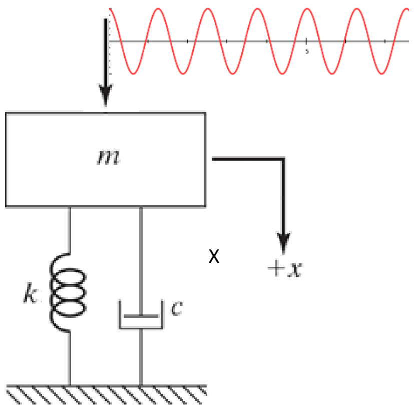
Characteristics of the Magnification factor $\left(M = \frac{X}{\delta_{st}}\right)$

$$\boldsymbol{M} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

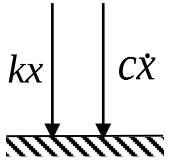
1. For an undamped system (ζ=0),and $\boldsymbol{M} \to \infty \; as \quad r \to 1$
2. Any amount of damping (ζ>0) reduces the magnification factor ($\boldsymbol{M}$) for all values of the forcing frequency.
3. In the degenerate case of a constant force (when r=0), the value of $\boldsymbol{M} = 1$.
4. The reduction in $\boldsymbol{M}$ in the presence of damping is very significant at or near resonance.
5. The amplitude of forced vibration becomes smaller with increasing values of the forcing frequency (i.e., $\boldsymbol{M} \to \boldsymbol{0} \; as \quad r \to \infty$)
6. The maximum value of $\boldsymbol{M}$ occurs when $r_{crit} = \sqrt{1-2\zeta^2}$, valid for values $0 < \zeta < 1/\sqrt{2}$
7. The maximum value of $\boldsymbol{M} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$, which occurs at r_{crit}
8. For $r = 1 \quad \boldsymbol{M} = \frac{1}{2\zeta}$
9. For values $\zeta > 1/\sqrt{2}$ the graph of $\boldsymbol{M}$ monotonically decreases with increasing values of r

Force transmitted to the base of a viscous damped system

$F(t) = F_0 \cos \Omega t$



$$F_T(t) = c\dot{x} + kx$$



- The **steady-state** response $x_p(t) = \frac{F_0}{k}\boldsymbol{M}\cos(\Omega t - \varphi)$ $\qquad \boldsymbol{M} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$ $\qquad \varphi = \tan^{1}\frac{2\zeta r}{1-r^2}$

- The spring and the damper transmit force to the base:

$$F_T(t) = kx + c\dot{x} = k\frac{F_0}{k}\boldsymbol{M}\,cos(\Omega t - \varphi) - c\frac{F_0}{k}\boldsymbol{M}\Omega\sin(\Omega t - \varphi) = F_0\boldsymbol{M}\left[\cos(\Omega t - \varphi) - \frac{c}{k}\Omega\sin(\Omega t - \varphi)\right]$$

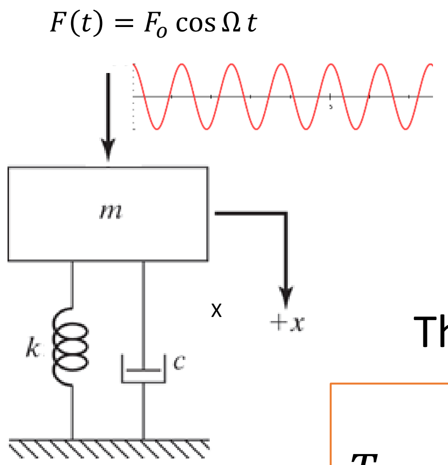
$$\frac{c}{k} = \frac{2\zeta}{\omega_n}$$

- We can write this sum of *cos* and *sin* in a single *cos* function with a phase angle

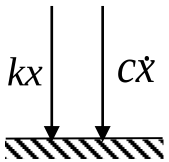
$$\boxed{F_T(t) = F_0\frac{\sqrt{(2\zeta r)^2 + 1}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\cos(\Omega t - \varphi_1 - \alpha)}$$

$$\boxed{\varphi_1 = \tan^{-1}\frac{2\zeta r}{1-r^2}} \qquad \boxed{\alpha = \tan^{-1}(2\,\zeta r)} \qquad \boxed{\varphi = \varphi_1 + \alpha = \tan^{-1}\left(\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right)}$$

Force transmitted to the base of a viscous damped system



$$F_T(t) = c\dot{x} + kx$$



The ratio of the amplitude

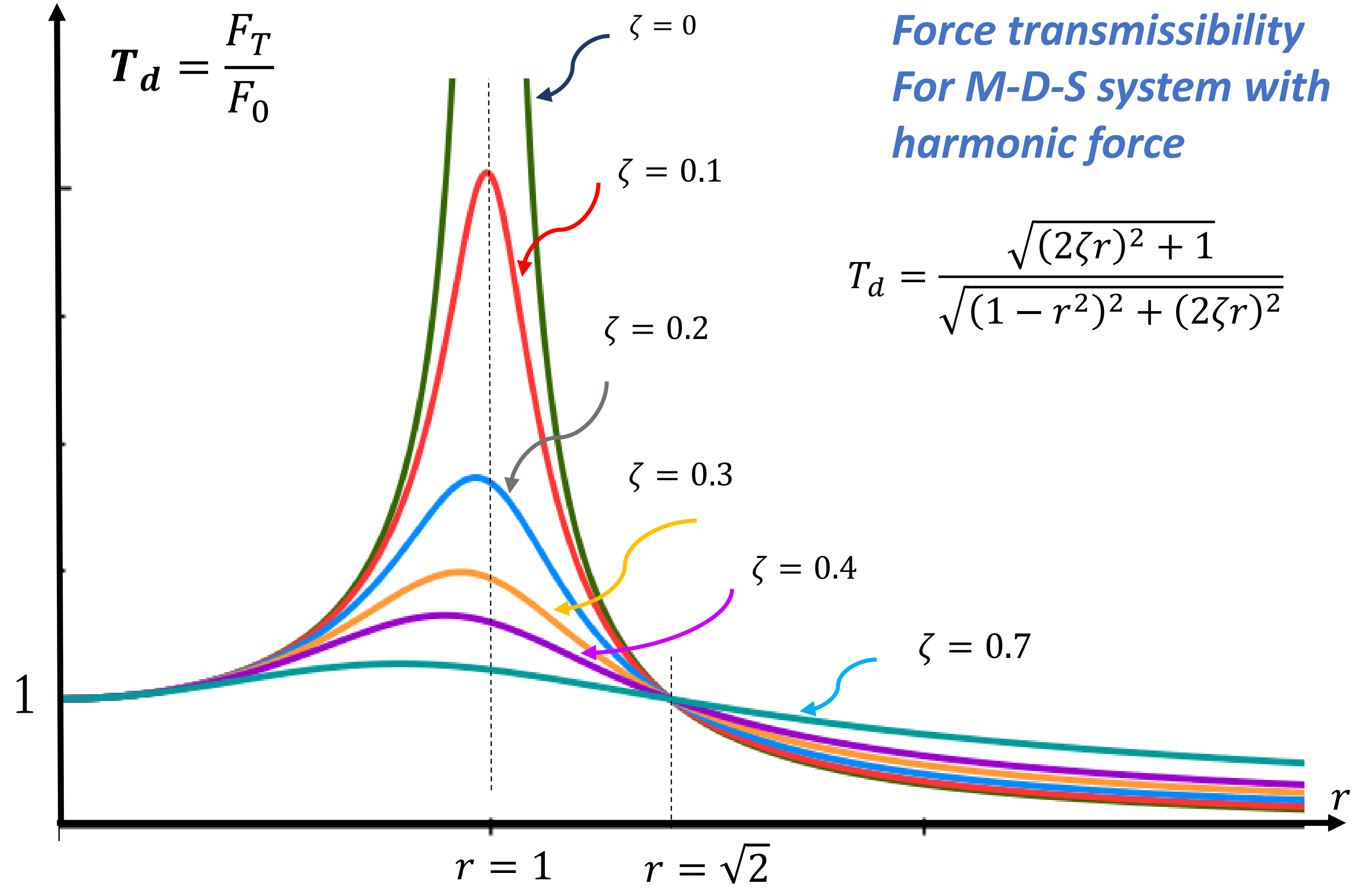
$$T_d = \frac{F_T}{F_0} = \frac{\sqrt{(2\zeta r)^2 + 1}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

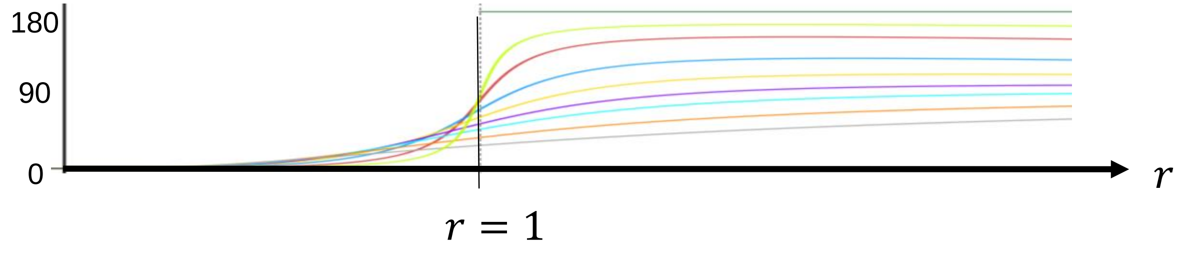
$$\varphi = \tan^{-1}\left(\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right)$$

$$F_T(t) = F_0 \frac{\sqrt{(2\zeta r)^2 + 1}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \cos(\Omega t - \varphi)$$

Force transmissibility For M-D-S system with harmonic force

$$T_d = \frac{\sqrt{(2\zeta r)^2 + 1}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$





Characteristics of the transmissibility factor (T_d)

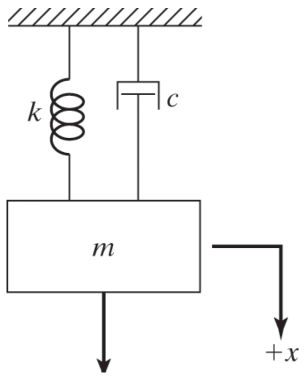
$$T_d = \frac{\sqrt{(2\zeta r)^2+1}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

1. The value of T_d is unity at $r=0$ and close to unity for small values of r.
2. For an undamped system (ζ=0), $T_d \rightarrow \infty$ at resonance (r=1).
3. The value of T_d is less than unity (T_d <1) for values of $\mathrm{r} > \sqrt{2}$ (for any amount of damping ζ).
4. The value of T_d is unity for all values of ζ at $\mathrm{r} = \sqrt{2}$.
5. For $\mathrm{r} < \sqrt{2}$, smaller damping ratios lead to larger values of T_d On the other hand, for $r > \sqrt{2}$, smaller values of damping ratio lead to smaller values of T_d.
6. The transmissibility factor, T_d, attains a maximum for 0<ζ<1 at the frequency ratio $r = r_m$ given by

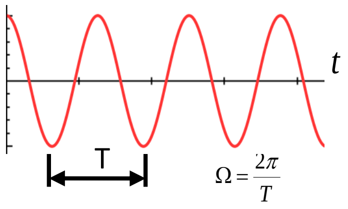
$$r_m = \frac{1}{2\zeta}\left[\sqrt{1+8\zeta^2}-1\right]^2$$

CASES TO STUDY:

Case 1: Harmonic force



$$F(t) = F_o \cos \Omega t$$



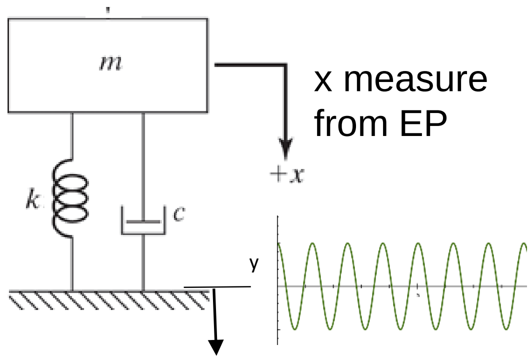
Absolute response:

$x(t)$

Force Transmitted to the base:

$$\boldsymbol{F_T(t) = c\dot{x} + kx}$$

Case 2: Moving base



$$y(t) = Y_o \sin \Omega t$$

Absolute response:

$\boldsymbol{x(t)}$

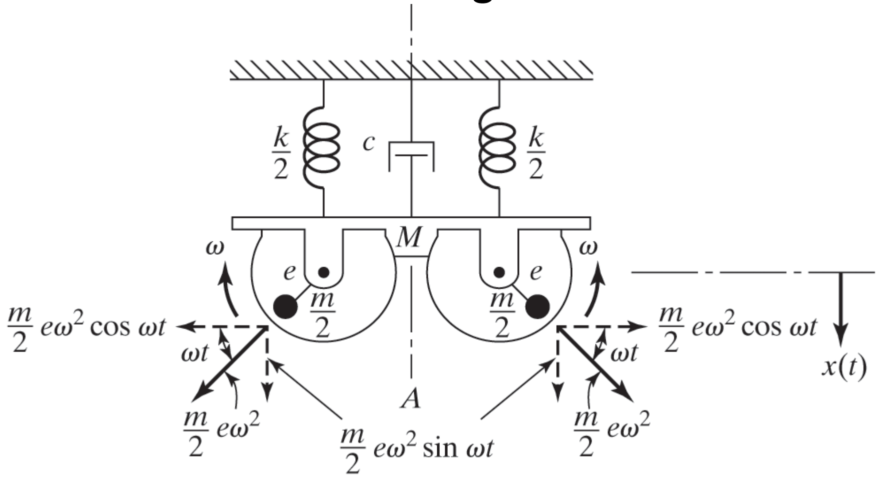
Relative Response:

$\mathbf{z}(\boldsymbol{t}) = (\boldsymbol{x - y})$

Force Transmitted to the base:

$$\boldsymbol{F_T(t) = c\dot{x} + kx}$$

Case 3: Rotating unbalance



$$F(t) = me\omega^2 \sin(\omega t)$$

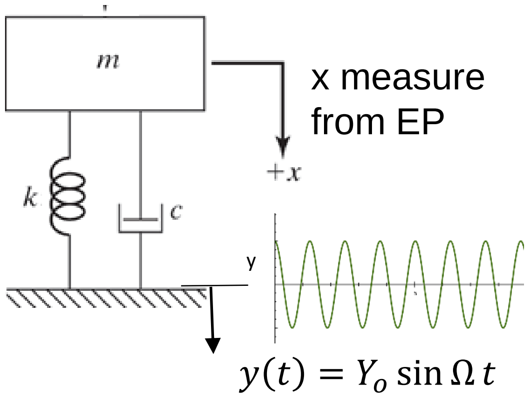
Absolute response:

$\boldsymbol{x(t)}$

Force Transmitted to the base:

$$\boldsymbol{F_T(t) = c\dot{x} + kx}$$

Response of a Damped System Under the Harmonic Motion of the Base



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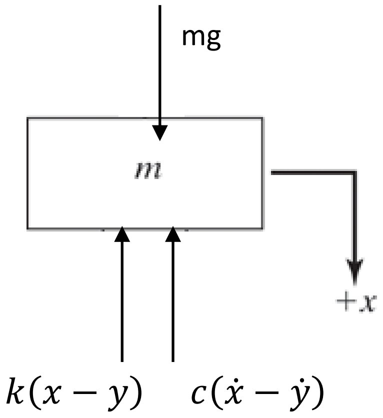


Forces in vertical direction:

$$c(\dot{x}-\dot{y}) + k(x-y) = -m\ddot{x}$$

$$y(t) = Y_o \sin \Omega\, t$$

$$\dot{y}(t) = Y_o \Omega \cos \Omega\, t$$

Governing equation:

$$M\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$$

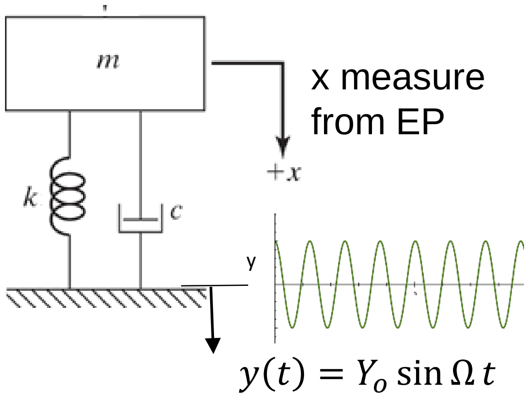
$$M\ddot{x} + c\dot{x} + kx = cY_o\Omega \cos \Omega\, t + kY_o \sin \Omega\, t$$

$$M\ddot{x} + c\dot{x} + kx = \left[Y_o\sqrt{(c\Omega)^2 + (k)^2}\right] cos(\Omega t - \varphi_1)$$

$$F(t) = Y_o\sqrt{(c\Omega)^2 + (k)^2}\, cos(\Omega t - \varphi_1)$$

$$\boxed{\varphi_1 = \tan^{-1}\frac{2\zeta r}{1-r^2}}$$

Response of a Damped System Under the Harmonic Motion of the Base



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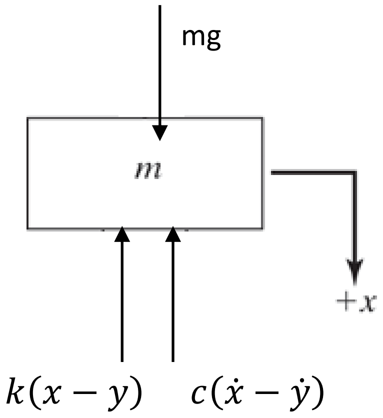


Governing equation: $M\ddot{x} + c\dot{x} + kx = F(t)$

$$F(t) = Y_o\sqrt{(c\Omega)^2 + (k)^2}\, cos(\Omega t - \varphi_1) \qquad \varphi_1 = \tan^{-1}\frac{2\zeta r}{1-r^2}$$

The particular solution

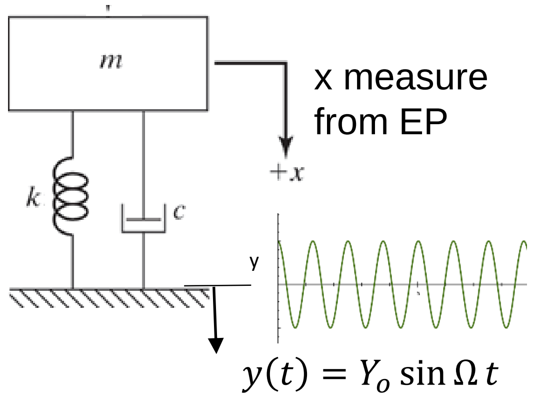
$$x_p(t) = \frac{F_o}{k}\boldsymbol{M}\cos(\Omega t - \varphi_1 - \alpha)$$

$$x_p(t) = \frac{Y_o\sqrt{(c\Omega)^2 + (k)^2}}{k}\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\cos(\Omega t - \varphi_1 - \alpha) \qquad \boxed{\alpha = \tan^{-1}(2\zeta r)}$$

$$\boxed{x_p(t) = Y_o\frac{\sqrt{(2\zeta r)^2 + 1}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\cos(\Omega t - \varphi)} \qquad \boxed{\varphi = \tan^{-1}\left(\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right)}$$

The ratio of the amplitude of the response to that of the base motion y(t), is called the ***displacement transmissibility***.

Response of a Damped System Under the Harmonic Motion of the Base



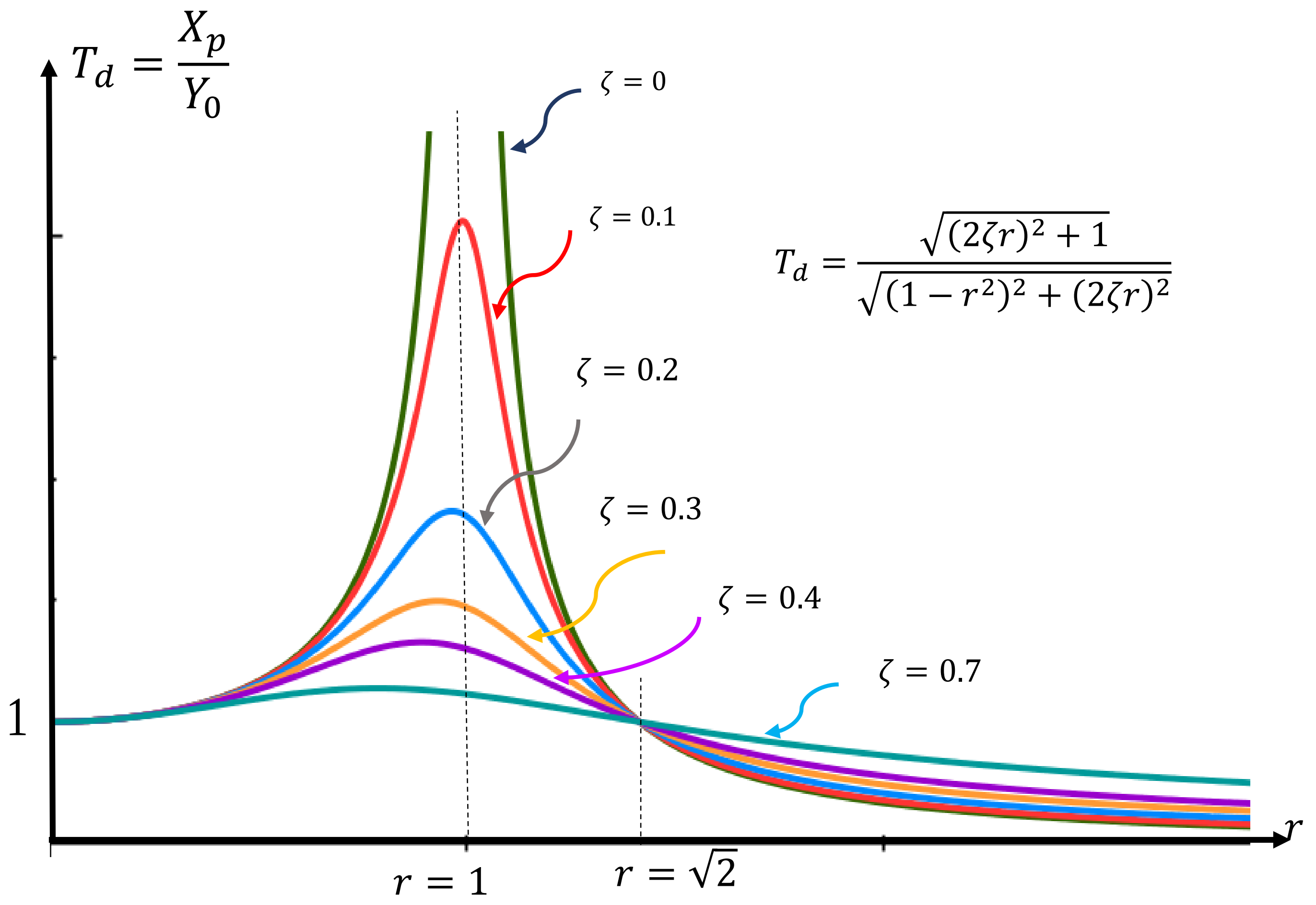
The ratio of the amplitude

$$T_d = \frac{X_p}{Y_0} = \frac{\sqrt{(2\zeta r)^2 + 1}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

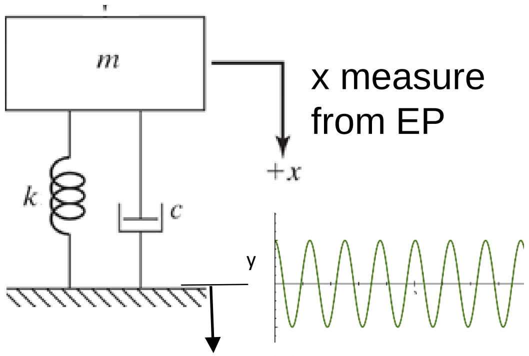
$$\varphi = \tan^{-1}\left(\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right)$$

$$x_p(t) = Y_o \frac{\sqrt{(2\zeta r)^2 + 1}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \cos(\Omega t - \varphi)$$

Displacement transmissibility. For M-D-S system, under harmonic motion of base

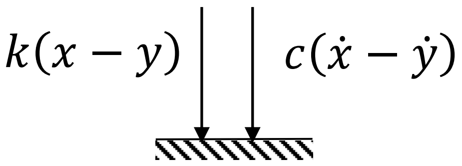


Force transmitted to the moving base of a viscous damped system



$y(t) = Y_o \sin \Omega\, t$

FBD



$$F_T(t) = c(\dot{x}-\dot{y}) + k(x-y)$$

$$F_T(t) = -m\ddot{x}$$

$$x_p(t) = Y_o \frac{\sqrt{(2\zeta r)^2+1}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \cos(\Omega t - \varphi)$$

$$\ddot{x}_p(t) = -\Omega^2 Y_o \frac{\sqrt{(2\zeta r)^2+1}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \cos(\Omega t - \varphi)$$

$$F_T(t) = -m\ddot{x}_p = m\Omega^2 Y_o \frac{\sqrt{(2\zeta r)^2+1}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \cos(\Omega t - \varphi)$$

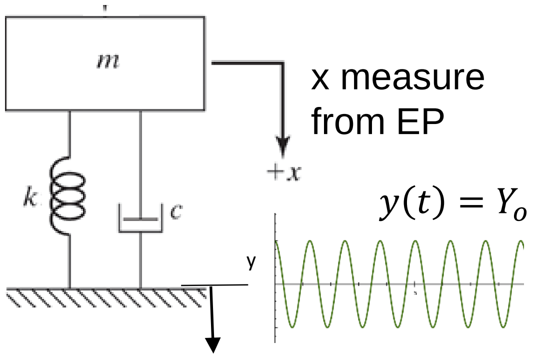
$$\boxed{F_T(t) = kY_o r^2 \frac{\sqrt{(2\zeta r)^2+1}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \cos(\Omega t - \varphi)}$$

The amplitude or maximum value of the force transmitted to the base is called ***Force transmissibility*** , and is in phase with the motion of the mass *x*(*t*)

$$\boxed{\frac{F_T}{kY_o} = r^2 \frac{\sqrt{(2\zeta r)^2+1}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}}$$

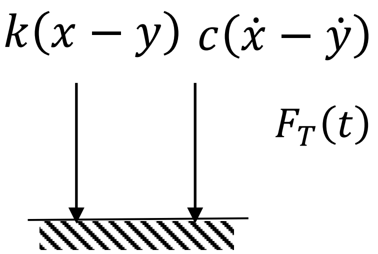
$$\boxed{\varphi = \tan^{-1}\left(\frac{2\zeta r^3}{1+(4\zeta^2-1)r^2}\right)}$$

Force transmitted to the moving base of a viscous damped system



$$y(t) = Y_o \sin \Omega t$$

$$F_T(t) = c(\dot{x} - \dot{y}) + k(x - y)$$

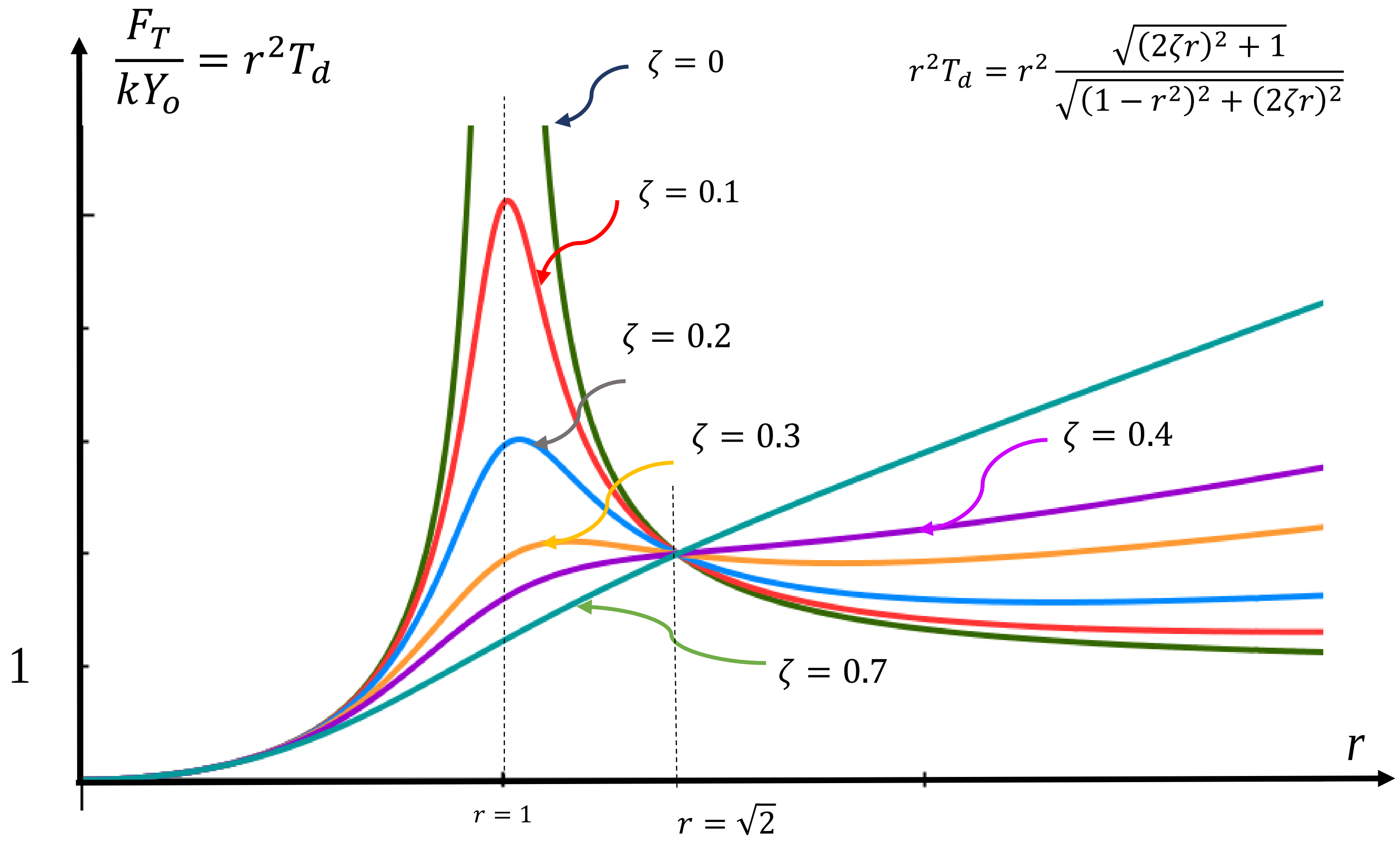


$$F_T(t) = -m\ddot{x}$$

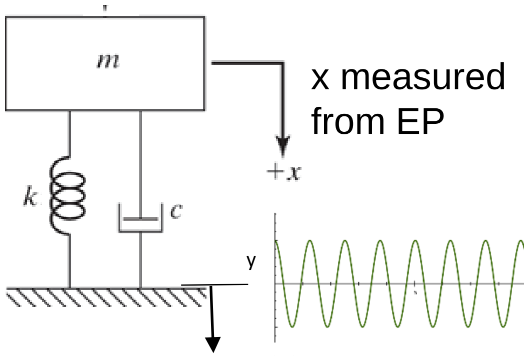
$$\boxed{\frac{F_T}{kY_o} = r^2 \frac{\sqrt{(2\zeta r)^2 + 1}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}}$$

Force transmissibility
For M-D-S system, under harmonic motion of base

$$r^2 T_d = r^2 \frac{\sqrt{(2\zeta r)^2 + 1}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

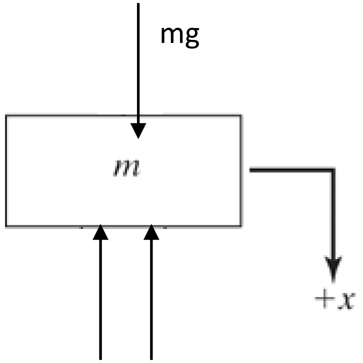


Response of a Damped System Under the Harmonic Motion of the Base (relative motion z)



$y(t) = Y_o \sin \Omega\, t$

FBD



$k(x - y)$ $\quad c(\dot{x} - \dot{y})$

$\text{z} = (x - y)$

$\text{x} = (z + y)$

Forces in vertical direction:

$$c(\dot{x} - \dot{y}) + k(x - y) = -m\ddot{x}$$

$$c(\dot{z}) + k(z) = -m(\ddot{z} + \ddot{y})$$

$$m\ddot{z} + c\dot{z} + kz = -m(\ddot{y})$$

Governing equation: $\quad m\ddot{z} + c\dot{z} + kz = -m(\ddot{y})$

with: $\quad \ddot{y}(t) = -\Omega^2 Y_o \sin \Omega\, t$

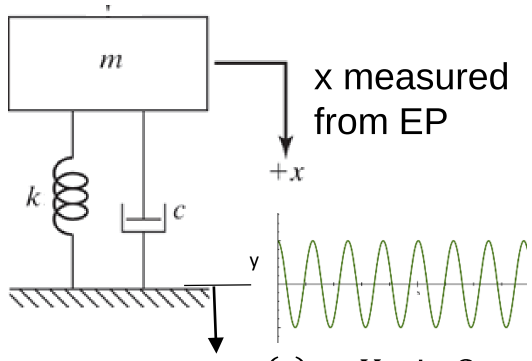
$$m\ddot{z} + c\dot{z} + kz = m\Omega^2 Y_o \sin \Omega\, t$$

$$m\ddot{z} + c\dot{z} + kz = F(t)$$

$$F(t) = m\Omega^2 Y_o \sin \Omega\, t$$

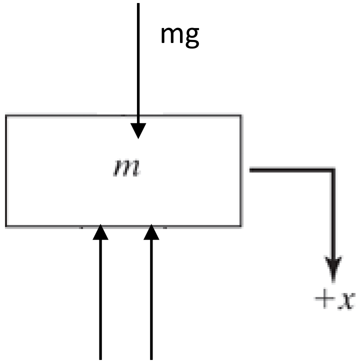
Figures and content adapted from Textbook:
Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition

Response of a Damped System Under the Harmonic Motion of the Base (relative motion z)



$y(t) = Y_o \sin \Omega\, t$

FBD



$k(x-y)$ $c(\dot{x}-\dot{y})$

$z = (x-y)$

$x = (z+y)$

Governing equation:

$$m\ddot{z} + c\dot{z} + kz = F(t)$$

$$F(t) = m\Omega^2 Y_o \sin \Omega\, t$$

The steady state solution

$$z(t) = \frac{F_o}{k} \boldsymbol{M} \sin(\Omega t - \varphi_1)$$

$$z(t) = \frac{m\Omega^2 Y_o}{k} \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\Omega t - \varphi_1)$$

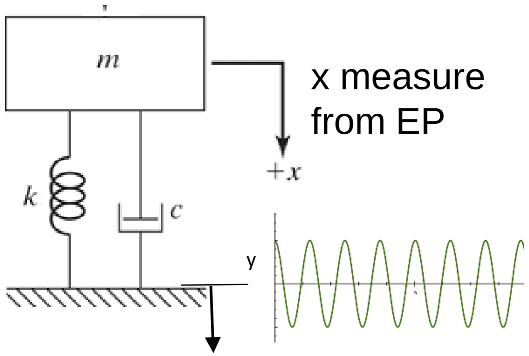
$$\boxed{z(t) = Y_o \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\Omega t - \varphi_1)}$$

$$\boxed{\varphi_1 = \tan^{-1} \frac{2\zeta r}{1-r^2}}$$

The amplitude or maximum value can be expressed as

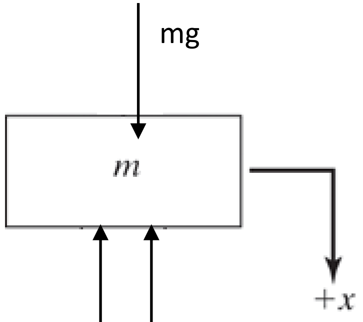
$$\boxed{\frac{Z}{Y_0} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = r^2 \boldsymbol{M}}$$

Response of a Damped System Under the Harmonic Motion of the Base (relative motion z)



$$y(t) = Y_o \sin \Omega\, t$$

FBD



$$k(x-y) \qquad c(\dot{x}-\dot{y})$$

$$z = (x-y)$$

$$x = (z+y)$$

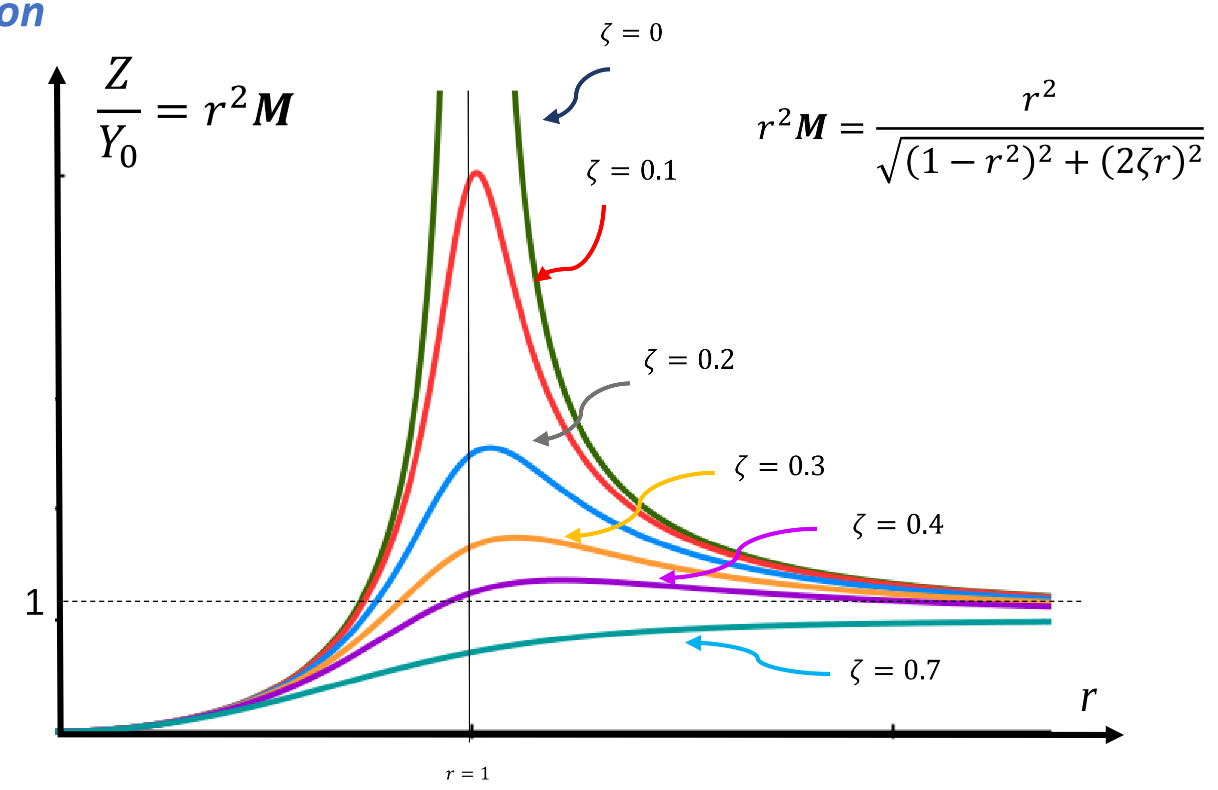
Relative amplitude variation for a M-D-S system, under harmonic motion of base

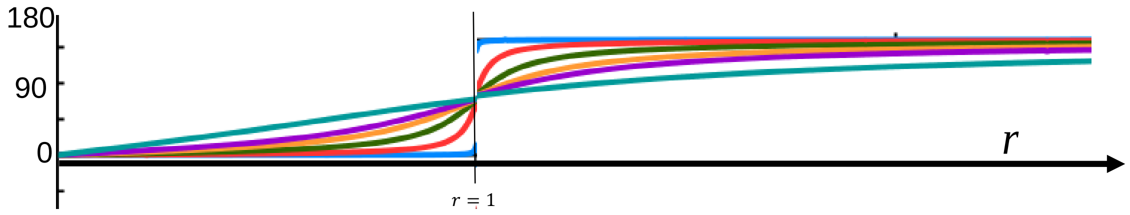
- The maximum take place at :

$$\frac{\partial r^2 \boldsymbol{M}}{\partial r} = 0$$

$$r_{crit} = \frac{1}{\sqrt{1-2\zeta^2}}$$

$$\varphi = \tan^{1} \frac{2\zeta r}{1-r^2}$$





Characteristics of the factor $\left(\frac{Z}{Y_0} = r^2\boldsymbol{M}\right)$

$$r^2\boldsymbol{M} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

1. All the curves begin at zero amplitude.

2. The amplitude near resonance ($\omega = \omega_n$) is markedly affected by damping. Thus if the machine is to be run near resonance, damping should be introduced purposefully to avoid dangerous amplitudes.

3. At very high speeds (ω large, $r \rightarrow \infty$), $\frac{MX}{me}$ is almost unity, and the effect of damping is negligible.

4. The maximum of $r^2\boldsymbol{M}$ occurs when $r_{crit} = \frac{1}{\sqrt{1-2\zeta^2}}$, valid for values $0 < \zeta < 1/\sqrt{2}$

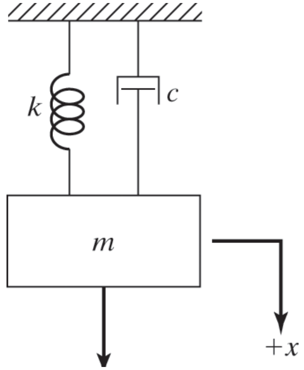
5. The maximum value of $r^2\boldsymbol{M} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$, which occurs at r_{crit}

6. For values $\zeta > 1/\sqrt{2}$ the graph of $r^2\boldsymbol{M}$ does not attain a maximum, its value grows from 0 to 1 when $r \rightarrow \infty$.

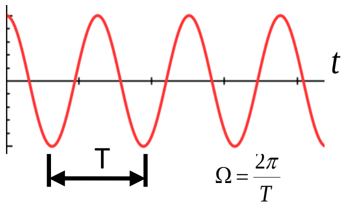
7. The transmitted force

CASES TO STUDY:

Case 1: Harmonic force



$$F(t) = F_o \cos \Omega\, t$$



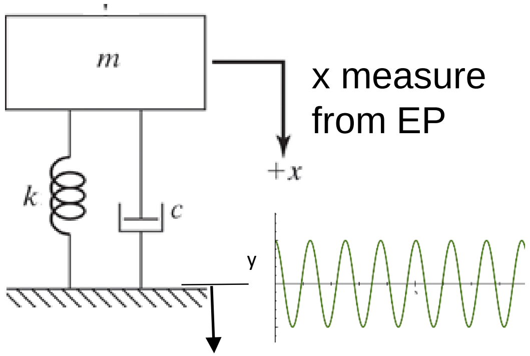
Absolute response:

$x(t)$

Force Transmitted to the base:

$$\boldsymbol{F_T(t) = c\dot{x} + kx}$$

Case 2: Moving base



$$y(t) = Y_o \sin \Omega\, t$$

Absolute response:

$\boldsymbol{x(t)}$

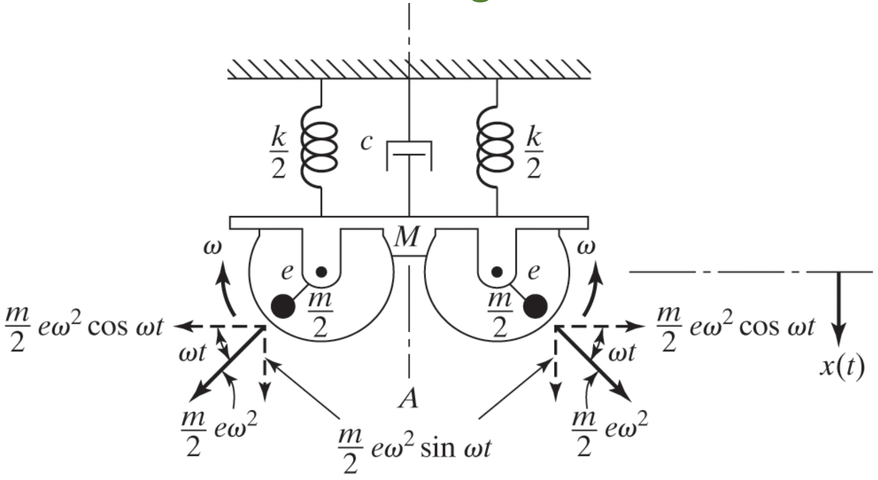
Relative Response:

$\mathbf{z}(\boldsymbol{t}) = (\boldsymbol{x} - \boldsymbol{y})$

Force Transmitted to the base:

$$\boldsymbol{F_T(t) = c\dot{x} + kx}$$

Case 3: Rotating unbalance



$$F(t) = me\omega^2 \sin(\omega t)$$

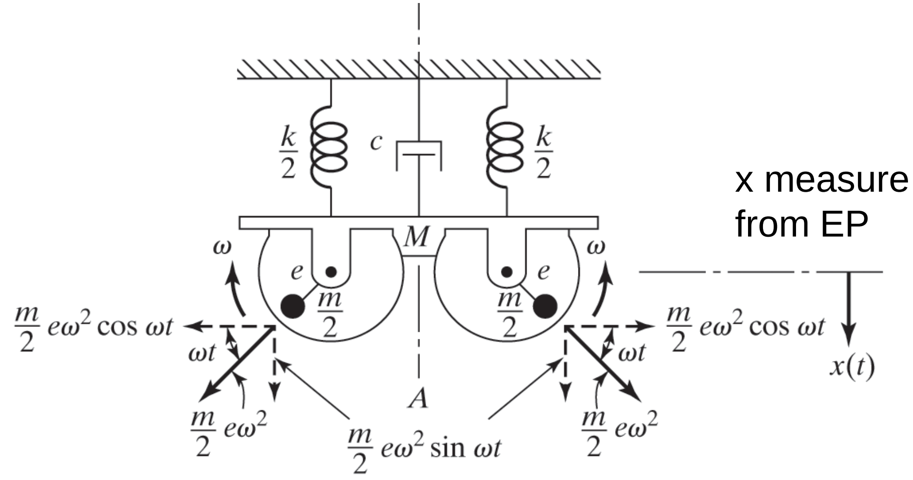
Absolute response:

$\boldsymbol{x(t)}$

Force Transmitted to the base:

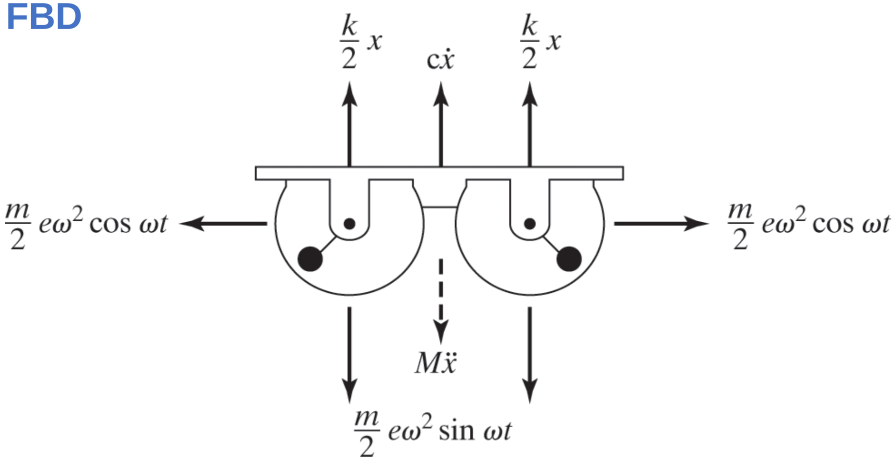
$$\boldsymbol{F_T(t) = c\dot{x} + kx}$$

Response of a Damped System Under rotating unbalance



x measure from EP

FBD



In steady-state the angular acceleration of the unbalance masses is zero

Forces in vertical direction:

$$c\dot{x} + kx - me\omega^2 \sin(\omega t) = -M\ddot{x}$$

Governing equation:

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin(\omega t)$$

$$F(t) = me\omega^2 \sin(\omega t)$$

The steady state solution

$$x(t) = \frac{me\omega^2}{k} \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega t - \varphi_1)$$

Dividing and multiplying by the mass M:

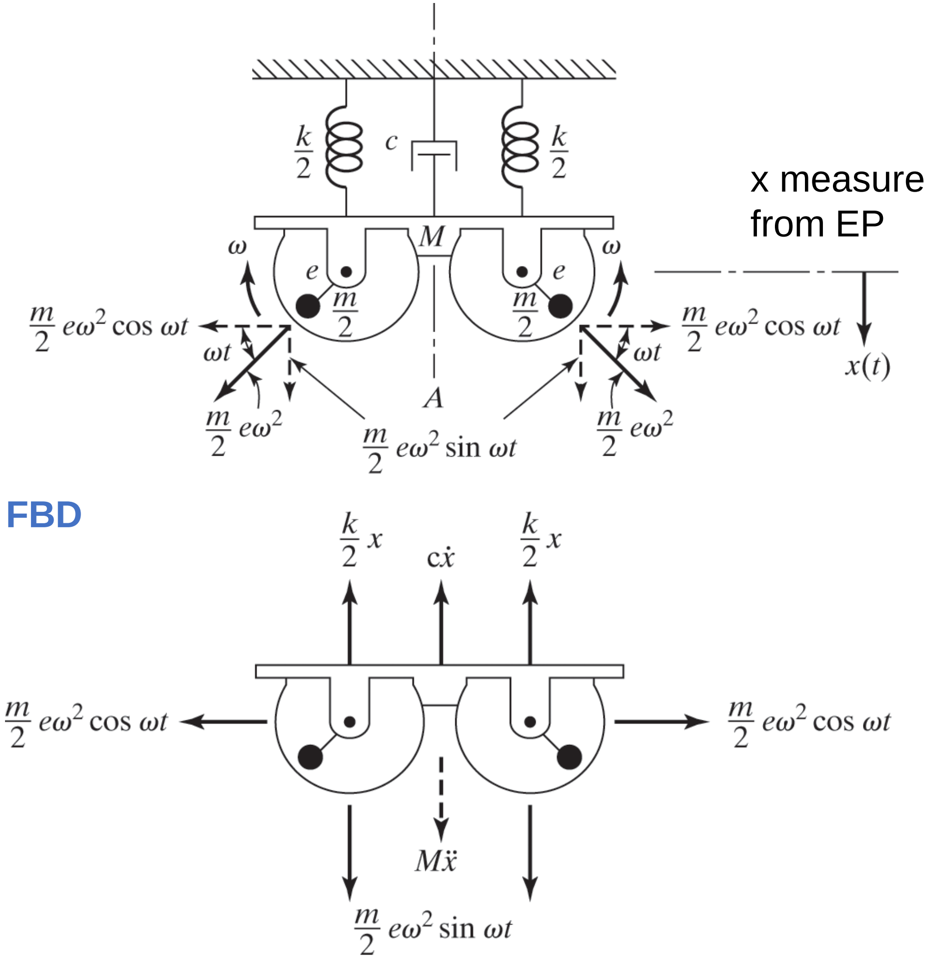
$$x(t) = \frac{me\omega^2 M}{Mk} \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega t - \varphi_1)$$

The amplitude or maximum value can be expressed as

$$\boxed{X = \frac{me}{M} \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{me}{M} r^2 \boldsymbol{M}}$$

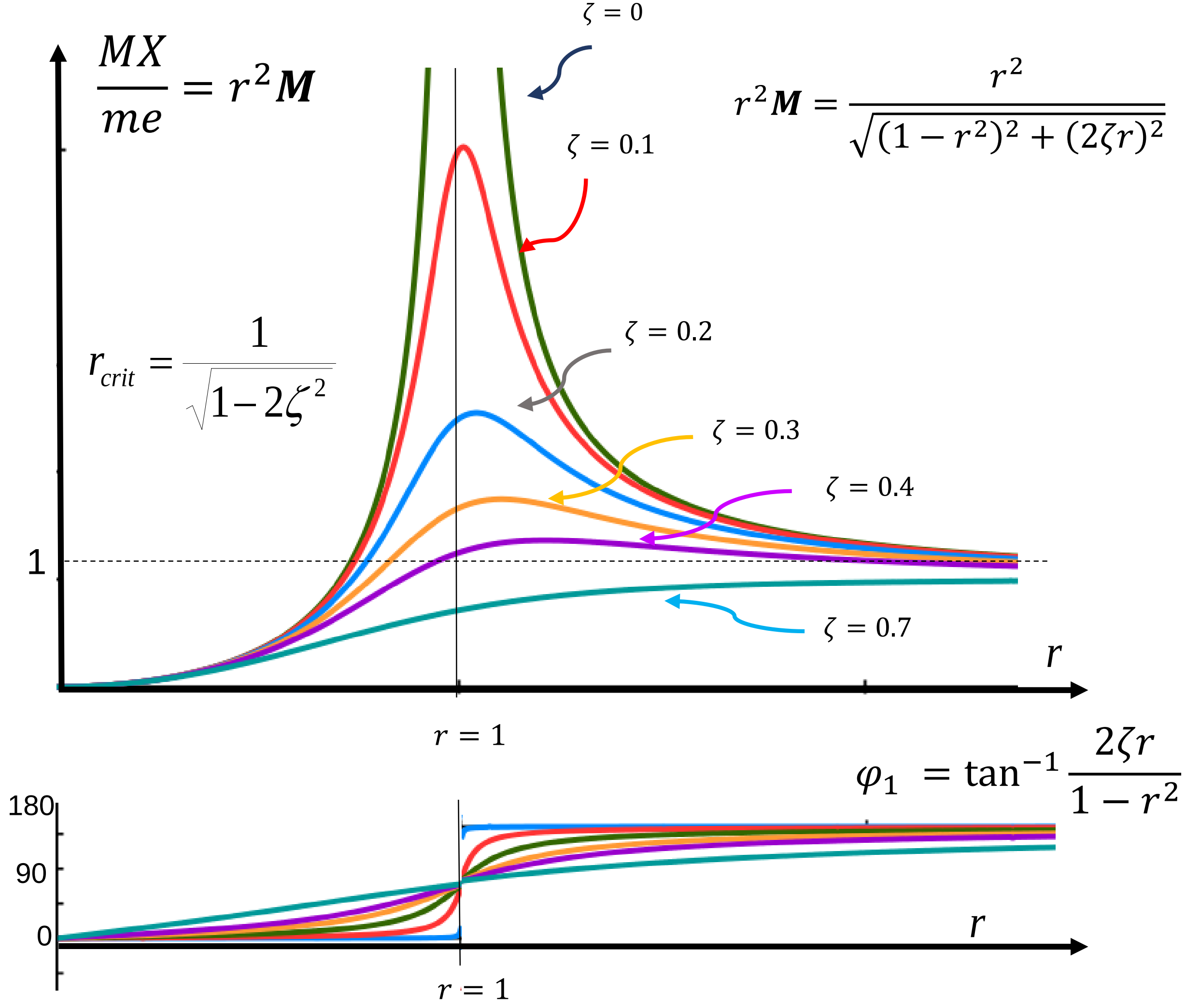
$$\boxed{\varphi_1 = \tan^{-1} \frac{2\zeta r}{1-r^2}}$$

Response of a Damped System Under rotating unbalance



x measure from EP

In steady-state the angular acceleration of the unbalance masses is zero



Figures and content adapted from Textbook:
Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition

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Characteristics of the factor $\left(\frac{Z}{Y_0} = r^2\boldsymbol{M} \; ; \; \frac{MX}{me} = r^2\boldsymbol{M}\right)$

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4. The maximum of $r^2\boldsymbol{M}$ occurs when $r_{crit} = \frac{1}{\sqrt{1-2\zeta^2}}$, valid for values $0 < \zeta < 1/\sqrt{2}$

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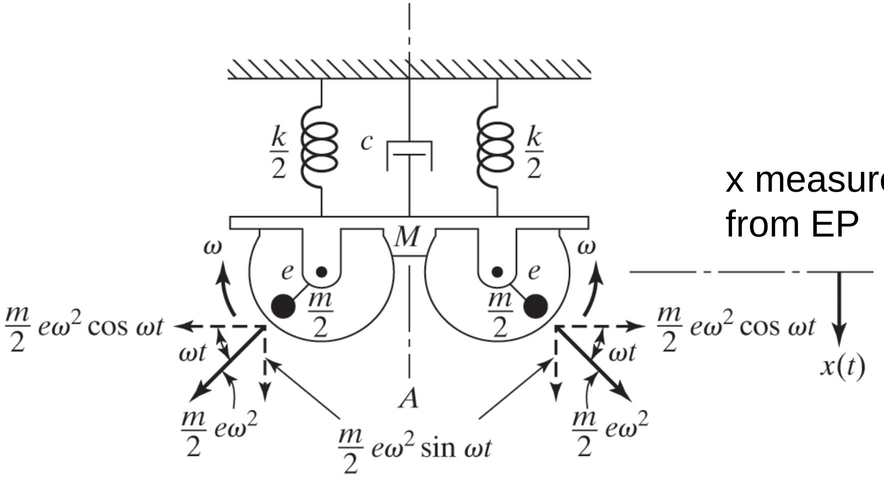
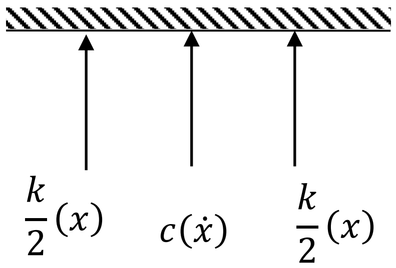
Transmitted force of Damped System Under rotating unbalance



x measure from EP

FBD



$$F_T(t) = c\dot{x} + kx$$

$$x(t) = \frac{me}{M}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t - \varphi)$$

$$F_T(t) = c\dot{x} + kx = c\frac{me}{M}r^2\boldsymbol{M}\cos(\omega t - \varphi_1) + k\frac{me}{M}r^2\boldsymbol{M}\sin(\omega t - \varphi_1)$$

$$F_T(t) = \frac{me}{M}r^2\boldsymbol{M}\left[\sqrt{(c\omega)^2 + (k)^2}\right]\sin(\omega t - \varphi_1 - \alpha)$$

$$F_T(t) = \frac{mek}{M}r^2\boldsymbol{M}\left[\sqrt{(2\zeta r)^2 + 1}\right]\sin(\omega t - \varphi_1 - \alpha)$$

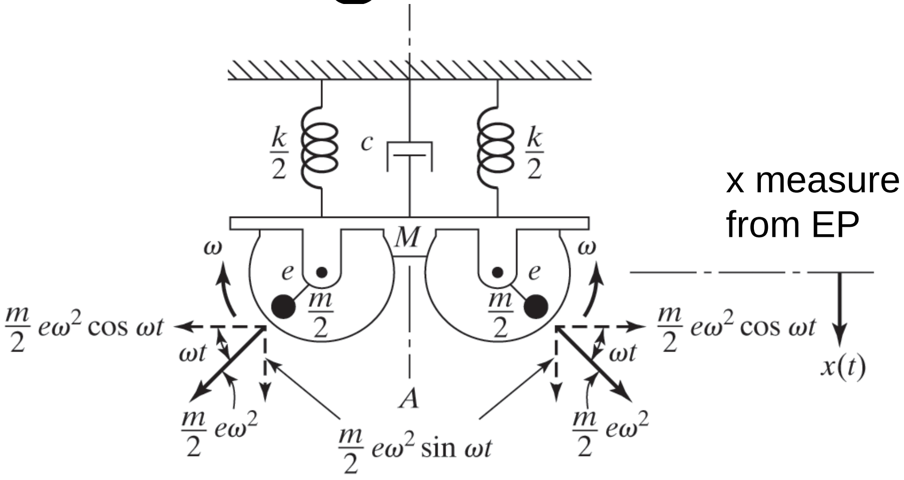
$$\boxed{F_T(t) = me{\omega_n}^2 r^2\frac{\sqrt{(2\zeta r)^2+1}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t - \varphi_1 - \alpha)}$$

The amplitude or maximum value of the force transmitted to the base is called again ***Force transmissibility***

$$\boxed{\frac{F_T}{me{\omega_n}^2} = r^2\frac{\sqrt{(2\zeta r)^2+1}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = r^2 T_d}$$

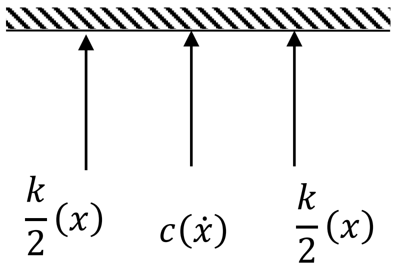
$$\boxed{\varphi = \tan^{-1}\left(\frac{2\zeta r^3}{1+(4\zeta^2-1)r^2}\right)}$$

Force transmitted viscous damped system under rotating unbalance



x measure from EP

FBD

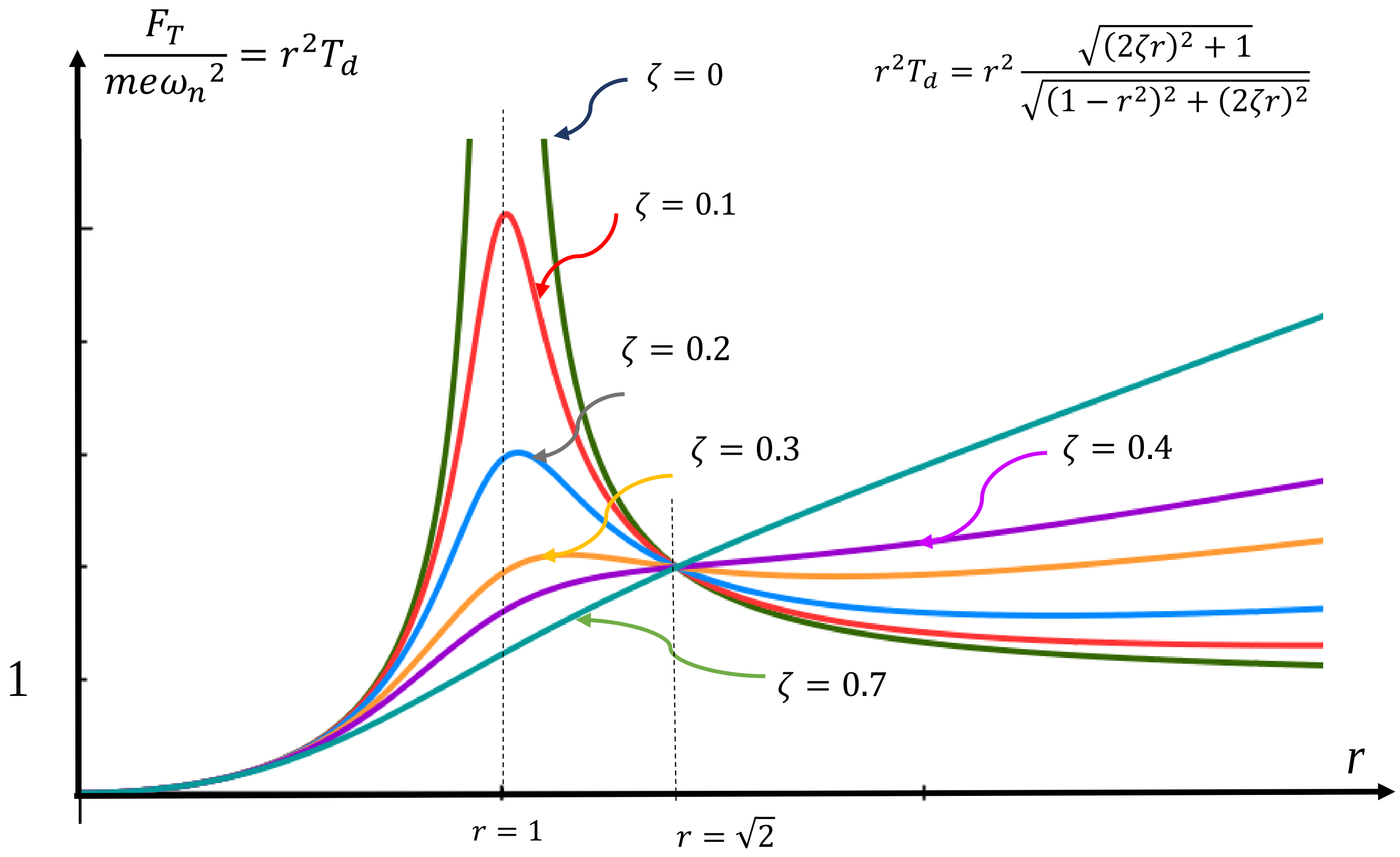


$$F_T(t) = c\dot{x} + kx$$

$$x(t) = \frac{me}{M}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t - \varphi)$$

Force transmissibility For M-D-S system, under rotating unbalance

$$r^2T_d = r^2\frac{\sqrt{(2\zeta r)^2+1}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$



Formula Sheet for Harmonilly excited Vibration Steady state response

Non-dimentional parameters

$$r = \frac{\Omega}{\omega_n}$$

Magnification factor

$$\boldsymbol{M} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Transmissibility Coefficient

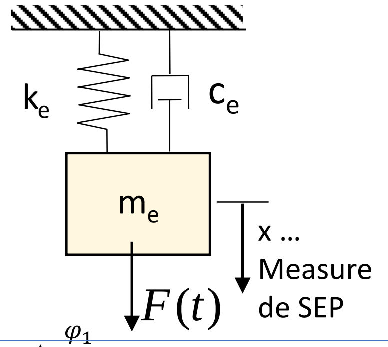
$$\boldsymbol{T_d} = \frac{\sqrt{(2\zeta r)^2 + 1}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$m_e\ddot{x} + c_e\dot{x} + k_e x = F(t)$$

$$x(t) = x_h(t) + x_p(t)$$

Steady-state response

$$x(t) = x_p(t)$$



For a simple harmonic force:

$$F(t) = F_o \cos \Omega\, t$$

Response:

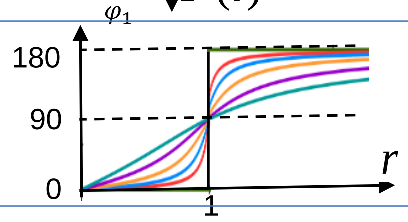
$$x_p(t) = \frac{F_0}{k_e}\boldsymbol{M}\cos(\Omega t - \varphi_1)$$

Force transmitted to the base

$$F_T(t) = F_0 \boldsymbol{T_d} \cos(\Omega t - \varphi_1 - \alpha)$$

$$\varphi_1 = \tan^{-1}\frac{2\zeta r}{1-r^2}$$

$$\alpha = \tan^{-1}(2\zeta r)$$



Harmonic Motion of the Base

$$F(t) = cY_o\Omega \cos \Omega\, t + kY_o \sin \Omega\, t$$

Absolute response:

$$X_p = Y_o \boldsymbol{T_d} \sin(\Omega t - \varphi_1 - \alpha)$$

Force transmitted to the base

$$F_T(t) = kY_o \boldsymbol{r^2} \boldsymbol{T_d} \cos(\Omega t - \varphi)$$

Relative Response:

$$y_p(t) = Y_o \boldsymbol{r^2} \boldsymbol{M} \sin(\Omega t - \varphi_1)$$

Rotating unbalance *me*

$$F(t) = me\Omega^2 \sin \Omega\, t$$

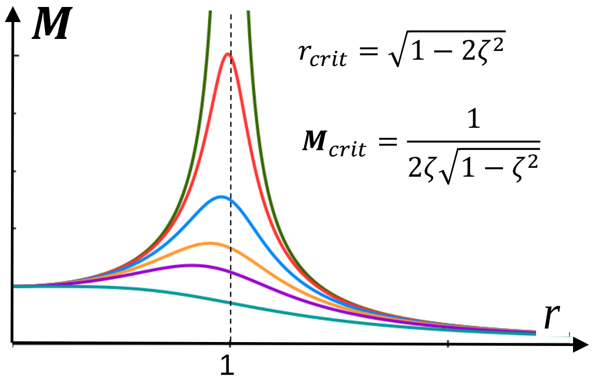
Response:

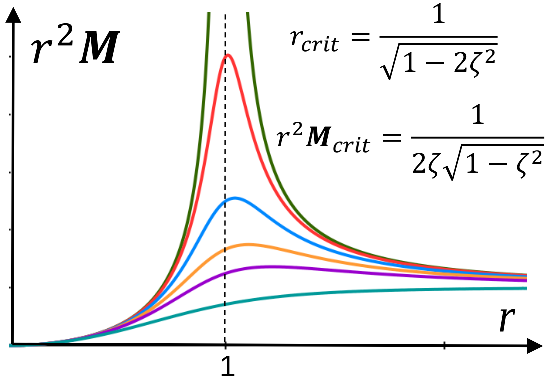
$$x_p(t) = \frac{me}{M}\boldsymbol{r^2}\boldsymbol{M}\sin(\Omega t - \varphi_1)$$

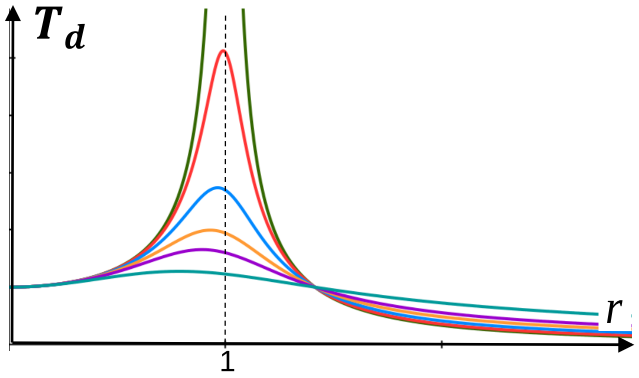
Force transmitted to the base

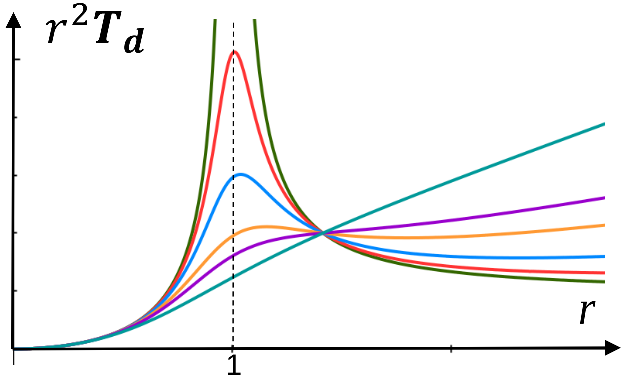
$$F_T(t) = me\omega_n^2 \boldsymbol{r^2} \boldsymbol{T_d} \sin(\Omega t - \varphi_1 - \alpha),$$

$$\varphi = \tan^{-1}\left(\frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2}\right)$$









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Formula Sheet for Harmonilly excited Vibration Total response

Non-dimentional parameters

$$r = \frac{\Omega}{\omega_n}$$

Magnification factor

$$\boldsymbol{M} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Static deflection

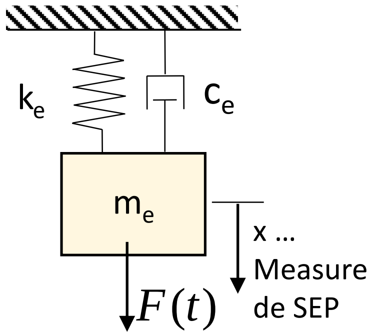
$$\delta_{st} = \frac{Fo}{k}$$

$$m_e\ddot{x} + c_e\dot{x} + k_e x = F(t)$$

$$x(t) = x_h(t) + x_p(t)$$

Steady-state response

$$x(t) = x_p(t)$$



Governing equation

$$m_e\ddot{x} + c_e\dot{x} + k_e x = F(t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = F(t)/m$$

Force : $F(t) = F_o \cos \Omega\, t$

Response : $x(t) = x_h(t) + x_p(t)$

$$x_p(t) = X\cos(\Omega t - \varphi_1)$$

$$X = \frac{F_0}{k_e}\boldsymbol{M} \qquad \varphi_1 = \tan^{-1}\frac{2\zeta r}{1-r^2}$$

Underdamped systems

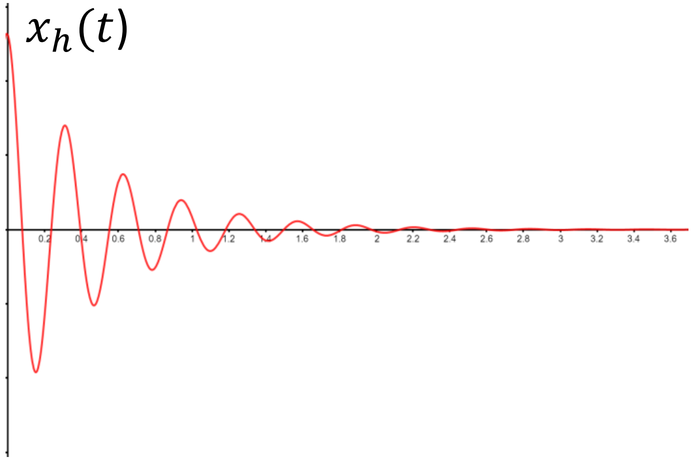
$0 < \zeta < 1$ $\quad x(t) = X_0 e^{-\zeta\omega_n t}\cos\left(\omega_d t - \varphi_0\right) + X\cos(\Omega t - \varphi_1)$

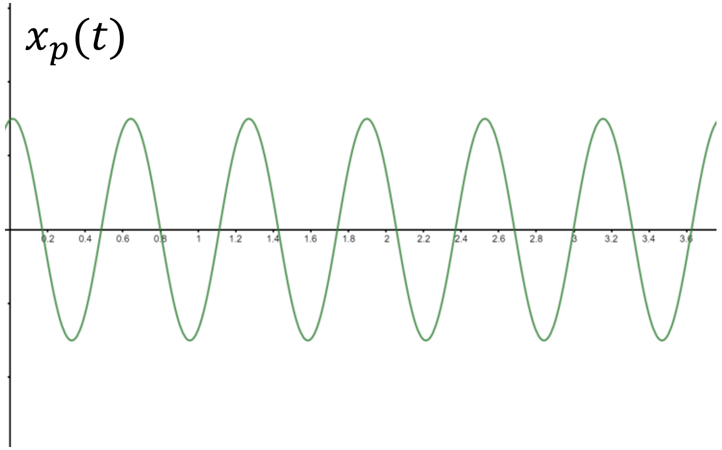
Initial conditions x_0, v_0

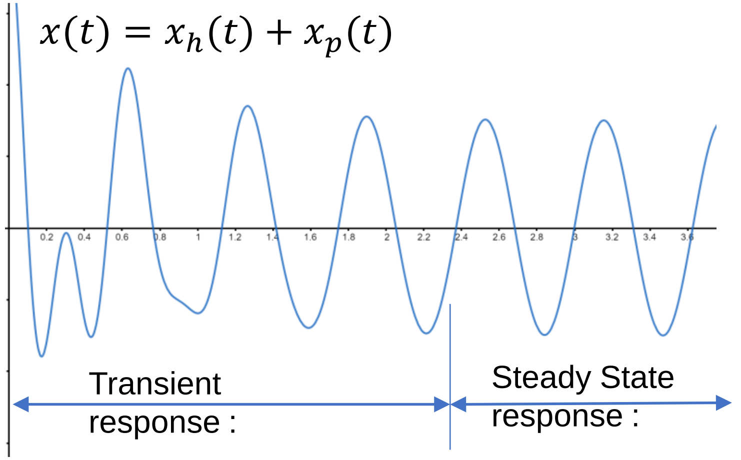
$$\left.\begin{aligned} x_0 &= X_0\cos\varphi_0 + X\cos\varphi_1 \\ v_0 &= -\zeta\omega_n X_0\cos\varphi_0 + \omega_d X_0\sin\varphi_0 + \Omega X\sin\varphi_1 \end{aligned}\right\}$$

$$X_0 = \left[(-X_0\cos\varphi_0)^2 + \frac{1}{{\omega_d}^2}(\zeta\omega_n x_0 + v_0 - \zeta\omega_n X\cos\varphi_1 - \Omega X\sin\varphi_1)^2\right]^{\frac{1}{2}}$$

$$\varphi_0 = tan^{-1}\left[\frac{\zeta\omega_n x_0 + v_0 - \zeta\omega_n X\cos\varphi_1 - \Omega X\sin\varphi_1}{\omega_d(x_0 - X\cos\varphi_1)}\right]$$







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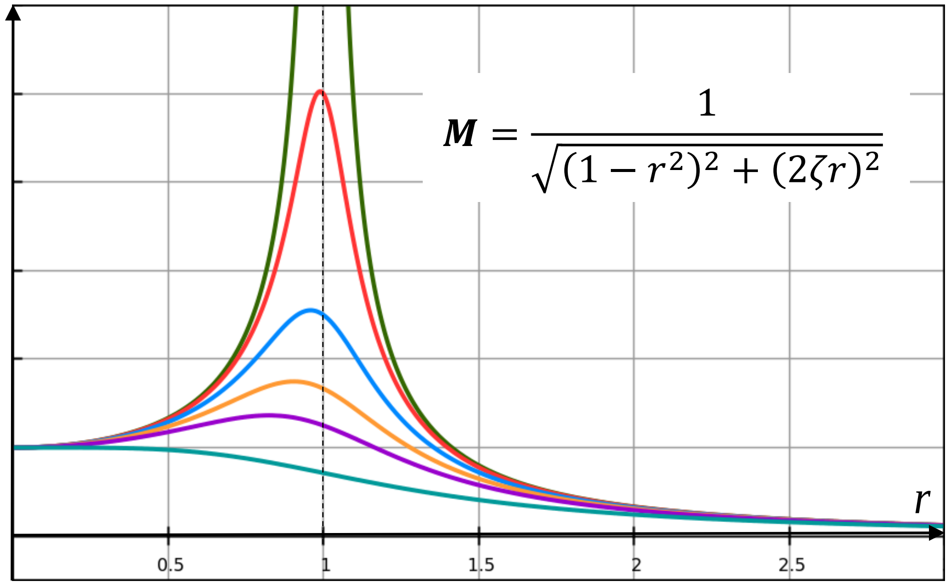
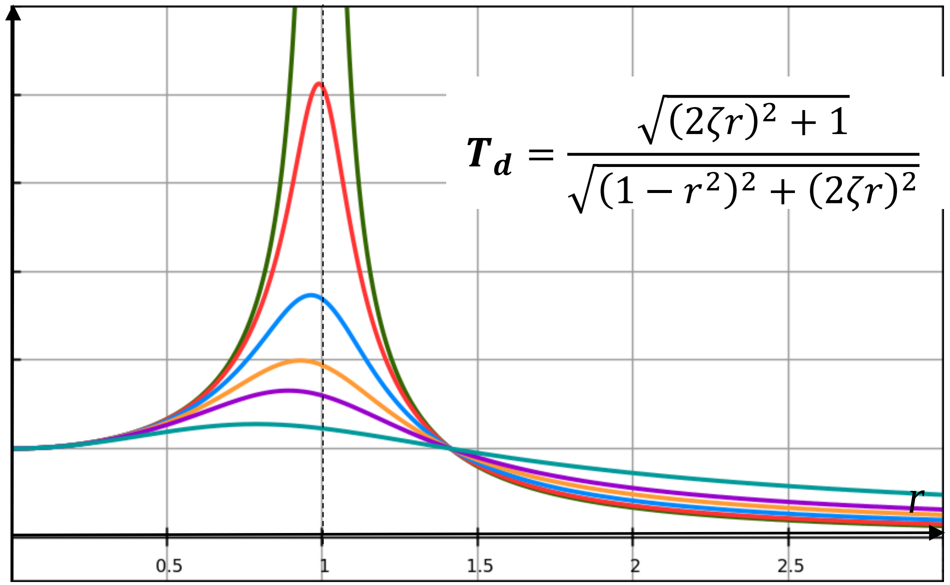
Graphs:

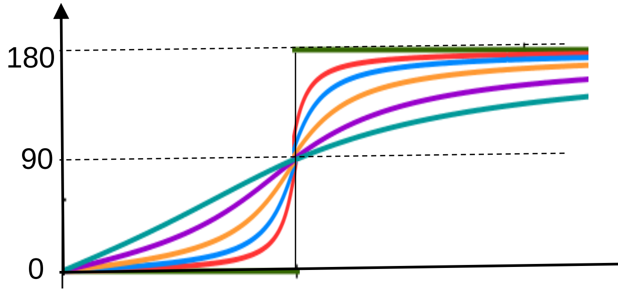
Magnification factor (M)

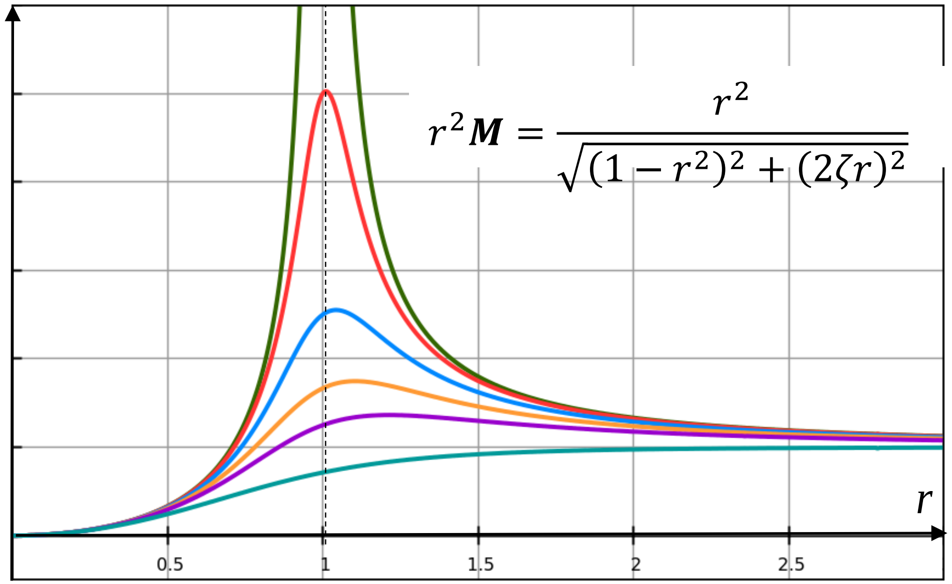
Transmissibility Coefficient (T_d)

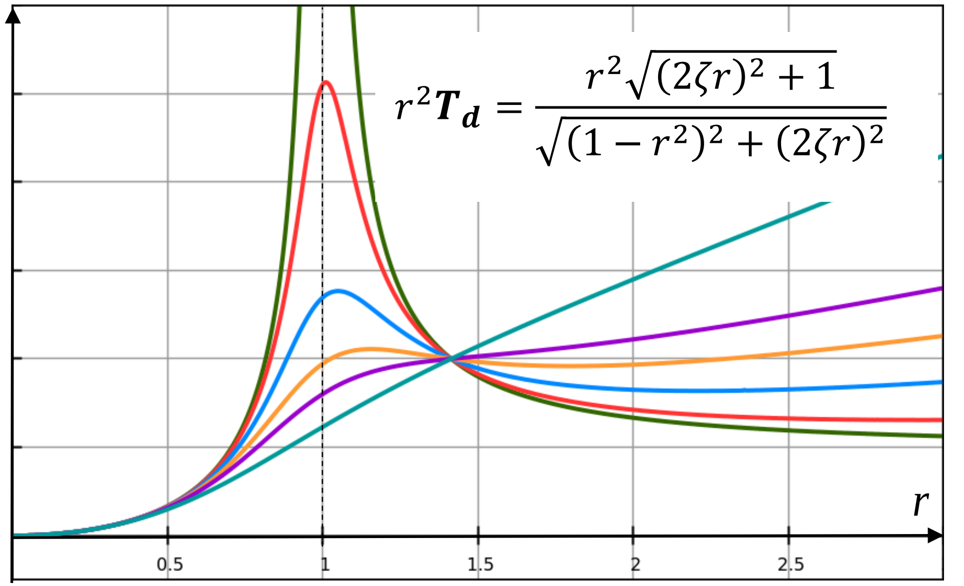




$$\varphi_1 = \tan^{1} \frac{2\zeta r}{1 - r^2}$$







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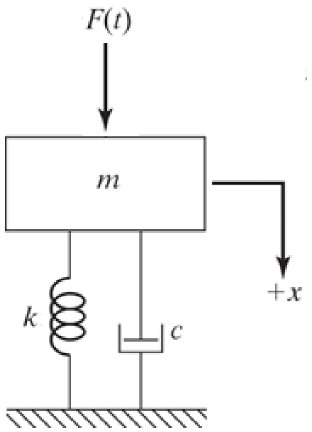
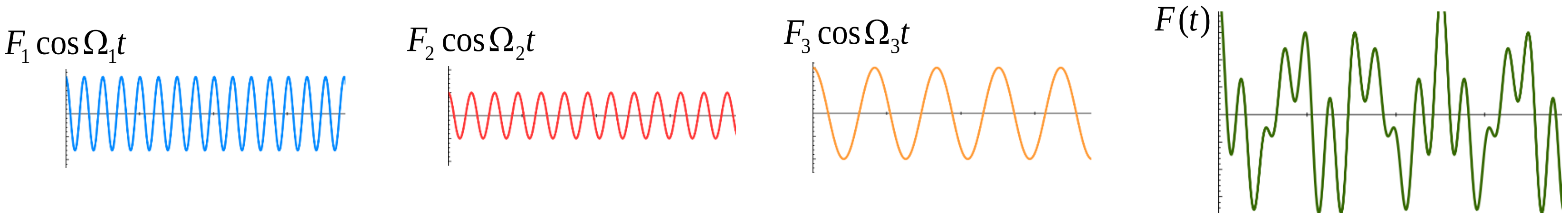
Response of a Damped System composited forces



Governing equation:

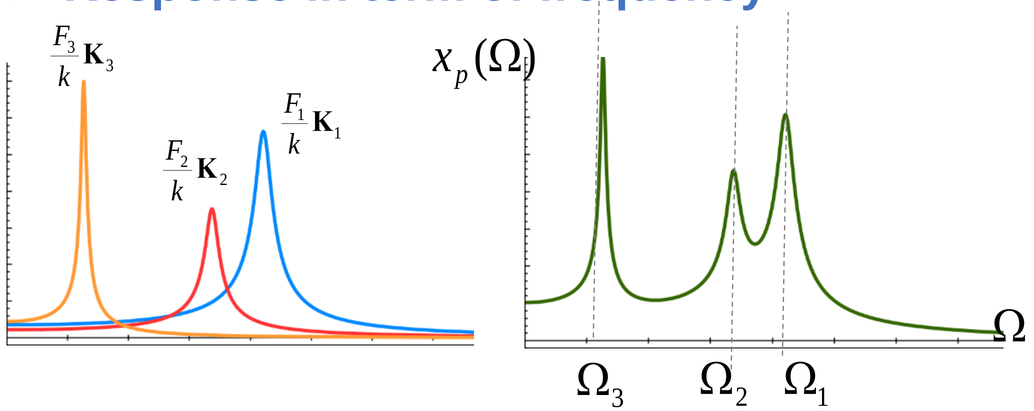
$$m\ddot{x} + c\dot{x} + kx = F(t) \qquad F(t) = F_1 \cos \Omega_1\, t + F_2 \cos \Omega_2\, t + F_3 \cos \Omega_3\, t$$



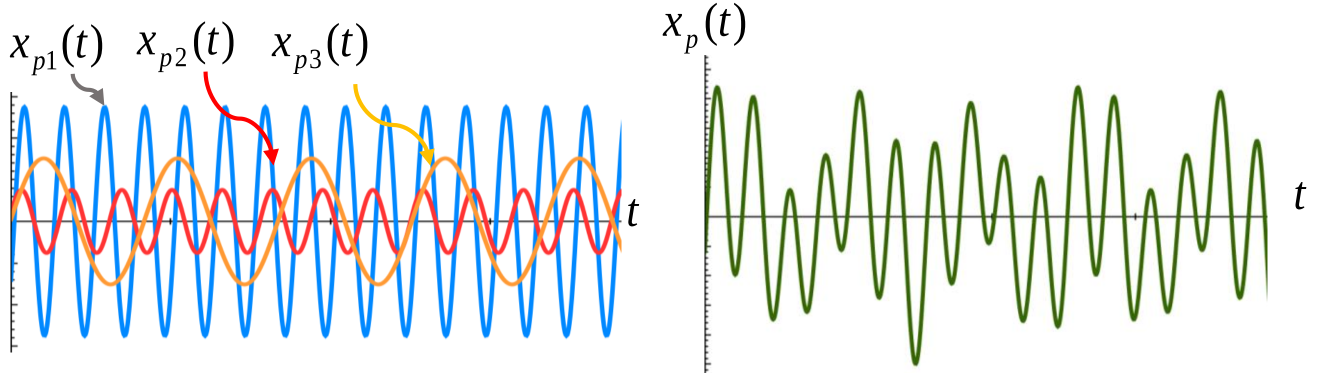
When the system is linear for small displacements, the **steady state solution** becomes the sumation of all the individual responses to each force.

$$x_p(t) = \frac{F_1}{k} \boldsymbol{M}_1 \cos(\Omega_1 t - \varphi_1) + \frac{F_2}{k} \boldsymbol{M}_2 \cos(\Omega_2 t - \varphi_2) + \frac{F_3}{k} \boldsymbol{M}_3 \cos(\Omega_3 t - \varphi_3) \qquad \boldsymbol{M}_i = \frac{1}{\sqrt{(1 - r_i^2)^2 + (2\zeta r_i)^2}}, \qquad \phi_i = \tan^{-1} \frac{2\zeta r_i}{1 - r_i^2} \qquad i = 1,2,3$$

- **Response in term of frequency**

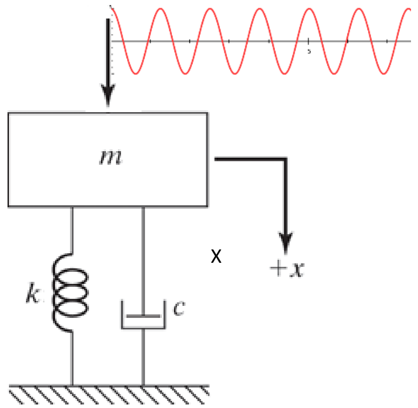


- **Response in term of time**



Quality factor (Q) and Bandwith

$$F(t) = F_o \cos \Omega\, t$$



- The **steady-state** response

$$x_p(t) = X_P \cos(\Omega t - \varphi)$$

$$X_P = = \delta_{st} \boldsymbol{M}$$

$$\delta_{st} = \frac{F_0}{k}$$

$$\varphi = \tan^1 \frac{2\zeta r}{1 - r^2}$$

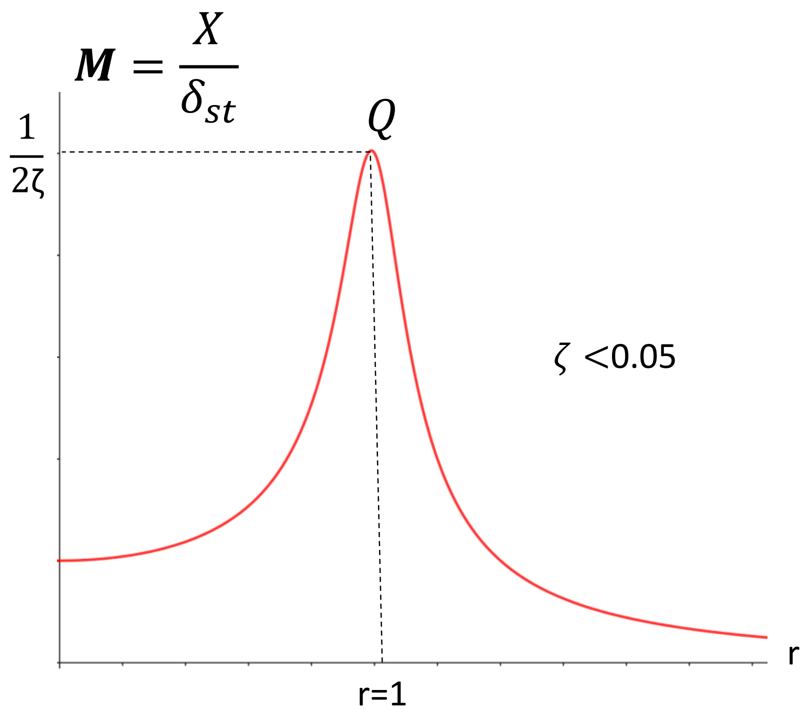
- The value of the amplitud ratio at resonance (r=1) is also called **Q factor** or **Quality factor**

$$\boxed{Q = \left(\frac{X}{\delta_{st}}\right)_{r_{crit}} = (\boldsymbol{M})_{max}}$$

$$\boldsymbol{M} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

- For small values of damping (ζ<0.05) we can take $r_{crit} \cong 1$

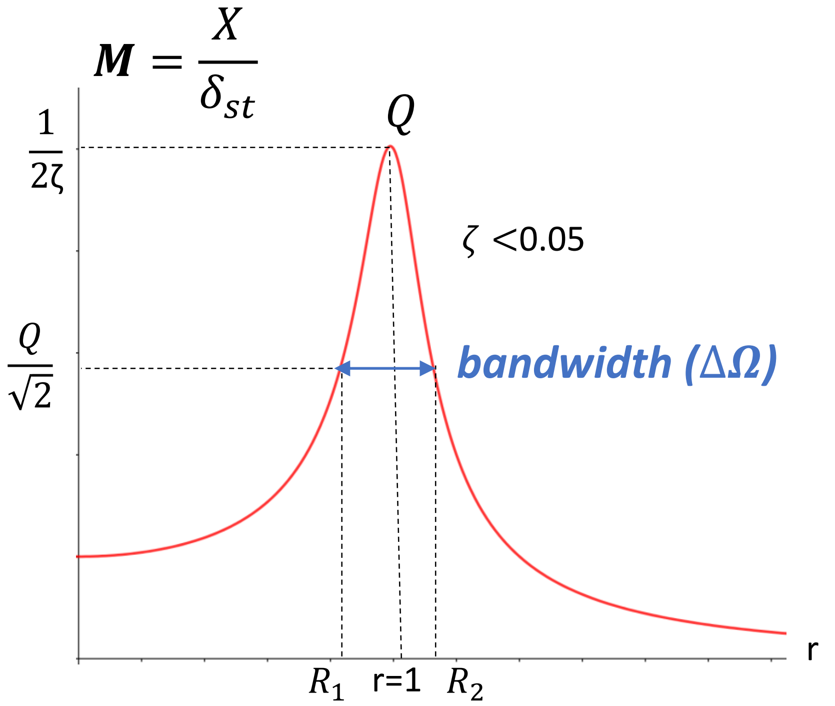
$$\left(\frac{X}{\delta_{st}}\right)_{max} \cong \left(\frac{X}{\delta_{st}}\right)_{\Omega = \omega_n} = (\boldsymbol{M})_{r=1} = \frac{1}{2\zeta} = Q$$



Quality factor (Q) and Bandwith

- The points R1 and R2, called ***half-power points***, are set where the amplification factor falls to $\frac{Q}{\sqrt{2}}$, this is because the power absorbed ($\Delta W = \pi c \Omega X^2$) by the damper, responding harmonically at a given frequency, is proportional to the square of the amplitude. They can be approximated

$$R_1^2 = 1 - 2\zeta \qquad\qquad R_2^2 = 1 + 2\zeta$$



- The difference between the frequencies associated with the half-power points R1 and R2 is called the ***bandwidth (ΔΩ),*** and can be calculated as:

$$\Delta\Omega \cong \Omega_2 - \Omega_1 \cong 2\zeta\omega_n$$

- The quality factor Q can be used for estimating the equivalent viscous damping in a mechanical system, can also be calculated using the following expression:

$$Q \cong \frac{1}{2\zeta} \cong \frac{\omega_n}{\Omega_2 - \Omega_1}$$

Mechanical Vibrations Measurement instruments

Prof. Carmen Muller-Karger, PhD
Florida International University

Figures and content adapted from
Textbook: Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition.
Chapter 10, section 10.3: Vibration Pickups

Importance of measuring vibration

- Measurement of vibration ensures adequate safety margins.
- Selecting the operational speeds of nearby machinery to avoid resonant conditions.
- Mechanical models may not represent actual values due to assumptions.
- Frequencies of vibration and the forces are necessary in the design and operation of active vibration-isolation systems
- Identification of the characteristics of a system in terms of its mass, stiffness, and damping.
- Information about ground vibrations due to earthquakes, fluctuating wind velocities on structures, random variation of ocean waves, and road surface roughness are important in the design of structures, machines, oil platforms, and vehicle suspension systems.

Vibration Measurement Scheme

Vibrating Machine or structure

The motion (or dynamic force) of the vibrating body is converted into an electrical signal by the vibration transducer or pickup.

Vibration transducer or pickup

Transducer transforms changes in mechanical quantities (such as displacement, velocity, acceleration, or force) into changes in electrical quantities (such as voltage or current).

Signal Conversion instrument

A signal conversion instrument is used to amplify the signal to the required value. Usually the output signal (voltage or current) of a transducer is too small to be recorded directly.

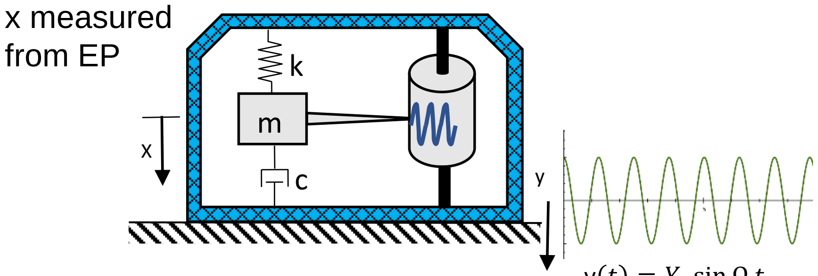
Display unit, recorder, or computer

The output from the signal conversion instrument can be presented on a display unit for visual inspection, or recorded by a recording unit, or stored in a computer for later use.

Data analysis

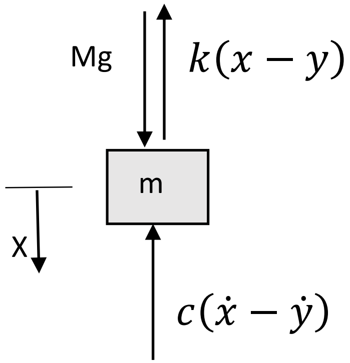
The data can then be analyzed to determine the desired vibration characteristics of the machine or structure.

Vibration Pickups



$y(t) = Y_o \sin \Omega\, t$

FBD



Governing equation:

$$c(\dot{x}-\dot{y}) + k(x-y) = -m\ddot{x}$$

Relative motion : $z = (x-y)$

$$c(\dot{z}) + k(z) = -m(\ddot{z}+\ddot{y})$$

$$\boxed{m\ddot{z} + c\dot{z} + kz = -m(\ddot{y})}$$

- Are instruments to measure vibration commonly known as seismic instrument.
- They consists of a mass-spring-damper system mounted on the vibrating body.
- The vibratory motion is measured by finding the displacement of the mass relative to the base.
- The bottom ends of the spring and the dashpot will have the same motion as the cage (to be measured).
- *x* denotes the vertical displacement of the suspended mass

Figures and content adapted from Textbook:
Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition
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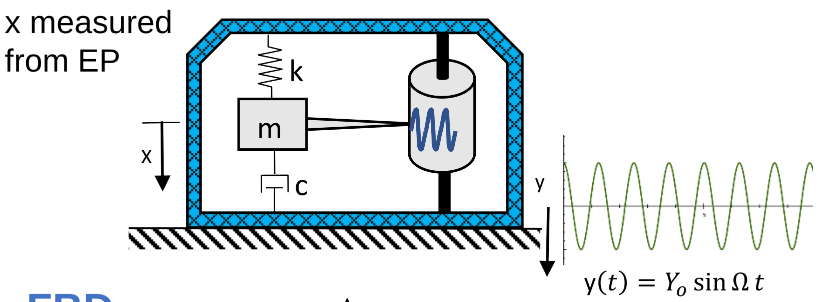
Vibration Pickups



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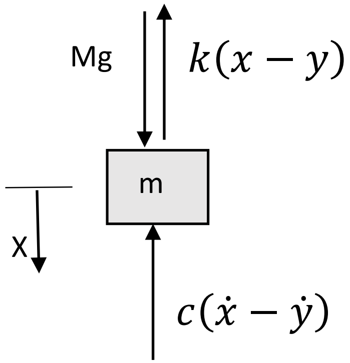


Governing equation:

$$c(\dot{x}-\dot{y}) + k(x-y) = -m\ddot{x}$$

Relative motion : $z = (x-y)$

$$c(\dot{z}) + k(z) = -m(\ddot{z}+\ddot{y})$$

$$\boxed{m\ddot{z} + c\dot{z} + kz = -m(\ddot{y})}$$

Governing equation:

$$m\ddot{z} + c\dot{z} + kz = m\Omega^2 Y_o \sin \Omega\, t \qquad \ddot{y} = -m\Omega^2 Y_o \sin \Omega\, t$$

The steady state solution

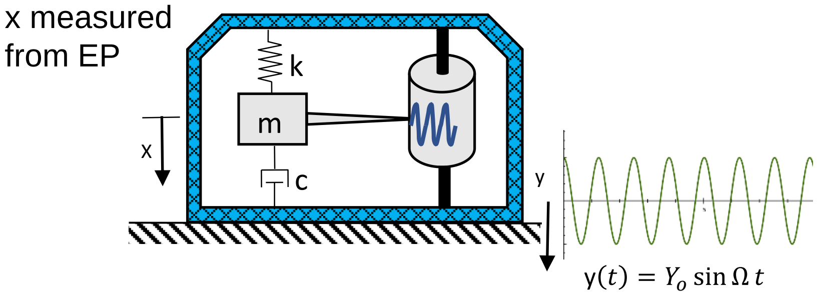
$$z(t) = \frac{m\Omega^2 Y_o}{k} \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\Omega t - \varphi_1)$$

$$\boxed{z(t) = Y_o \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\Omega t - \varphi_1)} \qquad \boxed{\varphi_1 = \tan^{-1}\frac{2\zeta r}{1-r^2}}$$

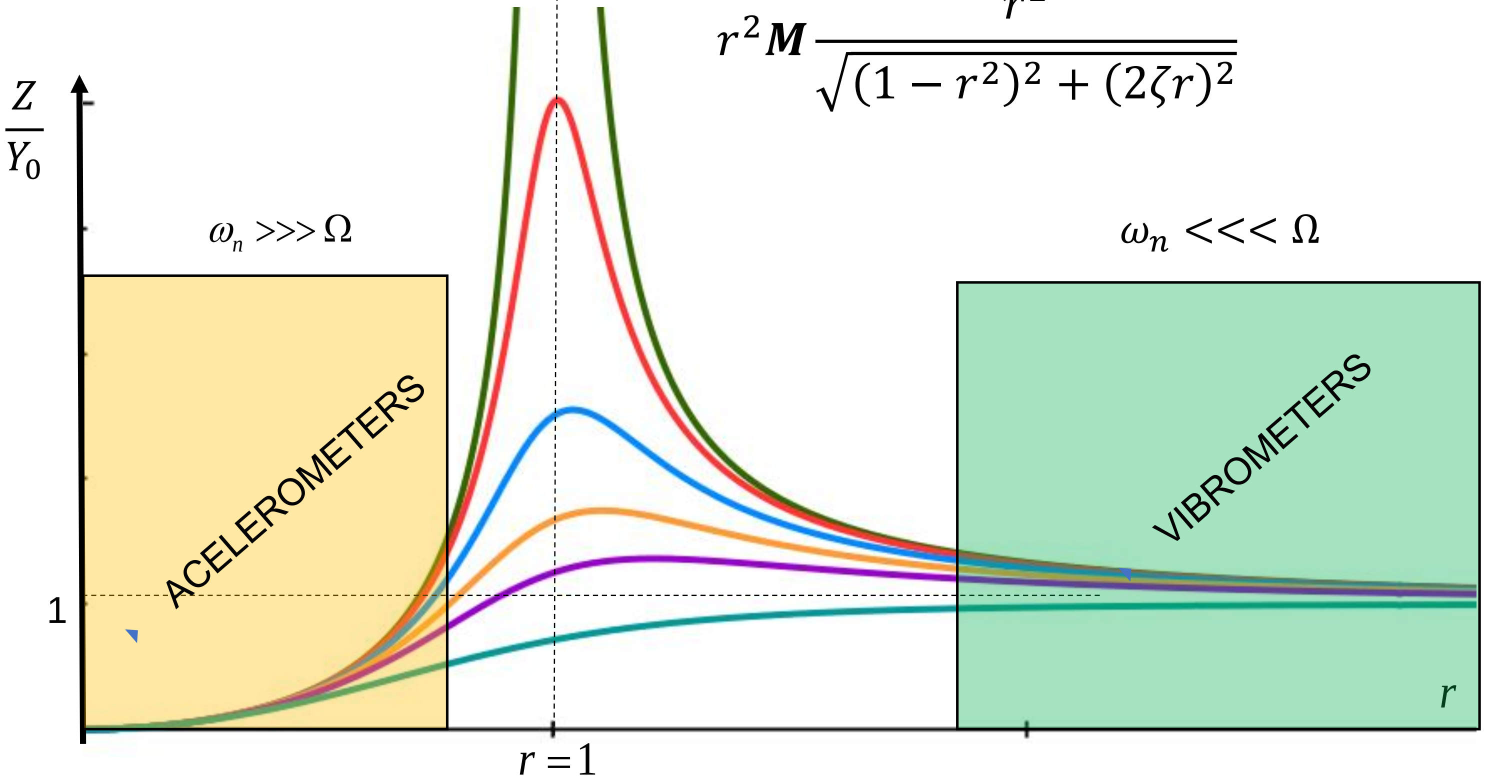
The amplitude or maximum value can be expressed as

$$\boxed{\frac{Z}{Y_0} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = r^2 \boldsymbol{M}}$$

Vibration Pickups



$$r^2 \boldsymbol{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$



The steady state solution

$$z(t) = Y_o r^2 \mathbf{K} \sin(\Omega t - \varphi)$$

$$\varphi = \tan^{1} \frac{2\zeta r}{1 - r^2}$$

$$\boxed{\frac{Z}{Y_0} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = r^2 \boldsymbol{M}}$$

- ACELEROMETERS measure the acceleration of a vibrating body. The natural frequency of the instrument is much higher than the frequency to measure

$r \Rightarrow 0 \qquad \omega_n >>> \Omega \qquad \boldsymbol{M} \Rightarrow 1$

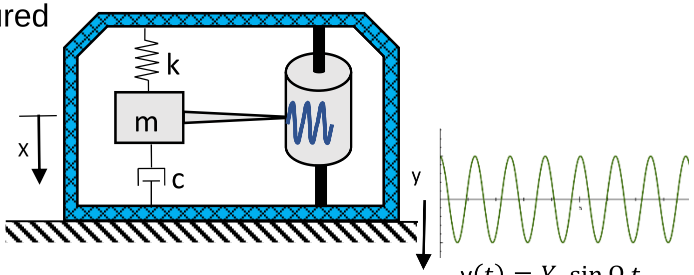
$$\boxed{Z = Y_o r^2 = \frac{1}{{\omega_n}^2} Y_o \Omega^2}$$

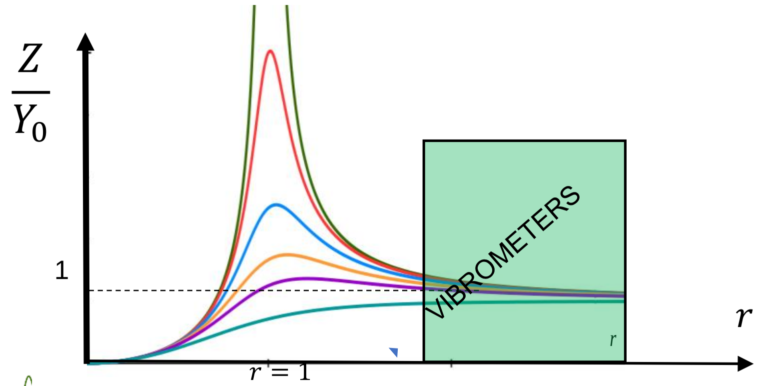
- VIBROMETERS, measure the displacement of a vibrating body. The natural frequency of the instrument is much less than the frequency to measure.

$r \Rightarrow \infty \qquad \omega_n <<< \Omega \qquad r^2 \boldsymbol{M} \Rightarrow 1$

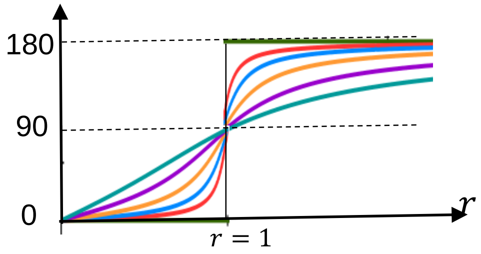
$$\boxed{Z = Y_o}$$

Vibrometers





$\varphi = \tan^1 \frac{2\zeta r}{1}$



$$z(t) = Y_o r^2 \mathbf{K} \sin(\Omega t - \varphi)$$

$$\varphi = \tan^1 \frac{2\zeta r}{1 - r^2}$$

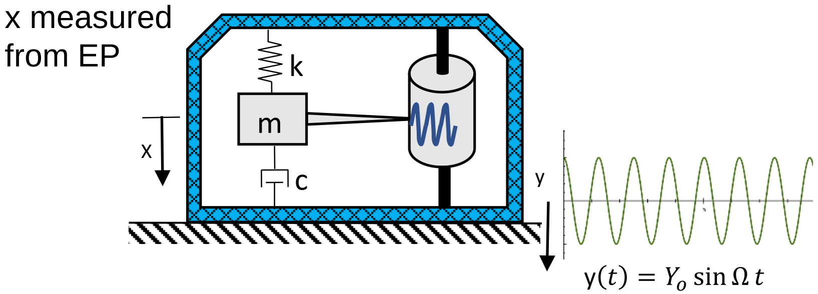
$$\boxed{\frac{Z}{Y_0} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = r^2 \boldsymbol{M}}$$

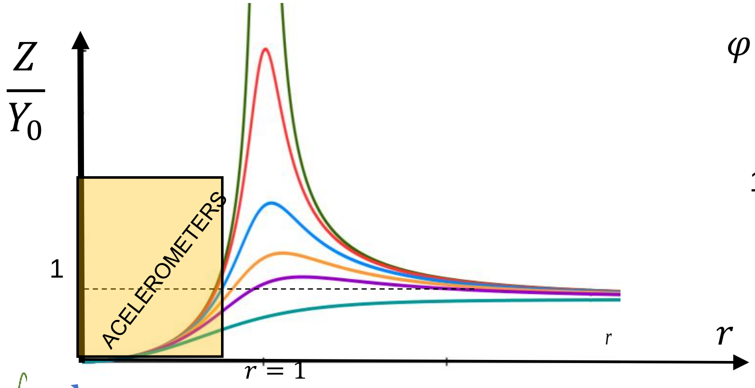
$r \Rightarrow \infty \qquad \omega_n <<< \Omega \qquad r^2 \boldsymbol{M} \Rightarrow 1$

$$\boxed{Z = Y_o}$$

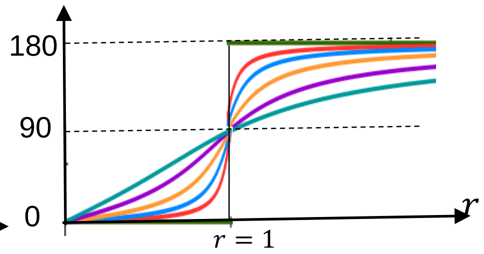
- Measures the displacement of a vibrating body.
- The frequency to measure is very large relative to the natural frequency of the instrument (at least r>3).
- The relative displacement between the mass and the base is essentially the same as the displacement of the base for $r \Rightarrow \infty$.
- To obtain a low natural frequency, them mass must be large and the spring a low stiffness, this result in a bulky instrument.
- If the value of *r* is not sufficiently height the Z value measured is not equal to Y_o, in which case you have to use the whole equation $r^2 \boldsymbol{M}$.
- The phase lag can be seen to be equal to 180° for ζ=0. Thus the recorded displacement $z(t)$ lags behind the displacement being measured $y(t)$ by time t′= φ / Ω. This time lag is not important if the base displacement $y(t)$ consists of a single harmonic component.

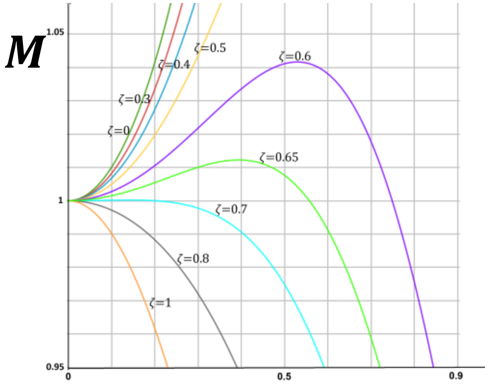
Accelerometer





$$\varphi = \tan^{1} \frac{2\zeta r}{1 - r^2}$$





$$z(t) = Y_o r^2 \mathbf{K} \sin(\Omega t - \varphi)$$

$$\varphi = \tan^{1} \frac{2\zeta r}{1 - r^2}$$

$$\boxed{\frac{Z}{Y_0} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = r^2 \boldsymbol{M}}$$

$$r \Rightarrow 0 \qquad \omega_n >>> \Omega \qquad \boldsymbol{M} \Rightarrow 1$$

$$\boxed{y_p = Z_o r^2 = \frac{1}{{\omega_n}^2} Z_o \Omega^2}$$

- It is use to measure vibration with much lower frequency than the natural frequency of the instrument.
- For small values of r, 0≤r≤0.60 and values of ζ between 0.65 and 0.7, $\boldsymbol{M}$ lies between of 0.96 and 1.04.
- Measure the acceleration of a vibrating body, except for the phase lag φ. The time by which the record lags the acceleration is given by $t' = \varphi / \Omega$. If the acceleration $\ddot{y}$ consists of a single harmonic component, the time lag will not be of importance.
- Since r is small, the natural frequency of the instrument has to be large compared to the frequency of vibration to be measured. The mass needs to be small and the spring needs to have a large value of k (i.e., short spring), so the instrument will be small in size. Due to their small size and high sensitivity, accelerometers are preferred in vibration measurements.

Vibration Exciters

- The vibration exciters or shakers can be used in several applications such as determination of the dynamic characteristics of machines and structures and fatigue testing of materials. The vibration exciters can be mechanical, electromagnetic, electrodynamic, or hydraulic type.