Mechanical Vibrations

Free vibrations of a SDOF System

Prof. Carmen Muller-Karger, PhD
Florida International University

Figures and content adapted from
Textbook: Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition.
Chapter 2: Free vibrations of a single degree of freedom system

Learning Objectives

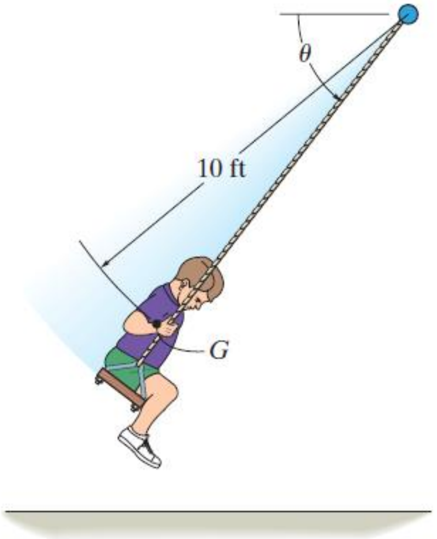
- Define Free Vibrations
- Derive the equation of motion of a single-degree-of-freedom system using different approaches as Newton’s second law of motion and the principle of conservation of energy.
- Linearize a nonlinear equation of motion.
- Solve a spring-mass-damper system for different types of free-vibration response depending on the amount of damping.
- Compute the natural frequency, damping ratio, and frequency of damped vibration.
- Find the responses of systems with Coulomb and hysteretic damping.
- Determine the stability of a system.

Free Vibration

A system is said to undergo free vibration when it oscillates only under an initial disturbance with no external forces acting afterward.

Examples:

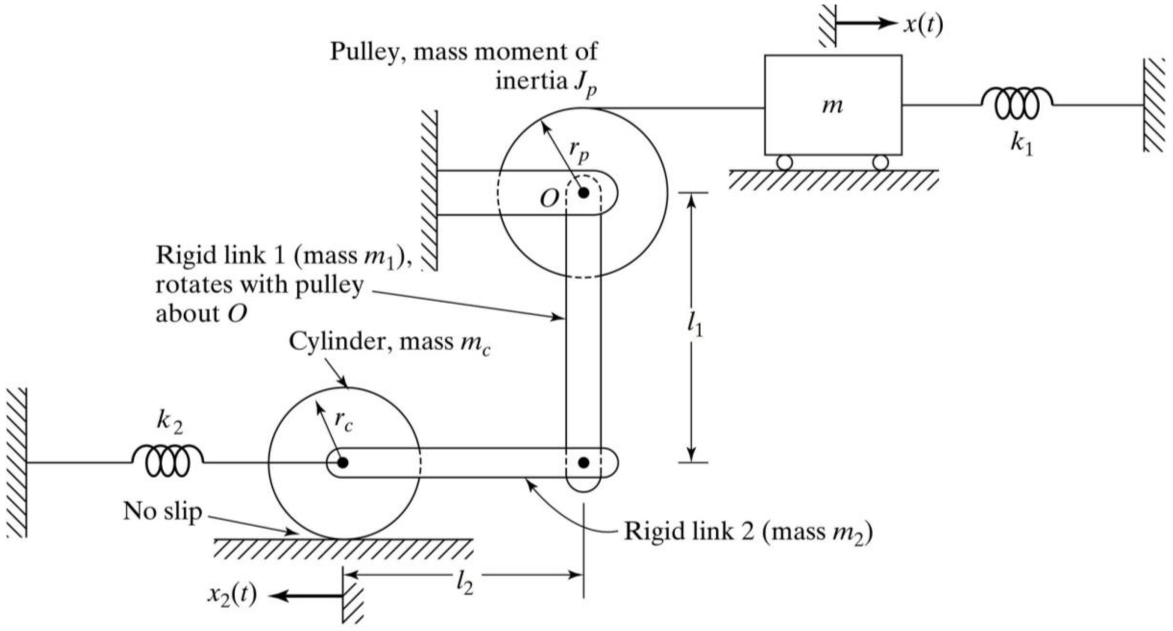
- A child in a swing
- A Pendulum or
- Inverted pendulum



Single Degree of Freedom (SDOF) system

- One coordinate (*x*) is sufficient to specify the position of the mass at any time.
- Several mechanical and structural systems can be idealized as single-degree-of-freedom systems. In many practical systems, the mass is distributed, but for a simple analysis, it can be approximated by a single point mass.
- The study of the free vibration of undamped and damped single-degree-of-freedom systems is fundamental to the understanding of more advanced topics in vibrations.

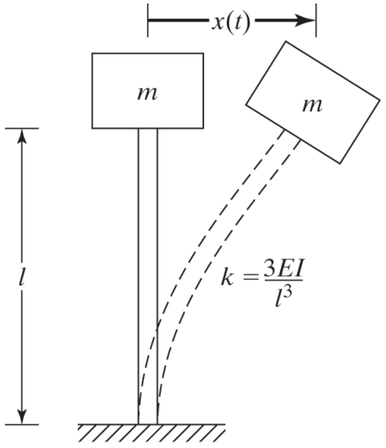
- EXAMPLE: All parameter in term of x.



$$\theta_p = \frac{x}{r_p} \qquad \theta_1 = \frac{x}{r_p} \qquad x_2 = \frac{x l_1}{r_p} \qquad \theta_C = \frac{x l_1}{r_p r_C}$$

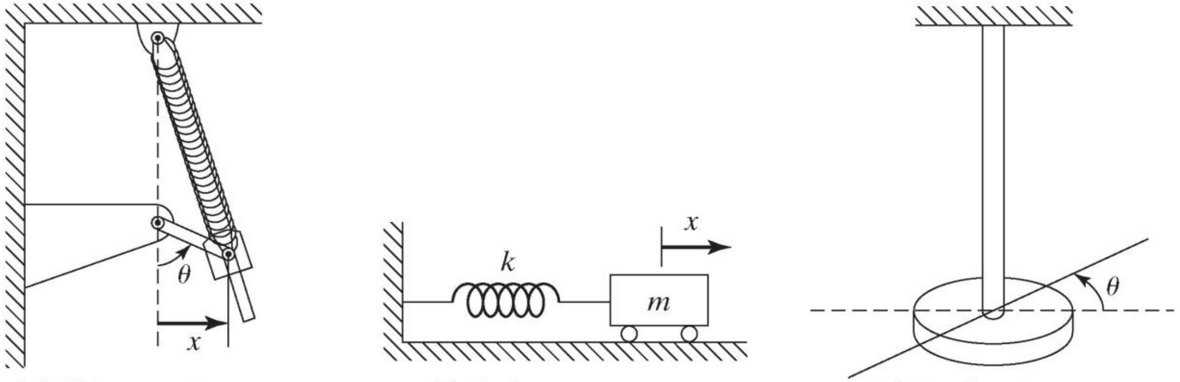
Undamped SDOF system

When there is no element that causes dissipation of energy during the motion of the mass:



(a) Idealization of the tall structure

- The amplitude of motion remains constant with time.
- The system vibrates at its natural frequency



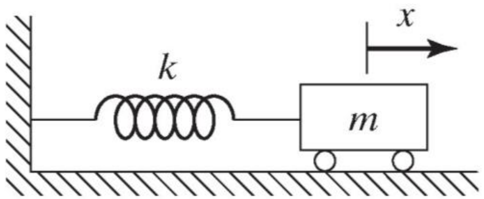
(a) Slider-crank-spring mechanism (b) Spring-mass system (c) Torsional system

Governing equation of an undamped SDOF system using equation of motion

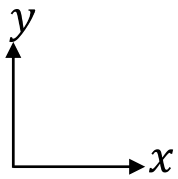
$$\sum F_x = \mathrm{m}a_x$$

$$\sum F_y = \mathrm{m}a_y$$

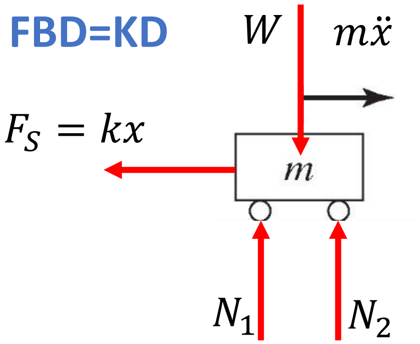
$$\sum \bar{M}_{pz} = I_{pzz}\,\bar{\alpha}_z + [m\bar{r}_G] \times \bar{a}_p$$



(b) Spring-mass system

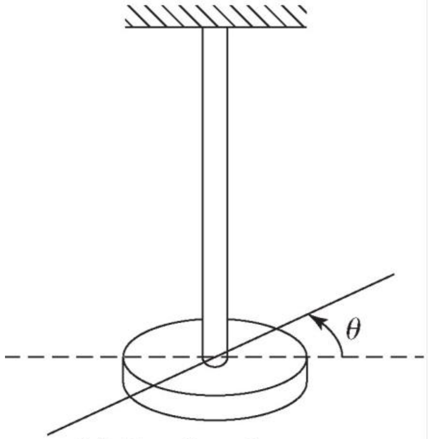


FBD=KD

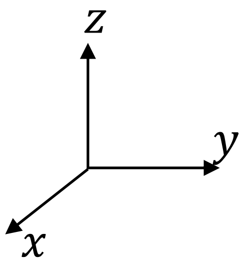


$$\sum F_x = -kx = m\ddot{x}$$

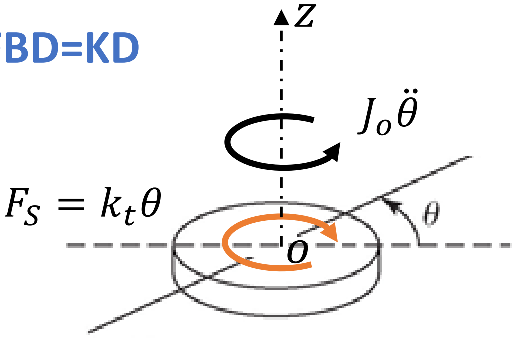
$$m\ddot{x} + kx = 0$$



(c) Torsional system



FBD=KD



(c) Torsional system

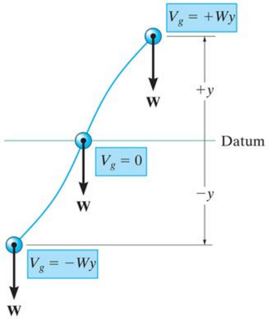
$$\sum M_z = -k_t\theta = J_o\ddot{\theta}$$

$$J_o\ddot{\theta} + k_t\theta = 0$$

Governing equation using Principle of Conservation of Energy

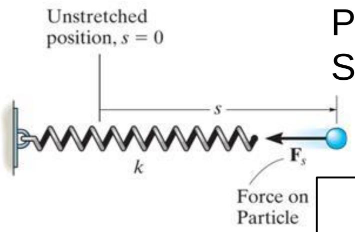
$$T + U = \text{constant}$$

$$\frac{d}{dt}(T+U) = 0$$



Potential energy for Weight

$$U_{1-2} = W\Delta y$$

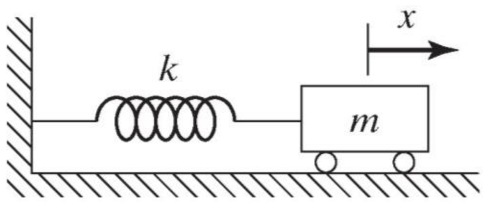


Potential energy for Spring

$$U_{1-2} = \frac{1}{2}k(s_2)^2 - \frac{1}{2}k(s_1)^2$$

Kinetic energy

$$T = \frac{1}{2}m(\bar{v}_p)^2 + \frac{1}{2}I_{pzz}(\omega_z)^2 + \bar{v}_p \cdot (\bar{\omega}_z \times m\bar{r}_G)$$



(b) Spring-mass system

$$U = \frac{1}{2}k(x)^2 \qquad \frac{dU}{dt} = \frac{2}{2}k(x)\dot{x}$$

$$T = \frac{1}{2}m(\dot{x})^2 \qquad \frac{dT}{dt} = \frac{2}{2}m(\dot{x})\ddot{x}$$

$$m(\dot{x})\ddot{x} + k(x)\dot{x} = 0$$

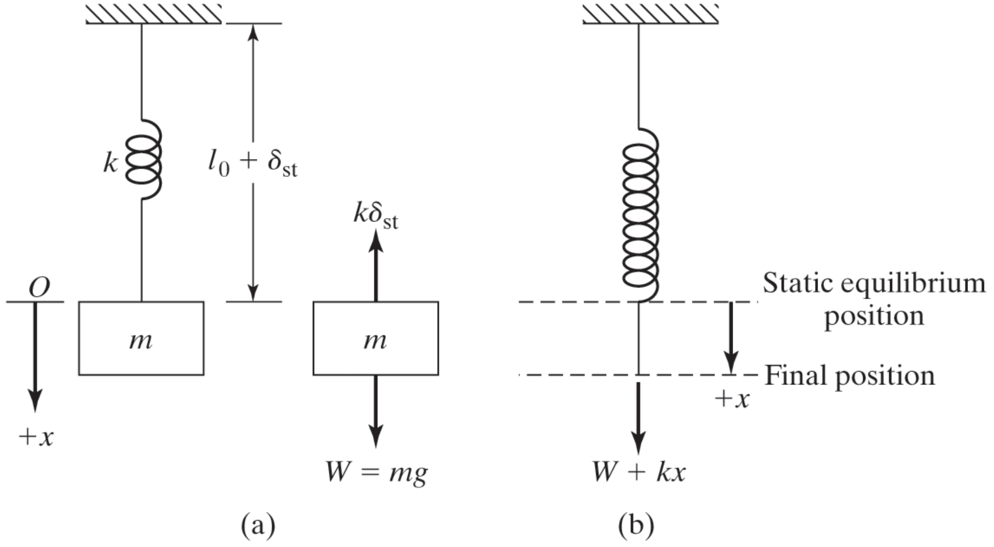
$$[m\ddot{x} + kx]\,\dot{x} = 0$$

Since: $\dot{x} \neq 0$

$$m\ddot{x} + kx = 0$$

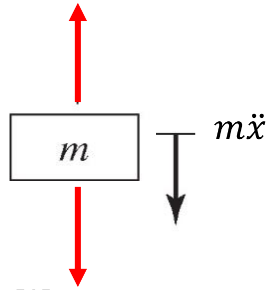
Equation of Motion of a Spring-Mass System in Vertical Position

At rest, the mass will hang in a position called the *static equilibrium position.*
In this position the length of the spring is $l_o + \delta_{st}$, where δ_{st} is the static deflection—the elongation due to the weight W of the mass m.



FBD=KD

$F_S = k(\delta_{st} + x)$



$W = mg$

Using equation of motion

$$\sum F_x = k(\delta_{st} + x) - mg = m\ddot{x}$$

Since

$$k(\delta_{st}) = mg$$

Using energy method

$$U = \frac{1}{2}k(\delta_{st} + x)^2 - mg(\delta_{st} + x)$$

$$\frac{dU}{dt} = k(\delta_{st} + x)\dot{x} - mg(\dot{x}) = [kx + \delta_{st} - mg]\dot{x}$$

$$T = \frac{1}{2}m(\dot{x})^2$$

$$\frac{dT}{dt} = m(\dot{x})\ddot{x}$$

$$[m\ddot{x} + kx]\,\dot{x} = 0$$

$$m\ddot{x} + kx = 0$$

With x measure from static equilibrium position

Solution to the equation of motion

$$m\ddot{x} + kx = 0$$

- The solution of this second order differential equation can be found by assuming

$$x = Ce^{st}$$
$$\dot{x} = sCe^{st}$$
$$\ddot{x} = s^2Ce^{st}$$

$$\left(ms^2 + k\right)Ce^{st} = 0$$

- Since $C\,e^{st}$ cannot be zero, what is in parenthesis which is the characteristic equation is zero. The solution represent the *eigenvalues of the equation*

- Characteristic equation:

$$\left(ms^2 + k\right)=0 \Rightarrow s_{1,2} = \pm\sqrt{-\frac{k}{m}} \Rightarrow s_{1,2} = \pm i\sqrt{\frac{k}{m}} \Rightarrow s_{1,2} = \pm i\omega_n$$

- We define the natural frequency as :

$$\omega_n = \sqrt{\frac{k}{m}}$$

- The solution becomes:

$$x = C_1e^{i\omega_n t} + C_2e^{-i\omega_n t}$$

Solution to the equation of motion

$m\ddot{x} + kx = 0$

$\omega_n = \sqrt{\frac{k}{m}}$

- Recall Euler formula that establishes the fundamental relationship between the trigonometric functions and the complex exponential function:

$$x = Ce^{is} = C(\cos(s) + i\sin(s))$$

- The solution becomes:

$$x = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t} = C_1(cos\omega_n t + i\sin\omega_n t) + C_2(cos\omega_n t - i\sin\omega_n t)$$

$$x = (C_1 + C_2)\,(cos\omega_n t) + (C_1 - C_2)i(\sin\omega_n t)$$

$C_1 = \mathrm{a} + \mathrm{ib}$ $\qquad C_1 + C_2 = 2\mathrm{a}$

$C_2 = \mathrm{a} - \mathrm{ib}$ $\qquad (C_1 - C_2)i = 2\mathrm{b}i^2$

If $\quad A_1 = 2\mathrm{a}$

$A_2 = -2\mathrm{b}$

Then $\quad x(t) = A_1 cos\omega_n t + A_2 \sin\omega_n t \quad$ or $\quad x(t) = Acos(\omega_n t - \varphi)$

$$\boldsymbol{A = \sqrt{A_1^2 + A_2^2}}$$

$$\boldsymbol{\varphi = tan^{-1}\left(\frac{A_2}{A_1}\right)}$$

Solution to the equation of motion

$$m\ddot{x} + kx = 0$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$x(t) = A_1 cos\omega_n t + A_2 \sin\omega_n t$$ or $$x(t) = Acos(\omega_n t - \varphi)$$

- For particular initial conditions: $x(o) = x_o$ $\dot{x}(o) = v_o$

$x(0) = A_1$ $A_1 = x_o$

$\dot{x}(t) = -\omega_n A_1 sin\omega_n t + \omega_n A_2 \cos\omega_n t$

$\dot{x}(0) = \omega_n A_2$ $A_2 = v_o/\omega_n$

$$x(t) = x_o cos\omega_n t + (v_o/\omega_n)\sin\omega_n t$$

or

$$x(t) = \left(x_o{}^2 + (v_o/\omega_n)^2\right)^{1/2} cos(\omega_n t - \varphi)$$

$$\boldsymbol{\varphi = tan^{-1}\left(\frac{v_o}{x_o\omega_n}\right)}$$

Different ways to write the solution

$m\ddot{x} + kx = 0$

$\omega_n = \sqrt{\frac{k}{m}}$

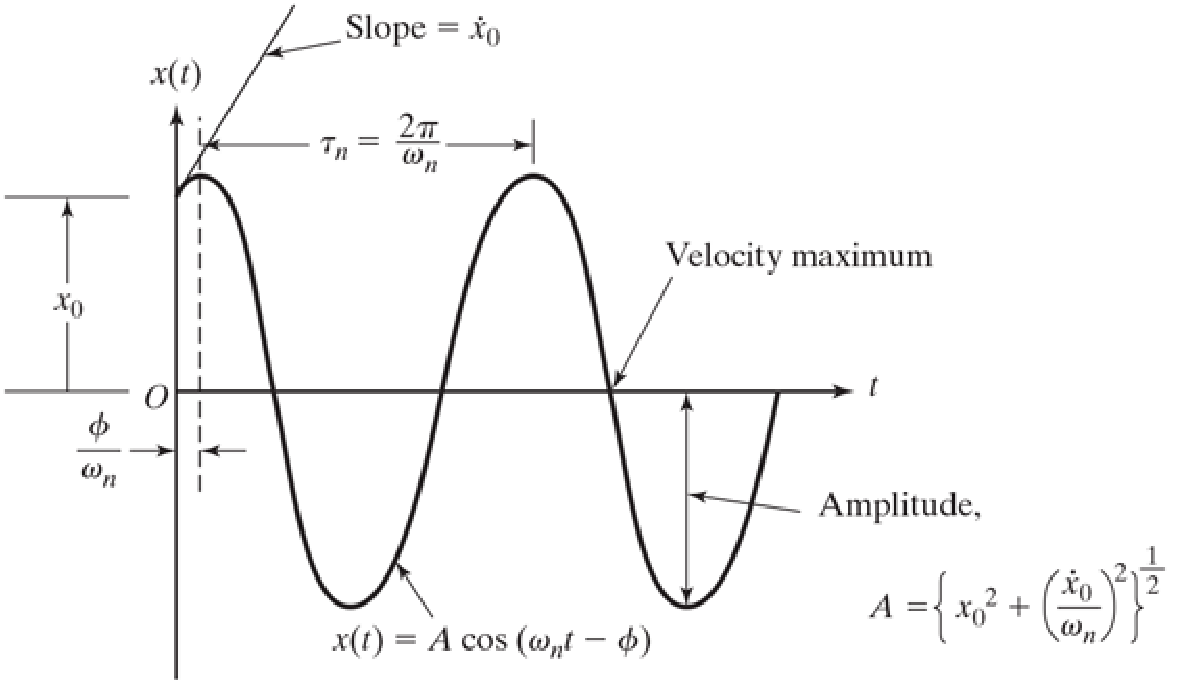
1. $$x(t) = x_o cos\omega_n t + \left(\frac{v_o}{\omega_n}\right) \sin\omega_n t$$

2. $$x(t) = \left(x_o{}^2 + \left(\frac{v_o}{\omega_n}\right)^2\right)^{\frac{1}{2}} cos(\omega_n t - \varphi)$$

$$\varphi = tan^{-1}\left(\frac{v_o}{x_o\omega_n}\right)$$

3. $$x(t) = \left(x_o{}^2 + \left(\frac{v_o}{\omega_n}\right)^2\right)^{\frac{1}{2}} sin(\omega_n t + \varphi')$$

$$\varphi' = tan^{-1}\left(\frac{x_o\omega_n}{v_o}\right)$$



$\tau_n = \frac{2\pi}{\omega_n}$

Period

$f_n = \frac{1}{\tau_n}$

Frequency

$\omega_n = 2\pi f_n$

Circular Frequency

Equation of Motion of a Spring-Mass System in Vertical Position

At rest, the mass will hang in a position called the *static equilibrium position.*

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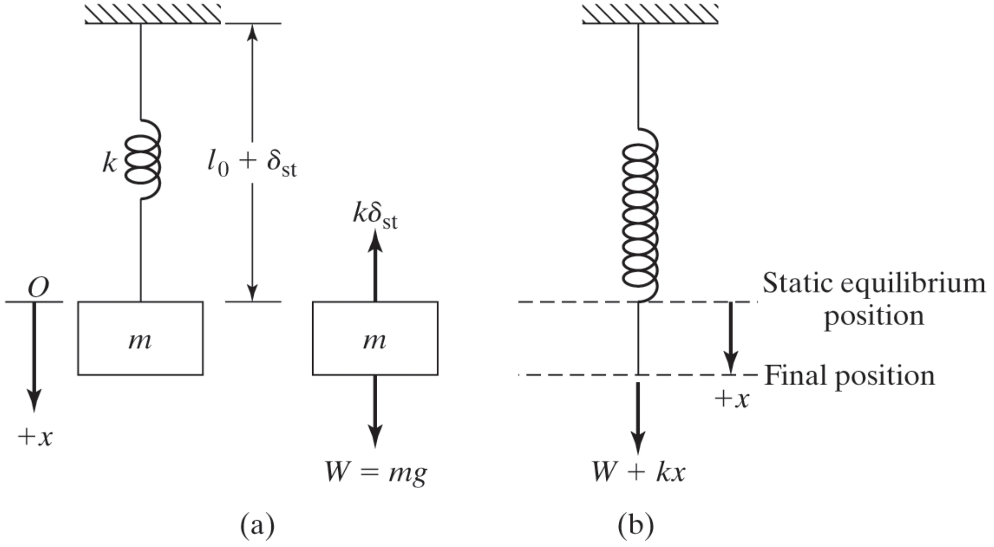
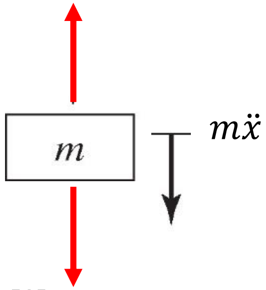


FBD=KD

$F_S = k(\delta_{st} + x)$



$W = mg$

With x measure from static equilibrium position

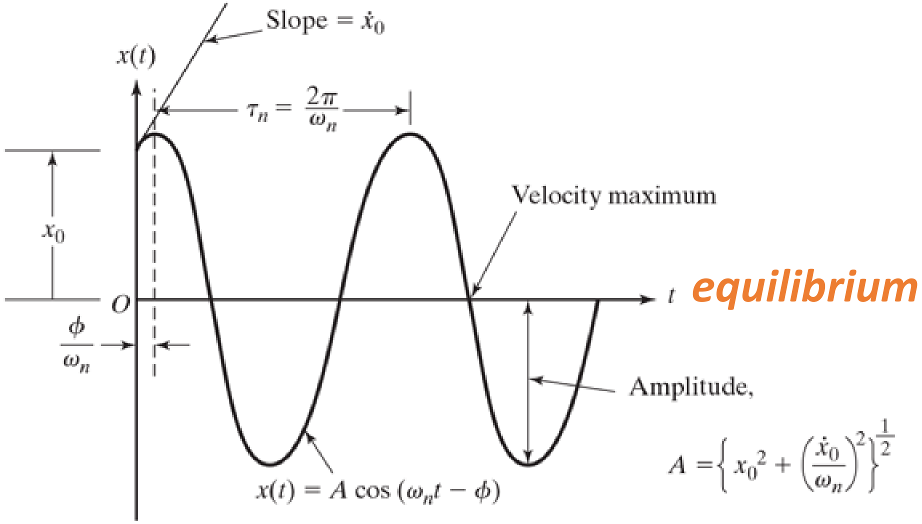
$k(\delta_{st}) = mg$

$$m\ddot{x} + kx = 0$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}}$$

$$f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta_{st}}}$$

$$\tau_n = \frac{2\pi}{\omega_n} \qquad f_n = \frac{1}{\tau_n}$$



equilibrium position

Position, velocity and Acceleration

$m\ddot{x} + kx = 0$

$\omega_n = \sqrt{\frac{k}{m}}$

1. Position

$$x(t) = A\cos(\omega_n t - \varphi)$$

2. Velocity

$$\dot{x}(t) = -\omega_n A\sin(\omega_n t - \varphi)$$

$$\dot{x}(t) = \omega_n A\cos\left(\omega_n t - \varphi + \frac{\pi}{2}\right)$$

3. Acceleration

$$\ddot{x}(t) = -\omega_n^2 A\cos(\omega_n t - \varphi)$$

$$\ddot{x}(t) = \omega_n^2 A\cos(\omega_n t - \varphi + \pi)$$

The velocity leads the displacement by $\frac{\pi}{2}$ and

the acceleration leads the displacement by π.

Particular cases:

$$x(t) = x_o cos\omega_n t + \left(\frac{v_o}{\omega_n}\right) \sin\omega_n t$$

$$x(t) = \left(x_o{}^2 + \left(\frac{v_o}{\omega_n}\right)^2\right)^{\frac{1}{2}} cos(\omega_n t - \varphi)$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

$$\boldsymbol{\varphi = tan^{-1}}\left(\frac{v_o}{x_o\omega_n}\right)$$

1. If the initial displacement x_o is zero, the solution becomes

$$x(t) = \frac{v_o}{\omega_n} cos\left((\omega_n t - \frac{\pi}{2}\right)$$

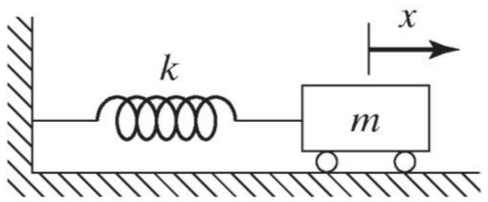
2. If the initial velocity v_o is zero, the solution becomes

$$x(t) = x_o cos(\omega_n t)$$

3. The value of the phase angle φ given, needs to be calculated with care. Tan φ can be positive when both x_o and $\frac{v_o}{\omega_n}$ are either positive or negative.

Thus, we need to use the first quadrant value of φ when both x_0 and $\frac{v_o}{\omega_n}$ are positive and the third quadrant value of φ when both x_0 and $\frac{v_o}{\omega_n}$ are negative. Similarly, since tan φ can be negative when x_0 and $\frac{v_o}{\omega_n}$ have opposite signs, we need to use the second quadrant value of φ when x_0 is negative and $\frac{v_o}{\omega_n}$ is positive and the fourth quadrant value of φ when x_0 is positive and · $\frac{v_o}{\omega_n}$ is negative.

Natural frequency for Equivalent systems



Spring-mass system

$$m\ddot{x} + kx = 0$$

- We define the natural frequency as :

$$\omega_n = \sqrt{\frac{k}{m}}$$

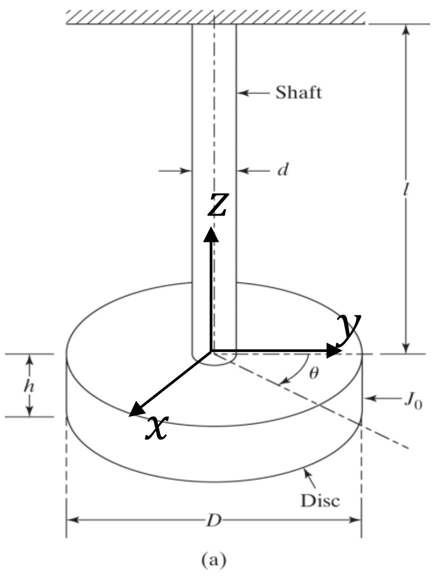
- For any other system, we will find the governing equation and if we are able to write it in the following form:

$$m_{eq}\ddot{x} + k_{eq}x = 0$$

- We define the natural frequency as :

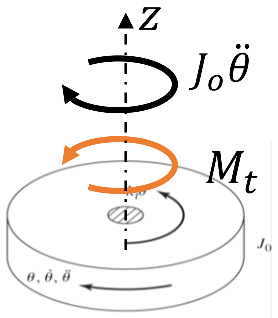
$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}}$$

Free Vibration of an Undamped Torsional System



Also called *torsional pendulum.*

FBD=KD



$$J_o\ddot{\theta} + k_t\theta = 0$$

$$\omega_n = \sqrt{\frac{k_t}{J_o}}$$

$$\omega_n = \sqrt{\frac{G\pi d^4}{lWD^2}}$$

From the theory of torsion of circular shaft, we have the relation

$$M_t = k_t\theta = \frac{GI_o}{l}\theta$$

where M_t is the torque that produces the twist θ, G is the shear modulus, l is the length of the shaft, I_o is the polar moment of inertia of the cross section of the shaft, and *d* is the diameter of the shaft.

$$I_o = \frac{\pi d^4}{32}$$

torsional spring with a torsional spring constant

$$k_t = \frac{M_t}{\theta} = \frac{GI_o}{l} = \frac{G\pi d^4}{32l}$$

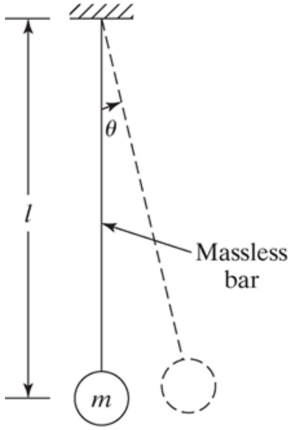
The polar mass moment of inertia of a disc is given by

$$J_o = \frac{\rho h\pi D^4}{32} = \frac{WD^2}{32}$$

where ρ is the mass density, h is the thickness, D is the diameter, and W is the weight of the disc

Free Vibration of an simple Pendulum

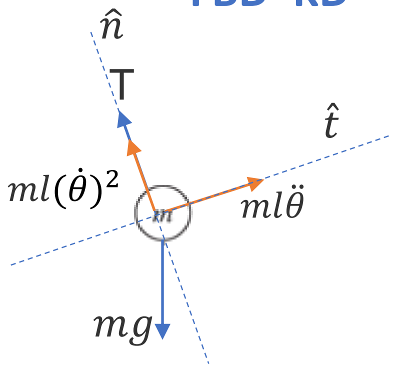
Simple Pendulum



(a)

Using Newton's Law:

FBD=KD



$$\sum F_t = -mgsin\theta = ml\ddot{\theta}$$

$$ml\ddot{\theta} + mgsin\theta = 0$$

Using Conservation of Energy:

$$U = mgl(1 - cos\theta) \qquad \frac{dU}{dt} = mgl(sin\theta)\dot{\theta}$$

$$T = \frac{1}{2}m(l\dot{\theta})^2 \qquad \frac{dT}{dt} = \frac{2}{2}ml^2(\dot{\theta})\ddot{\theta}$$

$$ml^2(\dot{\theta})\ddot{\theta} + mgl(sin\theta)\dot{\theta} = 0$$

$$[ml\ddot{\theta} + mg(sin\theta)]l\dot{\theta} = 0$$

Since: $\dot{\theta} \neq 0$

$$ml\ddot{\theta} + mgsin\theta = 0$$

for small angular displacements, we linearize the equation using :

$$\sin(\theta) \approx \theta$$

Linearized equation of motion:

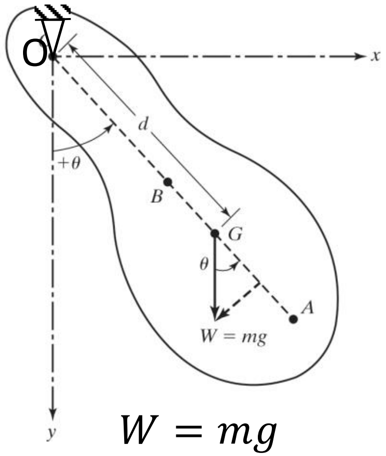
$$l\ddot{\theta} + g\theta = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{g}{l}}$$

Free Vibration of an Compound Pendulum

Compound Pendulum



$W = mg$

Any rigid body pivoted at a point other than its center of mass will oscillate about the pivot point under its own gravitational force

Using Newton's Law:

$$\sum M_{oz} = -Wd\sin\theta = J_o\ddot{\theta}$$

$$J_o\ddot{\theta} + mgd\sin\theta = 0$$

Linearized equation of motion:

$$J_o\ddot{\theta} + mgd\theta = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{mgd}{J_o}}$$

In terms of radius of gyration:

$$\omega_n = \sqrt{\frac{gd}{k_o^2}}$$

Equivalent length of a compound pendulum compared to a simple pendulum :

$$l = \frac{k_o^2}{d}$$

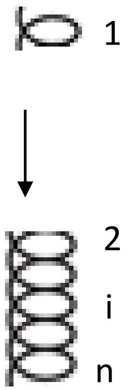
Effect of Mass of a Spring

- **Static analysis:** we will assume that we have n spring on series

$$\frac{1}{k_T} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} = n\frac{1}{k_i}$$

$$k_i = nk_T$$

$$m_i = \frac{m_s}{n}$$



$$\delta_1 = \frac{m_s g(n-1)}{n}\frac{1}{nk_T}$$ Deflection of first spiral due to the weight of the rest of the spring below

$$\delta_2 = \frac{m_s g(n-2)}{n}\frac{1}{nk_T}$$ Deflection of second spiral due to the weight of the rest of the spring below

$$\delta_i = \frac{m_s g(n-i)}{n}\frac{1}{nk_T}$$ Deflection of i spiral due to the weight of the rest of the spring below

$$\delta_n = 0$$ Last spiral does not have any deflection

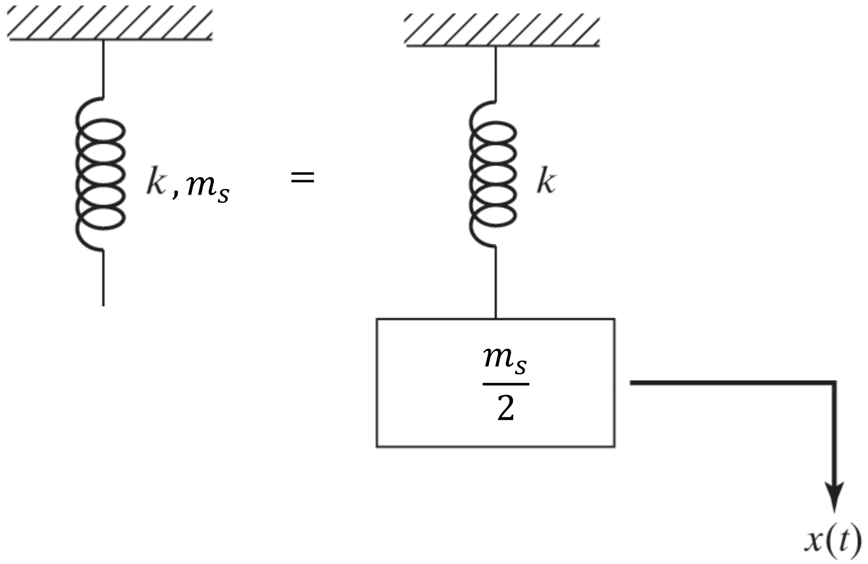
$$\delta_T = \sum_{i=1}^{n}\frac{m_s g(n-i)}{n}\frac{1}{nk_T} = \frac{m_s g}{n^2 k_T}\sum_{i=1}^{n}(n-i)$$

$$\sum_{i=1}^{n}(n-i) = \frac{n(a_1+a_1)}{2} = \frac{n[(n-1)+0]}{2} = \frac{n(n-1)}{2}$$

$$\delta_T = \frac{m_s g}{2k_T}\frac{n(n-1)}{n^2} \rightarrow \log_{n\to\infty}\frac{(n-1)}{n} = 1 \rightarrow \boxed{\delta_T = \frac{m_s g}{2k_T}}$$

For a limit of ∞ spirals the deflection half the mass. Therefore would be equivalent to place a concentrated mass of 1/2 the mass of the spring at the end.

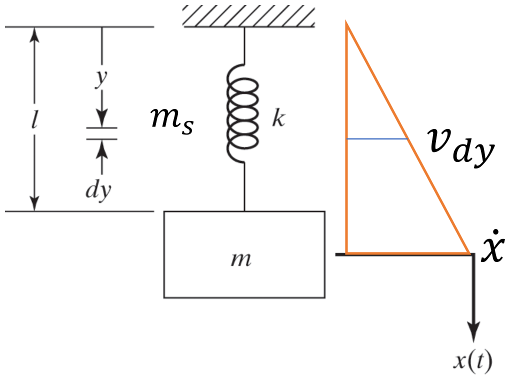
$$\delta_T = \frac{m_s g}{2k_T} \qquad m_{eq} = \frac{m_s}{2}$$



Effect of Mass of a Spring

- **Dynamic analysis:** we will assume a differential of mass dm_s at dy

$$dm_s = \frac{m_s}{l} dy$$



Would be equivalent to place a concentrated mass of 1/3 the mass of the spring at the end.

$$m_{eq} = m + \frac{m_s}{3}$$

- We assume linear velocity along the spring, therefore the velocity of the differential dy is:

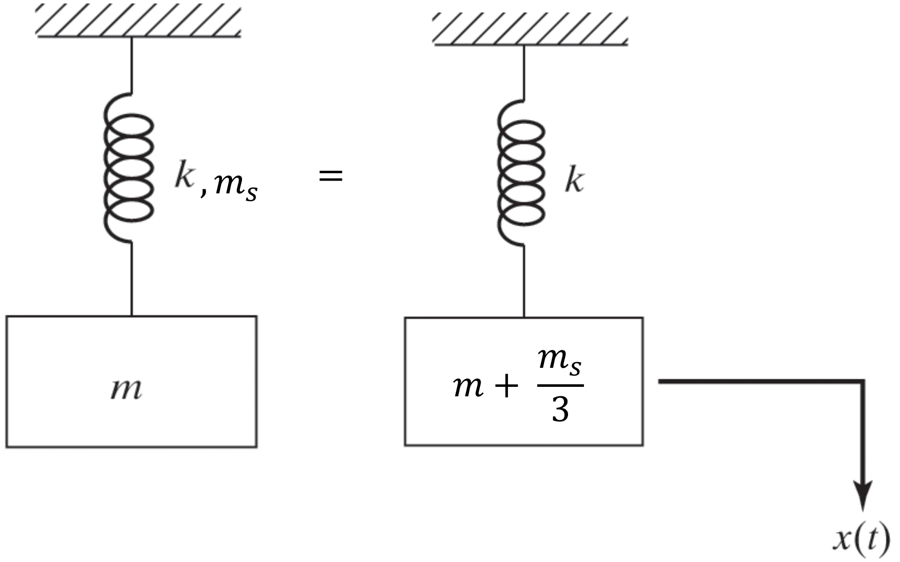
$$v_{dy} = \frac{y}{l}\dot{x}$$

- The kinetic energy :

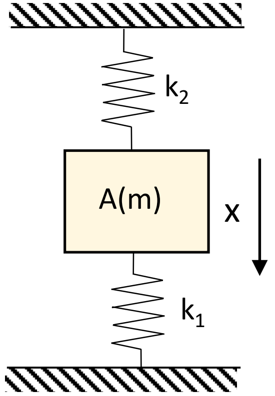
$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\int_0^l \left(\frac{y}{l}\dot{x}\right)^2 dm_s = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\int_0^l \left(\frac{y}{l}\dot{x}\right)^2 \frac{m_s}{l} dy$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{m_s\dot{x}^2}{l^3}\frac{y^3}{3}\Big|_0^l = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}\frac{m_s}{l^3}\frac{(l)^3}{3}\dot{x}^2$$

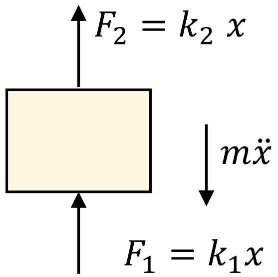
$$T = \frac{1}{2}\left[m + \frac{m_s}{3}\right]\dot{x}^2$$



Examples of natural frequency



FBD and KD

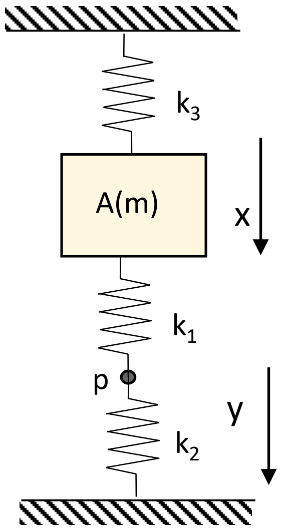


Equation of Motion

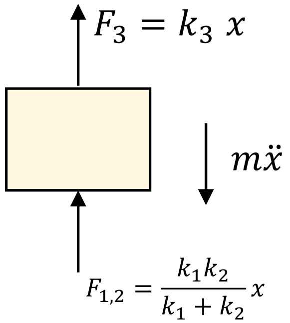
$$m\ddot{x} + (k_1 + k_2)x = 0$$

$$k_{eq} = k_1 + k_2$$

$$\omega_n = \sqrt{\frac{k_1 + k_2}{m}}$$



FBD and KD



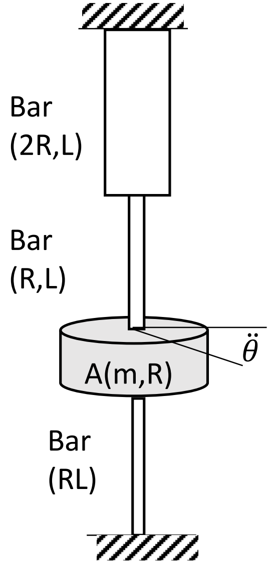
Equation of Motion

$$m\ddot{x} + \left(\frac{k_1 k_2}{k_1 + k_2} + k_3\right) x = 0$$

$$k_{eq} = \frac{k_1 k_2 + k_3 k_1 + k_3 k_2}{k_1 + k_2}$$

$$\omega_n = \sqrt{\frac{k_1 k_2 + k_3 k_1 + k_3 k_2}{(k_1 + k_2)m}}$$

For a torsional system



$$k_{eq} = \frac{k_{t1} k_{t2}}{k_{t1} + k_{t2}} + k_{t3}$$

$$k_{ti} = \frac{G_i I_{pi}}{L_i}$$

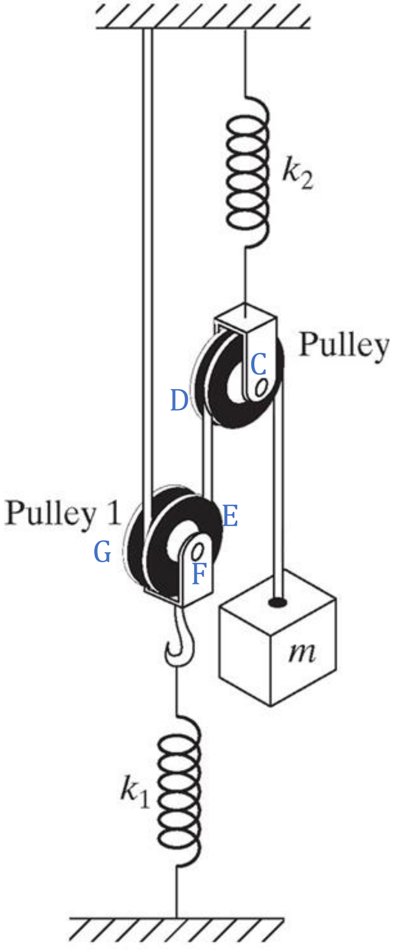
Equation of Motion

$$J_0\ddot{\theta} + \left(\frac{k_{t1} k_{t2}}{k_{t1} + k_{t2}} + k_{t3}\right)\theta = 0$$

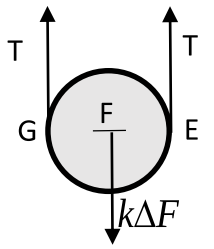
$$k_{eq} = \frac{k_{t1} k_{t2} + k_{t3} k_{t1} + k_{t3} k_{t2}}{k_{t1} + k_{t2}}$$

$$\omega_n = \sqrt{\frac{k_{t1} k_{t2} + k_{t3} k_{t1} + k_{t3} k_{t2}}{(k_{t1} + k_{t2}) J_0}}$$

Natural frequency of pulley system



FBD 1

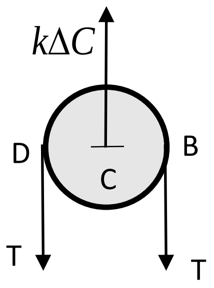


Ideal pulleys have no mas and no friction

$$F_Y^E = 2T - k_1 \Delta F = 0$$

$$\Delta F = \frac{2T}{k_1}$$

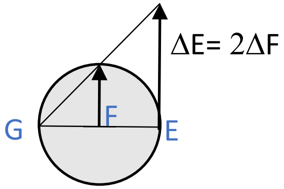
FBD 2

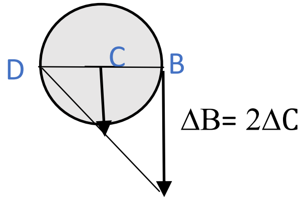


$$F_Y^E = -2T + k_2 \Delta C = 0$$

$$\Delta C = \frac{2T}{k_2}$$

Analysis of motion, G is a fixed point.





x=ΔB= 2ΔF+2ΔC

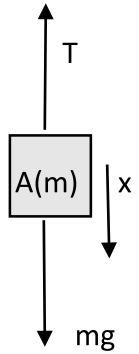
Relation of displacement of mass respect to center of pulleys

$$x = 2\left(\frac{2T}{k_1}\right) + 2\left(\frac{2T}{k_2}\right) = \frac{4T(k_1 + k_1)}{k_1 k_2}$$

Solving for Tension T

$$T = \frac{x(k_1 k_2)}{4(k_1 + k_1)}$$

FBD 3



$$F_Y^E = T - mg = -m\ddot{x}$$

$$F_Y^E = \frac{(k_1 k_2)}{4(k_1 + k_1)} x - mg = -m\ddot{x}$$

Equation of Motion

$$m\ddot{x} + \frac{(k_1 k_2)}{4(k_1 + k_1)} x = mg$$

mg cancel out if you take x respect to the equilibrium position

Natural Frequency

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} \Rightarrow \omega_n = \sqrt{\frac{(k_1 k_2)}{4(k_1 + k_1)m}}$$

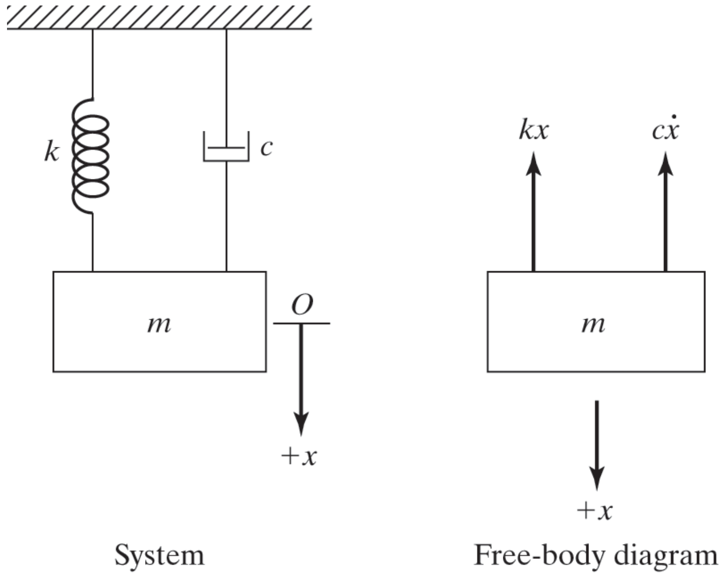
Mechanical Vibrations

Free vibration with viscous damping

Prof. Carmen Muller-Karger, PhD
Florida International University

Figures and content adapted from
Textbook: Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition.
Chapter 2: Free vibrations of a sigle degree of freedom system

Free vibration with viscous damping



The viscous damping force *F* is proportional to the velocity and can be expressed as

$$F = -cv = -c\dot{x}$$

where *c* is the damping constant or coefficient of viscous damping and the negative sign indicates that the damping force is opposite to the direction of velocity.

Using Newton's Law

$$\sum F_x = k(\delta_{st} + x) + c\dot{x} - mg = m\ddot{x} \qquad k(\delta_{st}) = mg$$

Equation of Motion

$$m\ddot{x} + c\dot{x} + kx = 0$$

With x measure from static equilibrium position (EP)

Solution to the equation of motion

$m\ddot{x} + c\dot{x} + kx = 0$

- The solution of this second order differential equation can be found by assuming

$$x = Ce^{st}$$
$$\dot{x} = sCe^{st}$$
$$\ddot{x} = s^2Ce^{st}$$

$$(ms^2 + cs + k)Ce^{st} = 0$$

- Since $C\,e^{st}$ cannot be zero, we have the characteristic equation, which solution represent the *eigenvalues of the equation*

- Characteristic equation:

$$(ms^2 + cs + k) = 0 \Rightarrow s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} \Rightarrow$$

$$s_1 = \frac{-c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$s_2 = \frac{-c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

- The solution becomes:

$$x = C_1e^{s_1t} + C_2e^{s_2t}$$

$$x = C_1e^{\left\{\frac{-c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t} + C_2e^{\left\{\frac{-c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\}t}$$

Critical damping constant, damping ratio

The critical damping c_c is defined as the value of the damping constant *c* for which the radical becomes zero:

$$m\ddot{x} + c\dot{x} + kx = 0$$

$$s_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

- Natural frequency

$$c_c = 2\sqrt{km}$$

- Critical damping constant

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}}$$

- Damping ratio

- Divide by the mass

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$$

- Some algebra:

$$\frac{c}{m} = \frac{2c\sqrt{k}}{2\sqrt{m}\sqrt{m}\sqrt{k}} = 2\zeta\omega_n$$

- The equation and solution in term of ω_n and ζ becomes:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

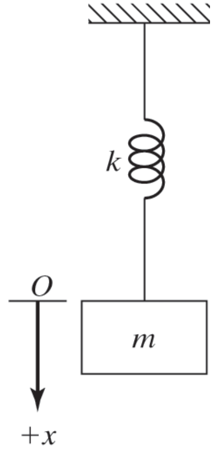
$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

$$x = C_1 e^{\left\{-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right\}t} + C_2 e^{\left\{-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right\}t}$$

Types of solution will depends upon the magnitude of damping

CASES	TYPE OF SYSTEMS	COEF. ζ	TYPE OF SOLUTION	VALUE OF THE ROOTS	TYPE OF MOTION
1	Undamped	$\zeta = 0$	Conjugate imaginary roots, no real part in the solution	$s_{1,2} = \pm\omega_n i$	Oscillatory
2	Underdamped	$\zeta < 1$	Conjugate imaginary roots, with real part in the solution	$s_{1,2} = -\zeta\omega_n \pm \omega_n i\sqrt{1-\zeta^2}$	Oscillatory
3	Critically damped system	$\zeta = 1$	Both roots real and equal	$s_{1,2} = -\zeta\omega_n$	No oscillatory
4	Overdamped system	$\zeta > 1$	Two different real roots	$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$	No oscillatory

Case 1. Undamped System $\zeta = 0$



Equation of Motion *x measure from static equilibrium position (EP)*

$$m\ddot{x} + kx = 0$$

Value for the roots: $s_{1,2} = \pm\omega_n i$

Solution to the EoM: $x = ae^{i\omega_n t} + be^{-i\omega_n t}$

$x = a(\cos\omega_n t + i\sin\omega_n t) + b(\cos\omega_n t - i\sin\omega_n t)$ ❯ $x = (a+b)\cos\omega_n t + (a-b)i\sin\omega_n t$

If we name A and B as

a and b are complex number: $a = c + di, b = c - di$ ❯ A= $(a+b) = 2c,\quad B = (a-b)i = 2di.i$

Solution can be written as a sum of cos and sin or a cos with a phase angle

$$x = A\cos\omega_n t + B\sin\omega_n t$$

$$A = X_0\cos\phi$$

$$B = X_0\sin\phi$$

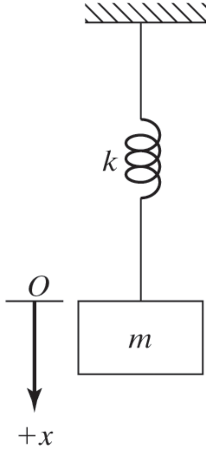
$$x = X_0\cos(\omega_n t - \phi)$$

$$X_0 = \sqrt{A^2 + B^2}$$

$$\phi = \tan^{-1}\frac{B}{A}$$

Case 1. Undamped System

$\zeta = 0$ **EoM** $m\ddot{x} + kx = 0$



$x = A\cos\omega_n t + B\sin\omega_n t$

$x = X_0\cos(\omega_n t - \phi)$

When applying initial conditions x_0 and v_0

$\dot{x}(t) = -A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t$

$x_0 = x(0) = A\cos\omega_n 0 + B\sin\omega_n 0$

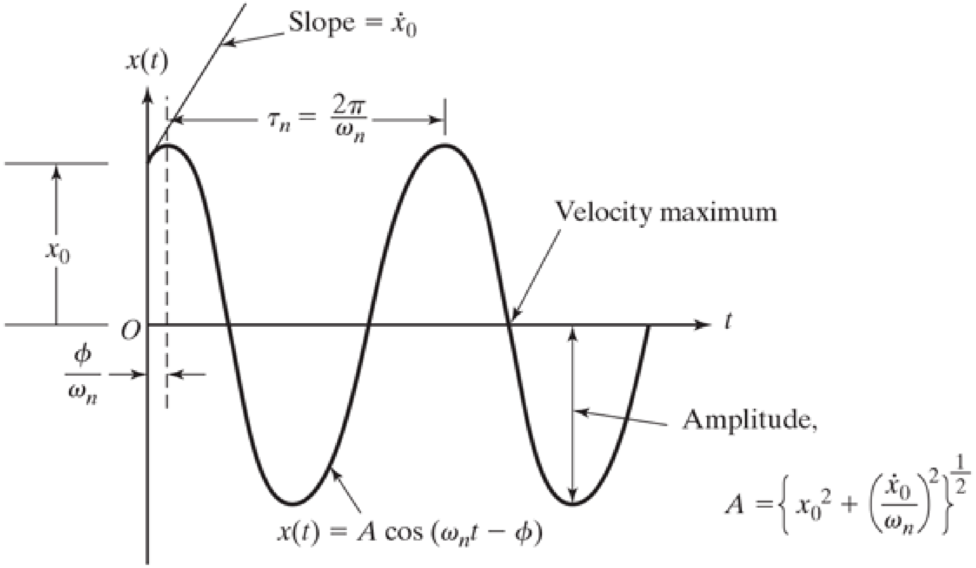
$v_o = \dot{x}(0) = -A\omega_n \sin\omega_n 0 + B\omega_n \cos\omega_n 0$

$A = x_0$

$B = \dfrac{v_o}{\omega_n}$

$X_0 = \sqrt{{x_0}^2 + \left(\dfrac{v_0}{\omega_n}\right)^2}$

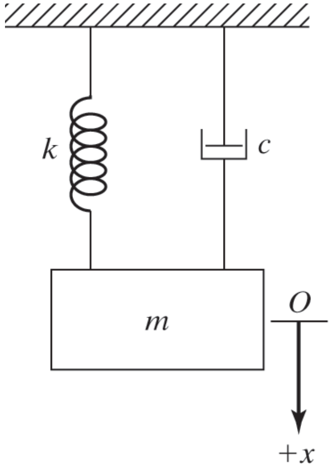
$\phi = \tan^{-1}\dfrac{v_0}{x_0\omega_n}$



System oscillates at its natural frequency

$\omega_n = \dfrac{2\pi}{\tau_n}$

Case 2. Underdamped System $\zeta < 1$



Equation of Motion *x measure from static equilibrium position (EP)*

$$m\ddot{x} + c\dot{x} + kx = 0$$

Value for the roots: $s_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$

Damped frequency:

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Solution to the EoM: $x = ae^{(-\zeta\omega_n+\omega_d i)t} + be^{(-\zeta\omega_n\pm\omega_d i)t}$

$$x = ae^{-\zeta\omega_n t}(cos\omega_d t + isin\omega_d t) + be^{-\zeta\omega_n t}(\cos\omega_d t - isin\omega_d t)$$

$$x = (a+b)\, e^{-\zeta\omega_n t}\cos\omega_d t + (a-b)i\, e^{-\zeta\omega_n t}\sin\omega_d t$$

a and b are complex number:

$$a = c + di, b = c - di$$

A= $(a+b) = 2c,\quad B = (a-b)i = 2di.i$

Solution can be written as a sum of cos and sin or a cos with a phase angle

$$x = Ae^{-\zeta\omega_n t}\cos\omega_d t + Be^{-\zeta\omega_n t}\sin\omega_d t$$

$$A = X_0\cos\phi$$

$$B = X_0\sin\phi$$

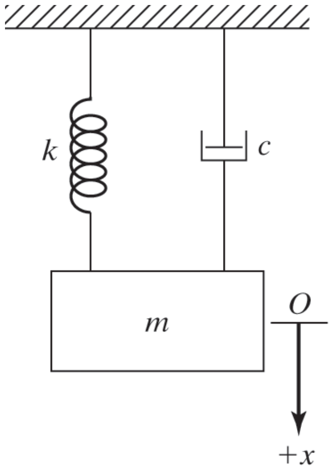
$$x = X_0\, e^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

$$X_0 = \sqrt{A^2 + B^2}$$

$$\phi = \tan^{-1}\frac{B}{A}$$

Case 2. Underdamped System $\zeta < 1$

EoM $m\ddot{x} + c\dot{x} + kx = 0$



$$x = Ae^{-\zeta\omega_n t}\cos\omega_d t + Be^{-\zeta\omega_n t}\sin\omega_d t$$

When applying initial conditions x_0 and v_0

$$x_0 = x(0) = Ae^{-\zeta\omega_n 0}\cos\omega_d 0 + Be^{-\zeta\omega_n 0}\sin\omega_d 0 = x_0$$

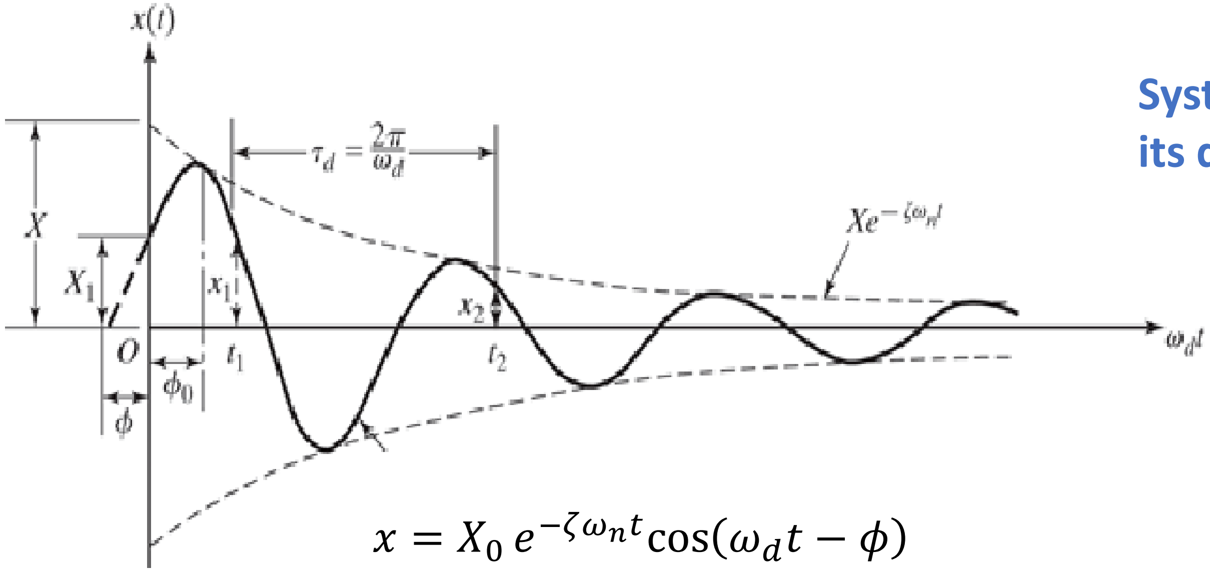
$A = x_0$

$$\dot{x}(t) = -A\zeta\omega_n e^{-\zeta\omega_n t}\cos\omega_d t - Ae^{-\zeta\omega_n t}\omega_d\sin\omega_d t - B\zeta\omega_n e^{-\zeta\omega_n t}\sin\omega_d t + Be^{-\zeta\omega_n t}\omega_d\cos\omega_d t$$

$$v_o = \dot{x}(0) = -A\zeta\omega_n e^{-\zeta\omega_n 0}\cos\omega_d 0 - Ae^{-\zeta\omega_n 0}\omega_d\sin\omega_n 0 - B\zeta\omega_n e^{-\zeta\omega_n 0}\sin\omega_d 0 + Be^{-\zeta\omega_n 0}\omega_d\cos\omega_d 0 = v_0$$

$$A = x_0$$

$$B = \frac{v_0 + x_0\zeta\omega_n}{\omega_d}$$



$$x = X_0\, e^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

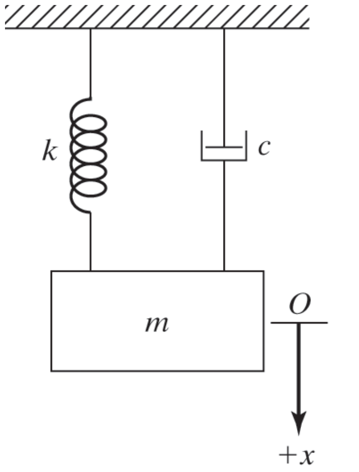
System oscillates at its damped frequency

$$\omega_d = \frac{2\pi}{\tau_d}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Case 2. Underdamped System $\zeta < 1$

EoM $m\ddot{x} + c\dot{x} + kx = 0$



$$x = Ae^{-\zeta\omega_n t}\cos\omega_d t + Be^{-\zeta\omega_n t}\sin\omega_d t$$

$$A = x_0$$

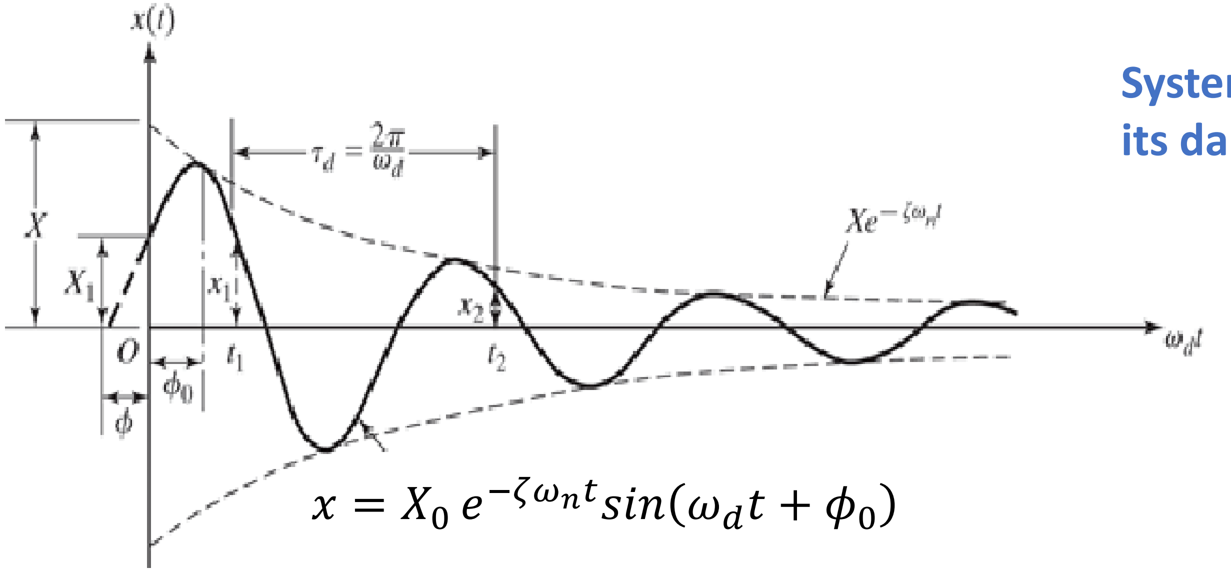
$$B = \frac{v_0 + x_0\zeta\omega_n}{\omega_d}$$

We can also represent the solution by a single cos

$$x = X_0\, e^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

$$X_0 = \sqrt{{x_0}^2 + \left(\frac{v_0 + x_0\zeta\omega_n}{\omega_d}\right)^2}$$

$$\phi = \tan^{-1}\left(\frac{v_0 + x_0\zeta\omega_n}{x_0\omega_d}\right)$$



$$x = X_0\, e^{-\zeta\omega_n t}sin(\omega_d t + \phi_0)$$

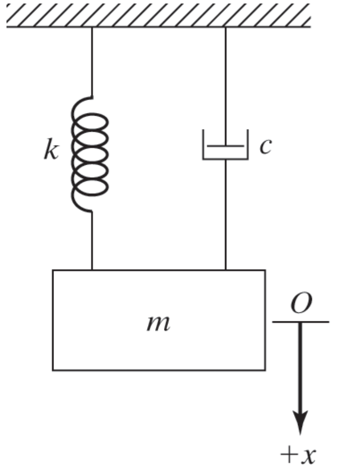
System oscillates at its damped frequency

$$\omega_d = \frac{2\pi}{\tau_d}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Case 2. Underdamped System $\zeta < 1$

EoM $m\ddot{x} + c\dot{x} + kx = 0$



$$x = Ae^{-\zeta\omega_n t}\cos\omega_d t + Be^{-\zeta\omega_n t}\sin\omega_d t$$

$$A = x_0$$

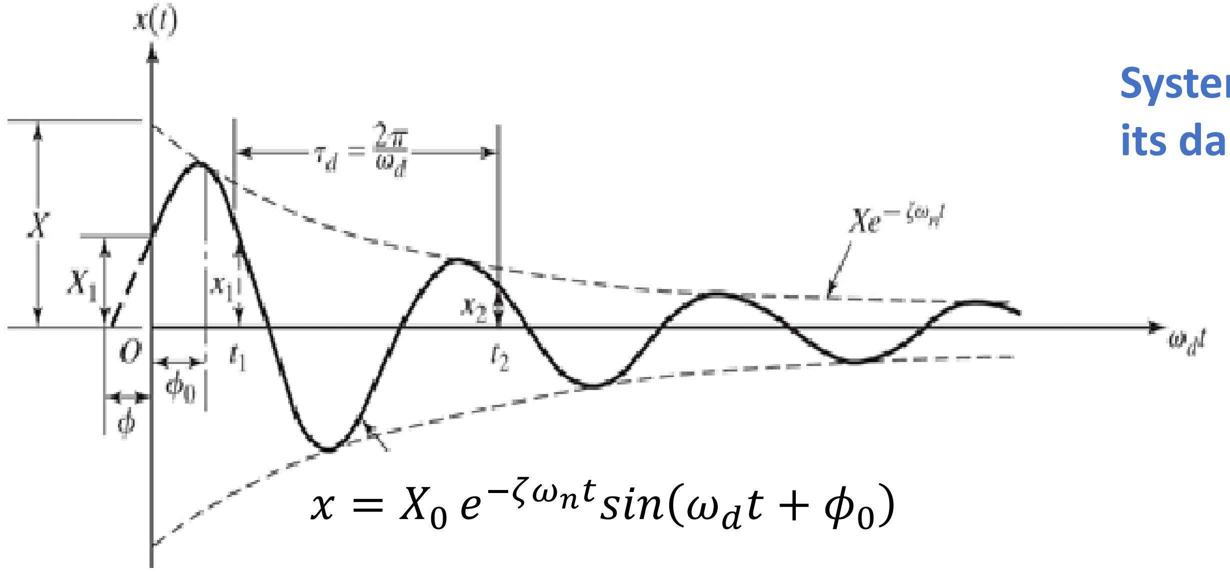
$$B = \frac{v_0 + x_0\zeta\omega_n}{\omega_d}$$

We can also represent the solution by a single sin

$$x = X_0\, e^{-\zeta\omega_n t} sin(\omega_d t + \phi_0)$$

$$X_0 = \sqrt{{x_0}^2 + \left(\frac{v_0 + x_0\zeta\omega_n}{\omega_d}\right)^2}$$

$$\phi_0 = \tan^{-1}\left(\frac{x_0\omega_d}{v_0 + x_0\zeta\omega_n}\right)$$



$$x = X_0\, e^{-\zeta\omega_n t} sin(\omega_d t + \phi_0)$$

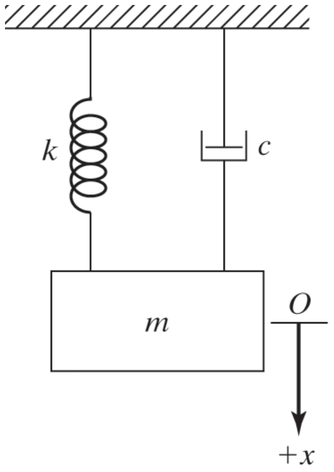
System oscillates at its damped frequency

$$\omega_d = \frac{2\pi}{\tau_d}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Case 2. Underdamped System
Logarithmic Decrement

$\zeta < 1$



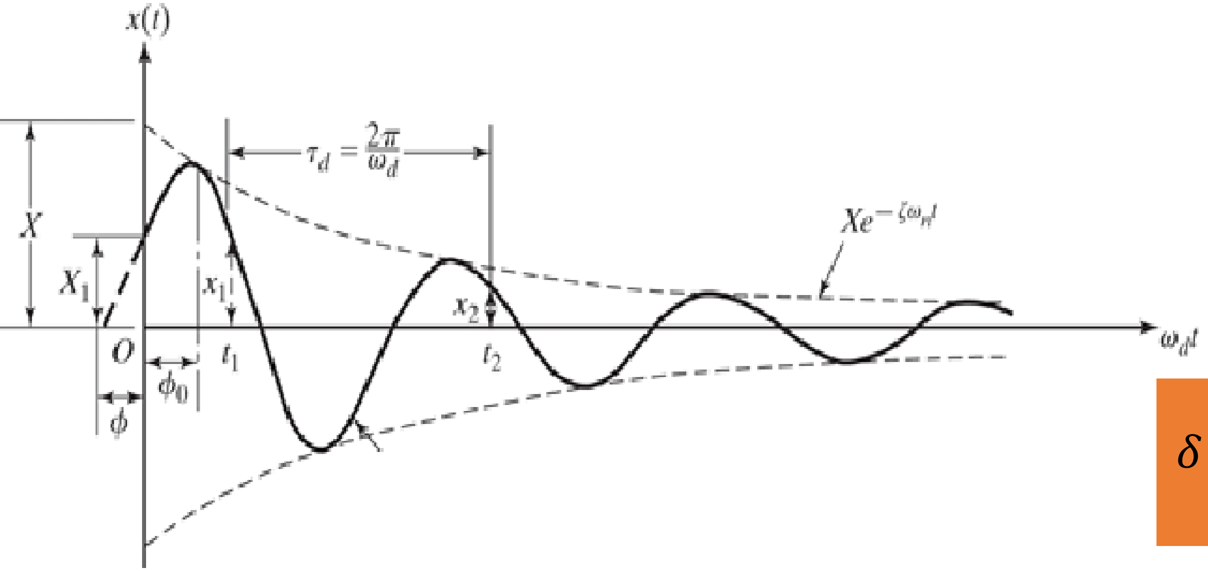
The logarithmic decrement represents the rate at which the amplitude of a free-damped vibration decreases

$$x(t_1) = X_0 e^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi) \qquad t_2 = t_1 + t_d \qquad \cos(\omega_d t_1 - \phi) = \cos(\omega_d t_2 - \phi)$$

$$x(t_2) = X_0 e^{-\zeta\omega_n t_2}\cos(\omega_d t_2 - \phi) \qquad \frac{x(t_1)}{x(t_2)} = \frac{X_0 e^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi)}{X_0 e^{-\zeta\omega_n t_1} e^{-\zeta\omega_n T_d}\cos(\omega_d t_2 - \phi)} = \frac{1}{e^{-\zeta\omega_n(T_d)}} = e^{\frac{\zeta\omega_n 2\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

The logarithmic decrement δ can be obtained, and we can solve for ζ in term of δ

$$\delta = \ln\frac{x(t_1)}{x(t_2)} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$



Also can be found by two displacements separated by any number of complete cycles.

$$\delta = \ln\frac{x(t_1)}{x(t_{n+1})} = \frac{2\pi\zeta(n)}{\sqrt{1-\zeta^2}} \qquad \delta = \frac{1}{n}\ln\frac{x(t_1)}{x(t_{n+1})} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

Case 2. Underdamped System Logarithmic Decrement

$\zeta < 1$

For small damping: $\zeta \ll 1$

$$\delta = 2\pi\zeta$$

$$\zeta = \frac{\delta}{2\pi}$$

The logarithmic decrement is dimensionless and is another form of the dimensionless damping ratio ζ

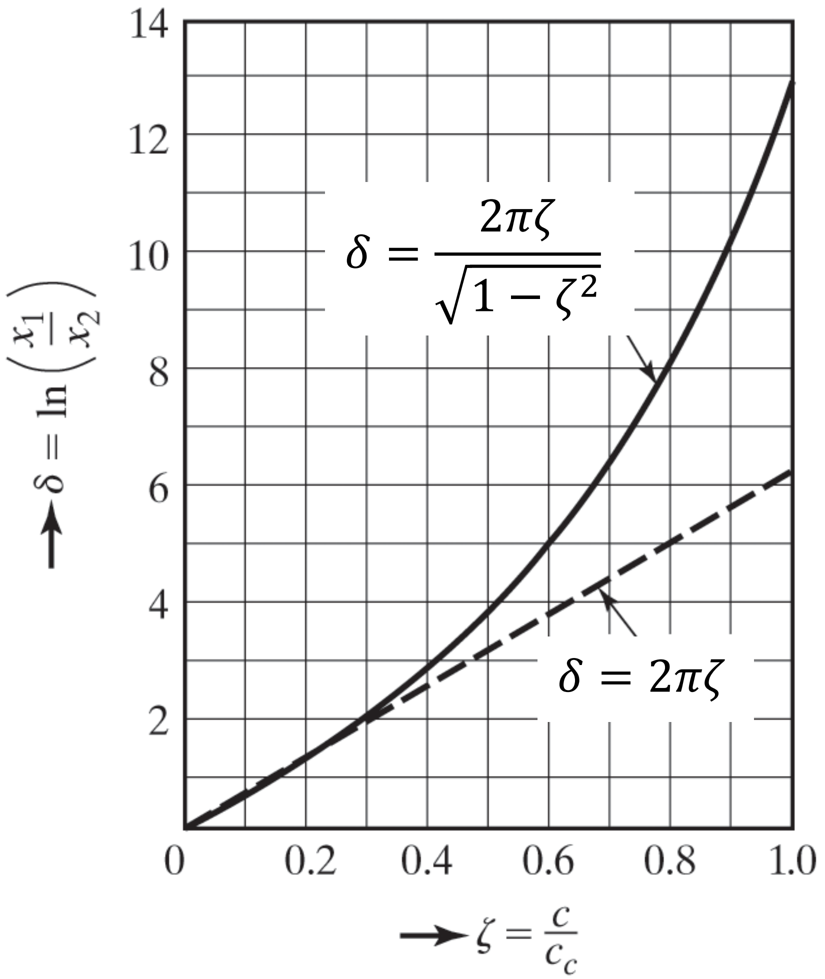
$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 - \delta^2}}$$

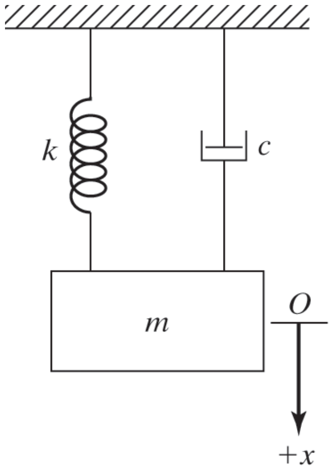
recall: $\omega_d = \omega_n\sqrt{1-\zeta^2}$ $\quad \frac{c}{m} = 2\zeta\omega_n$

The logarithmic decrement can also be written as:

$$\delta = \frac{2\pi}{\omega_d}\frac{c}{2m}$$



Case 3. Critically damped system $\zeta = 1$



Equation of Motion *x measure from static equilibrium position (EP)* $m\ddot{x} + c\dot{x} + kx = 0$

Value for the roots: $s_{1,2} = -\omega_n$

Solution to the EoM: $x = Ae^{-\omega_n t} + Bte^{-\omega_n t}$

When applying initial conditions x_0 and v_0

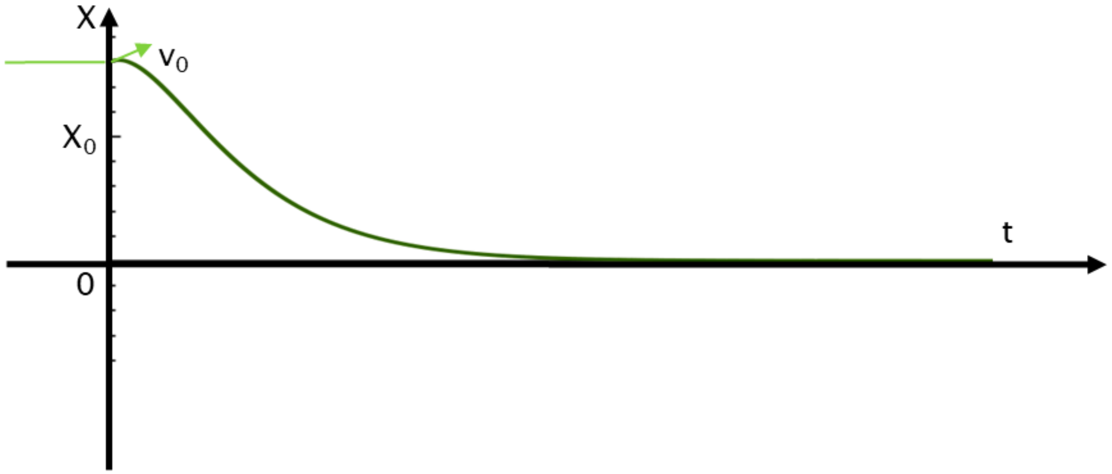
$x_0 = x(0) = Ae^{-\omega_n 0}$ ❯ $A = x_0$

$\dot{x}(t) = -A\omega_n e^{-\omega_n t} - Bt\omega_n e^{-\omega_n t} + Be^{-\omega_n t}$

$v_0 = \dot{x}(0) = -A\omega_n e^{-\omega_n 0} - B0\omega_n e^{-\omega_n 0} + Be^{-\omega_n 0}$ ❯ $v_0 = -A\omega_n + B$ ❯ $B = v_0 + x_0\omega_n$

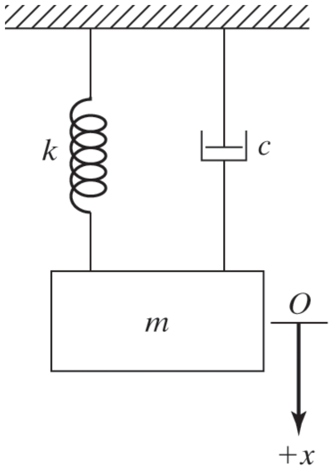
The solution becomes:

$x = x_0 e^{-\omega_n t} + (v_0 + x_0\omega_n)te^{-\omega_n t}$



The motion represented is aperiodic, eventually diminish to zero

Case 4. Overdamped system $\zeta > 1$



Equation of Motion *x measure from static equilibrium position (EP)*

$$m\ddot{x} + c\dot{x} + kx = 0$$

Value for the roots: $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

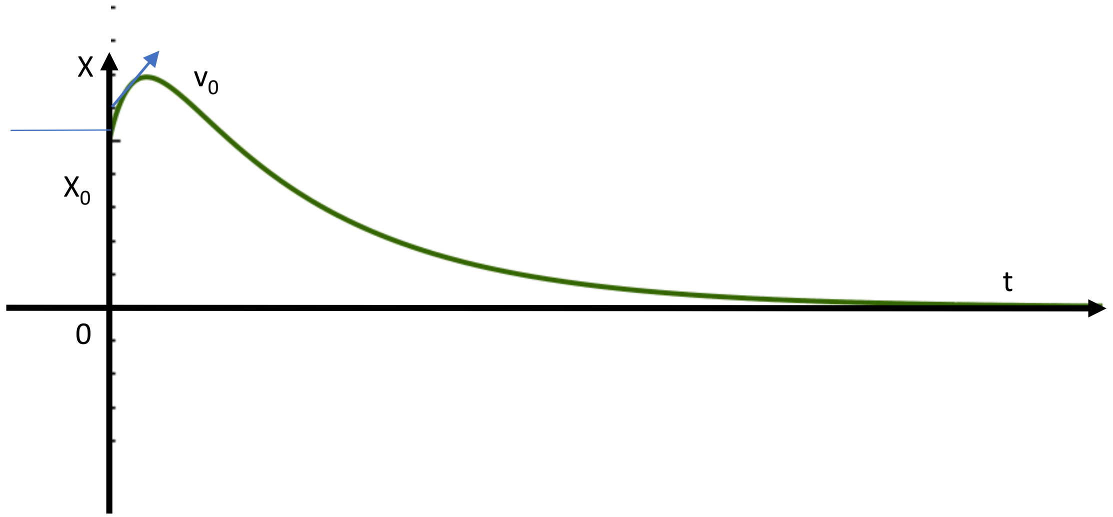
Solution to the EoM:

$$x = Ae^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} + Be^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}$$

When applying initial conditions x_0 and v_0

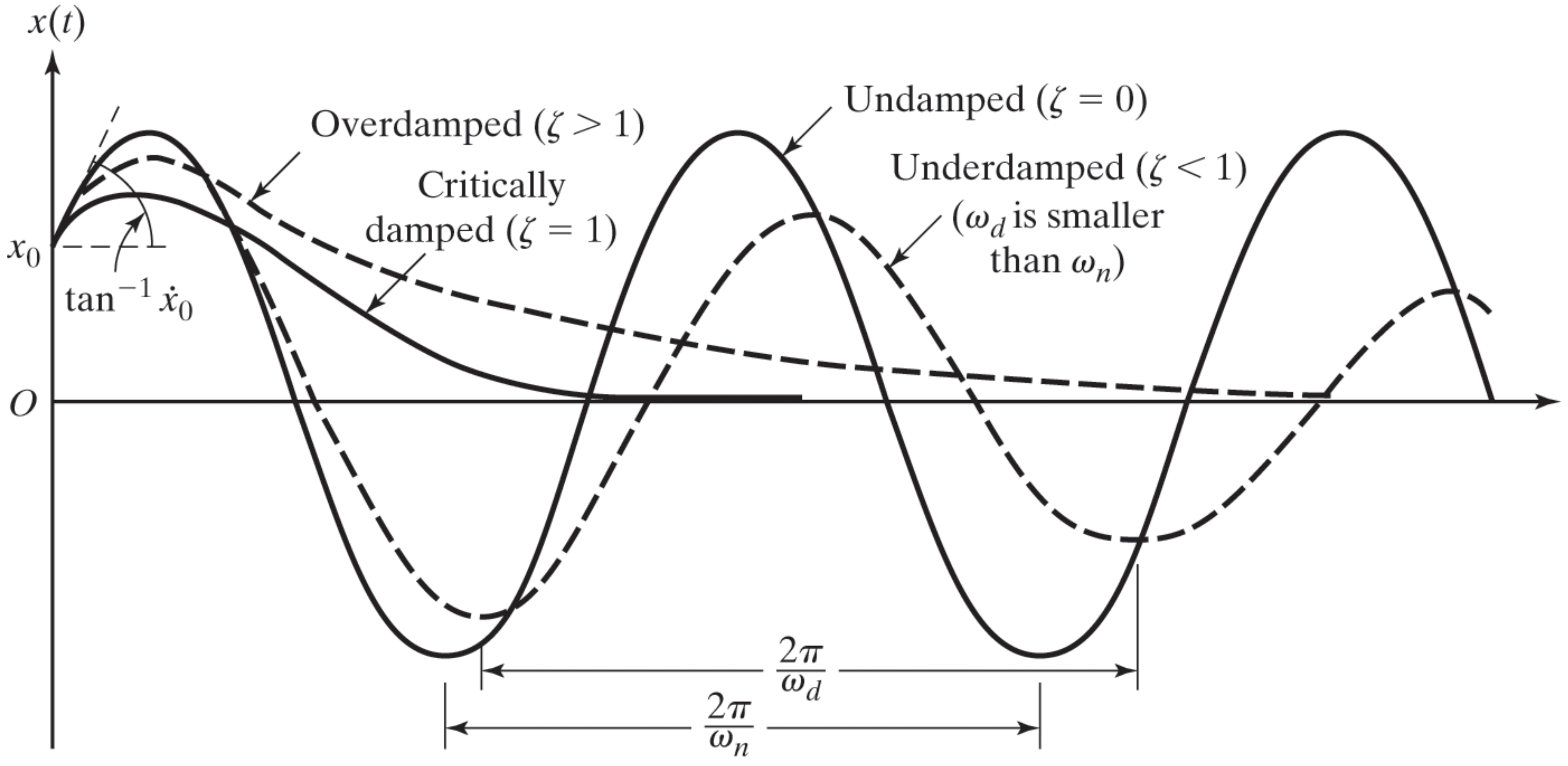
$$A = \frac{v_0 + \left(\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$B = \frac{v_0 - \left(\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$



The motion represented is aperiodic, eventually diminish to zero but much slower than critically damped system

Comparison of motion with different types of damping



Figures and content adapted from Textbook:
Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition

Energy dissipated in viscous damping

In a viscously damped system, the rate of change of energy with time (dW/dt) is given by

$$\frac{dW}{dt} = \text{force x velocity} = Fv = -(cv)v = -c\left(\frac{dx}{dt}\right)^2$$

In the case of a damped system, simple harmonic motion, $x = X\,sin(\omega_d t)$

$$\Delta W = \int_0^{2\pi/\omega_d} -c\left(\frac{dx}{dt}\right)^2 dt = \int_0^{2\pi/\omega_d} -c(\omega_d \cos(\omega_d t))^2 dt = \int_0^{2\pi} -c(\omega_d)(\cos(\omega_d t))^2 d(\omega_d t)$$

$$\Delta W = \pi c\,\omega_d X^2$$

This shows that the energy dissipated is proportional to the square of the amplitude of motion and ω_d.

The energy loss in each cycle can be compute dividing by the maximum kinetic or potential energy

$$\frac{\Delta W}{W} = \frac{\pi c\,\omega_d X^2}{\frac{1}{2} m\omega_d^2 X^2} = 2\left(\frac{2\pi}{\omega_d}\right)\left(\frac{c}{2m}\right) = 4\pi\zeta \approx 2\delta$$

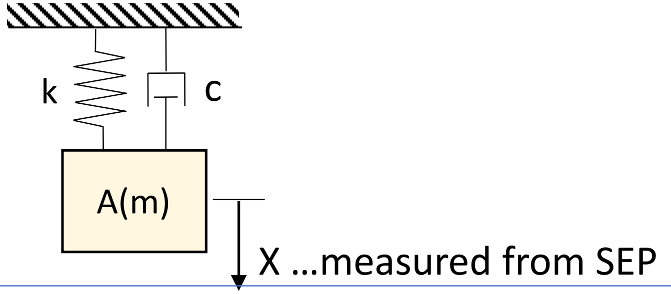
This term is called *specific damping capacity*

Formula Sheet for Free vibration

Important parameters

$$\omega_n = \sqrt{\frac{k_e}{m_e}}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad \zeta = \frac{c_e}{2\sqrt{m_e k_e}}$$



Governing equation

$$m_e\ddot{x} + c_e\dot{x} + k_e x = f(t)$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = f(t)/m$$

Response : $x(t) = x_h(t) + x_p(t)$

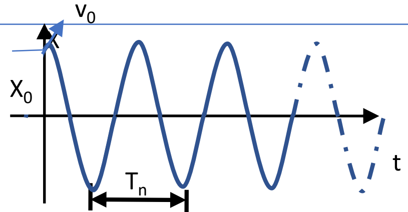
$$\omega_n = 2\pi f_n \qquad \omega_n = \frac{2\pi}{T_n}$$

Undamped systems

$\zeta = 0$

$$x = A\cos\omega_n t + B\sin\omega_n t \qquad A = x_0 \qquad B = v_0/\omega_n$$

$$x = X_0\cos(\omega_n t - \varphi) \qquad X_0 = \sqrt{A^2+B^2}, \varphi = \tan^{-1} B/A$$

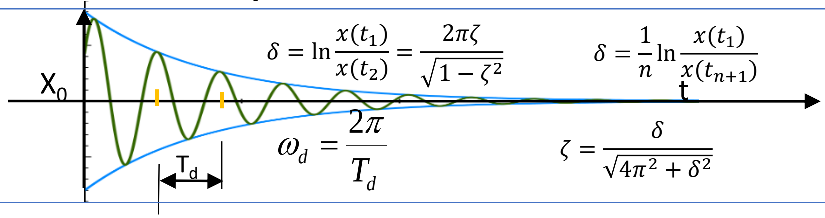


Underdamped systems

$\zeta < 1$

$$x = Ae^{-\zeta\omega_n t}\cos\omega_d t + Be^{-\zeta\omega_n t}\sin\omega_d t \qquad A = x_0 \qquad B = (v_0 + x_0\zeta\omega_n)/\omega_d$$

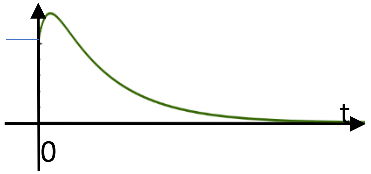
$$x = X_0 e^{-\zeta\omega_n t}\cos(\omega_d t - \varphi) \qquad X_0 = \sqrt{A^2+B^2}, \varphi = \tan^{-1} B/A$$



Critically damped systems

$\zeta = 1$

$$x = Ae^{-\omega_n t} + Bte^{-\omega_n t} \qquad A = x_0 \qquad B = v_0 + x_0\omega_n$$

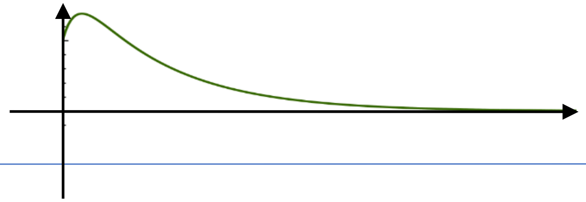


Overdamped systems

$\zeta > 1$

$$x = Ae^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} + Be^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}$$

$$A = \frac{v_0 + \left(\zeta + \sqrt{\zeta^2-1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}} \qquad B = \frac{v_0 - \left(\zeta - \sqrt{\zeta^2-1}\right)\omega_n x_0}{2\omega_n\sqrt{\zeta^2-1}}$$



Dissipated energy, viscous damping system

$$\Delta W = \pi c\,\omega_d X^2$$

specific damping capacity $\frac{\Delta W}{W} = 4\pi\zeta \approx 2\delta$

Prof. Carmen Muller-Karger, PhD

Mechanical Vibrations

Coulomb and Hysteretic Damping

Prof. Carmen Muller-Karger, PhD
Florida International University

Figures and content adapted from
Textbook: Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition.
Chapter 2: Free vibrations of a sigle degree of freedom system

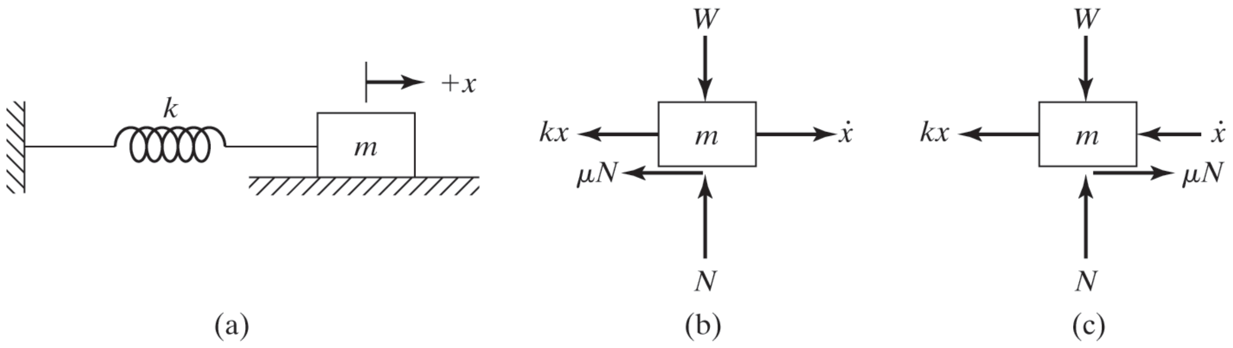
Free Vibration with Coulomb Damping

Coulomb damping arises when bodies slide on dry surfaces.

The force required to produce sliding is proportional to the normal force acting in the plane of contact.

$$F = \mu N = \mu W = \mu mg$$

The value of the coefficient of friction depends on the materials in contact and the condition of the surfaces in contact.



Equation of Motion is a piecewise function

The friction force acts in a direction opposite to the direction of velocity.

$\dot{x}$ positive → $m\ddot{x} + kx = -\mu N$

$\dot{x}$ negarive ← $m\ddot{x} + kx = \mu N$

can be expressed as a single equation (using signum function)

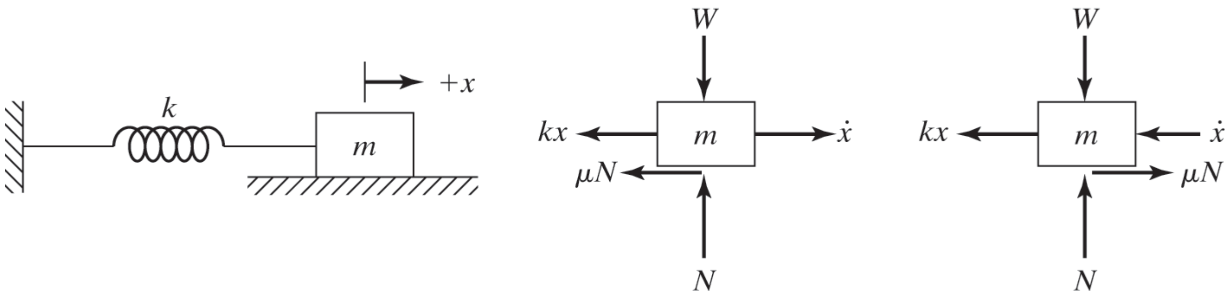
$$m\ddot{x} + \mu mg\ sign(\dot{x}) + kx = 0$$

Free Vibration with Coulomb Damping

The equation of motion:

$$m\ddot{x} + \mu mg\ sign(\dot{x}) + kx = 0$$

where $sign(\dot{x})$ value is defined as:
1 for $\dot{x}$ >0, -1 for $\dot{x}$ <0, and 0 for $\dot{x}$ =0



(a) (b) (c)

For the solution we will assume the equation of motion is a piecewise function

1. When $\dot{x}$ >0, the sign function is positive and the equation becomes,

$$m\ddot{x} + kx = -\mu mg$$

and the solution is a harmonic motion plus a constant:

$$x = A_1 \cos \omega_n t + A_2 \sin \omega_n t - \frac{\mu mg}{k}$$

2. When $\dot{x}$ <0, the sign function is negative and the equation becomes,

$$m\ddot{x} + kx = \mu mg$$

and the solution is a harmonic motion plus a constant:

$$x = A_3 \cos \omega_n t + A_4 \sin \omega_n t + \frac{\mu mg}{k}$$

Free Vibration with Coulomb Damping

If we solve the equation for initial conditions $\mathrm{x}(0) = x_0$ and $\boldsymbol{v}_0 = 0$.
Since the mass started with an initial displacement, it moves from right to left with a negative velocity. Starting in case 2:

$$\mathrm{x}(0) = x_0 = A_3 \cos\omega_n(0) + A_4 \sin\omega_n(0) - \frac{\mu N}{k}$$

$$\dot{x}(0) = -A_3\omega_n \sin\omega_n(0) + A_4\omega_n cos\omega_n(0)$$

$$A_3 = x_0 - \frac{\mu N}{k}$$

$$A_4 = 0$$

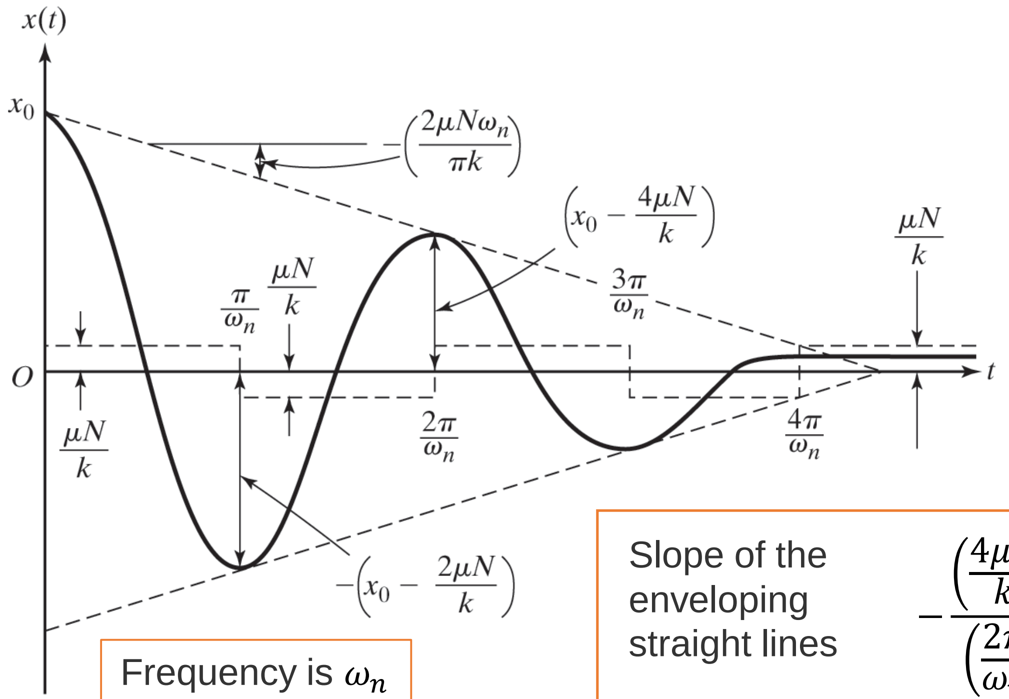
This solution is valid for half the cycle only—that is, for $0 < t < \frac{\pi}{\omega_n}$

When $t = \frac{\pi}{\omega_n}$, the mass will be at its extreme left position and its displacement from equilibrium position can be found from

$$t = \frac{\pi}{\omega_n} \qquad x\left(\frac{\pi}{\omega_n}\right) = \left(x_0 - \frac{\mu N}{k}\right)\cos\omega_n\left(\frac{\pi}{\omega_n}\right) + \frac{\mu N}{k} = -\left(x_0 - \frac{2\mu N}{k}\right)$$

Since the motion started with a displacement of x_0 and, in a half cycle, the value of *x* became $-\left(x_0 - \frac{2\mu N}{k}\right)$, the reduction in magnitude of x in time $\frac{2\mu N}{k}$, it can be demonstrated that for the other half to the cycle the reduction is $\frac{4\mu N}{k}$

Free Vibration with Coulomb Damping important equations:



Frequency is ω_n

Slope of the enveloping straight lines $-\frac{\left(\frac{4\mu N}{k}\right)}{\left(\frac{2\pi}{\omega_n}\right)} = \left(\frac{2\mu N\omega_n}{\pi k}\right)$

The amplitude reduces linearly with Coulomb damping and amount of: $\frac{4\mu N}{k}$

Therefore: $x_m = x_{m-1} - \frac{4\mu N}{k}$

The motion stops when $x_n < \frac{\mu N}{k}$, since the restoring force exerted by the spring (kx) will then be less than the friction force μN. Thus the number of cycles (n) that elapse before the motion ceases is given by

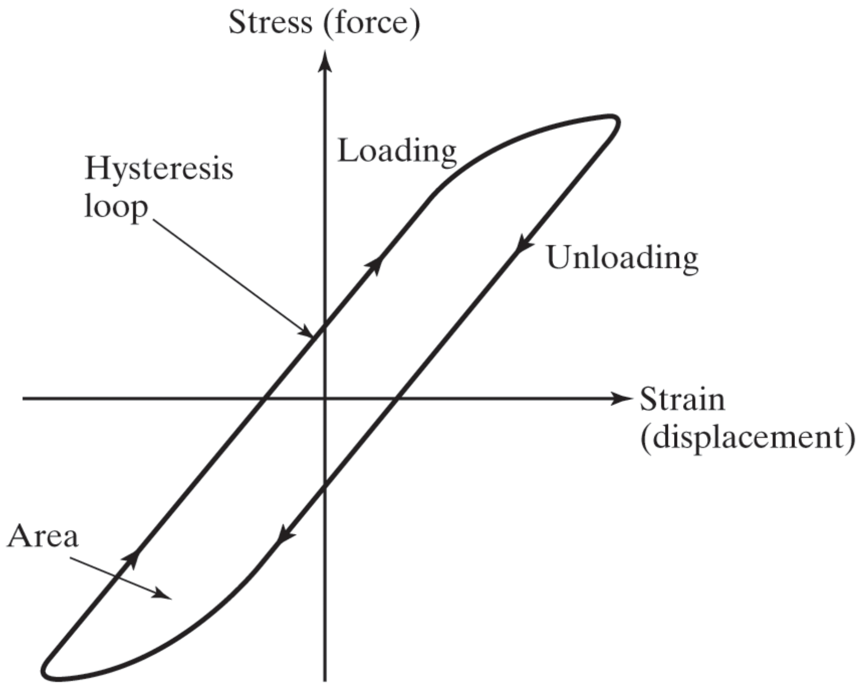
$$x_n = x_0 - n\frac{4\mu N}{k} \leq \frac{\mu N}{k}$$

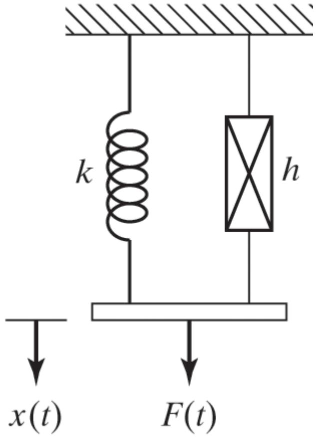
Number of cycles to stop $n \geq \left\{\frac{x_0 - \frac{\mu N}{k}}{\frac{4\mu N}{k}}\right\}$

Time to stop $\Delta t_{stop} = n\tau_n = n\frac{2\pi}{\omega_n}$

Free Vibration with Hysteretic Damping

- Also called solid or structural damping, is caused by the friction between the internal planes that slip or slide inside the material.
- This causes a hysteresis loop to be formed in the stress-strain or force-displacement curve. The energy loss in one loading and unloading cycle is equal to the area enclosed by the hysteresis loop.
- It was found experimentally that the energy loss per cycle due to internal friction is independent of the frequency but approximately proportional to the square of the amplitude.





Free Vibration with Hysteretic Damping

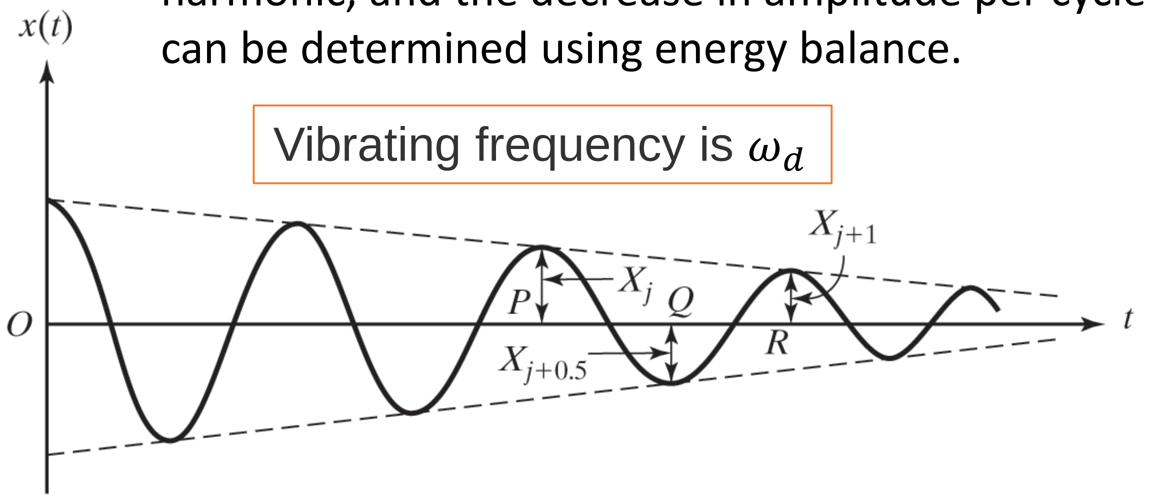
The damping coefficient c is assumed to be inversely proportional to the frequency, where h is called the hysteresis damping constant.

$$c = \frac{h}{\omega}$$

$$m\ddot{x} + \frac{h}{\omega}\dot{x} + kx = 0$$

The motion can be considered to be nearly harmonic, and the decrease in amplitude per cycle can be determined using energy balance.

Vibrating frequency is ω_d



$$x = x_0 e^{-\zeta\omega_n t}\cos\omega_d t + \frac{v_0 + x_0\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t$$

Energy loss for viscous damping $\Delta W = \pi c\, \omega_d X^2$

In term of h: $\Delta W = \pi h\, X^2$

Another dimensionless constant used to describe the hysteric damping is

$$\beta = \frac{h}{k}$$

Energy loss in term of β $\Delta W = \pi k \beta\, X^2$

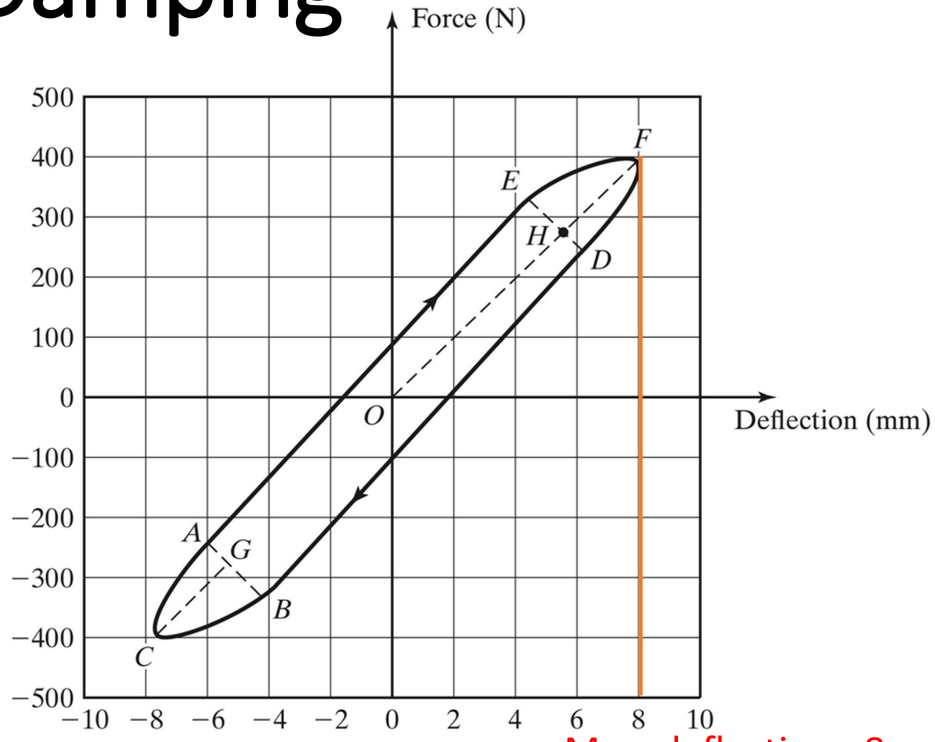
Logarithmic decrement $\delta = \ln\left(\frac{X_j}{X_{j+1}}\right) \approx \ln(1 + \pi\beta) \approx \pi\beta \approx 2\pi\zeta_{eq} = \frac{\pi h}{k}$

The equivalent viscous damping is $\zeta_{eq} = \frac{\beta}{2} = \frac{h}{2k}$

Free Vibration with Hysteretic Damping

Characteristics of the hysteretic loop:

- The graph force- deflection is usually obtained from experimental measurements on a structure.
- The energy dissipated ΔW in a cycle is the area enclosed by the hysteresis loop.
- The constant of the spring k is the slope of the force-deflection curve.
- The graph give information about the maximum deflection of the response.
- Using the equation for work we can related the energy loss with the damping constant and the logarithmic decrement.
- Under hysteretic damping the system behaves as underdamped and the response is similar to the a viscous damping system.



(b)

Max. deflection= 8 mm

Approximate the area using a square and 2 triangles.

$$AREA = \Delta W$$

Free Vibration with Hysteretic Damping, important equations:

Damping coefficient: $c = \frac{h}{\omega}$

Dimensionless damping constant: $\beta = \frac{h}{k}$

Energy loss : $\Delta W = \pi h\, X^2 = \pi k\beta\, X^2$ ΔW = Area in hysteretic loop

Equivalent spring constant: k= Slope of hysteretic loop

When the system is underdamped the answer would same as a viscous damping system: $x = x_0 e^{-\zeta\omega_n t}\cos\omega_d t + \frac{v_0 + x_0\zeta\omega_n}{\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t$

Logarithmic decrement: $\frac{X_j}{X_{j+1}} = \frac{2+\pi\beta}{2-\pi\beta} \cong 1 + \pi\beta$ $\delta = ln\left(\frac{X_j}{X_{j+1}}\right) \cong ln(1+\pi\beta) \cong \pi\beta = \frac{\pi h}{k}$ $\delta = \frac{1}{n} ln\left(\frac{X_o}{X_n}\right)$

Damping ratio: $\delta \cong 2\pi\zeta_{eq} \cong \pi\beta$ $\zeta_{eq} = \frac{\beta}{2} = \frac{h}{2k}$ $c_{eq} = c_c\zeta_{eq} = 2\sqrt{mk}\frac{\beta}{2} = \beta\sqrt{mk} = \frac{h}{k}\sqrt{mk} = \frac{h}{\omega}$

Mechanical Vibrations

Stability on Vibrating Systems

Prof. Carmen Muller-Karger, PhD
Florida International University

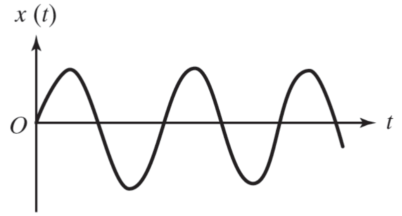
Figures and content adapted from

Textbook: Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition.

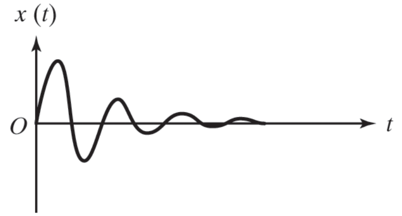
Chapter 2: Free vibrations of a sigle degree of freedom system

Stability of vibrating systems

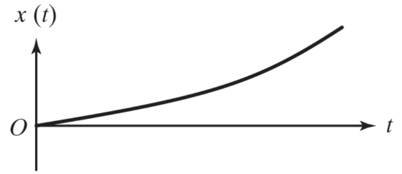
- A system is said to be *stable* if its free-vibration response neither decays nor grows, but remains constant or oscillates as time approaches infinity.
- A system is defined to be *asymptotically stable* if its free-vibration response approaches zero as time approaches infinity.
- A system is considered to be *unstable* if its free-vibration response grows without bound as time approaches infinity.
- An unstable system can cause damage to the system, adjacent property, or human life.



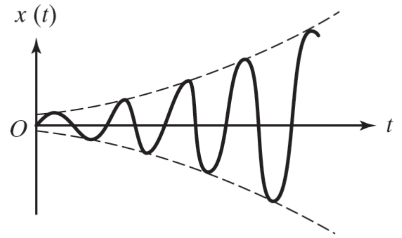
Stable system
(a)



Asymptotically stable system
(b)

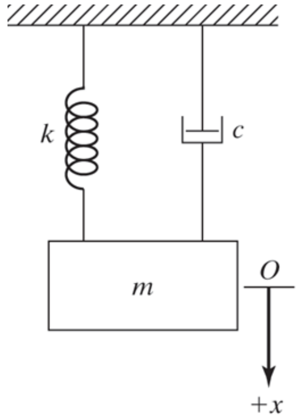


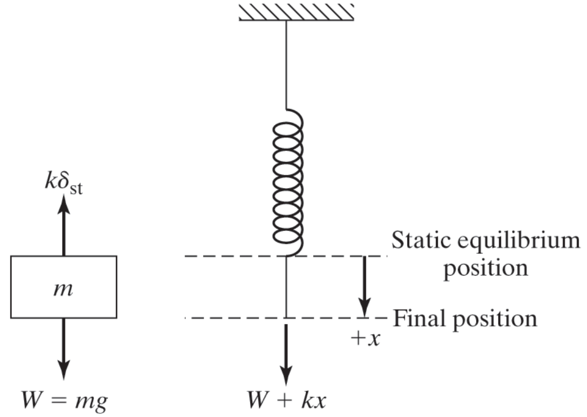
Unstable system (with divergent instability)
(c)



Unstable system (with flutter instability)
(d)

Stability of vibrating systems





- The *static equilibrium position* of a system can be found by setting velocity and acceleration equals to zero in the equation of motion: $\ddot{x} = 0, \dot{x} = 0$

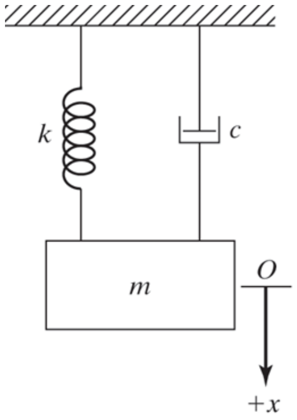
$$m\ddot{x} + c\dot{x} + k(x) - mg = 0 \Rightarrow \boxed{x = \delta_{st} = \frac{mg}{k}}$$

- At the equilibrium position the potential energy is minimum, therefore *static equilibrium position* of a system can be found by setting the derivative of the potential energy respect to position equal to zero:

$$\left.\frac{dU}{dx}\right|_{\delta_{st}} = 0$$

$$U = \frac{1}{2}k(x)^2 - mg(x) \Rightarrow \frac{dU}{dx} = \frac{2}{2}k(x) - mg = 0 \Rightarrow \boxed{x = \delta_{st} = \frac{mg}{k}}$$

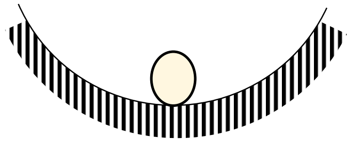
Stability of vibrating systems



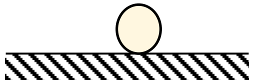
Stability of a system can be explained in terms of its energy. According to this scheme, a system is considered to be asymptotically stable, stable, or unstable if its energy decreases, remains constant, or increases, respectively, with time.

The *static equilibrium position* will be stable following the behavior of the second derivative of the potential energy respect to position:

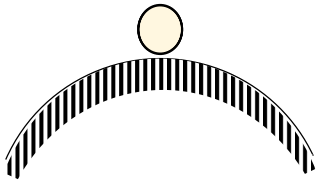
$$\frac{\partial^2 U}{\partial x^2} > 0$$



$$\frac{\partial^2 U}{\partial x^2} = 0$$



$$\frac{\partial^2 U}{\partial x^2} < 0$$



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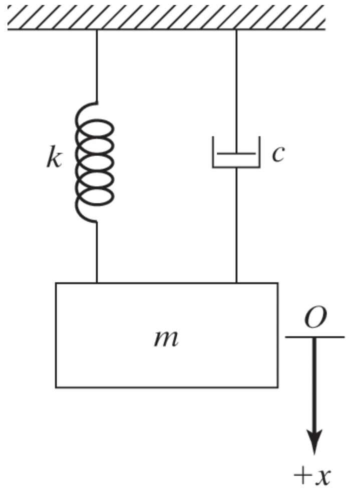
Stability of vibrating systems



- We can also describe the stability of the system according to the signs of the coefficients of the characteristic equation

Governing Equation

$$m\ddot{x} + c\dot{x} + kx = 0$$

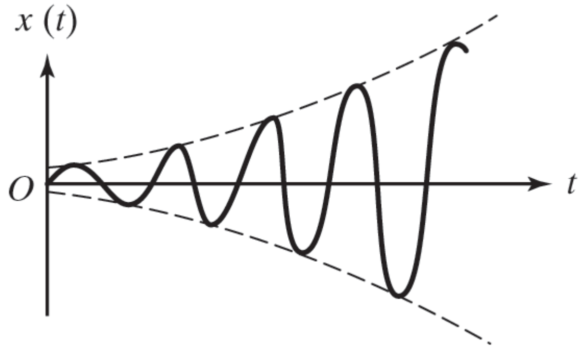
Solution is of the form:

$$x(t) = Ce^{st}$$

Characteristic equation :

$$\left(ms^2 + cs + k\right) = 0 \quad \Rightarrow \quad s_{1,2} = \frac{-c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \quad \Rightarrow \quad x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$

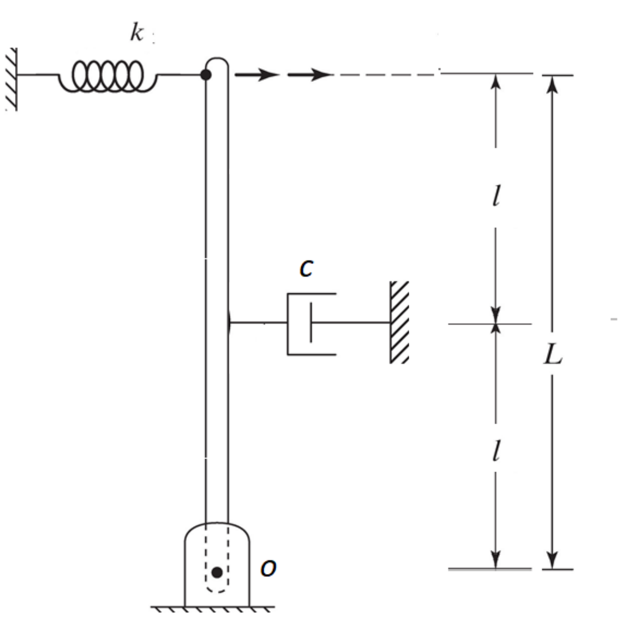
- If the exponential is positive the response may grow without bounds.

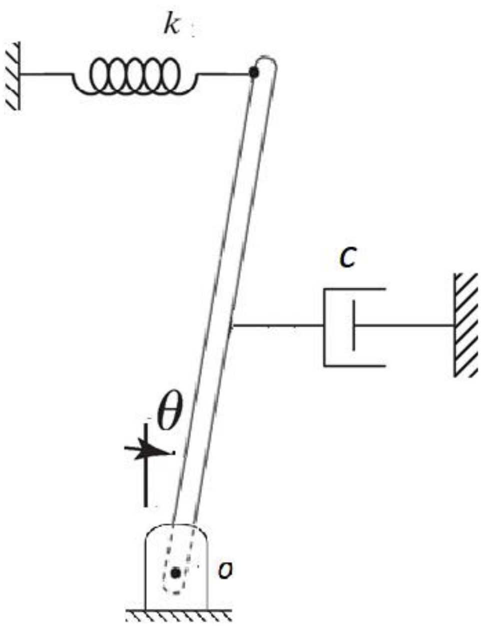


Unstable system (with flutter instability)

Example

Consider a uniform rigid bar, of mass m and length L, pivoted at one end and connected by one spring at the other end and one damper at the middle of the bar. Assuming that the spring is unstretched when the bar is vertical, derive the equation of motion of the system for small angular displacements (θ) of the bar about the pivot point, and investigate the stability behavior of the system.
For small angular displacements the spring and the damper are considered to be always horizontal.



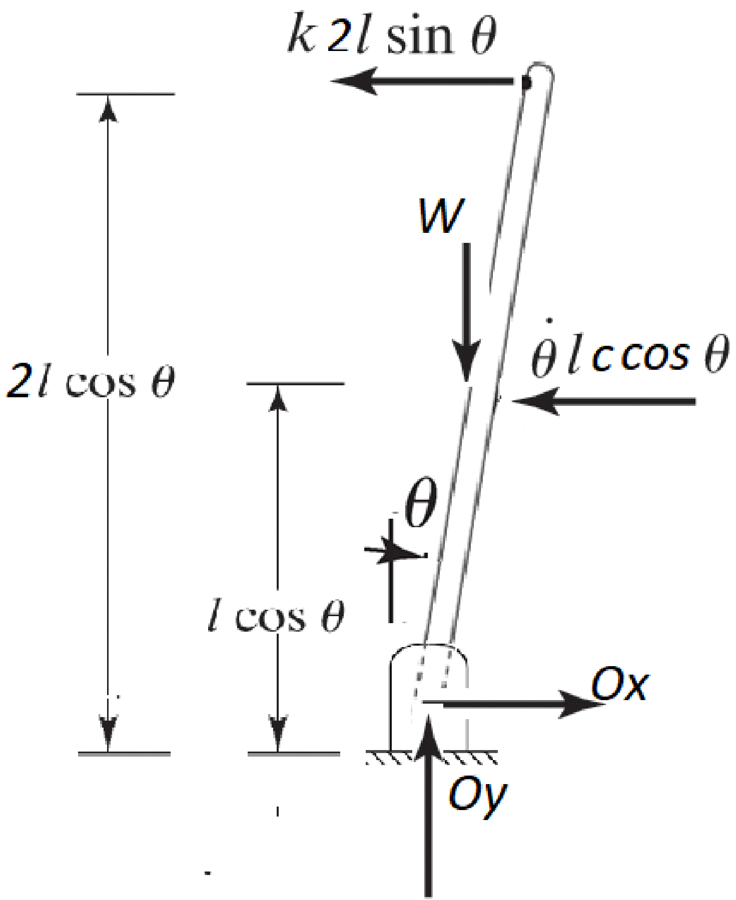


STEPS FOR THE ANALYSIS:

1. Derive the equation of motion of the system for small angular displacements (θ)
2. Find the equilibrium position,
3. Analysis of stability

Example (cont.)

FBD



Applying Equation of Motion. Moment respect to point "0"

$$\sum M_{oz} = -mgl sin\theta + \dot\theta c l cos\theta l cos\theta + k2l\sin\theta\, 2l cos\theta = -J_o\ddot\theta$$

$$\dot\theta l cos\theta l cos\theta + k4l\sin\theta\, l cos\theta - mgl sin\theta = -\left(\frac{1}{3}m(2l)^2\right)\ddot\theta$$

Governing Equation

$$\left(\frac{4}{3}ml^2\right)\ddot\theta + \dot\theta c l^2 (cos\theta)^2 + 4kl^2\sin\theta\, cos\theta - mgl sin\theta = 0$$

Equilibrium positions $\ddot\theta = 0, \dot\theta = 0$

$$4kl^2\sin\theta\, cos\theta - mgl sin\theta = 0$$

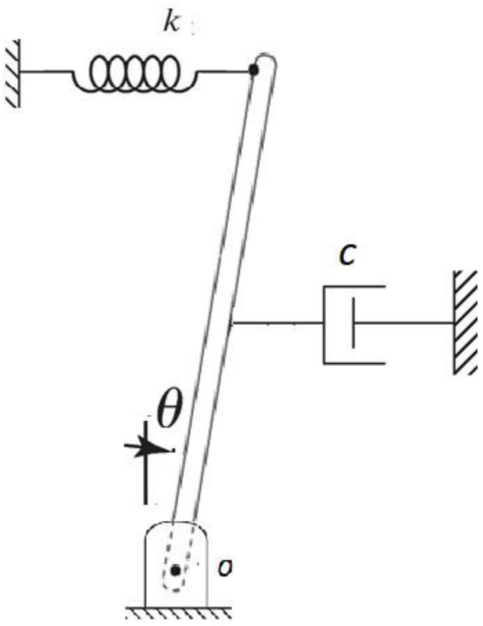
$$[4kl^2 cos\theta - mgl] sin\theta = 0$$

$$\begin{cases} \theta_1 = 0° \\ \theta_2 = 180° \\ \theta_{3,4} = \pm\cos^{-1}\left(\frac{mg}{4kl}\right) \end{cases}$$

Example (cont.)

$\boldsymbol{\theta = \theta_1 = 0°}$

Equilibrium position



This is a nonlinear governing equation:

$$\left(\frac{4}{3}ml^2\right)\ddot{\theta} + \dot{\theta}cl^2(cos\theta)^2 + 4kl^2\sin\theta\, cos\theta - mglsin\theta = 0$$

For small rotational displacements: $\begin{cases}\sin\theta \approx \theta \\ \cos\theta \approx 1\end{cases}$

The equation of motion becomes linear for equilibrium position $\boldsymbol{\theta = \theta_1 = 0°}$

$$\left(\frac{4}{3}ml^2\right)\ddot{\theta} + \dot{\theta}cl^2 + \left(4kl^2 - mgl\right)\theta = 0$$

$$m_{eq} = \frac{4}{3}ml^2 \qquad c_{eq} = cl^2 \qquad k_{eq} = \left(4kl^2 - mgl\right)$$

The equation can be written as the typical 2nd order differential equation:

$$m_{eq}\ddot{\theta} + c_{eq}\dot{\theta} + k_{eq}\theta = 0$$

The solution has the form:

$$\theta(t) = Ce^{st}$$

The characteristic polynomial:

$$\left(m_{eq}s^2 + c_{eq}s + k_{eq}\right)Ce^{st} = 0$$

Roots of the polynomial:

$$s_{1,2} = \frac{-c_{eq}}{2m_{eq}} \pm \sqrt{\left(\frac{c_{eq}}{2m_{eq}}\right)^2 - \frac{k_{eq}}{m_{eq}}}$$

Figures and content adapted from Textbook:
Singiresu S. Rao. Mechanical Vibration, Pearson sixth edition

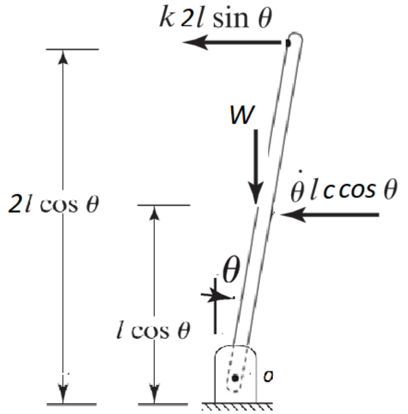
Example (cont.) $\boldsymbol{\theta = \theta_1 = 0°}$

Equation of motion for Equilibrium position $\theta_1 = 0$

$$\left(\frac{4}{3}ml^2\right)\ddot{\theta} + \dot{\theta}cl^2 + \left(4kl^2 - mgl\right)\theta = 0$$

$$\left(m_{eq}s^2 + c_{eq}s + k_{eq}\right)Ce^{st} = 0$$

$$s_{1,2} = \frac{-c_{eq}}{2m_{eq}} \pm \sqrt{\left(\frac{c_{eq}}{2m_{eq}}\right)^2 - \frac{k_{eq}}{m_{eq}}}$$



Solution CASE 1 : Radical is negative, $s_{1,2}$ are complex , the system is STABLE and oscillates around the equilibrium position

$$\frac{k_{eq}}{m_{eq}} = \frac{3\left(4kl^2 - mgl\right)}{4ml^2} > 0 \quad \Rightarrow \quad 4kl^2 > mgL$$

The spring is capable to overcome the weight.

$$\theta(t) = Ae^{-\zeta\omega_n t}\cos\omega_d t + Be^{-\zeta\omega_n t}\sin\omega_d t$$

Solution CASE 2 : Radical is zero, $s_{1,2}$ are real and negative , the system is STABLE the system is in critical damping.

$$s_{1,2} = \frac{-c_{eq}}{2m_{eq}} \quad \Rightarrow \quad \theta(t) = Ae^{-\omega_n t} + Bte^{-\omega_n t}$$

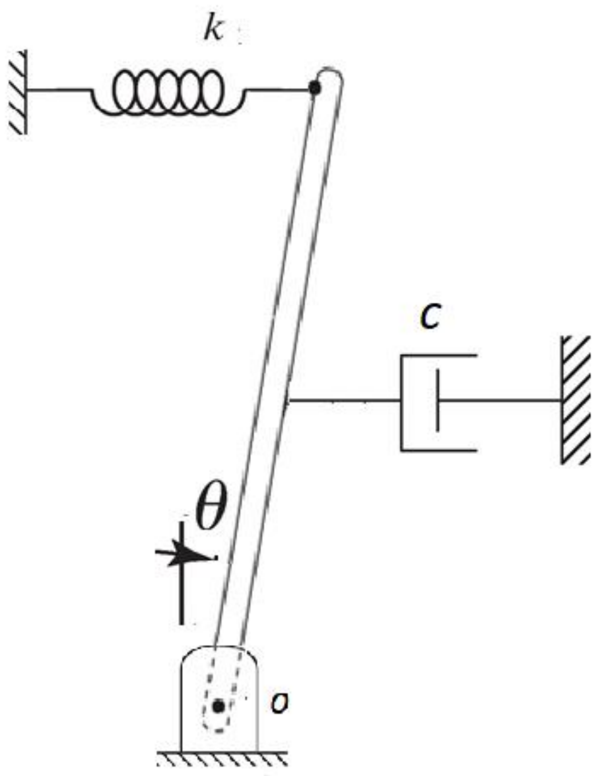
Solution CASE 3 : Radical is positive, $s_{1,2}$ are real and one is positive, the system is UNSTABLE.

$$\frac{k_{eq}}{m_{eq}} = \frac{3\left(4kl^2 - mgl\right)}{4ml^2} < 0 \quad \Rightarrow \quad 4kl^2 < mgl$$

$$\theta(t) = Ae^{s_1 t} + Be^{s_2 t}$$

The spring is NOT capable to overcome the weight, $\theta(t)$ increases exponentially .

Example (cont.)



Applying Equation of Motion. Moment respect to point "0"

$$\sum M_{oz} = -mglsin\theta + \dot{\theta}clcos\theta lcos\theta + k2l\sin\theta\, 2lcos\theta = -J_o\ddot{\theta}$$

$$\dot{\theta}lcos\theta lcos\theta + k4l\sin\theta\, lcos\theta - mglsin\theta = -\left(\frac{1}{3}m(2l)^2\right)\ddot{\theta}$$

Governing Equation

$$\left(\frac{4}{3}ml^2\right)\ddot{\theta} + \dot{\theta}cl^2(cos\theta)^2 + 4kl^2\sin\theta\, cos\theta - mglsin\theta = 0$$

Equilibrium positions $\ddot{\theta} = 0, \dot{\theta} = 0$

$$4kl^2\sin\theta\, cos\theta - mglsin\theta = 0$$

$$[4kl^2cos\theta - mgl]sin\theta = 0$$

$$\theta_1 = 0°$$

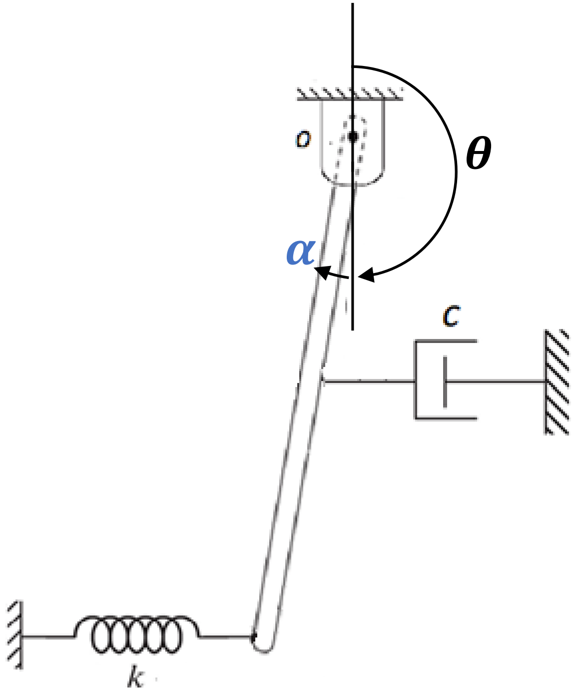
$$\theta_2 = 180°$$

$$\theta_{3,4} = \pm\cos^{-1}\left(\frac{mg}{4kl}\right)$$

Example

$\theta = \theta_2 = 180° = \pi$

Equilibrium position



This is a nonlinear governing equation:

$$\left(\frac{4}{3}ml^2\right)\ddot{\theta} + \dot{\theta}cl^2(cos\theta)^2 + 4kl^2 \sin\theta \, cos\theta - mglsin\theta = 0$$

Change of variable $\theta = \alpha + \pi, \ddot{\alpha} = \ddot{\theta}, \ \dot{\alpha} = \dot{\theta},$ The equation can be written in term of α:

$$\left(\frac{4}{3}ml^2\right)\ddot{\alpha} + \dot{\alpha}cl^2(\cos(\alpha + \pi))^2 + 4kl^2 \sin(\alpha + \pi)\, cos(\alpha + \pi) - mglsin(\alpha + \pi) = 0$$

For small rotational displacements of α respect to the equilibrium position:

$$\begin{cases} \sin\alpha \approx \alpha \\ \cos\alpha \approx 1 \end{cases}$$

$$\sin(\alpha + \pi) = \sin(\alpha)\cos(\pi) + \cos(\alpha)\sin(\pi) = -\sin(\alpha) \approx -\alpha$$

$$\cos(\alpha + \pi) = \cos(\alpha)\cos(\pi) - \sin(\alpha)\sin(\pi) = -\cos(\alpha) \approx -1$$

$$\left(\frac{4}{3}ml^2\right)\ddot{\alpha} + \dot{\alpha}cl^2(-1)^2 + 4kl^2(-\alpha)\,(-1) - mgl\,(-\alpha) = 0$$

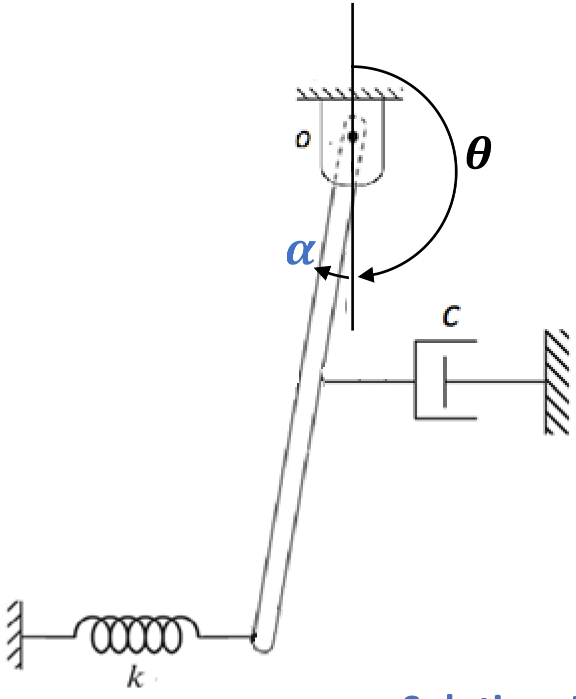
The equation of motion becomes linear for equilibrium position, in term of α:

$$\left(\frac{4}{3}ml^2\right)\ddot{\alpha} + (c\, l^2)\dot{\alpha} + [4kl^2 + mgl](\alpha) = 0$$

Example

$\theta = \theta_2 = 180° = \pi$

Equilibrium position



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$$m_{eq} = \frac{4}{3}ml^2 \qquad c_{eq} = cl^2 \qquad k_{eq} = \left(4kl^2 + mgl\right)$$

The equation can be written as the typical 2nd order differential equation:

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The solution has the form:

$$\theta(t) = Ce^{st}$$

The characteristic polynomial:

$$\left(m_{eq}s^2 + c_{eq}s + k_{eq}\right)Ce^{st} = 0$$

Roots of the polynomial:

$$s_{1,2} = \frac{-c_{eq}}{2m_{eq}} \pm \sqrt{\left(\frac{c_{eq}}{2m_{eq}}\right)^2 - \frac{k_{eq}}{m_{eq}}}$$

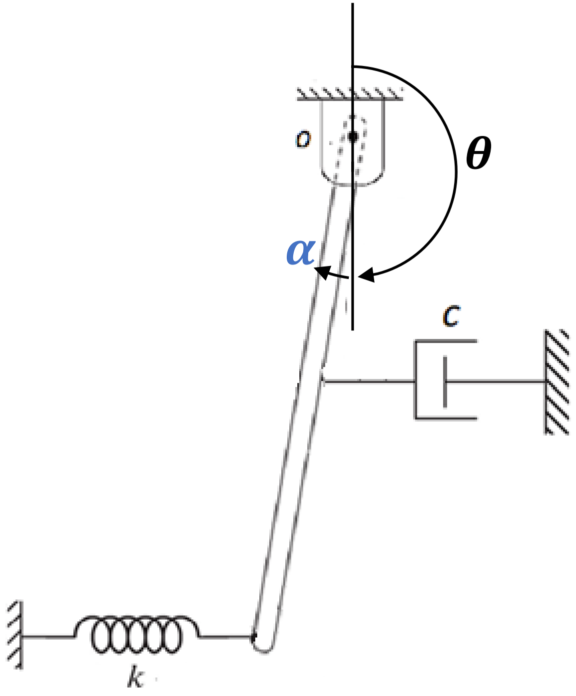
Solution: Radical is always less than first term, the system is STABLE and oscillates around the equilibrium position

$$\frac{k_{eq}}{m_{eq}} = \frac{3\left(4kl^2 + mgl\right)}{4ml^2} > 0 \;\Rightarrow\; \frac{-c_{eq}}{2m_{eq}} > \sqrt{\left(\frac{c_{eq}}{2m_{eq}}\right)^2 - \frac{k_{eq}}{m_{eq}}} \;\Rightarrow\; s_{1,2} = both\ negative$$

Example

$\theta = \theta_2 = 180° = \pi$

Equilibrium position



The equation of motion becomes linear for equilibrium position, in term of α:

$$\left(\frac{4}{3}ml^2\right)\ddot{\alpha} + \left(c\, l^2\right)\dot{\alpha} + \left[4kl^2 + mgl\right](\alpha) = 0$$

$$m_{eq} = \frac{4}{3}ml^2 \qquad c_{eq} = cl^2 \qquad k_{eq} = \left(4kl^2 + mgl\right)$$

With the definitions:

$$\zeta = \frac{c_{eq}}{2\sqrt{k_{eq}m_{eq}}} \qquad \omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2}$$

The solution is STABLE and could be any of the following depending on the parameters of the system :

$$x(t) = \begin{cases} \zeta = 0, x(t) = X_0\cos(\omega_n t - \phi) \\ \zeta < 1, x(t) = X_0 e^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \\ \zeta = 1, x(t) = C_1 e^{-\omega_n t} + C_2 t e^{-\omega_n t} \\ \zeta > 1, x(t) = C_1 e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} + C_2 e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t} \end{cases}$$