Probl...

4-32 A machine having a total weight of 96.5 lb is mounted on a spring of modulus 900 lb/in and is connected to a dashpot having a damping ratio of 0.25. The machine contains an unbalance of $(W_0 e)$ 5 lb in. If the speed of rotation is 401.1 rpm find (a) the amplitude of steady state motion (b) the max. dynamic force transmitted to the foundation (c) the angular position of the arm when the structure goes thru its neutral position

Solution $m = \frac{W}{g} = \frac{96.5}{32.2} = 0.25 \ \frac{\text{lb-sec}^2}{\text{ft}}$; $k = 900 \frac{\text{lb}}{\text{in}} = 10800 \frac{\text{lb}}{\text{ft}}$

$\zeta = \frac{c}{c_{cr}} = 0.25$ given $W_0 e = 5$ lb-in $f_f = 401.1$ rpm

$$\left.\begin{aligned} \omega_f &= 401.1 \cdot \frac{2\pi}{60} = 42 \text{ rad/s} \\ \omega_n &= \sqrt{\frac{k}{m}} \cong 60 \text{ rad/s} \end{aligned}\right\} \ r = \frac{\omega_f}{\omega_n} = 0.7$$

$$X_{RU} = \overset{W_0 e/W}{\left(\frac{m_0 e}{m}\right)} \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{5 \text{ lb-in}}{96.5 \text{ lb}} \frac{(0.7)^2}{\sqrt{(1-0.7^2)^2+(2\cdot 0.25 \cdot 0.7)^2}}$$

$= 0.041$ in

$$\tan\psi = \frac{2\zeta r}{1-r^2} = 0.6863$$

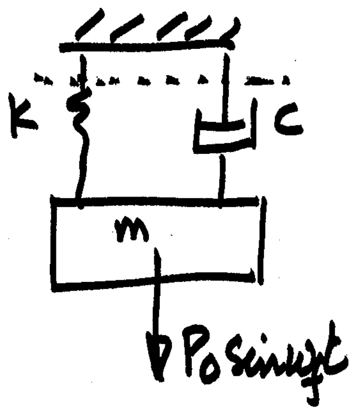
$\psi = 34.46^\circ = \omega_f t$ when $(m-m_0)$-main mass passes through $x = 0$.

(b)

$$F_T = k X_{RU}\sqrt{1+(2\zeta r)^2} = 900\,\frac{\text{lb}}{\text{in}}\cdot(0.041\text{ in})\sqrt{1+(2\cdot 0.25\cdot 0.7)^2}$$

$= \underline{\quad 39.095 \quad}$ lb force.

Force transmitted to support (General Case)



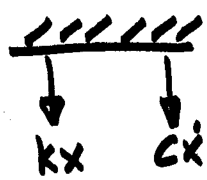
$$m\ddot{x} + c\dot{x} + kx = P_0 \sin\omega_f t$$

$$x_p = \frac{P_0}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}\sin(\omega_f t - \psi)$$

s.s.

$$= \boxed{\frac{X_0}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}}\sin(\omega_f t - \psi)$$

X – amplitude of forced vibration



$$x_p = X\sin(\omega_f t - \psi)$$

$$= kX\sin(\omega_f t - \psi) + c\omega_f X\cos(\omega_f t - \psi)$$

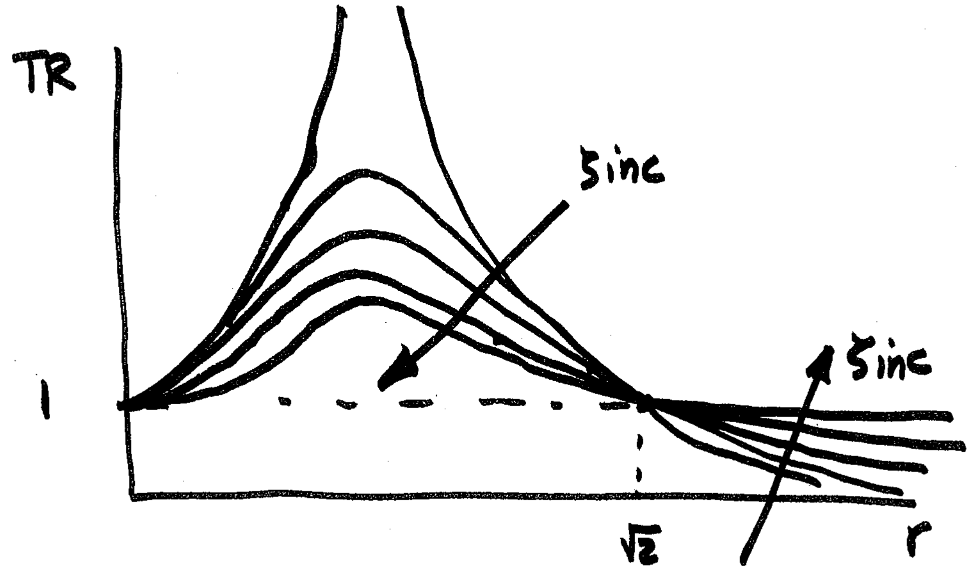
$$x_p = \underbrace{\boxed{X\sqrt{k^2+(c\omega_f)^2}}}_{F_T}\sin(\omega_f t - \psi - \beta)$$

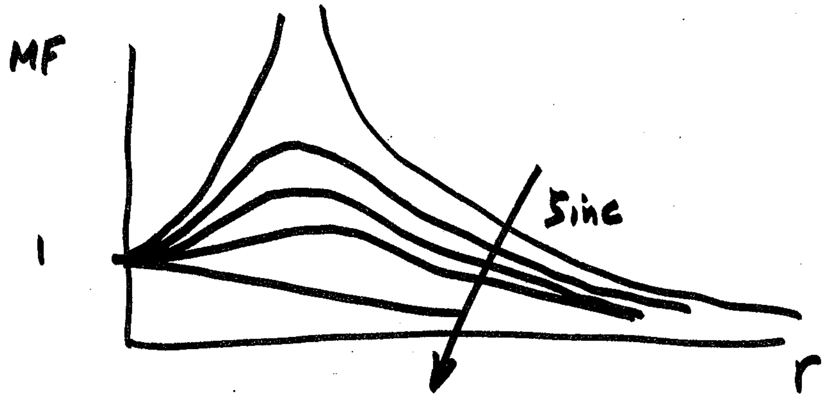
$$\tan\beta = -\frac{c\omega_f}{k}$$

$$F_T = \frac{P_0\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}} = \frac{P_0\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

TR = $F_T/P_0 = \dfrac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

transmissibility





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in rotating unbalance case

$$F_{T,RU} = \frac{\frac{m_0 e}{m} k r^2 \sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \quad ; \quad \frac{F_{T,RU}}{\frac{m_0 e}{m} k} = \frac{r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$F = X\left[k\sin(\omega_f t-\psi) + c\omega_f\cos(\omega_f t-\psi)\right].$$

$$F = X\,C\sin(\omega_f t-\psi-\beta)$$

let $C\sin\beta = -c\omega_f$, $C\cos\beta = k$ $\Big\}$ $C=\sqrt{k^2+(c\omega_f)^2}$

$\tan\beta = \dfrac{-c\omega_f}{k} = -2\zeta r$

$$= X\,C\sin(\omega_f t-\gamma) \qquad \gamma=\psi+\beta$$

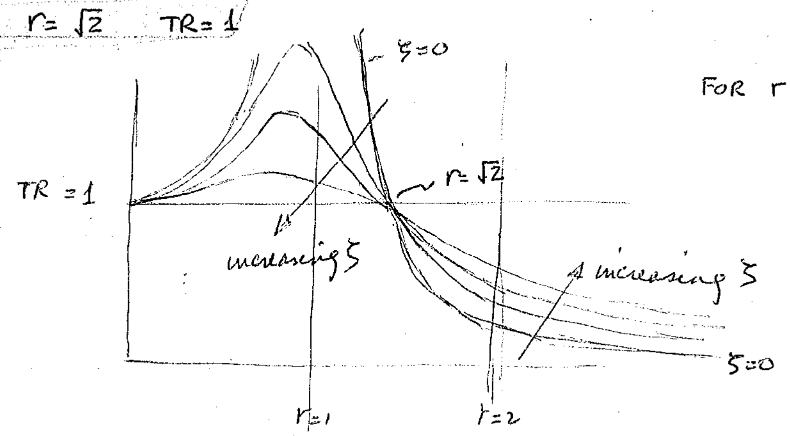
$$= F_{max}\sin(\omega_f t-\gamma) = F_T\sin(\omega_f t-\gamma)$$

$F_{MAX} = XC = X\sqrt{k^2+(c\omega_f)^2}$: max. transmitted force F_T

$$= \frac{X_0\, k\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = P_0\frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = P_0\frac{\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}$$

$$F_T/P_0 = TR \text{ (transmissibility)} = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

- WHEN $r=0$ $TR=1$ NO FORCING FN. FREQ. — APERIODIC $\forall\zeta$
- WHEN $r=\sqrt{2}$ $TR=1$ $\forall\zeta$.



FOR $r>>1$ $TR \sim \dfrac{2\zeta r}{r^2} \sim \dfrac{2\zeta}{r}$

AS $\zeta\uparrow$ TR $\uparrow$; AS $r\uparrow$ TR $\to 0$

- WHEN $r\to\infty$ $TR\to 0$ $\forall\zeta$ VIBRATIONS OCCUR SO FAST CHANGE IN DISP & VEL $\to 0$

- DAMPING reduces peak force at resonance, but increases F_T for $r>\sqrt{2}$
- NOTE THAT for $r>1$ increase in $\zeta \Rightarrow X\downarrow$

- PEAK OCCURS AT $r=\dfrac{\sqrt{(1+8\zeta^2)^{1/2}-1}}{2\zeta}$ <1 TO LEFT OF RESONANCE

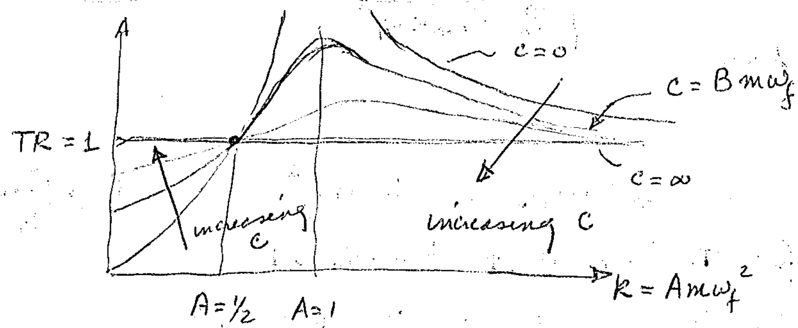
- CAVEAT AGAIN – THAT CHANGE IN $r \Rightarrow$ CHANGE ONLY IN ω_f

- TO LOOK AT CHANGE IN k & m better to use

$$TR = \frac{F_T}{P_0} = \frac{\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}$$

IF we take $\boxed{\dfrac{dTR}{dk} = 0 \Rightarrow \quad k \geqslant m\omega_f^2}$ take $k = Am\omega_f^2 \quad c = Bm\omega_f$

then $\boxed{TR = \dfrac{F_T}{P_0} = \dfrac{m\omega_f^2\sqrt{A^2+B^2}}{m\omega_f^2\sqrt{(A-1)^2+B^2}}}$ $\quad$ $\boxed{\text{as } k \to \infty \Rightarrow A \to \infty \quad TR \to 1 \quad \forall B \ (c)}$



- $\boxed{\text{for } c \neq 0 \quad \text{peak value of TR is when } k = m\omega_f^2/2 + \sqrt{(m\omega_f^2/2)^2 + (c\omega_f)^2}}$ $> m\omega_f^2$ PEAK TO right of ~~mass~~ k value at resonance
- TR is reduced only when $k << m\omega_f^2/2$ ie for light damping & small spring const.

EXAM INVOLVES

1. equiv spring equiv mass
2. free vibs w/o damping
3. " w damping
4. forced vibs w/ or w/o damping

- WE CAN DETERMINE mass variations also

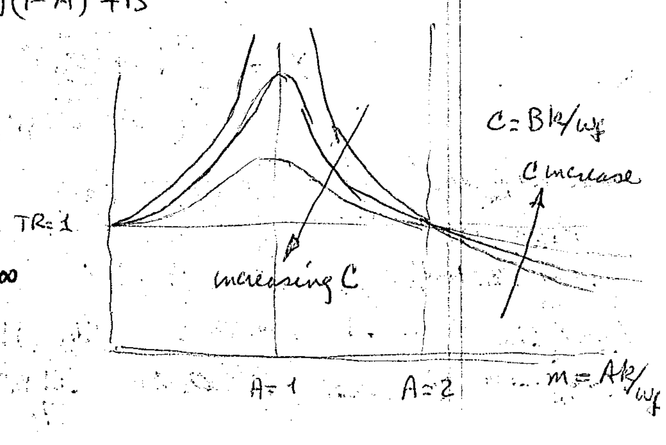
$$\frac{dTR}{dm} = 0 \Rightarrow m = k/\omega_f^2 \qquad \text{if} \quad m = Ak/\omega_f^2 \qquad c = Bk/\omega_f$$

$$\Rightarrow TR = \frac{F_T}{P_0} = \frac{k\sqrt{1+B^2}}{k\sqrt{(1-A)^2+B^2}} = \frac{\sqrt{1+B^2}}{\sqrt{(1-A)^2+B^2}}$$

when $A = 0 \Rightarrow m = 0 \quad TR = 1 \quad \forall c$

$A = 2 \Rightarrow m = \dfrac{2k}{\omega_f^2} \quad TR = 1 \quad \forall c$

$A \to \infty \quad TR \to 0 \quad \forall c$



ie TR decreases for very large mass $\Rightarrow \delta_{STATIC} = \dfrac{W}{k} \to \infty$

ALL PEAKS occur at $m = k/\omega_f^2 \Rightarrow A = 1$

$$(TR)_{max} = \frac{\sqrt{1+(2\zeta)^2}}{2\zeta} = \frac{\sqrt{k^2+(c\omega_f)^2}}{c\omega_f}$$

in 4th ed
9.29

Rao 3rd ed
9.27 $m = 500\,kg$ $m_0 e = 50\,kg\text{-}cm = .5\,kg\text{-}m$ $f_f = 300$ rpm $\Rightarrow$ $300 \cdot \frac{2\pi}{60} = 31.42$ rad/s

a) k $c = 0$

b) $\zeta = .1$ $k =$ small

$\left.\right\}\ P_0 = m_0 e \omega_f^2$

4th ed
9.26 9.28
$\omega_f = 62.83\ \frac{rad}{s}$
$TR = 2.5$ @ $r = 1$
find curve where ζ has max
$TR = 2.5 \Rightarrow \zeta = .2182$
@ operating speed
$Tr = 0.1$ for $\zeta = 0.2182$
$\Rightarrow$ get $r = 4.985$
$r^4 - 20.96 r^2 - 99 = 0 \rightarrow r^2 = 24.84$, $r^2 = 3.986$
$\omega_n r = \omega_f$
$\omega_n = \omega_f / r = 12.61$
$k = m\omega_n^2 = 13000\ N/m$
$c = 2\zeta m \omega_n = 449\ \frac{Ns}{m}$

$TR^2 = \frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}$

$2.5^2 = \frac{1 + 4\zeta^2}{4\zeta^2} \Rightarrow \zeta$

a) $\Rightarrow$ For $c = 0$ TR max occurs at $k = m\omega_f^2$ $TR = \frac{F_{max}}{P_0}$

note that for k large TR $\downarrow$ but still > 1

note that for k small $TR < 1$ if $k < m\omega_f^2$

choose $TR = .1$ $\Rightarrow$ $TR = \frac{1}{|1 - r^2|}$ $\Rightarrow$ $r^2 = 11$

$r^2 = \frac{\omega_f^2 m}{k} = 11 \Rightarrow k = \frac{1}{11} m\omega_f^2$ ✓ $r = 3.32$

$\omega_n = \frac{\omega_f}{r} = 9.47$ rad/sec & $k = \omega_n^2 m = 44863\ N/m$

$\cancel{\delta_{st} = \frac{W}{k} = \frac{500(9.81)}{44863} \cong .1\ m}$ $X_0 = \frac{P_0}{k} = .011\ m$

$X = \frac{r^2\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \cdot \frac{m_0 e}{m} = \frac{11 \cdot 1}{10}\left(\frac{.5}{500}\right) = 1.1\ (.001\ m) = .0011 m$

case (2) ~~...~~

$\cancel{TR = .1 = \ldots}$ choose $TR = .1$ $\zeta = .1$

$\cancel{X = \frac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}}$ $0.1 = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$ $\Rightarrow r^2 \cong 13.37,\ -7.5$
$r \cong 3.66$ rad/sec

$\omega_n = \frac{\omega_f}{r} = 8.59$ rad/sec $k = m\omega_n^2 = 36919\ N/m$

$\delta_{st} = \frac{W}{k} = \frac{500\,(9.81)}{36919} \cong .125\ m$ $X_0 = \frac{P_0}{k} = .013\ m$

$X = \frac{r^2\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \frac{m_0 e}{m}$ $.00108 m$

since X shock absorber smaller, it is preferred but since X_k also smaller $\rightarrow$ leads to cheaper design

Choose $k = \frac{1}{100} m\omega_f^2$
$r^2 = 100$ $r = 10$
$\omega_n = \frac{\omega_f}{10} = 3.14$ rad/sec
$\delta_{st} = \frac{W}{k} \cong 1 m$
$X_0 = P_0/k = .1\,m$

$TR = \frac{\sqrt{k^2 + (c\omega_f)^2}}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}$ $\zeta = .1 = \frac{c}{c_c} = \frac{c}{2\omega_n m} = \frac{.1\, m\omega_f}{2 m\omega_n} = \frac{1}{20} r$

$r = 2 \Rightarrow \omega_n = \frac{1}{2}\omega_f = 15.7$ rad/s

Choose .1 solve for c $.01\,(m^2\omega_f^4 + c^2\omega_f^2) - c^2\omega_f^2 = 0$

$X \cong .00236$ $TR \cong .236$ $\frac{.01}{.99} m^2\omega_f^2 = c^2$ $c \cong \frac{1}{10} m\omega_f$

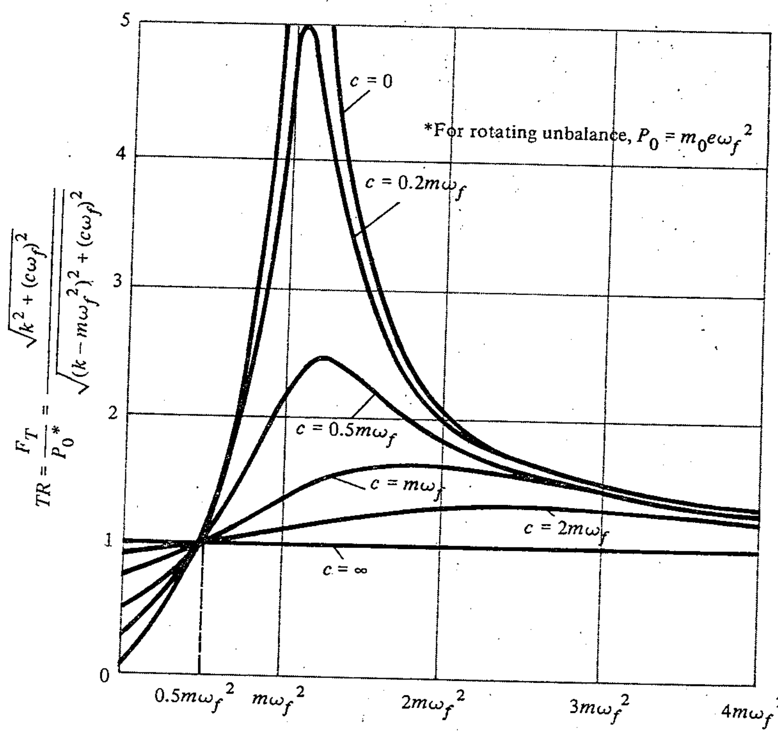


Figure 4-23

damping cuts down the peak force transmitted in the resonant region, it also results in greater force transmission for $r > \sqrt{2}$. This latter effect is opposed to the influence which the damping increase has in reducing the displacement amplitude for large values of r, as indicated by Fig. 4-15. It can be shown that the peak of the curve occurs at

$$r = \frac{\sqrt{-1 + (1 + 8\zeta^2)^{1/2}}}{2\zeta} < 1 \tag{4-91}$$

Consideration of Fig. 4-22 should be limited to conditions for which P_0 is constant and r is varied by changing ω_f only. If it is desired to observe the effect of varying the spring constant k, Eq. 4-88 can be used to plot TR against k for various values of the damping constant c. The transmissibility is then expressed as

$$TR = \frac{F_T}{P_0} = \frac{\sqrt{k^2 + (c\omega_f)^2}}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \tag{4-92}$$

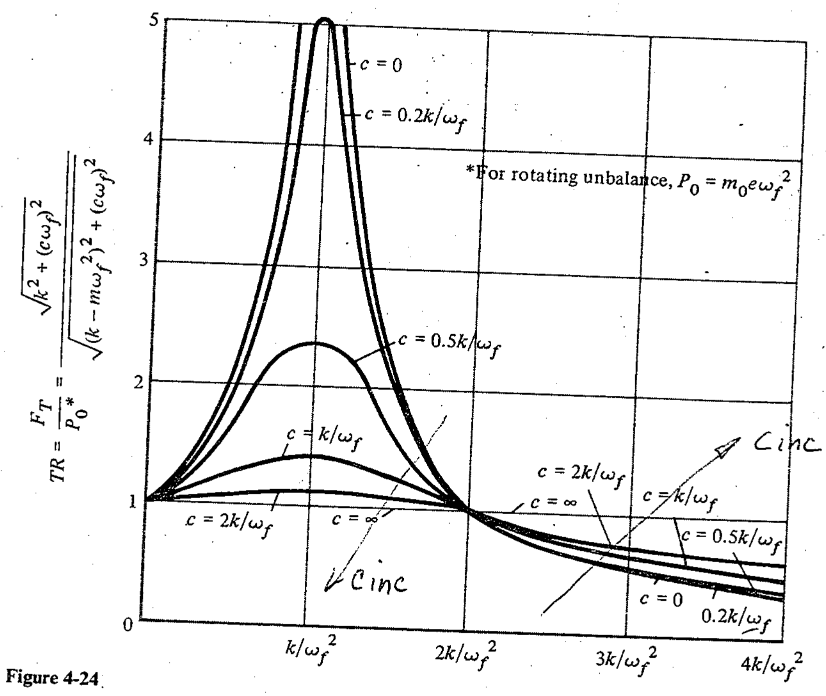


Figure 4-24

The resulting family of curves is shown in Fig. 4-23. The initial value of TR and the crossover point can be readily determined, as well as the trend of the curves in approaching $TR = 1$ for large values of k. The peak value occurs at $k = m\omega_f^2/2 + \sqrt{(m\omega_f^2/2)^2 + (c\omega_f)^2}$. This will be greater than $m\omega_f^2$ if $c \neq 0$. Thus the peak occurs to the right of the $k = m\omega_f^2$ value. An appreciable reduction in the transmitted force is achieved only in the region for which $k < m\omega_f^2/2$ by light damping in addition to a small spring constant.

The effect of varying the mass can be also determined, from Eq. 4-92, by plotting TR against m for various values of c, resulting in the family of curves shown in Fig. 4-24. Here, the peak value occurs at $m = k/\omega_f^2$ and is given by

$$(TR)_{\max} = \frac{\sqrt{k^2 + (c\omega_f)^2}}{c\omega_f}$$

A reduction in the transmitted force occurs only in the region where m is large. This might not be expected. In this connection it is important to note that an increase in the mass m will, however, also result in increasing the static force carried by the support.

The force transmitted for the case of a rotating eccentric mass should also be investigated. The relation for F_T can be obtained by substituting

$$\zeta = \frac{c}{c_c} = \frac{0.60}{3.0} = 0.20$$

$$X_0 = \frac{P_0}{k} = \frac{25}{45} = 0.556 \text{ in.}$$

$$X_{\text{res}} = \frac{X_0}{2\zeta} = \frac{0.556}{2 \times 0.2} = 1.39 \text{ in.}$$

$$X_{\max} = \frac{X_0}{2\zeta\sqrt{1-\zeta^2}} = \frac{0.556}{2 \times 0.2\sqrt{1-(0.2)^2}} = \frac{1.39}{0.980} = 1.42 \text{ in.}$$

4-11. PHASE ANGLE

The value of the phase angle, as defined by Eq. 4-50, depends on the damping factor ζ and the frequency ratio r. This can be studied by plotting ψ against r for various values of ζ. The resulting family of curves is shown in Fig. 4-16. For no damping, $\psi = 0$ from $r = 0$ to $r < 1$, $\psi = 90$ degrees for $r = 1$, and $\psi = 180$ degrees for $r > 1$. This agrees with the analysis and discussion of Section 4-2. For small values of ζ, these same conditions are approximated; that is, the curve approaches the curve for the zero-damping case. All curves go through the point of $\psi = 90$ degrees for $r = 1$. Note that for $\zeta = 0.707$ the ψ curve is approximately linear from $r = 0$ through $r = 1$.

4-12. INFLUENCE OF MASS AND ELASTICITY ON AMPLITUDE

In determining the effect of varying r on the steady-state amplitude, recall that $r = \omega_f\sqrt{m/k}$, and hence r can be varied by changing k or m as well as ω_f. However, if either k or m is changed, this will alter ζ (as $\zeta = c/2\sqrt{mk}$) and distort the interpretation of Fig. 4-15, since a different ζ curve would then have to be used. In addition, altering k will change the reference value X_0. If it is desired to study the effect of varying k, the amplitude relation (Eq. 4-49) can be written in the form

$$X = \frac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \tag{4-71}$$

and X can then be plotted against k for several values of the damping constant c. The resulting family of curves is shown in Fig. 4-17. It should be noted that P_0, m, and ω_f are constant in this consideration. Maximum and minimum points on the curves can be obtained by setting $dX/dk = 0$. From this, it is found that the maximum point occurs for $k = m\omega_f^2$ and is defined by

$$X_{\max} = \frac{P_0}{c\omega_f} \tag{4-72}$$

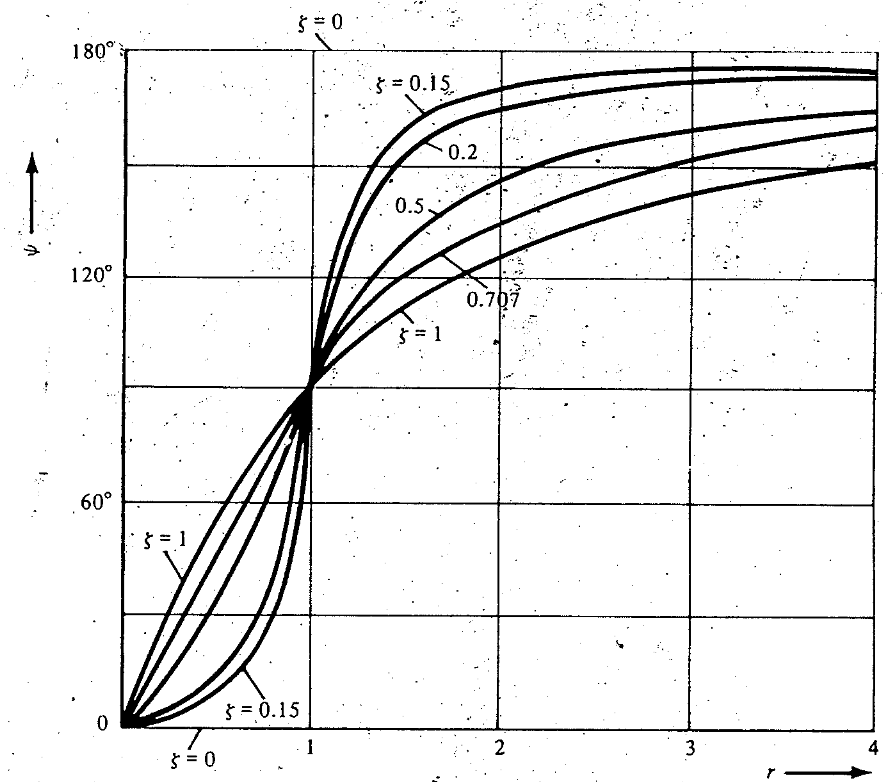


Figure 4-16

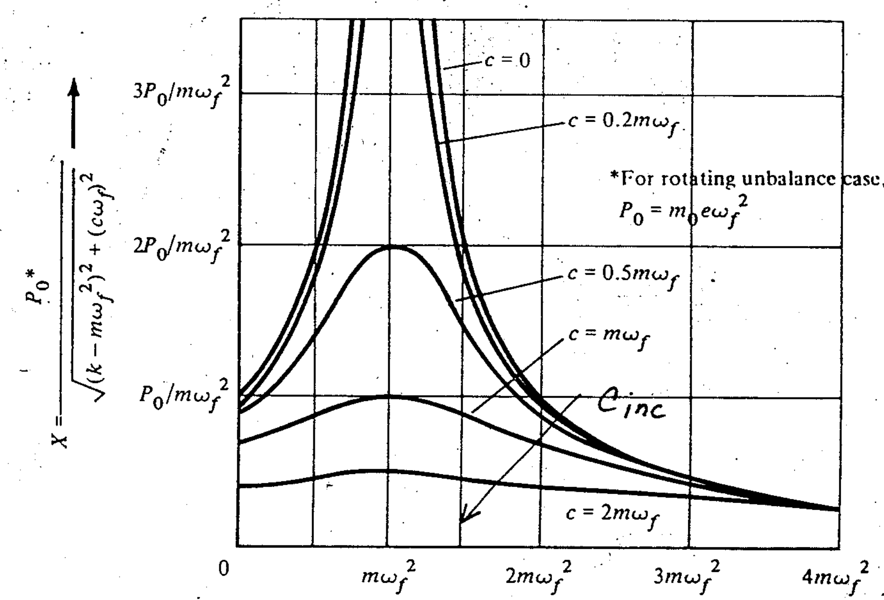


Figure 4-17

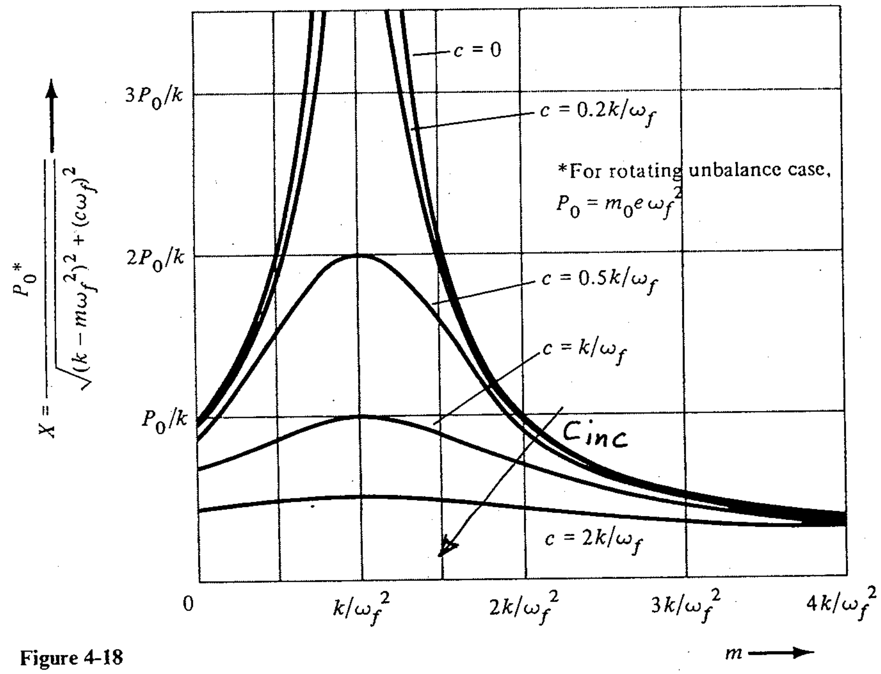


Figure 4-18

In addition, all the curves approach zero as k becomes large. The initial point (for $k = 0$) is given by

$$X = \frac{P_0}{\sqrt{(m\omega_f^2)^2 + (c\omega_f)^2}} \tag{4-73}$$

Reduction in amplitude is attained only as k becomes large. This means, as would be expected, that stiff springs will result in a small amplitude of motion for a given system.

The amplitude relation (Eq. 4-71) can also be used to observe the effect of varying m on the amplitude. In this case X can be plotted against m for various values of c, with P_0, k, and ω_f being taken as constant. The resulting family of curves is shown in Fig. 4-18. Maximum and minimum points on the curves can be determined by setting $dX/dm = 0$. The maximum point occurs at $m = k/\omega_f^2$ and is given by

$$X_{\max} = \frac{P_0}{c\omega_f} \tag{4-74}$$

All the curves approach zero as m becomes large. The initial point (for $m = 0$) is given by

$$X = \frac{P_0}{\sqrt{k^2 + (c\omega_f)^2}} \tag{4-75}$$

As might be anticipated, large values of m result in a reduction in the amplitude.

EXAMPLE 4-4 A machine weighing 19.3 lb is subjected to a harmonic force having a maximum value of 12 lb and a frequency of 300 cycles/min. The clearance for the vibrational movement of the machine is 1 in. Design a lightly damped elastic-support system for the machine, so that the machine does not collide in its movement.

SOLUTION In order to properly limit the movement of the machine, the allowable movement is set at one-half the actual clearance. Thus

$$X = 0.5 \text{ in.}$$

Also,

$$\omega_f = \frac{300}{60} \times 2\pi = 10\pi \text{ rad/sec}$$

For small damping, from Fig. 4-17, the value of c is selected as

$$c = 0.1m\omega_f = 0.1 \times \frac{19.3}{386} \times 10\pi = 0.05\pi \text{ lb sec/in.}$$
$$= 0.15708 \text{ lb sec/in.}$$

Then from Eq. 4-71,

$$X = \frac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}$$

$$0.5 = \frac{12}{\sqrt{[k - (19.3/386)(10\pi)^2]^2 + (0.05\pi \times 10\pi)^2}}$$

Expanding and rearranging gives

$$k^2 - 98.6960k + 1883.58 = 0$$

which has the single positive root

$$k = 25.861 \text{ lb/in.}$$

and the design is composed of an elastic support and damping device having the values of k and c determined.

EXAMPLE 4-5 A damped system is driven by the force $P = 0.54 \sin 12t$, where P is in newtons and t is in seconds. The system has a mass of 0.1 kg, and the damping constant is 0.24 N · s/m. (a) Obtain the steady-state amplitude for spring-constant k values of 2, 25, and 90 N/m. (b) Determine the spring constant that will produce the maximum amplitude, and calculate this amplitude.

$$\zeta = \frac{c}{c_c} = \frac{0.60}{3.0} = 0.20$$

$$X_0 = \frac{P_0}{k} = \frac{25}{45} = 0.556 \text{ in.}$$

$$X_{res} = \frac{X_0}{2\zeta} = \frac{0.556}{2 \times 0.2} = 1.39 \text{ in.}$$

$$X_{max} = \frac{X_0}{2\zeta\sqrt{1-\zeta^2}} = \frac{0.556}{2 \times 0.2\sqrt{1-(0.2)^2}} = \frac{1.39}{0.980} = 1.42 \text{ in.}$$

4-11. PHASE ANGLE

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4-12. INFLUENCE OF MASS AND ELASTICITY ON AMPLITUDE

In determining the effect of varying r on the steady-state amplitude, recall that $r = \omega_f\sqrt{m/k}$, and hence r can be varied by changing k or m as well as ω_f. However, if either k or m is changed, this will alter ζ (as $\zeta = c/2\sqrt{mk}$) and distort the interpretation of Fig. 4-15, since a different ζ curve would then have to be used. In addition, altering k will change the reference value X_0. If it is desired to study the effect of varying k, the amplitude relation (Eq. 4-49) can be written in the form

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and X can then be plotted against k for several values of the damping constant c. The resulting family of curves is shown in Fig. 4-17. It should be noted that P_0, m, and ω_f are constant in this consideration. Maximum and minimum points on the curves can be obtained by setting $dX/dk = 0$. From this, it is found that the maximum point occurs for $k = m\omega_f^2$ and is defined by

$$X_{max} = \frac{P_0}{c\omega_f} \tag{4-72}$$

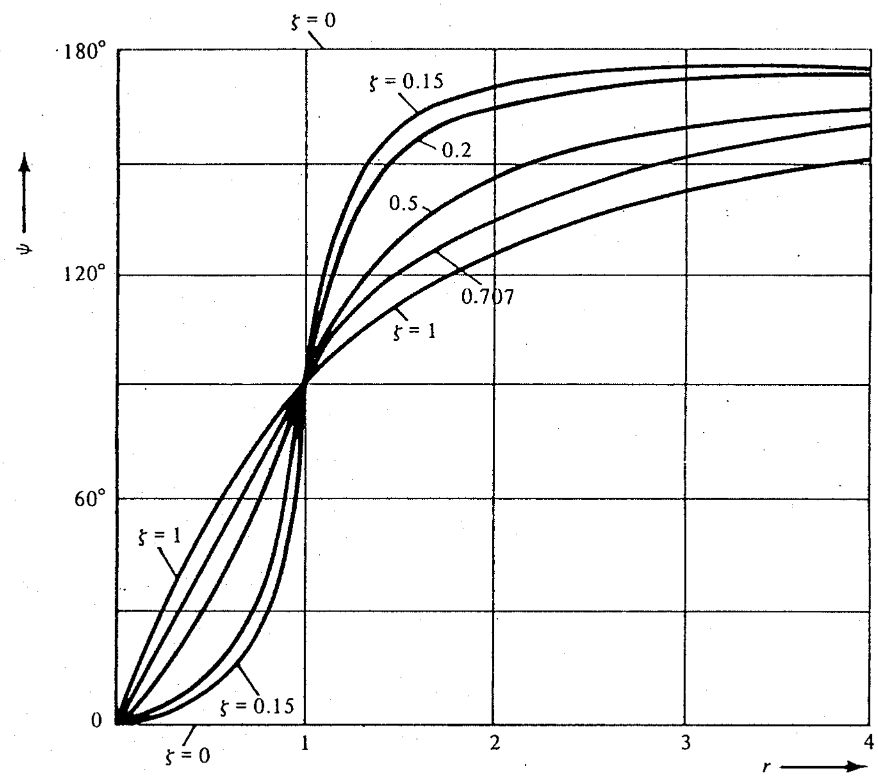

Figure 4-16

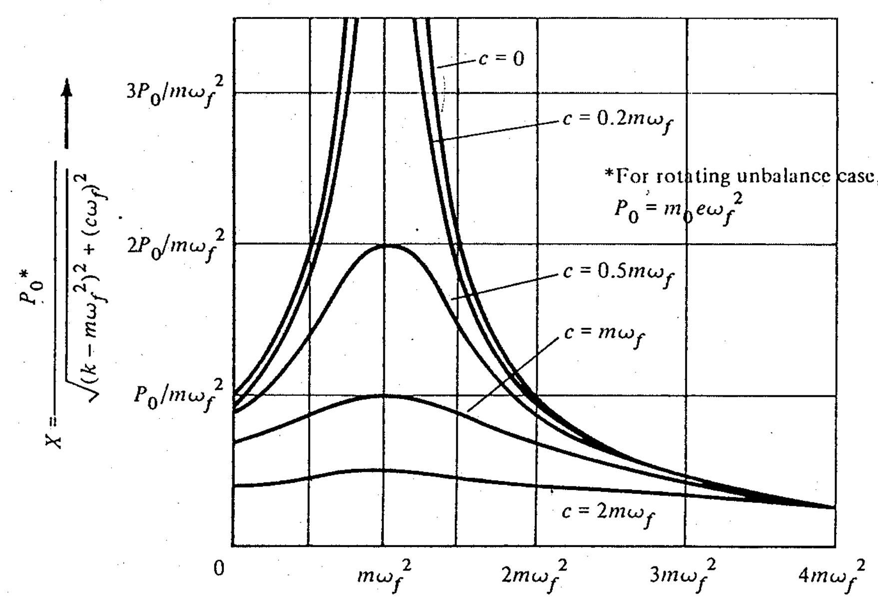

Figure 4-17

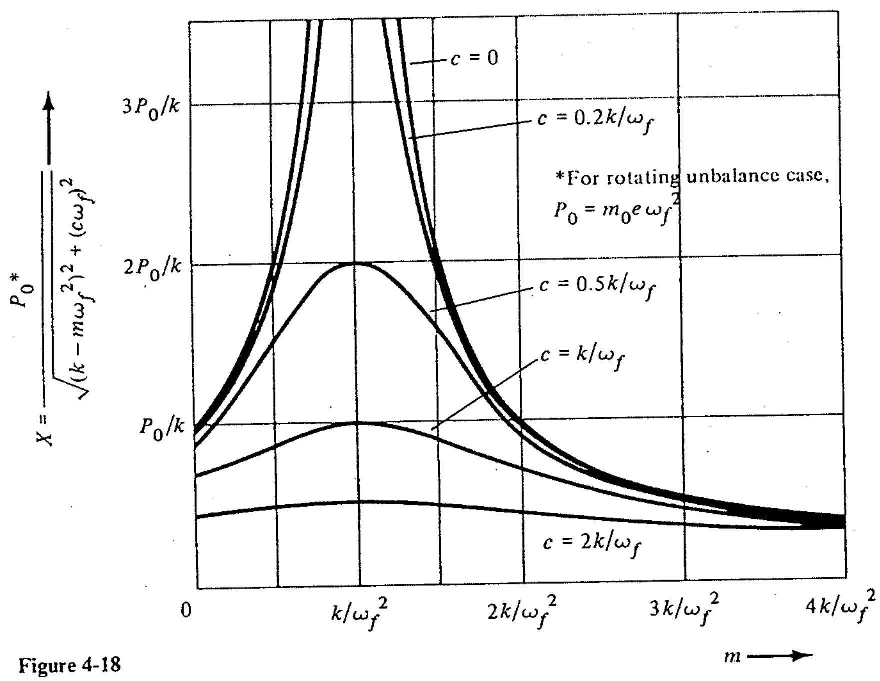

Figure 4-18

In addition, all the curves approach zero as k becomes large. The initial point (for $k = 0$) is given by

$$X = \frac{P_0}{\sqrt{(m\omega_f^2)^2 + (c\omega_f)^2}} \tag{4-73}$$

Reduction in amplitude is attained only as k becomes large. This means, as would be expected, that stiff springs will result in a small amplitude of motion for a given system.

The amplitude relation (Eq. 4-71) can also be used to observe the effect of varying m on the amplitude. In this case X can be plotted against m for various values of c, with P_0, k, and ω_f being taken as constant. The resulting family of curves is shown in Fig. 4-18. Maximum and minimum points on the curves can be determined by setting $dX/dm = 0$. The maximum point occurs at $m = k/\omega_f^2$ and is given by

$$X_{\max} = \frac{P_0}{c\omega_f} \tag{4-74}$$

All the curves approach zero as m becomes large. The initial point (for $m = 0$) is given by

$$X = \frac{P_0}{\sqrt{k^2 + (c\omega_f)^2}} \tag{4-75}$$

As might be anticipated, large values of m result in a reduction in the amplitude.

EXAMPLE 4-4 A machine weighing 19.3 lb is subjected to a harmonic force having a maximum value of 12 lb and a frequency of 300 cycles/min. The clearance for the vibrational movement of the machine is 1 in. Design a lightly damped elastic-support system for the machine, so that the machine does not collide in its movement.

SOLUTION In order to properly limit the movement of the machine, the allowable movement is set at one-half the actual clearance. Thus

$$X = 0.5 \text{ in.}$$

Also,

$$\omega_f = \frac{300}{60} \times 2\pi = 10\pi \text{ rad/sec}$$

For small damping, from Fig. 4-17, the value of c is selected as

$$c = 0.1m\omega_f = 0.1 \times \frac{19.3}{386} \times 10\pi = 0.05\pi \text{ lb sec/in.}$$
$$= 0.15708 \text{ lb sec/in.}$$

Then from Eq. 4-71,

$$X = \frac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}$$
$$0.5 = \frac{12}{\sqrt{[k - (19.3/386)(10\pi)^2]^2 + (0.05\pi \times 10\pi)^2}}$$

Expanding and rearranging gives

$$k^2 - 98.6960k + 1883.58 = 0$$

which has the single positive root

$$k = 25.861 \text{ lb/in.}$$

and the design is composed of an elastic support and damping device having the values of k and c determined.

EXAMPLE 4-5 A damped system is driven by the force $P = 0.54 \sin 12t$, where P is in newtons and t is in seconds. The system has a mass of 0.1 kg, and the damping constant is 0.24 N · s/m. (a) Obtain the steady-state amplitude for spring-constant k values of 2, 25, and 90 N/m. (b) Determine the spring constant that will produce the maximum amplitude, and calculate this amplitude.

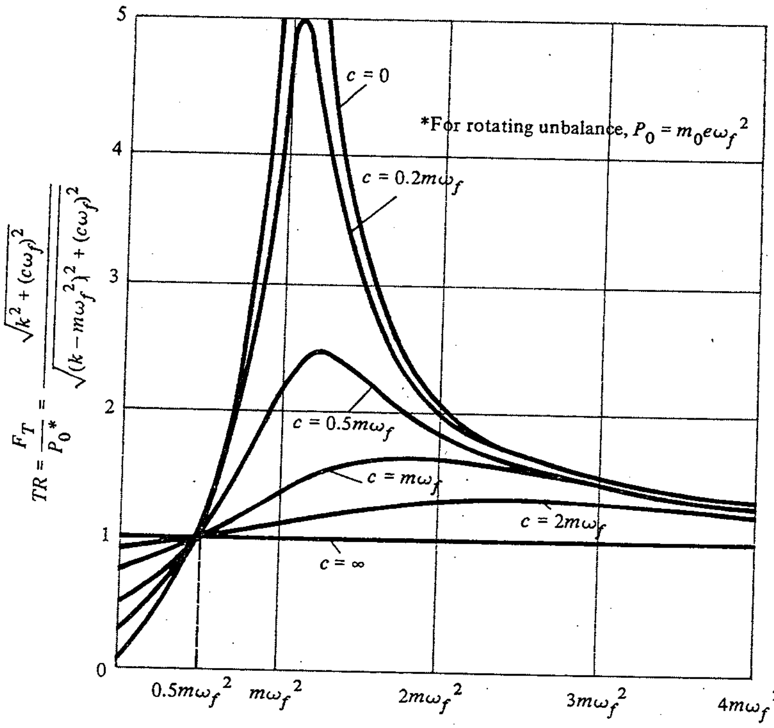

Figure 4-23

damping cuts down the peak force transmitted in the resonant region, it also results in greater force transmission for $r > \sqrt{2}$. This latter effect is opposed to the influence which the damping increase has in reducing the displacement amplitude for large values of r, as indicated by Fig. 4-15. It can be shown that the peak of the curve occurs at

$$r = \frac{\sqrt{-1 + (1 + 8\zeta^2)^{1/2}}}{2\zeta} < 1 \tag{4-91}$$

Consideration of Fig. 4-22 should be limited to conditions for which P_0 is constant and r is varied by changing ω_f only. If it is desired to observe the effect of varying the spring constant k, Eq. 4-88 can be used to plot TR against k for various values of the damping constant c. The transmissibility is then expressed as

$$TR = \frac{F_T}{P_0} = \frac{\sqrt{k^2 + (c\omega_f)^2}}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \tag{4-92}$$

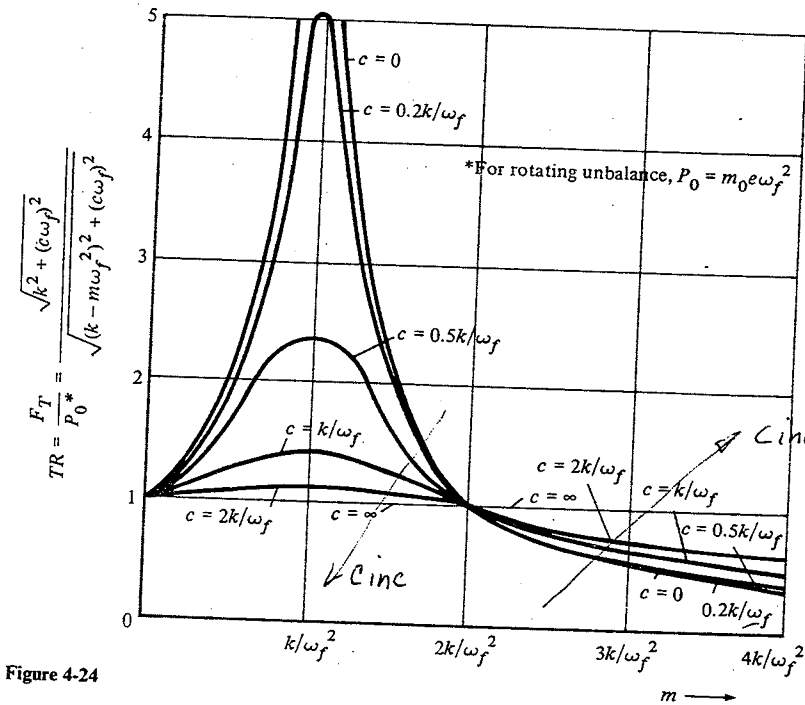

Figure 4-24

The resulting family of curves is shown in Fig. 4-23. The initial value of TR and the crossover point can be readily determined, as well as the trend of the curves in approaching $TR = 1$ for large values of k. The peak value occurs at $k = m\omega_f^2/2 + \sqrt{(m\omega_f^2/2)^2 + (c\omega_f)^2}$. This will be greater than $m\omega_f^2$ if $c \neq 0$. Thus the peak occurs to the right of the $k = m\omega_f^2$ value. An appreciable reduction in the transmitted force is achieved only in the region for which $k < m\omega_f^2/2$ by light damping in addition to a small spring constant.

The effect of varying the mass can be also determined, from Eq. 4-92, by plotting TR against m for various values of c, resulting in the family of curves shown in Fig. 4-24. Here, the peak value occurs at $m = k/\omega_f^2$ and is given by

$$(TR)_{\max} = \frac{\sqrt{k^2 + (c\omega_f)^2}}{c\omega_f}$$

A reduction in the transmitted force occurs only in the region where m is large. This might not be expected. In this connection it is important to note that an increase in the mass m will, however, also result in increasing the static force carried by the support.

The force transmitted for the case of a rotating eccentric mass should also be investigated. The relation for F_T can be obtained by substituting

SOLUTION

$$X = \frac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} = \frac{0.54}{\sqrt{[k - 0.1 \times (12)^2]^2 + (0.24 \times 12)^2}}$$

$$= \frac{0.54}{\sqrt{(k - 14.4)^2 + (2.88)^2}}$$

a. For $k = 2$, $X = 0.042\,42$ m $= 4.242$ cm
For $k = 25$, $X = 4.916$ cm
For $k = 90$, $X = 0.7138$ cm

b. $X_{\max} = \dfrac{P_0}{c\omega_f} = \dfrac{0.54}{0.24 \times 12} = 0.1875$ m $= 18.75$ cm

for $k = m\omega_f^2 = 0.1 \times (12)^2 = 14.4$ N/m

4-13. ROTATING UNBALANCE

A common source of forced vibration is caused by the rotation of a small eccentric mass such as that represented by m_0 in Fig. 4-19(a). This condition results from a setscrew or a key on a rotating shaft, crankshaft rotation, and many other simple but unavoidable situations. Rotating unbalance is inherent in rotating parts, because it is virtually impossible to place the axis of the mass center on the axis of rotation.

For the system shown, the total mass is m and the eccentric mass is m_0, so the mass of the machine body is $(m - m_0)$. The length of the eccentric arm, or the eccentricity of m_0, is represented by e. If the arm rotates with an angular velocity ω_f rad/sec, then the angular position of the arm is defined by $\omega_f t$ with respect to the indicated horizontal reference, where t is time, in seconds. The free-body diagram for this system is shown in Fig. 4-19(b), positive x having been taken as upward. The horizontal motion of $(m - m_0)$ is considered to be prevented by guides. The vertical displacement of m_0 is $(x + e \sin \omega_f t)$. From Eq. 1-8, the differential equation of motion can then be written as

$$(m - m_0)\frac{d^2x}{dt^2} + m_0 \frac{d^2}{dt^2}(x + e \sin \omega_f t) = -kx - c\frac{dx}{dt} \tag{4-76}$$

which can be rearranged in the form

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = m_0 e\omega_f^2 \sin \omega_f t \tag{4-77}$$

Examination of this and comparison to the differential equation (Eq. 4-38) for motion forced by $P = P_0 \sin \omega_f t$ enable the steady-state solution to be set down, from Eq. 4-48, as

$$x = X \sin (\omega_f t - \psi) \tag{4-78}$$

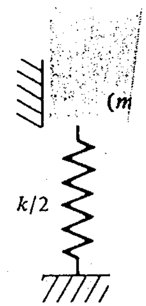

Figure 4-

where

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3.42

$k = 10^6\ N/m \qquad c = 2000\ \frac{N\text{-}sec}{m} \qquad m_0 = 0.1\,kg \qquad m = 100\,kg$

$f_f = 3000\ rpm \qquad e = 10cm = .1m$

$$m\ddot{x} + c\dot{x} + kx = m_0 e \omega_f^2 \sin \omega_f t$$

$$x = \frac{m_0 e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \psi)$$

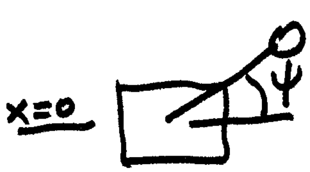
$$= \frac{m_0 e \omega_f^2}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \sin(\omega_f t - \psi)$$

$$\tan \psi = \frac{2\zeta r}{1 - r^2} = \frac{c\omega_f}{k - m\omega_f^2}$$

$$\omega_f = \frac{3000\ rpm \times 2\pi\ rad/rev}{60\ sec/min} = 100\pi = 314.16\ rad/sec$$

$$x = \frac{(0.1\,kg)(0.1\,m)(314.16)^2}{\sqrt{\left(10^6 - 100(314.16)^2\right)^2 + (2000 \cdot 314.16)^2}} \sin(314.16t - \psi$$

$$= \underbrace{\boxed{110.996 \times 10^{-6}\ m}}_{X_{RU}} \sin(\omega_f t - \psi)$$



$$\psi = \tan^{-1} \frac{c\omega_f}{k - m\omega_f^2} = \tan^{-1} \frac{(2000 \cdot 314.16)}{10^6 - 100(314.16)^2} = -.0707\,rad$$

$$= -4.05^\circ$$

Figure P7.29

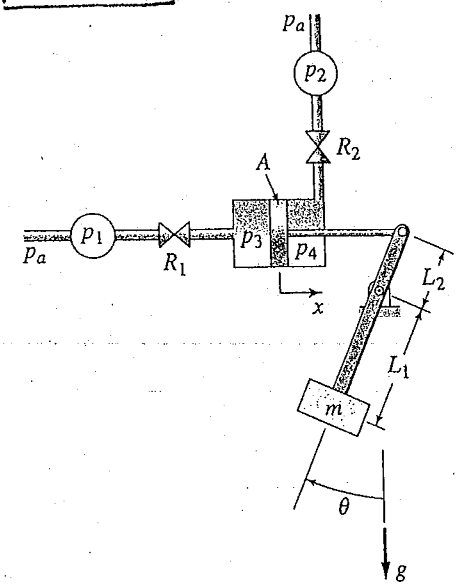


Figure P7.30

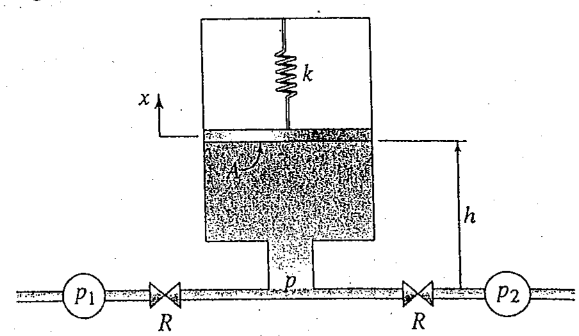
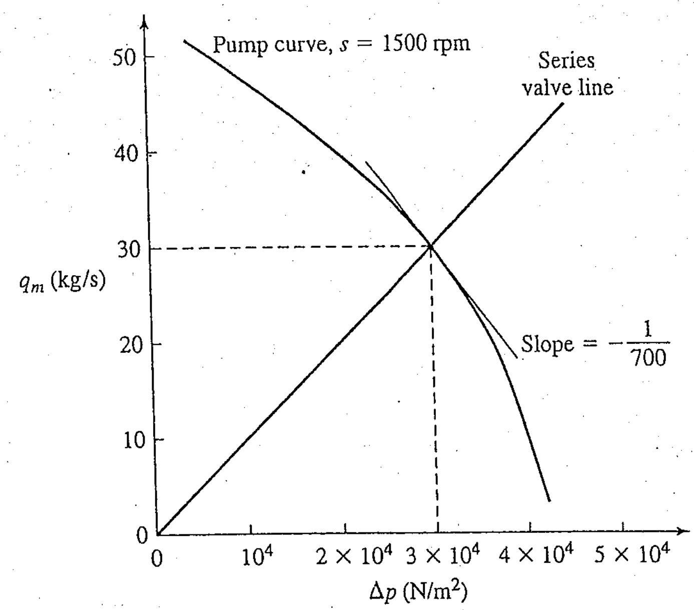


Figure P7.32



7.33 Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.33. The outlet resistance is linear. Derive the dynamic model of the height h.

7.34 Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.34. The outlet is an orifice of area A_o and discharge coefficient C_d. Derive the dynamic model of the height h.

7.35 Consider the cylindrical container treated in Problem 7.8. In Figure P7.35 the tank is shown with a valve outlet at the bottom of the tank. Assume that the flow through the valve is turbulent with a resistance R. Derive the dynamic model of the height h.

7.27 An electric motor is sometimes used to move the spool valve of a hydraulic motor. In Figure P7.27 the force f is due to an electric motor acting through a rack-and-pinion gear. Develop a model of the system with the load displacement y as the output and the force f as the input. Consider two cases: (a) $m_1 = 0$ and (b) $m_1 \neq 0$.

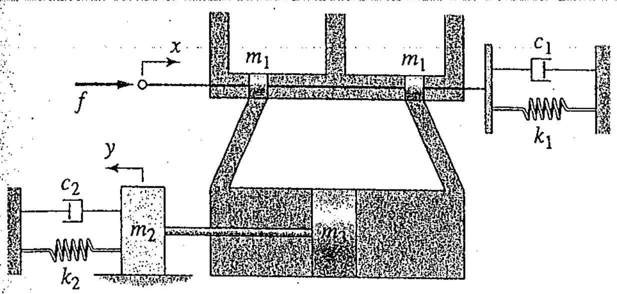


Figure P7.27

7.28 In Figure P7.28 the piston of area A is connected to the axle of the cylinder of radius R, mass m, and inertia I about its center. Develop a dynamic model of the axle's translation x, with the pressures p_1 and p_2 as the inputs.

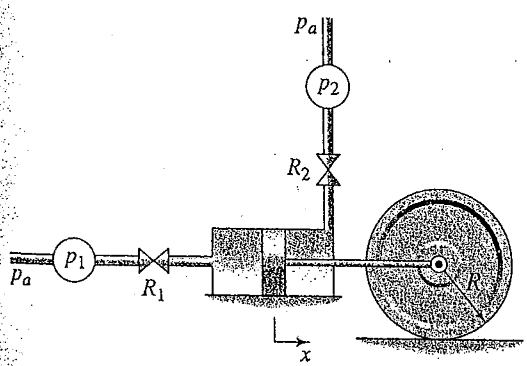


Figure P7.28

7.29 Figure P7.29 shows a pendulum driven by a hydraulic piston. Assuming small angles θ and a concentrated mass m a distance L_1 from the pivot, derive the equation of motion with the pressures p_1 and p_2 as inputs.

7.30 Figure P7.30 shows an example of a *hydraulic accumulator*, which is a device for reducing pressure fluctuations in a hydraulic line or pipe. The fluid density is ρ, the plate mass is m, and the plate area is A. Develop a dynamic model of the pressure p with the pressures p_1 and p_2 as the given inputs. Assume that $m\ddot{x}$ of the plate is small, and that the hydrostatic pressure ρgh is small.

7.31 Design a hydraulic accumulator of the type shown in Figure P7.30. The liquid volume in the accumulator should increase by 30 in.^{3} when the pressure p increases by 1.5 lb/in.^{2}. Determine suitable values for the plate area A and the spring constant k.

7.32 Consider the liquid-level system treated in Example 7.4.8 and shown in Figure 7.4.9. The pump curve and the line for the steady-state flow through both valves are shown in Figure P7.32. It is known that the bottom area of the tank is 2 m^2 and the outlet resistance is $R_2 = 400\ 1/(\mathrm{m}\cdot\mathrm{s})$. (a) Compute the pump resistance R_1 and the steady-state height. (b) Derive a linearized dynamic model of the height deviation δh in the tank.

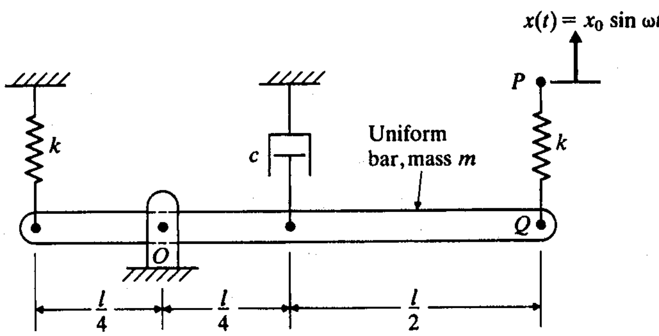


FIGURE 3.53

given by 10^6 N/m and 2000 N-s/m, respectively. If the unbalance of the compressor is equivalent to a mass 0.1 kg located at the end of the crank (point A), determine the response of the compressor at a crank speed of 3000 rpm. Assume $r = 10$ cm and $l = 40$ cm.

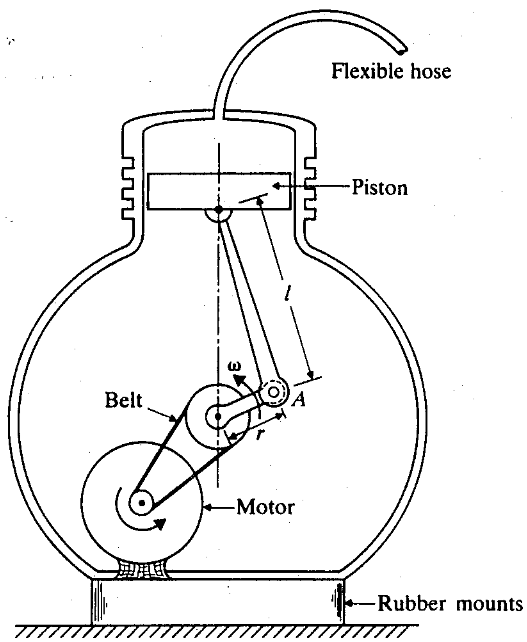


FIGURE 3.54

3.43 One of the tail rotor blades of a helicopter has an unbalanced mass of $m = 0.5$ kg at a distance of $e = 0.15$ m from the axis of rotation, as shown in Fig. 3.55. The tail section has a length of 4 m, a mass of 240 kg, a flexural stiffness (EI) of 2.5

MN – m^2, and a damping ratio of 0.15. The mass of the tail rotor blades, including their drive system, is 20 kg. Determine the forced response of the tail section when the blades rotate at 1500 rpm.

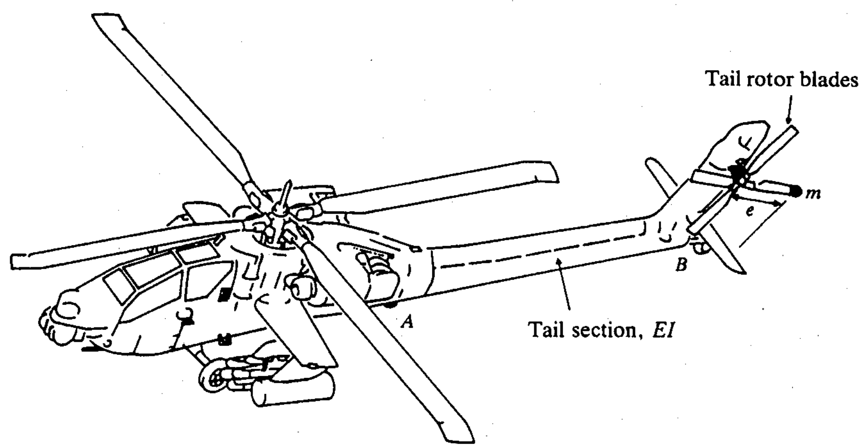


FIGURE 3.55

3.44 When an exhaust fan of mass 380 kg is supported on springs with negligible damping, the resulting static deflection is found to be 45 mm. If the fan has a rotating unbalance of 0.15 kg-m, find (a) the amplitude of vibration at 1750 rpm, and (b) the force transmitted to the ground at this speed.

3.45 A fixed-fixed steel beam, of length 5 m, width 0.5 m, and thickness 0.1 m, carries an electric motor of mass 75 kg and speed 1200 rpm at its mid-span, as shown in Fig. 3.56. A rotating force of magnitude $F_0 = 5000$ N is developed due to the unbalance in the rotor of the motor. Find the amplitude of steady-state vibrations by disregarding the mass of the beam. What will be the amplitude if the mass of the beam is considered?

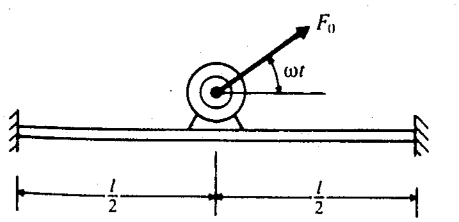


FIGURE 3.56

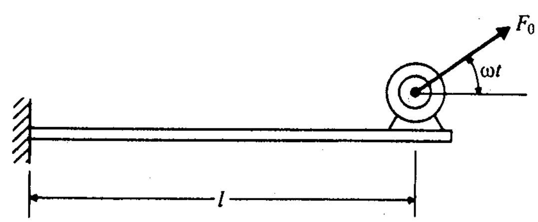


FIGURE 3.57

3.46* If the electric motor of Problem 3.45 is to be mounted at the free end of a steel cantilever beam of length 5 m (Fig. 3.57), and the amplitude of vibration is to be limited to 0.5 cm, find the necessary cross-sectional dimensions of the beam. Include the weight of the beam in the computations.

* (3.42) Equation of motion: $M\ddot{x} + c\dot{x} + kx = m e \omega^2 \sin \omega t$

where $\omega_f = \dfrac{3000\,(2\pi)}{60} = 314.16$ rad/sec, $M = 100$ kg, $c = 2000$ N-s/m, $k = 10^6$ N/m

$m = 0.1$ kg and $e = r = 0.1$ m. Steady state response is:

$$x_p(t) = X \sin(\omega t - \phi)$$

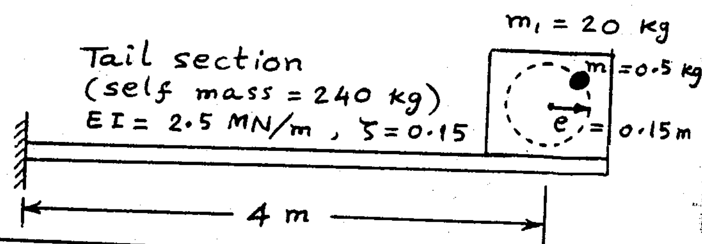
.spwhere $X = \dfrac{m e \omega_f^2}{\left[(k - M\omega_f^2)^2 + (c\,\omega_f)^2\right]^{\frac{1}{2}}}$

$$= \frac{0.1\,(0.1)\,(314.16^2)}{\left[\left\{10^6 - 100\,(314.16^2)\right\}^2 + (2000\,(314.16))^2\right]^{\frac{1}{2}}} = 110.9960\,(10^{-6})\text{ m}$$

and $\phi = \tan^{-1}\left(\dfrac{c\,\omega}{k - M\omega^2}\right) = \tan^{-1}\left(\dfrac{2000\,(314.16)}{10^6 - 100\,(314.16^2)}\right)$

$= -\,0.07072$ rad $= -\,4.0520°$

(3.43) k = spring constant of cantilever beam



$$= \frac{3EI}{l^3} = \frac{3\,(2.5\times 10^6)}{4^3}$$

$= 0.1172 \times 10^6$ N/m

$$\omega_n = \sqrt{\frac{k}{m_1 + 0.25\, m_b}} = \sqrt{\frac{0.1172\times 10^6}{20 + 0.25\,(240)}} = 38.2753 \text{ rad/sec}$$

$\omega = 2\pi\,(1500)/60 = 157.08$ rad/sec

$r = \omega/\omega_n = 157.08/38.2753 = 4.1040$, $r^2 = 16.8428$

Forced response is given by Eq. (3.79) :

$$x_p(t) = X \sin(\omega t - \phi)$$

where

$$X = \frac{m e}{m_1} \cdot \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$= \frac{(0.5)(0.15)}{20}\;\frac{16.8428}{\sqrt{(1 - 16.8428)^2 + (2\times 0.15\times 4.1040)^2}}$$

$= 3.9747 \times 10^{-3}$ m $= 3.9747$ mm

Problem ~~3.43~~ 3.52 Erratum

- The main mass for this problem is the blades drive system + the rotor blades LESS the eccentric mass, i.e., $20\,kg - 0.5\,kg = 19.5\,kg = (m - m_0)$
- The eccentric mass $m_0 = 0.5\,kg$, $e = 0.15\,m$
- $\zeta_{TAIL} = 0.15$

The tail section ONLY acts to support the drive system and blades and thus provides the elasticity (spring equivalent) and the damping (viscous equivalent) for the mass (main + eccentric). Tail information is used to get k

$$k_{eq} = \frac{3EI}{\ell^3} = \frac{3\,(2.5\times10^6\,N\text{-}m^2)}{(4)^3\ m^3} = 1.172 \times 10^5\ \frac{N}{m}$$

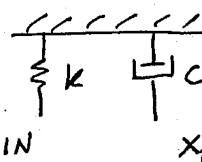
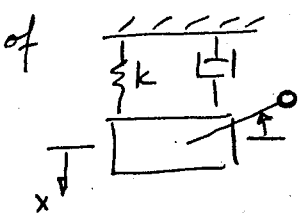
- the tail has weight and so it acts like a spring having weight. This is important for $\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}}$

$$m_{eq} = m_{\substack{drivesys\\+blades}} + 0.25\, m_{tail} = 20 + 0.25(240) = 80\,kg$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{1.172\times10^5}{80}} = 38.27\ rad/sec$$

$$\omega_f = 1500\,\frac{rev}{min} \times \frac{2\pi\,rad}{rev} \times \frac{1\,min}{60\,sec} = 50\pi\ rad/sec = 157.08\ rad/sec$$

$$r = \frac{\omega_f}{\omega_n} = \frac{157.08}{38.27} = 4.1$$

- This work so far is like finding  part of 
- THE MASS OF TAIL DOES NOT PLAY A PART IN x_p
- Now to find the actual forced response, we need to use

$$x_p = \frac{m_0 e}{m} \cdot \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\ \sin(\omega_f t - \psi)$$

where $m = m_{drivesy+blade} = 20\,kg$
$m_0 = 0.5\,kg$ $\quad r = 4.1$
$e = 0.15\,m$ $\quad \zeta = 0.15$
$\longrightarrow\ X_{RU} = 0.003975\,m$

when $\omega_f \to \infty \Rightarrow r \to \infty$ $\quad X_{RU} = .00375\,m = \frac{m_0 e}{m}$

$$\psi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = -4.45^\circ$$

WHEN MAIN MASS IS AT X=0, ECCENTRICITY WILL BE AT AN ANGLE OF 4.45° BELOW HORIZONTAL

3.17 $W = 500\,lb$ $F(t) = 200 \sin 100\pi t$ $X < .1\,in$

Choose $X = .05\,in$

$$X_{max} = \frac{X_0}{2\zeta\sqrt{1-\zeta^2}} = .05\,in \qquad X_0 = P_0/K$$

Chose $\zeta = 0.01$ Then $X_0 = \frac{P_0}{k_{eq}} = \frac{200}{k_{eq}}$

$$X_{max} = \frac{X_0}{2(.01)\sqrt{1-(.01)^2}} = \frac{X_0}{.02} = \frac{200}{k_{eq}(.02)} = .05\,in \therefore\ k_{eq} = \frac{2000}{.05(.02)} = 20\times10^4\,\frac{lb}{in}$$

$$\frac{k_{eq}}{3} = k = 6.667\times10^4\,\frac{lb}{in}$$ since 3 springs

$$\zeta = \frac{c_{eq}}{c_{cr}} \qquad c_{cr} = 2\sqrt{k_{eq}m} = 2\sqrt{\left(20\times10^4\,\frac{lb}{in}\right)\left(\frac{500\,lb}{\underbrace{386.4\,\frac{in}{sec^2}}_{32.2\times12}}\right)} = \cancel{7195}\ 719.5\ \frac{lb\text{-}sec}{in}$$

$$c_{eq} = \zeta c_c = .01(\cancel{7195}\ 719.5) = 7.195\,\frac{lb\text{-}sec}{in}$$

$$c = \frac{c_{eq}}{3} = 2.4\,\frac{lb\text{-}sec}{in}$$ since 3 absorbers

Appendix: Nomenclature

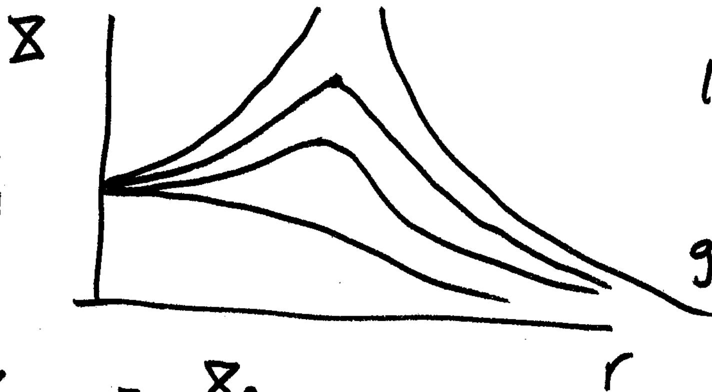
b	width of the beam
C_1	$= E_1h_1 + E_3h_3 + E_a^*h_4$
C_3	$= E_3h_3 + E_a^*h_4$
d	$= h_2 + (h_1 + h_3)/2$
d_1	$= h_2 + h_3 + (h_1 + h_4)/2$
D_t	$= (E_1h_1^3 + E_3h_3^3)/12$
E_a^*	Young's modulus of the shape memory layer
E_i	Young's modulus of the ith layer (i = 1,3)
G	$= G_o(1 + j\eta_c)$, complex shear modulus of the damping layer
G_o	real shear modulus of damping layer
h_i	thickness of the ith layer (i = 1,2,3,4)
H_i	dimensionless thickness equal to h_i / h_3 (i = 1,2,4)
j	$= \sqrt{-1}$
k, k_i	complex characteristic value
L	length of the beam
m	mass per unit length of the beam
q	applied transverse loading
S	shear force
t	time
u_i	axial displacement of the ith layer (i = 1,3,4)
w	transverse displacement for the system
x	axial or longitudinal coordinate

Greek Symbols

η	system loss factor
η_c	material loss factor of damping layer
ρ_i	mass density of the ith layer
σ_r^*	stress of shape memory layer
Ω, ω	complex and real resonance frequency of the system

3.17 $W = 500\text{ lb}$ $\quad F(t) = 200 \sin 100\pi t = P_0 \sin \omega_f t$ $\quad X \le 0.1\text{ in}$

in order to insure $X < 0.1\text{in}$ take $\underline{X_{max} = 0.05\text{in}}$



local max if $\zeta < .707$

go to zero as $r \to \infty$

$$X_{max} = \frac{X_0}{2\zeta\sqrt{1-\zeta^2}} = 0.05\text{in}$$

Choose $\zeta = 0.01$ then

$$X_{max} = 0.05 = \frac{X_0}{2(.01)\sqrt{1-(.01)^2}} \cong \frac{X_0}{.02} = \frac{P_0/k_{eq}}{.02} = \frac{200}{.02k}$$

$$k_{eq} = \frac{10000}{.05} = 20 \times 10^4 \frac{\text{lb}}{\text{in}}$$

$$K = \frac{k_{eq}}{3} = 6.667 \times 10^4 \frac{\text{lb}}{\text{in}} \quad \text{since 3 springs}$$

$$\zeta = \frac{c_{eq}}{c_c} \qquad c_c = 2\sqrt{k_{eq} m} = 2\sqrt{20\times10^4 \frac{\text{lb}}{\text{in}} \cdot \frac{500\text{ lb}}{386.4 \frac{\text{in}}{\text{sec}^2}}}$$

$$= 719.5 \frac{\text{lb-sec}}{\text{in}}$$

$$c_{eq} = \zeta c_c = 0.01\, c_c = 7.195 \frac{\text{lb-sec}}{\text{in}}$$

$$c = \frac{c_{eq}}{3} \cong 2.4 \frac{\text{lb-sec}}{\text{in}}$$

3.44 $\delta_{st} = \frac{45}{1000}\, m = \frac{Mg}{k} = \frac{380 \times 9.81}{k}$

i.e., $k = 82,840$ N/m

$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{82,840}{380}} = 14.7648$ rad/sec ; $\omega = \frac{2\pi(1750)}{60} = 183.26$ rad/sec

$r = \frac{\omega}{\omega_n} = \frac{183.26}{14.7648} = 12.412$; $r^2 = 154.0566$

(i) Amplitude of vibration

$$X = \frac{me}{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.15}{380} \frac{154.0566}{\sqrt{(153.0566)^2 + 0}}$$

$= 3.9732 \times 10^{-4}$ m

(ii) Force transmitted to ground

$= kX = (82840)(3.9732 \times 10^{-4}) = 32.9140$ N

3.45* $I = \frac{1}{12}(0.5)(0.1)^3 = 0.4167 \times 10^{-4}$ m^4

$k = \frac{192\, EI}{l^3} = \frac{192\,(2.07 \times 10^{11})(0.4167 \times 10^{-4})}{(5)^3} = 1.3248 \times 10^7$ N/m

(a) $\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{13.248 \times 10^6}{75}} = 420.2856$ rad/sec

$\omega_f = 2\pi(1200)/60 = 125.664$ rad/sec

$r = \omega/\omega_n = 125.664/420.2856 = 0.299$, $r^2 = 0.0894$

Amplitude of steady-state vibration is given by Eq. (3.30) with $\zeta = 0$:

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k|(r^2-1)|} = \frac{5000}{(1.3248 \times 10^7)(0.9106)}$$

$= 0.4145 \times 10^{-3}$ m

$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = 0$ since no damping

(b) Using the effective mass due to self weight of beam (for a cantilever) to be valid here also,

$$\omega_n = \sqrt{\frac{k}{M + 0.2357\, m}}$$

where M = mass of motor = 75 kg , and

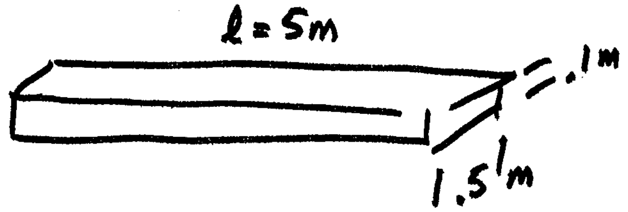
m = mass of beam = $\underset{\text{Vol}}{(5 \times 0.5 \times 0.1)}\left(\frac{76.5 \times 10^3}{9.81}\right) = 1949.5313$ kg

$$\omega_n = \sqrt{\frac{13.248 \times 10^6}{75 + (1949.5313)(0.2357)}} = 157.4339 \text{ rad/sec}$$

$r = \omega/\omega_n = 125.664/157.4339 = 0.7982$, $r^2 = 0.6371$

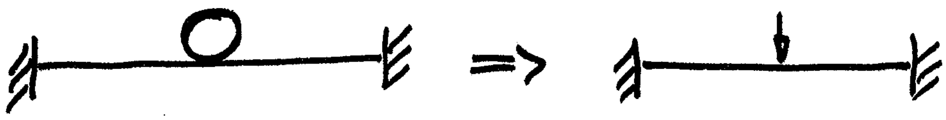
$$X = \frac{F_0/k}{|r^2 - 1|} = \frac{5000}{(1.3248 \times 10^7)(.3629)} = 1.04 \times 10^{-3} \text{ m}$$

3.45



$$I = \frac{1}{12} bh^3 = \frac{1}{12}(.5)(.1)^3 = 0.417 \times 10^{-4}\, m^4$$

$m = 75\,kg \quad f_f = 1200\, rpm \quad F_0 = 5000\, N = m_0 e \omega_f^2$



$$k_{eq} = \frac{192 EI}{l^3} = \frac{192\,(2.07 \times 10^{11}\, N/m^2)(0.417 \times 10^{-4}\, m^4)}{5^3\, m^3}$$

$$= 1.325 \times 10^7\, N/m$$

$$x = X_{RU} \sin(\omega_f t - \psi) = \frac{m_0 e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \psi)$$

when no damping $\quad x = \frac{m_0 e}{m} \cdot \frac{r^2}{|r^2 - 1|} \sin(\omega_f t - \psi)$

$r = ? = \frac{\omega_f}{\omega_n}\ ; \quad \omega_f = 1200 \times \frac{2\pi}{60} = 125.66\, rad/s$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1.325 \times 10^7}{75}} = 420.29\, rad/s$$

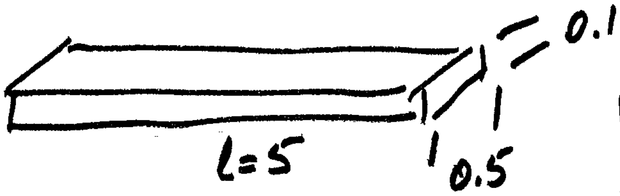
$r < 1 \cong 0.3$

$$X_{RU} = \frac{m_0 e}{m} \frac{r^2}{1 - r^2} = \frac{m_0 e \omega_f^2}{k(1 - r^2)} = \frac{F_0/k}{1 - r^2} = \frac{5000\, N/k}{1 - (.3)^2}$$

$$= \frac{5000}{(1.325 \times 10^7)(.91)} = 0.415 \times 10^{-3}\, m$$

since no damping $\psi = 0$

results without including mass of beam



Volume $= 5 \times 0.1 \times 0.5 = .25 \, m^3$

$\rho \cong 7.6 \times 10^3 \, \frac{kg}{m^3}$

$m_{beam} = 1949.5 \, kg$

before

$$\omega_n = \sqrt{\frac{k}{m_{rotor}}}$$

now

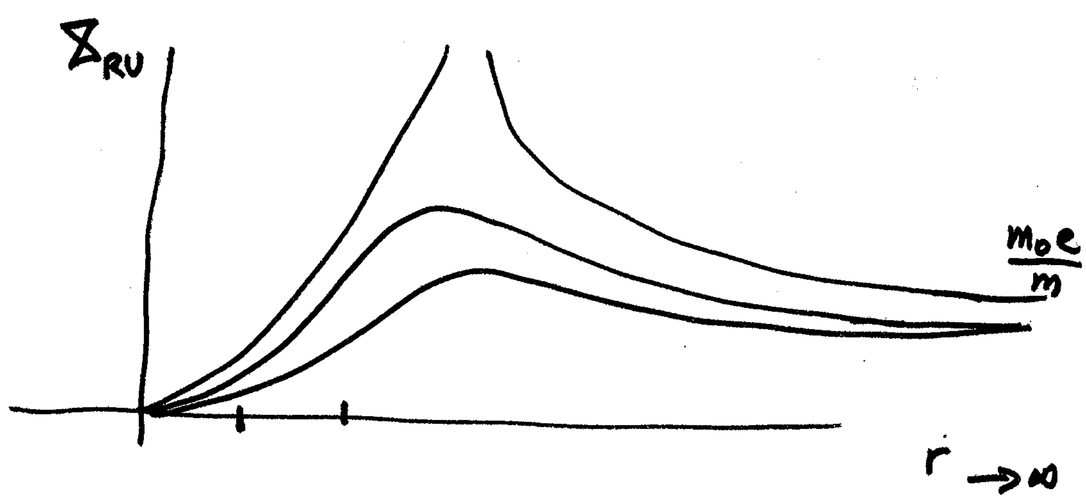
$$\omega_n = \sqrt{\frac{k}{m_{rotor} + 0.2357 \, m_{beam}}}$$

$$= \sqrt{\frac{1.325 \times 10^7}{75 \, kg + 0.2357 \, (1949.53)}}$$

$$= 157.43 \text{ rad/sec}$$

$$r = \frac{\omega_f}{\omega_n} = \frac{125.66}{157.43} = .798$$

$$X_{RU} = \frac{F_0}{k \, |1 - r^2|} = \frac{5000}{(1.325 \times 10^7)(.363)} = 1.04 \times 10^{-3} \, m$$



FOR 4-32

$$F_T = \frac{P_o \sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = m_o e \omega_f^2 \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$\omega_f = 42.003$ rad/sec $\quad \zeta = .25 \quad m_o e = \frac{W_o e}{g} = \frac{5 \text{ lb-in}}{32.2 \frac{\text{lb}}{\text{sec}^2} \cdot 12 \frac{\text{in}}{\text{ft}}} = .01294$ lb-sec^2

$P_o = m_o e \omega_f^2 = 22.83$ lb $\qquad F_T = 39.10$ lb

SESSION # 15 EXAM

SESSION # 16 REVIEW OF EXAM + THIS

- For the rotating unbalance remember that $P_o = m_o e \omega_f^2$

$$X = \frac{P_o}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}} = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

- we also said

$$F_T = \frac{P_o \sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{m_o e k}{m} \frac{r^2 \sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$P_o = m_o e \frac{\omega_f^2}{\omega_n^2} \omega_n^2 = \frac{m_o e k}{m} r^2$$

- if we wanted to plot $\left(F_T / \frac{m_o e k}{m}\right)$ vs. r

NOTE FOR $\zeta \neq 0$, $r \to \infty$ $\quad \sim \frac{r^2 \cdot O(2\zeta r)}{O(r^2)} \to \infty$ as $r \to \infty$

FOR $\zeta = 0$ $\quad \sim \frac{r^2}{O(r^2)} \to 1$ as $r \to \infty$

NOTE HERE THAT DAMPING <u>INCREASES</u> THE TRANSMITTED FORCE & MORE RAPIDLY

- <u>CAVEAT</u> r VARIATION $\Rightarrow$ ω_f variation k, m are fixed

- NOTE THAT WHEN $r = \sqrt{2}$ $\quad \frac{r^2 \sqrt{}}{\sqrt{}} = 2 \quad \forall \zeta$

- NOTE THAT $r = 0$ $\quad \left(F_T / \frac{m_o e k}{m}\right) = 0 \quad \forall \zeta$

LOOK AT P. 125 $\quad \dfrac{d}{dr}\left(F_T \Big/ \dfrac{m_o e k}{m}\right) = 0$

- $r=0$ is a relative minimum
- $0 < \zeta < \sqrt{2}/4$ $\quad$ FOR $0 < r < \sqrt{2}$ RELATIVE MAX
  $\quad\quad r > \sqrt{2}$ " MIN
- $\zeta > \sqrt{2}/4$ $\quad$ NO RELATIVE MAX ONLY AN ABSOLUTE MAX AT $r = \infty$

- FIGURE ON P. 124 IS PLOT OF $\left(F_T \Big/ \dfrac{m_o e k}{m}\right)$ VS. r

  NOTE IN PRACTICE $r \geq 10$ $\zeta \neq 0$ WE HAVE LARGE TRANSMITTED FORCE

- IF WE VARY k or m. LOOK AT $\dfrac{F_T}{m_o e \omega_f^2} = \dfrac{\sqrt{k^2 + (c\omega_f)^2}}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} = \dfrac{F_T}{P_o}$

  THIS IS SAME AS WHAT WE DID EARLIER FOR TRANSMITTED FORCE P122 / 123

SESSION #17

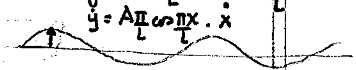
OSCILLATING SUPPORT

- COMMON SOURCE OF VIBS

EXAMPLES $\quad$ VEHICLES $\quad$ NO STREET IS FLAT

$y = A\sin\frac{\pi x}{L} = A\sin\frac{\pi v}{L}t$

$\dot{y} = A\frac{\pi}{L}\cos\frac{\pi x}{L} \cdot \dot{x}$

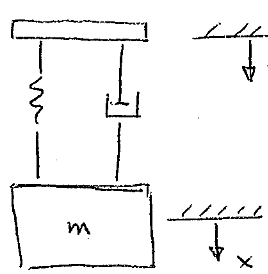


SHIPS $\quad$ OCEAN DUE TO TROUGHS & CRESTS

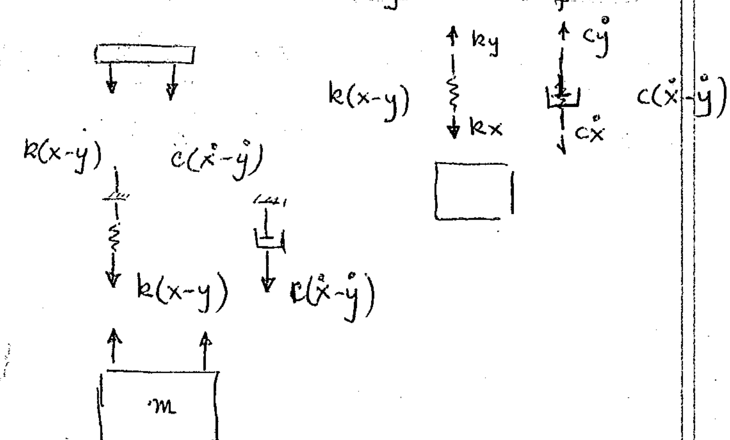
AIRCRAFT

SUPPORT

Mass



ABSOLUTE DISP. OF SUPPORT $\quad y = Y\sin\omega_f t$



$x-y$ IS RELATIVE MOTION

$$m\ddot{x} = \sum \text{forces} = -k(x-y) - c(\dot{x}-\dot{y}) \quad \text{or} \quad m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$

$$= kY\sin\omega_f t + c\omega_f Y\cos\omega_f t$$

$$= Y\left[k\sin\omega_f t + c\omega_f\cos\omega_f t\right]$$

$$\ddot{y} = mY\omega_f^2 \sin \omega_f t$$

$$z\omega_n^2 = \frac{Y\omega_f^2}{\sqrt{\quad}} \sin(\omega_f t - \psi)$$

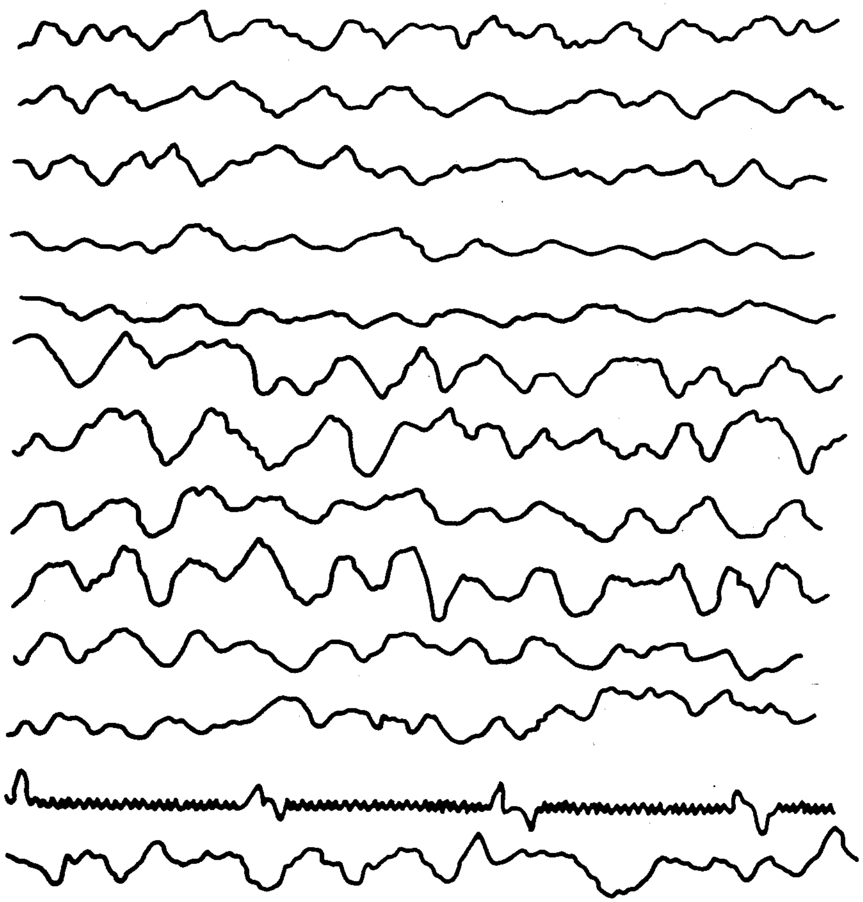


Figure 6-9

traces as well as for displacement records. In such analyses, various methods† are employed, including numerical procedures, superposition, the envelope method, and so forth. Mechanical and other aids are also used to facilitate these determinations.

PROBLEMS

6-1. Determine the percent error of a vibrometer for the frequency ratios of 3, 5, and 10 for each of the following damping factors: (a) $\zeta = 0$, (b) $\zeta = 0.6$, and (c) $\zeta = 0.7$.

6-2. For a vibrometer, determine the greatest percent error that occurs for the range of frequency ratio from $r = 3$ to $r = \infty$. Also obtain the corresponding value of r at which this occurs. Do this for each of the following damping factors: (a) $\zeta = 0$, (b) $\zeta = 0.6$, and (c) $\zeta = 0.7$.

† See R. G. Manley, *Waveform Analysis* (New York: John Wiley, 1945).

6-3. Determine the percent error of an accelerometer for the frequency ratios of 0, 0.2, 0.4, and 0.6 for each of the following damping ratios: (a) $\zeta = 0$, (b) $\zeta = 0.66$, and (c) $\zeta = 0.71$.

6-4. For an accelerometer, determine the greatest percent error in measurement that occurs for the range of the frequency ratio from $r = 0$ to $r = 0.6$. Also obtain the value of r at which this occurs. Carry this out for (a) $\zeta = 0$, (b) $\zeta = 0.66$, and (c) $\zeta = 0.71$.

6-5. An accelerometer is composed of weight $W = 0.042$ lb suspended by a spring for which $k = 5$ lb/in., and the fluid damping factor for the instrument is $\zeta = 0.65$. When used on a machine vibrating at a frequency of 13.647 Hz, the instrument reading for the maximum acceleration is 50 in./sec^2. Determine the correct value of the maximum acceleration of the machine.

6-6. A vibrometer is composed of a spring for which $k = 28.8$ lb/in., a suspended weight $W = 19.3$ lb, and a viscous damper for which $c = 0.96$ lb sec/in. The amplitude indicated by the instrument is 0.73 in. when attached to a structure oscillating with a frequency of 14 Hz. Determine the correct value for the amplitude of displacement of the structure.

6-7. An accelerometer is composed of weight $W = 0.0965$ lb suspended by a spring for which $k = 0.355$ lb/in., and the fluid damping factor for the instrument is $\zeta = 0.65$. When used on a machine vibrating at a frequency of 2.4 Hz, the instrument reading for the maximum acceleration is 20 in./sec^2. Determine the correct value for the maximum acceleration of the machine.

6-8. A vibrometer is composed of a spring for which $k = 1.44$ lb/in., a suspended weight of $W = 3.86$ lb, and a viscous damper for which $c = 0.096$ lb sec/in. The amplitude indicated by the instrument is 0.5 in. when attached to a structure oscillating with a frequency of 7 Hz. Determine the correct value for the amplitude of displacement of the structure.

6-9. A vibrometer is to be designed, based on a maximum error of 3% when used for frequencies of 1200 cycles/min and above. The instrument mass is to weigh 1 lb, and the damping factor is to be 0.6. Determine the required spring modulus and damping constant for the instrument.

6-10. An accelerometer is composed of a suspended weight of 0.193 lb and a spring having a modulus of 5 lb/in., but the damping is not known. When tested on a vibration having a frequency of 382 cycles/min, the acceleration measurement is found to be 1% greater than the correct value. Determine the damping constant for the instrument.

6-11. It is desired to design an accelerometer that will have an optimum *range* of frequency ratio for a maximum accelerometer error of 1%. (a) Calculate the required damping factor. (b) Determine this optimum range of frequency ratio.

6-12. An accelerometer is to be designed that will have an optimum *range* of frequency ratio for a maximum accelerometer error of 2%. (a) Calculate the required damping factor. (b) Determine this optimum range of frequency ratio.

6-13. Design an accelerometer based on a maximum error of 1% when used for vibration measurements having frequencies in the range from 0 to 1800 cycles/min. The instrument mass is to have a weight of 0.05 lb. Determine the required spring modulus and damping constant for the instrument. (Suggestion: Refer to Prob. 6-11.)

6-14. Design an accelerometer based on a maximum error of 2% when used for vibration measurements having frequencies in the range from 0 to 6000

6-6

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{28.8 \times 12}{19.3/32.2}} = 24 \text{ rad/s}$$

$$r = \frac{\omega_f}{\omega_n} = \frac{2\pi \cdot 14}{24} = 3.665$$

$$c_c = 2m\omega_n = 2 \times \frac{19.3}{32.2 \cdot 12} \cdot 24 = 2.4 \ \frac{\text{lb-s}}{\text{in}}$$

$$\zeta = c/c_c = \frac{0.96}{2.4} = .4$$

for a vibrometer $\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = 1.052 = 1+\epsilon$

ie 5.2% high

$$\therefore Y = \frac{Z}{1.052} = \frac{.73}{1.052} = .694 \text{ in}$$

β, ψ $\quad z = \frac{r^2 Y}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t - \psi - \beta)$

$$\psi = \tan^{-1}\left[\frac{2\zeta r}{1-r^2}\right]$$

$$\beta = \tan^{-1}(-2\zeta r)$$

6.10

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{(5)(12)}{.193/32.2}} = 100 \ \frac{\text{rad}}{\text{s}}$$

$$\omega_f = 382 \ \frac{\text{rev}}{\text{min}} \cdot \frac{2\pi}{60} = 40 \text{ rad/s}$$

$$r = \frac{\omega_f}{\omega_n} = .4$$

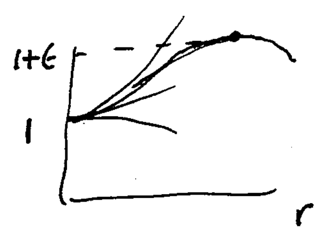
for accel $\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = 1.01$ $\nearrow$ $\zeta = .655$

$$c_c = 2m\omega_n = 2\frac{(.193)}{32.2} \cdot 100 = 1.2 \ \frac{\text{lb-s}}{\text{ft}}$$

$$c = c_c\zeta = \cancel{.4795}\ .786 \ \frac{\text{lb-s}}{\text{ft}}$$

$$\left\{\frac{1}{4r^2}\left[\left(\frac{1}{1.01}\right)^2 - (1-r^2)^2\right]\right\}^{1/2} = \zeta$$

To find max error $\quad \frac{Z}{Y_{max}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1+\epsilon$



solve to determine range $\quad 1-\epsilon = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ $\quad$ solve for r

let $k = C\cos\beta$ let $-c\omega_f = C\sin\beta$ thus $\tan\beta = -\frac{c\omega_f}{k} = -2\zeta r$

$C = \sqrt{k^2+(c\omega_f)^2} = k\sqrt{1+(2\zeta r)^2}$

and $\boxed{m\ddot{x} + c\dot{x} + kx = Y\sqrt{k^2+(c\omega_f)^2}\,\sin(\omega_f t-\beta)}$

- as before if we let $P_o = Y\sqrt{k^2+(c\omega_f)^2}$ then

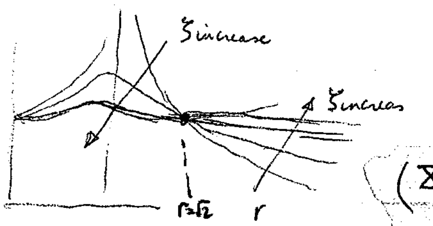
$$x_p = X\sin(\omega_f t-\gamma) = \frac{P_o}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}\sin(\omega_f t-\gamma)$$

LIKE THE FORCE TRANSMISSION

$$X = \frac{Y\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}} = \frac{Y\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$\gamma = \beta+\psi$ $\tan\psi = \frac{2\zeta r}{1-r^2}$

$\tan\beta = -2\zeta r$



for $r=\sqrt{2}$, $r=0$ $\frac{X}{Y}=1$ $\forall\zeta$

as $\zeta\uparrow$ $X\uparrow$

$$\boxed{(X/Y) = F_T/P_o = TR}$$

transmissibility $= \frac{\text{motion of mass}}{\text{motion of base}}$

$(X/Y)_{max}$ occurs @ $r = \frac{\sqrt{(1+8\zeta^2)^{1/2}-1}}{2\zeta}$

Force on support is

$$F = k(x-y) + c(\dot{x}-\dot{y}) = -m\ddot{x}$$

$$= \frac{m\omega_f^2 Y\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}\sin(\omega_f t-\gamma)$$

$$= \boxed{m\omega_f^2 X}\sin(\omega_f t-\gamma) = \boxed{F_{TRANS}}\sin(\omega_f t-\gamma)$$

NOTE THAT

F is in phase w/ mass

$$= Yk\,\frac{r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega_f t-\gamma)$$

PLOTTING $\boxed{\frac{F_{MAX}}{Yk} \text{ vs. } r}$ same graph as $\boxed{F_T\Big/\frac{m_o e k}{m}}$ vs. r

AS $r\to\infty$ $X\to 0$ but $F\to\infty$ since velocity $\to\infty$

damper transmits F

$\boxed{\text{IF we let } z = x-y}$
RELATIVE DISP.

$m\ddot{x} = -kz - c\dot{z}$

$m(\ddot{z}+\ddot{y}) = -kz - c\dot{z}$ or $m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$

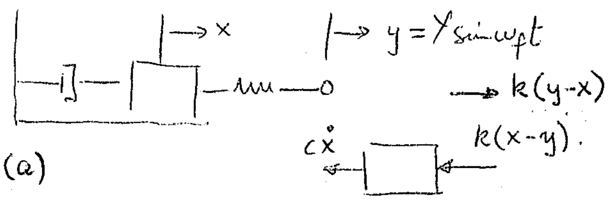
$= m\omega_f^2 Y\sin\omega_f t$

$\boxed{\text{let } m\omega_f^2 Y = P_o}$

$z = Z\sin(\omega_f t-\psi)$ $Z = \frac{P_o}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}} = \frac{mY\omega_f^2}{\sqrt{}} = \frac{Yr^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

$$Z/Y = \frac{X}{m_o e/m} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad \tan\psi = \frac{2\zeta r}{1-r^2}$$

Problem 4-16



(a) $F = k(x-y)$

Find DE for m (b) $m\ddot{x} = -c\dot{x} - k(x-y)$ or $m\ddot{x} + kx + c\dot{x} = ky = kY \sin\omega_f t$

what is ~~force~~ displ of mass due to $y = Y\sin\omega_f t$ (c) $x = X \sin(\omega_f t - \psi)$ $\quad X = \dfrac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} = \dfrac{kY}{\sqrt{}} = \dfrac{Y}{\sqrt{(1-r^2)^2 + (2\zeta}}$

$$\tan\psi = \frac{2\zeta r}{1-r^2}$$

what is F on mass due to spring (d)

$$\boxed{F = k(x-y)} = -[m\ddot{x} + c\dot{x}] = \omega_f^2 X m \sin(\omega_f t - \psi) - c\omega_f X \cos(\omega_f t - \psi)$$

$$= \omega_f X \{ m\omega_f \sin(\omega_f t - \psi) - c \cos(\omega_f t - \psi) \}$$

$$= \omega_f X \, \mathcal{C} \sin(\omega_f t - \psi - \beta)$$

$$\mathcal{C}\cos\beta = \omega_f m \qquad \mathcal{C}\sin\beta = c$$

$$\mathcal{C} = \sqrt{m^2\omega_f^2 + c^2} \qquad \tan\beta = \frac{c}{\omega_f m} = \frac{2\zeta r}{r^2} = \frac{2\zeta}{r}$$

$$= \omega_f \sqrt{m^2\omega_f^2 + c^2}\; X \sin(\omega_f t - \psi - \beta)$$

$$= m\omega_f^2 \sqrt{1 + \left(\frac{2\zeta}{r}\right)^2}\; X \sin(\omega_f t - \gamma) \qquad \gamma = \psi + \beta$$

what is F at support (e) $\boxed{F = c\dot{x}} = c\omega_f X \cos(\omega_f t - \psi) = c\omega_f X \sin(\omega_f t - \psi + \pi/2)$

READ SECTIONS 4-16, 4-17

SELF-EXCITED VIBRATION & INSTABILITY

- CONSIDERED FORCING FUNC. EXTERNAL TO SYSTEM & INDEPENDENT OF MOTION
  i.e IF SYSTEM IS CLAMPED, FORCING FUNC. WILL CONTINUE
- CONSIDER CASE WHEN $P = P(x)$, $P = P(\dot{x})$ or $P(\ddot{x})$
  → SELF EXCITED MOTION IF SYSTEM IS CLAMPED ⇒ $P = 0$.

$a_0 = m \qquad a_1 = c - P_0 \qquad a_2 = k$

$a_0 > 0 \Rightarrow m > 0$ ✓ $\begin{vmatrix} a_1 & 0 \\ a_0 & a_2 \end{vmatrix}$

$a_1 > 0 \Rightarrow$

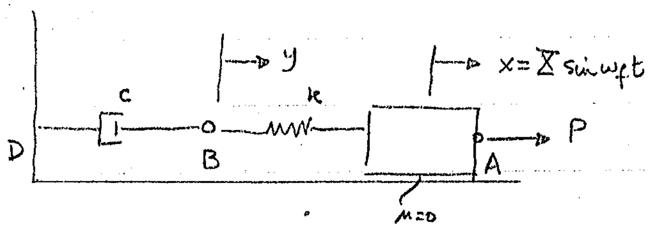
$c > P_0$

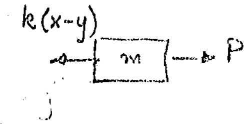
$a_1 a_2 > 0 \Rightarrow a_2 > 0 \quad k > 0$ ✓

HW #11

4-20, 42, 43

4-20



a) 

b) $m\ddot{x} = -k(x-y) + P$ or $\boxed{m\ddot{x} + kx = ky + P}$

c) $c\dot{y} \leftarrow \circ \rightarrow k(x-y)$

d) $m_B\ddot{y} = 0 = k(x-y) - c\dot{y} \Rightarrow \boxed{kx = c\dot{y} + ky = kX\sin\omega_f t}$

e) $\dot{y} + \frac{k}{c}y = \frac{k}{c}X\sin\omega_f t$; let $y = A\cos\omega_f t + B\sin\omega_f t$ & put into D.E.

$\Rightarrow (-A\omega_f + \frac{k}{c}B)\sin\omega_f t + (B\omega_f + \frac{k}{c}A)\cos\omega_f t = \frac{k}{c}X\sin\omega_f t$. Solve for A & B

$\Rightarrow \boxed{y = \frac{\frac{k}{c}X}{\omega_f^2 + k^2/c^2}\left[\frac{k}{c}\sin\omega_f t - \omega_f\cos\omega_f t\right]}$ or $\frac{k/c\,X}{\sqrt{\omega_f^2 + k^2/c^2}}\sin(\omega_f t - \lambda) = y$

with $\lambda = \tan^{-1}\frac{c\omega_f}{k}$; now let $y = Y\sin(\omega_f t - \lambda)$ where $Y = \frac{k/c\,X}{\sqrt{\omega_f^2 + k^2/c^2}}$

f) from (b), (e) and the defn of $x = X\sin\omega_f t$

$P = (-m\omega_f^2 + k)X\sin\omega_f t + \frac{k^2/c\,X}{\sqrt{\omega_f^2 + k^2/c^2}}\sin(\omega_f t - \lambda)$

g) @ D $F_T = c\dot{y}$

$\therefore F_T = \frac{\omega_f k X}{\sqrt{\omega_f^2 + k^2/c^2}}\cos(\omega_f t - \lambda) = c\omega_f Y\cos(\omega_f t - \lambda)$ with $\lambda = \tan^{-1}\frac{c\omega_f}{k}$.

4-42

Given $f = 75$ cy/min $\Rightarrow X = .6$ in & $(F_T)_{max} = 48.75$ lb. ; $k = 13$ lb/in

$c = 0.454$ lb sec/in find ω, m and Y

b). now $x = X\sin(\omega_f t - \gamma) = \frac{Y\sqrt{k^2 + (c\omega_f)^2}}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}\sin\omega_f t$ eq (4-100)

need to find m to get Y ; can get m by looking at transmitted force eqn

c) now $(F_T)_{max} = m\omega_f^2 X$ eqn (4-104) $\Rightarrow m = \frac{(F_T)_{max}}{\omega_f^2 X} = \frac{48.75\text{lb}}{(7.854\frac{\text{rad}}{\text{sec}})^2(.6\text{in})} = 1.317\frac{\text{lb sec}^2}{\text{in}}$

a) $\omega = \sqrt{\frac{k}{m}} = \pi$ rad/sec $= 3.141592653$ rad/sec

b) $Y = X\frac{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}{\sqrt{k^2 + (c\omega_f)^2}} = 3.042$ in

3.30 $m = 100$ kg, $F_0 = 100$ N, $X_{max} = 0.005$ m at $\omega = 300$ rpm $= 31.416$ rad/sec.
Equations (3.33) and (3.34) yield:

$$\omega_f = \omega_n \sqrt{1 - 2\zeta^2} = \sqrt{\frac{k}{m}} \sqrt{1 - 2\zeta^2} = 31.416$$

$$\text{or} \quad k\,(1 - 2\zeta^2) = (100)\,(31.416^2) = 98{,}696.5056 \qquad (1)$$

$$\text{and} \quad X_{max} = \delta_{st} \frac{1}{2\zeta\sqrt{1-\zeta^2}} = \frac{F_0}{k} \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 0.005$$

$$\text{or} \quad k\,\zeta\sqrt{1-\zeta^2} = \frac{F_0}{2\,(0.005)} = 10{,}000.0 \qquad (2)$$

Divide Eq. (1) by (2):

$$\frac{1 - 2\zeta^2}{\zeta\sqrt{1-\zeta^2}} = 9.8696 \qquad (3)$$

Squaring Eq. (3) and rearranging leads to:

$$101.4090\,\zeta^4 - 101.4090\,\zeta^2 + 1 = 0 \quad \text{or} \quad \zeta = 0.0998,\ 0.9950$$

Using $\zeta = 0.0998$ in Eq. (1), we obtain

$$k = \frac{98696.5056}{1 - 2\,(0.0998^2)} = 100{,}702.4994 \text{ N/m}$$

Since $\zeta = \frac{c}{2\,m\,\omega_n}$, we find

$$c = 2\,m\,\omega_n\,\zeta = 2\,(100)\sqrt{\frac{100702.4944}{1000}}\,(0.0998) = 633.4038 \text{ N–s/m}$$

3.50 For each spring,

$$k = \frac{G d^4}{64\, n\, R^3} = \frac{(11.5385 \times 10^6)\,(0.25)^4}{64\,(8)\,(1.5)^3} = 26.083 \text{ lb/in}$$

Total $k = 4\,(26.083) = 104.332$ lb/in since 4 springs

$$\omega_f = \frac{2\pi\,(1800)}{60} = 188.496 \text{ rad/sec}$$

$m_0 = 100/386.4$ lb-s²/in, $M = 750/386.4$ lb-s²/in, $\zeta = 0$

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{104.332}{(750/386.4)}} = 7.3316 \text{ rad/sec}$$

$r = 188.496/7.3316 = 25.7102$, $r^2 = 661.0144$

$$X = \frac{m_0 e}{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{100\,(0.01)}{750}\left(\frac{661.0144}{660.0144}\right)$$

$$= 1.3354 \times 10^{-3} \text{ in.}$$

$F_{trans} = kX = 104.332\,(1.3354 \times 10^{-3}) = .138$ lb since no damping

$= k\sqrt{1 + (2\zeta r)^2}\,X$ if there was damping

HW - Problems 3.25, 3.30, 3.50

~~3.25~~ 3.33 pg290

Complete solution is $x(t) = X_0 e^{-\zeta\omega_n t}\cos(\omega_d t + \phi_0) + X\cos(\omega t - \phi)$

$\omega_f = 2\pi(3.5) = 21.9912$ rad/sec, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{10}} = 15.8114$ rad/sec

$X_0 = \frac{F_0}{k} = \frac{180}{2500} = 0.072$ m

$\zeta = \frac{c}{2m\omega_n} = \frac{45}{2(10)(15.8114)} = 0.1423$, $r = \frac{\omega_f}{\omega_n} = \frac{21.9912}{15.8114} = 1.3908$

$\zeta\omega_n = 2.25$, $\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 15.6505$

$$X = \frac{X_0}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.072}{\left[(1-1.3908^2)^2 + (2\times 0.1423\times 1.3908)^2\right]^{1/2}}$$

$= 0.07095$ m

$$\psi\ \cancel{\phi} = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{0.3958}{-0.9343}\right) = -22.9591^\circ$$

$$x(t) = X_0\, e^{-2.25t}\cos(15.6505t + \phi_0) + 0.07095\cos(21.9912t + 22.9591^\circ)$$

$$\dot{x}(t) = -2.25\, X_0\, e^{-2.25t}\cos(15.6505t + \phi_0) - 15.6505\, X_0\, e^{-2.25t}\sin(15.6505t + \phi_0)$$
$$- 21.9912\,(0.07095)\sin(21.9912t + 22.9591^\circ)$$

$x(0) = 0.015 = X_0\cos\phi_0 + 0.07095\cos 22.9591^\circ$

$X\cos\phi_0 = -0.05033$ ---- (E_1)

$\dot{x}(0) = 5 = -2.25\, X_0\cos\phi_0 - 15.6505\, X_0\sin\phi_0 - 1.5603\sin 22.9591^\circ$

$$X\sin\phi_0 = \frac{-0.6086 - 2.25\, X_0\cos\phi_0 - 5}{15.6505} = -0.3511 \quad ----(E_2)$$

Eqs. (E_1) and (E_2) give

$X_0 = \{(-0.05033)^2 + (-0.3511)^2\}^{1/2} = 0.3547$

$\phi_0 = \tan^{-1}\left(\frac{0.3511}{0.05033}\right) = \tan^{-1}(6.9760) = 81.8423^\circ$

in our case when we use $x(t) = X_0 e^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + X\sin(\omega_f t - \psi)$

we can still use these results but ϕ here $= \phi_0 + 90^\circ = 171.84^\circ$ and ψ here $= \phi + 90^\circ = 67.0409$

$\therefore\ x(t) = 0.3547\, e^{-2.25t}\sin(15.6505t + 171.84^\circ) + 0.07095\sin(21.9912t - 67.0409^\circ)$

$\dot{x}(t) = X\left[-\zeta\omega_n e^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + \omega_d e^{-\zeta\omega_n t}\cos(\omega_d t + \phi)\right] + X\omega_f\cos(\omega_f t - \psi)$

$x(0) = X_0\sin\phi + X\sin\psi = 0.015 = X_0\sin\phi + .07095\sin(+22.9591^\circ) = .015$

$\dot{x}(0) = X_0\left[-\zeta\omega_n\sin\phi + \omega_d\cos\phi\right] + X\omega_f\cos\psi$

$= X_0\left[-2.25\sin\phi + 15.6505\cos\phi\right] + 0.07095(22.9912)\cos(22.9591) = 5$

$X\sin\phi = 0.015 - .07095\sin(22.9591) = \cancel{.04267}\ -.01268$

$X\cos\phi = \left(5 - 0.07095(22.9912)\cos(22.9591) + 2.25\left[.015 + .07095\sin(22.9591)\right]\right)/15.6505$

$= .22168$

$X = \sqrt{(X\sin\phi)^2 + (X\cos\phi)^2} = X = .2221$ $\phi = -3.27^\circ \Leftarrow \frac{X\sin\phi}{X\cos\phi} = \tan\phi$

Based on the value of m, c, ω_f, k, X obtained we can now find Y

$$Y = X \frac{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}{\sqrt{k^2+(c\omega_f)^2}} = 3.042 \text{ in}$$

4-43

$W = mg = 20 \text{ lb}$ $k = 14 \text{ lb/in}$; if $f_f = 300 \text{ cy/min} = 5 \text{ Hz} \Rightarrow X = .793 \text{ in}$

a) Find $(F_T)_{max}$; b) if $\omega_f = \omega$ and $Y_{res} = 1.121 \text{ in} \Rightarrow X_{res} = 3.069 \text{ in}$; find c

$y = Y \sin \omega_f t$

a) now $F = k(x-y) + c(\dot{x}-\dot{y}) = -m\ddot{x} = m\omega_f^2 X \sin \omega_f t$

$$(F_T)_{max} = m\omega_f^2 X = \frac{W}{g}\omega_f^2 X = \frac{20 \text{ lb}}{386.4 \text{ in/sec}^2}(10\pi)^2(.793 \text{ in}) = 40.51 \text{ lb.}$$

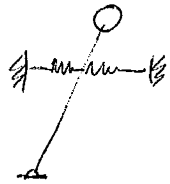
b) when $\omega_f = \omega \Rightarrow r = 1$

$$X_{res} = Y_{res}\frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\Bigg|_{r=1} = \frac{Y_{res}\sqrt{1+4\zeta^2}}{2\zeta}$$

now $2\zeta X_{res} = Y_{res}\sqrt{1+4\zeta^2} \Rightarrow 4\zeta^2(X_{res}^2 - Y_{res}^2) = Y_{res}^2$ or $\zeta = \frac{1}{2}\frac{Y_{res}}{\sqrt{X_{res}^2 - Y_{res}^2}} = .1962$.

now $c = c_c\zeta$ and $c_c = 2m\omega = 2\sqrt{mk}$, $\frac{W}{g} = m = .0518 \frac{\text{lb sec}^2}{\text{in}}$

$c = .334 \frac{\text{lb sec}}{\text{in}} \Leftarrow c_c = 1.703 \frac{\text{lb sec}}{\text{in}} \Leftarrow$



NON RESTORING ~~FORCE~~ MOMENT $Wl\theta$ was a function of θ

- WILL CONCENTRATE ON CASE WHERE FORCE IS FUNCTION OF VELOC.
  - EXAMPLE IS WING FLUTTER, NOSE-GEAR SHIMMY
  - AUTO WHEEL SHIMMY
  - AERODYNAMIC INDUCED MOTION OF BRIDGES

- SUPPOSE $m\ddot{x} + c\dot{x} + kx = P_0\dot{x} = P$

$\Rightarrow \quad \ddot{x} + (c - P_0)\frac{\dot{x}}{m} + \frac{kx}{m} = 0 \qquad$ let $\quad x = Ce^{st}$

- CHAR EQ IS

$$S_{1,2} = \frac{P_0 - c}{2m} \pm \sqrt{\left(\frac{P_0 - c}{2m}\right)^2 - \frac{k}{m}}$$

- SUPPOSE $P_0 > c \Rightarrow$ negative damping
- IF $\left(\frac{P_0-c}{2m}\right)^2 > \frac{k}{m} \Rightarrow S_1, S_2$ are >0 & real

$\Rightarrow x \uparrow$ as $t \uparrow \qquad x = C_1 e^{s_1 t} + C_2 e^{s_2 t}$ DIVERGENT APERIODIC

- IF $\left(\frac{P_0-c}{2m}\right)^2 < \frac{k}{m} \Rightarrow$ conjugate pairs $\quad S_{1,2} = \frac{P_0 - c}{2m} \pm i\sqrt{\frac{k}{m} - \left(\frac{P_0-c}{2m}\right)^2}$

$\Rightarrow \quad x = Xe^{\frac{P_0-c}{2m}t} \sin\left[\sqrt{\frac{k}{m} - \left(\frac{P_0-c}{2m}\right)^2}\,t + \phi\right]$ DIVERGENT OSCILLATORY

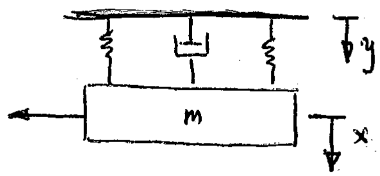
- SUPPOSE $P_0 < c \Rightarrow$ positive damping: same as damped free vibs

- $P_0 = c \Rightarrow \ddot{x} + \frac{kx}{m} = 0 \Rightarrow$ undamped free vibs

HW 4-20, 4-42, 4-43 Due Monday 17 March

VIBRATION MEASURING INSTRUMENTS

- PURPOSE - INDICATE AN OUTPUT BASED ON AN INPUT
  - TYPICAL INPUTS $x, \dot{x}, \ddot{x}$
- WANT OUTPUT TO BE AS CLOSE TO INPUT
- SEE P. 150 RAO FOR AN EXAMPLE



$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \qquad y = Y\sin\omega t \qquad z = x - y$$
$$= +m\omega^2 Y\sin\omega t$$

$$\therefore \quad z_p = Z\sin(\omega t - \psi) \quad \text{and} \quad Z = \frac{Y\, r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Vibrometer (Seismometer) want to measure displacement of vibrating body so that

- $\frac{Z}{Y} \sim 1 \Rightarrow$ r must be large $\Rightarrow$ ω_n must be low
  $\Rightarrow$ k small or m is large
- RECORD LAGS REAL EVENT BY $t = \psi/\omega$ $\qquad \psi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$ $\qquad$ as $r \to \infty$ $\frac{2\zeta r}{1-r^2} \to -0$ $\psi \to +180°$
- LEADS TO BULKY INSTRUMENTS IN GENERAL
- NOT USED AS OFTEN

$\sin 90 - \cos$ ...

ACCELEROMETER - measure acceleration want to mimic accel of base $\ddot{y} = -Y\omega_f^2 \sin\omega_f t$

IF $\quad z_p = Z \sin(\omega_f t - \psi) = \frac{Y\, r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega_f t - \psi)$

~~$-z_p\omega_n^2 =$~~

multiply by ω_n^2

$$-z_p\omega_n^2 = \frac{-Y\omega_f^2 \sin(\omega_f t - \psi)}{\sqrt{}}$$

if $\quad \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \cong 1 \quad \Rightarrow \quad -z_p\omega_n^2 = -Y\omega_f^2 \sin(\omega_f t - \psi)$

also $\quad \ddot{y}(t) = -Y\omega_f^2 \sin(\omega_f t) \leftarrow$ accel of base

- OUTSIDE THE PHASE LAG $\quad \ddot{y} = -z_p\omega_n^2$ $\qquad \psi = \tan^{-1}(2\zeta r) \cong 2\zeta r = \text{const} \cdot r = \text{const} \cdot \omega$ $\quad \sin(\omega t - C\omega) = \sin(\omega[t - C])$
- $\Rightarrow$ r must be small for large ζ $\Rightarrow$ ω_n must be high $\qquad$ p152 $0 \le r \le .6$, $\zeta = .7$
- $\Rightarrow$ m small & k high $\Longrightarrow$ small instrument
- preferred instrument

VELOMETERS — $\boxed{\text{measure velocity of a body}}$

$$y = Y\sin\omega t \qquad \dot{y} = Y\omega\cos\omega t$$

$$z_p = Z\sin(\omega t - \psi) \qquad \dot{z}_p = Z\omega\cos(\omega t-\psi) = \frac{Y\omega r^2}{\sqrt{\quad}}\cos(\omega t-\psi)$$

if $\dfrac{r^2}{\sqrt{\quad}} \sim 1 \Rightarrow \boxed{\dot{z}_p \sim \dot{y} \text{ outside of the phase difference}}$

$\boxed{\Rightarrow r \text{ must be large}}$

$\boxed{\Rightarrow \text{same problems as the vibrometer.}}$

(**) FOR vibrometer & velocity; phase shift causes $z = -y$, $\dot{z} = -\dot{y}$,

FOR errors < 30% $\zeta \in [.6-1]$ and $-Z\omega_n^2 = \ddot{y}(t)$ (τ)

READ 3.11 & 3.12

$\tau = t - \frac{\text{const}}{\omega_n}$

3.54 given $x(t) = 20\sin \overset{f_1}{6.5\pi t} + 5\sin \overset{3f_1}{19.5\pi t}$

$$\ddot{x} = -\left[20(6.5\pi)^2\sin 6.5\pi t + 5(19.5\pi)^2\sin(19.5\pi t)\right] \quad mm/s^2$$

$$= -8339.8547\sin(6.5\pi t) - 18764.673\sin 19.5\pi t$$

Given $\zeta = 0.6$ $\omega_n = 45$ rad/s.

for each one $r_1 = \dfrac{\omega_1}{\omega_n} = \dfrac{6.5\pi}{45} = .4538$ $Y_1 = 20$

$r_2 = \dfrac{\omega_2}{\omega_n} = \dfrac{19.5\pi}{45} = 1.3614 = 3r_1$ $Y_2 = 5$

$$\ddot{z}_p = \frac{-Y_1\omega_1^2\sin(\omega_1 t - \psi_1)}{\sqrt{(1-r_1^2)^2+(2\zeta r_1)^2}} + \frac{Y_2\omega_2^2\sin(\omega_2 t - \psi_1)}{\sqrt{(1-r_2^2)^2+(2\zeta r_2)^2}}.$$

$$= -8661.7957\sin(6.5t - \psi_1) - 10180.750\sin(19.5t - \psi_2)$$

$$\psi_1 = \tan^{-1}\left(\frac{2\zeta r_1}{1-r_1^2}\right) = 34.4418^\circ \qquad \psi_2 = \tan^{-1}\left(\frac{2\zeta r_2}{1-r_2^2}\right) = \tan^{-1}(-1.9143)$$
$$= 117.5818^\circ$$

10.9 Vibrometer

~~3.58~~ $f_n = 5Hz$ $\zeta = 0.5$ FOR A VIBR $\dfrac{Z}{Y} \cong 1$

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

Note $\left(\dfrac{Z}{Y}\right)_{max}$ occurs at $r = \dfrac{1}{\sqrt{1-2\zeta^2}}$ and $\left(\dfrac{Z}{Y}\right)_{max} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.1547$

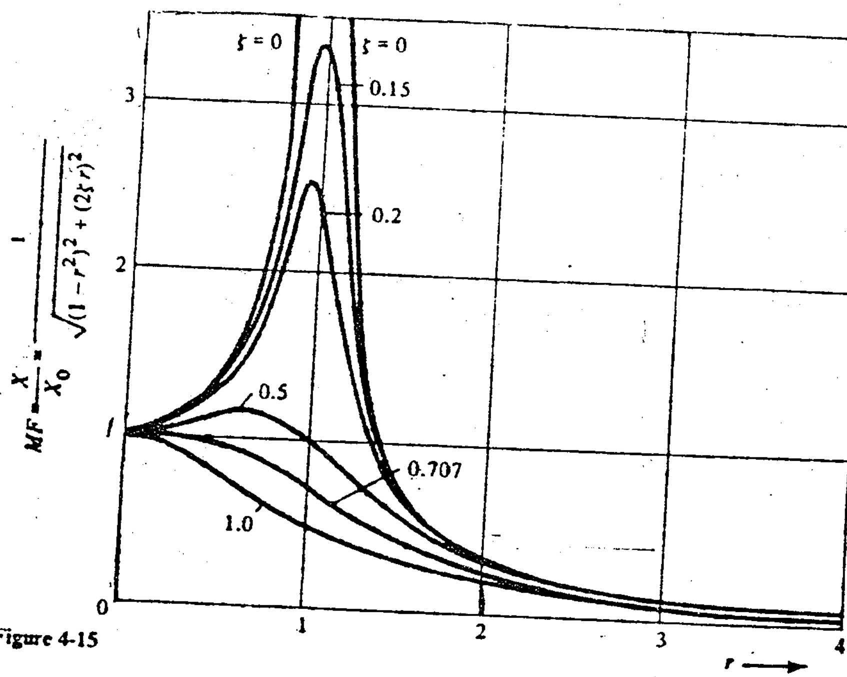


Figure 4-15

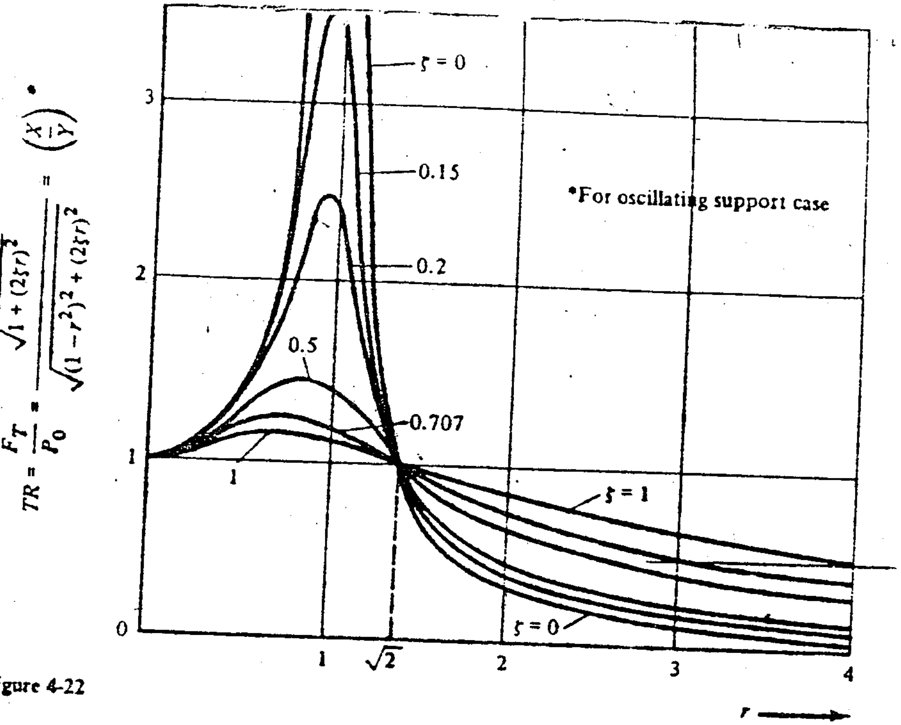


Figure 4-22

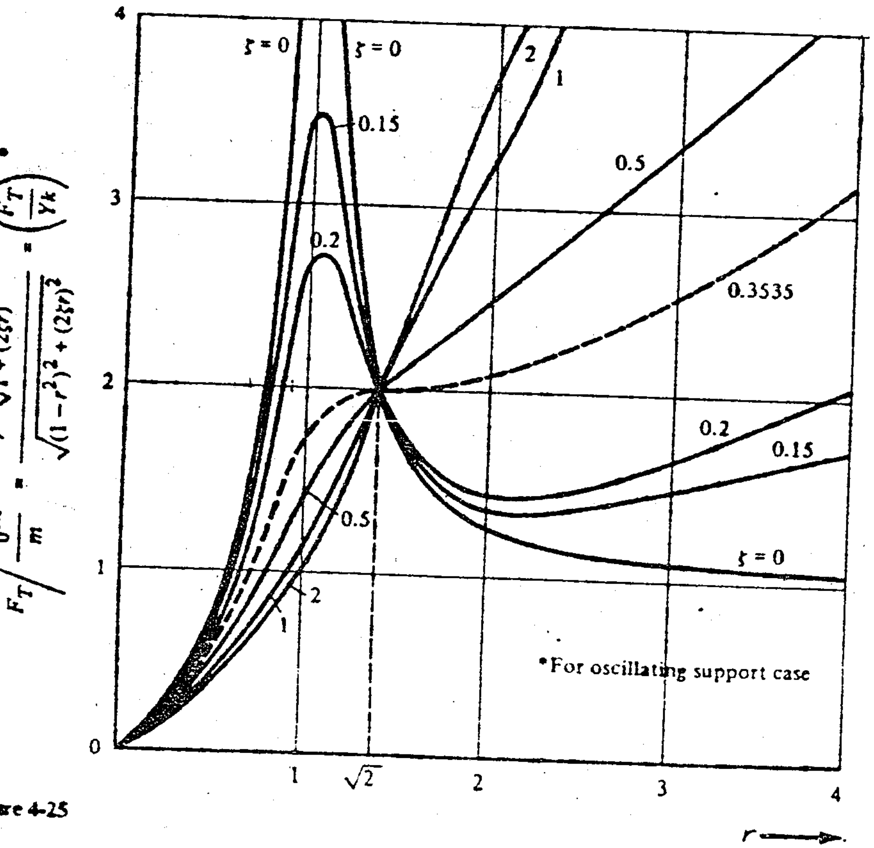
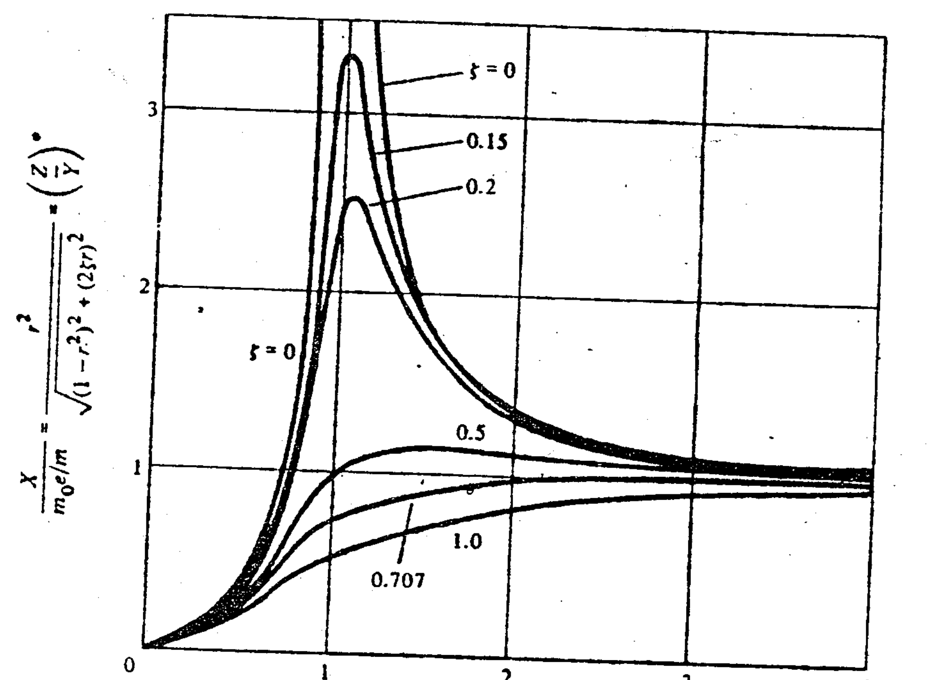


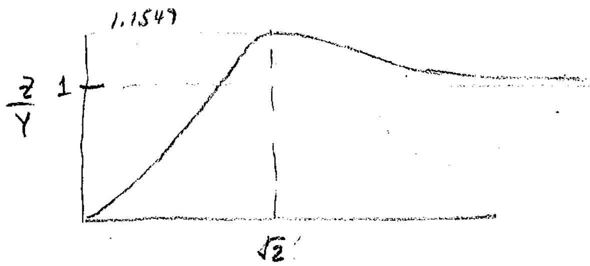
Figure 4-25



*For oscillating support case.

Figure 4-20

$\therefore$



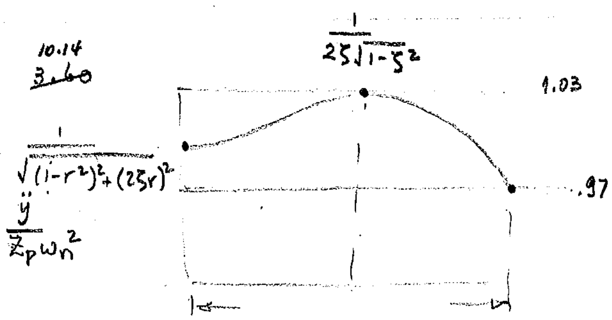
$\therefore \frac{Z}{Y} \to 1+$

for a 1% error $\frac{Z}{Y} = 1.01$

$\therefore \quad 1.01 = \dfrac{r^2}{\left[(1-r^2)^2 + r^2\right]^{1/2}} \quad \Rightarrow \quad r = 1.0101 \ , \quad 7.0527 = \frac{\omega}{\omega_n} = \frac{f}{f_n}$

$5f_n = f = 5.0505\ Hz \ , \quad \underline{f = 35.2635\ Hz}$

remember $r \geq 3$



$r^* = \sqrt{1-2\zeta^2}$
$= .49$

$\frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.03$ gives $\zeta = .6164$

for this ζ

$\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = .97$ gives $r_{opt.}$

$r_{opt} = .7662$

TO FIND OPTIMAL r & ζ FOR ACCELEROMETER GIVEN required error.

$0 \leq r \leq .7662 \qquad \therefore \quad \omega_n = \omega_f / r \Rightarrow \frac{100\ Hz}{.7662} = f_n \cong 130.5\ Hz \ ; \ \omega_n = 8$

$2\pi f_n = \omega_n$ & $2m\omega_n = C_{crit} = 205.\ \frac{N\text{-}s}{m}$ $\therefore$ $c = 2m\omega_n \zeta = 2(.05)(820)(.6164) = 126.4\ \frac{N\text{-}s}{m}$ & $k = \omega_n^2 m = 33625\ N/m$

An accelerometer is to be designed for optimum range of freq ratio for max error of 3%. Determine ζ and range.

For the particular solutions choose $x_1 = B_1 \sin 4\pi t$ $\qquad x_2 = B_2 \sin 4\pi t$

for $t < .5\text{sec}$; note $F = 0$ after $t = 0.5\text{sec}$

Put this into the original ODE

$$m_1\ddot{x}_1 + k_1 x_1 - k_1 x_2 = F \quad \Rightarrow \left[-m_1(4\pi)^2 + k_1\right]B_1 - k_1 B_2 = F_0$$

$$m_2\ddot{x}_2 + (k_1+k_2)x_2 - k_1 x_1 = 0 \Rightarrow \left[-(4\pi)^2 m_2 + (k_1+k_2)\right]B_2 - k_1 B_1 = 0$$

and solve for B_1 and B_2

$$\therefore \quad B_1 = \frac{F_0\left[-(4\pi)^2 m_2 + k_1 + k_2\right]}{\left[-m_1(4\pi)^2 + k_1\right]\left[-(4\pi)^2 m_2 + k_1 + k_2\right] - k_1^2} = \frac{10^5(\qquad)}{-5.3104\times 10^{14}} = -.0349$$

$$B_2 = \frac{-k_1 F_0}{\left[-m_1(4\pi)^2 + k_1\right]\left[-(4\pi)^2 m_2 + k_1 + k_2\right] - k_1^2} = .02825$$

$\therefore$

$$x_{1T} = x_{1h} + x_{1p} = A_1 \sin(12.2474t + \varphi) + \tilde{A}_1 \sin(38.7298t + \tilde{\varphi}) - .0349 \sin 4\pi t$$

$$x_{2T} = x_{2h} + x_{2p} = .8A_1 \sin(12.2474t + \varphi) - \tilde{A}_1 \sin(38.7298t + \tilde{\varphi}) + .02825 \sin 4\pi t$$

to find $A_1, \varphi, \tilde{A}_1, \tilde{\varphi}$ assume $x_{1T}, x_{2T} = 0$ and $\dot{x}_{1T} = \dot{x}_{2T} = 0$ at $t = 0$

with these values

$x_{1T} = 0 = A_1 \sin\varphi + \tilde{A}_1 \sin\tilde{\varphi}$

$x_{2T} = 0 = .8A_1 \sin\varphi - \tilde{A}_1 \sin\tilde{\varphi}$

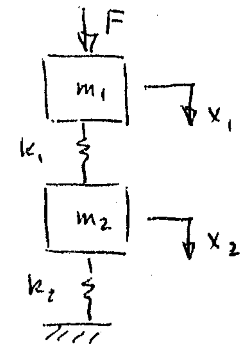
$\dot{x}_{1T} = 0 = A_1(12.2474)\cos\varphi + 38.7298\tilde{A}_1 \cos\tilde{\varphi} - (4\pi)(.0349)$

$\dot{x}_{2T} = 0 = A_1(.8)(12.2474)\cos\varphi - 38.7298\tilde{A}_1 \cos\tilde{\varphi} + (4\pi)(.02825)$

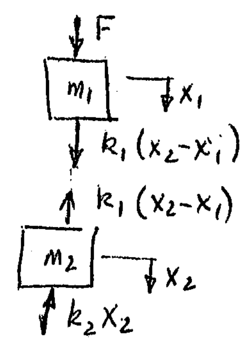
This is solvable by trial and error. Once $A_1, \varphi, \tilde{A}_1$ and $\tilde{\varphi}$ are found this is the solution for $t \le \tau$. After $t = 0.5\text{sec}\,(\tau)$, the equations are given by x_{1h} and x_{2h} with the initial conditions being

$$\left.\begin{aligned} x_{1T}(t=0.5) &= x_{1h}(t=0.5) \\ x_{2T}(t=0.5) &= x_{2h}(t=0.5) \\ \dot{x}_{1T}(t=0.5) &= \dot{x}_{1h}(t=0.5) \\ \dot{x}_{2T}(t=0.5) &= \dot{x}_{2h}(t=0.5) \end{aligned}\right\} \text{these will given } A_1, \varphi, \tilde{A}_1, \tilde{\varphi} \text{ for } t > 0.$$

5.40



Assume $x_2 > x_1$

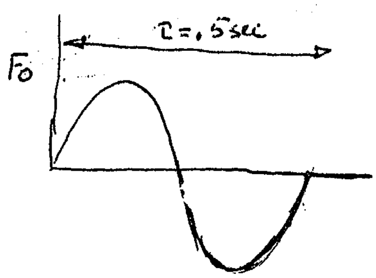


Eqns of Motion

$m_1\ddot{x}_1 = k_1(x_2-x_1) + F$

$m_2\ddot{x}_2 = -k_2x_2 - k_1(x_2-x_1)$

since $F = F_0 \sin \omega t$ w/ period of ½ second. Since $\omega t = 2\pi$ for 1 period



then $\omega = \frac{2\pi}{\tau} = 4\pi$. Therefore $F = F_0 \sin 4\pi t$

To find the natural frequencies, look at eqns of motion with $F = 0$

$\therefore$

$$\left.\begin{aligned} m_1\ddot{x}_1 + k_1x_1 - k_1x_2 &= 0 \\ m_2\ddot{x}_2 + (k_1+k_2)x_2 - k_1x_1 &= 0 \end{aligned}\right\} \text{ with } \begin{aligned} x_1 &= A_1\sin(\omega t+\varphi) \\ x_2 &= A_2\sin(\omega t+\varphi) \end{aligned}$$

thus the amplitude equations are obtained by substituting them into the DE's

$$(-m_1\omega^2 + k_1)A_1 - k_1A_2 = 0$$

$$(-m_2\omega^2 + k_1 + k_2)A_2 - k_1A_1 = 0$$

This leads to the following characteristic equation

$$\omega^4 m_1m_2 - \omega^2[m_1(k_1+k_2) + m_2k_1] + k_1k_2 = 0$$

Based on the values of $m_1 = 2\times10^5$ kg $m_2 = 2.5\times10^5$ kg $k_1 = 2k_2 = 150\times10^6$ N/m

$\omega_{n_1} = 12.2474$ rad/s and $\omega_{n_2} = 38.7298$ rad/s

With these values of ω_{n_1} & ω_{n_2} we find

$$\omega_{n_1} \Rightarrow \frac{A_2}{A_1} = \frac{-m_1\omega^2 + k_1}{k_1} = .8 \qquad \omega_{n_2} \Rightarrow \frac{\tilde{A}_2}{\tilde{A}_1} = -1$$

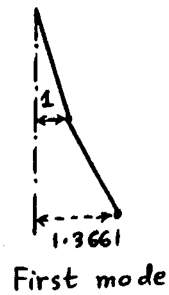
$\therefore$ the homogeneous solution is

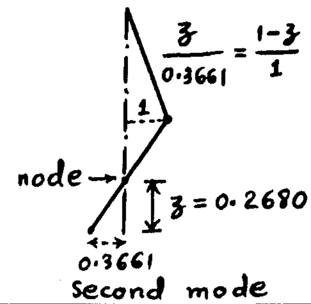
$$x_{1_h} = A_1\sin(12.2474t+\varphi) + \tilde{A}_1\sin(38.7298t+\tilde{\varphi})$$

$$x_{2_h} = .8A_1\sin(12.2474t+\varphi) - \tilde{A}_1\sin(38.7298t+\tilde{\varphi})$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-J_0\,\omega_1^2 + 3\,k_t}{2\,k_t} = 1.3661$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-J_0\,\omega_2^2 + 3\,k_t}{2\,k_t} = -0.3661$$





(5.25) Equation of motion of mass m: $\quad m\,\ddot{x} = -k_2\,(x - r\,\theta) \quad ---(E_1)$

Equation of motion of cylinder of mass m_0 and mass moment of inertia $J_0 = \frac{1}{2} m_0 r^2$: $\quad J_0\,\ddot{\theta} = -k_1 r^2 \theta - k_2\,(r\theta - x)\,r \quad ---(E_2)$

For $x(t) = X\cos(\omega t + \phi)$ and $\theta(t) = \Theta\cos(\omega t + \phi)$, Eqs. (E_1) and (E_2) give the frequency equation

$$\begin{vmatrix} -m\,\omega^2 + k_2 & -k_2\,r \\ -k_2\,r & -\frac{1}{2} m_0 r^2 \omega^2 + k_1 r^2 + k_2 r^2 \end{vmatrix} = 0$$

i.e. $$\omega^4 - \omega^2\left(\frac{k_2}{m} + \frac{2\{k_1 + k_2\}}{m_0}\right) + \frac{2\,k_1 k_2}{m_0\,m} = 0$$

$$\omega_1^2,\ \omega_2^2 = \frac{k_2}{2m} + \frac{(k_1 + k_2)}{m_0} \mp \sqrt{\frac{1}{4}\left(\frac{k_2}{m} + \frac{2\,k_1}{m_0} + \frac{2\,k_2}{m_0}\right)^2 - \frac{2\,k_1 k_2}{m\,m_0}}$$

(5.26) For $J_1 = J_0$, $J_2 = 2\,J_0$, $k_{t_1} = k_{t_2} = k_{t_3} = k_t$, and $M_{t1} = M_{t2} = 0$, Eqs. (5.20) give

$$J_0\,\ddot{\theta}_1 + 2\,k_t\,\theta_1 - k_t\,\theta_2 = 0$$

$$2\,J_0\,\ddot{\theta}_2 - k_t\,\theta_1 + 2\,k_t\,\theta_2 = 0$$

For harmonic motion, these equations give

$$\begin{bmatrix} -\omega^2 J_0 + 2\,k_t & -k_t \\ -k_t & -2\omega^2 J_0 + 2\,k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

from which the frequency equation can be obtained as

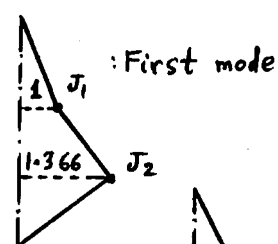
$$2\,J_0^2\,\omega^4 - 6\,J_0\,k_t\,\omega^2 + 3\,k_t^2 = 0$$

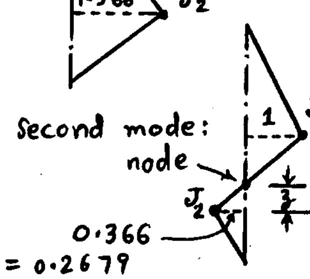
$$\omega_1^2,\ \omega_2^2 = \frac{1}{2}(3 \mp \sqrt{3})\,\frac{k_t}{J_0}$$

$$\therefore\ \omega_1 = 0.79623\sqrt{\frac{k_t}{J_0}}\ ; \qquad \omega_2 = 1.53819\sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 J_0 + 2\,k_t}{k_t} = 1.36603$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 J_0 + 2\,k_t}{k_t} = -0.36603$$





mode shapes are: $\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} X_1^{(1)}$, $\vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(2)}$

No node — one node at middle of the two masses

(5.19) For $m_1 = 3m$, $m_2 = m$, $k_1 = 3k$ and $k_2 = k$, Eq. (E_2) in solution of problem 5.17 gives

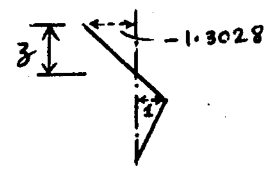
$$\omega^2 = \frac{(3mk + mk + 3mk) \pm \sqrt{(3mk+mk+3mk)^2 - 36k^2m^2}}{6m^2} = \frac{k}{6m}(7 \pm \sqrt{13})$$

$$\omega^2 = 0.5657\,\frac{k}{m}\ ,\quad 1.7676\,\frac{k}{m}$$

$$\omega_1 = 0.7521\sqrt{\frac{k}{m}}\ ,\qquad \omega_2 = 1.3295\sqrt{\frac{k}{m}}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-\omega_1^2 m_1 + k_1 + k_2}{k_2} = \frac{-0.5657(3) + 3 + 1}{1} = 2.3029$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-\omega_2^2 m_1 + k_1 + k_2}{k_2} = \frac{-1.7676(3) + 3 + 1}{1} = -1.3028$$

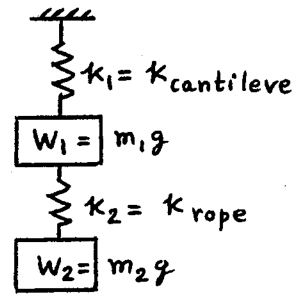


$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 2.3029 \end{Bmatrix} X_1^{(1)}\ ,\qquad \vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -1.3028 \end{Bmatrix} X_1^{(2)}$$

No node — one node

$$\frac{z}{1.3028} = \frac{1-z}{1}\ ;\ z = 0.5657$$

(5.20) $$k_1 = \frac{3EI}{b^3} = \frac{3E\left(\frac{1}{12}at^3\right)}{b^3} = \frac{Eat^3}{4b^3}$$



$$k_2 = \frac{AE}{l} = \frac{\pi d^2 E}{4l}$$

From problem 5.1,

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1k_2}{m_1m_2}}$$

$$= \left(\frac{Eat^3}{4b^3} + \frac{\pi d^2 E}{4l}\right)\frac{g}{2W_1} + \frac{\pi d^2 E g}{8 l W_2}$$

$$\mp \sqrt{\left[\frac{1}{4}\left\{\left(\frac{Eat^3}{4b^3} + \frac{\pi d^2 E}{4l}\right)\frac{g}{W_1} + \frac{\pi d^2 E g}{4 l W_2}\right\}^2 - \frac{E^2 a t^3 \pi d^2 g^2}{16 l b^3 W_1 W_2}\right]}$$

(5.21) From solution of Problem 5.1, we find that for $m_1 = m$, $m_2 = 2m$, $k_1 = k$ and $k_2 = 2k$,

$$\omega_1^2 = (2 - \sqrt{3})\,\frac{k}{m}\ ;\ \omega_2^2 = (2 + \sqrt{3})\,\frac{k}{m}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-m_2\,\omega_1^2 + k_2} = \frac{1}{-1 + \sqrt{3}}$$

the characteristic eqn comes from.

$$\begin{vmatrix} -m_1\omega^2 + k & -k \\ -k & -\omega^2\left(m_2 + \frac{J_0}{r^2}\right) + k \end{vmatrix} = 0$$

from which we find

$$\omega^4 \left(m_1 m_2 + \frac{m_1 J_0}{r^2}\right) - \omega^2\left((m_1+m_2)k + \frac{kJ_0}{r^2}\right) = 0$$

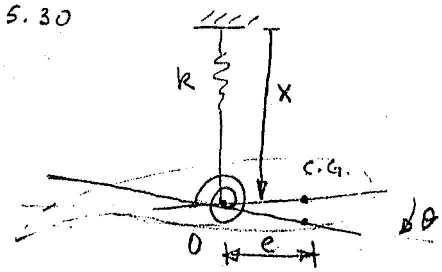
note

$$\omega = 0 \quad \text{and} \quad \omega = \sqrt{\frac{(m_1+m_2)k + \frac{J_0}{r^2}k}{m_1 m_2 + m_1 J_0/r^2}}$$

this is a degenerate case.

531 (ed. 3)

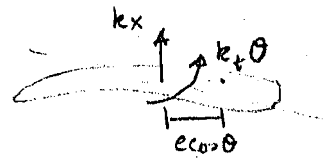
5.30



This problem is similar to that derived in class. Pick x & θ in such a manner like x_A & θ in our class problem.

$\therefore \quad \Sigma F = m\ddot{x}_{CG}$ with $x_{CG} = x + e\sin\theta \cong x + e\theta$

$\Sigma F = -kx = m\ddot{x} + me\ddot{\theta} \Rightarrow \quad m\ddot{x} + kx + me\ddot{\theta} = 0$ (1)



Also

$\Sigma T_{CG} = J_{CG}\ddot{\theta}$ with $J_{CG} + me^2 = J_0 \Rightarrow J_0 + me^2 = J_{CG}$

but $\Sigma T_{CG} = kxe\cos\theta - k_t\theta = (J_0 - me^2)\ddot{\theta} \Rightarrow J_0\ddot{\theta} + k_t\theta - kx_e - me$ (2)

Now let $\theta = \Theta \sin(\omega t + \varphi)$ and $x = A\sin(\omega t + \varphi)$

(1) becomes $(-m\omega^2 + k)A + me\omega^2\Theta = 0$

(2) becomes $\left[-\omega^2(J_0 - me^2) + k_t\right]\Theta - keA = 0$] amplitude equation

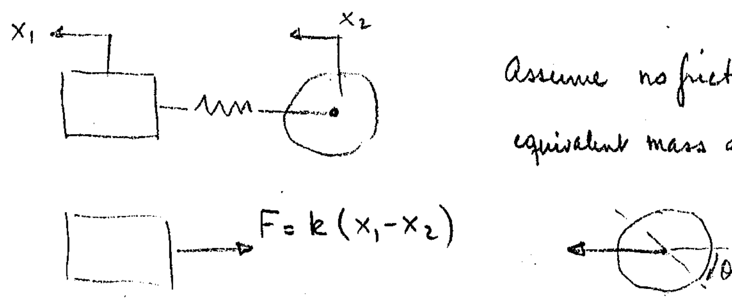
& the solution to

$$\begin{vmatrix} -m\omega^2 + k & -me\omega^2 \\ -ke & -\omega^2(J_0 - me^2) + k_t \end{vmatrix} = 0 \quad \text{(characteristic eqn)}$$

gives values of ω_n for system. Note the dynamic coupling in (1) & (2) & static coupling in (2)

(5.53 in 3rd ed)

5.41



Assume no friction & $x_1 > x_2$. Must find equivalent mass at x_2, ie $\frac{1}{2}m_{eq}\dot{x}_2^2 = \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}J_0\dot{\theta}^2$ where $\dot{\theta} = \frac{\dot{x}_2}{r}$

$\therefore \quad m_{eq} = m_2 + J_0/r^2$ (3)

$m_1\ddot{x}_1 = -k(x_1 - x_2)$ (1)

$J_0\ddot{\theta} = \Sigma T_0 = 0$ since F acts through center of mass

$m_{eq}\ddot{x}_2 = F = k(x_1 - x_2)$ (2)

with (1), (2), (3)

$m_1\ddot{x}_1 + kx_1 - kx_2 = 0$

$\left(m_2 + \frac{J_0}{r^2}\right)\ddot{x}_2 + kx_2 - kx_1 = 0$

Now let $x_1 = A_1\sin(\omega t + \varphi)$ $\quad x_2 = A_2\sin(\omega t + \varphi)$ then

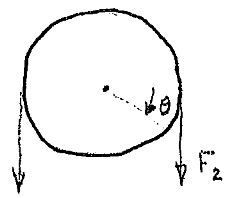
$(-m_1\omega^2 + k)A_1 \quad - kA_2 = 0$

$-kA_1 \quad + \left[-\omega^2\left(m_2 + \frac{J_0}{r^2}\right) + k\right]A_2 = 0$] amplitude eqn

Dynamic FBD

5.25

FBD of pulley



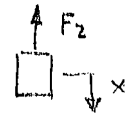
$F = k_1 r\theta$

spring k_2

if $x > r\theta$

$F_2 = k_2(x - r\theta)$

mass



$m\ddot{x} = -F_2 = -k_2(x-r$

$$\boxed{m\ddot{x} + k_2 x - k_2 r\theta = 0} \quad (1)$$

$$I_0\ddot{\theta} = \sum \text{Torques}$$

$$= F_2 r - k_1 r\theta \cdot r$$

Note $r\theta$ = arc length traveled by circumference of disk when disk rotates through θ. Since rope doesn't slip $\Rightarrow k_1$ spring stretches by $r\theta$.

$$I_0\ddot{\theta} + k_1 r^2\theta - F_2 r = 0$$

$$I_0\ddot{\theta} + k_1 r^2\theta - k_2(x - r\theta)r = 0 \quad \text{or} \quad \boxed{I_0\ddot{\theta} + (k_1+k_2)r^2\theta - k_2 x r = 0} \quad (2)$$

let $x = A\sin(\omega t + \varphi)$ and $\theta = \Theta\sin(\omega t + \varphi)$

$\therefore$ (1) $\Rightarrow$ $(-\omega^2 m + k_2)A - k_2 r\Theta = 0$

(2) $\Rightarrow$ $\left[-I_0\omega^2 + (k_1+k_2)r^2\right]\Theta - k_2 A r = 0$

$\}$ amplitude eqns.

finally

$$\begin{vmatrix} \omega^2 - \frac{k_2}{m} & +\frac{kr}{m} \\ \frac{k_2 r}{I_0} & \omega^2 - \frac{(k_1+k_2)r^2}{I_0} \end{vmatrix} \begin{pmatrix} A \\ \Theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\tilde{k}_1 + \tilde{k}_2 = -k_2$ $\quad \tilde{k}_2 = k_2 r$

$\tilde{k}_2 = -k_2 r$ $\quad \tilde{k}_2 + \tilde{k}_3 = -(k_1+k_2)r^2$

and the freq eqn is

$$\left(\omega^2 - \frac{k_2}{m}\right)\left(\omega^2 - \frac{k_1+k_2}{I_0}r^2\right) - \frac{k_2^2 r^2}{I_0 m} = 0 \quad \text{or} \quad \omega^4 - \omega^2\left[\frac{k_1+k_2}{I_0}r^2 + \frac{k_2}{m}\right] + \frac{k_2 k_1 r^2}{I_0 m} = 0$$

For a disk $I_0 = \frac{1}{2}m_0 r^2$ $\quad \therefore \quad \omega^4 - \omega^2\left[\frac{2(k_1+k_2)}{m_0} + \frac{k_2}{m}\right] + \frac{2k_2 k_1}{m_0 m} = 0$

b $\qquad$ c

and ω_1 & ω_2 can be found.

$$\omega^2 = \frac{b \pm \sqrt{b^2 - 4c}}{2}$$

and ω_1, ω_2 are the $\sqrt{}$ of each.

$r\Theta$

$$\text{or} \quad \left(\omega^2 - \frac{2k_1}{m}\right) A_1 + \frac{2k_1}{m} A_2 = 0 \quad (1) \qquad \text{Now let} \quad \frac{2k_1}{m} = 2\alpha \qquad \frac{2k_1}{m} = \gamma$$

$$\frac{2k_1}{M} A_1 + \left\{\omega^2 + \left(\frac{k_2 + 2k_1}{M}\right)\right\} A_2 = 0 \quad (2) \qquad \frac{k_2 + 2k_1}{M} = 2\beta \qquad \frac{2k_1}{M} = \varepsilon$$

$$\text{thus} \quad \alpha = \frac{k_1}{m} = \frac{2000}{2.5} = 800 \qquad \gamma = \frac{4000}{2.5} = 1600$$

$$\beta = \frac{k_2 + 2k_1}{2M} = \frac{8000}{10} = 1800 \qquad \varepsilon = \frac{2k_1}{M} = \frac{4000}{5} = 800$$

$$\omega_1 = \sqrt{(\alpha+\beta) - \sqrt{(\alpha-\beta)^2 + \gamma\varepsilon}} = 21.6478 \text{ rad/s}$$

$$\omega_2 = \sqrt{(\alpha+\beta) + \sqrt{(\alpha-\beta)^2 + \gamma\varepsilon}} = 52.2625 \text{ rad/s}.$$

with

$$\omega = \omega_1 \Rightarrow \frac{A_2}{A_1} = \frac{(\alpha-\beta) + \sqrt{(\alpha-\beta)^2 + \gamma\varepsilon}}{\gamma} = .7071 \Rightarrow \begin{matrix} x_1 = A_1 \sin(21.6478t + \varphi) \\ x_2 = .7071 A_1 \sin(\quad " \quad) \end{matrix}$$

$$\omega = \omega_2 \Rightarrow \frac{A_2}{A_1} = \frac{(\alpha-\beta) - \sqrt{\qquad}}{\gamma} = -.7071 \Rightarrow \begin{matrix} x_1 = A_1 \sin(52.2625t + \varphi) \\ x_2 = -.7071 A_1 \sin(\quad " \quad) \end{matrix}$$

m

$E_1 = 0 \quad PE_1 = (mgh)$

ref. line

$PE_2 \to 0$

$E_2 = \frac{1}{2}(m v^2)$ for each mass

When the system is dropped from the height of $1\,m \Rightarrow$ when the system hits ground assume that $x_1(t=0) = x_2(t=0) = 0$ (when vibration begin). Also since PE system is converted to KE $\Rightarrow \dot{x}_1(t=0) = \dot{x}_2(t=0) = \sqrt{2gh}$. thus with these data, we can find

$$0 = x_{1_T}(t=0) = A_1 \sin(21.6478 \cdot 0 + \varphi) + \tilde{A}_1 \sin(52.2625 \cdot 0 + \tilde{\varphi})$$

$$0 = x_{2_T}(t=0) = (.7071)\left[A_1 \sin(21.6478 \cdot 0 + \varphi) - \tilde{A}_1 \sin(52.2625 \cdot 0 + \tilde{\varphi})\right]$$

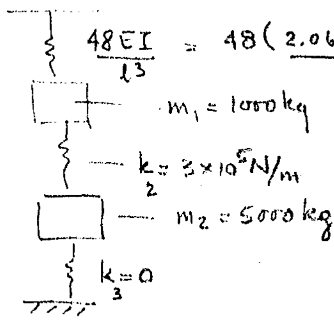
$$4.4294\,\frac{m}{s} = \begin{cases} \sqrt{2gh} = \dot{x}_{1_T}(t=0) = A_1\omega_1 \cos(\omega_1 \cdot 0 + \varphi) + \tilde{A}_1 \omega_2 \cos(\omega_2 \cdot 0 + \tilde{\varphi}) \\ \sqrt{2gh} = \dot{x}_{2_T}(t=0) = .7071\left[A_1 \omega_1 \cos(\omega_1 \cdot 0 + \varphi) - \tilde{A}_1 \omega_2 \cos(\omega_2 \cdot 0 + \tilde{\varphi})\right] \end{cases}$$

Using the eqns in the books (5.18) you can find $x_{1_{max}} = -.2470$

The assumption here is that each box doesn't start vibrating until the outer box & inner box have hit ground. This assumes that the mass m & mass M have the same volume, thus the centers of mass for each ~~reach the~~ ~~ground~~ travels the ~~center~~ same distance.

5.6 in 3rd ed

~~5.5~~ for a beam, simply supported (as in Ex. 1.1 p 18) $\quad K_b = \frac{48EI}{\ell^3}$



$$\frac{48EI}{\ell^3} = \frac{48(2.06\times10^{11})(.02)}{(40)^3} = 3.09\times10^6 \text{ N/m} = k_1 = k_b$$

We can use what we did in class w/ $k_3 = 0$

$\therefore\ 2\alpha = \frac{k_1+k_2}{m_1} = 3.39\times10^3 \qquad \gamma = \frac{k_2}{m_1} = 3\times10^2$

$2\beta = \frac{k_2}{m_2} = 60 \qquad \varepsilon = \frac{k_2}{m_2} = 60$

$$\omega_1 = \sqrt{(\alpha+\beta) - \sqrt{(\alpha-\beta)^2+\varepsilon\gamma}} = 7.38941 \text{ rad/s}$$

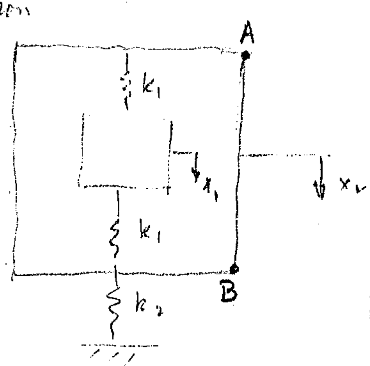
$$\omega_2 = \sqrt{(\alpha+\beta) + \sqrt{(\alpha-\beta)^2+\varepsilon\gamma}} = 58.27 \text{ rad/s}$$

if $\omega = \omega_1 \Rightarrow \frac{A_2}{A_1} = \frac{(\alpha-\beta)+\sqrt{(\alpha-\beta)^2+\gamma\varepsilon}}{\gamma} = 11.118 \Rightarrow$ $x_1 = A_1 \sin(7.38941t+\varphi)$, $x_2 = 11.118A_1 \sin(\ \text{"}\ )$

if $\omega = \omega_2 \Rightarrow \frac{A_2}{A_1} = \frac{(\alpha-\beta)-\sqrt{(\alpha-\beta)^2+\gamma\varepsilon}}{\gamma} = -.01799 \Rightarrow$ $x_1 = A_1 \sin(58.27t+\varphi)$, $x_2 = -.01799A_1 \sin(\ \text{"}\ )$

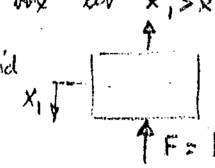
Not in 3rd edition

5.6



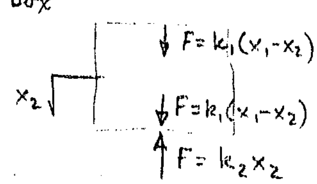
For the inner box let $x_1 > x_2$ at A & B displ $= x_2$

since outer box is rigid



$$\therefore\ m\ddot{x}_1 = -2k_1(x_1-x_2) \qquad (1)$$

For outer box



thus

$$M\ddot{x}_2 = 2k_1(x_1-x_2) - k_2x_2 \qquad (2)$$

$\therefore\quad m\ddot{x}_1 + 2k_1x_1 - 2k_1x_2 = 0 \qquad (1) \qquad$ let $x_1 = A_1\sin(\omega t+\varphi)$

$\quad M\ddot{x}_2 + (k_2+2k_1)x_2 - 2k_1x_1 = 0 \qquad (2) \qquad x_2 = A_2\sin(\omega t+\varphi)$

then $\quad (-m\omega^2 + 2k_1)A_1 - 2k_1A_2 = 0$

$\quad -2k_1A_1 + \left[-M\omega^2 + (k_2+2k_1)\right]A_2 = 0$ $\Big\}$ these are the amplitude eqns

$$\Rightarrow \quad m\ddot{x} = P \quad \text{or} \quad \int_{t<0}^{t=0} m\ddot{x}\,dt = \int_{t<0}^{t=0} P\,dt = 1 \Rightarrow m\dot{x}\Big|_{t<0}^{t^*=0} = 1 = m\dot{x}\Big|_{t=0}$$

- IF $t^* \sim 0$ LIKE SAYING $\dot{x}\big|_{t=0} = 1/m$
- now ~~$m\ddot{x}+kx=P$ has the complimentary soln~~ now since P acts over short duration governing eqn is $m\ddot{x}+kx=0$ whose soln is

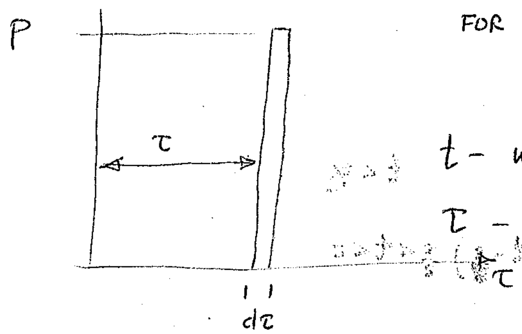
$$x = A\cos\omega t + B\sin\omega t \qquad \omega = \sqrt{\frac{k}{m}}$$

@ $t=0$ $x=0 \Rightarrow A=0$

@ $t=0$ $\dot{x} = B\omega = \frac{1}{m} \Rightarrow B = \frac{1}{m\omega} = \frac{1}{\sqrt{mk}}$

- thus $x = \frac{1}{\sqrt{mk}}\sin\omega(t-0)$ ⇐ response to unit impulse at time t when impulse is a $t=0$

- IF WE SAY $P(t)$ IS A SET OF IMPULSES HAVING IMPULSE $= P\,d\tau$ AREA UNDER GRAPH



FOR EACH IMPULSE $dx = \frac{P\,d\tau}{\sqrt{mk}}\sin\omega(t-\tau)$

t – when response is measured

τ – time when impulse took place (independent variable

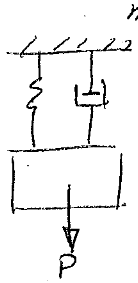
- $x_{Total} = \Sigma dx = \int dx = \Sigma \frac{P\,d\tau}{\sqrt{mk}}\sin\omega(t-\tau) = \int_0^t \frac{P}{\sqrt{mk}}\sin\omega(t-\tau)\,d\tau$

$$\frac{1}{m\omega_n} = \frac{1}{m\sqrt{\frac{k}{m}}} \qquad x = \int_0^t \frac{P}{\sqrt{mk}}\sin\omega(t-\tau)\,d\tau \quad \text{is Duhamel's integral convolution integral}$$

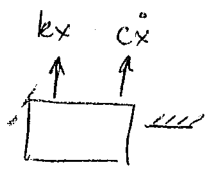
for nonzero x_0, $\dot{x}_0$ $\quad x = x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \int_0^t \frac{P}{\sqrt{mk}}\sin\omega_n(t-\tau)\,d\tau.$

FOR A VISCOUSLY DAMPED SYSTEM

$m\ddot{x} + c\dot{x} + kx = 0$ before the impulse & after impulse



IF mass at rest at $t<0$ $\quad x=0 \quad \dot{x}=0.$



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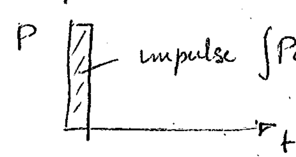
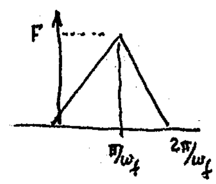


- $\int m\ddot{x}\,dt = \int P\,dt = m\dot{x}\Big/_{t<0}^{t=0} = 1 \quad \therefore \dot{x}\Big/_{t=0} = \frac{1}{m}$

- ALSO $x=0$ @ $t=0$

IF SYSTEM IS UNDERDAMPED $\quad x = e^{-\zeta\omega t}(A\cos\omega_d t + B\sin\omega_d t) \qquad \omega_d = \sqrt{1-\zeta^2}\,\omega$



for 5.28 steidel $t < \frac{2\pi}{\omega_f}$ $\qquad \frac{\omega_f F t}{\pi} \quad t < \frac{\pi}{\omega_f}$

$-\frac{\omega_f F}{\pi}\left(t - \frac{2\pi}{\omega_f}\right) \quad \frac{\pi}{\omega_f} < t < \frac{2\pi}{\omega_f} \qquad 2F - \frac{\omega_f F t}{\pi}$

$$\frac{\omega_f F}{\pi} \int_0^t \frac{(\tau)}{m\omega_n} \sin(\omega_n(t-\tau))\, d\tau \qquad t < \frac{\pi}{\omega_f}$$

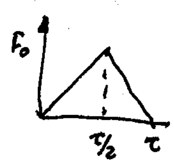
$$\frac{\omega_f F}{\pi m \omega_n^3} \int_0^{\bar{t}} s \sin(\bar{t} - s)\, ds = \frac{\omega_f F}{\pi m \omega_n^3}\left[\omega_n t - \sin \omega_n t\right]$$

$+ s\cos(\bar{t}-s) + \sin(\bar{t}-s)$

$s \sin(\bar{t}-s) + \cos(\ ) + \cos(\ )$

when $t = \frac{\pi}{\omega_f} \Rightarrow \frac{\omega_f F}{\pi m \omega_n^3}\left[\omega_n \frac{\pi}{\omega_f} - \sin \pi \frac{\omega_n}{\omega_f}\right]$

or



$\frac{2F_0}{\tau} t \qquad t < \tau/2$

$-\frac{2F_0}{\tau}\left(t - \frac{\tau}{2}\right) \quad \frac{\tau}{2} < t < \tau$

$$\frac{1}{m\omega_n} \cdot \frac{2F_0}{\tau} \int_0^t \bar{t} \sin(\omega_n[t - \bar{t}])\, d\bar{t} \qquad \text{let } \omega_n \bar{t} = s$$

$$\frac{2F_0}{m\omega_n \tau} \int_0 \frac{1}{\omega_n^2} \left\{ s \sin(\omega_n t - s)\, ds \right.$$

$\int u\,dv = uv - \int v\,du \qquad u = s \quad du = ds$

$dv = \sin(\ )ds \quad v = \cos(\omega_n t - s)$

$s\cos(\omega_n t - s) - \int \cos(\omega_n t - s)\, ds$

$-\left[-\sin(\omega_n t - s)\right]$

SESSION #18

CH 5

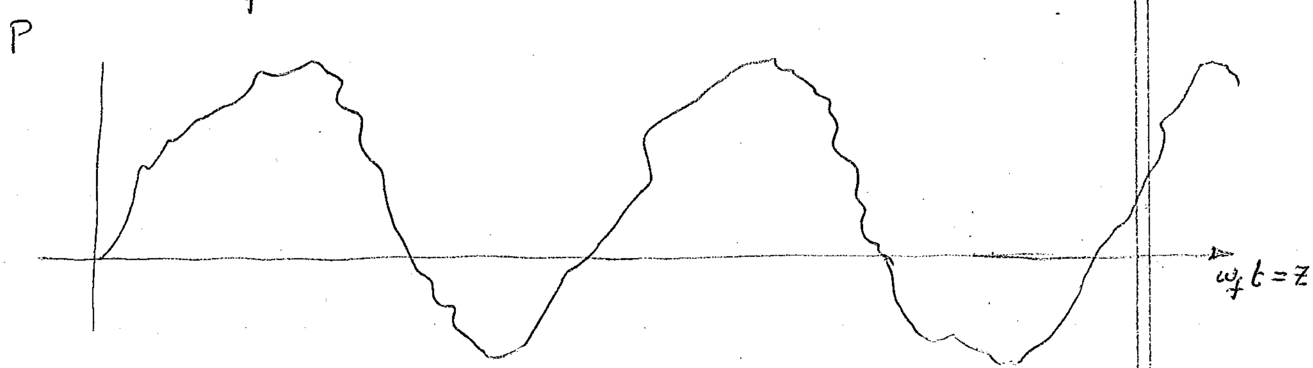
- HAVE STUDIED FORCING FUNCTIONS WHICH ARE HARMONIC PERIODIC FNS
- WHAT IF THE GENERAL FORCING FN IS PERIODIC ONLY

- ANY PERIODIC FN CAN BE WRITTEN AS A FOURIER SERIES

$$P(z) = \frac{a_0}{2} + a_1 \cos z + a_2 \cos 2z + \dots + a_k \cos kz + \dots$$
$$+ b_1 \sin z + b_2 \sin 2z + \dots + b_k \sin kz + \dots$$

or $$P(t) = \frac{a_0}{2} + a_1 \cos\omega_f t + a_2 \cos 2\omega_f t + \dots + b_1 \sin\omega_f t + b_2 \sin 2\omega_f t + \dots$$

$z = \omega_f t$



where $a_0 = \frac{1}{\pi}\int_0^{2\pi} P\,dz$ $\quad a_k = \frac{1}{\pi}\int_0^{2\pi} P\cos kz\,dz$ $\quad b_k = \frac{1}{\pi}\int_0^{2\pi} P\sin kz\,dz$

$= \frac{\omega_f}{\pi}\int_0^{\tau} P\,dt = \frac{2}{\tau}\int_0^{\tau} P\,dt$ $\quad \frac{\omega_f}{\pi}\int_0^{\tau} P\cos l\omega_f t\,dt$ $\quad$ but $2\pi f = \omega_f$ $\therefore$ $2f = \frac{\omega_f}{\pi} = \frac{2}{\tau}$ $\quad$ or $\frac{2}{\tau}\int_0^{\tau} P\cos l\omega_f t\,dt$

- THESE ARE THE REPRESENTATIONS OF THE FOURIER COEFFS

use l

- SUPPOSE SPRING-MASS-DASHPOT SYSTEM IS SUBJECTED TO A GENERAL FORCING FUNCTION

$$P = \sum_{k=0}^{\infty} a_k \cos kz + \sum_{k=0}^{\infty} b_k \sin kz \qquad z = \omega_f t$$

- THE PRINCIPLE OF SUPERPOSITION APPLIES
- STEADY STATE SOLUTION (PARTICULAR SOLN)

$$x_p = \sum_{k=0}^{\infty} x_{pk} + \sum_{m=0}^{\infty} x_{pk}$$

for the a_0 term $\quad m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} \quad x_{p_0} = \frac{a_0/k}{2}$

for the a_k term $\quad m\ddot{x} + c\dot{x} + kx = a_k \cos k\omega_f t \quad x_{pk} = X_k \cos(k\omega_f t - \psi_k)$

$$\omega_n = \sqrt{\frac{k}{m}} \;;\quad \bar{X}_n = \frac{a_n/k}{\sqrt{(1-r_n^2)^2 + (2\zeta r_n)^2}} \;;\quad r_k = \frac{k\omega_f}{\omega} = kr \;;\quad \tan\psi_k = \frac{2\zeta r_k}{1-r_k^2}$$

FOR THE b_ℓ term $\quad m\ddot{x} + c\dot{x} + kx = b_\ell \sin \ell\omega_f t \Rightarrow x_{P\ell} = X_\ell \sin(\ell\omega_f t - \psi_\ell)$

$$\omega = \sqrt{\frac{k}{m}} \; ; \quad X_\ell = \frac{b_\ell/k}{\sqrt{(1-r_\ell^2)^2 + (2\zeta r_\ell)^2}} \; ; \quad r_\ell = \frac{\ell\omega_f}{\omega} = \ell r \; ; \quad \tan\psi_\ell = \frac{2\zeta r_\ell}{1-r_\ell^2}$$

THUS

$$x_p = \frac{a_0}{2k} + \sum_{\ell=1}^{\infty} \frac{(a_\ell/k)\cos(\ell\omega_f t - \psi_n)}{\sqrt{(1-r_\ell^2)^2 + (2\zeta r_\ell)^2}} + \sum_{\ell=1}^{\infty} \frac{(b_\ell/k)\sin(\ell\omega_f t - \psi_\ell)}{\sqrt{(1-r_\ell^2)^2 + (2\zeta r_\ell)^2}}$$

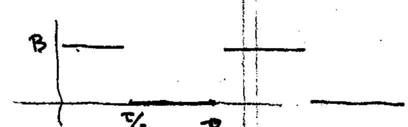
- WE CANNOT GIVE ANY INFO ABOUT X_p VS. P AS BEFORE / ie X_p is in/out of phase w/ P

REASONS {
- LEAD OR LAG ANGLE ψ_n, ψ_m and r_n, r_m WHICH DEPEND ON n, m
- AMPLITUDE is dependent on n

- AS $\begin{matrix} n \\ m \end{matrix} \to \infty \quad \frac{1}{\sqrt{(1-r_n^2)^2 + (2\zeta r_n)^2}} \to 0 \Rightarrow$ thus NEED ONLY FEW TERMS TO FIND X

- NOTE THIS IS ONLY FOR STEADY STATE — TO INCLUDE TRANSIENT ⇒ COMPLICATED

PROBLEM 5-1 $\qquad P(z) = \begin{cases} B & 0 \le z \le \pi \\ 0 & \pi \le z \le 2\pi \end{cases}$



Now $\quad a_0 = \frac{1}{\pi}\int_0^{2\pi} P\,dz = \frac{B}{\pi} z\Big|_0^{\pi} = \frac{B}{1} \qquad a_0 = \frac{2}{\tau}\int_0^{\tau/2} B\,dt = B$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} P\cos nz\,dz = \frac{1}{\pi}\int_0^{\pi} B\cos nz\,dz = \frac{B}{n\pi}\sin nz\Big|_0^{\pi} = 0$$

$a_n = \frac{2}{\tau}\int_0^{\tau/2} B\cos n\omega_f t\,dt = \frac{2}{\tau}\frac{B \sin n\omega_f t}{n\omega_f}\Big|_0^{\tau/2} = 0$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} P\sin nz\,dz = \frac{1}{\pi}\int_0^{\pi} B\sin nz\,dz = \frac{-B}{n\pi}\cos nz\Big|_0^{\pi} = \frac{-B}{n\pi}\left\{(-1)^n - 1\right\}$$

$b_n = \frac{2}{\tau}\int_0^{\tau/2} B\sin n\omega_f t\,dt = \frac{2}{\tau}\frac{B}{n\omega_f}\left(-\cos n\omega_f t\right)\Big|_0^{\tau/2} = \frac{-2B}{\tau n\omega_f}(\cos n\pi - 1) = \frac{-2B}{2n\pi}(\;\;)$

DO. 4.1 4.13 $\quad$ let $n = 2\ell - 1 \quad \ell = 1, 2, 3, \ldots \quad n = 1, 3, 5, \ldots$

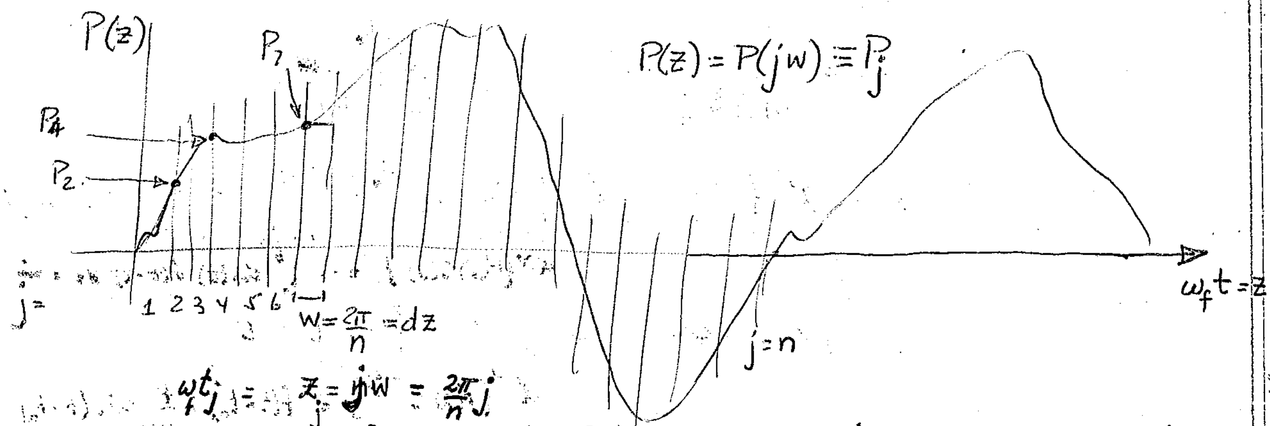
$$b_\ell = +\frac{2B}{(2\ell-1)\pi}$$

$$\therefore \quad x_p = \frac{B}{2k} + \sum_{\ell=1}^{\infty} \frac{2B/k}{(2\ell-1)\pi} \frac{\sin(\overbrace{(2\ell-1)}^{n}\omega_f t - \psi_\ell)}{\sqrt{(1 - \underbrace{(2\ell-1)^2 r^2}_{r_n^2})^2 + (2\zeta \underbrace{(2\ell-1)r}_{r_n})^2}} \; ; \qquad \tan\psi_\ell = \frac{2\zeta \overbrace{(2\ell-1)r}^{r_n}}{1 - \underbrace{(2\ell-1)^2 r^2}_{r_n^2}}$$

$$P = \frac{B}{2} + \sum_{\ell=1}^{\infty} \frac{2B}{(2\ell-1)\pi} \sin(\overbrace{(2\ell-1)}^{n}\omega_f t)$$

SECTION 5-5 RESPONSE UNDER PERIODIC FORCE WHICH IS IRREGULAR

- IF YOU REMEMBER THAT EACH INTEGRAL REPRESENTS AN AREA UNDER A CURVE
- IF GIVEN AN EXPERIMENTAL CURVE FOR P WHICH IS PERIODIC
  - BREAK PERIODIC INTERVAL INTO n segments SO THAT THE SEGMENT WIDTH $W = \frac{2\pi}{n}$
  - DEFINE FOR EACH SEGMENT $P(z) = P(jW) = P_j$



$\omega_f t_j = z_j = jW = \frac{2\pi}{n} j$

$\omega_f \tau = 2\pi$ or $\omega_f = 2\pi/\tau$ $\therefore$ $\omega_f t_j = \frac{2\pi}{\tau} t_j = \frac{2\pi}{n} j$ $\therefore$ $t_j/\tau = j/n$ $\qquad N\Delta t = \tau$

now $a_0 = \frac{1}{\pi}\int_0^{2\pi} P\,dz = \frac{1}{\pi}\sum_{j=1}^{n} P_j W = \frac{2}{n}\sum P_j \qquad \frac{2}{\tau}\int_0^{\tau} P\,dt = \frac{2}{N\Delta t}\sum P_j \Delta t = \frac{2}{N}\sum P_j$

for m^{th} constant: $a_m = \frac{1}{\pi}\int_0^{2\pi} P\cos mz\,dz = \frac{1}{\pi}\sum_{j=1}^{n} P_j \cos(mjW)\,W = \frac{2}{n}\sum_{j=1}^{n} P_j \cos(mjW)$ $\quad mj\cdot\frac{2\pi}{n}$, $\cos(2\pi m\cdot t_j/\tau)$

$b_m = \frac{1}{\pi}\int_0^{2\pi} P\sin mz\,dz = \frac{1}{\pi}\sum_{j=1}^{n} P_j \sin(mjW)\,W = \frac{2}{n}\sum_{j=1}^{n} P_j \sin(mjW)$ $\quad mj\cdot\frac{2\pi}{n}$, $\sin(2\pi m t_j/\tau)$

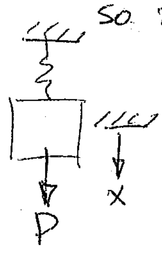
$a_m = \frac{2}{\tau}\int_0^{\tau} P\cos m\omega_f t\,dt$
$= \frac{2}{N\Delta t}\sum_{j=1}^{N} P_j \cos m\frac{2\pi}{\tau}t_j\,\Delta t = \frac{2}{N}\sum_j P_j \cos\frac{m2\pi j}{N}$

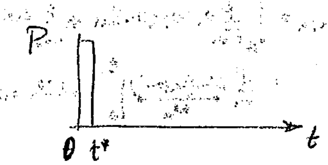
$\frac{m2\pi t_j}{\tau} = \frac{m\cdot 2\pi j}{N}$

- METHODS DISCUSSED SO FAR WILL BE GOOD FOR STEADY STATE
- TO INCLUDE TRANSIENT NEED DIFFERENT APPROACH OTHERWISE – IT WOULD BE CUMBERSOME
- IF P IS NOT PERIODIC PREVIOUS METHOD IS NOT GOOD
  - NEED NEW METHOD

CONSIDER A MASS-SPRING SYSTEM UNDER AN IMPULSIVE LOAD P

SO THAT $\int_{t<0}^{t^*=0} P\,dt = 1$





$m\ddot{x} + kx = P$

ASSUME THAT FOR $t<0$ MASS STARTS w/ $x = 0$, $\dot{x} = 0$ (AT REST) by Laplace Transforms.

$$\ddot{x} + \omega_n^2 x = \frac{P}{m}\delta(t-t_0) \qquad \int_0^\infty e^{-st}\delta(t-t_0)d\tau = e^{-st_0}$$

$x(t=0)=0$
$\dot{x}(t=0)=0$

$$s^2 X + \omega_n^2 X = \frac{P}{m}e^{-st_0}$$

$$\therefore \quad X = \frac{P}{m(s^2+\omega_n^2)}e^{-st} = \frac{\omega_n e^{-st_0}}{s^2+\omega_n^2}\cdot\frac{P}{m\omega_n}$$

$$x = \mathcal{L}^{-1}\{X\} = \mathcal{L}^{-1}\left\{\frac{\omega_n}{s^2+\omega_n^2}e^{-st_0}\right\}\cdot\frac{P}{m\omega_n} = \frac{P}{m\omega_n}\sin\omega_n(t-t_0)$$

~~$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t \sin(\tau)\,\delta(t-\tau)\,d\tau = \sin$~~

for $t > \tau$ & 0 for $t < \tau$

$$\mathcal{L}^{-1}\{f(s)e^{-t_0 s}\} = f(t-t_0) = \frac{P}{m\omega_n}\sin(t-t_0)$$

$s^2 X - s\dot{x} - \dot{x}$

$\therefore$

$(s^2+\omega_n^2)X = \frac{P}{m} + s\dot{x} + \dot{x}$

$$\int P\,\delta(t-\tau)\,d\tau = 1$$

$\mathcal{L}^{-1}\frac{s}{s^2+\omega_n^2} = \cos$

$\mathcal{L}^{-1}\frac{1}{s^2+\omega_n^2} = \frac{1}{\omega_n}\sin$

$$\ddot{x} + \omega_n^2 x = \frac{P(t)}{m}$$

$$s^2 X + \cancel{s(x(t=0))} + \cancel{\dot{x}(t=0)} + \omega^2 X = \frac{p(s)}{m}$$

$$X = \frac{1}{s^2+\omega_n^2}\,\frac{p(s)}{m} = \frac{1}{m\omega_n}\,\frac{\omega_n}{s^2+\omega_n^2}\cdot p(s)$$

$$\mathcal{L}^{-1}(X) = x \qquad = \mathcal{L}^{-1}\left[\qquad\qquad\right] = \frac{1}{m\omega_n}\mathcal{L}^{-1}\{f(s)g(s)\}$$

Leibnitz

$$\frac{d}{dx}\int_A^B f(x,t)\,dt = \int_A^B \frac{\partial f}{\partial x}dt + f(x,B)\frac{dB}{dx} - f(x,A)\frac{dA}{dx}$$

let $x = t$, $t = \tau$

$$\frac{d}{dt}\int_0^t \frac{P}{\sqrt{mk}}\sin\omega_n(t-\tau)d\tau = \int_A^B \frac{P}{\sqrt{mk}}\omega_n\cos(\omega_n(t-\tau))d\tau + \cancel{\frac{P}{\sqrt{mk}}\sin\omega_n(t-t)} - \cancel{\frac{P}{\sqrt{mk}}\sin\omega_n(t-0)\cdot 0}$$

$$= \frac{-P}{m}\frac{\sin(\omega_n(t-\tau))}{\omega_n}\Big|_0^t \quad \text{which} = 0 \text{ when } t=0$$

$f(t) = \sin\omega_n t$
$g(t) = p(t)$

$$\mathcal{L}^{-1}\{f(s)\,g(s)\} = \int_0^t p(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$= \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

FOR UNDAMPED SYSTEM $m\ddot{x} + kx = P(t)$ $P(t)$ general forcing function

$$x(t) = \int_0^t \frac{P(\tau)}{\sqrt{mk}} \sin \omega_n (t-\tau)\, d\tau$$

τ - integrating variable

FOR DAMPED SYSTEM $m\ddot{x} + c\dot{x} + kx = P(t)$

1) IF UNDERDAMPED $\zeta < 1$

$$x(t) = \frac{1}{m\omega_d} \int_0^t P(\tau)\, e^{-\zeta\omega_n (t-\tau)} \sin \omega_d (t-\tau)\, d\tau$$

2) IF CRITICALLY DAMPED $\zeta = 1$

$$x(t) = \frac{1}{m} \int_0^t P(\tau)\, [t-\tau]\, e^{-\omega_n (t-\tau)}\, d\tau$$

3) IF OVERDAMPED $\zeta > 1$

$$x(t) = \frac{1}{m\omega_n\sqrt{\zeta^2-1}} \int_0^t P(\tau)\, e^{-\zeta\omega_n (t-\tau)} \sinh\left[\omega_n\sqrt{\zeta^2-1}\,(t-\tau)\right] d\tau$$

Some formulas you can use. For example, if $z = t-\tau$ $dz = -d\tau$ and

$$\int e^{az}\, bz\, dz = be^{az}\left[\frac{z}{a} - \frac{1}{a^2}\right]$$

$$\int e^{az} \sinh bz\, dz = \frac{1}{2}\left[\frac{e^{(a+b)z}}{a+b} - \frac{e^{(a-b)z}}{a-b}\right] \quad \text{if } a \neq b, -b$$

$$\frac{1}{2}\left[\frac{e^{2az}}{2a} - z\right] \quad \text{if } a = b$$

$$\frac{1}{2}\left[z - \frac{e^{2az}}{2a}\right] \quad \text{if } a = -b$$

FOR UNDAMPED SYSTEM $m\ddot{x} + kx = P(t)$ $P(t)$ general forcing function

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Some formulas you can use. For example, if $z = t - \tau$ $dz = -d\tau$ and

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FOR DAMPED SYSTEM $m\ddot{x} + c\dot{x} + kx = P(t)$

1) IF UNDERDAMPED $\zeta < 1$

$$x(t) = \frac{1}{m\omega_d} \int_0^t P(\tau)\, e^{-\zeta\omega_n (t-\tau)} \sin \omega_d (t-\tau)\, d\tau$$

2) IF CRITICALLY DAMPED $\zeta = 1$

$$x(t) = \frac{1}{m} \int_0^t P(\tau)\, [t-\tau]\, e^{-\omega_n (t-\tau)}\, d\tau$$

3) IF OVERDAMPED $\zeta > 1$

$$x(t) = \frac{1}{m\omega_n \sqrt{\zeta^2 - 1}} \int_0^t P(\tau)\, e^{-\zeta\omega_n (t-\tau)} \sinh\left[\omega_n \sqrt{\zeta^2-1}\,(t-\tau)\right] d\tau$$

Some formulas you can use. For example, if $z = t - \tau$ $dz = -d\tau$ and

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$$\frac{1}{2}\left[z - \frac{e^{2az}}{2a}\right] \quad \text{if } a = -b$$

THUS $\quad A = 0 \qquad B = \frac{1}{m\omega_d}$

- Thus $x = \frac{1}{m\omega_d} e^{-\zeta\omega(t-0)} \sin \omega_d(t-0)$ response to unit impulse at $t=0$
- FOR AN IMPULSE $= P d\tau \Rightarrow dx = \frac{P d\tau}{m\omega_d} e^{-\zeta\omega(t-\tau)} \sin \omega_d (t-\tau)$

$\therefore$ TOTAL RESPONSE TO ANY FORCE P IS

- $$x = \frac{1}{m\omega_d} \int_0^t P e^{-\zeta\omega_n(t-\tau)} \sin \omega_d (t-\tau)\, d\tau$$

if $x_0, \dot{x}_0 \neq 0 \qquad x = x_h + \frac{1}{m\omega_d} \int_0^t P e^{-\zeta\omega_n(t-\tau)} \sin \omega_d (t-\tau)\, d\tau$

HW Prob 5-6, 5-28 Due Monday 17 March.

$$x_h = \left(x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \sin\omega_d t\right) e^{-\zeta\omega_n t}$$

SESSION #19

Chapter 5 in Rao

READ SECTIONS 4-16 THRU. 4-18 IF NOT ALREADY DONE SO

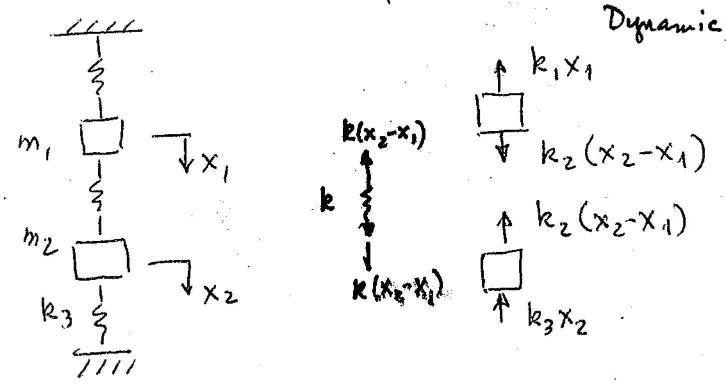
READ CH. 6

- SO FAR HAVE LOOKED AT SDOF systems
- FREE UNDAMPED VIBS $\longrightarrow$ HARMONIC OSCILLATIONS $\quad \omega_n = \sqrt{\frac{k}{m}}$
- ONE FREQ

- WANT TO LOOK AT 2 DOF SYSTEMS
- REQUIRES 2 INDEP COORDINATES
- LOOK AT UNDAMPED CASE FIRST
- FOR EACH DEGREE OF FREEDOM $\Rightarrow$ AN ω EXISTS

CONSIDER $\quad x_2 > x_1$

Dynamic FBD - measured from static equilb of system



$$m_1\ddot{x}_1 = k_2(x_2-x_1) - k_1 x_1$$

$$m_2\ddot{x}_2 = -k_3 x_2 - k_2(x_2-x_1)$$

$A_1 Z_1 \sin\varphi_1 = A_2 Z_2 \sin\varphi_2$ $\qquad$ $A_2 Z_3 \sin\varphi_2 = A_1 Z_4 \sin\varphi_1$

$$\frac{A_1 Z_1}{A_2 Z_2} = \frac{\sin\varphi_2}{\sin\varphi_1} = \frac{A_1 Z_4}{A_2 Z_3}$$

$$A_1 A_2 \left[Z_1 Z_3 - Z_2 Z_4 \right] = 0 \qquad \frac{Z_1}{Z_2} = \frac{Z_4}{Z_3}$$

$Z_1 \quad Z_2$

$Z_4 \quad Z_3$

$$A_1 \frac{\cancel{Z_2} Z_4}{Z_3} \sin\varphi_1 = A_2 \cancel{Z_2} \sin\varphi_2$$

$$\ddot{x}_1 + \left(\frac{k_1+k_2}{m_1}\right)x_1 - \frac{k_2}{m_1}x_2 = 0$$

HOMOG EQS – CONST COEFF

$$\ddot{x}_2 + \frac{(k_2+k_3)}{m_2}x_2 - \frac{k_2}{m_2}x_1 = 0$$

COUPLED EQNS

METHODS OF SOLUTION

A. ① TAKE 2 deriv of eq 1 TO FIND EQ OF $\ddddot{x}_1$ & $\ddot{x}_1$ IN TERMS OF $\ddot{x}_2$

② EQ. (1) GIVES x_2 IN TERMS OF $\ddot{x}_1, x_1$

} ELIMINATE x_2

B. LET $x_1 = A_1 \sin(\omega_1 t + \varphi_1)$
$x_2 = A_2 \sin(\omega_2 t + \varphi_2)$ } most general

$$\left[-A_1\omega_1^2 + \left(\frac{k_1+k_2}{m_1}\right)A_1\right]\sin(\omega_1 t+\varphi_1) - \frac{k_2}{m_1}A_2 \sin(\omega_2 t+\varphi_2) = 0 \qquad (1)$$
$$\left[-A_2\omega_2^2 + \frac{k_2+k_3}{m_2}A_2\right]\sin(\omega_2 t+\varphi_2) - \frac{k_2}{m_2}A_1 \sin(\omega_1 t+\varphi_1) = 0$$

- True $\forall t$ $\therefore$ pick $t=0$

$$\left[-A_1\omega_1^2 + \left(\frac{k_1+k_2}{m_1}\right)A_1\right]\sin\varphi_1 - \frac{k_2}{m_1}A_2\sin\varphi_2 = 0 \qquad (2)$$
$$\left[-A_2\omega_2^2 + \frac{k_2+k_3}{m_2}A_2\right]\sin\varphi_2 - \frac{k_2}{m_2}A_1\sin\varphi_1 = 0$$

TRIVIAL CASE

- IF φ_1 & φ_2 are independent $\Rightarrow A_1 = A_2 = 0$ if $\omega_1, k_1, k_2, k_3, \omega_2 \neq 0$
- IF $\varphi_1 = \varphi_2 \Rightarrow A_1\left[\left(\frac{k_1+k_2}{m_1}\right) - \omega_1^2\right] = \frac{k_2}{m_1}A_2$ &.

$$\frac{A_1}{A_2}\frac{[\quad]}{k_2/m_1} = \cancel{\text{const}} = \frac{\sin(\omega_2 t+\varphi_2)}{\sin(\omega_1 t+\varphi_1)} \quad \forall t \text{ from (1)}$$

- $\Rightarrow \omega_2 = \omega_1$ $\qquad \sin(\omega_2 t+\varphi_2) = \sin(\omega_1 t+\varphi_1) = \sin(\omega t+\varphi)$

$\Longrightarrow$

$$\left.\begin{aligned} x_1 &= A_1 \sin(\omega t+\varphi) \\ x_2 &= A_2 \sin(\omega t+\varphi) \end{aligned}\right\}$$

x_1, x_2 only vary in amplitude

BACK TO (1) $\Rightarrow$ Amplitude eqns.

$$\left\{A_1\left[\left(\frac{k_1+k_2}{m_1}\right) - \omega^2\right] - A_2\left[\frac{k_2}{m_1}\right]\right\}\sin(\omega t+\varphi) = 0$$
$$\left\{A_1\left[-\frac{k_2}{m_2}\right] + A_2\left[\left(\frac{k_2+k_3}{m_2}\right) - \omega^2\right]\right\}\sin(\omega t+\varphi) = 0$$

this is true $\forall t \Rightarrow$ either $A_1 = A_2 = 0$ TRIVIAL CASE

or

$$\begin{vmatrix} \left(\frac{k_1+k_2}{m_1}\right)-\omega^2 & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \left(\frac{k_2+k_3}{m_2}\right)-\omega^2 \end{vmatrix} = 0$$

$$m_1 m_2 \omega^4 - \left[m_2\left(\frac{k_1+k_2}{\cancel{m_1}}\right) + \left(\frac{k_2+k_3}{\cancel{m_2}}\right)m_1\right]\omega^2 + \left(\frac{k_1+k_2}{\cancel{m_1}}\right)\left(\frac{k_2+k_3}{\cancel{m_2}}\right) - \frac{k_2^2}{\cancel{m_1 m_2}} = 0$$

$$\omega^4 - [\qquad\qquad]\omega^2 + \frac{k_1k_2+k_2k_3+k_1k_3}{m_1 m_2} = 0$$

- THIS IN CHARACTERISTIC EQ IN ω^2 OR FREQ. EQN

THIS NORMALLY HAS 2 POSITIVE SOLUTIONS SINCE ALL k's, m's > 0

- LOOK AT SPECIAL CASE $k_1 = k_2 = k_3 = k$ $\quad m_1 = m_2 = m$

$$\omega^4 - \left[4\frac{k}{m}\right]\omega^2 + \frac{3k^2}{m^2} = 0$$

$$\left(\omega^2 - 3\frac{k}{m}\right)\left(\omega^2 - \frac{k}{m}\right) = 0 \quad \text{or} \quad \omega_2 = \sqrt{\frac{3k}{m}} \quad \omega_1 = \sqrt{\frac{k}{m}}$$

FUNDAMENTAL OR FIRST MODE

LET $\omega = \omega_1 = \sqrt{\frac{k}{m}}$ put into Amplitude Eqn

$$\left[\left(\frac{k_1+k_2}{m_1} - \omega^2\right)A_1 - \frac{k_2}{m_1}A_2 = 0 \right. \quad \Rightarrow \quad \left[\left(\frac{2k}{m} - \frac{k}{m}\right)A_1 - \frac{k}{m}A_2\right] = 0$$

$$\left(-\frac{k_2}{m_2}\right)A_1 + \left[\left(\frac{k_2+k_1}{m_2}\right) - \omega^2\right]A_2 = 0 \qquad \left[-\frac{k}{m}A_1 + \left(\frac{2k}{m} - \frac{k}{m}\right)A_2\right] = 0$$

NOTE THAT 1^{st} eq $= 2^{nd}$ eq $\Rightarrow$ DETERMINANT $= 0$ CAN'T ACTUALLY SOLVE FOR A_1 & A_2 BUT ONLY GET RELATION OF A_2 TO A_1

$$\frac{k}{m}A_1 - \frac{k}{m}A_2 = 0 \quad \Rightarrow \quad A_1 = A_2 \quad A_2/A_1 = 1 = \eta_1$$

$$\Rightarrow \quad x_1 = x_2 = A_1 \sin(\omega_1 t + \varphi_1)$$

both masses move in unison

A_1 & ϕ_1 come from IC

LET $\omega = \omega_2 = \sqrt{\frac{3k}{m}}$ put into amplitude eqn $\qquad \omega_2$ is freq for 2^{nd} mode of vib.

$$\left.\begin{aligned} &\left[\left(\frac{2k}{m} - \frac{3k}{m}\right)A_1 - \frac{k}{m}A_2\right] = 0 \\ &\left[-\frac{k}{m}A_1 + \left(\frac{2k}{m} - \frac{3k}{m}\right)A_2\right] = 0 \end{aligned}\right\} \Rightarrow -\frac{k}{m}A_1 - \frac{k}{m}A_2 = 0 \quad \text{or } A_1 = -A_2$$

$$\Rightarrow A_1/A_2 = -1 = \eta_2$$

$$\Rightarrow \text{ IF } \quad x_1 = A_1 \sin(\omega_2 t + \varphi_2)$$
$$x_2 = -A_1 \sin(\omega_2 t + \varphi_2)$$

masses move in opposite

A_1 & ϕ_2 come from IC

now $\tau_1 = 2\pi/\omega_1$ $\tau_2 = 2\pi/\omega_2$ $\therefore$ since $\omega_2 > \omega_1$
$\tau_2 < \tau_1$

LOOK AT p. 207/208 wave forms P 230 in RAO

LOOK AT p. 208/209 TO DEFINE GENERAL CASE

IF $2\alpha = \dfrac{k_1+k_2}{m_1}$ $\gamma = \dfrac{k_2}{m_1}$ $\Rightarrow \ddot{x}_1 + 2\alpha x_1 - \gamma x_2 = 0$

$2\beta = \dfrac{k_2+k_3}{m_2}$ $\varepsilon = \dfrac{k_2}{m_2}$ $\Rightarrow \ddot{x}_2 + 2\beta x_2 - \varepsilon x_1 = 0$

$\boxed{\text{the amplitude eqn.}}$

$$(2\alpha - \omega^2)A_1 - \gamma A_2 = 0$$
$$(-\varepsilon A_1 + (2\beta - \omega^2)A_2 = 0$$

$\boxed{\text{FREQ EQ.}}$

$$\omega^4 - 2(\alpha+\beta)\omega^2 + 4\alpha\beta - \varepsilon\gamma = 0$$

and

$$\boxed{\omega_1} = \sqrt{(\alpha+\beta) - \sqrt{(\alpha-\beta)^2 \cancel{4\alpha\beta} + \varepsilon\gamma}}$$
$$\boxed{\omega_2} = \sqrt{(\alpha+\beta) + \sqrt{(\alpha-\beta)^2 + \gamma\varepsilon}}$$

when $\omega = \omega_1 \Rightarrow \dfrac{A_2}{A_1} = \dfrac{(\alpha-\beta) + \sqrt{(\alpha-\beta)^2 + \gamma\varepsilon}}{\gamma}$

$= \eta_1$ $\eta_1 > 0$

$\left.\begin{array}{l} x_1 = A_1 \sin(\omega_1 t + \varphi_1) \\ x_2 = A_2 \sin(\omega_1 t + \varphi_1) = \eta_1 A_1 \sin(\ ) \end{array}\right\}$

when $\omega = \omega_2 \Rightarrow \dfrac{A_2}{A_1} = \dfrac{(\alpha-\beta) - \sqrt{(\alpha-\beta)^2 + \gamma\varepsilon}}{\gamma}$

$= -\eta_2$ $\eta_2 > 0$

$\left.\begin{array}{l} x_1 = \tilde{A}_1 \sin(\omega_2 t + \varphi_2) \\ x_2 = -\eta_2 \tilde{A}_1 \sin(\omega_2 t + \varphi_2) \end{array}\right\}$

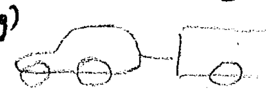
- BY LINEAR SUPERPOSITION

$$x_1 = A_1 \sin(\omega_1 t + \varphi_1) + \tilde{A}_1 \sin(\omega_2 t + \varphi_2)$$
$$x_2 = \eta_1 A_1 \sin(\omega_1 t + \varphi_1) - \eta_2 \tilde{A}_1 \sin(\omega_2 t + \varphi_2)$$

NOTE η_1, η_2 depend on k_1, k_2, k_3, m_1, m_2

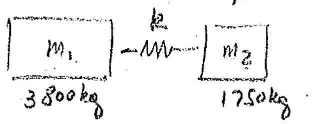
- THERE ARE 4 unknowns $\underline{A_1, \tilde{A}_1, \varphi_1, \varphi_2}$ $\Rightarrow$ need IC on x_1 & x_2

An auto, m_1, uses a bumper hitch to pull a loaded trailer m_2. The bumper acts like a spring k
The trailer (3800 kg) is hitched to auto (1750 kg)
hitch flexibility is 175 N/mm. Find ω's



9.4
Steidel

$= 175{,}000$ N/m



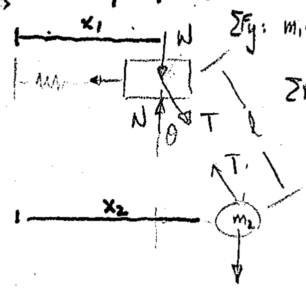
$k_1 = k_3 = 0$

semi definite

$m_1 m_2 \;\; \omega^4 - (m_1 + m_2) k \omega^2 + 0 = 0 \qquad \omega^2 = 0 \qquad \omega^2 = \frac{(m_1+m_2)k}{m_1 m_2} = \frac{5550}{3800(1750)} \cdot 175000 = 146.05$

$\omega = 12.08 \frac{rad}{sec} \cdot \frac{rev}{2\pi} = 1.923$ Hz

A mass m is constrained to move in a horiz smooth guide & is supported by an elastic spring k
9.13 a simple pendulum is supported by the mass. Both masses are 5 kg. $k = 400$ N/m & pendulum length $= 0.25$ m



$\Sigma F_y: \; m_1 g + m_2 g \cos\theta = N \cong (m_1 + m_2) g$

$\Sigma F_x: -kx_1 + T\sin\theta = m\ddot{x}_1 \;\; = \;\; -kx_1 + T\frac{(x_2 - x_1)}{\ell} = m\ddot{x}_1$

$m_2 \ddot{x}_2 = -T\sin\theta = -T\frac{(x_2 - x_1)}{\ell} \qquad or \qquad m_2 \ell^2 \ddot{\theta} - m_2 \ell g \sin\theta = 0$

$\therefore \qquad m_1 \ddot{x}_1 + \left(k + \frac{T}{\ell}\right) x_1 - \frac{T}{\ell} x_2 = 0$

$\Sigma F_r = 0 \Rightarrow T\cos\theta = m_2 g \Rightarrow T \cong m_2 g \qquad m_2 \ddot{x}_2 + \frac{T}{\ell} x_2 - \frac{T}{\ell} x_1 = 0$

let $k_2 = T/\ell \quad k_3 = 0 \quad k_1 = k$

$m_2 = 5$ kg
$k_1 = 400$ N
$\ell = 0.25$ m

$T = \overset{\sim m_2 g}{5(9.81)} = 49.05$ N $\qquad k_2 = \frac{49.05}{0.25} = 196.20$ N/m $\qquad k_3 = 0 \quad k_1 = 400$ N/m

$25\omega^4 - \omega^2\left[5(400 + 196.2) + 5(196.2 + 0)\right] + 400(196.2) = 0$

$\omega^4 - 158.48\omega^2 + 3139.2 = 0 \qquad \omega^2 = 135.27 \qquad \omega = 11.631 \frac{rad}{s}$ or 1.85 Hz

$\omega^2 = 23.21 \qquad \omega = 4.817 \frac{rad}{s}$ or .767 Hz

now $\frac{\tilde{A}_2}{\tilde{A}_1} = \frac{-m\omega^2 + (k_1 + k_2)}{k_2} = \frac{-5(135.27) + 596.2}{196.2} = -.4085$ for 1.85 Hz

$\frac{A_2}{A_1} = \frac{-5(23.21) + 596.2}{196.2} = 2.447$

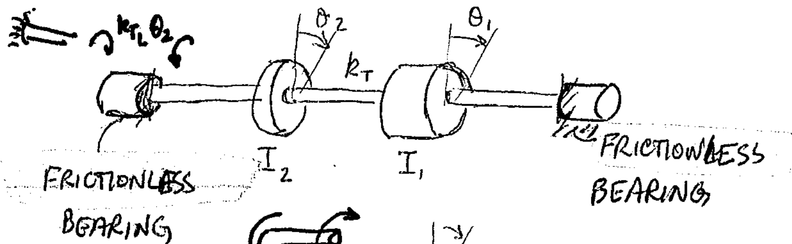
Thus $x_1 = A_1 \sin(4.817t + \phi_1) + \tilde{A}_1 \sin(11.631t + \phi_2)$

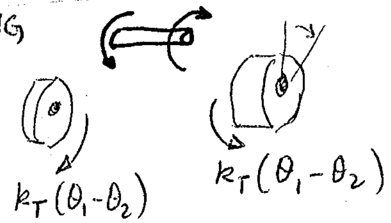
$x_2 = 2.447 A_1 \sin(4.817t + \phi_1) - .4085 \tilde{A}_1 \sin(11.631t + \phi_2)$

× $I\ddot{\theta} = -mg \cdot \ell \sin\theta$

SESSION # 20

FOR THE TORSIONAL SHAFT





if not frictionless

$$I_1\ddot{\theta}_1 = -k_T(\theta_1-\theta_2) - k_{T_R}\theta_1$$
$$I_2\ddot{\theta}_2 = k_T(\theta_1-\theta_2) - k_{T_L}\theta_2$$

thus $$\ddot{\theta}_1 + \frac{k_T}{I_1}\theta_1 - \frac{k_T}{I_1}\theta_2 = 0$$

$$\ddot{\theta}_2 + \frac{k_T}{I_2}\theta_2 - \frac{k_T}{I_2}\theta_1 = 0$$

CHOOSE $\left.\begin{array}{l}\theta_1 = A_1\sin(\omega_n t+\varphi)\\ \theta_2 = A_2\sin(\omega_n t+\varphi)\end{array}\right\} \Rightarrow \begin{bmatrix}\left(-\omega_n^2+\frac{k_T}{I_1}\right)A_1 - \frac{k_T}{I_1}A_2 \\ -\frac{k_T}{I_2}A_1 + \left(-\omega_n^2+\frac{k_T}{I_2}\right)A_2\end{bmatrix} = 0$

FREQ EQN.

$$\omega_n^4 - \left(\frac{k_T}{I_1}+\frac{k_T}{I_2}\right)\omega_n^2 + \cancel{\frac{k_T^2}{I_1 I_2}} - \cancel{\frac{k_T^2}{I_1 I_2}} = 0$$

$$\omega_n = 0 \quad \text{or} \quad \omega_n = \sqrt{K_T\left(\frac{1}{I_1}+\frac{1}{I_2}\right)}$$

FROM DE.

if $\omega_n = 0 \Rightarrow A_1 = A_2 \qquad \theta_1 = \theta_2 \Rightarrow I_1\ddot{\theta}_1 = 0$ or $\theta_1 = Ct + D$

$\theta_2 = Ct + D.$

NON OSCILLATORY MOTION HAVING CONST VEL $\left\{\begin{array}{l}\text{SAME VELOC}\\ \text{\& DISP.}\end{array}\right.$ THIS IS DEGENERATE MODE

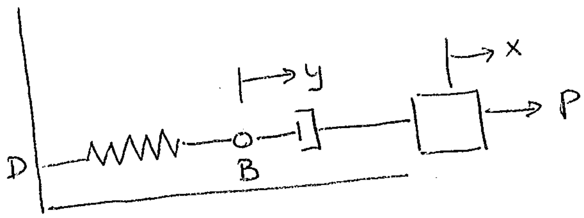
if $\omega_n = \sqrt{k_T\left(\frac{1}{I_1}+\frac{1}{I_2}\right)} \Rightarrow \frac{A_1}{I_1} = -\frac{A_2}{I_2}$ or $\frac{A_2}{A_1} = -\frac{I_2}{I_1}$

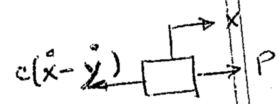
thus $$\theta_1 = A_1\sin(\omega_n t+\varphi)$$
$$\theta_2 = A_2\sin(\omega_n t+\varphi) = -\frac{I_2 A_1}{I_1}\sin(\omega_n t+\varphi).$$

HW. DO 7-1, 7-4, 7-7

TALK ABOUT <u>HW</u>

PROBLEM 4-19



a) 

b) $m\ddot{x} = P - c(\dot{x}-\dot{y})$

$m\ddot{x} + c\dot{x} = P + c\dot{y}$

@ B

c) $ky \leftarrow \bullet \rightarrow c(\dot{x}-\dot{y})$

d) $\cancel{m_B}\overset{0}{\ddot{y}} = 0 = c(\dot{x}-\dot{y}) - ky$ or $c\dot{x} = c\dot{y} + ky \Rightarrow cX\omega_f\cos\omega_f t = c\dot{y} + ky$

solve y in terms of k, c, X, ω_f : guess $y = A\cos\omega_f t + B\sin\omega_f t$

solve for A, B

Now

f) $P = m\ddot{x} + c\dot{x} - c\dot{y} = m\ddot{x} + ky$ where $x = X\sin\omega_f t$; y is from part d

g) $F_D = ky$ where y is from part d).

REMEMBER this gives only the particular solution

<u>PROBLEM 5-29</u>

$$x = \frac{1}{\sqrt{mk}}\int_0^t P(\tau)\sin\omega(t-\tau)\,d\tau$$

$$P(\tau) = \frac{B\tau}{2\pi/\omega} = \frac{B\omega\tau}{2\pi} \qquad 0 \le \tau \le 2\pi/\omega$$

$$= B\left[2 - \frac{\tau}{2\pi/\omega}\right] = \frac{B\omega}{2\pi}\left[\frac{4\pi}{\omega} - \tau\right] \qquad 0 \le \tau \le \frac{4\pi}{\omega}$$

$$= 0 \qquad \tau > \frac{4\pi}{\omega}$$

$0 < t < \frac{2\pi}{\omega}$ $\quad x = \frac{1}{\sqrt{mk}}\int_0^t P(\tau)\sin\omega(t-\tau)\,d\tau = \frac{1}{\sqrt{mk}}\int_0^t \frac{B\omega\tau}{2\pi}\sin\omega(t-\tau)\,d\tau$

$\frac{2\pi}{\omega} \le t < \frac{4\pi}{\omega}$ $\quad x = \frac{1}{\sqrt{mk}}\int_0^t P(\tau)\sin\omega(t-\tau)\,d\tau = \frac{1}{\sqrt{mk}}\int_0^{\frac{2\pi}{\omega}} \frac{B\omega\tau}{2\pi}\sin\omega(t-\tau)\,d\tau$

$$+ \frac{1}{\sqrt{mk}}\int_{\frac{2\pi}{\omega}}^t \frac{B\omega}{2\pi}\left[\frac{4\pi}{\omega} - \tau\right]\sin\omega(t-\tau)\,d\tau$$

$$\frac{1}{\sqrt{mk}}\int_0^{\frac{2\pi}{\omega}} B\omega\tau\sin\omega(t-\tau)\,d\tau + \frac{1}{\sqrt{mk}}\int_{2\pi/\omega}^{4\pi/\omega} \frac{B\omega}{2\pi}\left[\frac{4\pi}{\omega} - \tau\right]\sin\omega(t-\tau)\,d\tau +$$

$$\frac{1}{\sqrt{mk}} \int_{\frac{4\pi}{\omega}}^{t} \cancel{P}^{\,0} \sin(\omega[t-\tau])\,d\tau$$

since $t > \frac{4\pi}{\omega}$ and $P = \text{const} \Rightarrow x = \text{const}$

We can see this since $m\ddot{x} + kx = P$ and $P = \text{const} \Rightarrow$ particular solution $x_p = P/k$

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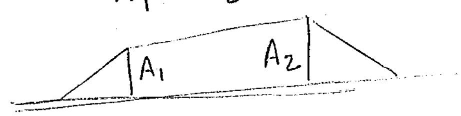
SESSION # 21

RETURNING TO p. 208 TALK ABOUT NODE

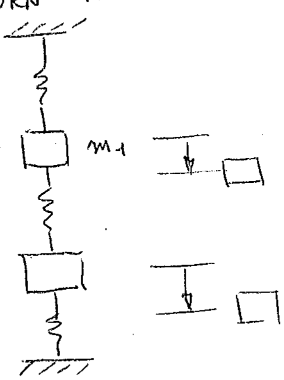
$x_1 = A_1 \sin(\omega_1 t + \varphi_1)$ $\quad x_2 = A_2(\omega_1 t + \varphi_1)$ is defined as the displacement ~~mode~~ equations for the masses

A_1 & A_2 found for ω_1 define the mode pattern

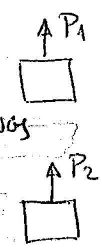


DISPLACEMENT METHOD — ANOTHER METHOD

RETURN TO

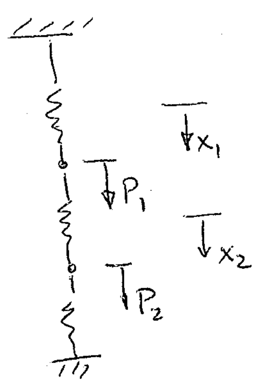


LOOK AT FBD OF mass m_1



TOTAL FORCE ON MASS M_1 ~~DUE~~ DUE TO SPRINGS $\quad m_1\ddot{x}_1 = -P_1 \quad (1)$

TOTAL FORCE ON MASS M_2 DUE TO SPRINGS $\quad m_2\ddot{x}_2 = -P_2 \quad (2)$



LOOK AT SPRINGS ALONE

P_1 — FORCE EXERTED BY MASS 1 ON SPRINGS

P_2 — " " " " 2 ON SPRINGS

— DEFORMATION IS PROPORTIONAL TO LOADS APPLIED

FOR SMALL DEFORMATIONS

$$x_1 = a_{11}P_1 + a_{12}P_2 \quad (3)$$

$$x_2 = a_{21}P_1 + a_{22}P_2 \quad (4)$$

where a_{ij} is a flexibility coeff

DISPL AT POINT i DUE TO FORCE AT POINT j

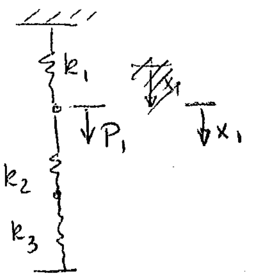
substitute (1) & (2) into (3) & (4) FOR P_1 and P_2

$\Rightarrow$

$$x_1 = a_{11}(-m_1\ddot{x}_1) + a_{12}(-m_2\ddot{x}_2) \Rightarrow a_{11}m_1\ddot{x}_1 + a_{12}m_2\ddot{x}_2 + x_1 = 0$$

$$x_2 = a_{21}(-m_1\ddot{x}_1) + a_{22}(-m_2\ddot{x}_2) \Rightarrow a_{21}m_1\ddot{x}_1 + a_{22}m_2\ddot{x}_2 + x_2 = 0$$

TO FIND THE FLEXIBILITY COEFF a_{11} assume $P_2 = 0$



FORCE P_1 applied at point 1

k_2 & k_3 are in series $\frac{1}{k'} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{k_3+k_2}{k_2k_3}$

and k_1 & k' are in parallel $k_1 + \frac{k_2k_3}{k_3+k_2} = \frac{k_1k_3+k_1k_2+k_2k_3}{k_3+k_2} = k''$

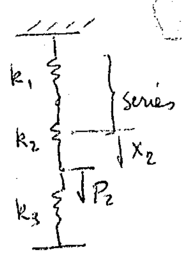
$x_1 = a_{11}P_1 + a_{12}\cancel{P_2}$, $P_2 = 0$

now $P_1 = k''x_1 \Rightarrow k'' = \frac{1}{a_{11}}$ or $\boxed{a_{11} = \frac{1}{k''}}$

$x_1 = a_{11}P_1$ & $P_1 = k_1x_1$

$P_2 = k''x_2$ DISPL IS $\propto$ LOAD

TO FIND a_{22} assume $P_1 = 0$



series $k''' = \frac{k_2k_1}{k_1+k_2}$ Parallel $k''' + k_3 = k^{IV} = \frac{k_1k_2+k_1k_3+k_2k_3}{k_1+k_2}$

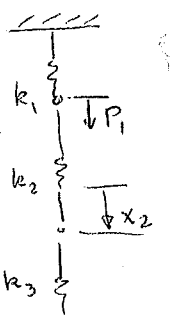
$x_2 = a_{21}\cancel{P_1} + a_{22}P_2$, $P_1 = 0$ $\Rightarrow k^{IV} = \frac{1}{a_{22}}$ $\quad a_{22} = \frac{1}{k^{IV}}$

TO FIND a_{21} assume $P_2 = 0 \Rightarrow x_2 = a_{21}P_1$

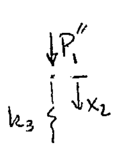
$P_1 = P_1' + P_1''$

$P_1' = k_1x_1$

$P_1'' = k'x_1$



since k_2 & k_3 in series each feels the force P_1''



$$\therefore \quad P_1 = \underbrace{k_1x_1}_{P_1'} + \underbrace{k'x_1}_{P_1''} = \frac{k_2k_3}{k_3+k_2}x_1 + k_1x_1 = \underbrace{k_3x_2}_{P_1''} + \underbrace{k_1x_1}_{P_1'} ; \quad x_2 = \frac{k_2}{k_3+k_2}x_1$$

but $P_1 = k''x_1$ or $x_1 = \frac{P_1}{k''} = P_1\left[\frac{k_3+k_2}{k_1k_3+k_1k_2+k_2k_3}\right]$

$$k'' = k_1 + \frac{1}{1/k_2 + 1/k_3} = k_1 + \frac{k_2k_3}{k_2+k_3}$$

$$x_2 = \frac{k_2}{k_2+k_3}x_1 = P_1\left[\frac{k_2}{k_1k_2+k_1k_3+k_2k_3}\right] = a_{21}P_1$$

TO FIND a_{12} assume $P_1 = 0 \Rightarrow x_1 = a_{12}P_2$

GOING THROUGH SAME THING WE CAN SHOW THAT

$$x_1 = a_{12}P_2 = P_2\left[\frac{k_2}{k_1k_2+k_1k_3+k_2k_3}\right]$$

- THIS RESULT SHOWS $a_{12} = a_{21}$ WHICH COMES FROM MAXWELL'S RECIPROCAL LAW

FOR $M_1 = M_2 = M$ & $k_1 = k_2 = k_3 = k$

$$a_{11} = \frac{2}{3}\frac{1}{k} \qquad a_{12} = \frac{1}{3k} = a_{21} \qquad a_{22} = \frac{2}{3k}$$

THUS

$$\frac{2}{3}\frac{m}{k}\ddot{x}_1 + \frac{1}{3}\frac{m}{k}\ddot{x}_2 + x_1 = 0$$
$$\frac{1}{3}\frac{m}{k}\ddot{x}_1 + \frac{2}{3}\frac{m}{k}\ddot{x}_2 + x_2 = 0$$

IF $x_1 = A_1 \sin(\omega t + \phi)$ $\quad x_2 = A_2 \sin(\omega t + \phi)$

$$\begin{bmatrix} (1 - \frac{2}{3}\frac{m}{k}\omega^2)A_1 & -\frac{1}{3}\frac{m}{k}\omega^2 A_2 \\ -\frac{1}{3}\frac{m}{k}\omega^2 A_1 & + (1 - \frac{2}{3}\frac{m}{k}\omega^2)A_2 \end{bmatrix} \begin{matrix} = 0 \\ = 0 \end{matrix}$$

$\omega^2 \neq \frac{k}{m}$ in general.

let

$\frac{k}{m\omega^2} = \lambda$

INVERSE FREQ. FACTOR

$$(\lambda - 2/3)A_1 \quad -\cancel{1/3}\, A_2 = 0$$
$$-1/3\, A_1 + (\lambda - 2/3)A_2 = 0$$

FOR $A_1 \neq A_2 \neq 0$ determinant $= 0$

$$\begin{vmatrix} \lambda - 2/3 & -1/3 \\ -1/3 & (\lambda - 2/3) \end{vmatrix} = 0 \quad \text{or} \quad \cancel{1 - \frac{4}{3}\lambda + \frac{1}{3}\lambda^2 = 0 \Rightarrow \lambda^2 - 4\lambda + 3 = 0}$$

$\cancel{1 - 1/3}$

$$\lambda^2 - 4/3\lambda + 1/3 = 0 \qquad (\lambda - 1/3)(\lambda - 1) = 0 \qquad \lambda = 1 \quad \lambda = 1/3$$

$\lambda = 1 \Rightarrow \omega^2 = k/m \qquad \omega = \sqrt{k/m}$

$\lambda = 1/3 \Rightarrow \omega^2 = 3k/m \qquad \omega = \sqrt{\frac{3k}{m}}$

if $\lambda = 1 \Rightarrow (1 - 2/3)A_1 - 1/3\, A_2 = 0 \qquad A_1 = A_2 \qquad \frac{A_2}{A_1} = 1$

IF $\lambda = 1/3 \qquad (\frac{1}{3} - 2/3)A_1 - 1/3\, A_2 = 0 \qquad A_1 = -A_2 \qquad A_2/A_1 = -1$

THESE ARE THE SAME AS WHAT WERE OBTAINED FOR THE DIRECT APPROACH

CAN DEFINE FOR SYMPLE PROBLEMS coordinate ~~mode shapes~~ that will uncouple the equations

ORIGINALLY

$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 = 0$$
$$m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2 x_1 = 0$$

STATIC OR ELASTIC COUPLING

COUPLING IS IN THE DISPL COORD.

INVERSE APPROACH

$$\frac{k_2+k_3}{k_1k_2+k_1k_3+k_2k_3} m_1\ddot{x}_1 + \frac{k_2}{k_1k_2+k_1k_3+k_2k_3} m_2\ddot{x}_2 + x_1 = 0$$

$$\frac{k_2}{k_1k_2+k_1k_3+k_2k_3} m_1\ddot{x}_1 + \frac{k_1+k_2}{k_1k_2+k_1k_3+k_2k_3} m_2\ddot{x}_2 + x_2 = 0$$

DYNAMIC OR INERTIA COUPLING - COUPLING IS IN THE ACCELERATION VARIABLE

TYPE OF COUPLING IS IN METHOD USED TO DERIVE EQS.

LET $m_1 = m_2 = m$ $\quad k_1 = k_2 = k_3 = k$

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$
$$m\ddot{x}_2 + 2kx_2 - kx_1 = 0$$

ADD $\quad m(\ddot{x}_1+\ddot{x}_2) + \cancel{2}k(x_1+x_2) = 0$

SUBTRACT $\quad m(\ddot{x}_1-\ddot{x}_2) + 3k(x_1-x_2) = 0$

LET $q_1 = x_1+x_2 \qquad q_2 = x_1 - x_2$ GENERALIZED COORD

$$m\ddot{q}_1 + kq_1 = 0 \quad \text{or} \quad \omega_1 = \sqrt{k/m} \qquad q_1 = A_1\sin(\omega_1 t+\phi_1)$$
$$m\ddot{q}_2 + 3kq_2 = 0 \quad \text{or} \quad \omega_2 = \sqrt{\frac{3k}{m}} \qquad q_2 = A_2\sin(\omega_2 t+\phi_2)$$

$q_1+q_2 = 2x_1 \qquad q_1 - q_2 = 2x_2$

if

$$x_1 = \left[A_1\sin(\omega_1 t+\phi_1) + A_2\sin(\omega_2 t+\phi_2)\right]/2$$
$$x_2 = \left[A_1\sin(\omega_1 t+\phi_1) - A_2\sin(\omega_2 t+\phi_2)\right]/2$$

SESSION #22

- ANOTHER METHOD TO FIND THE EQUATIONS - LAGRANGE'S METHOD
  - MAKES USE OF ENERGY OF SYSTEM
  - POTENTIAL ENERGY V IS A FN OF GENERALIZED COORDINATES q_k
  - KINETIC ENERGY T IS A FN OF GENERALIZED VELOCITY $\dot{q}_k$ and q_k
  - FOR A CONSERVATIVE SYSTEM   READ APPENDIX B

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} + \frac{\partial V}{\partial q_k} = 0$$

- GENERALIZED COORDINATE IS SUCH THAT A CHANGE IN ONE COORDINATE DOESN'T CAUSE OR DOESN'T REQUIRE A CHANGE IN ANY OTHER COORD.

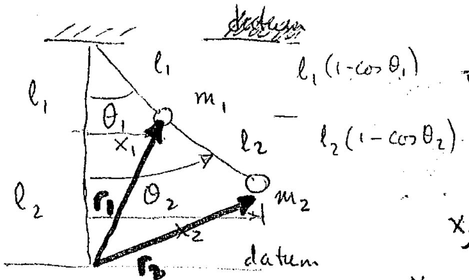
q
• MAY BE LINEAR, ANGULAR OR COMBO $\exists$ <u>1 for each DOF</u>

• REQUIRES KNOWLEDGE OF GENERALIZED COORD.

CAN DEFINE A LAGRANGIAN $L = T - V$ $\quad \dfrac{\partial L}{\partial \dot q} = \dfrac{\partial T}{\partial \dot q} \quad \dfrac{\partial L}{\partial q} = -\dfrac{\partial V}{\partial q} + \dfrac{\partial T}{\partial q}$

IF $T = T(q_k, \dot q_k)$ and $V = V(q_k)$

$$\therefore \frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_k}\right) + \frac{\partial V}{\partial q_k} - \frac{\partial T}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad q = 1, 2, 3, \ldots$$



$$V = m_1 g l_1(1-\cos\theta_1) + m_1 g l_2 + m_2 g\left[l_1(1-\cos\theta_1) + l_2(1-\cos\right.$$

$$T = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 = \tfrac{1}{2} m_1 (l_1 \dot\theta_1)^2 + \tfrac{1}{2} m_2 (\underbrace{l_2 \dot\theta_2 + l_1 \dot\theta_1}_{\text{relative velocity}}$$

$$\cancel{x_1 = l_1 \sin\theta_1 ;\ \dot x_1 = l_1 \cos\theta_1 \dot\theta_1 \cong l_1 \dot\theta_1}$$

$$\cancel{x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 ;\ \dot x_2 = l_1 \cos\theta \dot\theta_1 + l_2 \cos\theta \dot\theta_2 \cong l_1\dot\theta_1 + l_2 \dot\theta_2}$$

$q_1 = \theta_1 \qquad q_2 = \theta_2$

$$\frac{\partial L}{\partial \theta_1} = \frac{\partial (T-V)}{\partial \theta_1} = -\frac{\partial V}{\partial \theta_1} = -\left[m_1 g l_1 \overset{\theta_1}{\cancel{\sin\theta_1}} + m_2 g l_1 \overset{\theta_1}{\cancel{\sin\theta_1}}\right]$$

$$\frac{\partial L}{\partial \theta_2} = \frac{\partial (T-V)}{\partial \theta_2} = -\frac{\partial V}{\partial \theta_2} = -\left[m_2 g l_2 \overset{\theta_2}{\cancel{\sin\theta_2}}\right]$$

$$\frac{\partial L}{\partial \dot\theta_1} = \frac{\partial (T-V)}{\partial \dot\theta_1} = \frac{\partial T}{\partial \dot\theta_1} = m_1 l_1^2 \dot\theta_1 + m_2 l_1^2 \dot\theta_1 + m_2 l_1 l_2 \dot\theta_2$$

$$\frac{\partial L}{\partial \dot\theta_2} = \frac{\partial (T-V)}{\partial \dot\theta_2} = \frac{\partial T}{\partial \dot\theta_2} = m_2 (l_2 \dot\theta_2 + l_1 \dot\theta_1) l_2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_1}\right) - \frac{\partial L}{\partial \theta_1} = m_1 l_1^2 \ddot\theta_1 + m_2 l_1^2 \ddot\theta_1 + m_2 l_1 l_2 \ddot\theta_2 + m_1 g l_1 \theta_1 + m_2 g l_1 \theta_1 = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot\theta_2}\right) - \frac{\partial L}{\partial \theta_2} = m_2 l_2^2 \ddot\theta_2 + m_2 l_1 l_2 \ddot\theta_1 + m_2 g l_2 \theta_2 = 0$$

$\bar r_1 = l_1 \sin\theta_1 \bar\imath + [l_2 + l_1(1-\cos\theta_1)]\bar\jmath$

$\dot{\bar r}_1 = [l_1 \cos\theta_1 \bar\imath + l_1 \sin\theta_1 \bar\jmath]\dot\theta_1$

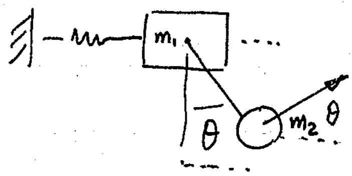
$|\dot{\bar r}_1|^2 = l_1^2 \dot\theta_1^2$

$\bar r_2 = (l_1 \sin\theta_1 + l_2 \sin\theta_2)\bar\imath + [l_1(1-\cos\theta_1) + l_2(1-\cos\theta_2)]$

$\dot{\bar r}_2 = (l_1 \cos\theta_1 \dot\theta_1 + l_2 \cos\theta_2 \dot\theta_2)\bar\imath + (l_1 \sin\theta_1 \dot\theta_1 + l_2 \sin\theta_2 \dot\theta_2)\bar\jmath$

$|\dot{\bar r}_2|^2 = l_1^2 \dot\theta_1^2 + l_2^2 \dot\theta_2^2 + 2 l_1 l_2 \dot\theta_1 \dot\theta_2 \cos(\theta_1 - \theta_2)$

$\cong (l_1 \dot\theta_1 + l_2 \dot\theta_2)^2$



$$V = \tfrac{1}{2}kx^2 + mgl(1-\cos\theta)$$
$$T = \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}\left[\dot{x} + l\dot{\theta}\right]^2 m_2$$

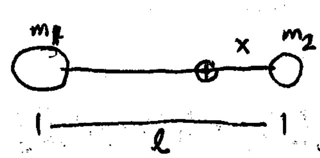
$$V_{m_2} = V_{m_1} + V_{m_2/m_1}$$
$$= \dot{x}\,\underset{\sim}{i} + l\dot{\theta}(\cos\theta\,\underset{\sim}{i} + \sin\theta\,\underset{\sim}{j})$$
$$= (\dot{x} + l\dot{\theta}\cos\theta)\,\underset{\sim}{i} + l\dot{\theta}(\sin\theta)\,\underset{\sim}{j}$$
$$V^2_{m_2} = \dot{x}^2 + 2l\dot{x}\dot{\theta}\cos\theta + l^2\dot{\theta}^2$$

$$V+T = \tfrac{1}{2}kx^2 + mgl(1-\cos\theta) + \tfrac{1}{2}m_1\dot{x}^2 + \tfrac{1}{2}\left[\dot{x}+l\dot{\theta}\right]^2 m_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} + \frac{\partial V}{\partial x} = \frac{d}{dt}\left[\cancel{\tfrac{1}{2}}m_1 \cancel{2}\dot{x}\right] + \frac{d}{dt}\left[\cancel{\tfrac{1}{2}}\cdot\cancel{2}\left[\dot{x}+l\dot{\theta}\right]\right]m_2 - 0 + \tfrac{1}{2}\cdot 2kx$$
$$m_1\ddot{x} + m_2\ddot{x} + m_2 l\ddot{\theta} + kx = 0$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} + \frac{\partial V}{\partial \theta} = \frac{d}{dt}\left[\cancel{\tfrac{1}{2}}\cdot\cancel{2}\left[\dot{x}+l\dot{\theta}\right]\cdot l\right]m_2 - 0 + m_2 g\, l\sin\theta = 0$$
$$(l\ddot{x} + l^2\ddot{\theta})m_2 + m_2 g l\sin\theta = 0$$

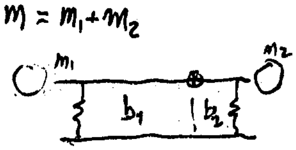
$$l\theta = x_2 - x_1$$
$$l\theta + x = x_2 - x_1 + x_1 = x_2$$
$$m_1\ddot{x}_1 + m_2\ddot{x}_2 + kx_1 = 0$$
$$l\left[\ddot{x} + l\ddot{\theta}\right] = l\ddot{x}_2 m_2 + m_2 g l\theta = 0$$
$$= m_2\ddot{x}_2 + m_2 g\sin\theta = 0 \qquad m_2\ddot{x}_2 = -\overset{T}{m_2 g}\sin\theta \checkmark$$
$$m_1\ddot{x}_1 - \underset{\sim T}{m_2 g}\sin\theta + kx_1 = 0 \quad \checkmark$$



$$\Sigma M_{CG} = 0$$
$$m_1 g(l-x) = m_2 g x$$
$$m_1 g l = (m_1+m_2)gx$$
$$x = \frac{m_1 l}{m_1+m_2}$$

 $r^2\rho\,dA$

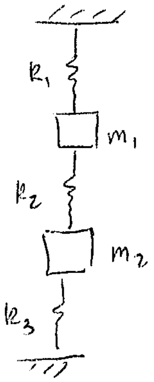
1. Determine CG
2. Determine $I_{CG} = m_1(l-x)^2 + m_2 x^2$
3. $m = m_1 + m_2$



MATRIX NOTATION

NOT NECESSARY TO DO

PASS



$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$$
$$m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2x_1 = 0$$

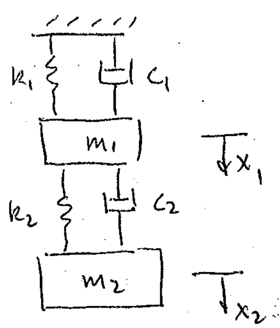
$$\underbrace{\begin{bmatrix} m_1 & 0 \\ 0 & m_1 \end{bmatrix}}_{\substack{\text{mass matrix} \\ \text{diagonal matrix}}} \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{pmatrix}}_{\substack{\text{stiffness matrix} \\ \text{symmetric matrix} \\ a_{12}=a_{21}}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

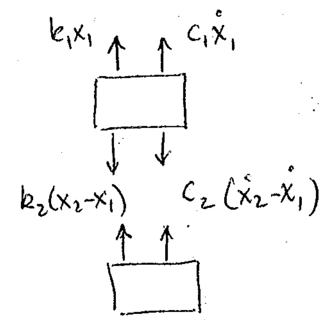
$$x_1 + m_1\ddot{x}_1 a_{11} + m_2\ddot{x}_2 a_{12} = 0$$
$$x_2 + m_1\ddot{x}_1 a_{21} + m_2\ddot{x}_2 a_{22} = 0$$

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\substack{\text{identity} \\ \text{matrix}}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}}_{\substack{\text{flexibility} \\ \text{matrix}}} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} = 0$$

PASS - NO

FREE VIBRATIONS OF A DAMPED SYSTEM





$$m_1\ddot{x}_1 = k_2(x_2-x_1) + c_2(\dot{x}_2-\dot{x}_1) - k_1x_1 - c_1\dot{x}_1$$

$$m_2\ddot{x}_2 = -k_2(x_2-x_1) - c_2(\dot{x}_2-\dot{x}_1)$$

JUST AS BEFORE pick $x_1 = D_1e^{st}$ $x_2 = D_2e^{\pm st}$

DAMPED SYSTEM FOR SOLNS $\to 0$ as $t \to \infty$

Auxiliary Eqns

$$[m_1s^2 + (k_2+k_1) + (c_2+c_1)s]D_1 + [-k_2 + c_2s]D_2 = 0$$
$$[m_2s^2 + k_2 + c_2s]D_2 + [-k_2 + c_2s]D_1 = 0$$

FOR NON TRIV SOLN $D_1 \neq D_2 = 0$

$$\begin{vmatrix} m_1s^2 + (k_2+k_1) + (c_2+c_1)s & -(k_2+c_2s) \\ -(k_2+c_2s) & m_2s^2+k_2+c_2s \end{vmatrix} = 0$$

PASS | NO

$m_1 m_2 s^4 + [m_1 c_2 + m_2(c_1+c_2)] s^3 + [m_1 k_2 + m_2(k_1+k_2) + c_1 c_2] s^2 + (k_1 c_2 + k_2 c_1) s + k_1 k_2 = 0$

4 roots (1) 2 complex conj pairs ie $+p_1 + iq_1 \quad +p_1 - iq_1 \quad +p_2 + iq_2 \quad +p_2 - iq_2 \quad = s$

(2) 1 complex pair + 2 real $+p_1 + iq_1 \quad +p_1 - iq_1 \quad +\sigma_3 \quad +\sigma_4 \quad = s$

(3) 4 real $+\sigma_1 \quad +\sigma_2 \quad +\sigma_3 \quad +\sigma_4 \quad = s$

if (1) $x_1 = D_{11} e^{-(p_1+iq_1)t} + D_{12} e^{-(p_1-iq_1)t} + D_{21} e^{-(p_2+iq_2)t} + D_{22} e^{-(p_2-iq_2)t}$

$e^{-p_1 t}\left[D_{11} e^{-iq_1 t} + D_{12} e^{iq_1 t}\right] + e^{-p_2 t}\left[D_{21} e^{-iq_2 t} + D_{22} e^{iq_2 t}\right]$

$x_1 = A_{11} \sin(q_1 t + \phi_{11}) e^{-p_1 t} + A_{12} e^{-p_2 t} \sin(q_2 t + \phi_{12})$

$x_2 = E_{11} e^{-(p_1+iq_1)t} + E_{12} e^{-(p_1-iq_1)t} + E_{21} e^{-(p_2+iq_2t)} + E_{22} e^{-(p_2-iq_2t)}$

$A_{21} e^{-p_1 t} \sin(q_1 t + \phi_{21}) + A_{22} e^{-p_2 t} \sin(q_2 t + \phi_{22})$

8 unknowns $A_{11}, A_{12}, A_{21}, A_{22}, \phi_{11}, \phi_{21}, \phi_{12}, \phi_{22}$

from frequency equation can relate A_{11} to A_{21} and ϕ_{11}, ϕ_{21}

A_{22} to A_{12} and ϕ_{22}, ϕ_{12}

$\Rightarrow$ 4 unknowns $A_{11}, A_{22}, \phi_{11}, \phi_{22}$

need 4 IC to get these.

SESSION # 24

EXAMPLE 7-42 $k_1 = 76.46$ lb/in $k_2 = 11.77$ lb/in $W_1 = W_2 = 38.6$ lb

$c_1 = .0034$ lbsec/in $c_2 = 0.0283$ lbsec/in

$m_1 = m_2 = .1$ lbsec²/in

if $x_1 = C_1 e^{-st}$ $x_2 = C_2 e^{-st}$

Auxillary Eqn $[.1s^2 + (88.23) - .0317s]\, C_1 + [-11.77 + .0283s]\, C_2 = 0$

$[-11.77 + .0283s]\, C_1 + [.1s^2 + 11.77 - .0283s]\, C_2 = 0$

Char eq is $.01 s^4 - [.006] s^3 + [10] s^2 - [2.20384] s + 899.93 = 0$

$s^4 - [.6] s^3 + [1000] s^2 - [220.38] s + 89993.42 = 0$

suppose 4 roots real

$$x_1 = C_{11} e^{-\sigma_1 t} + C_{12} e^{-\sigma_2 t} + C_{13} e^{-\sigma_3 t} + C_{14} e^{-\sigma_4 t}$$

$$x_2 = C_{21} e^{-\sigma_1 t} + C_{22} e^{-\sigma_2 t} + C_{23} e^{-\sigma_3 t} + C_{24} e^{-\sigma_4 t}$$

PASS

since $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ are ≥ 0 $\Rightarrow$ large damping

put into aux. equation σ_1 $\Rightarrow$ relation between C_{11} & C_{21}

σ_2 $\Rightarrow$ " " C_{12} & C_{22}

σ_3 $\Rightarrow$ " " C_{13} & C_{23}

σ_4 $\Rightarrow$ " " C_{14} & C_{24}

remaining unknowns need initial conditions

suppose 2 roots real + 2 complex

$$x_1 = C_{11} e^{-(p+iq)t} + C_{12} e^{-(p-iq)t} + C_{13} e^{-\sigma_3 t} + C_{14} e^{-\sigma_4 t}$$

$$x_2 = C_{21} e^{-(p+iq)t} + C_{22} e^{-(p-iq)t} + C_{23} e^{-\sigma_3 t} + C_{24} e^{-\sigma_4 t}$$

$$x_1 = e^{-pt}\left[A_{11} \sin(qt + \phi_{11})\right] + C_{13} e^{-\sigma_3 t} + C_{14} e^{-\sigma_4 t}$$

$$x_2 = e^{-pt}\left[A_{22} \sin(qt + \phi_{22})\right] + C_{23} e^{-\sigma_3 t} + C_{24} e^{-\sigma_4 t}$$

use auxil equation to get relation for $A_{11}, A_{22}, \phi_{11}, \phi_{22}$ for s_1, s_2

use aux. equation for σ_3 $\Rightarrow$ relation between C_{13} & C_{23}

σ_4 $\Rightarrow$ " " C_{14} & C_{24}

EXAMPLE

$$\left[m_1 s^2 + (c_1+c_2)s + (k_1+k_2)\right] C_{11} + (c_2 s - k_2) C_{21} = 0 \quad \text{for } s = p \pm iq$$

$$(c_2 s - k_2) C_{11} + (m_2 s^2 - c_2 s + k_2) C_{21} = 0$$

$$s^2 = p^2 \mp q^2 \pm 2ipq$$

$$\left[c_2(p \pm iq) - k_2\right] C_{11} + \left[m_2(p^2 \mp q^2 \pm 2ipq) - c_2(p \pm iq) + k_2\right] C_{21} = 0$$

$$(c_2 p - k_2) C_{11} + \left[m_2(p^2 \mp q^2) - c_2 p + k_2\right] C_{21} = 0$$

$$\pm c_2 q\, C_{11} + (\pm 2pq\, m_2 \mp c_2 q) C_{21} = 0$$

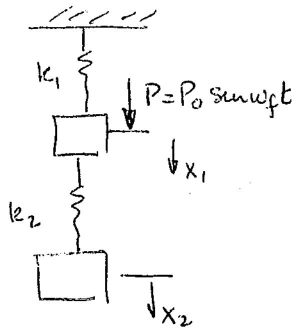
$$C_{21} = -\frac{c_2 q}{2pq m_2 - c_2 q} C_{11}$$

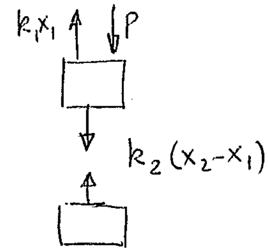
$$C_{12} = -\frac{c_2 q}{2pq m_2 - c_2 q} C_{22}$$

@ $t=0$ $C_{12} + C_{11} = A_{11} \sin\phi_{11}$

$C_{22} + C_{21} = A_{22} \sin\phi_{22}$

TWO DEGREE OF FREEDOM W/O DAMPING W/ FORCED VIBS.





$$m_1\ddot{x}_1 = k_2(x_2 - x_1) - k_1 x_1 + P$$

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$

$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 = P_0 \sin\omega_f t$$

$$m_2\ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$

$$x_i = x_{i_h} + x_{i_p} \qquad i = 1,2$$

solution to the homogeneous equations $x_{1_h} = A_1 \sin(\omega t + \phi)$ $\quad x_{2_h} = A_2 \sin(\omega t + \phi)$

solution to the forcing function choose

$$x_{1p} = X_1 \sin\omega_f t \qquad x_{2p} = X_2 \sin\omega_f t$$

$\div k_1$ $\quad -m_1 X_1 \omega_f^2 + (k_1+k_2)X_1 - k_2 X_2 = P_0$

$\div$ by k_2 $\quad -m_2 X_2 \omega_f^2 + k_2 X_2 - k_2 X_1 = 0 \qquad$ if $\sin\omega_f t \neq 0$

$$X_1\left[1 - r_1^2 + k_2/k_1\right] - k_2/k_1\, X_2 = P_0/k_1 = X_0 \qquad r_1 = \frac{\omega_f}{\omega_1} \qquad \omega_1 = \sqrt{\frac{k_1}{m_1}}$$

$$-1\, X_1 + (1 - r_2^2) X_2 = 0 \qquad r_2 = \omega_f/\omega_2 \qquad \omega_2 = \sqrt{\frac{k_2}{m_2}}$$

$\Rightarrow$

$$X_1/X_0 = \frac{(1-r_2^2)}{(1-r_2^2)(1-r_1^2+k_2/k_1) - k_2/k_1} \Rightarrow \frac{P_0(k_2 - m_2\omega_f^2)}{(k_2 - m_2\omega_f^2)(k_1 - m_1\omega_f^2 + k_2) - k_2^2} = X_1$$

$$X_2/X_0 = \frac{1}{(1-r_2^2)(1-r_1^2+k_2/k_1) - k_2/k_1} = \frac{P_0 k_2}{(k_2 - m_2\omega_f^2)(k_1 - m_1\omega_f^2 + k_2) - k_2^2} = X_2$$

DETERMINANT $\quad (1-r_2^2)(1-r_1^2+k_2/k_1) - k_2/k_1 = 0 \quad$ we have resonance or $m_1 m_2 \omega_f^4 - (k_1 m_2 + k_2 m_1 + k_2 m_2)\omega_f^2 + k_2 k_1 = 0$

IF $r_2 = 1$ ie $\omega_f = \omega_2 \Rightarrow X_1 = 0 \qquad X_2/X_0 = -\dfrac{k_1}{k_2} \Rightarrow X_2 = -\dfrac{k_1}{k_2} X_0 = -\dfrac{P_0}{k_2}$

IF $r_1 = 1$ ie $\omega_f = \omega_1 \Rightarrow X_1 = -\dfrac{(1-r_2^2)}{r_2^2}\dfrac{k_1}{k_2} X_0 = -\dfrac{(1-r_2^2)}{r_2^2}\dfrac{P_0}{k_2}$

$$X_2 = -\frac{1}{r_2^2}\frac{k_1}{k_2} X_0 = -\frac{1}{r_2^2}\frac{P_0}{k_2}$$

IF $\omega_f = 0 \quad r_1 = r_2 = 0 \quad X_1 = X_2 = X_0$

IF $\omega_f \to \infty \quad X_1/X_0 \to 0 \;\&\; X_2/X_0 \to 0$

9.44

9.52 pg 735

Fig 9.43

$\omega_1 = \omega_{o} = 1500 \cdot \frac{2\pi}{60} = 157.08 = \sqrt{\frac{k_1}{m_1}}$ this is natural freq of orig sys

$m_1 = m = 300$ kg

$k_1 = k_{beam} = \omega_1^2 m_1 = (157.08)^2 (300) = 7.4 \times 10^6$ N/m

$\omega_1 = \omega_2 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k_2}{m_2}}$ if we pick $\omega_f = \sqrt{\frac{k_2}{m_2}} \rightarrow X_1 = 0$

$k_2 = m_2 \omega_2^2 = 24674.2\, m_2$

but if $\omega_f = \sqrt{\frac{k_1}{m_1}}$ as well then we make $X_1 = 0$ at original natural freq

w/ absorber $\omega_f = .75\omega_f$ $\quad .75 = \frac{\omega_f}{\omega_f} = \frac{\omega_f}{\omega_2} = r_1$

for tuned absorber $\frac{\omega_1}{\omega_2} = 1$ +

$r_1^2 = (1+\frac{\mu}{2}) - \sqrt{(1+\mu/2)^2 - 1}$

$\mu = \frac{r_1^4 + 1}{r_1^2} - 2 = .3403 = \frac{m_2}{m_1}$

$\therefore \quad m_2 = .3403\, m_1 = .3403(300) = 102.1$ kg

$k_2 = m_2 \omega_2^2 = 24674.2(102.1) = 2.52 \times 10^6$ N/m

$X_2 = -\frac{P_0}{k_2} = -\frac{m_0 e \omega_f^2}{k_2} = \frac{-0.02(157.08)^2}{2.52\times10^6} = -.2\times10^{-3}$ m

Governing eqn

if $\omega_2 = \sqrt{\frac{k_2}{m_2}} \quad \omega_1 = \sqrt{\frac{k_1}{m_1}}$

then

$m_1 m_2 \omega_f^4 - (k_1 m_2 + k_2 m_1 + k_2 m_2)\omega_f^2 + k_1 k_2 = 0$

$\Rightarrow \left(\frac{\omega_f}{\omega_2}\right)^4 \left[\frac{\omega_2}{\omega_1}\right]^2 - \left(\frac{\omega_f}{\omega_2}\right)^2 \left[1 + \left(1 + \frac{m_2}{m_1}\right)\left(\frac{\omega_2}{\omega_1}\right)^2\right] + 1 = 0$

$\frac{\omega_f}{\omega_2} = \sqrt{\frac{\left[1 + (1+\frac{m_2}{m_1})(\frac{\omega_2}{\omega_1})^2\right] - \sqrt{[\quad]^2 - 4\cdot(\frac{\omega_2}{\omega_1})^2\cdot 1}}{2\left(\frac{\omega_2}{\omega_1}\right)^2}}$

$\Omega_f^4 - \Omega_f^2[2+\mu] + 1 = 0$

$r_1^2 + r_2^2 = 2+\mu$

$r_1^2 \cdot r_2^2 = 1$

$r_1^2 - (1+\mu/2) = -\sqrt{(\quad)^2 - 1}$

$r_1^4 - 2r_1^2(1+\mu/2) + \cancel{(1+\mu/2)^2} = \cancel{(1+\mu/2)^2} - 1$

$r_1^4 + 1 = 2r_1^2(1+\mu/2)$

$\frac{r_1^4+1}{2r_1^2} = 1 + \mu/2$

$\frac{r_1^4+1}{2r_1^2} - 1 = \mu/2$

$\frac{r_1^4+1}{r_1^2} - 2 = \mu$

Do 9.54, 57, 60

$r_2^2 = (1+\mu/2) + \sqrt{\quad}$

$r_2^2 - (1+\mu/2) = \sqrt{\quad}$

$r_2^4 - 2(1+\mu/2)r_2^2 + \cancel{(1+\mu/2)^2} = \cancel{(1+\mu/2)^2} - 1$

$\frac{r_2^4+1}{r_2^2} = 2(1+\mu/2)$

$= 2+\mu$

$\frac{r_2^4+1}{r_2^2} - 2 = \mu$

$\frac{r_1^4+1}{r_1^2} - 2 = \mu$

$r_1^2 + r_2^2 = 2+\mu$

$r_1^2 = \frac{1}{r_2^2}$

$1 + r_2^4 = (2+\mu)r_2^2$ ✓

$(A-B)(A+B)$

$A^2 - B^2$

$(1+\mu/2)^2 - [(1+\mu/2)^2 - 1] = 1$

denom is $m_2 m_1 \omega_f^4 - [m_1 k_2 + m_2(k_1+k_2)]\omega_f^2 + k_1 k_2$

~~9.44~~ in 3rd
9.52 in 4th

$m_o e = .02\ kg\text{-}m$

$\omega_f = 1500\ rpm \cdot \frac{2\pi}{60} = 50\pi = 157.08\ rad/s$

for tuned absorber $\sqrt{\frac{K_1}{m_1}} = \sqrt{\frac{K_2}{m_2}}$ but large vibs when $\omega_f = \omega_n = \sqrt{\frac{K_1}{m_1}}$

$$\frac{\Omega_1}{\omega_2} = r_1^2 = 1 + \mu/2 - \sqrt{(1+\frac{\mu}{2})^2 - 1}$$

$r_1 = 0.75$

or $\mu = \frac{r_1^4 + 1}{r_1^2} - 2 = \frac{.75^4 + 1}{.75^2} - 2 = 0.3403 = \frac{m_2}{m_1}$

Since $m_1 = 300\,kg$ $m_2 = 102.09\,kg$

$k_2 = \omega_f^2 m_2 = (157.08)^2 \cdot (102.09) = 2.519 \times 10^6\ \frac{N}{m}$

$X_2 = -P_o/k_2 = \frac{-m_o e \omega_f^2}{2.52 \times 10^6} = \frac{-0.2\ kg\text{-}m \cdot 157.08^2}{2.52 \times 10^6} = -0.196\ mm$

$$\frac{\Omega_2}{\omega_2} = r_2^2 = 1 + \mu/2 + \sqrt{1 + (\mu/2)^2 - 1}$$

now $r_2^2 + r_1^2 = 2 + \mu$

$\therefore r_2^2 = 2 + \mu - r_1^2 = 2.3403 - .75^2 = 1.7778$

$r = 1.331 = \frac{\Omega_2}{\omega_f}$ $\Omega_2 = 209.09\ \frac{rad}{s}$

---

in 4th 9.53

$\omega_1 = \sqrt{\frac{K_1}{m_1}} = 800 \times \frac{2\pi}{60} = 83.78\ rad/s$ $\omega_2 = \sqrt{\frac{K_2}{m_2}}$ for tuning let $\omega_f = \sqrt{\frac{K_2}{m_2}} = \sqrt{\frac{K_1}{m_1}}$

$$\left[\frac{\Omega}{\omega_2}\right]^2 = 1 + \mu/2 - \sqrt{(1+\mu/2)^2 - 1}$$

$\therefore \frac{\Omega}{\omega_2} = \frac{\Omega}{\omega_f} = \frac{750}{800} = .9375 = r_1$

$r_1^2 + r_2^2 = 2 + \mu$

$\mu = \frac{r_1^4 + 1}{r_1^2} - 2 = .0167 = \frac{m e'}{m_1} = \frac{1}{m_1}$

$m_1 = \frac{1}{.0167} = 59.87\,kg$

now new $\Omega = 700\ rpm.$ $r_1 = \frac{700}{800} = .875$ $\mu = \frac{r_1^4 + 1}{r_1^2} - 2 = .07175$

$\therefore \mu = \frac{m_2}{m_1} \Rightarrow m_2 = \mu m_1 = 4.3\ kg$

$r_2^2 = \left(\frac{\Omega}{\omega_2}\right)^2 = 1 + \mu/2 + \sqrt{(1+\mu/2)^2 - 1} = 2 + \mu - r_1^2 = 2 + .07175 - (.875)^2 = 1.306125$

$r_2 = 1.143$ or $\Omega = r_2 \omega_2 = 1.14286 \times 800 = 914.3$ rpm

so $k_2 = \omega_2^2 m_2 = 4.3 \times \left(800 \times \frac{2\pi}{60}\right)^2 = 30186.3\ N/m$

$r_2^2 = 1.137794$
$r_2 = 1.06667$
$r_2 = \frac{1000}{800} = 1.25$
$\mu = \frac{r_2^4 + 1}{r_2^2} - 2 = .2025$
$m_1 = \frac{1}{\mu} = 4.94\,kg$
$r_1 = \sqrt{2 + \mu - r_2^2} = .8$
or 640 rpm

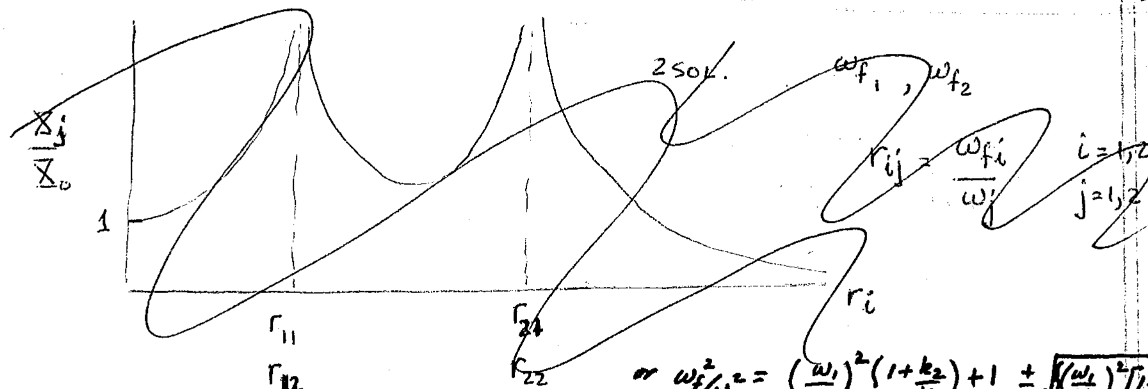
if we choose to move first Ω_{f_1} to 600 rpm
$r_1 = .75$ $\mu = .3403$
$r_2 = 1.333$ $\Omega_{f_2} = 1067$
$m_2 = \mu\,(59.87) = 20.3$
$k_2 = 143070\ N/m$

FROM RESONANCE EQ.

$$\omega_f^4 - \omega_f^2\left[\frac{k_2}{k_1}\omega_1^2 + \omega_1^2 + \omega_2^2\right] + \omega_1^2\omega_2^2 = 0$$

this has 2 roots for ω_f^2

$$\omega_f^2 = \frac{\omega_1^2(1+k_2/k_1) + \omega_2^2 \pm \sqrt{\left[\omega_1^2(1+k_2/k_1)+\omega_2^2\right]^2 - 4\omega_1^2\omega_2^2}}{2}$$



or $$\omega_f^2/\omega_2^2 = \frac{\left(\frac{\omega_1}{\omega_2}\right)^2\left(1+\frac{k_2}{k_1}\right)+1 \pm \sqrt{\left\{\left(\frac{\omega_1}{\omega_2}\right)^2\left[1+\frac{k_2}{k_1}\right]+1\right\}^2 - 4\left(\frac{\omega_1}{\omega_2}\right)^2}}{2}$$

FOR EQUAL MASSES & EQUAL SPRINGS. $k_2 = k_1 = k$ $\quad m_1 = m_2 = m$ $\quad \omega_1 = \omega_2 = \omega = \sqrt{\frac{k}{m}}$

$$r^4 - 3r^2 + 1 = 0 \qquad r^2 = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2} = .382,\ 2.618$$

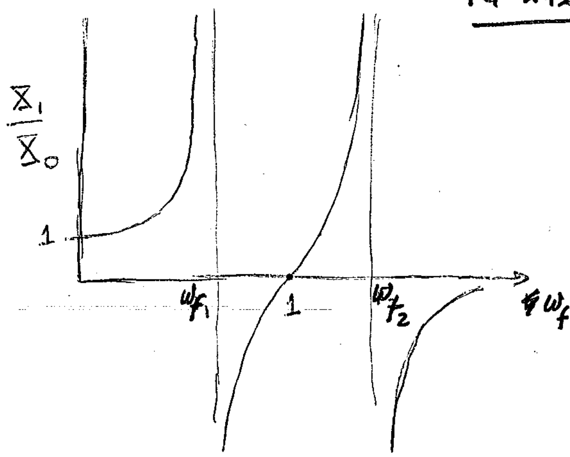
or $$\omega_f^2/\omega_2^2 = \frac{\left[1+(1+\mu)\left(\frac{\omega_2}{\omega_1}\right)^2\right] \pm \sqrt{\left[1+(1+\mu)\left(\frac{\omega_2}{\omega_1}\right)^2\right]^2 - 4\left(\frac{\omega_2}{\omega_1}\right)^2}}{2\left(\frac{\omega_2}{\omega_1}\right)^2}$$

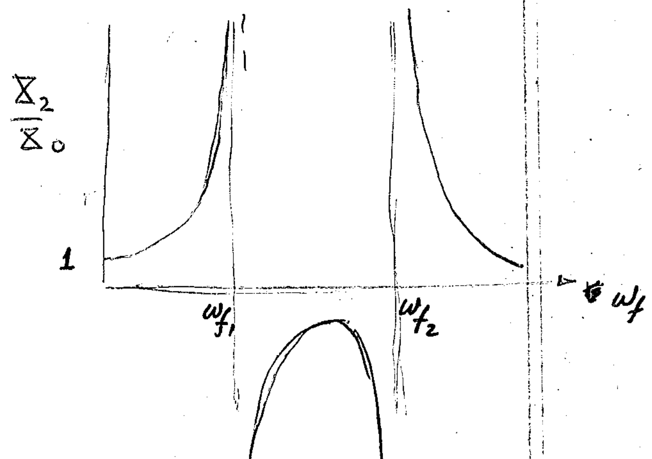
$r_1 = .618 \qquad r_2 = 1.618$

if $\omega_1 = \omega_2$ (tuned)

$$\omega_f^2/\omega_2^2 = \left[(1+\mu/2) \pm \sqrt{(1+\mu/2)^2 - 1}\right]$$

<u>PG 242 Rao.</u>

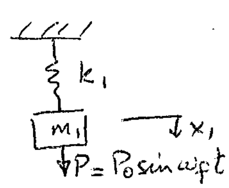




SESSION # 25

- UNDAMPED VIBRATION ABSORBER
  - FOR A MASS SPRING SYSTEM
    - $m_1\ddot{x}_1 + k_1 x_1 = P \Rightarrow x = X \sin\omega_f t$

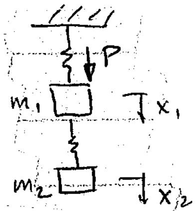
      $X = \frac{X_0}{1-r^2} \qquad r = \frac{\omega_f}{\omega} \qquad X_0 = \frac{P_0}{k_1}$



WHEN $r=1$ RESONANCE

- IF $r \approx 1$ $X \uparrow \infty$

- SUPPOSE WE LOOK AT



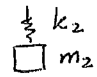
Assume

$x_1 = X_1 \sin \omega_f t$

$x_2 = X_2 \sin \omega_f t$

- WE FOUND
  - IF $r_2 = \dfrac{\omega_f}{\omega_2} = 1 \left(\omega_2 = \sqrt{\dfrac{k_2}{m_2}} \right)$ then $X_1 = 0$, $X_2 = -X_o \dfrac{k_1}{k_2} = -\dfrac{P_o}{k_2}$

- THIS ALLOWS FOR DESIGNING OF ABSORBER



  - Pick k_2, m_2 : knowing ω_f, P_o and X_2 (*perhaps* allowable travel of mass 2)
  - since

$$k_2 = \frac{P_o}{X_2} \quad \text{put into} \quad \overset{(r_2=1)}{\omega_2 = \omega_f} = \sqrt{\frac{k_2}{m_2}} \implies m_2$$

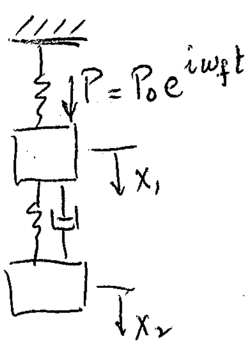
  - NORMALLY PICK $k_2 << k_1$ $m_2 << m_1$ $\implies X_2$ will be large
    since $X_2 = -P_o/k_2$

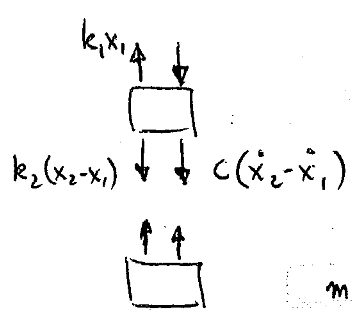
  - DISADV. EFFECTIVE ONLY AT $\omega_f = \omega_2$
    - IF ω_f VARIES A LOT $\implies$ WILL CAUSE TROUBLE
    - $\omega_2 - \omega_1$ small

  - GOOD FOR FIXED SPEED MACHINES BUT NOT AT STARTUP
    - MUST GO THROUGH LOWER RESONANT FREQ TO REACH OPERATING SP[…]

- CAN DO THE SAME FOR TORSIONAL ABSORBER

  - VIBS OF FORCED DAMPED SYSTEM





$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) + c(\dot{x}_2 - \dot{x}_1) + P$$

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1) - c(\dot{x}_2 - \dot{x}_1)$$

$$m_1\ddot{x}_1 + (k_2+k_1)x_1 + c\dot{x}_1 - k_2x_2 - c\dot{x}_2 = P$$
$$m_2\ddot{x}_2 + k_2x_2 + c\dot{x}_2 - k_2x_1 - c\dot{x}_1 = 0$$

- solution to homog gives transient ; solution to nonzero rhs leads to steady state

- choose $\left.\begin{array}{l} x_1 = \bar{X}_1 e^{i(\omega_f t-\psi)} = \bar{X}_1 e^{i\omega_f t} \\ x_2 = \bar{X}_2 e^{i(\omega_f t-\psi)} = \bar{X}_2 e^{i\omega_f t}\end{array}\right\}$ like SDOF

$P_0\cos\omega_f t = Re(\bar{X}e^{i\omega_f t})$
$= X_1\cos(\omega_f t-\psi)$

$$\left[-m_1\omega_f^2 + (k_2+k_1) + ic\omega_f\right]\bar{X}_1 - (k_2+ic\omega_f)\bar{X}_2 = P_0$$
$$\left[-m_2\omega_f^2 + ic\omega_f + k_2\right]\bar{X}_2 - (k_2+ic\omega_f)\bar{X}_1 = 0$$

Now
$$\bar{X}_1 = \frac{\begin{vmatrix} P_0 & -k_2+ic\omega_f \\ 0 & -m_2\omega_f^2+ic\omega_f+k_2\end{vmatrix}}{\begin{vmatrix} -m_1\omega_f^2+(k_2+k_1)+ic\omega_f & -(k_2+ic\omega_f) \\ -(k_2+ic\omega_f) & (-m_2\omega_f^2+ic\omega_f+k_2)\end{vmatrix}} = \frac{A+iB}{C+iD}$$

$$\bar{X}_2 = \frac{\begin{vmatrix} -m_1\omega_f^2+(k_2+k_1)+ic\omega_f & P_0 \\ -(k_2+ic\omega_f) & 0\end{vmatrix}}{\text{denom.}}$$

$\left\{\begin{array}{ll} A = -P_0k_2 & B = -P_0c\omega_f \\ C = & \end{array}\right\}$

$$\tan\psi = \frac{AD-BC}{AC+BD}$$

$A = P_0(k_2-m_2\omega_f^2) \quad B = P_0c\omega_f$
$C = (k_1-m_1\omega_f^2)(k_2-m_2\omega_f^2) - m_2k_2\omega_f^2$
$D = c\omega_f(-m_1\omega_f^2+k_1-m_2\omega_f^2)$

$$X_1 = \frac{P_0\sqrt{(k_2-m_2\omega_f^2)^2+(c\omega_f)^2}}{\sqrt{\left[(k_1-m_1\omega_f^2)(k_2-m_2\omega_f^2)-m_2k_2\omega_f^2\right]^2+(c\omega_f)^2(m_1\omega_f^2-k_1+m_2\omega_f^2)^2}}$$

$$X_2 = \frac{P_0\sqrt{k_2^2+(c\omega_f)^2}}{\sqrt{}}$$

if $P = P_0\cos\omega_f t \Rightarrow \left.\begin{array}{l} x_1 = X_1\cos(\omega_f t-\psi_1) \\ x_2 = X_2\cos(\omega_f t-\psi_2)\end{array}\right\}$ ψ_1,ψ_2 dependent of A

when $c \to \infty \quad \Rightarrow \quad X_1 = X_2 = \dfrac{\pm P_0}{m_1 \omega_f^2 - k_1 + m_2 \omega_f^2}$

for $c \neq 0$ we reduce resonant amplitude - AS IN SDOF

IF equal masses, equal springs and $c_c = 2m\omega$ $\quad \omega = \sqrt{\frac{k}{m}} \quad \zeta = \frac{c}{c_c}$

$$\frac{X_1}{X_0} = \frac{\sqrt{(r^2-1)^2 + (2\zeta r)^2}}{\sqrt{(2\zeta r)^2 (2r^2-1)^2 + (r^4 - 3r^2 + 1)^2}}$$

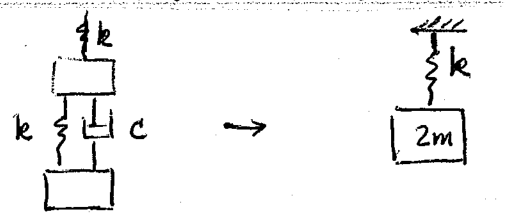
$$\frac{X_2}{X_0} = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(2\zeta r)^2 (2r^2-1)^2 + (r^4 - 3r^2 + 1)^2}}$$

FOR $\zeta \to \infty$

$$\frac{X_1}{X_0} \to \frac{2\zeta r}{(2\zeta r)(\pm(2r^2-1))} = \pm \frac{1}{1-2r^2}$$

$$\frac{X_2}{X_0} \to \frac{2\zeta r}{(2\zeta r)(\pm(2r^2-1))} = \pm \frac{1}{1-2r^2}$$

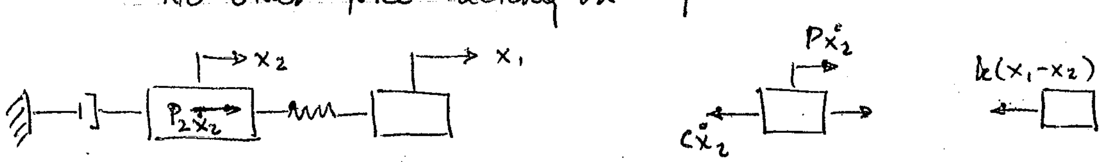
ie 2 masses are locked with a single spring acting ie $\omega = \sqrt{k/2m}$ $\quad r = \sqrt{2}\, r_1$



even though $\zeta = \infty$ we get resonance at $r = \frac{1}{\sqrt{2}}$

- Suppose we have a 2 degree of freedom system
  - subjected to self excitation
  - No other force acting on system

PASS



$m_1\ddot{x}_1 = -k(x_1 - x_2)$ $\qquad m_1\ddot{x}_1 + kx_1 - kx_2 = 0$

$m_2\ddot{x}_2 = k(x_1 - x_2) - c\dot{x}_2 + P\dot{x}_2$ $\qquad m_2\ddot{x}_2 + (c-P)\dot{x}_2 + kx_2 - kx_1 = 0$

PASS

- choose $x_1 = D_1 e^{-st}$ $\quad x_2 = D_2 e^{-st}$

$$(s^2 m_1 + k)D_1 - kD_2 = 0 \qquad ①$$

$$(-kD_1) + [m_2 s^2 + (c-P)s + k]D_2 = 0$$

$$\begin{vmatrix} s^2 m_1 + k & -k \\ -k & m_2 s^2 + (c-P)s + k \end{vmatrix} = 0$$

$$m_1 m_2 s^4 + m_1(c-P)s^3 + k(m_1+m_2)s^2 + k(c-P)s + \cancel{k^2} - \cancel{k^2} = 0$$

$s=0$ $\qquad m_1 m_2 s^3 + m_1(c-P)s^2 + (m_1+m_2)s + k(c-P) = 0$

- $s=0$ degenerate solution $\Rightarrow$ from ① $D_1 = D_2$
- 3 sign changes imply 3 positive roots

  1 positive root + 2 complex.

~~let $s_1 = p_1$ $\quad s_2 = p_2 + iq_2$ $\quad s_3 = p_2 - iq_2$~~

- if 3 roots s_1, s_2, s_3 $\Rightarrow$ $(s-s_1)(s-s_2)(s-s_3) = 0$

$$s^3 - a_2 s^2 + a_1 s - a_0 = s^3 - (s_1+s_2+s_3)s^2 + (s_1 s_2 + s_1 s_3 + s_2 s_3)s - s_1 s_2 s_3 = 0$$

$$\Rightarrow \quad \frac{c-P}{m_2} = s_1+s_2+s_3 = a_2 \qquad \frac{1}{m_1} + \frac{1}{m_2} = s_1 s_2 + s_1 s_3 + s_2 s_3 = a_1$$

$$\frac{k(c-P)}{m_1 m_2} = s_1 s_2 s_3 = a_0$$

- if 3 positive roots $\Leftrightarrow$ $c > P$ $\qquad$ stable system require ~~roots~~ coeffs all >0

- if 1 positive + 2 complex $\qquad (s-p_1)(s-(p_2+iq_2))(s-(p_2-iq_2)) = 0$

$$[s-p_1][s^2 - 2p_2 s + (p_2^2+q_2^2)]$$

- $$s^3 - \overset{a_2}{(p_1+2p_2)}s^2 + \overset{a_1}{[2p_1p_2 + p_2^2 + q_2^2]}s - \overset{a_0}{p_1(p_2^2+q_2^2)} = 0$$

$$\boxed{\text{if } p_1(p_2^2+q_2^2) > 0 \Rightarrow p_1 > 0} \qquad \text{if } a_0 > 0 \Rightarrow p_1 > 0$$

- ~~roots~~ coeff are all >0 if $p_1 > 0$ & $p_2 > 0$

$$x_1 = D_1 e^{-p_1 t} + D_1' e^{-p_2 t}\sin(q_2 t + \varphi_{11})$$

$$x_2 = D_2 e^{-p_1 t} + D_2' e^{-p_2 t}\sin(q_2 t + \varphi_{12})$$

PASS

- if $a_1, a_2, a_0 > 0 \not\Rightarrow p_2 > 0$
- if $p_2 > 0$ stable   if $p_2 < 0$ unstable

borderline is $p_2 = 0$

$$\left.\begin{aligned} a_2 &= p_1 \\ a_1 &= q_2^2 \\ a_0 &= p_1 q_2^2 \end{aligned}\right\} \longrightarrow \underline{a_2 a_1 = a_0}$$

Condition for stability is if $a_2 a_1 > a_0$

$$(p_1 + 2p_2)\left[2p_1 p_2 + p_2^2 + q_2^2\right] > p_1(p_2^2 + q_2^2)$$

$$2p_1^2 p_2 + 4p_1 p_2^2 + 2p_2(p_2^2 + q_2^2) > 0$$

if $p_2 > 0$ DIVIDE BY $2p_2$

$$\underline{p_1^2 + 2p_1 p_2 + p_2^2 + q_2^2 > 0}$$

(*) (0) if $a_1, a_2, a_0 > 0$ system may be stable, if not unstable

(1) if $a_0 > 0 \Rightarrow p_1 > 0$

(*) (2) if $a_1 a_2 > a_0 \Rightarrow p_2 > 0$

FOR A 4th degree characteristic equation $s^4 - a_3 s^3 + a_2 s^2 + a_1 s + a_0 = 0$

$a_0, a_1, a_2, a_3 > 0$ & if $a_1 a_2 a_3 > a_1^2 + a_3^2 a_0$

FOR our case $s^3 - \dfrac{(c-P)}{m_2} s^2 + \left(\dfrac{m_1 + m_2}{m_1 m_2}\right) s - \dfrac{k}{m_1 m_2}(c-P) = 0$

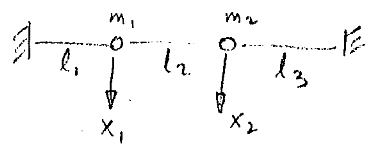
$$a_2 = \frac{c-P}{m_2} \qquad a_1 = \frac{m_1 + m_2}{m_1 m_2} \qquad a_0 = \frac{k}{m_1 m_2}(c-P)$$

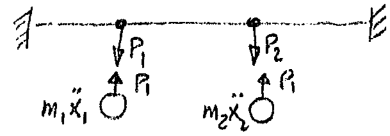
all are > 0 if $\underline{\underline{c > P}}$

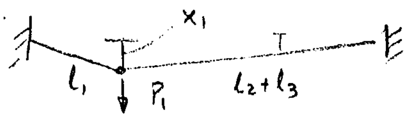
$$a_1 a_2 = \left(\frac{m_1 + m_2}{m_1 m_2}\right)\left(\frac{c-P}{m_2}\right) > a_0 = \frac{k}{m_1 m_2}(c-P)$$

$$\underline{\underline{\frac{m_1 + m_2}{m_2} > k}} \quad \text{for stability}$$

6.6 For the system shown below, we can replace it by the figure on the right.







for $P_2 = 0$

$P\sin\theta + P\sin\varphi = P_1$

$\sin\theta = \dfrac{x_1}{l_1}$ $\quad \sin\varphi = \dfrac{x_1}{l_2+l_3}$

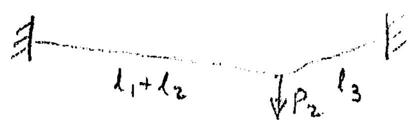
$\therefore \quad P_1 = Px_1\left[\dfrac{1}{l_1} + \dfrac{1}{l_2+l_3}\right] = Px_1 \dfrac{l_0}{l_1(l_2+l_3)}$

But $x_1 = a_{11}P_1 \Rightarrow \boxed{\dfrac{l_1(l_2+l_3)}{Pl_0} = a_{11}}$ where $l_0 = l_1 + l_2 + l_3$

by similar Δ's $\quad \dfrac{x_1}{l_2+l_3} = \dfrac{x_2}{l_3}$ (**) $\therefore \quad x_2 = \dfrac{x_1 l_3}{l_2+l_3} = \dfrac{P_1 l_1(l_2+l_3)}{Pl_0} \cdot \dfrac{l_3}{l_2+l_3} = \dfrac{P_1}{P}\dfrac{l_1 l_3}{Pl_0}$

but $x_2 = a_{21}P_1 \Rightarrow a_{21} = \dfrac{l_1 l_3}{Pl_0}$. By maxwell-betty reciprocity theorem

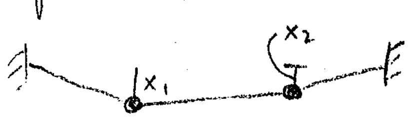
$a_{21} = a_{12} = \dfrac{l_3}{Pl_0}$. For $P_1 = 0$



$x_2 = a_{22}P_2$ by similar method to find a_{11} $\quad a_{22} = \dfrac{l_3(l_1+l_2)}{Pl_0}$

$\therefore \quad [a] = \dfrac{1}{Pl_0}\begin{bmatrix} l_1(l_2+l_3) & l_1 l_3 \\ l_1 l_3 & l_3(l_1+l_2) \end{bmatrix}$

To find [K] : look at a displaced system w/ $x_1 > x_2$:

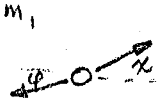


Now look at m_1

$-(P\sin\theta + P\sin\varphi) = $

$\sin\theta = \dfrac{x_1}{l_1}$ $\quad \sin\varphi = \dfrac{x_1 - x_2}{l_2}$

now look at m_1



$P\sin\varphi - P\sin\chi = m_2\ddot{x}_2$ $\quad \sin\varphi = \dfrac{x_1 - x_2}{l_2}$ $\quad \sin\chi = \dfrac{x_2}{l_3}$

$$\therefore \quad \begin{aligned} m_1\ddot{x}_1 + \left(\frac{P}{l_1} + \frac{P}{l_2}\right)x_1 - \frac{P}{l_2}x_2 = 0 \\ m_2\ddot{x}_2 + \left(\frac{P}{l_2} + \frac{P}{l_3}\right)x_2 - \frac{P}{l_2}x_1 = 0 \end{aligned} \Rightarrow [M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad [K] = \begin{bmatrix} P\left(\frac{1}{l_1} + \frac{1}{l_2}\right) & -\frac{P}{l_2} \\ -P/l_2 & P\left(\frac{1}{l_2} + \cdots\right. \end{bmatrix}$$

if $m_1 = m_2 = m$, $l_1 = l_2 = l_3 = l$ $\quad [M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$ $\quad [K] = \begin{bmatrix} 2P/l & -P/l \\ -P/l & 2P/l \end{bmatrix}$ $\quad [a] = \begin{bmatrix} \frac{2l}{3P} & l/3P \\ \frac{l}{3P} & \frac{2l}{3P} \end{bmatrix}$

show generalized coordinates ✓

do mixed coordinates ✓

do damped vibs

do 2DOF w/ forcing undamped ✓

vibration absorber

do 2DOF w/ damping

$$\begin{vmatrix} a_1 & a_3 & a_5 & a_7 \\ a_0 & a_2 & a_4 & \\ 0 & a_1 & a_3 & \\ & & & \end{vmatrix}$$

$$|a_1|$$

$$\begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4$$

$$\begin{vmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & a_4 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & a_4 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & & \\ & & \\ & & \end{vmatrix}$$

$$a_4 \begin{vmatrix} a_1 & a_3 & 0 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$

$a_0 \to a_n$ must be positive

$$a_4 \left[a_1 a_2 a_3 - a_3^2 a_0 - a_1^2 a_3 \right]$$

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + a_3 s^{n-3} + a_4 s^{n-4} + \dots + a_n s^0 = 0$$

$$\begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2m-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2m-2} \\ 0 & a_1 & a_3 & \cdots & a_{2m-3} \\ 0 & a_0 & a_2 & \cdots & \\ 0 & 0 & a_1 & \cdots & \end{vmatrix}$$

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

$$\begin{vmatrix} a_1 & a_3 & 0 & 0 \\ a_0 & a_2 & 0 & 0 \\ 0 & a_1 & a_3 & 0 \\ 0 & a_0 & a_2 & 0 \\ 0 & 0 & a_1 & a_3 \end{vmatrix}$$

$$a_0 s^2 + a_1 s + a_2$$

$$\begin{matrix} a_1 & 0 & 0 & 0 \\ a_0 & a_2 & 0 & 0 \\ 0 & a_1 & 0 & 0 \\ 0 & a_0 & a_2 & 0 \end{matrix}$$

$T_1 = a_1 > 0$

$T_2 = a_1 a_2 - a_0 a_3 > 0$

$T_3 = a_1 a_2 a_3 - a_1 a_3^2 > 0$

$T_1 = a_1 > 0$

$T_2 = a\,a > 0 \Rightarrow a > 0$

Florida International University

Office of the President

October 23, 1991

DOE PREP Program
Associated Western Universities
4190 South Highland Drive, Suite 211
Salt Lake City, Utah 48124

RE: FIU - PREP (Pre-Freshman Enrichment Program) Project

Dear Sirs:

I am writing to express my strong support for the above proposal, being submitted by the Department of Mechanical Engineering. Our College of Engineering and Design is the fastest growing Engineering school in the State of Florida with all programs fully accredited in record time. We are ranked among the leading national Engineering schools in research and scholarly productivity. Last year, the Department of Mechanical Engineering received $1,201,941 in research grants.

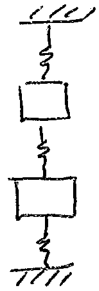
Faculty and staff enthusiasm and academic strength have been attracting increasing number of students into the Engineering bachelor's and master's programs. Minority enrollments are especially strong, as a matter of fact, we have the highest levels of minority enrollment in the State of Florida. Our Fall 1991 minority enrollments were as follows:

Blacks	9.3%
Hispanics	53.6%
Women	12.6%

Indeed, FIU's Hispanic Engineering enrollment is 49 times the national average. The College of Engineering has the largest percentage of Hispanic students in the nation.

University Park, Miami, Florida 33199 • (305) 348-2111

1. LETS ASSUME WE HAD



$$m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2x_2 = 0$$
$$m_2\ddot{x}_2 + (k_2+k_3)x_2 - k_2x_1 = 0$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{pmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{pmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$[M]\ddot{\underset{\sim}{x}} + [K]\underset{\sim}{x} = \underset{\sim}{0}$$

M is the mass matrix
K is the stiffness matrix

Notice coupling (static) produces a diagonal matrix [M]

remember when $x_1 = A_1\sin(\omega t+\varphi)$ $\quad x_2 = A_2\sin(\omega t+\varphi)$

$$\therefore \begin{pmatrix}\ddot{x}_1 \\ \ddot{x}_2\end{pmatrix} = -\omega^2\begin{pmatrix} A_1 \\ A_2\end{pmatrix}\sin(\omega t+\varphi) = -\omega^2\begin{pmatrix} x_1 \\ x_2\end{pmatrix}$$

$$\therefore \left\{-\omega^2[M] + [K]\right\}(\underset{\sim}{x}) = (\underset{\sim}{0})$$

BY SETTING

$$\det\left\{-\omega^2[M]+[K]\right\} = \begin{vmatrix} -\omega^2 m_1 + (k_1+k_2) & -k_2 \\ -k_2 & -\omega^2 m_2 + (k_2+k_3)\end{vmatrix} = 0$$

we get frequency equation

$$\det\left(-\omega^2[M]+[K]\right) = 0$$

Rayleigh's Method: for a conservative system $T_{max} = V_{max}$

FIND: $T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = \frac{1}{2}\dot{\underset{\sim}{x}}^T[M]\dot{\underset{\sim}{x}}$ $\qquad \dot{\underset{\sim}{x}}^T = [\dot{x}_1 \ \ \dot{x}_2]$

$V = \frac{1}{2}k_1(x_1)^2 + \frac{1}{2}k_2(x_1-x_2)^2 + \frac{1}{2}k_3x_2^2 = \frac{1}{2}\underset{\sim}{x}^T[K]\underset{\sim}{x}$

IF $\left.\begin{array}{l} x_1 = A_1\sin(\omega t+\varphi) \\ x_2 = A_2\sin(\omega t+\varphi)\end{array}\right\}$ $\quad \begin{pmatrix} x_1 \\ x_2\end{pmatrix} = \begin{pmatrix} A_1 \\ A_2\end{pmatrix}\sin(\omega_n t+\phi)$

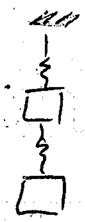
$$\therefore T = \left(\frac{1}{2}m_1\omega^2A_1^2 + \frac{1}{2}m_2\omega^2A_2^2\right)\cos^2(\omega t+\varphi) = \frac{1}{2}[A_1 \ A_2]\begin{bmatrix} m_1 & 0 \\ 0 & m_2\end{bmatrix}\begin{bmatrix} A_1 \\ A_2\end{bmatrix}\omega^2\cos^2(\omega t$$

$$V = \left[\frac{1}{2}k_1A_1^2 + \frac{1}{2}k_2(A_1-A_2)^2 + \frac{1}{2}k_3A_2^2\right]\sin^2(\omega t+\varphi)$$
$$= \frac{1}{2}[A_1 \ A_2]\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_3+k_2\end{bmatrix}\begin{bmatrix} A_1 \\ A_2\end{bmatrix}\sin^2($$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$A^{-1} = \frac{\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}}{\det A = a_{11} a_{22} - a_{12} \cdot a_{21}}$$

$$[a_{ij}]^{-1} = \frac{1}{\det A} \overset{\text{signed minor}}{} (-1)^{i+j}$$



k_1

$\downarrow 2W$

$$\delta_{st} = \frac{W_1 + W_2}{k_1} = \delta_1$$

$$\delta_{st_2} = \delta_1 + \frac{W_2}{k_2} = \delta_2$$

if $m_1 = m_2 = m$ & $k_1 = k_2 = k$

$$\delta_1 = \frac{2W}{k}$$

$$\delta_2 = \frac{2W}{k} + \frac{W}{k} = \frac{3W}{k}$$

$$\frac{W}{k}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \qquad \text{chose } \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

then
$$\omega^2 = \frac{\begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}}{\begin{bmatrix} 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix}} = \frac{14}{13}\frac{k}{m}$$

Rayleigh $\Rightarrow T_{max} = V_{max}$. $\therefore$ $\omega^2 \underset{\sim}{A}^T [M] \underset{\sim}{A} = \underset{\sim}{A}^T [K] \underset{\sim}{A}$

$$\therefore \quad \omega^2 = \frac{\underset{\sim}{A}^T [K] \underset{\sim}{A}}{\underset{\sim}{A}^T [M] \underset{\sim}{A}}$$

if $\underset{\sim}{A}$ is close to the mode shapes then ω_{RAYLEIGH} will be close to the value of ω obtained from the characteristic eqn. Actual $\underset{\sim}{A}$ minimizes value ω^2.

EXAMPLE if $m_1 = m_2 = m$ $\quad k_1 = k_2 = k = k_3$ $\quad [K] = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} k$

$$[M] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} m$$

If you are given $[A_1 \ A_2]^T = [1 \ 2]$

$$\omega^2 = \frac{[1 \ 2] \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} k}{[1 \ 2] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} m} = \frac{[1 \ 2] \begin{bmatrix} 0 \\ 3 \end{bmatrix} k = 6k}{[1 \ 2] \begin{bmatrix} 1 \\ 2 \end{bmatrix} m = 5m}$$

$$= 1.2 \frac{k}{m} \quad \text{or} \quad \omega_{\text{rayleigh}} \cong 1.1 \sqrt{\frac{k}{m}}$$

FOR ACTUAL $[A_1 \ A_2]^T = [1 \ 1] \Rightarrow \omega = \sqrt{\frac{k}{m}}$ $\quad$ error 10%

$\omega_{\text{rayleigh}} \geq \omega_{n_1}$

DUNKERLEY'S FORMULA

remember $\{-\omega^2 [M] + [K]\} \underset{\sim}{X} = \underset{\sim}{0}$

if $\det[K] \neq 0$ then $\{[K]^{-1}[M] - \frac{1}{\omega^2}[I]\} \underset{\sim}{X} = 0$

where $[K]^{-1} = [a] = \begin{bmatrix} \dfrac{k_2 + k_3}{k_1 k_3 + k_2 k_3 + k_1 k_2} & \dfrac{k_2}{k_1 k_3 + k_2 k_3 + k_1 k_2} \\ \dfrac{k_2}{k_1 k_3 + k_2 k_3 + k_1 k_2} & \dfrac{k_1 + k_2}{k_1 k_3 + k_1 k_2 + k_2 k_3} \end{bmatrix}$

where $[I] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad [a] = \begin{bmatrix} \frac{2}{3k} & \frac{1}{3k} \\ \frac{1}{\ } & \frac{2}{\ } \end{bmatrix}$ if $k_1 = k_2 = k_3$

$$\det\left\{[a][M]-\frac{1}{\omega^2}[I]\right\}=0 = \begin{bmatrix} a_{11}m_1-\frac{1}{\omega^2} & a_{12}m_2 \\ a_{21}m_1 & a_{22}m_2-\frac{1}{\omega^2}\end{bmatrix}$$

$$\frac{1}{\omega^4}-\underbrace{\left(a_{11}m_1+a_{22}m_2\right)}\frac{1}{\omega^2}+a_{11}m_1a_{22}m_2-a_{12}^2m_1m_2=0$$

(ROOTS)

$a_{11}m_1+a_{12}m_2$ represent the sum of the terms of the factors

$(x-s_1)(x-s_2)=0$

$x^2-(s_1+s_2)x+s_1s_2=0$

$$=\frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}$$

IN GENERAL if $\omega_1 \ll \omega_2 \Rightarrow \omega_1^2 \ll \omega_2^2$ and $\frac{1}{\omega_1^2} \gg \frac{1}{\omega_2^2}$

$$\Longrightarrow \frac{1}{\omega_1^2}+\frac{1}{\omega_2^2}\cong\frac{1}{\omega_1^2} \qquad \text{IF } \omega_2,\omega_3,\omega_4,\ldots \gg \omega_1$$

FOR OUR PROBLEM IF $m_1=m_2=m$ & $k_1=k_2=k$

$$\underset{a_{11}m_1}{\frac{2}{3k}m}+\underset{a_{22}m_2}{\frac{2}{3k}m}\cong\frac{1}{\omega_1^2}=\frac{4m}{3k} \qquad \therefore \omega_1\cong .866\sqrt{\frac{k}{m}}$$

NOTE THAT FOR $\omega_1 < \omega_{1\,REAL}$

WORKS BEST FOR VERY LARGE SYSTEMS - MULTIDEGREE SYSTEMS

~~6.34~~ 6.61

$$\omega^2=\frac{[.275\ \ .4\ \ .45]\begin{bmatrix}6&-4&0\\-4&10&0\\0&0&6\end{bmatrix}\begin{bmatrix}.275\\.4\\.45\end{bmatrix}}{[.275\ \ .4\ \ .45]\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\begin{bmatrix}.275\\.4\\.45\end{bmatrix}}=\frac{[.275\ .4\ .45]\begin{bmatrix}.05\\2.9\\2.7\end{bmatrix}=2.39}{[\quad]\begin{bmatrix}.275\\.8\\1.35\end{bmatrix}=1.003}=\frac{2.39}{}$$

$\omega=1.544$

$$\overset{[k]}{\left[\begin{array}{ccc|ccc}6&-4&0&1&0&0\\-4&10&0&0&1&0\\0&0&6&0&0&1\end{array}\right]}\rightarrow\left[\begin{array}{ccc|ccc}6&-4&0&1&0&0\\0&44&0&4&6&0\\0&0&1&0&0&\frac{1}{6}\end{array}\right]\rightarrow\left[\begin{array}{ccc|ccc}66&0&0&\frac{5}{22}&\frac{1}{11}&0\\0&44&0&\frac{4}{11}&\frac{3}{22}&0\\0&0&1&0&0&\frac{1}{6}\end{array}\right]\ [a]$$

$$\frac{5}{22}\cdot 1+\frac{3}{22}\cdot 2+\frac{1}{6}\cdot 3=1=\frac{1}{\omega_n^2} \qquad \omega_n=1$$

$$a_{11}m_1+a_{22}m_2+a_{33}m_3=\frac{17}{22}+2=2.7$$

$2.1+.06+.67 = 2.83$

$$\omega_n^2 = \frac{\begin{bmatrix} .7 & .3 & -1/3 \end{bmatrix}\begin{bmatrix} 6 & -4 & 0 \\ -4 & 10 & 0 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} .7 \\ .3 \\ -1/3 \end{bmatrix}}{\begin{bmatrix} .7 & .3 & -1/3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} .7 \\ .3 \\ -1/3 \end{bmatrix}} = \frac{\begin{bmatrix} .7 & .3 & -1/3 \end{bmatrix}\begin{bmatrix} 3 \\ .2 \\ -2 \end{bmatrix}}{\begin{bmatrix} .7 & .3 & -1/3 \end{bmatrix}\begin{bmatrix} .7 \\ .6 \\ -1 \end{bmatrix}} = \frac{2.83}{1} \cong 2.83$$

$.49+.18+.33$

$\omega_n = \sqrt{2.83}$

DO 6.6 but only for 2 masses find: $[k]$, $[m]$; find ω and modes

For $[A_1 \; A_2] = [1 \;\; 1]$ find ω using Rayleigh

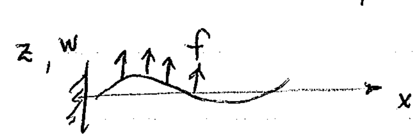
$[\quad\;] = [1 \;\; -1]$ " " " "

Find ω_1 using dunkerley

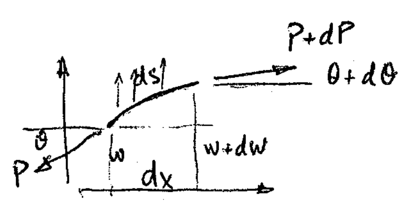
- DEALT WITH DISCRETE SYSTEMS ie mass, elasticity, damping at discrete pts
- SYSTEMS WHERE M, K, C occur at ∞ no. of pts = CONTINUOUS SYSTEMS
    - INFINITE DEGREES OF FREEDOM

- DISCRETE SYSTEMS ARE O.D.E's (LUMPED)
- CONTINUOUS SYSTEMS ARE P.D.E's
- CONTINUOUS SYSTEM IS MOST ACCURATE DESCRIPTION
- CHOICE OF LUMPED OR CONTINUOUS DEPENDENT ON WHY THE ANALYSIS

TRANSVERSE VIBS OF STRING



- CONSIDER A TAUT CABLE LENGTH l SUBJECTED TO A force $f(x,t)$ per unit length
- TRANSVERSE DISPLACEMENT IS $W(x,t)$ W is small wrt l



$$(P+dP)\sin(\theta+d\theta) - P\sin\theta + f\,ds = \rho\,ds\,\frac{\partial^2 w}{\partial t^2}$$

$ds = dx\sqrt{1+w'^2}$ $w' \ll 1 \Rightarrow dx = ds$

$$dP = \frac{\partial P}{\partial x}\,dx$$

$$\sin\theta = \frac{dw}{ds} = \frac{\partial w}{\partial x} \cong \tan\theta$$

$$\sin(\theta + d\theta) \cong \tan(\theta + d\theta) = \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2}\,dx$$

put into Diff Eqn.

$$\frac{\partial}{\partial x}\left[P\frac{\partial w}{\partial x}\right] + f = \rho\frac{\partial^2 w}{\partial t^2}$$

wave eqn.

if $P = \text{const}$ & $f = 0$ $\qquad P\frac{\partial^2 w}{\partial x^2} = \rho\frac{\partial^2 w}{\partial t^2} \Rightarrow \boxed{c^2 w_{xx} = w_{tt}} \quad c = \sqrt{P/\rho}$

B.C. $\quad w(x= \ ,t)$ — displacement of pt.

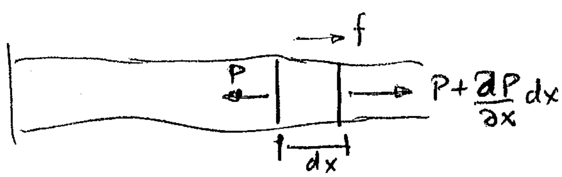
$\qquad \frac{\partial w}{\partial x}$ — angular

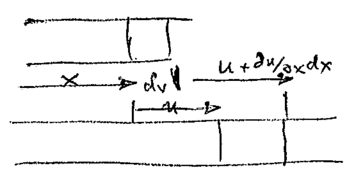
I.C. $\quad w(x, t=0)$ no displ.

$\qquad \dot{w}(x, t=0)$ no velocity

Solution Separation of Variables Let $w(x,t) = X(x)\,T(t)$

Longitudinal Vibs of a rod. $\qquad u(x,t)$ longitudinal displ.





elongation Δl

$\varepsilon = \frac{\Delta l}{l}$ $\quad l = dx$

$$\sigma = \text{stress} = E\,\varepsilon = E\frac{\partial u}{\partial x} \qquad P = \sigma A = EA\frac{\partial u}{\partial x}$$

$$P + dP - P + f\,dx = \underbrace{\rho A\,dx}_{\text{mass}}\frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial}{\partial x}\left[EA\frac{\partial u}{\partial x}\right]dx + f\,dx = \rho A\,dx\; u_{tt} \Rightarrow EA\,u_{xx} = \rho A\,u_{tt} \text{ if } f=0 \text{ \& } EA = \text{const}$$

$\frac{E}{\rho}u_{xx} = u_{tt}$ where $\quad c^2 = E/\rho \quad$ bar or rod velocity

TYPICAL BC are found on pg 372.

IC. $u(x, t=0)$

$\dot{u}(x, t=0) =$

Out of Vierck

HW #4	2-3, 10, 13, 23 ✓
HW #5	2-31, 32, 33 ✓
HW #6	3-9, 3-10, 17, 19 ✓
HW #7	3-26; 3-27, 3-30, 3-35
HW #8	4-2, 6, 13
HW #9	4-21, 4-27, 4-28, 4-29a find BANDWIDTH, MF_{max}, $MF_{resonant}$
HW #10	4-33, 4-35, 4-40, 4-41
HW #11	4-20, 4-42, 4-43
HW #12	5-6, 5-28
HW #13	7-1, 7-4, 7-7
HW #14	7-2, 7-5
HW #15	7-25, 7-29

HW #1

1. Solve $y' + \frac{1}{x} y = 3\cos 2x$ $\qquad p(x) = \frac{1}{x} \qquad g(x) = 3\cos 2x$

$$\mu(x) = c e^{\int^x \frac{1}{t} dt} = c e^{\ln x} = cx$$

$$y(x) = \frac{1}{cx}\left[\int^x ct \cdot 3\cos 2t \, dt + \text{const}\right]$$

$$= \frac{1}{cx}\left[\frac{c3x\sin 2x}{2} + \frac{c3\cos 2x}{4}\right] + \frac{\text{const}}{x}$$

$$y(x) = \frac{3}{2}\sin 2x + \frac{3}{4x}\cos 2x + \frac{\text{Const}}{x}$$ ✓ solution unbounded at $x=0$.

2. $y' = \frac{1}{e^y - x}$ ∴ let $x' = e^y - x$ thus $x' + x = e^y$ $\qquad p(y)=1 \quad g(y)=e^y$

$$\mu(y) = c e^{\int^y p(y')dy'} = c e^y$$

$$x(y) = \frac{1}{ce^y}\int^y c e^{y'} \cdot e^{y'} dy' + \frac{\text{Const}}{e^y}$$

$$x(y) = \frac{1}{e^y}\frac{e^{2y}}{2} + \frac{\text{Const}}{e^y} = \frac{e^y}{2} + \text{const}\, e^{-y}$$

thus $y(x) = \frac{e^x}{2} + \text{const}\, e^{-x}$ ✓

3. $xy' + 2y = x^2 - x + 1$ with $y(x=1) = 1$

$$y' + \frac{2}{x}y = x - 1 + \frac{1}{x} \qquad \mu(x) = c e^{\int^x \frac{2}{t}dt} = c e^{2\ln x} = cx^2$$

$$y(x) = \frac{1}{cx^2}\left[\int^x ct^2 \cdot \left(t - 1 + \frac{1}{t}\right)dt + \text{const}\right]$$

$$= \frac{\text{const}}{x^2} + \frac{1}{x^2}\int^x (t^3 - t^2 + t)\,dt = \frac{\text{const}}{x^2} + \frac{x^2}{4} - \frac{x}{3} + \frac{1}{2}$$

when $x=1$ $y=1$

$$1 = \frac{c}{1} + \frac{1}{4} - \frac{1}{3} + \frac{1}{2} = \frac{c}{1} + \frac{5}{12} \qquad c = \frac{7}{12}.$$

$$y(x) = \frac{7}{12x^2} + \frac{1}{12}(3x^2 - 4x + 6)$$ ✓

solution unbounded at $x=0$ & $x=\infty$

4. $y' + (\cot x)\,y = 2\csc x \qquad w/ \quad y(\pi/2) = 1$

$$\mu(x) = Ce^{\int^x p(t)dt} = Ce^{\int^x \frac{\cos t}{\sin t}dt} = Ce^{\ln \sin x} = C\sin x$$

$$y(x) = \frac{1}{C\sin x}\left[\int^x C\sin t \cdot 2\frac{1}{\sin t}\,dt + \text{const}\right]$$

$$= \frac{C}{\sin x} + \frac{1}{\sin x}\int^x 2\,dt = \frac{C}{\sin x} + \frac{2x}{\sin x}$$

@ $x = \pi/2 \quad y = 1 \qquad 1 = \frac{C}{1} + \frac{\cancel{2}\pi}{\cancel{2}} \qquad C = 1 - \pi$

$$y(x) = \frac{1-\pi}{\sin x} + \frac{2x}{\sin x}$$

solution unbounded at $n\pi = x \quad n = 0, \pm 1, \pm 2, \pm 3, \ldots$

$xy' + y = x^2 - x + 1 \qquad w/\ y(1) = 1$

$y' + \frac{y}{x} = x - 1 + \frac{1}{x} \qquad e^{\int \frac{1}{t}dt} = e^{\ln x} = cx$

$$y = \frac{1}{cx}\left[\int ct\left[t - 1 + \frac{1}{t}\right]dt + \text{const}\right]$$

$$= \frac{1}{x}\left[t^3/3 - t^2/2 + t\right]^x + \frac{\text{const}}{x}$$

$$y = x^2/3 - \frac{x}{2} + 1 + \frac{c}{x}$$

$$\cancel{1} = \frac{1}{3} - \frac{1}{2} + \cancel{1} + c \qquad c = +\frac{1}{6}$$

$$y = x^2/3 - \frac{x}{2} + 1 + \frac{1}{6x}$$

HW#2

1. $y''-y=0$ has char. eq $(r^2-1)=0$: solution is $r=\pm 1$ or

$$y = C_1 e^{-x} + C_2 e^{x}$$

$W(y_1, y_2)$ for $y_1 = e^{-x}$ $y_2 = e^{x}$

$$\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{x} \\ -e^{-x} & e^{x} \end{vmatrix} = 1-(-1) = 2 \neq 0 \quad \forall x$$

$\therefore$ y_1 & y_2 are independent & fundamental set.

2. give $y'' + p(x)y' + q(x)y = y'' + \frac{1}{x}y' + \left(1 - \frac{1}{4x^2}\right)y = 0$

$$y_1(x) = x^{-1/2}\sin x$$

$$y_1' = -\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x$$

$$y_1'' = \frac{3}{4}x^{-5/2}\sin x - 1\cdot x^{-3/2}\cos x - x^{-1/2}\sin x$$

$$x^2y'' + xy' + (x^2 - 1/4)y = \left(\cancel{\frac{3}{4}x^{-1/2}\sin x} - \cancel{x^{1/2}\cos x} - \cancel{x^{3/2}\sin x}\right) - \cancel{\frac{1}{2}x^{-1/2}\sin x} + \cancel{x^{1/2}\cos x}$$
$$+ \cancel{x^{3/2}\sin x} - \cancel{1/4\, x^{-1/2}\sin x} = 0$$

let $y_2 = V(x)\, y_1$ $\qquad p(x) = \frac{1}{x}$

$$V' = \frac{c}{y_1^2}\, e^{-\int^x p(t)dt} = \frac{cx}{\sin^2 x}\, e^{-\int^x \frac{1}{t}dt} = \frac{cx\, e^{-\ln x}}{\sin^2 x} = \frac{c}{\sin^2 x}$$

$$V = c\int^x \csc^2 s\; ds = -c\cot x. \qquad \frac{d\cot x}{dx} = -\frac{1}{\sin^2 x} = -\csc^2 x$$

$$y_2 = y_1 V = -x^{-1/2}\sin x \cdot c\,\frac{\cos x}{\sin x} = x^{-1/2}\cos x \;\checkmark$$

3. $6y'' - y' - y = 0$ has char eq $6r^2 - r - 1 = 0$

$$r = \frac{+1 \pm \sqrt{1+4(6)(1)}}{12} = \frac{1 \pm 5}{12} = \frac{1}{2}, -\frac{1}{3}$$

$$y = C_1 e^{x/2} + C_2 e^{-x/3} \;\checkmark$$

b. $4y'' + 4y' + y = 0$ has char eq $4r^2 + 4r + 1 = 0$

$$r = \frac{-4 \pm \sqrt{16 - 4(4)(1)}}{8} = -\frac{1}{2}, -\frac{1}{2}$$

$$\therefore \quad y(x) = (c_1 + c_2 x)e^{-x/2}$$

c. $y'' - 2y' + 6y = 0$ has char eq $r^2 - 2r + 6 = 0$

$$r = \frac{2 \pm \sqrt{4 - 4(1)(6)}}{2} = \frac{2 \pm 2i\sqrt{5}}{2} = 1 \pm i\sqrt{5}$$

$$y(x) = e^{x}\left[c_1 \sin\sqrt{5}x + c_2 \cos\sqrt{5}x\right]$$

d. $y'' + 2y' + 2y = 0$ has char eq $r^2 + 2r + 2 = 0$

$$r = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$$

$$y(x) = e^{-x}\left[c_1 \sin x + c_2 \cos x\right]$$

4. $u'' + \omega_0^2 u = \cos\omega t \qquad \omega \neq \omega_0$

$y_h = A\cos\omega_0 t + B\sin\omega_0 t$

let $y_p = A_1 \cos\omega t \qquad y_p' = -A_1\omega\sin\omega t \qquad y_p'' = -A_1\omega^2\cos\omega t$

$$y_p'' + \omega_0^2 y_p = -A_1\left[\omega^2\cos\omega t\right] + \omega_0^2 A_1\cos\omega t = \cos\omega t$$

$$= +A_1\left[\omega_0^2 - \omega^2\right]\cos\omega t = \cos\omega t \qquad \therefore \boxed{A_1 = \frac{1}{\omega_0^2 - \omega^2}}$$

Thus $y = y_h + y_p = A\cos\omega_0 t + B\sin\omega_0 t + \dfrac{1}{\omega_0^2 - \omega^2}\cos\omega t.$

if $\omega_0 = \omega$. Chose $y_p = A_1 t\sin\omega_0 t$

$y_p' = A_1\left[\sin\omega_0 t + \omega_0 t\cos\omega_0 t\right]$; $y_p'' = A_1\left[2\omega_0\cos\omega_0 t - \omega_0^2 t\sin\omega_0 t\right]$

$\therefore y_p'' + \omega_0^2 y_p = A_1\left[2\omega_0\cos\omega_0 t - \omega_0^2 t\sin\omega_0 t + \omega_0^2 t\sin\omega_0 t\right] = A_1 \cdot 2\omega_0\cos\omega_0 t = \cos\omega_0 t$

$\therefore \boxed{A_1 = \frac{1}{2\omega_0}}$

$y = A_1\sin\omega_0 t + B\cos\omega_0 t + \dfrac{t}{2\omega_0}\sin\omega_0 t$ as $t \to \infty$ amplitude of partic solution $\to \infty$

amplitude eqns $\left\{-\omega^2[M]+[K]\right\}\begin{pmatrix}A_1\\A_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \Rightarrow \begin{bmatrix}-\omega^2 m+\frac{2P}{\ell} & -P/\ell\\ -P/\ell & -\omega^2 m+\frac{2P}{\ell}\end{bmatrix}\begin{pmatrix}A_1\\A_2\end{pmatrix}$

characteristic eqn is $\omega^4 m^2 - \frac{4P}{\ell}\omega^2 m + \frac{4P^2}{\ell^2} - P^2/\ell^2 = \omega^4 - \frac{4P}{\ell}\frac{\omega^2}{m} + \frac{3P^2}{m^2\ell^2} = 0$

$$\left(\omega^2 - \frac{3P}{m\ell}\right)\left(\omega^2 - \frac{P}{m\ell}\right) = 0 \Rightarrow \omega = \sqrt{\frac{3P}{m\ell}}$$
$$\omega = \sqrt{\frac{P}{m\ell}}$$

if $\omega = \sqrt{P/m\ell}$ put into amplitude eqns. $\left(-\omega^2 m + \frac{2P}{\ell}\right)A_1 - P/\ell\, A_2 = \frac{P}{\ell}A_1 - P/\ell\, A_2 = 0$

$\Rightarrow A_2 = A_1$ & $A_2/A_1 = 1$; if $A_1 = 1 \Rightarrow \underset{\sim}{A}^T = [A_1 \;\; A_2] = [1 \;\; 1$

IF $\omega = \sqrt{\frac{3P}{m\ell}}$ put into $\left(-\omega^2 m + \frac{2P}{\ell}\right)A_1 - P/\ell\, A_2 = -\frac{P}{\ell}A_1 - P/\ell\, A_2 = 0$

$\Rightarrow A_2 = -A_1$ & $A_2/A_1 = -1$; if $A_1 = 1 \Rightarrow \underset{\sim}{A}^T = [A_1 \;\; A_2] = [1 \;\; -$

$\omega^2 = \dfrac{\underset{\sim}{A}^T[K]\underset{\sim}{A}}{\underset{\sim}{A}^T[M]\underset{\sim}{A}}$

Rayleigh's method. w/ $\underset{\sim}{A}^T = [1 \;\; 1]$

$$\omega^2 = \frac{[1 \;\; 1]\begin{bmatrix}2P/\ell & -P/\ell\\ -P/\ell & 2P/\ell\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}}{[1 \;\; 1]\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}} = \frac{[1 \;\; 1]\begin{bmatrix}P/\ell\\ P/\ell\end{bmatrix}}{[1 \;\; 1]\begin{bmatrix}m\\ m\end{bmatrix}} = \frac{2P/}{2m}$$

$$\omega^2 = \frac{P}{m\ell} \Rightarrow \omega = \sqrt{P/m\ell}$$

Rayleigh's method w/ $\underset{\sim}{A}^T = [1 \;\; -1]$

$$\omega^2 = \frac{[1 \;\; -1]\begin{bmatrix}2P/\ell & -P/\ell\\ -P/\ell & 2P/\ell\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}}{[1 \;\; -1]\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}} = \frac{[1 \;\; -1]\begin{bmatrix}3P/\ell\\ -3P/\ell\end{bmatrix}}{[1 \;\; -1]\begin{bmatrix}m\\ -m\end{bmatrix}} = \frac{6}{2}$$

$$\omega^2 = \frac{3P}{\ell m} \Rightarrow \omega = \sqrt{\frac{3P}{m\ell}}$$

Dunkerley's method $\quad a_{11}m_1 + a_{22}m_2 = \frac{1}{\omega_1^2}$

$$\frac{2\ell}{3P}m + \frac{2\ell}{3P}m = \frac{4\ell m}{3P} = \frac{1}{\omega_1^2} \Rightarrow \omega_1 = \sqrt{\frac{3}{4}}\sqrt{P/\ell m} = .845\sqrt{\frac{P}{\ell}}$$

SOLUTIONS TO

$+ \omega^2 x = \cos \omega_0 t$

$+ \omega_0^2 x = \cos \omega_0 t$

and

Problems $\overset{4}{\cancel{1}}$, $\overset{1.5}{\cancel{1.2}}$, $\overset{1.9}{\cancel{1.6}}$, $\overset{1.17}{\cancel{1.10}}$ from RAO Second Ed

3 PAGES

$\ddot{x} + \omega^2 x = \cos\omega_0 t$

Solution to the homogeneous equation is

$$x_h = A\cos\omega t + B\sin\omega t$$

Solution to the non-homogeneous equation

$$x_p = \frac{1}{\omega^2 - \omega_0^2}\cos\omega_0 t$$

$x_{TOT} = A\cos\omega t + B\sin\omega t + \frac{1}{\omega^2-\omega_0^2}\cos\omega_0 t$. If A & B and $\frac{1}{\omega^2-\omega_0^2}$ are finite (bounded values), then x_{TOT} is bounded. This is due to the fact that $\sin\omega t$, $\cos\omega t$ and $\cos\omega_0 t$ vary between -1 and $+1$. Thus x_{TOT} is bounded as $t \to \infty$

$\ddot{x} + \omega_0^2 x = \cos\omega_0 t$

Solution to the homogeneous equation is

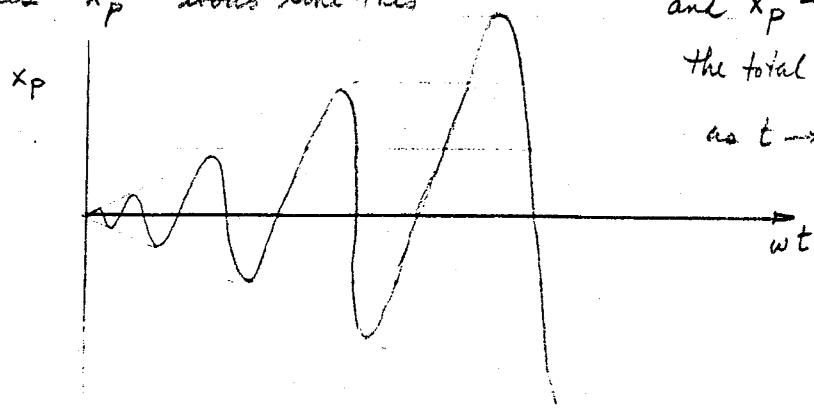
$$x_h = A\cos\omega_0 t + B\sin\omega_0 t$$

Solution to the nonhomogeneous equation is

$$x_p = \frac{t}{2\omega_0}\sin\omega_0 t$$

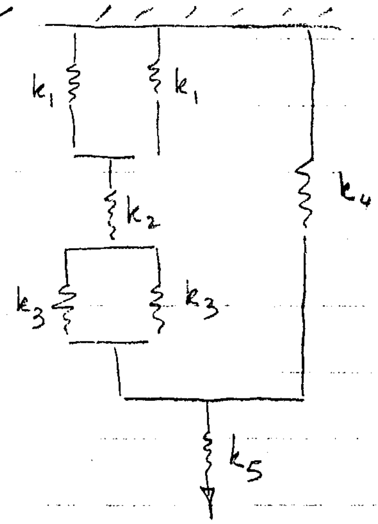
$x_{TOT} = x_h + x_p = A\cos\omega_0 t + B\cos\omega_0 t + \frac{t}{2\omega_0}\sin\omega_0 t$. If A & B are bounded x_h is bounded as $t \to \infty$. But $\frac{t}{2\omega_0} \to \infty$ as $t \to \infty$

Thus x_p looks like this

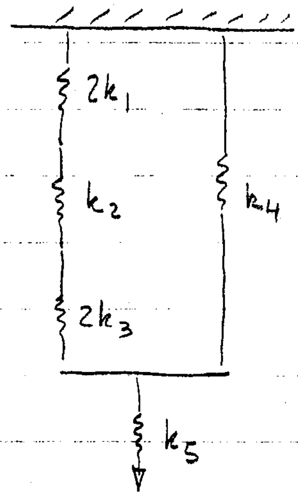


and $x_p \to \infty$ as $t \to \infty$. Thus the total solution is unbounded as $t \to \infty$.

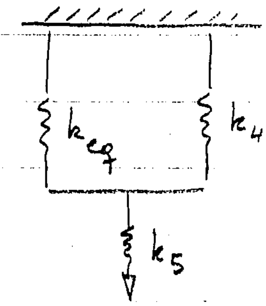
1.4 ~~1.1~~



EQUIV =



EQUIV =

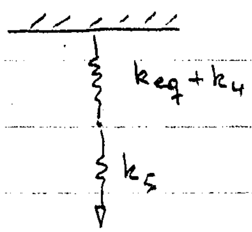


$$\frac{1}{k_{eq}} = \frac{1}{2k_1} + \frac{1}{k_2} + \frac{1}{2k_3}$$

$$= \frac{2k_3k_2 + 4k_1k_3 + 2k_1k_2}{4k_1k_2k_3}$$

$$\text{or } k_{eq} = \frac{4k_1k_2k_3}{2k_3k_2 + 4k_1k_3 + 2k_1k_2}$$

EQUIV TO



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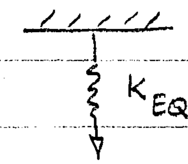
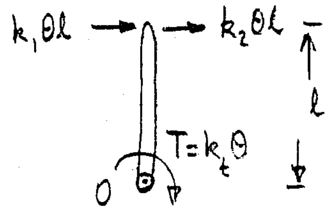


$$\frac{1}{K_{EQ}} = \frac{1}{k_{eq} + k_4} + \frac{1}{k_5}$$

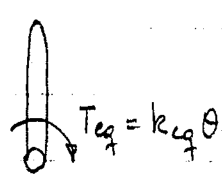
$$= \frac{k_5 + k_{eq} + k_4}{k_5 k_{eq} + k_5 k_4}$$

$$\text{or } K_{EQ} = \frac{k_5(k_{eq} + k_4)}{k_5 + k_4 + k_{eq}}$$

1.5 ~~1.2~~ movement of the rod of θ degrees causes the linear springs to move through ~~an~~ displacement of θl and forces of $k\theta l$.



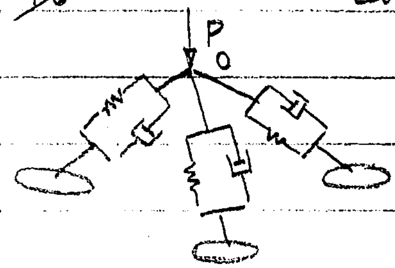
EQUIV =



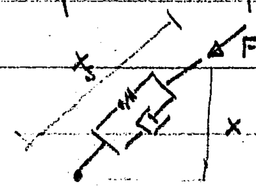
Total torque about O is $T_{TOT} = k_t\theta + k_1\theta l^2 + k_2\theta l^2 = T_{eq} = k_{eq}\theta \quad \therefore k_{eq} = (k_1 + k_2)l^2 + k_t$

NOTE: units of k_t = Force · length ; units of k_1 & k_2 = Force / length

1.9 ~~1.6~~



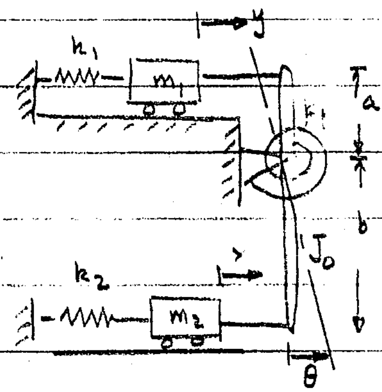
Let P be the force acting at point O and the F be the force acting on each spring-damper system



For the spring $F = k x_s$ with $x_s = x\cos\alpha$. Since $F_v = F\cos\alpha$ then $F_v = kx\cos^2\alpha$. F_v is the vertical component of F. But since this is the force only in one leg then in all 3 legs we have $3F_v$. Thus the vertical force $P = 3F_v = 3kx\cos^2\alpha$

if $P = k_{eq}x \Rightarrow k_{eq} = 3k\cos^2\alpha$ similarly $c_{eq} = 3c\cos^2\alpha$ for the dampers. This could have also been done by equivalent energies

1.17 ~~1.10~~



a positive θ movement of the rocker arm will move m_2 through a change of position $x = +b\theta$ and m_1 through a change of position $y = -a\theta$. Equivalent mass means find the total kinetic energy of the system.

Thus $KE = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{1}{2}J_o\dot\theta^2$

where $\dot\theta$ is $d\theta/dt$. Thus $v_1 = dy/dt = -a\dot\theta$ $\quad v_2 = \frac{dx}{dt} = b\dot\theta$ or

$$KE = \frac{1}{2}m_1(-a\dot\theta)^2 + \frac{1}{2}m_2(b\dot\theta)^2 + \frac{1}{2}J_o\dot\theta^2 = \frac{1}{2}\left[m_1a^2 + m_2b^2 + J_o\right]\dot\theta^2.$$

since we must relate this to x let us define $KE_{eq} = \frac{1}{2}m_{eq}\dot x^2 = \frac{1}{2}m_{eq}b^2\dot\theta^2$

thus $m_{eq} = m_1\left(\frac{a}{b}\right)^2 + m_2 + J_o\left(\frac{1}{b}\right)^2$

The total energy of the system is $KE + PE = KE + \frac{1}{2}k_1y^2 + \frac{1}{2}k_2x^2 + \frac{1}{2}k_t\theta^2$

$$= KE + \frac{1}{2}k_1a^2\theta^2 + \frac{1}{2}k_2b^2\theta^2 + \frac{1}{2}k_t\theta^2$$

* HW #2 - 3 PAGES

~~1.15~~, ~~1.24~~, ~~1.25~~, ~~1.29~~

1.20, 1.33, 1.30, 1.35 in Second Ed.

~~1-29~~ 1.35 Given $X = .05 m$ $f = 10 Hz$ find τ, $\dot{x}_{max}$, $\ddot{x}_{max}$

$2\pi f = \omega_n = 62.83$ rad/s ; $\tau = \frac{1}{f} = \frac{1}{10} = .1$ sec

$\dot{x}_{max} = X\omega_n = 3.142$ m/s

$\ddot{x}_{max} = X\omega_n^2 = (.05)(62.83)^2 = 197.39$ m/s^2

~~1.24~~ 1.30 $x = 10 \sin(\omega t + 60°)$ $x_1 = 5 \sin(\omega t + 30°)$ find x_2 so that

$x = x_1 + x_2$, Let $x_2 = A \sin(\omega t + \phi)$

$10 \sin(\omega t + 60°) = 10[\sin \omega t \cos 60° + \cos \omega t \sin 60°]$

$5 \sin(\omega t + 30°) = 5[\sin \omega t \cos 30° + \cos \omega t \sin 30°]$

$A \sin(\omega t + \phi) = A[\sin \omega t \cos \phi + \cos \omega t \sin \phi]$

for $x = x_1 + x_2$

$10 \cos 60° = 5 \cos 30° + A \cos \phi$ $\rightarrow$ $A \cos \phi = 5 - 4.33 = .67$

$10 \sin 60° = 5 \sin 30° + A \sin \phi$ $\rightarrow$ $A \sin \phi = 8.66 - 2.5 = 6.16$

$\frac{6.76}{.67} = \frac{A \sin \phi}{A \cos \phi} = \tan \phi$ $\underline{\phi = 83.79°}$, Also $(A \sin \phi)^2 + (A \cos \phi)^2 = A^2$

therefore $\underline{A = 6.197}$; thus $x_2 = 6.197 \sin(\omega t + 83.79°)$

~~1.25~~ 1.33 Given $x_1 = .5 \cos \frac{\pi}{2} t$ $x_2 = \sin \pi t$. Is $x_1 + x_2$ periodic? if yes what is period? Is $x_1 + x_2$ harmonic? If yes what is the period?

Write $x_2 = \sin \pi t = \sin(2 \cdot \frac{\pi}{2} t) = 2 \sin \frac{\pi}{2} t \cos \frac{\pi}{2} t$, Now $x_1 + x_2$ is

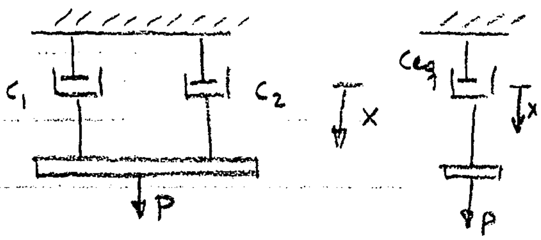
$(.5 + 2 \sin \frac{\pi}{2} t) \cos \frac{\pi}{2} t = A(t) \cos \frac{\pi}{2} t$. Note that $A(t)$

is periodic with the period $T = 4$. So is $\cos \frac{\pi}{2} t$ with the same period. Thus $x_1 + x_2$ is periodic. But $x_1 + x_2$ is not harmonic since $x_1 + x_2$ does not satisfy $\frac{d^2}{dt^2}(x_1 + x_2) + \omega^2 (x_1 + x_2) = 0$, ω = const.

This is because x_1 and x_2 have different ω_n's.

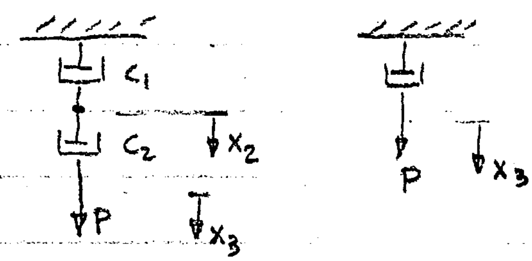
Now $x_1 + x_2$ is periodic but not harmonic and has period $\tau \approx 4$ sec.

~~1.15~~ 1.20 When two dampers are in parallel. If the bar remains parallel the displacement is given by x for both and the force in damper c_1 and c_2 is



$c_1 \dot{x}$ and $c_2 \dot{x}_2$ respectively. But the forces in the dampers must be equal to $P \Rightarrow P = c_1\dot{x} + c_2\dot{x} = c_{eq}\dot{x} \Rightarrow c_1 + c_2 = c_{eq}$

When the two dampers are in series

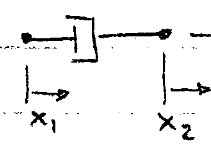


$$\left.\begin{aligned} P &= c_1 \dot{x}_2 \\ P &= c_2(\dot{x}_3 - \dot{x}_2) \end{aligned}\right\} \text{since } P \text{ is transmitted to both dampers}$$

$$P = c_2 \dot{x}_3 - c_2 \cdot \frac{P}{c_1} = c_{eq}\dot{x}_3$$

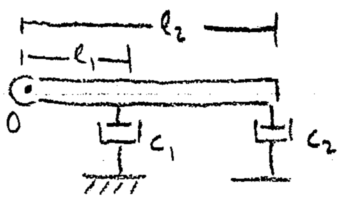
thus $P\left[1 + \frac{c_2}{c_1}\right] = c_2 \dot{x}_3 \Rightarrow P = \left[\frac{c_1 c_2}{c_1 + c_2}\right]\dot{x}_3 \Rightarrow c_{eq} = \frac{c_1 c_2}{c_1 + c_2} = \frac{1}{\frac{1}{c_1} + \frac{1}{c_2}}$

thus $\frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2}$. For the dampers the piston moves at ~~a~~ a different speed than the cup and in general



$\rightarrow P = c(\dot{x}_2 - \dot{x}_1)$

(iii)



The energy dissipated by the system is

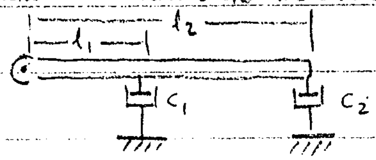
$$\pi c_1 \omega X_1^2 + \pi c_2 \omega X_2^2 \quad \text{where } X_1 = l_1\theta \quad X_2 = l_2\theta.$$

If we define an equiv system at pt 1 then $\pi c_{eq} \omega X_1^2 = \pi c_1 \omega X_1^2 + \pi c_2 \omega X_2^2$

$$X_2 = \frac{l_2}{l_1} X_1 \qquad \Rightarrow c_{eq} = c_1 + c_2\left(\frac{l_2}{l_1}\right)^2$$

For (iv) $\pi c_{t_1} \omega \theta_1^2 + \pi c_{t_2} \omega \theta_2^2$ is the total dissipated energy by the two rotational dampers. now $n_1\theta_1 = n_2\theta_2$ as shown in class. If we define an equiv system at pt 1 then $\pi c_{eq} \omega \theta_1^2 = \pi c_{t_1} \omega \theta_1^2 + \pi c_{t_2} \omega \theta_2^2 \Rightarrow c_{eq} = c_{t_1} + c_{t_2}\left(\frac{n_1}{n_2}\right)^2$

Problem iii can also be done by looking at the total torque about the pivot pt.



$$T_{TOT} = F_1 l_1 + F_2 l_2 = (c_1 l_1 \dot{\theta}) l_1 + (c_2 l_2 \dot{\theta}) l_2$$
$$= (c_1 l_1^2 + c_2 l_2^2)\dot{\theta}$$

An equivalent system at point 1 $\quad T_{eq} = (c_{eq}\, l_1 \dot{\theta})\, l_1 = T_{TOT}$

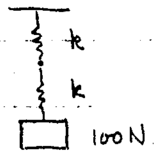
$$\Rightarrow c_{eq}\, l_1^2 = c_1 l_1^2 + c_2 l_2^2 \quad \Rightarrow \quad c_{eq} = c_1 + c_2\left(\frac{l_2}{l_1}\right)^2$$

SOLUTIONS TO HW #3

Problems 2.4, 2.5, like 2.6, like ~~2.12~~ 2.10 & like ~~2.13~~ 2.12

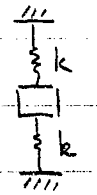
in 3 pages

2.4 Look at the overall spring for the first part as a series of springs



$$\frac{1}{k_{eq}} = \frac{1}{k} + \frac{1}{k} = \frac{2}{k} \quad \text{thus } k_{eq} = \frac{k}{2} \quad \text{now} \quad k_{eq} = \frac{W}{\delta_{ST}} \Rightarrow k = \frac{2W}{\delta_{ST}}$$

$$\text{or } k = \frac{200\text{ N}}{.01\text{ m}} = 20000\text{ N/m}$$



For the 2nd part

$$k_{eq} = 2k = 40000\text{ N/m} \quad \text{and} \quad \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{40000}{10}} = 63.25\text{ rad/s}$$

$$\text{since } 2\pi f = \omega_n \text{ and } \tau f = 1 \Rightarrow \tau = \frac{2\pi}{\omega_n} = \frac{2\pi}{63.25} = .0993\text{ sec}$$

2.5 if $x = X\sin(\omega_n t + \phi) \Rightarrow \dot{x} = X\omega_n\cos(\omega_n t + \phi)$; $\dot{x}_{max} = X\omega_n = 10\text{ cm/sec}$

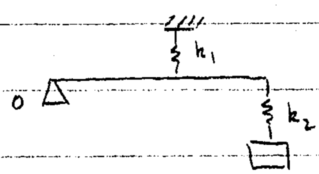
also given that $\tau = 2\text{sec} = 2\pi/\omega_n \Rightarrow \omega_n = 3.142\text{ rad/s} \Rightarrow X = \frac{10\text{ cm/s}}{\omega_n}$

or $\underline{X = 3.183\text{ cm}}$

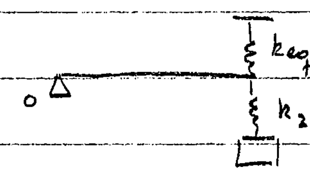
- when $t=0$ $\quad x = 2\text{ cm} = X\sin\phi \Rightarrow \sin\phi = \frac{2\text{ cm}}{3.183\text{ cm}} = .6283 \quad \underline{\underline{\phi = 38.93^\circ}}$
- when $t=0$ $\quad \dot{x} = X\omega_n\cos\phi = \underline{\underline{7.779\text{ cm/s}}}$

$\ddot{x} = -X\omega_n^2\sin(\omega_n t + \phi)$ the max accel $= +X\omega_n^2 = (X\omega_n)(\omega_n) = \underline{\underline{31.42\text{ cm/s}}}$

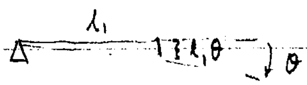
2.6



if we replace this system by



then the torque produced by k_1 is $(k_1 l_1 \theta)\, l_1$ = Force in in k_1 · moment arm



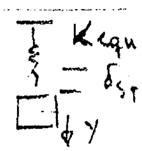
. The equivalent spring must produce an equivalent torque

thus $T_{eq} = (k_{eq}\, l_2\theta)\, l_2 = (k_1 l_1 \theta) l_1$ or $k_{eq} = k_1 \frac{l_1^2}{l_2^2}$

Now k_{eq} & k_2 are in series and $\quad \frac{1}{K_{eq}} = \frac{1}{k_{eq}} + \frac{1}{k_2} = \frac{l_2^2}{k_1 l_1^2} + \frac{1}{k_2} = \frac{k_2 l_2^2 + k_1 l_1^2}{k_1 k_2 l_1^2}$

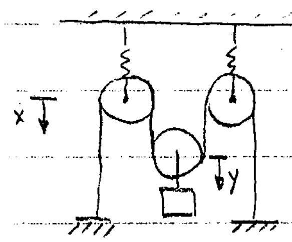
thus $K_{eq} = \frac{k_1 k_2 l_1^2}{k_2 l_2^2 + k_1 l_1^2}$ $\quad$ and the governing eq is $m\ddot{y} + K_{equ}\, y = 0$

is measured from the static displacement location of the spring K_{equ}

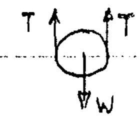


this is only correct for a rigid bar. If the bar were allowed to bend you would have to include the k_{eq} of a bending bar <u>in series</u> with the rest of the k's.

like 2.10
~~2.12~~

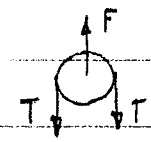


looking at the center pulley



$T = \frac{W}{2}$

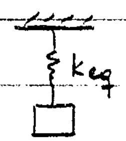
looking at the external pulleys



$F = W = k\,\Delta x$

but $T = \frac{W}{2} = \frac{k\,\Delta x}{2}$; hence when the spring statically stretches under load W, it stretches Δx but the center pulley (hence the mass) moves $\Delta y = \frac{\Delta x}{2}$

thus for an equivalent system



to undergo the same stretch $\Delta y \Rightarrow k_{eq} = \frac{k}{2}$

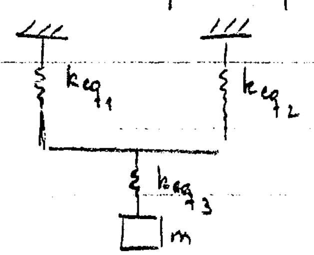
since $k_{eq}\,\Delta y = k\frac{\Delta x}{2}$. Hence $\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{2m}}$ and $m\ddot{y} + \frac{k}{2}y = 0$

You can also show by dependent motion that $\Delta y = \frac{\Delta x}{2}$.

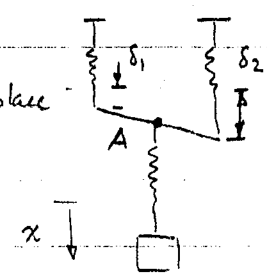
like 2.12
~~2.13~~ for bar 1 $k_{eq} = \frac{3E_1I_1}{l_1^3}$ for bar 2 $k_{eq} = \frac{3E_2I_2}{l_2^3}$

and bar 1 and 2 are in parallel. for bar 3 with the load in the center

$k_{eq} = \frac{48E_3I_3}{l_3^3}$ and the equiv. system looks like this



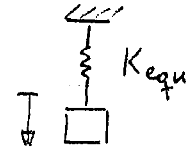
and the system will displace



where δ_1 and δ_2 are the displacements of k_{eq_1} & k_{eq_2} under the load $mg/2$ (by equilib)

The point A will displace by $\frac{\delta_1+\delta_2}{2}$ and the mass will displace by a distance $\frac{\delta_1+\delta_2}{2} + \delta_3$ where $\delta_3 = mg/k_{eq_3}$. Thus we have that

$$\frac{\delta_1+\delta_2}{2} + \delta_3 = \frac{W}{2} \cdot \frac{1}{2}\left[\frac{1}{k_{eq_1}} + \frac{1}{k_{eq_2}}\right] + \frac{W}{k_{eq_3}} = \frac{W}{K_{equ}}$$



where K_{equ} is the equivalent spring constant for the entire system

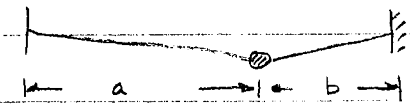
and $$\frac{1}{K_{equ}} = \frac{1}{4k_{eq_1}} + \frac{1}{4k_{eq_2}} + \frac{1}{k_{eq}}$$

and finally $\omega_n = \sqrt{\frac{K_{equ}}{m}}$. The governing equation will then be $m\ddot{x} + K_{equ}\,x = 0$ where x is measured from the location of the center of mass after all 3 beams have bent.

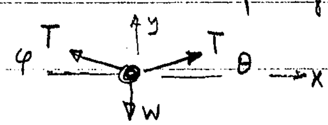
HW #4 → ~~2.17~~ 2.18, 2.23, ~~2.32~~ like 2.31, ~~2.42~~ like 2.40

in 3 pages

2.18



look at the Free Body Diagram (FBD) for static equil



$\sum F_y = 0 = T\sin\theta + T\sin\varphi - W$

$W = T\frac{\Delta y}{a} + T\frac{\Delta y}{b}$

$W = T\left[\frac{1}{a} + \frac{1}{b}\right]\Delta y$

$= k_{eq}\,\Delta y$

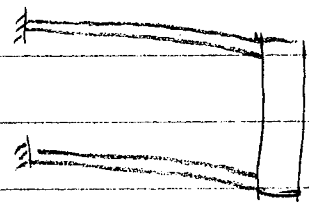
$\sum F_x = T\cos\theta - T\cos\varphi = 0 \Rightarrow \theta \cong \varphi$

From the above $k_{eq} = T\left[\frac{1}{a} + \frac{1}{b}\right]$. If we measure everything from static equilib position, then the DE $\Rightarrow m\ddot{y} + k_{eq}\,y = 0$ or $m\ddot{y} + T\left[\frac{1}{a} + \frac{1}{b}\right]y = 0$

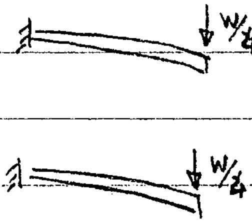
and $\omega_n = \sqrt{\frac{T}{m}\left[\frac{1}{a} + \frac{1}{b}\right]}$ $\quad f = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{T}{m}\left[\frac{1}{a} + \frac{1}{b}\right]}$

~~if measured from the horizontal:~~ $\cancel{m\ddot{y} + k_{eq}\,y + mg = 0}$

2.22



$\Rightarrow$

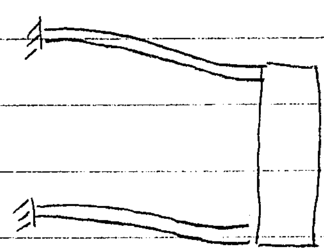


thus $\Delta x = \frac{W/4}{k_{eq}} = \frac{W/4}{3EI/\ell^3}$

$= \frac{W}{12EI/\ell^3}$

Now $\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_{eq}\,g}{W}} = \sqrt{\frac{g}{\Delta x}} = \sqrt{\frac{g\cdot 12EI}{mg\,\ell^3}} = \sqrt{\frac{12EI\,g}{W\ell^3}} = \sqrt{\frac{g}{\delta_{ST}}}$

and the DE $= m\ddot{x} + \frac{12EI}{\ell^3}x = 0$



The displacement fn $w(x)$ for a beam in bending is $w(x) = c_1x^3 + c_2x^2 + c_3x + c_4$ with $w(0) = w'(0) = 0$

and $w'(x) = 3c_1x^2 + 2c_2x + c_3$. for $w(0) = w'(0) = 0$

$\Rightarrow c_4 = c_3 = 0$. Also for no rotation at $\ell \Rightarrow w'(\ell) = 0$

$w'(\ell) = 3c_1\ell^2 + 2c_2\ell = 0$ or $c_2 = -\frac{3c_1\ell}{2}$

thus $w(x) = c_1x^3 - \frac{3}{2}c_1x^2\ell = c_1x^2\left[x - \frac{3}{2}\ell\right]$. Finally the shear must be $=$ to $\frac{W}{4}$ at $x = \ell$ $\Rightarrow V = -EI\,w'''(x) = -EI\left[6c_1\right] = \frac{W}{4} \Rightarrow c_1 = \frac{-W}{24EI}$

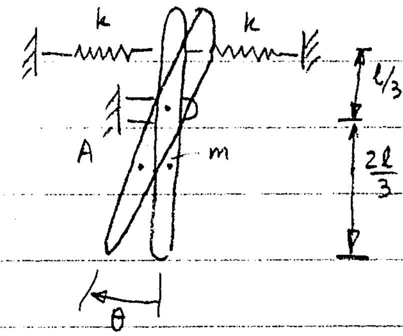
or $w(x) = -\frac{W}{24EI}x^2\left[x - \frac{3}{2}\ell\right]$ and $w(\ell) = -\frac{W\ell^2}{24EI}\left(-\frac{1}{2}\ell\right) = \frac{W\ell^3}{48EI} = \Delta x$

thus if $\Delta x = \frac{W}{k_{eq}} \Rightarrow k_{eq} = 48EI/\ell^3$

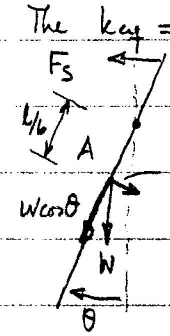
DE $\Rightarrow$ $m\ddot{x} + k_{eq}\,x = 0 \Rightarrow m\ddot{x} + (48EI/\ell^3)\,x = 0$

and $\omega_n = \sqrt{\dfrac{24FIg}{W\ell^3}} = \sqrt{\dfrac{24EI}{m\ell^3}}$ · based on $W_n = \sqrt{\dfrac{g}{\Delta x}} = \sqrt{\dfrac{g}{\delta_{ST}}}$

like 2.31
~~2.32~~



look at the FBD when system undergoes a rotation θ. The $k_{eq} = 2k$



Both F_s & $W\sin\theta$ provide restoring torques about A

Take $\Sigma T_A = -W\sin\theta\left[\frac{\ell}{2} - \frac{\ell}{3}\right] - F_s \cdot \frac{\ell}{3}\cos\theta \qquad F_s = k_{eq}\cdot\frac{\ell}{3}\sin\theta$

$= -W\sin\theta\left[\frac{\ell}{6}\right] - 2k\left(\frac{\ell}{3}\right)^2 \sin\theta\cos\theta$

inertia term $I_A\ddot{\theta} = \left[I_{CG} + m\left(\frac{\ell}{2}-\frac{\ell}{3}\right)^2\right]\ddot{\theta}$ and in class for a slender rod we showed $I_{CG} = \frac{m\ell^2}{12}$

$= \left[\frac{m\ell^2}{12} + \frac{m\ell^2}{36}\right]\ddot{\theta} = \frac{m\ell^2}{9}\ddot{\theta}$

∴ DE is $\frac{m\ell^2}{9}\ddot{\theta} + \frac{W\ell}{6}\sin\theta + \frac{2}{9}k\ell^2\sin\theta\cos\theta = 0$

for small vibs $\sin\theta \to \theta \quad \cos\theta \to 1 \quad \Rightarrow \quad \frac{m\ell^2}{9}\ddot{\theta} + \left(\frac{W\ell}{6} + \frac{2k\ell^2}{9}\right)\theta = 0$

$$\omega_n = \sqrt{\frac{\frac{W\ell}{6} + \frac{2k\ell^2}{9}}{m\ell^2/9}} = \sqrt{\frac{6W\ell + 8k\ell^2}{4m\ell^2}} = \sqrt{\frac{3W + 4k\ell}{2m\ell}}$$

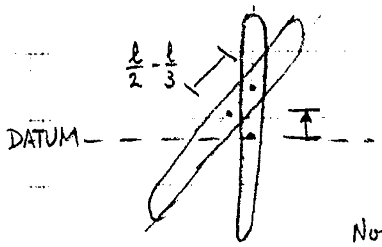
with the given data $\quad \omega_n = \sqrt{\dfrac{3(10)(9.81) + 4(2000)(5)}{2(10)(5)}} = 20.073$ rad/s

~~2.42~~ like 2.40

the KE for the system is $\frac{1}{2}I_A\dot{\theta}^2$

the PE for the springs is given by $2\left[\frac{1}{2}k\Delta x^2\right]$ where $\Delta x = \frac{\ell}{3}\sin\theta$

the PE for the rod is given by $W[\Delta y]$ where datum line is at the CG location when the rod hangs vertically



Δy for the weight is $\left(\frac{\ell}{2}-\frac{\ell}{3}\right)(1-\cos\theta)$

Now PE + KE = const $\Rightarrow \frac{1}{2}I_A\dot{\theta}^2 + 2\left[\frac{1}{2}k\left(\frac{\ell}{3}\sin\theta\right)^2\right] + W\left(\frac{\ell}{2}-\frac{\ell}{3}\right)(1-\cos\theta) = \mathscr{C}$

Now take $\frac{d}{dt}(PE+KE)=0 \Rightarrow \frac{1}{2}I_A\cdot 2\dot{\theta}\ddot{\theta} + 2\cdot\frac{1}{2}k\cdot 2\frac{\ell^2}{9}\sin\theta\cos\theta\,\dot{\theta} + W\left(\frac{\ell}{2}-\frac{\ell}{3}\right)(\sin\theta\,\dot{\theta})=0$

or

$$\dot{\theta}\left[I_A\ddot{\theta} + \frac{2k\ell^2}{9}\sin\theta\cos\theta + \frac{W\ell}{6}\sin\theta\right]=0 \qquad \text{where } I_A = \frac{m\ell^2}{9}$$

$\dot{\theta}=0$ only implies $\theta=$ const, & for this system to be in equilibrium $\theta=0 \Rightarrow$ rod remains vertical for all time. This is the trivial solution, so

$$I_A\ddot{\theta} + \frac{2k\ell^2}{9}\sin\theta\cos\theta + \frac{W\ell}{6}\sin\theta = 0 \quad \text{as before (in problem 2.32)}$$

To use Rayleigh's Method assume 1) vibrations are small and harmonic

thus $\quad KE = \frac{1}{2}I_A\dot{\theta}^2$

$\quad PE = 2\left[\frac{1}{2}k\left(\frac{\ell}{3}\sin\theta\right)^2\right] + W\left(\frac{\ell}{2}-\frac{\ell}{3}\right)(1-\cos\theta) \Rightarrow k\left(\frac{\ell}{3}\theta\right)^2 + \frac{W\ell}{6}\frac{\theta^2}{2}$

Note: $\cos\theta = 1-\frac{\theta^2}{2!}+\frac{\theta^4}{4!}+\ldots \qquad 1-\cos\theta = \frac{\theta^2}{2!}$ + higher order terms.

Now let $\quad \theta = \Theta\sin(\omega_n t+\phi)$

$\quad \dot{\theta} = \Theta\omega_n\cos(\omega_n t+\phi)$

$KE = \frac{1}{2}I_A\left(\omega_n^2\Theta^2\cos^2(\omega_n t+\phi)\right), \qquad KE_{max} = \frac{1}{2}I_A\Theta^2\omega_n^2$

$PE = \left[\frac{k\ell^2}{9}+\frac{W\ell}{12}\right]\theta^2 = \left(\frac{k\ell^2}{9}+\frac{W\ell}{12}\right)\left(\Theta^2\sin^2(\omega_n t+\phi)\right); \quad PE_{max} = \left(\frac{k\ell^2}{9}+\frac{W\ell}{12}\right)\Theta^2$

$$PE_{max} = KE_{max} = \frac{1}{2}\frac{m\ell^2}{9}\omega_n^2\Theta^2 = \left(\frac{k\ell^2}{9}+\frac{W\ell}{12}\right)\Theta^2$$

$$\text{or} \quad \omega_n = \sqrt{\frac{k\ell^2/9 + \frac{W\ell}{12}}{\frac{m\ell^2}{18}}} = \sqrt{\frac{4k\ell+3W}{2m\ell}} \quad \text{as before}$$

Solutions to HW# 5

like 2.43 2.50 2.51 like 2.73

~~2.48~~, ~~2.53~~, ~~2.54~~, ~~2.55~~

in 2 pages

like 2.43
~~2.48~~ Given $m = 4\,kg$ $f_n = 5\,Hz$ in vacuum $\Rightarrow \omega_n = 2\pi f_n = 31.416$ rad/s

$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = \omega_n^2 m = 3947.84$ N/m

Given $f_d = 4.5\,Hz$ in oil $\omega_d = 2\pi f_d = 28.274$ rad/s $= \omega_n\sqrt{1-\zeta^2}$

THUS $\left[1-(\omega_d/\omega_n)^2\right]^{1/2} = \zeta = .4359$

But $\zeta = \frac{c}{c_c}$ and $c_c = 2m\omega_n = 251.328\ \frac{N\text{-}s}{m} \Rightarrow c = \zeta c_c = 109.554\ \frac{N\text{-}s}{m}$

The answer in the back of the book is incorrect.

2.50
~~2.53~~ the torsional pendulum has a diff. eqn of $I\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0$

and for it $\omega_n = \sqrt{\frac{k_t}{I}}$ $c_t/c_{t_c} = \zeta$ and $c_{t_c} = 2I\omega_n = 2\sqrt{k_t I}$

given $f = 200$ cycles/min $\cdot \frac{1\,min}{60\,s} = 3.333\,Hz \Rightarrow \omega_n = 2\pi f_n = 20.944$ rad/s

in oil $f = 180$ cycle/min $\cdot \frac{1\,min}{60\,s} = 3\,Hz \Rightarrow \omega_d = 2\pi f_d = 18.8496$ rad/s

$\zeta = \left[1-(\omega_d/\omega_n)^2\right]^{1/2} = .4359$ and $c_{t_c} = 2I\omega_n = 8.378\ \frac{N\text{-}m\text{-}s}{rad}$

and $c_t = \zeta c_{t_c} = 3.652\ \frac{N\text{-}m\text{-}s}{rad}$

Since $\zeta < 1$ we have an underdamped system.

$$\delta = \ln\left(\frac{\theta_0}{\theta_1}\right) = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 3.0432 \Rightarrow \frac{\theta_0}{\theta_1} = 20.971$$

$$\theta_1 = \frac{\theta_0}{20.971} = \frac{2^\circ \cdot \pi/180}{20.971} = .001665 \text{ rad.} \Rightarrow .0954^\circ$$

2.51
~~2.54~~ Given $f_d = 5$ osc/sec $= 5\,Hz \Rightarrow \omega_d = 2\pi f_d = 31.416$ rad/s

also after 50 cycles $\delta = \frac{1}{50}\ln\left(\frac{X_0}{X_{50}}\right)$ $X_n = .1X_0$

$\delta = \frac{1}{50}\ln(10) = .04605$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = .00733.$$

Since we know ζ and its relation to ω_d: $\omega_d = \omega_n\sqrt{1-\zeta^2} \Rightarrow \omega_n = \omega_d/\sqrt{1-\zeta^2} = 31.417$ rad/s

Also $\tau_n = \frac{2\pi}{\omega_n}$ & $\tau_d = \frac{2\pi}{\omega_d} \Rightarrow$ % change is $\frac{\tau_n - \tau_d}{\tau_d} = \frac{2\pi\left[\frac{1}{\omega_n} - \frac{1}{\omega_d}\right]}{2\pi\ \frac{1}{\omega_d}} = \frac{\omega_d - \omega_n}{\omega_n}$

this is equal to $\frac{\sqrt{1-\zeta^2} - 1}{1} = -.00003$ or .003% change

Solutions to HW # 6

like 2.56   2.57 like 2.60

~~2.60~~, ~~2.62~~ to ~~2.64~~

in 3 pages

~~3.60~~ also 2.56. From the trace there are 4 cycles/sec $= 4\,Hz = f \Rightarrow \omega_n = 8\pi$ rad/s $= 25.133$ rad/s. The decrement $\frac{4\mu N}{k} = \frac{X_0 - X_{final}}{4\text{ cycles}}$ where $X_{final} = .5$ mm

This value is the final displacement for which the system comes to a stop.

Thus $\frac{4\mu N}{k} = \frac{30 - .5}{4} = 7.375$ mm

Now $N =$ weight and $\frac{4\mu W}{k} = \frac{4\mu N}{k} = \frac{4\mu mg}{k} = \frac{4\mu}{\omega_n^2} g = 7.375\text{mm} = .007375\text{m}$

now $\mu = .007375\left(\frac{\omega_n^2}{4g}\right) = .1187$

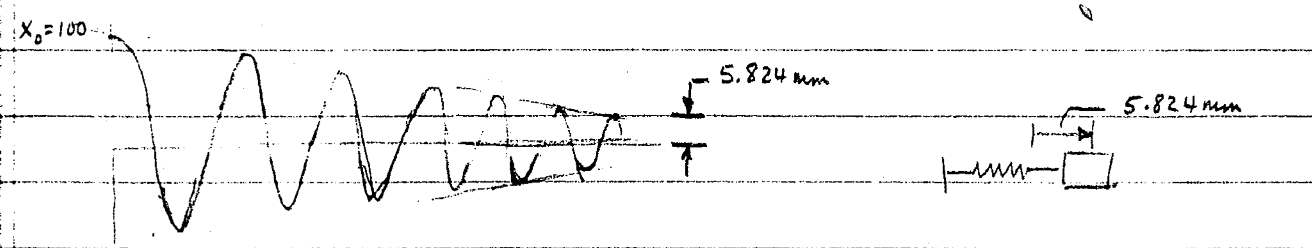
2.63 Given $m = 10$ kg $k = 3000$ N/m and $X_0 = 100$ mm $\dot{X}_0 = 0$, $\mu = .12$

Now $\frac{f}{k} = \frac{\mu N}{k} = \frac{\mu W}{k} = \frac{\mu mg}{k} = \frac{(.12)(10)(9.81)}{3000} = .003924$ m or 3.924 mm

This is the displacement at which motion stops and the number of cycles required are $\frac{4\mu N}{k} = \frac{X_0 - X_{final}}{n} \Rightarrow n = \frac{100 - 3.924}{4(3.924)} = 6.121$ cycles

Thus at the end of 6 cycles the displacement is given by $-n\left[\frac{4\mu N}{k}\right] + X_0 = X_6$ where $n = 6$

$X_6 = 5.824$ mm also $\dot{X}_6 = 0$ since X_6 represents the maximum displacement



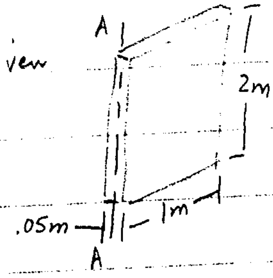
Thus at the start of the 6th cycle the system is at 5.824 mm (measured from the unstretched length of the spring). Thus in the seventh cycle the mass stops once $f = \mu N$ is $> kx$, ie when $x = 3.924$ mm

~~3.62~~ 2.57 Given $k = 10$ N/mm $= 10000$ N/m, $m = 20$ kg and $X_4 = 100$ mm. If $X_0 = 150$ mm what is μ and how long has it taken to reach X_4.

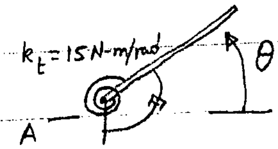
$T = 2\pi/\omega_n$ $\omega_n = \sqrt{\frac{k}{m}} = 22.36$ rad/s $T = .281$ sec $\Rightarrow$ time to X_4 is $4T = 1.124$ sec

like 2.73

~~2.55~~ Given



$m = 50\,kg$.



The differential equation is given by

$I_A \ddot{\theta} + c_t \dot{\theta} + k_t \theta = 0$ where I_A is the polar mass moment of inertia of the door about pt A ; and the door as looked upon from above, looks like a rod of length $l = 1$. From previous work we have shown $I_{CG} = \frac{1}{12} m l^2$ and $I_A = I_{CG} + md^2$ where $d = l/2$ $\therefore$ $I_A = \frac{1}{3} m l^2 = 16.6667\ kg\text{-}m^2$

→ for critical damping we showed in problem 2.53 that $c_{t_c} = 2 I_A \omega_n$ and

$\omega_n = \sqrt{\frac{k_t}{I_A}}$ thus $c_{t_c} = 2\sqrt{k_t I_A} = 2\sqrt{(15)(16.6667)} = \underline{31.623\ N\text{-}m\text{-}s/rad}$

and $\omega_n = \sqrt{\frac{15}{16.667}} = .9487$ rad/s. For critical damping $\theta = (C_1 + C_2 t) e^{-\omega_n t}$ as shown in class. Also given that $\theta(t=0) = 75^\circ = 1.309$ rad and $\dot{\theta}(t=0) = 0$

thus $C_1 = 1.309$ rad and $C_2 = \omega_n C_1 = 1.242 \frac{rad}{s}$ and

$\theta(t) = (1.309 + 1.242\,t)\, e^{-.9487t}$. We want to find the time so that $\theta(t_{close})$ is $= 5^\circ$ or $.08727$ rad. By trial & error we find

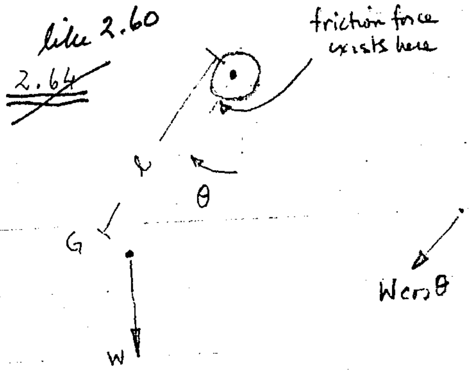
$\theta(t = 4s) = .1412$
$\theta(t = 5s) = .0655$

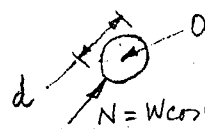
} answer is about 4.63 sec ⇒ $\theta(4.63) = .08732$ rad. by interpolation

also $\frac{4\mu N}{k} = \frac{X_0 - X_4}{4}$ or $\mu = \frac{k}{4N}\frac{X_0 - X_4}{4} = \frac{k}{4 \cdot mg}\frac{X_0 - X_4}{4} = 0.1593$

remember to convert X_0 & X_4 to meters if you use $g = 9.81\ m/s^2$

like 2.60 ~~2.64~~





N = Wcosθ on the upper bearing and

$f = -\mu N$ when movement is clockwise

$+\mu N$ when counterclockwise. Thus

by taking Σ Torques about the center pin bearing

O. For $\dot\theta > 0$ $I_0\ddot\theta = -W\sin\theta\, l - f d/2$

$\dot\theta < 0$ $I_0\ddot\theta = -W\sin\theta\, l + f d/2$

Thus $I_0\ddot\theta + W\sin\theta\, l = \mp\mu W\cos\theta \frac{d}{2}$ and for small oscillations $\cos\theta \cong 1$, $\sin\theta \cong \theta$

thus $I_0\ddot\theta + Wl\theta = \mp\mu W d/2 \implies A_1\ddot\theta + A_2\theta = \mp A_3$

In our Coulomb equation we had $m\ddot x + kx = \mp f$ and we found the displacement

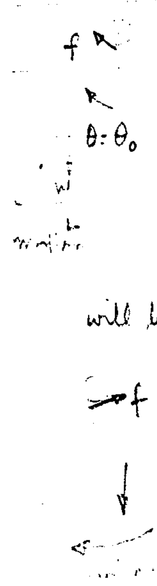
decrease $\frac{4f}{k} = 4\frac{A_3}{A_2}$ thus for this case $\frac{4A_3}{A_2} = \frac{4\mu d}{2l} = \frac{2\mu d}{l}$ per period

Thus (a) = $\frac{2\mu d}{l}$.

Also $\omega_n = \sqrt{\frac{Wl}{I_0}}$

(b) if $\theta = \theta_0$ and $\dot\theta = 0$ then the solution for $I_0\ddot\theta + Wl\theta = +fd/2$ as motion will be

counterclockwise. Solution will be $\theta(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{\mu d}{2l}$ under the



conditions $\theta(t=0) = \theta_0$ and $\dot\theta(t=0) = 0 \implies B = 0$ and $A = \theta_0 - \frac{\mu d}{2l}$

$\therefore \theta(t) = \left(\theta_0 - \frac{\mu d}{2l}\right)\cos\omega_n t + \frac{\mu d}{2l}$

this equation governs the motion until $\dot\theta = 0$ which occurs when $\omega_n t = \pi$

at that time $\theta(t = \pi/\omega_n) = -\left(\theta_0 - \frac{\mu d}{2l}\right) + \frac{\mu d}{2l} = -\left(\theta_0 - \frac{\mu d}{l}\right)$ and this

will become the initial condition for the clockwise motion of the connecting rod

Here $I_0\ddot\theta + Wl\theta = -\frac{fd}{2}$ and the solution is

$\theta(\tilde t) = A\cos\omega_n\tilde t + B\sin\omega_n\tilde t - \frac{\mu d}{2l}$ where $\tilde t = t - \pi/\omega_n$ and

$\theta(\tilde t = 0 \Rightarrow t = \pi/\omega_n) = -\left(\theta_0 - \frac{\mu d}{l}\right)$ and $\dot\theta(\tilde t = 0 \Rightarrow t = \pi/\omega_n) = 0 \implies B = 0$ $A = -\left(\theta_0 - \frac{3d\mu}{2l}\right)$

$\theta(\tilde t) = -\left(\theta_0 - \frac{3d\mu}{2l}\right)\cos\omega_n\tilde t - \mu d/2l$

(c) For the motion to stop we must have that $\dfrac{\theta_0 - \theta_{final}}{\#\text{ of cycles}} = \dfrac{2\mu d}{l}$

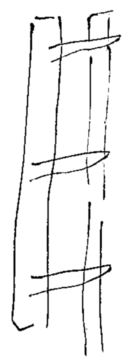
$\theta_{final} = \frac{1}{4}\left(\frac{2\mu d}{l}\right) = \frac{\mu d}{2l}$ thus $\#\text{cycles} = \dfrac{\theta_0 - \mu d/2l}{2\mu d/l}$

HW #7

like like

3.1 , 3.2 , 3.4 , 3.6

in 1 page



3.1 $W = 50N$ $k = 4000\ N/m$ $P_o = 60N$ $f = 6\ Hz \Rightarrow \omega = 2\pi f = 37.7\ rad/s$

a) $\delta_{ST} = \dfrac{W}{k} = .0125\ m = 12.5\ mm$ b) $X_o = \dfrac{P_o}{k} = .015\ m = 15\ mm$

c) $\omega_n = \sqrt{\dfrac{kg}{W}} = 28.0143\ rad/s \Rightarrow \omega/\omega_n = 1.3457$

$$X = \frac{X_o}{r^2 - 1} = .0185\ m = 18.5\ mm$$

3.2. $f = 39.8\ Hz$ $f_n = 40.0\ Hz$ $f_n - f_2 = .1\ Hz \Rightarrow \dfrac{\omega_n - \omega}{2} = 2\pi(.1) = .628\ rad/s$

$$\tau_b = \frac{\pi}{(\omega_n - \omega)/2} = \frac{\pi}{.628} = 5\ sec.$$

like 3.4 $k = 4000\ N/m$ $P_o = 50N$ $f = 4\ Hz \Rightarrow \omega = 2\pi f = 25.133\ rad/s$

$X = 20mm = .02\ m = \dfrac{X_o}{1-r^2}$ or $\dfrac{X_o}{r^2-1} \Rightarrow X_o = \dfrac{P_o}{k} = .0125\ m$

$\dfrac{X}{X_o} = 1.6 \Rightarrow 1 - r^2 = \dfrac{1}{1.6}$ or $r = .6124 = \dfrac{\omega}{\omega_n} \Rightarrow \omega_n = 41.042\ rad/s$

$\omega_n = \sqrt{\dfrac{k}{m}} \Rightarrow \underline{m = 2.375\ kg.}$

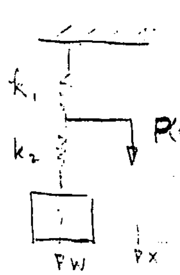
But can also mean $\dfrac{X}{X_o} = 1.6 \Rightarrow r^2 - 1 = \dfrac{1}{1.6} \Rightarrow r = \cancel{1.6125}\ 1.2748 = \dfrac{\omega}{\omega_n} \Rightarrow \omega_n = \cancel{15.587}\ 19.72\ rad/s$

$\omega_n = \sqrt{\dfrac{k}{m}} \Rightarrow \underline{m = \cancel{\ }\ 10.29\ kg}$

Thus either mass can cause this vibration. The book picks the smaller mass.

like 3.6 ~~3.6~~

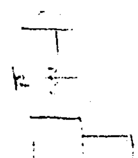
in series



Let the entire spring have spring constant k. Thus each half has spring constant $k_1 = k_2 = 2k$. Spring 1 will displace $X_o = P_o/k_1$ under the force of P_o, where $P(t) = P_o \sin\omega t$

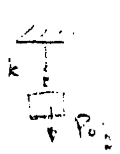
If we define an equivalent force $P_{eq}(t)$ placed at the mass then the spring displaces $X_{oeq} = P_{oeq}/k = \dfrac{2P_{oeq}}{k_1}$

For the two systems to be the same



$P_{eq} = P_{oeq} \sin\omega t$ $X_{oeq} = X_o \Rightarrow P_o = 2P_{oeq} \Rightarrow P_{oeq} = P_o/2$

thus we must have that $m\ddot{x} + kx = P_{eq}(t)$



$\Rightarrow m\ddot{x} + kx = \dfrac{P_o}{2}\sin\omega t$. If $X_o = P_o/k_1 = P_o/2k$

$\Rightarrow X_{particular} = \dfrac{1}{2}\dfrac{P_o/k}{1-r^2}\sin\omega t = \dfrac{X_o}{1-r^2}\sin\omega t$

HW #8

like 3.10 like 3.15 like 3.14

Problems ~~3.7~~, 3.8, ~~3.14~~, ~~3.16~~

in 2 pages

like 3.10

~~3.7~~ Given at $r=1$ $\quad \bar{X} = .015\,m = \dfrac{\bar{X}_o}{2\zeta}$; when $\dfrac{\omega}{\omega_n} = .85 = r$, $\bar{X} = .012\,m$

now in general

$$\bar{X} = \frac{\bar{X}_o}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \Rightarrow \frac{\bar{X}_o}{\sqrt{(1-(.85)^2)^2 + \left(\frac{\bar{X}_o}{.015}\cdot .85\right)^2}} = .012$$

solving for $\bar{X}_o \Rightarrow \dfrac{\bar{X}_o}{\sqrt{.0770062 + \bar{X}_o^2(3211.111)}} = .012 \Rightarrow \bar{X}_o = .004517\,m$

since $\quad .015 = \dfrac{\bar{X}_o}{2\zeta} = \dfrac{.004517}{2\zeta} \Rightarrow \zeta = .1514$

3.8 Given $W = 20N$ for a mass-spring dashpot system. Find ζ & c when $P_o = 30N$ causes $\bar{X} = .015\,m$, $\tau_f = .2\,s$ and $r = 1$.

$\tau_{forcing} = .2\,s = \dfrac{1}{f_f} = \dfrac{2\pi}{\omega} \Rightarrow \omega = 10\pi = 31.416\ rad/s = \omega_n$ since $r=1$

$\therefore\ P = P_o \sin\omega t = 30\sin(31.416t)$

since $\omega_n = \sqrt{\dfrac{k}{m}} \Rightarrow k = m\omega_n^2 = \dfrac{W}{g}\omega_n^2 = 2012.16\ N/m$

$\Rightarrow \bar{X}_o = P_o/k = .01491\ m$ and $\bar{X}\big|_{r=1} = \dfrac{\bar{X}_o}{2\zeta} \Rightarrow \zeta = \dfrac{\bar{X}_o}{2\bar{X}\big|_{r=1}} = \dfrac{.01491}{.03}$

THUS $\underline{\zeta = .497}$

ALSO $\quad c_c = 2m\omega_n = 128.098\ \dfrac{N\text{-}s}{m}$ and $c = \zeta c_c = 63.665\ \dfrac{N\text{-}s}{m}$

like 3.15

~~3.14~~ Given $m = 10\,kg$ $\quad k = 2500\ N/m$ $\quad c = 45\ N\text{-}s/m$ $\quad P_o = 180N$ $\quad f_{forc} = 3.5\,Hz$.

if $x(t=0) = .015\,m$ $\quad \dot{x}(t=0) = 5\,m/s$ find $x = x_h + x_p$

with $2\pi f_{forc} = \omega = 7\pi\ rad/s$ $\qquad P = P_o \sin\omega t \qquad \omega = 21.99\ rad/s$

Need to find ζ & r since $x_p = \dfrac{\bar{X}_o}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t - \psi)$ with $\psi = \tan^{-1}\dfrac{2\zeta r}{1-r^2}$

$\bar{X}_o = P_o/k = \dfrac{180}{2500} = .072\ m \qquad \omega_n = \sqrt{\dfrac{k}{m}} = 15.81\ rad/s$; $\quad r = \dfrac{\omega}{\omega_n} = 1.3909$

also $c_c = 2m\omega_n = 2(10)(15.81) = 316.23\ N\text{-}s/m$; $\quad \zeta = \dfrac{c}{c_c} = .1423$.

$\psi = \tan^{-1}\left(\dfrac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\dfrac{.4236}{-1}\right) = -22.96^\circ = -.4007$ rad

$$\frac{\bar{X}_o}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{.072}{1.0134} = .07105\,m$$

$\therefore \quad x_p = .07105 \sin(21.99t + 22.96°)$

For x_h : since $\zeta < 1 \qquad x_h = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$

where $\zeta\omega_n = 2.25\,\frac{rad}{s}$; $\omega_d = \omega_n\sqrt{1-\zeta^2} = 15.65$ rad/s. To find C & ϕ we must define $x = x_h + x_p$

$$x(t) = Ce^{-2.25t}\sin(15.65t+\phi) + .07105\sin(21.99t + 22.96°)$$

$$\dot{x}(t) = C\left[-2.25e^{-2.25t}\sin(15.65t+\phi) + 15.65e^{-2.25t}\cos(15.65t+\phi)\right] + .07105(21.99)\cdot\cos(21.99t+22.96°)$$

since $x(t=0) = .015\,m = C\sin\phi + .07105\sin(+22.96°) \Rightarrow C\sin\phi = -.01272$

since $\dot{x}(t=0) = 5\,m/s = C\left[-2.25\sin\phi + 15.65\cos\phi\right] + .07105(21.99)\cos(+22.96°)$

$\Rightarrow C\cos\phi = .2257$ after substituting for $C\sin\phi$

$\therefore \quad \tan\phi = \dfrac{C\sin\phi}{C\cos\phi} = -.0564 \Rightarrow \phi = -3.23°$

$\Rightarrow C = .2257$

$\therefore \quad x(t) = .2257\, e^{-2.25t}\sin(15.65t - 3.23°) + .07105\sin(21.99t + 22.96°)$

$x(t) = .3547\, e^{-2.25t}\cos(15.65t + 81.8423°) + .071\cos(21.99t + 22.96°)$

like 3.14
~~3.16~~ $I_o = 6\,kg\text{-}m^2$, $k_t = 14000$ N-m/rad, $c_t = 210$ N-m-s/rad. If $T = T_o\sin\omega t$ where $T_o = 450$ N-m and $\theta = 2° = .03491$ rad find ω.

Here $\theta = \dfrac{\theta_o}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ where $\theta_o = \dfrac{T_o}{k_t}$ and $c_{t_c} = 2I_o\omega_n$ & $\omega_n = \sqrt{\dfrac{k_t}{I_o}}$

$\theta_o = \dfrac{450}{14000} = .03214$ rad $\qquad \omega_n = 48.305$ rad/s $\qquad c_{t_c} = 579.66$ N-m-s/rad &

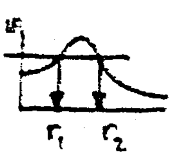
$\zeta = \dfrac{c_t}{c_{t_c}} = \dfrac{210}{579.66} = .3623$. Thus

$\theta = \dfrac{\theta_o}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ can be solved for r^2:

$$\left(\frac{\theta_o}{\theta}\right)^2 = (1-r^2)^2 + (2\zeta r)^2 = 1 - 2r^2 + r^4 + 4\zeta^2 r^2 \Rightarrow r^4 + (4\zeta^2-2)r^2 + \left[1 - \left(\frac{\theta_o}{\theta}\right)^2\right] = 0$$

Thus $r^4 - 1.475r^2 + .1524 = 0 \qquad r^2 = \dfrac{1.475 \pm \sqrt{(1.475)^2 - 4(1)(.1524)}}{2}$

thus $r^2 = 1.3632$ or $r^2 = .1118$



$r = 1.1676$ or $r = .3344 \Rightarrow \underline{\omega = 56.4 \text{ rad/s}}$ or $\underline{16.153 \text{ rad/s}}$

This is because the MF vs r curve for $\theta/\theta_o > 1$ occurs at 2 values of r one > 1 and one < 1. See pg 125 for $\zeta = .3623$ (interpolate) & $\theta/\theta_o = 1.086$

HW # 9 - in 3 pages

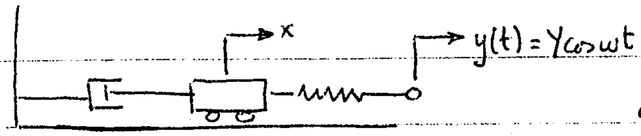
like 3.20

~~3.10~~, 3.12, 3.20, ~~3.22~~

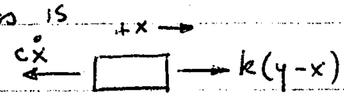
like 3.12 ↓ ↓

1ST ED 1ST ED

like 3.12
~~3.10~~



a) if $y > x$, the free body diagram of the mass is



thus $m\ddot{x} = \sum \text{forces} = k(y-x) - c\dot{x} \Rightarrow m\ddot{x} + c\dot{x} + kx = ky = kY\cos\omega t$

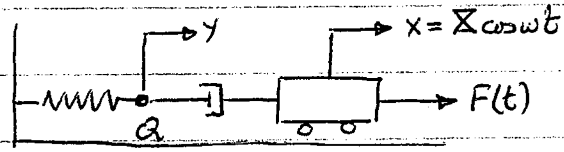
b) As in class the solution to this is $x_p = \dfrac{Y}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\cos(\omega t - \psi)$; $\tan\psi = \dfrac{2\zeta r}{1-r^2}$

c) Force transmitted at P: $F = c\dot{x}_p = \dfrac{-c\omega Y}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t - \psi) = \dfrac{-2\zeta r\, kY\sin(\omega t-\psi)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

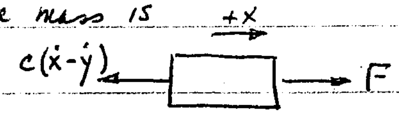
$\left|\dfrac{F_{max}}{kY}\right| = \dfrac{2\zeta r}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$; now $-\sin(\omega t-\psi) = \cos(\omega t - \psi + \pi/2)$

$\Rightarrow$ F & x_p are out of phase by $\pi/2$ radians or 90°

3.12 1st edition



a) if $x > y$, the free body diagram for the mass is



thus $m\ddot{x} = \sum \text{forces} = F - c(\dot{x}-\dot{y})$ or $m\ddot{x} + c\dot{x} = F + c\dot{y}$

b) at Q the free body diagram is as follows

$ky \leftarrow \underset{Q}{\bullet} \xrightarrow{y} c(\dot{x}-\dot{y}) \Rightarrow m\ddot{y} = \sum \text{forces} = c(\dot{x}-\dot{y}) - ky$. But m at Q = 0

since both the spring & damper have no mass, anything connecting the will have no mass (unless otherwise specified). Thus $c\dot{x} = c\dot{y} + ky = -c\omega X\sin\omega t$

If we choose a solution for y in the form $y = A\sin\omega t + B\cos\omega t$ and put into the equation then

$$-c\omega X\sin\omega t = [c\omega A\cos\omega t - c\omega B\sin\omega t] + k[A\sin\omega t + B\cos\omega t]$$

$$\Rightarrow \left.\begin{aligned} -c\omega X &= -c\omega B + kA \\ 0 &= c\omega A + kB \end{aligned}\right\} \rightarrow \quad A = \frac{kc\omega X}{-[k^2+(c\omega)^2]} = \frac{-2\zeta r\, X}{1+(2\zeta r)^2} \qquad B = \frac{(c\omega)^2 X}{k^2+(c\omega)^2} = \frac{(2\zeta r)^2 X}{1+(2\zeta r)^2}$$

thus $y = \dfrac{X\,2\zeta r}{1+(2\zeta r)^2}\left[-\sin\omega t + 2\zeta r\cos\omega t\right]$

c) For the force transmitted at P : $F_P = ky = k X \frac{(2\zeta r)}{1+(2\zeta r)^2}\left[-\sin\omega t + 2\zeta r \cos\omega t\right]$

or $\boxed{F_P = \frac{kc\omega X}{k^2 + c^2\omega^2}\left[-k\sin\omega t + c\omega\cos\omega t\right]}$. If we write $F_P = F_{max}\cos(\omega t + \phi$

then $\tan\phi = +\frac{1}{2\zeta r}$ and $F_{max} = \frac{kX\,(2\zeta r)}{\sqrt{1+(2\zeta r)^2}}$. This shows that F_P leads

x by an angle $\phi = \tan^{-1}\left(\frac{1}{2\zeta r}\right)$. Also $y = F_P/k = \frac{X(2\zeta r)}{\sqrt{1+(2\zeta r)^2}}\cos(\omega t + \phi) = Y\cos(\omega t + \phi$

thus y and F_P are in phase. Note that $\frac{F_{max}}{kX} = \frac{Y}{X} = \frac{2\zeta r}{\sqrt{1+(2\zeta r)^2}}$

If we wanted F(t) acting on the mass $F = m\ddot{x} + c\dot{x} - c\dot{y} = m\ddot{x} + ky$ from part 2.

thus $F = -m\omega^2 X\cos\omega t + \frac{kX(2\zeta r)}{\sqrt{1+(2\zeta r)^2}}\cos(\omega t + \phi)$

first edition

3.20 Given that a mass-spring-dashpot system in free vibration (ie $m\ddot{x} + c\dot{x} + kx = 0$) undergoes damping such that $X_0 = 30$ mm $X_{10} = 10$ mm $n = 10$ cycles. Find max amplitude of beam at resonance if $y(t) = 2\sin\omega t$ (y is motion of the base)

From what we've done in class if the base moves as $y = Y\sin\omega t$

then $x_p = \frac{Y\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\sin(\omega t - \beta - \psi)$

and the max amplitude of the beam is $\frac{Y\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$. At resonance $r = 1$

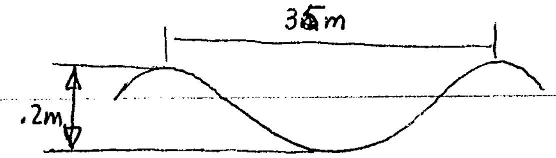
the other info will allow us to find ζ. Remember that δ (log. decrement) $= \frac{1}{n}\ln\left(\frac{X_0}{X_n}\right)$

but $\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$ $\Rightarrow$ $\zeta = .01748$ and Max amplitude $= \frac{Y\sqrt{1+(2\zeta)^2}}{2\zeta} = 57.24$ mm

or .05724 m. Here $Y = 2$ mm

like 3.20

~~3.22~~ Given $f_n = 2$ Hz $\Rightarrow \omega_n = 2\pi f_n = 12.5664$ rad/s

and 

and $V = 60 \frac{km}{hr} = 16.67\ m/s$

IT TAKES THE CAR $\frac{35m}{16.67\ m/s} = 2.1$ sec TO GO from peak to peak. This is the period and $\underline{\omega = \frac{2\pi}{\tau} = 2.992\ rad/s}$

Thus $y = .1m \sin(2.992t)$ where $Y = .1\ m$

we want to find max amplitude ie $\frac{Y\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ given $Y = .1m$

$\zeta = 0.15$

$r = \frac{\omega}{\omega_n} = \frac{2.992}{12.566}$

$= .2381$

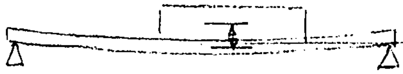
Thus max amplitude is $.10598\ m$.

Y730

HW # 10 in 2 pages

3.26, ~~3.27~~ like 3.29, 3.30, ~~3.31~~ like 3.30 other 2 From first edition

Y730

3.26  $\Delta_{STATIC} = .004\,m \quad \therefore \quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}} = \sqrt{\frac{g}{\Delta_{ST}}} =$ 49.523 rad/s

Given: $m = 50$ kg, $m_o = 10$ kg, $e = .012$m and $\omega = 1750$ rpm $\times \frac{2\pi}{60} = 183.26$ rad/s.

$\Rightarrow r = \frac{\omega}{\omega_n} = 3.7005$. Also given for the free vibrations part that $X_o = 40$mm and $X_n = 2$mm and for n cycles it takes 1 sec ($= \Delta\tau$)

$\therefore \quad 1\,\text{sec} = \Delta\tau = \frac{1}{\zeta\omega_n} \ln\left(\frac{X_o}{X_n}\right) \quad \text{or} \quad \zeta = \frac{1}{\omega_n}\ln\left(\frac{X_o}{X_n}\right) = .06049$

With r & ζ we can now find the amplitude given by

$$X = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{10(.012)}{50} \frac{(3.7005)^2}{\sqrt{[1-(3.7005)^2]^2 + [2\cdot(.06049)(3.7005)]^2}} = .002588\,m$$

To do this problem you needed r & ζ. r was found from ω_n & ω (ω_n was found from Δ_{ST}) and ζ came from the information of the free vibrations part.

like 3.29
~~3.27~~ Given $X\big|_{r=1} = .012\,m = \frac{m_o e}{m}\frac{1}{2\zeta}$. As $r \to \infty$ $X\big|_{r\to\infty} = .001\,m = \frac{m_o e}{m}$

$$\Rightarrow \quad 2\zeta = \frac{1}{12} \quad \text{or} \quad \zeta = .0417 \qquad \left(\frac{X_{r\to\infty}}{X_{r=1}} = \frac{1}{12} = 2\zeta\right)$$

3.30 From the free vibrations part get the logarithmic decrement $\delta = \frac{1}{n}\ln\left(\frac{X_o}{X_n}\right) = \frac{1}{10}\ln\left(\frac{45}{2}\right)$

$\therefore \quad \delta = .31135 \quad$ also $\quad \zeta = \frac{\delta}{\sqrt{\delta^2 + 4\pi^2}} = .04949$.

Also from above $1\,\text{sec} = \Delta\tau = \frac{1}{\zeta\omega_n}\ln\left(\frac{X_o}{X_n}\right)$ or $\omega_n = \frac{1}{\zeta}\ln\left(\frac{X_o}{X_n}\right) = 62.912$ $\frac{rad}{s}$

or from diagram which shows $f_d = 10\,Hz$ (10 cycles/s) $\Rightarrow 20\pi$ rad/s $= \omega_d$ & $\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}$

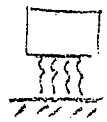
i) at resonance $\quad \frac{X m}{m_o e} = \frac{1}{\sqrt{(1-1)^2 + [2(0.04949)\cdot 1]^2}} = \frac{1}{2\zeta} = 10.1031$

ii) at 1750 rpm $\quad$ From problem 3.26 $\quad$ 1750 rpm $= 183.26$ rad/s $= \omega$

and $r = \frac{\omega}{\omega_n} = \frac{183.26}{62.912} = 2.913$ and $\frac{X m}{m_o e} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 1.1328$

Y730

like 3.30

3.31 

$\Delta_{ST} = .045\,m$ $\quad m = 380\,kg$ and $\zeta \cong 0$. Given $m_o e = 0.15$ kg-m

find X @ 1750 rpm and $F_{TRANS} = kX$

From 3.26 $\quad 1750$ rpm $= 183.26$ rad/s $= \omega$ $\quad$ From Δ_{ST} $\quad \omega_n = \sqrt{\frac{g}{\Delta_{ST}}} = 14.765\,\frac{rad}{s}$

$r = \frac{\omega}{\omega_n} = 12.412$. $\quad X\Big|_{\zeta=0} = \frac{m_o e}{m}\left|\frac{r^2}{1-r^2}\right| = \frac{0.15}{380}\;\frac{(12.412)^2}{1-(12.412)^2} = .0003973\,m$

$F_{TRANS} = kX = \omega_n^2 m X = 32.914\,N$

HW # 11 in 2 pages

like 10.16 like 10.10 like 10.14 10.7

~~3.56~~, ~~3.57~~, ~~3.59~~, ~~3.61~~

like 10.16

~~3.56~~ Given an accel. w/ $f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 80 Hz$ and $c = 8$ N-s/m. Given structure w/ $Y\omega^2 = \ddot{y} = 7.5\,m/s^2$ and $\frac{\omega}{2\pi} = f = 50 Hz$ $\quad z_p \omega_n^2 = 8\,m/s^2$. Find k & m

Now $\left|z_p \omega_n^2\right| = Y\omega^2 \cdot \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$ and $\frac{z_p \omega_n^2}{Y\omega^2} = \frac{z_p}{Y}\frac{1}{r^2} = \frac{8}{7.5} = 1.067 = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

but $r = \frac{\omega}{\omega_n} = \frac{f}{f_n} = \frac{50}{80} = .625$. So we can solve for $\zeta = \sqrt{\left[\left(\frac{1}{1.067}\right)^2 - (1-r^2)^2\right]\frac{1}{4r^2}} = .57$

but $\zeta = \frac{c}{c_c}$ $\therefore$ $c_c = \frac{c}{.57} = 14.035\ \frac{N\text{-}s}{m} \Rightarrow$ $c_c = 2m\omega_n$ or $m = \frac{c_c}{2\omega_n}$

$\omega_n = 2\pi f_n = 502.655$ rad/s $\quad \therefore\ m = \frac{c_c}{2\omega_n} = \underline{.014\,kg}$ and $k = \omega_n^2 m = \underline{3527.4\ N/m}$

like 10.10

~~3.57~~ Given $\Delta_{st} = \frac{W}{k} = .02\,m$. Vibro. records $Z = .00002\,m$ with $f = \frac{\omega}{2\pi} = 100 Hz$ find $Y, Y\omega, Y\omega^2$.

$2\pi f = \omega = 628.3185$ rad/s $\qquad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}} = \sqrt{\frac{g}{\Delta_{st}}} = 22.147$ rad/s $\qquad r = \frac{\omega}{\omega_n} = 28.37$

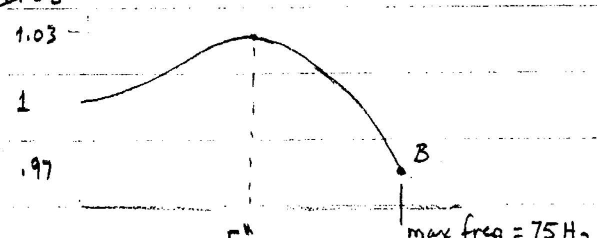
Choose $\zeta = 0$ then

$$Z = Y\,\frac{r^2}{r^2-1} = Y(1.001) \Rightarrow Y = 1.9975 \times 10^{-5}\,m$$

$Y\omega = .01255$ m/s and $Y\omega^2 = 7.886\ m/s^2$

like 10.14

~~3.59~~



From the data given the $\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ curve for the range of frequencies must be such that if it peaks, it must peak within the 3% error band. And the max frequency must also lie within the band (See fig 3.27 & Examples 3.10, 3.11)

If the peak of the curve is such that it is at 3% of $\left|\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\right| = 1$ then $\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ is a max at $r^* = \sqrt{1-2\zeta^2}$ and the max of this function is $\frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.03$

from this $\zeta = .6114$ or $.7874$. Only $\zeta = .6114$ will produce an $r^* = \sqrt{1-2\zeta^2}$ which is real

and $r^* = .49$. For this value of $\zeta = .6164$ use $\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = .97$ to find r_{max}

THIS REPRESENTS PT B ON GRAPH

this gives $r = .7662$ as the only solution. The 75 Hz is the frequency of the oscillating body $\therefore f/f_n = \omega/\omega_n = r \Rightarrow f_n = f/r = 97.886$ Hz. $\Rightarrow \omega_n = 615.0352$ rad/s

Since $\zeta = \frac{c}{c_c} = .6164$ and $c = 50$ N-s/m $\Rightarrow c_c = c/\zeta = 81.12 \frac{N\text{-}s}{m} = 2m\omega_n$

$\therefore m = \frac{c_c}{2\omega_n} = .066$ kg and $\omega_n^2 m = k = 24944.65$ N/m

~~3.61~~ like 10.7 Since $500 <$ speed $< 2000 \Rightarrow \omega_1 = \frac{500 \times 2\pi}{60} < \omega < \frac{2000 \times 2\pi}{60} = \omega_2$ and for a vibrometer r must be large $\therefore \frac{\omega_1}{\omega_n} = r_1 < r_2 = \frac{\omega_2}{\omega_n}$. So if $\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sim 1$ for r_1, then it will certainly satisfy this requirement at r_2

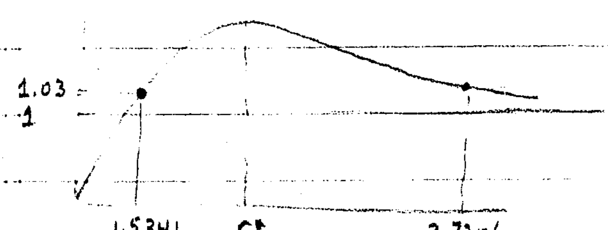
For the error to be less than 3% let $\frac{Z}{Y} = 1.03$

i) for $\zeta = 0$ $\quad \frac{Z}{Y} = \frac{r_1^2}{r_1^2 - 1} = 1.03$ since $r >> 1$ solving for $r_1 = 5.8595$

now $r = \frac{\omega_1}{\omega_n}$ or $\underline{\omega_n} = \frac{\omega_1}{r} = \frac{500 \times 2\pi}{60\ (5.8595)} = \underline{8.9359 \text{ rad/s}}$ for $\zeta = 0$

ii) for $\zeta = 0.6$ $\quad \frac{Z}{Y} = 1.03 = \frac{r_1^2}{\sqrt{(1-r_1^2)^2+(2\zeta r_1)^2}}$ solving for $r_1 = 1.5341$ and $r_1 = 2.7206$

now $\left(\frac{Z}{Y}\right)_{max}$ occurs at $r^* = \frac{1}{\sqrt{1-2\zeta^2}} = 1.8898$. Thus the value of r_1 must be $> r^*$ since $\left(\frac{Z}{Y}\right)_{r^*}$ is > 1.03 violating the 3% requirement.



Since $r_1 = 2.7206 = \frac{\omega_1}{\omega_n} \Rightarrow \omega_n = \frac{\omega_1}{r_1}$

$\underline{\omega_n = 19.24571 \text{ rad/s}}$

(3.28) Given: $m = 1000\,lb \times \frac{1}{32.2 \times 12} = 2.588\ \frac{lb\,sec^2}{in}$

$\omega_f = 1500\ RPM \times \frac{1}{60} \times 2\pi = 157.08\ Rad/sec$

$k_1 = 45000\ lb/in$

$k_2 = 45,000\ lb/in$

$\zeta = 0.15$

Required: Select the best possible isolation System for the compressor

$$F_t = \frac{F_o\,(k^2 + \omega_f^2 C^2)^{1/2}}{\left[(k - m_f\omega^2)^2 + \omega_f^2 C^2\right]^{1/2}}$$

$$T_r = \frac{F_t}{F_o} = \left\{\frac{k^2 + \omega_f^2 C^2}{(k - m\omega^2)^2 + \omega_f^2 C^2}\right\}^{1/2} = \left\{\frac{1 + (2\zeta r)^2}{[1 - (r)^2]^2 + (2\zeta r)^2}\right\}^{1/2}$$

① $k = 45,000\ lb/in$ (one spring)

No damping

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{45,000}{2.588}} = 131.86\ rad/sec; \quad r = \frac{\omega_f}{\omega_n} = 1.19$$

$$T_r = \left[\frac{1}{[1 - r^2]^2}\right]^{1/2} = 2.40$$

② $k = 45,000\ lb/in$ (one spring)

$\zeta = 0.15$

$$T_r = \left[\frac{1 + [2(0.15)(1.19)]^2}{[1 - r^2]^2 + [2(0.15)(1.19)]^2}\right]^{1/2} = 1.937$$

③ $k = 90,000\ lb/in$ (Two springs in parallel)

No damping

$$\omega_n = \sqrt{\frac{90,000}{2.588}} = 186.48; \quad r = .842$$

$$T_r = \left[\frac{1}{[1 - r^2]^2}\right]^{1/2} = 3.43$$

④ $k = 22,500\ lb/in$ (two springs in series)

no damping

$$\omega_n = \sqrt{\frac{22,500}{2.588}} = 93.24; \quad r = 1.685$$

$$T_r = \left[\frac{1}{[1 - r^2]^2}\right]^{1/2} = 0.544$$

⑤ $k = 22,500\ lb/in$ (Two springs in series)

$\zeta = 0.15$

$$T_r = \left[\frac{1 + [2(0.15)(1.685)]^2}{[1 - (1.685)^2]^2 + [2(0.15)(1.685)]^2}\right]^{1/2} = 0.587$$

(c) $k = 90{,}000$ lb/in (Two springs in parallel)
$\zeta = 0.15$

$$T_r = \frac{1 + [2(0.15)(.842)]^2}{[1-(.842)^2]^2 + [2(0.15)(0.842)]^2} = 7.16$$

Two springs in series without will give the best transmissibility. If deflection is not a problem this would be the best choice. If deflection is a problem, one 45,000 lb/in spring with damping would be the best choice.

10 ✓

excellent.

10.7 was 3.61 on the handout yesterday

10.10 Given $\omega_f \geq 100\ Hz$ $\quad \dfrac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = 1 \pm \epsilon \quad \epsilon = .02$

at $\omega_f = 100\ Hz \quad Z = 1\ mm$

$\zeta \cong 0 \quad k = 4000\ N/m$ $\quad$ Find m:

for $\zeta \cong 0$: $\dfrac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \dfrac{r^2}{r^2-1} = 1+\epsilon = 1.02$ $\quad$ or $r^2 = 51$ & $r = 7.1414$

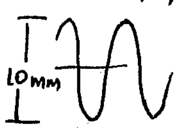
this is min value of r $\therefore$ $\dfrac{\omega_f}{\omega_n} = 7.1414$ & $\omega_n \Rightarrow \dfrac{100\ Hz}{7.1414} = 14.003\ Hz \ \left(87.983 \frac{rad}{sec}\right)$

$\omega_n = \sqrt{\dfrac{k}{m}}$ $\quad \therefore \quad \underline{m = k/\omega_n^2 = .5167\ kg}$

If you wanted Y, then $\quad Z/Y\Big|_{r=7.1414} = 1.02$ $\quad$ & $\underline{Y = \left(\dfrac{1.02}{1\ mm}\right)^{-1} = .98\ mm}$

10.15 Given $m = .1\ kg \quad k = 10{,}000\ N/m \quad \omega_n = \sqrt{\dfrac{k}{m}} = 316.228\ rad/sec \quad c \cong 0 \Rightarrow \zeta = 0$

Peak to peak travel



$\therefore$ $Z = \frac{1}{2}$ peak to peak $= 5\ mm$ @ $\phi_f = 1000\ rpm$

$\omega_f = 104.72\ rad/sec$

Find Y, $\underbrace{Y\omega_f}_{V_{max}}$, $\underbrace{Y\omega_f^2}_{a_{max}}$:

$$\frac{Z\omega_n^2}{Y\omega_f^2} = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$ for accelerometer here $r < 1$

$$\frac{Z}{Y} = \frac{r^2}{1-r^2} = .1232$$

$r = \dfrac{\omega_f}{\omega_n} = \dfrac{104.72}{316.228} = .3312$ $\quad \therefore \quad Y = \dfrac{Z}{.1232} = 40.58\ mm$

$V_{max} = Y\omega_f = 40.58\,(104.72) = 4.25\ m/sec$

$a_{max} = Y\omega_f^2 = 40.58\,(104.72)^2 = 445.06\ m/sec^2$

remember $y = Y \sin \omega_f t$ is displacement of found.

$\dot{y} = Y\omega_f \cos\omega_f t$ is veloc. of found

$\ddot{y} = -Y\omega_f^2 \sin\omega_f t$ is accel of foundation

Longitudinal Vibration and Damping Analysis of Adhesively Bonded Double-Strap Joints

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A mathematical model to study the longitudinal vibration of an adhesively bonded double-strap joint is presented in this paper. Energy method and Hamilton's principle are used to derive the governing equations of motion and natural boundary conditions of the joint system. The adhesive is modeled as a viscoelastic material using complex modulus approach. Both the shear and longitudinal deformation in the adhesive layer are included in the analysis. The equations to predict the system resonance frequencies and loss factors are derived from the system natural and forced boundary conditions for the case of simply supported boundary conditions. A special searching strategy for finding the zeros of a complex determinant has been utilized to obtain the numerical results. The effects of the adhesive shear modulus and structural parameters such as lap ratio, adhesive and strap thickness on the system resonance frequencies and loss factors are also studied.

1 Introduction

Many complex structures used in commercial, military, and aerospace applications are joint dominated. These structures should possess sufficient inherent damping capacity to keep vibration and acoustic response caused by external disturbances within acceptable limits. The current trend in the designed-in incorporation of viscoelastic materials in the joints for passive vibration control has resulted in many innovative means to enhance the inherent damping in structures subjected to dynamic loading. The combination of high damping capability of viscoelastic materials with predominantly friction damping mechanisms at joints and supports provides a promising opportunity to maximize the damping capacity of the structure. This approach, however, involves certain penalties in other structural properties such as stiffness, strength, and weight. Damping is now recognized as a design parameter in all stages of design and development of structures and joints subjected to dynamic loading. The design goal is to develop passively damped joints which would give favorable trade-offs between damping benefits and associated stiffness and strength penalties. In order to accomplish the above goal, accurate models to predict the damping capacity of various joint configurations are needed. The purpose of this paper is to develop one such model for the longitudinal vibration of a double-strap joint system.

The classic work in the area of vibration of sandwich beams incorporating viscoelastic damping materials is reported by Ross, Ungar, and Kerwin (1959), DiTaranto (1965), Mead and Markus (1969), and Miles and Reinhall (1986). Although lot of research work has been done on the stress analysis and dynamic modeling of single lap joints [see, for example, references by Goland and Reissner (1944), Renton and Vinson (1977), Delale and Erdogan (1981), Hart-Smith (1973, 1974), Saito and Tani (1984), Rao (1990)], work on other types of joints, such as the double-strap joints considered here is relatively unexplored. Hart-Smith (1973) was the first investigator to study the static load carrying capacity and failure modes of double-strap adhesively bonded joints using continuum mechanics approach. Prucz (1985) has developed a model for the quasi-static analysis of a double-strap bonded joint. He incorporated the viscoelastic behavior of the adhesive layers in the joint and utilized a quasi-static analysis of constrained layer damping treatment to evaluate the joint damping properties. Prucz's model is reasonable for structures where low vibration frequencies are expected to be dominant so that inertia effects may be neglected without series errors. In this paper, however, the authors have developed a more general dynamic model to study the damping under longitudinal vibration of a similar double-strap joint system. The Energy method and Hamilton's principle is used to derive the governing equations of motion. The system resonance frequencies along with modal loss factors are obtained through the forced boundary conditions, natural boundary conditions, and continuity conditions as discussed in the following sections. A parametric study has been conducted to study the effects of various design parameters on the system resonance frequencies and damping capacity.

2 Formulation of the Analytical Model

Figure 1 shows the configuration of the double-strap joint. There are four basic assumptions that pertain to this model:

(*1*) The two primary beams and the upper and lower constraining beams are isotropic, while the adhesive layers are modeled as linear viscoelastic materials.

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HW # 16

problem 6.6 Rao in 2pages

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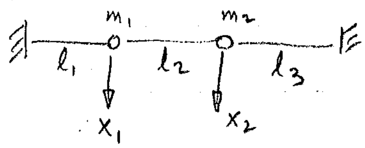
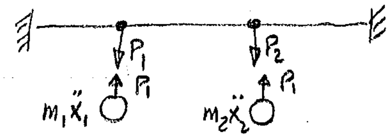
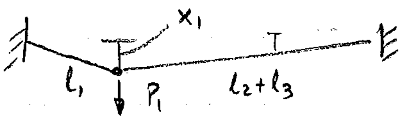
6.6 For the system shown below, we can replace it by the figure on the right.







for $P_2 = 0$

$P\sin\theta + P\sin\varphi = P_1$

$\sin\theta = \dfrac{x_1}{l_1} \quad \sin\varphi = \dfrac{x_1}{l_2+l_3}$

$\therefore \quad P_1 = Px_1\left[\dfrac{1}{l_1} + \dfrac{1}{l_2+l_3}\right]$

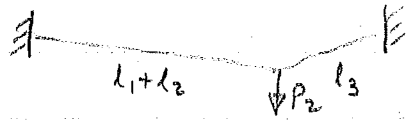
$= Px_1 \dfrac{l_0}{l_1(l_2+l_3)}$

But $x_1 = a_{11}P_1 \Rightarrow \boxed{\dfrac{l_1(l_2+l_3)}{Pl_0} = a_{11}}$ where $l_0 = l_1+l_2+l_3$

by similar Δ's $\quad \dfrac{x_1}{l_2+l_3} = \dfrac{x_2}{l_3}$ (**) $\therefore \quad x_2 = \dfrac{x_1 l_3}{l_2+l_3} = \dfrac{P_1 l_1(l_2+l_3)}{Pl_0}\cdot\dfrac{l_3}{l_2+l_3} = \dfrac{P_1}{P}\dfrac{l_1 l_3}{Pl_0}$

but $x_2 = a_{21}P_1 \Rightarrow a_{21} = \dfrac{l_1 l_3}{Pl_0}$. By maxwell-betty reciprocity theorem

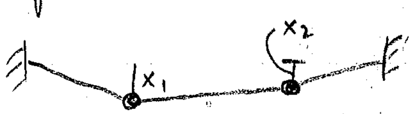
$a_{21} = a_{12} = \dfrac{l_3}{Pl_0}$, For $P_1 = 0$



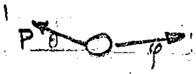
$x_2 = a_{22}P_2$ by similar method to find a_{11} $\quad a_{22} = \dfrac{l_3(l_1+l_2)}{Pl_0}$

$$\therefore \quad [a] = \frac{1}{Pl_0}\begin{bmatrix} l_1(l_2+l_3) & l_1 l_3 \\ l_1 l_3 & l_3(l_1+l_2) \end{bmatrix}$$

To find $[K]$: look at a displaced system w/ $x_1 > x_2$:



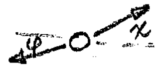
Now look at m_1



$-(P\sin\theta + P\sin\varphi) = m_1\ddot{x}_1$

$\sin\theta = \dfrac{x_1}{l_1} \quad \sin\varphi = \dfrac{x_1 - x_2}{l_2}$

now look at m_1



$P\sin\varphi - P\sin\chi = m_2\ddot{x}_2 \qquad \sin\varphi = \dfrac{x_1 - x_2}{l_2} \qquad \sin\chi = \dfrac{x_2}{l_3}$

$$\therefore \quad \begin{aligned} m_1\ddot{x}_1 + \left(\frac{P}{l_1} + \frac{P}{l_2}\right)x_1 - \frac{P}{l_2}x_2 &= 0 \\ m_2\ddot{x}_2 + \left(\frac{P}{l_2} + P/l_3\right)x_2 - \frac{P}{l_2}x_1 &= 0 \end{aligned} \Rightarrow [M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \quad [K] = \begin{bmatrix} P\left(\frac{1}{l_1}+\frac{1}{l_2}\right) & -\frac{P}{l_2} \\ -P/l_2 & P\left(\frac{1}{l_2}+\frac{1}{l_3}\right) \end{bmatrix}$$

if $m_1 = m_2 = m$, $l_1 = l_2 = l_3 = l$ $\quad [M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad [K] = \begin{bmatrix} 2P/l & -P/l \\ -P/l & 2P/l \end{bmatrix} \quad [a] = \begin{bmatrix} \frac{2l}{3P} & l/3P \\ \frac{l}{3P} & \frac{2l}{3P} \end{bmatrix}$.

amplitude eqns $\left\{-\omega^2[M] + [K]\right\}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{bmatrix} -\omega^2 m + \frac{2P}{l} & -P/l \\ -P/l & -\omega^2 m + \frac{2P}{l} \end{bmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

characteristic eqn is $\omega^4 m^2 - \frac{4P}{l}\omega^2 m + \frac{4P^2}{l^2} - P^2/l^2 = \omega^4 - \frac{4P}{l}\frac{\omega^2}{m} + \frac{3P^2}{m^2 l^2} = 0$

$$\left(\omega^2 - \frac{3P}{ml}\right)\left(\omega^2 - \frac{P}{ml}\right) = 0 \Rightarrow \omega = \sqrt{\frac{3P}{ml}}$$

$$\omega = \sqrt{\frac{P}{ml}}$$

if $\omega = \sqrt{P/ml}$ put into amplitude eqns. $\left(-\omega^2 m + \frac{2P}{l}\right)A_1 - P/l\, A_2 = \frac{P}{l}A_1 - P/l\, A_2 = 0$

$\Rightarrow A_2 = A_1$ & $A_2/A_1 = 1$; if $A_1 = 1 \Rightarrow \underset{\sim}{A}^T = [A_1 \;\; A_2] = [1 \;\; 1]$

if $\omega = \sqrt{\frac{3P}{ml}}$ put into $\left(-\omega^2 m + \frac{2P}{l}\right)A_1 - P/l\, A_2 = -\frac{P}{l}A_1 - P/l\, A_2 = 0$

$\Rightarrow A_2 = -A_1$ & $A_2/A_1 = -1$; if $A_1 = 1 \Rightarrow \underset{\sim}{A}^T = [A_1 \;\; A_2] = [1 \;\; -1]$

$\omega^2 = \dfrac{\underset{\sim}{A}^T[K]\underset{\sim}{A}}{\underset{\sim}{A}^T[M]\underset{\sim}{A}}$

Rayleigh's method, w/ $\underset{\sim}{A}^T = [1 \;\; 1]$

$$\omega^2 = \frac{[1 \;\; 1]\begin{bmatrix} 2P/l & -P/l \\ -P/l & 2P/l \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}{[1 \;\; 1]\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix}} = \frac{[1 \;\; 1]\begin{bmatrix} P/l \\ P/l \end{bmatrix}}{[1 \;\; 1]\begin{bmatrix} m \\ m \end{bmatrix}} = \frac{2P/l}{2m}$$

$$\omega^2 = \frac{P}{ml} \Rightarrow \omega = \sqrt{P/ml}$$

Rayleigh's method w/ $\underset{\sim}{A}^T = [1 \;\; -1]$

$$\omega^2 = \frac{[1 \;\; -1]\begin{bmatrix} 2P/l & -P/l \\ -P/l & 2P/l \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}}{[1 \;\; -1]\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = \frac{[1 \;\; -1]\begin{bmatrix} 3P/l \\ -3P/l \end{bmatrix}}{[1 \;\; -1]\begin{bmatrix} m \\ -m \end{bmatrix}} = \frac{6P/l}{2m}$$

$$\omega^2 = \frac{3P}{lm} \Rightarrow \omega = \sqrt{\frac{3P}{ml}}$$

Dunkerley's method
$$a_{11}m_1 + a_{22}m_2 = \frac{1}{\omega_1^2}$$
$$\frac{2l}{3P}m + \frac{2l}{3P}m = \frac{4lm}{3P} = \frac{1}{\omega_1^2} \Rightarrow \omega_1 = \sqrt{\frac{3}{4}}\sqrt{\frac{P}{lm}} = .845\sqrt{\frac{P}{lm}}$$

dunkerley produces a 15-16% error in lowest ω

1+7

Complete Dunkerley $\frac{1}{\omega_1^2} + \frac{1}{\omega_2^2} = \underbrace{a_{11}m_1}_{\frac{1}{\omega_{1i}^2}} + \underbrace{a_{22}m_2}_{\frac{1}{\omega_{2i}^2}} \cong \frac{1}{\omega_1^2}$

Do a dunkerley Problem 7.4/7.5

7.4 Do a rayleigh problem

$$\omega_{1i} = \sqrt{\frac{1}{a_{11}m_1}} = \sqrt{\frac{k_{11\,eq}}{m_1}}$$

$$\sqrt{\frac{k_{22}}{m_2}}$$

$$\omega_{1i} = \sqrt{\frac{k_{11}}{m_1}}$$

Copy

$$\left(\frac{m}{k}\right)$$

$$\omega_{2i} = \sqrt{\frac{k_{22}}{m_2}}$$

$$\sqrt{\frac{k}{m}}$$

$$\begin{vmatrix} \frac{2}{3k} & \frac{1}{3k} \\ \frac{1}{3k} & \frac{2}{3k} \end{vmatrix} = A \qquad \begin{bmatrix} & \\ -k & 2k \end{bmatrix} = K.$$

$$\sqrt{\frac{3k}{2m}} = \omega_{1n}$$

$k_{eq_{11}} = \frac{3k}{2}$ not to be confused w $k_{11} = 2k$

$$\sqrt{\frac{3k}{2m}} = \omega_{2n}$$

$$\frac{1}{\omega_1^2} \cong \frac{1}{\omega_{2n}^2} + \frac{1}{\omega_{1n}^2} = \frac{2m}{3k} + \frac{2m}{3k} = \frac{4m}{3k}$$

$$\frac{1}{\omega_1^2} = \frac{4m}{3k}$$

$$\omega_1 = \sqrt{.75}\sqrt{\frac{k}{m}} = .866\sqrt{\frac{k}{m}}$$

actual is $\sqrt{\frac{k}{m}}$

Florida International University

MEMORANDUM

TO: All M.E. Faculty

FROM: Ali Ebadian Ali

SUBJECT: Fall 1991 Teaching Evaluations

DATE: November 12, 1991

Our teaching evaluations are scheduled for the week of November 18-22, 1991. Please announce this in your class so we may have the maximum participation by the students. As usual, the ASME club will be conducting the evaluations.

Thank you for your cooperation.

Office of the Chairman • Department of Mechanical Engineering • College of Engineering and Design
University Park, Miami, Florida 33199 • (305) 348-2569 • FAX (305) 348-1932

$l_{TOT} =$

$$w\Big|_{l=1m} = P$$

$$w_1' = \overset{P_1}{200} \cdot \frac{272.5/200}{EI} \qquad a_{11} = \frac{1.36}{EI} \qquad \frac{1.39}{EI} \frac{ft^3}{lb\text{-}ft^2} = \frac{ft}{lb}$$

$$w_2' = P_1 \cdot \frac{377.9/200}{EI} \qquad a_{21} = \frac{1.89}{EI} \qquad \frac{1.72}{EI}$$

$$w_1'' = P_2 \cdot \frac{844.75/500}{EI} \qquad a_{21} = \frac{1.89}{EI} \qquad \frac{1.72}{EI}$$

$$w_2'' = P_2 \cdot \frac{1744/500}{EI} \qquad a_{22} \cong \frac{3.48}{EI} \qquad \frac{3.56}{EI}$$

$$\begin{bmatrix} \frac{1.36}{EI} & \frac{1.39}{EI} \\ \frac{1.39}{EI} & \frac{3.48}{EI} \end{bmatrix} \qquad \frac{1}{\omega_1^2} \cong \frac{1.39 m_1}{EI} + \frac{3.56\, m_2}{EI}$$

$$\cong \frac{27.78}{EI} + \frac{177.78}{EI}$$

$n \times n \cdot n \times 1 = n \times 1$

$$(x_1 \quad x_2 \quad \dots \quad x_n) \left[\begin{array}{l} k_{11}x_1 + k_{12}x_2 + k_{13}x_3 + \cdots \\ k_{21}x_1 + k_{22}x_2 + k_{23}x_3 + \cdots \\ \vdots \\ k_{n1}x_1 + k_{n2}x_2 + \cdots \end{array} \right.$$

$$\frac{1}{\omega_1^2} \cong \frac{205.56}{EI}$$

$$\omega_1 = \sqrt{.00486}\sqrt{EI}$$

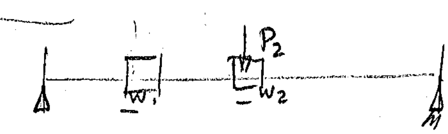
$$\cong .0\cancel{9}\sqrt{EI}$$

$$.06975\sqrt{EI}$$

$k_{11}x_1^2$

$$\frac{lb\cdot ft^2}{ft^3 \cdot lb\cdot sec^2/ft} \qquad \frac{EI}{1.39\, m_1} \qquad k_{11_{eq}} = .7194\, EI$$

$$V_{max} = \frac{1}{2}\left[P_1 W_1 + P_2 W_2 + \dots \right]$$



$$T_{max} = \frac{1}{2}\omega^2 \left[m_1 W_1^2 + m_2 W_2^2 + \dots \right]$$

$$\cancel{P_1 = m_1 g = k_{11_{eq}} W_1 \quad \therefore \quad k_{11_{eq}} = m_1 g / W_1}$$

for example $\cancel{k_{11_{eq}} = \frac{m_1 g}{W_1} = \frac{20 \times 9.81}{1117.25} EI = .1756 EI}$

$$\cancel{\omega_{1n} = \sqrt{\frac{k_{11_{eq}}}{m_1}} = \sqrt{\frac{9.81}{1117.25}}\sqrt{EI} = .0937\sqrt{EI}}$$

Florida International University

MEMORANDUM

TO : All ME Faculty

FROM : T. C. Yih T.C.Y.

DATE : November 15, 1991

SUBJECT : Photograph for ME Research Handbook

The schedule of taking photograph for this year's *ME research Handbook* is listed below. Please contact Gloria at **Media Service (AT 139 - inside Library)** on time. Thank you for your cooperation.

TIME	Nov. 19 (Tue.)	Nov. 20 (Wed.)
10:00 - 10:15		Orozco
10:15 - 10:30		Perl/Levy
10:30 - 10:45		Hopkins/Ebadian
10:45 - 11:00		Cherepanov/Bigzadeh
11:00 - 11:15		El-sayed
11:15 - 11:30		Wu/Tansel
14:00 - 14:15	Munroe	Chellaiah/Richard
14:15 - 14:30	Swift	Jones/Jiang
14:30 - 14:45	Radin	Yang/Yih
14:45 - 15:00		
15:00 - 15:15		
15:15 - 15:30		

$$\int dx = x + C$$

$$\int cy dx = c \int y dx$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + C \qquad n \neq -1$$

$$\int x^{-1} dx = \ln x + C$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + C \tag{1.6.11}$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{1}{4} \sin 2x + C$$

$$\int u \, dv = uv - \int v \, du .$$

This last integral is often referred to as "integration by parts." If the integrand (the coefficient of the differential) is not one of the above, then, in the last integral, $\int v \, du$ may in fact, be integrable.

1.7 DIFFERENTIAL EQUATIONS

A differential equation is *linear* if no term contains the dependent variable to a power other than one (terms that do not contain the dependent variable are not considered in the test of linearity). For example,

$$y'' + 2xy' - y \sin x = 3x^2 \tag{1.7.1}$$

is a linear differential equation; the dependent variable is y and the independent variable is x. If a term contained y'^2, or $y^{1/2}$, or $\sin y$ the equation would be nonlinear.

A differential equation is *homogeneous* if all of its terms contain the dependent variable. Eq. (1.7.1) is nonhomogeneous because of the term $3x^2$.

The *order* of a differential equation is established by its highest order derivative. Eq. (1.7.1) is a second order differential equation.

The general solution of a differential equation involves a number of arbitrary constants equal to the order of the equation. If conditions are specified, the arbitrary constants may be calculated.

1.7.1 FIRST ORDER

A first order differential equation is *separable* if it can be expressed as:

$$M(x)\,dx + N(y)\,dy = 0. \tag{1.7.2}$$

The solution follows by integrating each of the terms.

If $M = M(x,y)$ and $N = N(x,y)$, the solution $F(x,y) = C$ can be found if Eq. (1.7.2) is *exact*, that is, $\partial M/\partial y = \partial N/\partial x$; then $M = \partial F/\partial x$ and $N = \partial F/\partial y$. Note that Eq. (1.7.2) is, in general, nonlinear.

The linear, first order differential equation,

$$y' + h(x)\,y = g(x) \tag{1.7.3}$$

has the solution,

$$y(x) = \frac{1}{u}\int u\,g(x)dx + \frac{C}{u}\;, \tag{1.7.4}$$

where

$$u(x) = e^{\int h(x)dx} \tag{1.7.5}$$

The function, $u(x)$, is called an integrating factor.

1.7.2 SECOND ORDER, LINEAR, HOMOGENOUS, WITH CONSTANT COEFFICIENTS

The general form of a second order, linear, homogeneous differential equation with constant coefficients is:

$$y'' + Ay' + By = 0. \tag{1.7.6}$$

To find a solution, we must first solve the characteristic equation,

$$m^2 + Am + B = 0. \tag{1.7.7}$$

If $m_1 \neq m_2$, and both are real, the general solution is:

$$y(x) = c_1 e^{m_1 x} + c_2 e^{m_2 x}. \tag{1.7.8}$$

If $m_1 = m_2$, the general solution is:

$$y(x) = c_1 e^{m_1 x} + c_2 x e^{m_2 x}. \tag{1.7.9}$$

Finally, if $m_1 = a + ib$ and $m_2 = a - ib$, the general solution is:

$$y(x) = (c_1 \sin bx + c_2 \cos bx)\, e^{ax}. \tag{1.7.10}$$

$$a = \frac{-A}{2} \qquad b = \frac{\sqrt{4B - A^2}}{2}$$

NOTE: Eqs. (1.7.8) - (1.7.10) are the homogenous solutions, $y_h(x)$, used in the next section.

1.7.3 LINEAR, NONHOMOGENEOUS, WITH CONSTANT COEFFICIENTS

If Eq. (1.7.6) was nonhomogeneous, it would be written as:

$$y'' + Ay' + By = g(x). \tag{1.7.11}$$

The general solution is found by finding the solution, $y_h(x)$, to the homogenous equation (simply let the right-hand side be zero and solve the equation as in the Section 1.7.2) and adding to it a particular solution, $y_p(x)$, found by using the following Table 1.2:

TABLE 1.2 Particular Solutions

g(x)	$y_p(x)$	Provisions
a	C	
$ax + b$	$Cx + D$	
$ax^2 + bx + c$	$Cx^2 + Dx + E$	
e^{ax}	Ce^{ax}	if m_1 or $m_2 \neq a$
	Cxe^{ax}	if m_1 or $m_2 = a$
$b \sin ax$	$C \sin ax + D \cos ax$	if $m_{1,2} \neq \pm ai$
	$Cx \sin ax + Dx \cos ax$	if $m_{1,2} = \pm ai$
$b \cos ax$	(same as above)	

NOTE: b,a are general constants; not the same as those in Eq. (1.7.10).

1.8 PROBABILITY AND STATISTICS

Events are independent if the probability of occurrence of one event does not influence the probability of occurrence of other events. The number of permutations of n things taken r at a time is:

$$p(n,r) = \frac{n!}{(n-r)!}. \tag{1.8.1}$$

If the starting point is unknown, as in a ring, the *ring permutation* is — ORDER DEPENDENT

$$p(n,r) = \frac{(n-1)!}{(n-r)!} \tag{1.8.2}$$

The number of *combinations* of n things taken r at a time (it is not an order-conscious arrangement) is given by,

$$C(n,r) = \frac{n!}{r!(n-r)!}. \tag{1.8.3}$$

For independent events of two sample groups, A_i and B_i, the following rules are necessary:

1. The probability of A_1 or A_2 occurring equals the sum of the probability of occurrence of A_1 and the probability of occurrence of A_2; that is,

PART I -- SERVICE LOADING

Historically, the most important automobile structural design criteria have been those attributed to service loads. The reason for this is simply that the loads sustained during normal operation were extremely severe prior to the development of an extensive paved highway system. Field failures were due to extreme and/or repeated terrain roughness loads and were more common in the suspension components than in the body structure. Of course, the forces were greater in the suspension particularly as body isolation methods improved; however, another reason for the relatively low frequency of failure in the body structure was the requirement for overall body stiffness which was employed by body development engineers. Body stiffness criteria were imposed (and still are) to ensure minimum deflection during jacking of the bumpers, towing, and other static load conditions. The static stiffness criteria normally insured that the body could adequately sustain dynamic service loads [2.1].* In addition, the increased stiffness criteria inherently reduced unwanted passenger compartment vibration and vehicle shake. Nevertheless, the virtual elimination of automobile structural failure which has evolved over the years can be attributed largely to the utilization of comprehensive criteria for vehicle service loading. These criteria, which include terrain and maneuver loading conditions, will be discussed in detail. Prior to that discussion, however, the operating conditions of the automobile must be considered.

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*Numbers in brackets designate References at end of chapter.

USAGE SPECTRUM

Characterizing the automobile operating spectrum in a statistically accurate sense is a uniquely complex task, if it is at all possible. The operating spectrum for a commercial or military aircraft is rigidly specified by governmental agencies, and the performance limits are carefully observed by trained pilots. The usage spectrum of the automobile, on the other hand, is as diverse as are its drivers and its operating environment. For this reason an extensive effort has been made over the years to determine the automobile usage spectrum with adequate statistical confidence. For instance, automotive engineers have conducted several field test programs recently in which thousands of privately-owned automobiles, taxicabs, and police cars have been instrumented and pertinent data recorded during tens of thousands of miles of normal operation [2.2-2.4]. Recorded data include strain and acceleration at critical locations as well as more general usage variables such as vehicle speed, trip duration, number of brake applications, number of turns, and so forth. Results show that there are two primary modes of vehicle operation: highway (both rural and city expressways) and non-highway (arteries through and around cities and suburbs). Highway driving is characterized by few stops, little idle time and moderate speed; it occurs approximately 41% of the time. Nonhighway driving is characterized by moderate stopping, lower speed, and increased idle time; it occurs 59% of the total driving time. Table 2.1 shows the average operating conditions which were recorded during the spring and summer of 1974 and reported in [2.2]. It is also interesting to note that the distribution of vehicle speed is bimodal (Fig. 2.1), reflecting the two primary modes of vehicle operation, highway and non-highway.

From the operating conditions given in Table 2.1, we can conclude that the average automobile will experience about 325,000 braking applications and almost 100,000 turns during a service life of 160,900 km (100,000 miles). Although necessary, this information, in itself, is not sufficient to guide the structural design of an automobile. Additional parameters are required to define loading magnitudes, frequency content, and directions for the various maneuver and terrain loading conditions; vehicle specific data are also required including statistical information describing, for instance, probabilities of load exceedance.

With sophisticated measuring and recording equipment, load amplitudes and number of occurrences have been monitored during normal automobile operation. These data are obtained by measuring strain or acceleration at various locations on the vehicle. Usually, time histories of the response are recorded on magnetic tape and later digitized and analyzed with sophisticated time series analysis methods.

For assessing structural durability, load exceedance behavior is a key parameter. The fatigue damage caused by repeated dynamic loading depends upon the number of cycles as well as the frequency content. This information can be extracted from the time history by measuring the number of times the signal crosses discrete threshold levels. The data are frequently displayed as load exceedance curves or as number of crossings per unit distance traveled. Examples are shown in Figs. 2.2 and 2.3 [2.4]. Figure 2.2 shows vehicle lateral acceleration as miles traveled at various acceleration levels, per 1609 km (1000 test miles). The figure shows that accelerations (from maneuvering) of 0.35 g or less are observed 80% of the time. Figure 2.3 shows the dynamic loading in a

steering component -- the number of times that various force amplitudes are exceeded are indicated for three different drivers.

Data of the type described above have been in use in the automobile industry for decades. The efficiency of collecting such data has, of course, improved substantially in recent years; therefore, the availability of load spectra has also increased. The most exacting method of utilizing this information to eliminate service failure is experimental evaluation of the vehicle prior to its introduction into the market. Thus, field data have been collected and used in extensive "proof test" programs. Major structure and suspension components are subjected to laboratory durability tests in which the life cycle loading is applied at an accelerated rate. In addition, in order to ensure vehicle integrity, prototypes are subjected to severe durability schedules at a proving ground. With the application of more accurate load spectra such as those described previously, the automobile has become a very durable means of transportation.

VEHICLE OPERATION MODEL

Butkunas and Bussa [2.5] have described the use of a "vehicle operation model" in determining the external forces which act upon the vehicle system. The approach requires calculation of vehicle response for an envelope or spectrum of vehicle usage. In addition, characteristics of the vehicle environment, notably the terrain condition, must be specified. In the following, vehicle operation will be discussed (for instance, in braking and turning maneuvers) in application to the load analysis. We will begin, however, by describing the general requirements of the vehicle operation model. The first requirement is a description of the rigid body vehicle dynamic parameters.

Historically, the rigid body dynamic parameters have been separated into two groups: the sprung mass parameters and the unsprung mass parameters. The sprung weight is defined as the portion of the automobile which is supported by the suspension, including portions of the weight of the suspension members.* The unsprung weight is the remaining weight not carried by the suspension system, but supported by the tires and assumed to move with them. This division of the system is convenient for the structural engineer as well as the ride and handling engineer because it permits convenient application of the system synthesis approach. For a particular vehicle, the unsprung weight is essentially constant for all load cases; however, the sprung weight includes passenger and cargo and, therefore, is variable. For load analysis purposes, two total vehicle weight conditions are of primary concern. These are the curb weight and the maximum design weight or gross vehicle weight. The curb weight is the basic vehicle weight with standard equipment (including fuel). The maximum design weight includes also the weight of all available optional equipment (e.g., air conditioning system), passenger weight (at 68 kg per passenger) and the maximum weight recommended for cargo and luggage.

In addition to the vehicle weight, the mass moments of inertia of major components are required in defining the basic vehicle operation model. More specifically, the moments of inertia, I_{xx} , I_{yy} , I_{zz} , about the longitudinal, lateral, and vertical axis are required for the combined sprung mass and unsprung mass items. The standard SAE direc-

---

*Vehicle Dynamics Terminology, SAE 7670d, 1975. For brevity, much of the automotive engineering terminology will not be defined in the text -- the reader should consult this SAE reference.

tional control axis system to which the subscripts x , y , and z refer, is shown in Fig. 2.4. The inertia parameters for the larger components are generally obtained from laboratory tests. Moments of inertia for the total system, referred to the center of gravity of the entire system, are computed using the parallel axis theorem.

Dimensional parameters required in our simple operation model are wheelbase, front and rear track (lateral distance between tire contact centers of a pair of wheels), and location of the centers of gravity of the sprung mass, unsprung mass components, and the total vehicle for various loading conditions.

Definition of our simple operation model requires characterizing the elastic behavior of the components which connect the sprung and unsprung masses. The typical suspension of the present day automobile is an extremely complicated system composed of coil springs (or leaf springs), shock absorbers, control arms, "bump stops" (which prohibit metal-to-metal contact between the sprung and unsprung masses), and a variety of other components (see for example, Ref. 2.6). For small values of vertical wheel travel, however, the behavior of the suspension is adequately represented by a simple linear elastic spring. For a given wheel and load then, the change in wheel load, at the tire contact center, per unit vertical displacement (of the sprung mass relative to the wheel) is called the "wheel rate." Its value, which normally differs for front and rear wheels on a particular automobile, is a standard design parameter.

In many respects, the pneumatic tire is the most important element in the automotive dynamic system. Unfortunately, the tire is also one of the most complicated elements. To date, many of the tire dynamic

characteristics cannot be predicted from basic tire construction parameters, but must be directly measured. Indeed, measurement of the tire properties is itself a very complex undertaking. As in the case of the suspension, the tire behavior will be greatly oversimplified for our vehicle operation model. For a thorough introduction to the behavior of pneumatic tires, the reader is referred to the excellent monograph edited by S. K. Clark [2.7]. Initially, our interest will focus on the vertical stiffness of the rolling tire. Measurements made on rolling tires indicate that for small values of wheel vertical displacement, the effective tire spring rate behaves in a linear fashion although the nonrolling tire does not [2.8]. The vertical rolling rate has been measured on several hundred automobile tire designs -- it has been found that the rate depends upon basic construction, inflation pressure, rim width, and preload.

In summary, we have gathered sufficient vehicle information to describe the mass, stiffness, and dimensional characteristics of our simple vehicle operation model. In addition, we have sufficient information to compute some rather important vibration resonance characteristics for our model. These are the vibration modes of the sprung mass supported on the suspension and the vibration modes of the suspension system. The frequencies and mode shapes can be determined by solving the eigenvalue problem associated with our dynamic model, a schematic representation of which is shown in Fig. 2.5. Our model has seven degrees of freedom: sprung mass bounce, pitch, and roll, and vertical motion for each tire. The matrix equations which must be solved for the system frequencies are of the form:

$$[M]\,\{\ddot{X}\} + [K]\,\{X\} = \{0\} \qquad (2.1)$$

where the mass and stiffness matrices are readily derived using a Lagrangian formulation [2.9].

As an example, the parameters required to define a vehicle model for a production compact size car are given in Table 2.2. For this same model, the eigenvalue problem was solved on the computer to determine the modes of vibration of the spring-mass representation of the car. The results are shown in Table 2.3. The terminology used in defining the modes is of common automotive usage (Vehicle Dynamics Terminology, SAE J670d, 1975). The descriptions are self-explanatory with the exception of "tramp" -- "tramp" describes the antisymmetric motion of a pair of wheels (moving vertically out of phase).

The significance of these vibration modes will be brought out in sections which follow. For example, we will show that peak terrain loading normally occurs for excitation wavelengths which excite the suspension or sprung mass resonances. Of course, the vibration modes of the elastic structure also can contribute to the loading response -- these structural modes are discussed in greater detail in other chapters.

DYNAMIC LOADS ANALYSIS

We are now prepared to calculate vehicle dynamic loads using the operating model, the terrain environment and usage spectrum information which has been presented. Let us consider the vehicle loading which results when an automobile is driven over a rough road.

We assume to begin that the automobile can be reasonably accurately modeled as a linear dynamic system. It is well known from random vibration theory, that for a linear system, the input and output power

spectral density matrices are given by:

$$[\Phi_{XX}(\omega)] = [H^*(\omega)]\ [\Phi_{FF}(\omega)]\ [H(\omega)]^T \qquad (2.2)$$

where the elements in $H(\omega)$ are the frequency response functions and $*$ and T denote conjugation and transposition, respectively. The root-mean-square value of the response quantity, X_1, is obtained by integration:

$$\sigma_1 = \text{RMS }(X_1) = \int_o^\infty \sigma_{X_1X_1}(\omega)\ d\omega \qquad (2.3)$$

The response PSD and RMS value are important vehicle loading parameters. Other important parameters which can be derived from Eq. (2.2) under certain simplifying assumptions, are the threshold crossing or exceedance statistics. The exceedance statistics are indicative of the fatigue damage incurred by the structural members. For instance, the fatigue crack growth as a function of time in service is, in functional form [2.10]:

$$A(t) = f(A_o,\ t,\ c,\ N_o,\ \sigma_1) \qquad (2.4)$$

where A_o is the initial crack size, c is a material constant, and N_o is the zero level crossing rate.

It may be instructive at this point to consider typical dynamic analysis results. Our vehicle operation model of the compact unit body car was used to compute statistics of vehicle response for travel over a rough road, after adding representative system damping coefficients. The power spectral density function for acceleration response at the floor is shown in Fig. 2.6. Two curves are included. The broken line shows rigid body response for the curb weight configuration with one occupant, the driver; the solid line shows the response when the first

eight flexural body modes plus engine modes are included (all resonances up to 40 Hz). As anticipated, we see that the most severe vehicle loading occurs at the resonant frequencies, in particular the body bounce and pitch, and the suspension resonances. The significance of the system resonances in design criteria is also quite evident. Clearly, it is desirable to avoid coincidence of the resonant frequencies in order to minimize the vehicle loading. It is standard design procedure to require that the first body flexural resonance occur at a frequency well above the wheel-hop frequency.

The linear analysis which has just been described can be used to evaluate vehicle response for the average driving condition. The linear analysis finds its most significant design impact in the evaluation of vehicle shake, noise, and other aspects of passenger comfort (these topics are considered in a separate section). However, during the life of almost any automobile, unusually severe terrain loads (caused by potholes, for example) are encountered. These extreme load conditions must be analyzed using a dynamic analysis which accounts for system nonlinearities. It can be shown (e.g., Ref. 2.11) that the peak dynamic loads design a significant percentage of the major load-carrying structure.

The modifications required to convert our vehicle operation model to a simple nonlinear load model are not terribly extensive. We must include suspension nonlinearities which arise from excessive vertical travel of the wheels. The effective stiffness and damping of the tire-suspension system become nonlinear for the large displacement problem. These nonlinearities were modeled for our compact car and used in tran-

sient analysis simulations. (More sophisticated nonlinear analysis methods which are used for suspension design have been described in [2.6].) A special digital computer program was written for performing the nonlinear transient analysis [2.12]. For estimating peak dynamic forces acting on the wheels and suspension, it may be necessary to neglect the higher frequency elastic modes to keep computer costs at a minimum.

For the transient simulation, we are faced with the problem of specifying the terrain roughness. Two approaches find frequent use in nonlinear transient simulations. One approach is to generate on the computer a sample profile which has a specified power spectral density function. This technique has been used to generate "random profiles" for various types of excitation [2.13]. The other approach is to use discrete harmonic profiles of various amplitude and wavelength. The latter method has been recommended for use in establishing the taxi load criteria for military aircraft [2.14]. Both methods have been used in the automobile industry. In the following we will discuss the results of a nonlinear load analysis for which the input was a "1-cosine bump." It should be recognized that with this type of analysis we do not attempt to simulate an actual road condition; instead, the purpose is to sweep through excitation frequencies and identify the most severe realizable load conditions. For discrete input, maximum bump heights of 20 cm have been recommended for automobile design [2.15,2.16]. However, in the present analysis, bump height was selected according to the wavelength of the discrete profile. Figure 2.7 shows the input amplitudes which were arbitrarily chosen for our example.

For a large number of discrete profile and vehicle speed combinations, simulations were performed using a nonlinear version of our previously described seven-degree-of-freedom compact car model. The maximum load values of various vehicle components were automatically detected and stored during each transient simulation. From these data, it was possible to plot an "envelope" of maximum load values for each component as a function of input profile wavelength normalized by vehicle wheelbase. Figure 2.8 shows, for instance, the peak loading in the front coil spring resulting from travel over 1-cosine dips or depressions at 50 mph. The effective input frequency is equal to vehicle speed divided by bump wavelength. It is not difficult to show that the vehicle speed and bump wavelength combinations which cause the two peaks in Fig. 2.8 are those which yield excitation frequencies close to the vehicle resonant frequencies.

An extensive parametric study of the type described produced a number of curves resembling the one in Fig. 2.8. Study conclusions based on the compact vehicle analysis are:

1. The maximum vertical wheel spindle loads frequently exceeded three times the deadweight load or 3.0 g. This total load includes a 2.0 g dynamic load superimposed on a 1.0 g gravity load. With very few exceptions, the peak loads were less than 3.5 g.
2. The peak loads generally occurred for input wavelength and vehicle velocity combinations corresponding to effective excitation frequencies of:

a. from 1.0 to 2.0 Hz corresponding to rigid body bounce and pitch modes, and

b. from 10.0 to 13.0 Hz corresponding to wheel hop and tramp modes.

3. Two consecutive 1-cosine dips generally produced more severe peak loads than did two consecutive 1-cosine bumps.

4. Load severity generally increased with increasing vehicle velocity.

The conditions are not necessarily applicable to other vehicles; however, there is evidence in the literature that the 3.0 g load factor is somewhat consistent across vehicle types. To conclude the discussion of terrain loading, vertical load criteria recommended by other investigators are listed below.

Recommended Vertical Load Criteria

Wheel Jounce	Wheel Rebound	Reference
4.0	4.0	Schilling, 1951 [2.17]
3.0	3.0	Garrett, 1953 [2.18]
3.0	3.0	Johnson, 1956 [2.1]
2.0	2.5	Yamamoto, 1961 [2.15]
2.5	2.5	Pawlowski, 1964 [2.16]
3.0	3.0	Skattum, Harris, Howell, 1975 [2.11]

Having discussed the vertical loads caused by road roughness, we will turn our attention to the lateral and fore-aft loads incurred during braking and maneuvering. These maneuver loads have long been recognized for their importance in design criteria (e.g., Refs. 2.17 and 2.18). Unlike the terrain loading, however, the maximum attainable values of the maneuver loads are physically limited by the tire traction characteristics. That is, maximum values of the forces transmitted to the vehicle structure are limited by the traction coefficient between tire

and road surface. Thus, the analysis of the maneuver loading does not require as sophisticated a vehicle operation model as is needed in the terrain analysis.

It should also be noted that the lateral and fore-aft loads are more directly regulated by the driver than are the vertical loads. It has been shown in numerous vehicle handling studies that the average driver rarely, if ever, utilizes the limit handling capabilities of his automobile (see, for example, the study by Rice, Dell'Amico, and Rasmussen, [2.19]). It was previously shown in Fig. 2.5 that lateral accelerations achieved during normal driving are 0.35 g or less, 80% of the total driving time. This value is well below the maximum lateral acceleration which results during an extremely severe turn (e.g., Ref. 2.20). Indeed, during most maneuvers, the lateral acceleration does not exceed 0.25 g; below this value the behavior of the car can be accurately described by linear equations of motion. It is for this reason that the linearized handling theories, such as have been described in [2.21,2.22], provide accurate methods for vehicle handling design and analysis. The peak loads may be estimated from tire traction considerations with conservative results. For braking, for instance, studies of tire traction have shown that the longitudinal force at the contact patch varies with tire slip. The percent slip defines the proportion of tire sliding to tire rolling. The maximum braking force occurs at about 20% slip. Although the braking force varies considerably with tire type and road surface characteristics (e.g., wet surfaces, dry pavement, etc.), the maximum value for optimum stopping conditions is approximately the normal force value or 1.0 g. Experimental studies with a variety of

vehicles used in full-scale tests show that a 1.0 g deceleration is a conservative estimate [2.20].

It can be assumed, therefore, that the maximum fore-aft maneuver load condition for the vehicle is 1.0 g as incurred during a severe stop on dry pavement. In the final stage of braking, the body pitching is balanced by a reaction at the wheels. For a vehicle with four-wheel brakes, this load transfer produces front and rear vertical tire forces given by [2.15]:

$$F_f = \frac{b + \mu h}{\ell} W$$
$$F_r = \frac{a - \mu h}{\ell} W \qquad (2.5)$$

where a and b are the distances from the center of gravity to the front and rear wheels, respectively, ℓ is the wheelbase, and W is the vehicle weight. The traction coefficient μ is recommended by Yamamoto to be "taken as 0.6 or more."

The horizontal tire force which can be achieved during a turn is also limited by the traction coefficient. Road tests have shown that in severe steering maneuvers, the maximum lateral acceleration is approximately 1.0 g [2.20]. Therefore, the peak lateral force transmitted to suspension and body structure is approximately equal to the tire normal load. Accounting for load transfer, the vertical loads in the outer wheels during a sharp turn are

$$F_{fo} = \frac{1}{2} F_{fs} + \frac{nwh}{t_f + \frac{a}{b} t_r}$$
$$F_{ro} = \frac{1}{2} F_{rs} + \frac{nwh}{\frac{b}{a} t_f + t_r} \qquad (2.6)$$

where t_f and t_r are the front and rear track, the subscript s denotes static and n is the load factor. (The other symbols have the same meaning they had in Eq. 2.5.) The lateral force at the vehicle center of gravity is nw. Yamamoto has recommended using 0.6 for the load factor, n.

To conclude, we have summarized recommended criteria for maneuver loading which have been cited in the literature -- these data are given in the table below:

Recommended Maneuver Load Criteria

Lateral	Fore-Aft	Reference
0.6	1.0	Schilling, 1951 [2.17]
1.0	1.0	Garrett, 1953 [2.18]
0.6	0.6 or greater	Yamamoto, 1961 [2.15]
0.7-1.0	0.7-1.0	Pawlowski, 1964 [2.16]
1.0	2.0	Skattum, Harris, Howell, 1975 [2.11]

The manner in which these and the previous criteria are used in vehicle design has been described in [2.23].

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Problem #1 - Vibrometers measure relative amplitude Z not X

$k = 28.8$ lb/in ; $c = .96$ lb sec/in ; $W = 19.3$ lb ; $f_f = 14$ Hz

Amplitude indicated by the instrument is 0.73 in $= Z$

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}} = 24.012 \text{ rad/sec} \qquad f = \frac{\omega}{2\pi} = \underline{3.82 \text{ Hz}}$$

$$C_c = 2m\omega = \frac{2W\omega}{g} = \underline{2.399} \text{ lb-sec/in} \Rightarrow \zeta = \frac{c}{C_c} = \underline{.4}$$

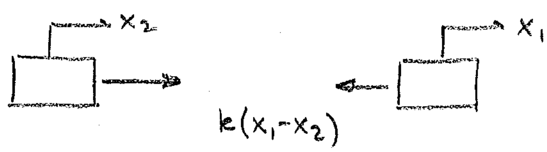
$$r = \frac{\omega_f}{\omega} = \frac{f_f}{f} = \underline{3.663}$$

with $Z = 0.73$ in, $r = 3.663$, $\zeta = .4$, use $Z = \dfrac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} Y$ to find Y

$\underline{Y = .694 \text{ in}}$

To find the force transmitted $F_T = \dfrac{Yk\sqrt{1+(2\zeta r)^2}\, r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = Zk\sqrt{1+(2\zeta r)^2} = \underline{65.1 \text{ lb}}$

Problem # 2



iF $x_1 > x_2$

(1) $m_1\ddot{x}_1 = -k(x_1 - x_2)$ (2) $m_2\ddot{x}_2 = k(x_1 - x_2)$

Choose $A_1 \sin(\omega t + \phi) = x_1$ $x_2 = A_2 \sin(\omega t + \phi)$ and put into (1) + (2)

TO GET $\begin{bmatrix} k - m_1\omega^2 & -k \\ -k & k - m_2\omega^2 \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This gives the frequency eqn.

$$\underline{m_2 m_1 \omega^4 - k(m_1 + m_2)\omega^2 = 0}$$

- This gives $\omega_1 = 0$ $\omega_2 = \sqrt{\dfrac{k(m_1+m_2)}{m_1 m_2}}$. $\underline{\omega_1 = 0}$ is a degenerate case meaning both masses move together with increasing time ie $x = At + B$

for the case where $k_1 = 750$ lb/in $W_1 = 9000$ lb $W_2 = 4000$ lb

$$\omega = \sqrt{\frac{k(m_1+m_2)}{m_1 m_2}} = \sqrt{\frac{k(W_1+W_2)g}{W_1 W_2}} = 10.23 \text{ rad/sec} \quad \text{and } f = \frac{\omega}{2\pi} = 1.628$$

Problem # 3

- with the original mass of 50 kg we can find the equivalent spring constant since $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \Rightarrow k = (2\pi f)^2 m$ where $f = 8\,Hz$. This is why the 50 kg mass and the natural frequency of 8 Hz were given.
- When the blade breaks off the system is considered unbalanced with the unbalanced mass m_0 being the remaining 3 blades or $\underline{12\,kg = m_0}$
- The total mass now is 50 kg − 4 kg (weight of lost blade) = $\underline{46\,kg = m}$
- The blades rotate at 1200 rpm = $\underline{125.664\ rad/sec = \omega_f}$
- From above $\underline{k = (2\pi f)^2 m = 126330.94\ N/m}$. Also $\zeta = .15$ was given to you
- with the new mass figure out the new $\underline{\omega} = \sqrt{\frac{k}{m}} = \sqrt{\frac{126.331\ kN/m}{46\ kg}} = \underline{52.40\ rad/sec}$

Note that k represents the equivalent spring that replaces the beam's reaction to the mass and is independent of what happens to the rotor.

- with $e = 0.15\,m$ (given to you) use this, r, m, m_0, ζ in

$$X = \frac{m_0 e}{m}\,\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{m_0 e}{m}(1.197) = .047\,m$$

- To find the location of the unbalanced mass use $\tan\psi = \frac{2\zeta r}{1-r^2}$ or

$$\psi = -8.61^\circ \text{ or } -.1497\text{ rad.}$$

Note that nowhere on this problem did I ask for the force transmitted to the tail section. What was asked was the forced response of the tail section which is the displacement.

2.60. From the trace there are 4 cycles/sec = 4 Hz = f $\Rightarrow$ $\omega_n = 8\pi$ rad/s = 25.133 rad/s. The decrement $\frac{4\mu N}{k} = \frac{X_0 - X_{final}}{4\text{ cycles}}$ where $X_{final} = .5$ mm

This value is the final displacement for which the system comes to a stop.

Thus $$\frac{4\mu N}{k} = \frac{30 - .5}{4} = 7.375 \text{ mm}$$

Now N = weight and $$\frac{4\mu W}{k} = \frac{4\mu N}{k} = \frac{4\mu mg}{k} = \frac{4\mu}{\omega_n^2} g = 7.375\text{ mm} = .007375 \text{ m}$$

now $$\mu = .007375\left(\frac{\omega_n^2}{4g}\right) = .1187$$

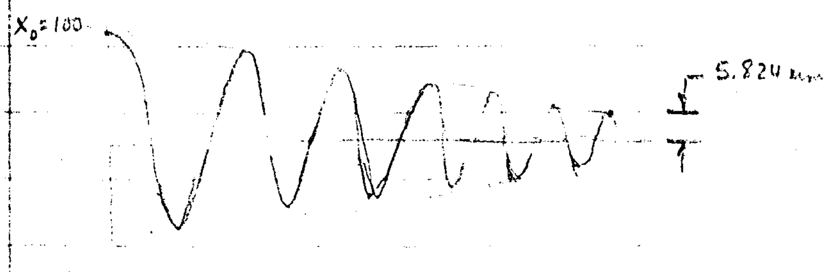
2.63 Given $m = 10$ kg $k = 3000$ N/m and $X_0 = 100$ mm $\dot{X}_0 = 0$, $\mu = .12$

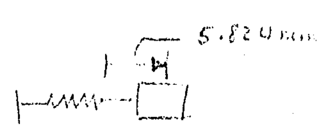
Now $$\frac{f}{k} = \frac{\mu N}{k} = \frac{\mu W}{k} = \frac{\mu mg}{k} = \frac{(.12)(10)(9.81)}{3000} = .003924 \text{ m or } 3.924 \text{ mm}$$

This is the displacement at which motion stops and the number of cycles required are $$\frac{4\mu N}{k} = \frac{X_0 - X_{final}}{n} \Rightarrow n = \frac{100 - 3.924}{4(3.924)} = 6.121 \text{ cycles}$$

Thus at the end of 6 cycles the displacement is given by $-n\left[\frac{4\mu N}{k}\right] + X_0 = X_6$ where $n = 6$

$X_6 = 5.824$ mm also $\dot{X}_6 = 0$ since X_6 represents the maximum displacement.





Thus at the start of the 6th cycle the system is at 5.824 mm (measured from the unstretched length of the spring). Thus in the seventh cycle the mass stops - since $f = \mu N$ is $> kx$, ie when $x = 3.924$ mm

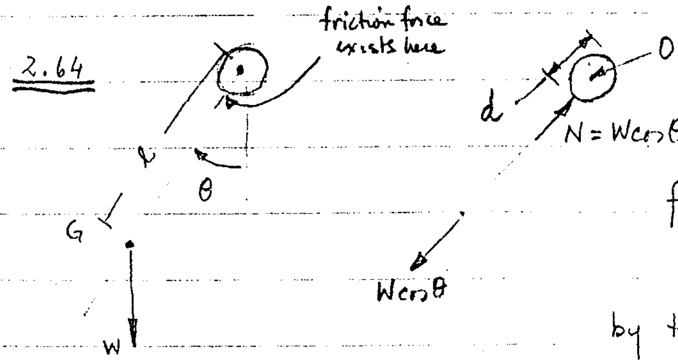
2.62 Given $k = 10$ N/mm $= 10000$ N/m, $m = 20$ kg and $X_4 = 100$ mm. If $X_0 = 150$ mm what is μ and how long is it taken to reach X_4.

$\tau = 2\pi/\omega_n$ $\omega_n = \sqrt{\frac{k}{m}} = 22.36$ rad/s $\tau = .281$ sec $\Rightarrow$ time to X_4 is $4\tau = 1.124$ sec

also $\frac{4\mu N}{k} = \frac{X_0 - X_4}{4}$ or $\mu = \frac{k}{4N}\frac{X_0 - X_u}{4} = \frac{k}{4 \cdot mg \cdot 4} X_0 - X_4 = 0.1593$

remember to convert X_0 & X_4 to meters if you use $g = 9.81\ m/s^2$

2.64



$N = W\cos\theta$ on the upper bearing and $f = -\mu N$ when movement is clockwise $+\mu N$ when counterclockwise. Thus by taking Σ Torques about the center pin bearing O. For $\dot\theta > 0$ $I_0\ddot\theta = -W\sin\theta\, l - fd/2$

$\dot\theta < 0$ $I_0\ddot\theta = -W\sin\theta\, l + fd/2$

Thus $I_0\ddot\theta + W\sin\theta\, l = \mp\mu W\cos\theta \frac{d}{2}$ and for small oscillations $\cos\theta \cong 1$, $\sin\theta \cong \theta$

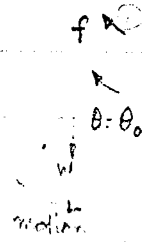
thus $I_0\ddot\theta + Wl\theta = \mp\mu W d/2 \implies A_1\ddot\theta + A_2\theta = \mp A_3$

In our Coulomb equation we had $m\ddot x + kx = \mp f$ and we found the displacement decrease $\frac{4f}{k} = 4\frac{A_3}{A_2}$ thus for this case $4\frac{A_3}{A_2} = \frac{4\mu d}{2l} = \frac{2\mu d}{l}$ per period

Thus (a) $= \frac{2\mu d}{l}$

Also $\omega_n = \sqrt{\frac{Wl}{I_0}}$

(b) if $\theta = \theta_0$ and $\dot\theta = 0$ then the solution for $I_0\ddot\theta + Wl\theta = +fd/2$ as motion will be counterclockwise. Solution will be $\theta(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{\mu d}{2l}$ under the



conditions $\theta(t=0) = \theta_0$ and $\dot\theta(t=0) = 0 \implies B = 0$ and $A = \theta_0 - \frac{\mu d}{2l}$

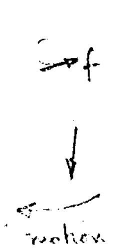
$\therefore\ \theta(t) = \left(\theta_0 - \frac{\mu d}{2l}\right)\cos\omega_n t + \frac{\mu d}{2l}$

this equation governs the motion until $\dot\theta = 0$ which occurs when $\omega_n t = \pi$

at that time $\theta(t = \pi/\omega_n) = -\left(\theta_0 - \frac{\mu d}{2l}\right) + \frac{\mu d}{2l} = -\left(\theta_0 - \frac{\mu d}{l}\right)$ and this will become the initial condition for the clockwise motion of the connecting rod

Here $I_0\ddot\theta + Wl\theta = -\frac{fd}{2}$ and the solution is

$\theta(\tilde t) = A\cos\omega_n\tilde t + B\sin\omega_n\tilde t - \frac{\mu d}{2l}$ where $\tilde t = t - \pi/\omega_n$ and



$\theta(\tilde t = 0 \Rightarrow t = \pi/\omega_n) = -\left(\theta_0 - \frac{\mu d}{l}\right)$ and $\dot\theta(\tilde t = 0 \Rightarrow t = \pi/\omega_n) = 0 \implies B = 0$ A: [illegible]

Thus $\theta(\tilde t) = -\left(\theta_0 - \frac{3d\mu}{2l}\right)\cos\omega_n\tilde t - \mu d/2l$

(c) For the motion to stop we must have that $\dfrac{\theta_0 - \theta_{final}}{\#\text{ of cycles}} = \dfrac{2\mu d}{l}$

$\theta_{final} = \frac{1}{4}\left(\frac{2\mu d}{l}\right) = \frac{\mu d}{2l}$ thus $\#\text{ cycles} = \dfrac{\theta_0 - \mu d/2l}{2\mu d/l}$

3.1 $W = 50N$ $k = 4000\ N/m$ $P_0 = 60N$ $f = 6Hz \Rightarrow \omega = 2\pi f = 37.7\ rad/s$

a) $\delta_{ST} = \frac{W}{k} = .0125\ m = 12.5\ mm$ b) $X_0 = \frac{P_0}{k} = .015m = 15\ mm$

c) $\omega_n = \sqrt{\frac{kg}{W}} = 28.0143\ rad/s \Rightarrow \omega/\omega_n = 1.3457$

$$X = \frac{X_0}{r^2 - 1} = .0185\ m = 18.5\ mm$$

3.2. $f = 39.8\ Hz$ $f_n = 40.0\ Hz$ $\frac{f_n - f}{2} = .1\ Hz \Rightarrow \frac{\omega_n - \omega}{2} = 2\pi(.1) = .628\ rad/s$

$$\tau_b = \frac{\pi}{(\omega_n - \omega)/2} = \frac{\pi}{.628} = 5\ sec.$$

3.4 $k = 4000\ N/m$ $P_0 = 50N$ $f = 4Hz \Rightarrow \omega = 2\pi f = 25.133\ rad/s$

$X = 20mm = .02\ m = \frac{X_0}{1 - r^2}$ or $\frac{X_0}{r^2 - 1} \Rightarrow X_0 = \frac{P_0}{k} = .0125\ m$

$\frac{X}{X_0} = 1.6 \Rightarrow 1 - r^2 = \frac{1}{1.6}$ or $r = .6124 = \frac{\omega}{\omega_n} \Rightarrow \omega_n = 41.042\ rad/s$

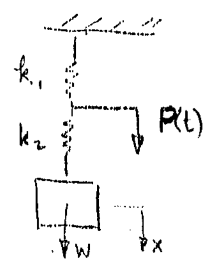
$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow \underline{m = 2.375\ kg}.$

But can also mean $\frac{X}{X_0} = 1.6 \Rightarrow r^2 - 1 = \frac{1}{1.6} \Rightarrow r = \cancel{1.6125}\ 1.2748 = \frac{\omega}{\omega_n} \Rightarrow \omega_n = \cancel{15.587}\ 19.72\ rad/s$

$\omega_n = \sqrt{\frac{k}{m}} \Rightarrow \underline{m = \cancel{16.464}\ 10.29\ kg}$

Thus either mass can cause this vibration. The book picks the <u>smaller</u> mass.

3.6

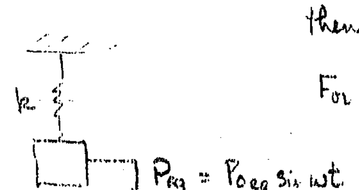


in series

Let the entire spring have spring constant k. Thus each half has spring constant $k_1 = k_2 = 2k$. Spring 1 will displace $X_0 = P_0/k_1$ under the force of P_0, where $P(t) = P_0 \sin \omega t$

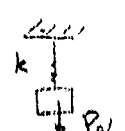
If we define an equivalent force $P_{eq}(t)$ placed at the mass then the spring displaces $X_{0eq} = P_{0eq}/k = \frac{2P_{0eq}}{k_1}$

For the two systems to be the same



$X_{0eq} = X_0 \Rightarrow P_0 = 2P_{0eq} \Rightarrow P_{0eq} = P_0/2$

Thus we must have that $m\ddot{x} + kx = P_{eq}(t)$



$\Rightarrow m\ddot{x} + kx = \frac{P_0}{2} \sin \omega t$. If $X_0 = P_0/k_1 = P_0/2k$

$\Rightarrow X_{particular} = \frac{1}{2} \frac{P_0/k}{1 - r^2} \sin \omega t = \frac{X_0}{1 - r^2} \sin \omega t$

3.7 Given at $r=1$ $\quad \bar{X} = .015m = \frac{\bar{X}_o}{2\zeta}$; when $\frac{\omega}{\omega_n} = .85 = r$, $\bar{X} = .012m$

now in general

$$\bar{X} = \frac{\bar{X}_o}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \Rightarrow \frac{\bar{X}_o}{\sqrt{(1-(.85)^2)^2+\left(\frac{\bar{X}_o}{.015}\cdot .85\right)^2}} = .012$$

solving for $\bar{X}_o \Rightarrow \frac{\bar{X}_o}{\sqrt{.0770062+\bar{X}_o^2(3211.111)}} = .012 \Rightarrow \bar{X}_o = .004517\,m$

since $\quad .015 = \frac{\bar{X}_o}{2\zeta} = \frac{.004517}{2\zeta} \Rightarrow \zeta = .1514$

3.8 Given $W = 20N$ for a mass-spring dashpot system. Find ζ & c when $P_o = 30N$ causes $\bar{X} = .015\,m$ $\quad \tau_f = .2\,s$ and $r = 1$.

$$\tau_{forcing} = .2s = \frac{1}{f_f} = \frac{2\pi}{\omega} \Rightarrow \omega = 10\pi = 31.416 \text{ rad/s} = \omega_n \text{ since } r=1$$

$\therefore\ P = P_o \sin\omega t = 30\sin(31.416t)$

since $\omega_n = \sqrt{\frac{k}{m}} \Rightarrow k = m\omega_n^2 = \frac{W}{g}\omega_n^2 = 2012.16\ N/m$

$\Rightarrow \bar{X}_o = P_o/k = .01491\ m$ and $\bar{X}\big|_{r=1} = \frac{\bar{X}_o}{2\zeta} \Rightarrow \zeta = \frac{\bar{X}_o}{2\bar{X}\big|_{r=1}} = \frac{.01491}{.03}$

THUS $\underline{\zeta = .497}$

ALSO $\quad c_c = 2m\omega_n = 128.098\ \frac{N\text{-}s}{m}$ and $c = \zeta c_c = 63.665\ \frac{N\text{-}s}{m}$

3.14 Given $m = 10kg$ $\quad k = 2500\ N/m$ $\quad c = 45\ N\text{-}s/m$ $\quad P_o = 180N$ $\quad f_{forc} = 3.5\,Hz$.

if $x(t=0) = .015m$ $\quad \dot{x}(t=0) = 5m/s$ find $x = x_h + x_p$

with $2\pi f_{force} = \omega = 7\pi$ rad/s $\quad P = P_o\sin\omega t \quad \omega = 21.99$ rad/s

Need to find ζ & r since $x_p = \frac{\bar{X}_o}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t-\psi)$ with $\psi = \tan^{-1}\frac{2\zeta r}{1-r^2}$

$\bar{X}_o = P_o/k = \frac{180}{2500} = .072\ m$; $\omega_n = \sqrt{\frac{k}{m}} = 15.81$ rad/s ; $r = \frac{\omega}{\omega_n} = 1.3909$

also $c_c = 2m\omega_n = 2(10)(15.81) = 316.23\ N\text{-}s/m$; $\zeta = \frac{c}{c_c} = .1423$.

$\psi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{.4236}{-1}\right) = -22.96^\circ = -.4007$ rad

$$\frac{\bar{X}_o}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{.072}{1.0134} = .07105m$$

$\therefore \quad x_p = .07105 \sin(21.99t + 22.96°)$

For x_h : since $\zeta < 1$ $\quad x_h = Ce^{-\zeta \omega_n t} \sin(\omega_d t + \phi)$

where $\zeta\omega_n = 2.25 \frac{rad}{s}$; $\omega_d = \omega_n\sqrt{1-\zeta^2} = 15.65$ rad/s. To find C & ϕ we must define $x = x_h + x_p$

$$x(t) = Ce^{-2.25t}\sin(15.65t+\phi) + .07105 \sin(21.99t + 22.96°)$$

$$\dot{x}(t) = C\left[-2.25e^{-2.25t}\sin(15.65t+\phi) + 15.65e^{-2.25t}\cos(15.65t+\phi)\right] + .07105(21.99)\cdot \cos(21.99t+22.96°)$$

since $x(t=0) = .015m = C\sin\phi + .07105 \sin(+22.96°) \Rightarrow C\sin\phi = -.01272$

since $\dot{x}(t=0) = 5 m/s = C\left[-2.25\sin\phi + 15.65\cos\phi\right] + .07105(21.99)\cos(+22.96°)$

$\Rightarrow C\cos\phi = .2257$ after substituting for $C\sin\phi$

$\therefore \quad \tan\phi = \frac{C\sin\phi}{C\cos\phi} = -.0564 \Rightarrow \phi = -3.23°$

$\Rightarrow C = .2257$

$\therefore \quad x(t) = .2257\, e^{-2.25t}\sin(15.65t - 3.23°) + .07105 \sin(21.99t + 22.96°)$

$x(t) = .3547\, e^{-2.25t}\cos(15.65t + 81.8423°) + .071\cos(21.99t + 22.96°)$

3.16 $\quad I_o = 6\, kg\text{-}m^2$, $k_t = 14000$ N-m/rad , $c_t = 210$ N-m-s/rad . If $T = T_o \sin\omega t$ where $T_o = 450$ N-m and $\theta = 2° = .03491$ rad find ω.

Here $\Theta = \frac{\Theta_o}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$ where $\Theta_o = \frac{T_o}{k_t}$ and $c_{t_c} = 2I_o\omega_n$ & $\omega_n = \sqrt{\frac{k_t}{I_o}}$

$\Theta_o = \frac{450}{14000} = .03214$ rad $\quad \omega_n = 48.305$ rad/s $\quad c_{t_c} = 579.66$ N-m-s/rad &

$\zeta = \frac{c_t}{c_{t_c}} = \frac{210}{579.66} = .3623$. Thus

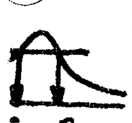
$\theta = \frac{\theta_o}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$ can be solved for r^2:

$$\left(\frac{\theta_o}{\theta}\right)^2 = (1-r^2)^2 + (2\zeta r)^2 = 1 - 2r^2 + r^4 + 4\zeta^2 r^2 \Rightarrow r^4 + (4\zeta^2 - 2)r^2 + \left[1 - \left(\frac{\theta_o}{\theta}\right)^2\right] = 0$$

Thus $r^4 - 1.475r^2 + .1524 = 0 \qquad r^2 = \frac{1.475 \pm \sqrt{(1.475)^2 - 4(1)(.1524)}}{2}$

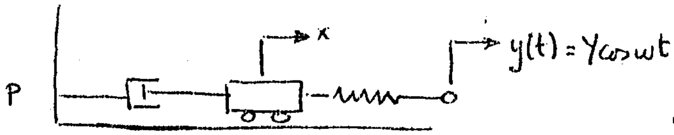
thus $r^2 = 1.3632$ or $r^2 = .1118$

$r = 1.1676$ or $r = .3344 \Rightarrow \underline{\omega = 56.4 \text{ rad/s}}$ or $\underline{16.153 \text{ rad/s}}$

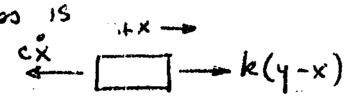


This is because the MF vs r curve for $\theta/\theta_o > 1$ occurs at 2 values of r one > 1 and one < 1. See pg 125 for $\zeta = .3623$ (interpolate) & $\theta/\theta_o = 1.086$

3.10



a) if $y > x$, the free body diagram of the mass is



thus $m\ddot{x} = \sum \text{forces} = k(y-x) - c\dot{x} \Rightarrow m\ddot{x} + c\dot{x} + kx = ky = kY\cos\omega t$

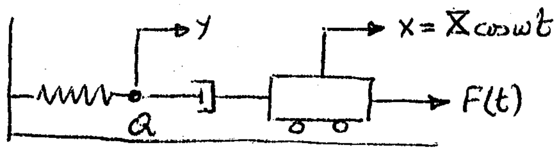
b) As in class the solution to this is $x_p = \dfrac{Y}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\cos(\omega t - \phi)$; $\tan\phi = \dfrac{2\zeta r}{1-r^2}$

c) Force transmitted at P: $F = c\dot{x}_p = \dfrac{-c\omega Y}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t-\phi) = -\dfrac{2\zeta r\, kY\sin(\omega t-\phi)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

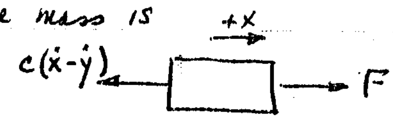
$\left|\dfrac{F_{max}}{kY}\right| = \dfrac{2\zeta r}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$; now $-\sin(\omega t-\phi) = \cos(\omega t - \phi + \pi/2)$

$\Rightarrow$ F & x_p are out of phase by $\pi/2$ radians or $90°$

3.12

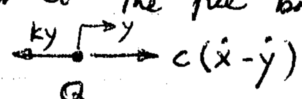


a) if $x > y$, the free body diagram for the mass is



Thus $m\ddot{x} = \sum \text{forces} = F - c(\dot{x}-\dot{y})$ or $m\ddot{x} + c\dot{x} = F + c\dot{y}$

b) at Q the free body diagram is as follows



$\Rightarrow m\ddot{y} = \sum \text{forces} = c(\dot{x}-\dot{y}) - ky$. But m at $Q = 0$ since both the spring & damper have no mass, anything connecting the will have no mass (unless otherwise specified). Thus $c\dot{x} = c\dot{y} + ky = -c\omega X\sin\omega t$

If we choose a solution for y in the form $y = A\sin\omega t + B\cos\omega t$ and put into the equation then

$$-c\omega X\sin\omega t = [c\omega A\cos\omega t - c\omega B\sin\omega t] + k[A\sin\omega t + B\cos\omega t]$$

$$\Rightarrow \left.\begin{aligned} -c\omega X &= -c\omega B + kA \\ 0 &= c\omega A + kB \end{aligned}\right\} \rightarrow A = \frac{kc\omega X}{-[k^2+(c\omega)^2]} = \frac{-2\zeta r\, X}{1+(2\zeta r)^2} \quad B = \frac{(c\omega)^2 X}{k^2+(c\omega)^2} = \frac{(2\zeta r)^2 X}{1+(2\zeta r)^2}$$

thus $y = \dfrac{X\,2\zeta r}{1+(2\zeta r)^2}\left[-\sin\omega t + 2\zeta r\cos\omega t\right]$

c) For the force transmitted at P : $F_p = ky = kX \frac{(2\zeta r)}{1+(2\zeta r)^2}\left[-\sin\omega t + 2\zeta r\cos\omega t\right]$

or $\boxed{F_p = \frac{kc\omega X}{k^2+c^2\omega^2}\left[-k\sin\omega t + c\omega\cos\omega t\right]}$. If we write $F_p = F_{max}\cos(\omega t+\phi)$

then $\tan\phi = +\frac{1}{2\zeta r}$ and $F_{max} = \frac{kX(2\zeta r)}{\sqrt{1+(2\zeta r)^2}}$. This shows that F_p leads

x by an angle $\phi = \tan^{-1}\left(\frac{1}{2\zeta r}\right)$. Also $y = F_p/k = \frac{X(2\zeta r)}{\sqrt{1+(2\zeta r)^2}}\cos(\omega t+\phi) = Y\cos(\omega t+\phi)$

thus y and F_p are in phase. Note that $\frac{F_{max}}{kX} = \frac{Y}{X} = \frac{2\zeta r}{\sqrt{1+(2\zeta r)^2}}$

If we wanted $F(t)$ acting on the mass $F = m\ddot{x} + c\dot{x} - c\dot{y} = m\ddot{x} + ky$ from part a.

then $F = -m\omega^2 X\cos\omega t + \frac{kX(2\zeta r)}{\sqrt{1+(2\zeta r)^2}}\cos(\omega t+\phi)$

3.20 Given that a mass-spring-dashpot system in free vibration (ie $m\ddot{x}+c\dot{x}+kx=0$) undergoes damping such that $X_0 = 30$ mm $X_{10} = 10$ mm $n = 10$ cycles. Find max amplitude of beam at resonance if $y(t) = 2\sin\omega t$ (y is motion of the base)

From what we've done in class if the base moves as $y = Y\sin\omega t$

then $x_p = \frac{Y\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t - \beta - \psi)$

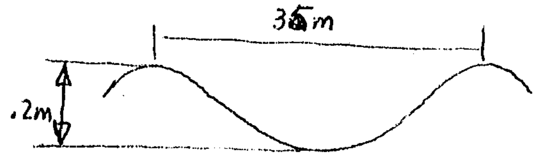
and the max amplitude of the beam is $\frac{Y\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$. At resonance $r=1$

the other info will allow us to find ζ. Remember that δ (log. decrement) $= \frac{1}{n}\ln\left(\frac{X_0}{X_n}\right)$

but $\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \zeta = .01748$ and max amplitude $= \frac{Y\sqrt{1+(2\zeta)^2}}{2\zeta} = 57.24$ mm

or $.05724$ m. Here $Y = 2$ mm

3.36
~~3.22~~ Given $f_n = 2$ Hz $\Rightarrow \omega_n = 2\pi f_n = 12.5664$ rad/s

and 

and $V = 60 \frac{km}{hr} = 16.67\ m/s$

IT TAKES THE CAR $\frac{35m}{16.67\ m/s} = 2.1$ sec TO GO from peak to peak. This is the period and $\underline{\omega = \frac{2\pi}{\tau} = 2.992\ rad/s}$

Thus $y = .1m \sin(2.992t)$ where $Y = .1m$

we want to find max amplitude ie $\frac{Y\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ given $Y = .1m$

$\zeta = 0.15$

$r = \frac{\omega}{\omega_n} = \frac{2.992}{12.566}$

$= .2381$

thus max amplitude is $.10598\ m$.

3.26 $\omega = 1750 \text{ rpm} \times \frac{2\pi}{60} = 183.26$ rad/s.

Given: $m = 50$ kg, $m_o = 10$ kg, $e = .012$m and $\omega = 1750$ rpm $\times \frac{2\pi}{60} = 183.26$ rad/s, Also given for the free vibrations part that $X_o = 40$mm and $X_n = 2$mm

$\Rightarrow r = \frac{\omega}{\omega_n} = 3.7005$, and for n cycles it takes 1 sec $(= \Delta\tau)$

$\therefore$ 1 sec $= \Delta\tau = \frac{1}{\zeta\omega_n} \ln\left(\frac{X_o}{X_n}\right)$ or $\zeta = \frac{1}{\omega_n}\ln\left(\frac{X_o}{X_n}\right) = .06049$

With r & ζ we can now find the amplitude given by

$$X = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{10(.012)}{50} \frac{(3.7005)^2}{\sqrt{[1-(3.7005)^2]^2+[2\cdot(.06049)(3.7005)]^2}} = .002588 \text{ m}$$

To do this problem you needed r & ζ. r was found from ω_n & ω (ω_n was found from Δ_{ST}) and ζ came from the information of the free vibration part.

3.27 Given $X|_{r=1} = .012$ m $= \frac{m_o e}{m}\frac{1}{2\zeta}$. As $r \to \infty$ $X|_{r\to\infty} = .001$ m $= \frac{m_o e}{m}$

$\Rightarrow 2\zeta = \frac{1}{12}$ or $\zeta = .0417$ $\left(\frac{X_{r\to\infty}}{X_{r=1}} = \frac{1}{12} = 2\zeta\right)$

3.30 From the free vibrations part get the logarithmic decrement $\delta = \frac{1}{n}\ln\left(\frac{X_o}{X_n}\right) = \frac{1}{10}\ln\left(\frac{45}{2}\right)$

$\therefore \delta = .31135$ also $\zeta = \frac{\delta}{\sqrt{\delta^2+4\pi^2}} = .04949$.

Also from above 1 sec. $= \Delta\tau = \frac{1}{\zeta\omega_n}\ln\left(\frac{X_o}{X_n}\right)$ or $\omega_n = \frac{1}{\zeta}\ln\left(\frac{X_o}{X_n}\right) = 62.912$

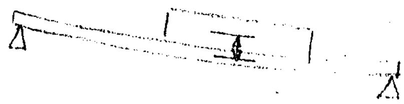
or from diagram which shows $f_d = 10$ Hz (10 cycles/s) $\Rightarrow 20\pi$ rad/s $= \omega_d$ & $\omega_n = \frac{\omega_d}{\sqrt{1-\zeta}}$

i) at resonance $\frac{X m}{m_o e} = \frac{1}{\sqrt{(1-1)^2+[2(0.04949)\cdot 1]^2}} = \frac{1}{2\zeta} = 10.1031$

ii) at 1750 rpm From problem 3.26 1750 rpm $= 183.26$ rad/s $= \omega$.

and $r = \frac{\omega}{\omega_n} = \frac{183.26}{62.912} = 2.913$ and $\frac{X m}{m_o e} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = 1.137$

3.26



$\Delta_{STATIC} = .004\,m \quad \therefore \quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}} = \sqrt{\frac{g}{\Delta_{ST}}} = 49.523\ rad/s$

Given: $m = 50\ kg$, $m_o = 10\ kg$, $e = .012m$ and $\omega = 1750\ rpm \times \frac{2\pi}{60} = 183.26\ rad/s$.

$\Rightarrow r = \frac{\omega}{\omega_n} = 3.7005$. Also given for the free vibrations part that $X_o = 40mm$ and $X_n = 2mm$ and for n cycles it takes 1 sec ($= \Delta\tau$)

$\therefore \quad 1\,sec = \Delta\tau = \frac{1}{\zeta\omega_n} \ln\left(\frac{X_o}{X_n}\right)$ or $\zeta = \frac{1}{\omega_n} \ln\left(\frac{X_o}{X_n}\right) = .06049$

With r & ζ we can now find the amplitude given by

$$X = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{10(.012)}{50} \frac{(3.7005)^2}{\sqrt{[1-(3.7005)^2]^2 + [2\cdot(.06049)(3.7005)]^2}} = .002588\,m$$

To do this problem you needed r & ζ. r was found from ω_n & ω (ω_n was found from Δ_{ST}) and ζ came from the information of the free vibrations part.

3.27 Given $X\big|_{r=1} = .012\,m = \frac{m_o e}{m}\frac{1}{2\zeta}$. As $r \to \infty$ $X\big|_{r\to\infty} = .001\,m = \frac{m_o e}{m}$

$\Rightarrow 2\zeta = \frac{1}{12}$ or $\zeta = .0417$ $\left(\frac{X_{r\to\infty}}{X_{r=1}} = \frac{1}{12} = 2\zeta\right)$

3.30 From the free vibrations part get the logarithmic decrement $\delta = \frac{1}{n} \ln\left(\frac{X_o}{X_n}\right) = \frac{1}{10} \ln\left(\frac{45}{2}\right)$

$\therefore \quad \delta = .31135$ also $\zeta = \frac{\delta}{\sqrt{\delta^2 + 4\pi^2}} = .04949$.

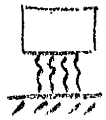
Also from above $1\,sec = \Delta\tau = \frac{1}{\zeta\omega_n} \ln\left(\frac{X_o}{X_n}\right)$ or $\omega_n = \frac{1}{\zeta} \ln\left(\frac{X_o}{X_n}\right) = 62.912\ \frac{rad}{s}$

or from diagram which shows $f_d = 10\,Hz$ (10 cycles/s) $\Rightarrow 20\pi\ rad/s = \omega_d$ & $\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}}$

i) at resonance $\frac{X m}{m_o e} = \frac{1}{\sqrt{(1-1)^2 + [2(0.04949)\cdot 1]^2}} = \frac{1}{2\zeta} = 10.1031$

ii) at 1750 rpm From problem 3.2[illegible] 1750 rpm = 183.26 rad/s = ω

and $r = \frac{\omega}{\omega_n} = \frac{183.26}{[illegible]} = 2.9[illegible]$

3.31 

$\Delta_{ST} = .045\,m$ $m = 380\,kg$ and $\zeta \cong 0$. Given $m_0 e = 0.15\ kg\cdot m$

find X @ 1750 rpm and $F_{TRANS} = kX$

From 3.26 1750 rpm = 183.26 rad/s = ω From Δ_{ST} $\omega_n = \sqrt{\dfrac{g}{\Delta_{ST}}} = 14.765$

$r = \dfrac{\omega}{\omega_n} = 12.412$. $X\Big|_{\zeta=0} = \dfrac{m_0 e}{m}\left|\dfrac{r^2}{1-r^2}\right| = \dfrac{0.15}{380}\ \dfrac{(12.412)^2}{1-(12.412)^2} = .0003973\,m$

$F_{TRANS} = kX = \omega_n^2 m X = 32.914\,N$

3.56 Given an accel. w/ $f_n = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 80\,Hz$ and $c = 8$ N-s/m. Given structure w/ $Y\omega^2 = \ddot{y} = 7.5\ m/s^2$ and $\frac{\omega}{2\pi} = f = 50\,Hz$ $\quad z_p\omega_n^2 = 8\ m/s^2$. Find k & m.

Now $\left|z_p\omega_n^2\right| = Y\omega^2 \cdot \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ and $\frac{z_p\omega_n^2}{Y\omega^2} = \frac{z_p}{Y}\frac{1}{r^2} = \frac{8}{7.5} = 1.067 = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

but $r = \frac{\omega}{\omega_n} = \frac{f}{f_n} = \frac{50}{80} = .625$. So we can solve for $\zeta = \sqrt{\left[\left(\frac{1}{1.067}\right)^2 - (1-r^2)^2\right]\frac{1}{4r^2}} = .57$

but $\zeta = \frac{c}{c_c}$ $\quad\therefore\ c_c = \frac{c}{.57} = 14.035\ \frac{N\text{-}s}{m} \Rightarrow$ $\quad c_c = 2m\omega_n$ or $m = \frac{c_c}{2\omega_n}$

$\omega_n = 2\pi f_n = 502.655$ rad/s $\quad\therefore\ m = \frac{c_c}{2\omega_n} = \underline{.014\ kg}$ and $k = \omega_n^2 m = \underline{3527.4\ N/m}$

3.57 Given $\Delta_{sr} = \frac{W}{k} = .02\,m$. Vibro. records $Z = .00002\ m$ with $f = \frac{\omega}{2\pi} = 100\,Hz$
find Y, $Y\omega$, $Y\omega^2$.

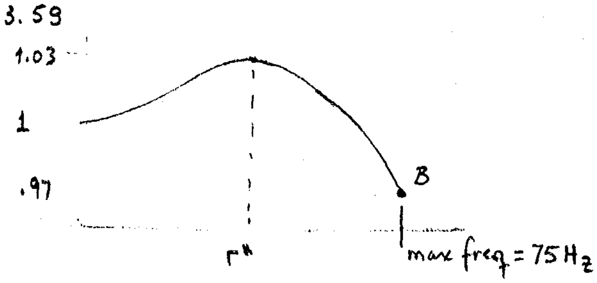
$2\pi f = \omega = 628.3185$ rad/s $\quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}} = \sqrt{\frac{g}{\Delta_{sr}}} = 22.147$ rad/s $\quad r = \frac{\omega}{\omega_n} = 28.37$

Choose $\zeta = 0$ then

$Z = Y\frac{r^2}{r^2-1} = Y(1.001) \Rightarrow Y = 1.9975\times10^{-5}\ m$

$Y\omega = .01255\ m/s$ and $Y\omega^2 = 7.886\ m/s^2$

3.59



From the data given the $\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ curve for the range of frequencies must be such that if it peaks, it must peak within the 3% error band. And the max frequency must also lie within the band (See fig 3.27 & Examples 3.10, 3.11)

If the peak of the curve is such that it is at 3% of $\left|\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\right| = 1$ then

$\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ is a max at $r^* = \sqrt{1-2\zeta^2}$ and the max of this function is $\frac{1}{2\zeta\sqrt{1-\zeta^2}} = 1.03$

from this $\zeta = .6164$ or $.7874$. Only $\zeta = .6164$ will produce an $r^* = \sqrt{1-2\zeta^2}$ which is real

and $r^* = .49$. For this value of $\zeta = .6164$ use $\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = .97$ to find r_{max}

THIS REPRESENTS PT B ON GRAPH

this gives $r = .7662$ as the only solution. The 75 Hz is the frequency of the oscillating body $\therefore f/f_n = \omega/\omega_n = r \Rightarrow f_n = f/r = 97.886$ Hz. $\Rightarrow \omega_n = 615.0352$ rad/s

Since $\zeta = \frac{c}{c_c} = .6164$ and $c = 50$ N-s/m $\Rightarrow c_c = c/\zeta = 81.12 \frac{N\text{-}s}{m} = 2m\omega_n$

$\therefore m = \frac{c_c}{2\omega_n} = .066$ kg and $\omega_n^2 m = k = 24944.65$ N/m.

3.61 Since $500 < \text{speed} < 2000 \Rightarrow \omega_1 = \frac{500 \times 2\pi}{60} < \omega < \frac{2000 \times 2\pi}{60} = \omega_2$ and for a vibrometer r must be large $\therefore \frac{\omega_1}{\omega_n} = r_1 < r_2 = \frac{\omega_2}{\omega_n}$. So if $\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sim 1$ for r_1, then it will certainly satisfy this requirement at r_2

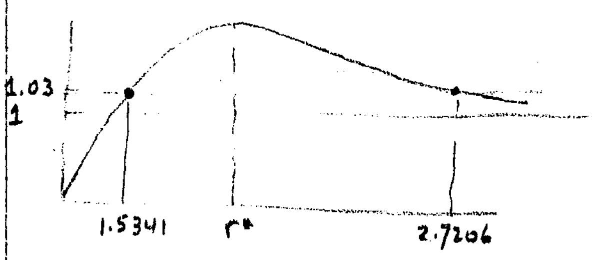
For the error to be less than 3% let $\frac{Z}{Y} = 1.03$

i) for $\zeta = 0$ $\quad \frac{Z}{Y} = \frac{r_1^2}{r_1^2 - 1} = 1.03$ since $r \gg 1$ solving for $r_1 = 5.8595$

now $r = \frac{\omega_1}{\omega_n}$ or $\underline{\omega_n} = \frac{\omega_1}{r} = \frac{500 \times 2\pi}{60\,(5.8595)} = \underline{8.9359 \text{ rad/s}}$ for $\zeta = 0$

ii) for $\zeta = 0.6$ $\quad \frac{Z}{Y} = 1.03 = \frac{r_1^2}{\sqrt{(1-r_1^2)^2+(2\zeta r_1)^2}}$ solving for $r_1 = 1.5341$ and $r_1 = 2.7206$

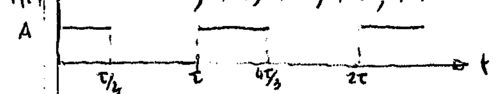
now $\left(\frac{Z}{Y}\right)_{max}$ occurs at $r^* = \frac{1}{\sqrt{1-2\zeta^2}} = 1.8898$. Thus the value of r_1 must be $> r^*$ since $\left(\frac{Z}{Y}\right)_{r^*}$ is > 1.03 violating the 3% requirement.



Since $r_1 = 2.7206 = \frac{\omega_1}{\omega_n} \Rightarrow \omega_n = \frac{\omega_1}{r_1}$

$\underline{\omega_n = 19.24571 \frac{rad}{s}}$

4.2, 4.5, 4.8, 4.15, 4.17



4.2 Since it takes τ seconds for 1 period then $\omega = \frac{2\pi}{\tau}$ and $z = \omega t = \frac{2\pi}{\tau} t$

Since
$$P(t) = \begin{cases} A & 0 \le t < \tau/3 \quad \text{or } 0 \le z \le \frac{2\pi}{3} \\ 0 & \frac{\tau}{3} < t \le \tau \quad \frac{2\pi}{3} \le z \le 2\pi \end{cases}$$

then

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} P\,dz \quad ; \text{ when } t = \tau/3 \quad z = \frac{2\pi}{\tau}\cdot\frac{\tau}{3} = \frac{2\pi}{3}$$

$$\therefore \quad a_0 = \frac{1}{2\pi}\int_0^{\frac{2\pi}{3}} A\,dz = \frac{A}{2\pi} z\Big|_0^{\frac{2\pi}{3}} = \frac{A}{3}$$

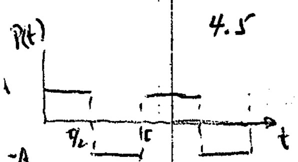
$$a_l = \frac{1}{\pi}\int_0^{2\pi} P\cos lz\,dz = \frac{1}{\pi}\int_0^{\frac{2\pi}{3}} \frac{A}{l}\cos(lz)\,(l\,dz) = \frac{1}{\pi}\frac{A}{l}\sin lz\Big|_0^{\frac{2\pi}{3}} = \frac{A}{\pi l}\sin\frac{2\pi l}{3}$$

$$b_l = \frac{1}{\pi}\int_0^{2\pi} P\sin lz\,dz = \frac{1}{\pi}\int_0^{\frac{2\pi}{3}} \frac{A}{l}\sin(lz)\,(l\,dz) = \frac{-1}{\pi}\frac{A}{l}\cos lz\Big|_0^{\frac{2\pi}{3}} = \frac{A}{\pi l}\left[1-\cos\frac{2\pi}{3}l\right]$$

Now
$$P(t) = a_0 + \sum a_l \cos lz + \sum b_l \sin lz = A\left[\frac{1}{3} + \sum\frac{A}{\pi l}\sin\frac{2\pi l}{3}\cos lz + \sum\frac{A}{\pi l}\left[1-\cos\frac{2\pi l}{3}\right]\cdot \sin lz\right]$$

$$\therefore \quad x(t) = \frac{a_0}{k} + \sum\frac{a_l/k}{\sqrt{(1-r_l^2)^2+(2\zeta r_l)^2}}\cos(l\omega t - \psi_l) + \sum\frac{b_l/k}{\sqrt{(1-r_l^2)^2+(2\zeta r_l)^2}}\sin(l\omega t - \psi_l)$$

where
$$\tan\psi_l = \frac{2\zeta r_l}{1-r_l^2}$$



4.5 Using the same idea as above $z = \frac{2\pi}{\tau} t$

Since
$$P(t) = \begin{cases} A & 0 \le t \le \tau/2 \Rightarrow 0 \le z \le \pi \quad @\, t = \tau/2 \;\; z = \frac{2\pi}{\tau}\cdot\frac{\tau}{2} = \pi \\ -A & \tau/2 \le t \le \tau \Rightarrow \pi \le z \le 2\pi \quad @\, t = \tau \;\; z = \frac{2\pi}{\tau}\cdot\tau = 2\pi \end{cases}$$

since P(t) is an odd fn $\Rightarrow$ all the a_l terms are zero

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} P\,dz = \frac{1}{2\pi}\int_0^{\pi} A\,dz + \frac{1}{2\pi}\int_{+\pi}^{2\pi} -A\,dz = \frac{1}{2\pi}Az\Big|_0^{\pi} - \frac{1}{2\pi}Az\Big|_{\pi}^{2\pi} = 0 \checkmark$$

$$a_l = \frac{1}{\pi}\int_0^{2\pi} P\,dz\cos lz = \frac{1}{\pi}\int_0^{\pi} A\cos lz\,dz - \frac{1}{\pi}\int_{\pi}^{2\pi} A\cos lz\,dz = \frac{A}{\pi l}\left[\sin lz\Big|_0^{\pi} - \sin lz\Big|_{\pi}^{2\pi}\right] = 0 \checkmark$$

$$b_l = \frac{1}{\pi}\int_0^{2\pi} P\,dz\sin lz = \frac{1}{\pi}\int_0^{\pi} A\sin lz\,dz - \frac{1}{\pi}\int_{\pi}^{2\pi} A\sin lz\,dz = -\frac{A}{\pi l}\left[\cos lz\Big|_0^{\pi} - \cos lz\Big|_{\pi}^{2\pi}\right]$$

$$= -\frac{A}{\pi l}\left[(-1)^n - 1 - (1-(-1)^n)\right] = +\frac{2A}{\pi l}\left[1-(-1)^n\right]$$

$$\therefore \quad P(t) = \sum_{l=1,3,5,7\ldots} \frac{2A}{\pi l} \cdot 2 \sin lz \qquad \text{or if } l = 2\nu - 1$$

or $$P(t) = \sum_{\nu=1}^{\infty} \frac{4A}{\pi[2\nu-1]} \sin\left[(2\nu-1)\omega t\right] \qquad \text{since } z = \omega t = \frac{2\pi t}{\tau}$$

Now $$x(t) = \sum_{l=1,3,5\ldots} \frac{b_l/k}{\sqrt{(1-r_l^2)^2 + (2\zeta r_l)^2}} \sin(l\omega t - \psi_l)$$

$$= \frac{4A}{\pi k} \sum_{l=1,3,5\ldots} \frac{1}{l} \frac{1}{\sqrt{(1-r_l^2)^2 + (2\zeta r_l)^2}} \sin(l\omega t - \psi_l)$$

4.8 $\omega = 3000 \text{ rpm} \times \frac{2\pi}{60} \frac{\text{rad/s}}{\text{rpm}} = 314.159 \text{ rad/s}$ and period $\tau = \frac{2\pi}{\omega} = .02$ sec

Since the power is generated only by $11/12$ of cycle. Average torque = $\frac{\text{Power}}{\text{angular veloc.}}$
[Remember $Fv = \text{Power} = Fr \cdot \frac{v}{r} = M\omega$] Thus $T_{AVE} = \frac{500 \text{ kW}}{314.159 \text{ rad/s}} = 1591.5457$ N-m
with 1 kW = 1000 W = 1000 N·m/s. This torque is not the total torque $T_0 = \frac{12}{11} T_{AVE} = 1736.232$ N-m

$$T = a_0 + \sum a_{\bar{l}} \cos \bar{l} z + \sum b_{\bar{l}} \sin \bar{l} z \quad \text{with } z = \omega t, \; T = \begin{cases} T_0 & 0 \le t \le \frac{11}{12}\tau \\ 0 & \frac{11}{12}\tau \le t \le \tau \end{cases}$$

$a_0 = \frac{1}{2\pi}\int_0^{2\pi} T\,dz$ where $0 \le t \le \frac{11}{12}\tau \Rightarrow 0 \le z \le \frac{11}{6}\pi$

$\frac{11}{12}\tau \le t \le \tau \Rightarrow \frac{11}{6}\pi \le z \le 2\pi$

thus $$a_0 = \frac{1}{2\pi}\int_0^{\frac{11}{6}\pi} T_0\,dz = \frac{11}{12} T_0$$

$$a_{\bar{l}} = \frac{1}{\pi}\int_0^{2\pi} T \cos \bar{l} z\,dz = \frac{1}{\pi}\int_0^{\frac{11}{6}\pi} T_0 \cos \bar{l} z\,dz = \frac{1}{\pi \bar{l}} T_0 \sin \bar{l} z \Big|_0^{\frac{11\pi}{6}} = \frac{1}{\pi \bar{l}} T_0 \sin \frac{11\pi \bar{l}}{6}$$

$$b_{\bar{l}} = \frac{1}{\pi}\int_0^{2\pi} T \sin \bar{l} z\,dz = \frac{1}{\pi}\int_0^{\frac{11}{6}\pi} T_0 \sin \bar{l} z\,dz = \frac{-1}{\pi \bar{l}} T_0 \cos \bar{l} z \Big|_0^{\frac{11\pi}{6}} = \frac{T_0}{\pi \bar{l}}\left[1 - \cos \frac{11\pi \bar{l}}{6}\right]$$

here $\bar{l}$ is the integer counter. Now $I_0 \ddot{\theta} + k_t \theta = T$ $\quad \omega_n = \sqrt{\frac{k_t}{I_0}}$
$k_t = \frac{JG}{l} = \frac{\pi d^4 G}{32\, l}$ with $G_{steel} = 79.3 \times 10^9$ N/m². This gives $k_t = 7.7853 \times 10^5$ N-m
put into above to get ω_n with $I_0 = 1$ N-m-s² and $\omega_n = 882.3435$ rad/s

SEE PAGE 4 FOR REST

4.17 From our work in class

$$x(t) = \frac{1}{m\omega_n}\int_0^t P(\tau)\sin\omega_n(t-\tau)\,d\tau \qquad \text{where } P(t) = F_0 \quad 0\le t\le t_0$$
$$P(t) = 0 \quad t_0 \le t \le \infty$$

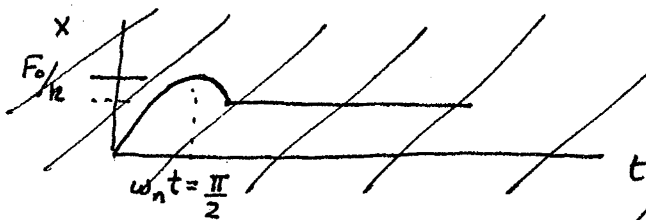
$$x(t) = \frac{1}{m\omega_n}\int_0^t F_0 \sin\omega_n(t-\tau)\,d\tau \qquad \text{for } \underline{t\le t_0}$$

$$= \frac{1}{m\omega_n}\,\frac{+F_0}{\omega_n}\cos\omega_n(t-\tau)\Big|_0^t = \frac{F_0}{m\omega_n^2}\left[1-\cos\omega_n t\right] = \frac{F_0}{k}\left[1-\cos\omega_n t\right]$$

for $t > t_0$

$$x(t) = \frac{1}{m\omega_n}\int_0^{t_0} F_0 \sin\omega_n(t-\tau)\,d\tau + \frac{1}{m\omega_n}\int_{t_0}^t 0\cdot\sin\omega_n(t-\tau)\,d\tau.$$

$$= \frac{1}{m\omega_n^2}F_0\left[\overbrace{}^{\cos\omega_n(t-t_0)} - \cos\omega_n t\right].$$



~~This is an example where $\frac{\pi}{2\omega_n} \le t_0 \le \frac{\pi}{\omega_n}$. If t_0 where not in that range, the results would be different.~~

4.15 From ~~above~~ work in class the governing equation is $m\ddot{x}+kx=0 \quad t<0$
and the mass is at rest $x=0,\ \dot{x}=0$ for $t<0$ before the force is applied
thus the solution for is the above

$$x(t) = \frac{1}{m\omega_n}\int_0^t P(\tau)\sin\omega_n(t-\tau)\,d\tau \qquad P(t) = \begin{cases}\frac{F_0}{2}(1-\cos\omega t) & 0\le t\le \frac{\pi}{\omega}\\ F_0 & t>\pi/\omega\end{cases}$$

for $t > \pi/\omega$

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^{\pi/\omega}\frac{F_0}{2}(1-\cos\omega\tau)\sin\omega_n(t-\tau)\,d\tau + \int_{\pi/2\omega}^t F_0\sin\omega_n(t-\tau)\,d\tau\right]$$

$$= \frac{F_0}{2m\omega_n}\int_0^{\pi/\omega}\Big\{\underbrace{\sin\omega_n(t-\tau)\,d\tau}_{①} - \cos\omega\tau\underbrace{\sin\omega_n(t-\tau)\,d\tau}_{②}\Big\} + \frac{F_0}{m\omega_n}\int_{\pi/\omega}^t \underbrace{\sin\omega_n(t-\tau)\,d\tau}_{③}$$

terms ① and ③ are easy to integrate: $\int \sin\omega_n(t-\tau)\,d\tau = \frac{\cos\omega_n(t-\tau)}{\omega_n}$

the second term: use the fact that $\sin\omega_n(t-\tau) = \sin\omega_n t\cos\omega_n\tau - \sin\omega_n\tau\cos\omega_n t$

note that $\sin\omega_n t$ & $\cos\omega_n t$ are constants here since the variable of integration is τ.

$$\therefore \cos\omega\tau \sin\omega_n(t-\tau)\,d\tau = \left[\sin\omega_n t \cos\omega_n\tau\cos\omega\tau - \cos\omega_n t \sin\omega_n\tau\cos\omega\tau\right]d\tau$$

and $$\cos\omega_n\tau\cos\omega\tau = \frac{1}{2}\left[\cos(\omega_n+\omega)\tau + \cos(\omega_n-\omega)\tau\right]$$

$$\sin\omega_n\tau\cos\omega\tau = \frac{1}{2}\left[\sin(\omega_n+\omega)\tau + \sin(\omega_n-\omega)\tau\right]$$

$$\therefore \int \cos\omega\tau \sin\omega_n(t-\tau)\,d\tau = \frac{\sin\omega_n t}{2}\underbrace{\left[\frac{\sin(\omega_n+\omega)\tau}{\omega_n+\omega} + \frac{\sin(\omega_n-\omega)\tau}{\omega_n-\omega}\right]}_{A} + \frac{\cos\omega_n t}{2}\underbrace{\left[\frac{\cos(\omega_n+\omega)\tau}{\omega_n+\omega} + \frac{\cos(\omega_n-\omega)\tau}{\omega_n-\omega}\right]}_{B}$$

$$\text{thus}\quad x(t) = \frac{F_0}{2m\omega_n^2}\cos\omega_n(t-\tau)\Big|_0^{\pi/\omega} + \frac{F_0}{m\omega_n^2}\cos\omega_n(t-\tau)\Big|_{\pi/\omega}^{t} - \frac{F_0}{2m\omega_n}\left\{\frac{\sin\omega_n t}{2}\left[A\right] + \frac{\cos\omega_n t}{2}\left[B\right]\right\}_0^{\pi/\omega}$$

if $\omega=\omega_n$, the second terms in A and B will exhibit resonance

if $\omega\neq\omega_n$

$$x = \frac{F_0}{2k}\left[\cos\omega_n(t-\pi/\omega) - \cos\omega_n t\right] + \frac{F_0}{m\omega_n^2}\left[1-\cos\omega_n(t-\pi/\omega)\right] - \frac{F_0\sin\omega_n t}{4m\omega_n}\left[\frac{\sin\pi(\omega_n+\omega)/\omega}{\omega_n+\omega} + \frac{\sin\pi(\omega_n-\omega)/\omega}{\omega_n-\omega}\right]$$

$$- \frac{F_0}{4m\omega_n}\cos\omega_n t\left[\frac{\cos\pi(\omega_n+\omega)/\omega}{\omega_n+\omega} + \frac{\cos\pi(\omega_n-\omega)/\omega}{\omega_n-\omega} - \left(\frac{1}{\omega_n+\omega} + \frac{1}{\omega_n-\omega}\right)\right]$$

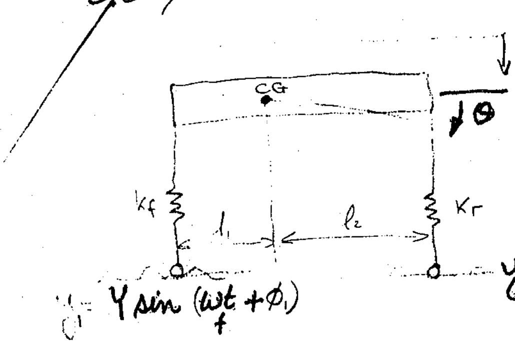
4.8 (CONT) FOR THE DISPLACEMENT

$$x = \frac{a_0}{k_t} + \sum \frac{a_i/k_t \cos(i\omega t - \psi_i)}{\sqrt{(1-r_i^2)^2 + (2\zeta r_i)^2}} + \sum \frac{b_i/k_t \sin(i\omega t - \psi_i)}{\sqrt{(1-r_i^2)^2 + (2\zeta r_i)^2}}$$

where $\zeta = 0$: $\psi_i = 0$

$$\therefore\quad x = \frac{a_0}{k_t} + \frac{1}{k_t}\sum \frac{a_i\cos(i\omega t) + b_i\sin(i\omega t)}{1 - r_i^2} \qquad \text{where } r_i = i\frac{\omega}{\omega_n}$$

5/34



$y_2 = Y\sin(\omega_f t + \phi_2)$

$m = 1000\,kg$

$I = mr^2 = 1000(.9)^2 = 810$

$Y = 0.05$

$l_1 = 1m$

$l_2 = 1.5m$

$k_f = 18000\ N/m$

$k_r = 22000\ N/m$

$m\ddot{x} = -k_f(x - l_1\theta) - k_r(x + l_2\theta) + F_1(t) + F_2(t)$

$I\ddot{\theta} = k_f(x - l_1\theta)l_1 - k_r(x + l_2\theta)l_2 + T_1(t) + T_2(t)$

$F_i(t) = k_i y_i$

$T_i(t) = k_i y_i l_i (-1)^i$

$\lambda = 10m$ period of ground variation

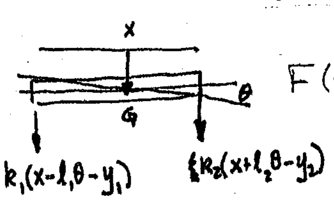
$V = 50\,km/h = 50\times10^3 \frac{m}{h}\left(\frac{h}{3600s}\right) = 13.889\ m/s$ ✓

period to travel on cycle $\tau = \frac{\lambda}{V} = \frac{10m}{13.889\,m/s} = 0.720$ sec. $\omega_f = \frac{2\pi}{\tau} = 8.727$ rad/sec ✓

$y_1 = 0.05\sin(8.727t)$ ✓ $\phi_1 = 0$ ✓

$y_2 = 0.05\sin(8.727t + \phi_2)$ $\phi_2 = \frac{2\pi}{\lambda/(l_1+l_2)} = \frac{2\pi}{10/(1+1.5)} = \pi/2 = 1.5708$ ✓

$y_2 = 0.05\sin(8.727t - 1.5708)$ ✓ since rear wheels motion lags front wheel motion



$F(t) = k_f y_1 + k_r y_2$

$= (18000)(0.05\sin(8.727t)) + (22000)(0.05\sin(8.727t - 1.5708))$

$= 900\sin(8.727t) + 1100\sin(8.727t - 1.5708)$ ✓

$T(t) = -k_f y_1 l_1 + k_r y_2 l_2$

$= -(18000)(0.05\sin 8.727t)(1) + (22000)(0.05\sin(8.727t - 1.5708))(1.5)$

$= 1650\sin(8.727t - 1.5708) - 900\sin 8.727t$ ✓

$m\ddot{x} + k_f(x - l_1\theta) + k_r(x +$ $F(t)$

$m\ddot{x} + k_f x - k_f l_1\theta + k_r x +$ $F(t)$

$m\ddot{x} + (k_f + k_r)x + (k_r l_2 - l_1 k_f$ $F(t)$

$1000\ddot{x} + (18000 + 22000)x$ $000(1.5) - 18000(1))\theta = F(t)$

$$\boxed{1000\ddot{x} + 40000x + 15000\theta \quad 900\sin(8.727t) + 1100\sin(8.727t - 1.5708)}$$

✓

$I\ddot{\theta} - k_f(x - l_1\theta)l_1 + k_r(x + l_2)l_2 = T(t)$

$I\ddot{\theta} - k_f x l_1 + k_f l_1^2\theta + k_r x l_2 + k_r\theta = T(t)$

$I\ddot{\theta} + (k_r l_2 - k_f l_1)x + (k_f l_1^2 + k_r l_2^2)\theta = T(t)$

$810\ddot{\theta} + (22000(1.5) - 18000(1))x + (18000(1)^2 + 22000(1.5)^2)\theta = 1650\sin(8.727t - 1.5708)$
$-900\sin(8.727t)$

$$\boxed{810\ddot{\theta} + 15000x + 67500\theta = 1650\sin(8.727t - 1.5708) - 900\sin(8.727t)}$$

✓

1. $u'' + \mu u' + \omega_0^2 u = \cos\omega t$ w/. $\mu^2 - 4\omega_0^2 < 0$ let $\lambda = \frac{\sqrt{4\omega_0^2 - \mu^2}}{2}$

let $y_h = ce^{rt}$ char eq is $r^2 + \mu r + \omega_0^2 = 0$

$$r = \frac{-\mu \pm \sqrt{\mu^2 - 4\omega_0^2}}{2} = -\frac{\mu}{2} \pm i\lambda$$

thus $y_h = e^{-\mu t/2}\left[c_1 \cos\lambda t + c_2 \sin\lambda t\right]$

for y_p let $\lambda \neq \omega$. $y_p = A_1 \cos\omega t + A_2 \sin\omega t$

$y_p' = \omega\left[-A_1 \sin\omega t + A_2 \cos\omega t\right]$ $y_p'' = -\omega^2\left[A_1 \cos\omega t + A_2 \sin\omega t\right]$

$$y_p'' + \mu y_p' + \omega_0^2 y_p = \cos\omega t\underbrace{\left[-\omega^2 A_1 + \mu\omega A_2 + \omega_0^2 A_1\right]}_{=1} + \sin\omega t\underbrace{\left[-\omega^2 A_2 - \mu\omega A_1 + \omega_0^2 A_2\right]}_{=0}$$

$A_2\left[\omega_0^2 - \omega^2\right] - \mu\omega A_1 = 0$

$A_2 \mu\omega + \left[\omega_0^2 - \omega^2\right]A_1 = 1$

$$A_1 = \frac{\begin{bmatrix} 0 & \omega_0^2 - \omega^2 \\ 1 & \mu\omega \end{bmatrix}}{\begin{bmatrix} -\mu\omega & \omega_0^2 - \omega^2 \\ \omega_0^2 - \omega^2 & \mu\omega \end{bmatrix}} = \frac{+(\omega_0^2 - \omega^2)}{+(\omega_0^2 - \omega^2)^2 + \mu^2\omega^2}$$

$$A_2 = \frac{\begin{bmatrix} -\mu\omega & 0 \\ \omega_0^2 - \omega^2 & 1 \end{bmatrix}}{-(\omega_0^2 - \omega^2)^2 + \mu^2\omega^2} = \frac{\mu\omega}{(\omega_0^2 - \omega^2)^2 + \mu^2\omega^2}$$

$$\therefore \quad y_p = \frac{\mu\omega \sin\omega t + (\omega_0^2 - \omega^2)\cos\omega t}{(\omega_0^2 - \omega^2)^2 + \mu^2\omega^2}$$

$$y = y_p + y_h = e^{-\mu t/2}\left[c_1 \cos\lambda t + c_2 \sin\lambda t\right] + \frac{\mu\omega \sin\omega t + (\omega_0^2 - \omega^2)\cos\omega t}{(\omega_0^2 - \omega^2)^2 + \mu^2\omega^2}$$

2. a) $y'' - 5y' + 6y = 2e^x$ char eq for homog ODE is $r^2 - 5r + 6 = 0 \Rightarrow (r-3)(r-2)$

$\therefore \quad y_h = c_1 e^{2x} + c_2 e^{3x}$

let $y_p = u_1 y_1 + u_2 y_2$

$u_1' e^{2x} + u_2' e^{3x} = 0$

$2u_1' e^{2x} + 3u_2' e^{3x} = 2e^x$

$$\Rightarrow u_1' = \frac{\begin{vmatrix} 0 & e^{3x} \\ 2e^x & 3e^{3x} \end{vmatrix}}{\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}} = \frac{-2e^{4x}}{3e^{5x} - 2e^{5x}} = \frac{-2e^{4x}}{1e^{5x}} = -2e^{-x}$$

$$u_1 = \int^x -2e^{-t}\,dt = +2e^{-x}$$

$$u_2' = \frac{\begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 2e^{x} \end{vmatrix}}{1\, e^{5x}} = \frac{2e^{3x}}{e^{5x}} = 2e^{-2x} \qquad u_2 = -e^{-2x}$$

$$\therefore \quad y_p = u_1 y_1 + u_2 y_2 = 2e^{-x}\left[e^{2x}\right] - e^{-2x}\left[e^{3x}\right]$$

$$= 2e^{x} - e^{x} = e^{x}$$

$$\therefore \quad y = y_p + y_h = e^{x} + C_1 e^{2x} + C_2 e^{3x}$$

2b) $y'' + y = \sec x \qquad y_h = C_1 \cos x + C_2 \sin x$

let $y_p = u_1 \cos x + u_2 \sin x$

$$u_1' \cos x + u_2' \sin x = 0$$

$$-u_1' \sin x + u_2' \cos x = \sec x$$

$$u_1' = \frac{\begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix}}{\begin{vmatrix} c & s \\ -s & c \end{vmatrix}} = \frac{-\tan x}{1} \qquad u_1 = \ln \cos x$$

$$u_2' = \frac{\begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix}}{1} = 1 \qquad u_2 = x$$

$$\therefore \quad y_p = (\ln \cos x)\cos x + x \sin x$$

$$y = y_p + y_h = C_1 \cos x + C_2 \sin x + \cos x(\ln \cos x) + x \sin x$$

3. $y'' - 5y' + 6y = g(x)$ the homog ODE has a solution of $y = C_1 e^{2x} + C_2 e^{3x}$ from (2a)

thus $y_p = u_1 e^{2x} + u_2 e^{3x}$

$$u_1' e^{2x} + u_2' e^{3x} = 0$$

$$2u_1' e^{2x} + 3u_2' e^{3x} = g(x)$$

thus $W(e^{2x}, e^{3x}) = e^{5x}$

$$u_1' = \frac{\begin{vmatrix} 0 & e^{3x} \\ g & 3e^{3x} \end{vmatrix}}{e^{5x}} = \frac{-g(x)e^{3x}}{e^{5x}} = -g(x)e^{-2x}$$

$$u_2' = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & g \end{vmatrix} = g(x)e^{2x} = g(x)e^{-3x}$$

Thus

$$u_1 = \int^x -g(t)e^{-2t}dt \qquad u_2 = \int^x g(t)\, e^{-3t} dt$$

$$y_p = e^{2x}\int^x -g(t)e^{-2t}dt + e^{3x}\int^x e^{-3t}g(t)dt$$

$$y_p = \int^x g(t)\left[e^{3(x-t)} - e^{2(x-t)}\right]dt$$

let $g(x) = 2e^x \qquad g(t) = 2e^t$

$$y_p = 2\int^x e^t\left[e^{3(x-t)} - e^{2(x-t)}\right]dt$$

$$= 2\int^x\left[e^{3x-2t} - e^{2x-t}\right]dt$$

$$2\left[-\frac{e^{3x-2t}}{2} + e^{2x-t}\right]^x = 2\left[-\frac{e^x}{2} + e^x\right] = 2\frac{e^x}{2} = e^x \checkmark$$

4. if $c=0 \qquad m\ddot{x} + kx = F_0\cos\omega t \implies \ddot{u} + \frac{k}{m}u = \frac{F_0}{m}\cos\omega t \implies \ddot{u} + \omega_0^2 u = \frac{F_0}{m}\cos\omega t$

The homog eq is

$$u_h = C_1\cos\omega_0 t + C_2\sin\omega_0 t$$

from HW #2, #4 $\quad y_p = A\cos\omega t \qquad A = \frac{F_0}{m(\omega_0^2-\omega^2)}$

$$\therefore \quad u = C_1\cos\omega_0 t + C_2\sin\omega_0 t + \frac{F_0}{m(\omega_0^2-\omega^2)}\cos\omega t$$

$u(0)=0 \implies C_1 = -\frac{F_0}{m(\omega_0^2-\omega^2)}$

$\dot{u}(0)=0 \implies C_2 = 0$

$$\therefore \quad u(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}\left[\cos\omega t - \cos\omega_0 t\right] \quad ; \qquad \cos A - \cos B = 2\sin\frac{(A+B)}{2}\sin\frac{(B-A)}{2}$$

$$= \frac{2F_0}{m(\omega_0^2-\omega^2)}\left[\sin\frac{(\omega_0+\omega)}{2}t\,\sin\frac{(\omega_0-\omega)}{2}t\right]$$

let $\omega_0 = 1.1\omega \qquad \omega_0+\omega = 2.1\omega \qquad \omega_0-\omega = .1\omega$

$$u = \frac{2F_0}{m(.21\omega^2)}\left[\sin(1.05\omega t)\sin(.05\omega t)\right]$$

let $2\pi = .05\omega t$ $\qquad \frac{2\pi}{.05\omega} = t \longrightarrow \frac{2\pi}{.05\omega}(1.05\omega) = 1.05\omega t$

$$\underline{2\pi \cdot 21 = 1.05\omega t} \qquad \sin(1.05\omega t) = 0$$

let $\pi = .05\omega t$ $\qquad \frac{\pi}{.05\omega} = t \longrightarrow \frac{\pi}{.05\omega}(1.05\omega) = 1.05\omega t$

$$\underline{21\pi = 1.05\omega t} \qquad \sin(1.05\omega t) = 0$$

let $\frac{3\pi}{2} = .05\omega t$ $\qquad \frac{3\pi}{2(.05\omega)} = t \longrightarrow \frac{3\pi}{2(.05\omega)}(1.05\omega) = 1.05\omega t$

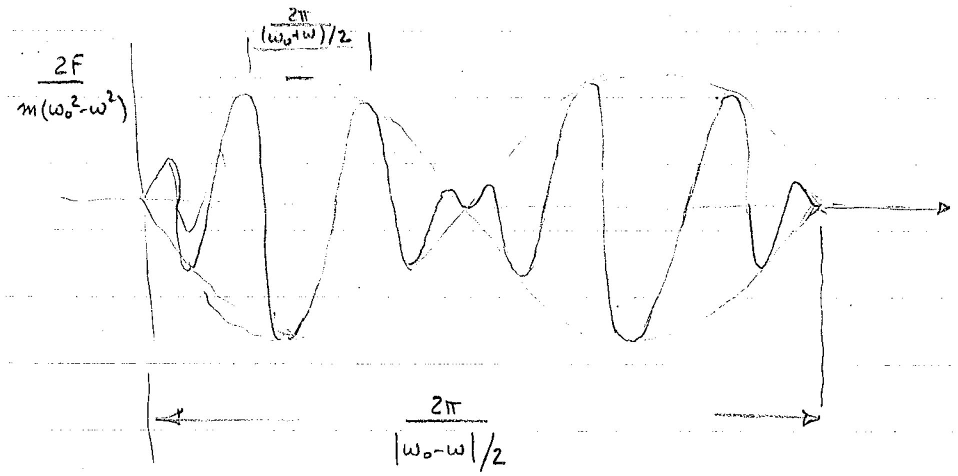
$$\underline{\frac{63\pi}{2} = 1.05\omega t} \qquad \sin(1.05\omega t) = \sin\frac{3\pi}{2} = -1$$

let $\pi/2 = .05\omega t$ $\qquad \frac{\pi}{2(.05\omega)} = t \longrightarrow \frac{\pi}{2(.05\omega)}(1.05\omega) = 1.05\omega t$

$$\frac{21\pi}{2} = 1.05\omega t. \Rightarrow \sin(1.05\omega t) = \sin\pi/2 = +1$$

for. $1.05\omega t = 2\pi \qquad t = \frac{2\pi}{1.05\omega} \qquad \omega t = \frac{2\pi}{1.05} \cong 2\pi = \frac{40}{21}\pi \qquad$ 1 cycle every $\cong 2\pi$

$.05\omega t = 2\pi \qquad \omega t = \frac{2\pi}{.05} \qquad \omega t = 40\pi$ – repeats. $\qquad$ 1 cycle every $\frac{1}{20}$

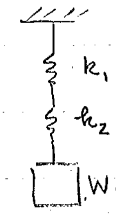


#4

2-3 $\quad m = 9\ kg \qquad k = 800\ N/m. \qquad \omega = \sqrt{\frac{k}{m}} = 9.43\ rad/sec$

$$\tau = \frac{1}{f} = \frac{2\pi}{\omega} = \frac{2\pi}{9.43} = .667\ sec$$

2-10

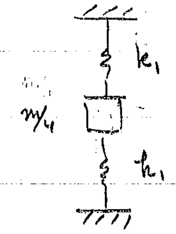


since in series $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} \Rightarrow k = \frac{k_1 k_2}{k_1 + k_2}$

since same material $\boxed{k = \frac{k_1}{2}}$

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 90\ cy/min = 1.5\ Hz$$

$$k_1 = 2[2\pi(1.5)]^2 m$$

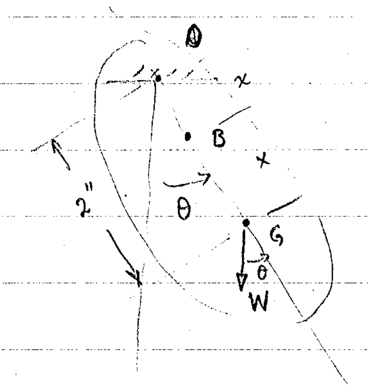


since is parallel $\qquad k' = 2k_1$

$$f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k'}{m'}} = \frac{1}{2\pi}\sqrt{\frac{2k_1}{m/4}} = \frac{1}{2\pi}\sqrt{8\frac{k_1}{m}} = \frac{1}{2\pi}\sqrt{16(2\pi)^2(1.5)^2}$$

$$= 4(1.5) = 6\ Hz$$

2-13.



$I_o\ddot{\theta} + W\sin\theta\, d = 0 \qquad d = 2''$

for small $\theta \qquad I_o\ddot{\theta} + Wd\theta = 0$

$$\omega = \sqrt{\frac{Wd}{I_o}}$$

$$f = \frac{1}{2\pi}\sqrt{\frac{Wd}{I_o}} = 54\ cy/min = .9\ Hz$$

$$\therefore \frac{W}{I_o} = \frac{[2\pi(.9)]^2}{d} = \frac{4\pi^2(.81)}{2} = 2\pi^2(.81)\ \frac{1}{sec^2}$$

$$\cong 15.989\ \frac{Hz^2}{in}$$

$I_o = I_G + md^2 \qquad \frac{I_o}{W} = \frac{I_G}{W} + \frac{d^2}{g}$

$$.0625\ \frac{in}{Hz^2} = \frac{I_G}{W} + .0104\ \frac{in}{Hz^2} \qquad \frac{I_G}{W} = .052192\ \frac{in}{Hz^2}$$

From the second condition $\quad f' = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{Wx}{I_B}} = 36\ \frac{cy}{min} = .6\ Hz\ ;\ I_B\ddot{\theta} + (W\sin\theta)x = 0.$

$$\therefore \frac{W}{I_B} = [2\pi(.6)]^2 = 14.21223\ Hz^2 \quad or \quad \frac{I_B}{W} = .07036x\ in$$

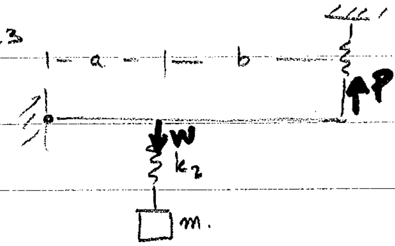
$$I_B = I_G + mx^2 \quad \text{or} \quad \frac{I_B}{W} = \frac{I_G}{W} + \frac{x^2}{g}$$

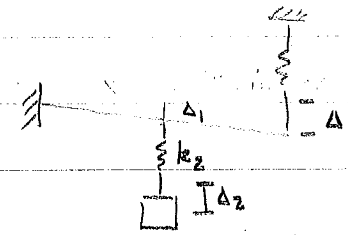
$$.07036\,x = .052192 + \frac{x^2}{(32.2)\,12}$$

or $$x^2 - 27.18785\,x + 20.167 = 0$$

Solving for x gives $\quad x = .7632\text{ in} \qquad 2 - x = \overline{OB} = \underline{\underline{1.2368\text{ in}}}$

2-23

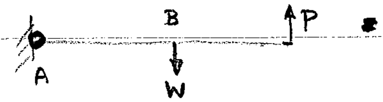


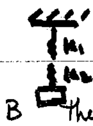


$\Delta_1 = \dfrac{a\Delta}{l}$

FOR STATIC EQUILIB:

$\sum M_A = 0 \Rightarrow \quad Pl - Wa = 0 \quad \text{or} \quad P = \dfrac{Wa}{l}$; but $P = k_1 \Delta \quad \therefore \quad \Delta = \dfrac{Wa}{k_1 l}$





Now @ B there is a Δ_1 due to the load P

$$\therefore \quad \Delta_1 = \frac{a\Delta}{l} = \frac{Wa^2}{k_1 l^2}$$

Due to the mass itself we have a static displ $= \dfrac{W}{k_2} = \Delta_2$

Thus the total static disp is $\Delta_1 + \dfrac{W}{k_2}$

$$\Delta_{static} = \Delta_1 + \frac{W}{k_2} = W\left[\frac{1}{k_2} + \frac{1}{k_1}\frac{a^2}{l^2}\right] = \frac{W}{l^2}\left[\frac{k_1 l^2 + k_2 a^2}{k_1 k_2}\right]$$

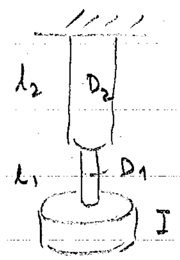
but $\Delta_{static} = \dfrac{W}{k_{EQ}} \quad \therefore \quad k_{EQ} = l^2 \dfrac{k_1 k_2}{k_1 l^2 + k_2 a^2}$

Thus $\quad m\ddot{x} + k_{EQ}x = 0 \quad \text{or} \quad \dfrac{\omega}{2\pi} = f = \dfrac{1}{2\pi}\sqrt{\dfrac{k_{EQ}}{m}} = \dfrac{l}{2\pi}\sqrt{\dfrac{k_1 k_2}{m(k_1 l^2 + k_2 a^2)}}$

where x measured from static equil position $(\Delta_1 + \Delta_2)$

or $\quad m\ddot{x} + k_{EQ}\,x = \dfrac{W}{l^2}\left[\dfrac{k_1 l^2 + k_2 a^2}{k_1 k_2}\right]$ if x is measured from horizontal

2-31



since the system is in series

$$1/k_T = 1/k_1 + 1/k_2 \qquad \text{where } k_i = \frac{J_i G}{l_i} \qquad J_i = \frac{\pi D_i^4}{32}$$

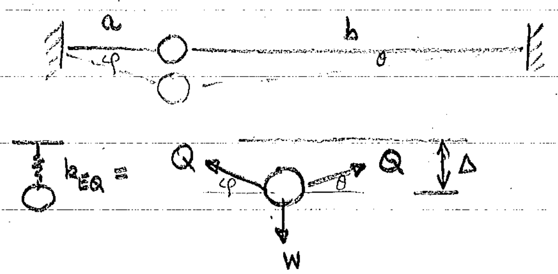
$$= \frac{l_1 \cdot 32}{\pi D_1^4 G} + \frac{32 l_2}{\pi D_2^4 G} = \frac{32}{\pi G}\left[\frac{l_1}{D_1^4} + \frac{l_2}{D_2^4}\right]$$

$$k_T = \frac{\pi G}{32}\left[\frac{D_1^4 D_2^4}{l_1 D_2^4 + l_2 D_1^4}\right] \qquad \omega = \sqrt{\frac{k_T}{I}} = \sqrt{\frac{\pi G}{32 I}\left[\frac{D_1^4 D_2^4}{l_1 D_2^4 + l_2 D_1^4}\right]}$$

DE: $I\ddot{\theta} + \omega^2\theta = 0$

$$\tau = \frac{2\pi}{\omega} = 8\sqrt{\frac{2\pi I}{G}\left[\frac{l_1 D_2^4 + l_2 D_1^4}{(D_1 D_2)^4}\right]}$$

2-32



Since deflections are the same we can find an equivalent spring system by looking at the vertical equilib. equs.

thus $Q\sin\varphi + Q\sin\theta = W$; $\sin\theta \cong \frac{\Delta}{b}$ $\sin\varphi \cong \frac{\Delta}{a}$ since $\Delta \ll a, b$

$$Q\Delta\left[\frac{1}{a} + \frac{1}{b}\right] = W = k_{EQ}\,\Delta \qquad (*)$$

THUS

$$k_{EQ} = Q\left[\frac{b+a}{ab}\right] \quad \text{and} \quad \omega = \sqrt{\frac{k_{EQ}}{m}} = \sqrt{\frac{Q}{m}\left[\frac{b+a}{ab}\right]}$$

HENCE

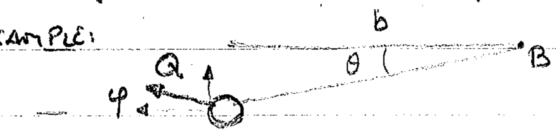
$\ddot{x} + \omega^2 x = 0$ where x is measured from the equilib. position

and

$$\tau = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{\ }\,\frac{ab}{\ }}$$

You can also obtain this result by cutting the cord at either side of the mass and looking at the torques produced about B

FOR EXAMPLE:



restoring torques $\quad (Q\cos\varphi)\Delta + (Q\sin\varphi)b \cong Q\cdot\Delta \quad + Q\frac{\Delta}{a}\cdot b$

non restoring torques $\quad Wb$

thus with $\Delta = b\theta$

$$I_B\ddot{\theta} = -Q\Delta\left[1+\frac{b}{a}\right] + Wb \qquad \text{where } I_B \cong mb^2 \text{ if } I_G \cong 0$$

then $\quad mb^2\ddot{\theta} + Qb\theta\left[1+\frac{b}{a}\right] = Wb \Rightarrow \ddot{\theta} + \frac{Q}{m}\left[\frac{a+b}{ab}\right]\theta = \frac{W}{mb}$

but from (*) on the previous page

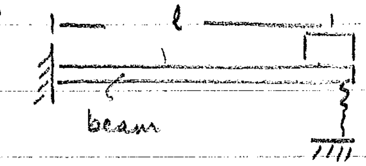
$$Q\Delta\left[\frac{b+a}{ab}\right] = W \quad \& \quad \Delta = b\theta \quad \Rightarrow Q\theta\left[\frac{b+a}{a}\right] = W$$

This value of $\theta = \theta_{STATIC\ EQUIL}$ which is related to $\Delta_{static} = b\,\theta_{STATIC}$.

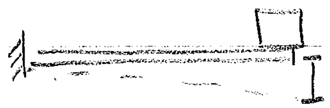
now let $\theta = \bar{\theta} + \theta_{STATIC}$ where $\bar{\theta}$ is measured from static equilib position. This then leads to

$$\ddot{\bar{\theta}} + \frac{Q}{m}\left[\frac{a+b}{ab}\right]\bar{\theta} = 0 \qquad \text{where } \omega = \sqrt{\frac{Q}{m}\left[\frac{a+b}{ab}\right]} \text{ as before.}$$

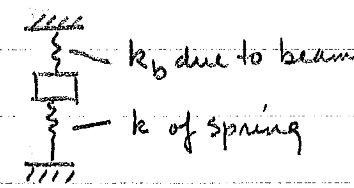
2-33



This can be likened to an equivalent system



$\Delta = \frac{Pl^3}{3EI}$; $k_b = \frac{P}{\Delta} = \frac{3EI}{l^3}$



Since the springs are in parallel since they experience the same displacement

thus $k_E = k_b + k \qquad$ and $\qquad \omega = \sqrt{\frac{k_b + k}{m}} = \sqrt{\frac{3EI + kl^3}{ml^3}}$

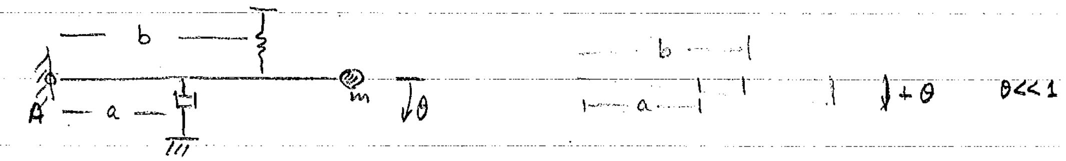
and $\ddot{x} + \omega^2 x = 0$, x being measured from the static equilib position

3-9 $\quad f_1 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$

$f_2 = \frac{1}{2\pi}\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \qquad \therefore \quad 4\pi f_2^2 = 4\pi f_1^2 - \frac{c^2}{(2m)^2}$

$\therefore \quad \frac{c^2}{(2m)^2} = 4\pi\left[f_1^2 - f_2^2\right] \qquad c = 4\pi m\sqrt{f_1^2 - f_2^2}$

3-10



1. Torque due to damper $\quad F_d \cdot a \qquad F_d = c\dot{x}_a = ca\dot{\theta} \qquad \therefore$ Torque $= ca^2\dot{\theta}$

Torque due to spring $\quad F_s \cdot b \qquad F_s = kx_b = kb\theta \qquad \therefore$ Torque $= kb^2\theta$

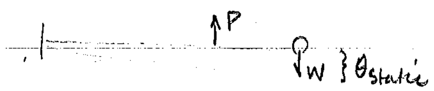
Torque due to mass $\quad W\ell$

$\therefore \quad I\ddot{\theta} = -kb^2\theta - ca^2\dot{\theta} + W\ell \quad$ for $\quad \theta \ll 1$

$I = m\ell^2 + I_G \qquad I_G$ is assumed to be ~ 0 since characteristic dimensions of the mass << dimensions of the problem

$\therefore \quad m\ell^2\ddot{\theta} + kb^2\theta + ca^2\dot{\theta} = W\ell \qquad (*)$

2. looking only at static equilib. $\quad \Sigma M_A = 0 = W\ell - Pb \quad \therefore \quad P = \frac{W\ell}{b}$



$P = k\Delta_{static} = kb\theta_{static}$

$\therefore \quad \theta_{static} = \frac{W\ell}{kb^2}$

3 but the particular solution to $(*)$ is $\theta_{part} = \frac{W\ell}{kb^2} = \theta_{static}$

Define $\quad \theta = \bar{\theta} + \theta_{static}$

then $\quad m\ell^2\ddot{\bar{\theta}} + kb^2\bar{\theta} + ca^2\dot{\bar{\theta}} = 0 \qquad$ where $\bar{\theta}$ is measured about the static equilib position

this is in the form $\quad m_{EQ}\ddot{\bar{\theta}} + c_{EQ}\dot{\bar{\theta}} + k_{EQ}\bar{\theta} = 0 \quad$ for critical damping constant

$\frac{c_{EQ}}{2m_{EQ}} = \sqrt{\frac{k_{EQ}}{m_{EQ}}} = \sqrt{\frac{kb^2}{m\ell^2}} = \frac{ca^2}{2m\ell^2}$

now $\frac{1}{2\pi}\sqrt{\frac{-c^2a^4+4mkl^2b^2}{4m^2l^4}} = \frac{\omega_d}{2\pi} = f_d$

$f_d = \frac{1}{4\pi m l^2}\sqrt{4mkl^2b^2-c^2a^4}$

3-17. $W = 9.65$ lb $k = 30$ lb/in w/ critically damped system $\zeta = 1$

$m = .3$ slugs $= .3\ \frac{lb\text{-}sec^2}{ft} = .025\ \frac{lb\text{-}sec^2}{in}$ $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{30}{.025}} = 34.64\ \frac{rad}{sec}$ $c_c = 2m\omega = 1.732\ \frac{lb\text{-}sec}{in}$

$\therefore\ x(t) = (A+Bt)e^{-\omega t}$

Given $\dot{x}(0) = 0$ $x(0) = 2\,in$ $\Rightarrow$ $A = 2\,in;\ B = A\omega = 69.28\ in/sec$

thus $x(t) = (2+69.28t)e^{-34.64t}$

@ $t = .01$ sec $x = 1.904$ in

@ $t = .1$ $x = .2795$ in

@ $t = 1$ $x = 6.44\times10^{-14}$ in $\approx 0.$

3-19 $k = 7875$ N/m $m = 8.75$ kg $\zeta = .1$ $x(0) = 10$ cm $\dot{x}(0) = 0$

$\omega = \sqrt{\frac{k}{m}} = 30$ rad/sec $c = 2m\omega = 525\ kg/_{sec} = 525\ \frac{N\text{-}sec}{m}$

now $n = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta}\ln\left(\frac{x_0}{x_n}\right)$ $\Rightarrow$ $\ln\left(\frac{x_0}{x_n}\right) = \frac{2\pi n\zeta}{\sqrt{1-\zeta^2}}$ $\Rightarrow$ $x_n = x_0 e^{-\frac{2\pi n\zeta}{\sqrt{1-\zeta^2}}}$

for $n = \frac{1}{2}$ $x_{\frac{1}{2}} = -10\,cm\ e^{-\frac{\pi(.1)}{\sqrt{.99}}} = -7.2925$ cm

$n = 1$ $x_1 = 10\,cm\ e^{-\frac{2\pi(.1)}{\sqrt{.99}}} = 5.318$ cm

or if $x = Ce^{-\zeta\omega t}\sin(\omega_d t+\phi)$ $\dot{x} = C\left[-\zeta\omega e^{-\zeta\omega t}\sin(\ \ ) + e^{-\zeta\omega t}\omega_d\cos(\ \ )\right]$

$\omega_d = \sqrt{1-\zeta^2}\,\omega = 29.85$ rad/sec ; when $\dot{x} = 0$ then $\tan\phi = \frac{\omega_d}{\zeta\omega} = \frac{29.85}{3} = 9.95$; $\phi = 84.2$

$10 = x = Ce^{-0}\sin\phi = C(.995)$ $C = 10.05$ cm

HW #7 3-26, 27, 30, 35

3-26 $k = 45$ lb/in $W = 19.3$ lb $c = .057$ lb sec/in $X_0 = .3$ in find X_{12}

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}} = 30.016 \text{ rad/sec}$$

$$c_c = 2m\omega = 2\,(19.3 \text{ lb})(30.016)/(32.2 \times 12) = 2.998 \text{ lb-sec/in}$$

4 $$\zeta = \frac{c}{c_c} = \frac{.057}{3} = .019$$

$$n = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta} \ln\left(\frac{X_0}{X_n}\right) \Rightarrow X_n = X_0\, e^{-\frac{2\pi\zeta n}{\sqrt{1-\zeta^2}}} \text{; for } n = 12 \quad X_{12} = 3e^{-1.4328} = .7159 \text{ in}$$

3-27 For the no. of cycles in the half-life

$$n = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta} \ln\left(\frac{X_0}{X_n}\right) \quad \text{where } X_n = \tfrac{1}{2}X_0$$

$$n = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta} \ln 2 = \frac{1}{.1194} \ln 2 = 5.81 \text{ cycles}$$

For the half-life

$$\Delta t = n\tau = n \cdot \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega} = \frac{1}{\zeta\omega}\ln 2 = \frac{1}{(.019)(30)} \ln 2 = \frac{\ln 2}{.57} = 1.215 \text{ sec}$$

3-30 $m = 5$ kg $k = 45$ N/m $\zeta < 1$ $x = x_0 \neq 0$ $\dot{x} = 0$ @ $t = 0$

$X_{overshoot} = .25X_0$; the time period for this to happen is ½ the period of one cycle $\Rightarrow$ thus

$$\frac{X_0}{X_{overshoot}} = 4 = e^{\zeta\omega\tau/2} \quad \& \quad \omega = \sqrt{\frac{k}{m}} = 3 \text{ rad/sec}$$

$$\frac{X_0}{X_{overshoot}} = 4 = e^{\pi\zeta/\sqrt{1-\zeta^2}}$$

$$\text{thus} \quad \left(\frac{\ln 4}{\pi}\right)^2 [1-\zeta^2] = \zeta^2 \Rightarrow \frac{\left(\frac{\ln 4}{\pi}\right)}{\sqrt{1+\left(\frac{\ln 4}{\pi}\right)^2}} = \zeta = .4037$$

$$\omega = \frac{c_c}{2m} \Rightarrow c_c = 2m\omega = 2(5)(3) = 30 \text{ kg/sec}$$

$$\zeta = \frac{c}{c_c} \Rightarrow c = c_c\zeta = 12.11 \text{ kg/sec}$$

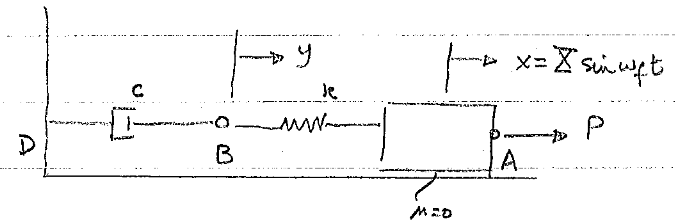
3-35 $W = 47$ lb $k = 30$ lb/in Δx between cycles decrease linearly $\Rightarrow$ Coulomb

damping $\therefore$ $\Delta x = 4F_c$ = .05 in $\Rightarrow$ F_c = .05 in (lb) = .375 lb

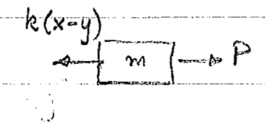
HW #11

4-20, 42, 43

4-20



a)



b) $m\ddot{x} = -k(x-y) + P$ or $\boxed{m\ddot{x} + kx = ky + P}$

c) $\xleftarrow{c\dot{y}} \circ \xrightarrow{k(x-y)}$

d) $m_B\ddot{y} = 0 = k(x-y) - c\dot{y} \Rightarrow \boxed{kx = c\dot{y} + ky = kX \sin\omega_f t}$

e) $\dot{y} + \frac{k}{c}y = \frac{k}{c}X \sin \omega_f t$; let $y = A\cos\omega_f t + B \sin\omega_f t$ & put into D.E.

$\Rightarrow (-A\omega_f + \frac{k}{c}B)\sin\omega_f t + (B\omega_f + \frac{k}{c}A)\cos\omega_f t = \frac{k}{c}X \sin\omega_f t$. Solve for A & B

$$\Rightarrow \boxed{y = \frac{\frac{k}{c}X}{\omega_f^2 + k^2/c^2}\left[\frac{k}{c}\sin\omega_f t - \omega_f \cos\omega_f t\right]} \text{ or } \frac{k/c\, X}{\sqrt{\omega_f^2 + k^2/c^2}}\sin(\omega_f t - \lambda) = y$$

with $\lambda = \tan^{-1}\frac{c\omega_f}{k}$; now let $y = Y\sin(\omega_f t - \lambda)$ where $Y = \frac{k/c\, X}{\sqrt{\omega_f^2 + k^2/c^2}}$

f) from (b), (e) and the defn of $x = X\sin\omega_f t$

$$P = (-m\omega_f^2 + k)X\sin\omega_f t + \frac{k^2/c\, X}{\sqrt{\omega_f^2 + k^2/c^2}}\sin(\omega_f t - \lambda)$$

g) @ D $\quad F_T = c\dot{y}$

$$\therefore F_T = \frac{\omega_f k X}{\sqrt{\omega_f^2 + k^2/c^2}}\cos(\omega_f t - \lambda) = c\omega_f Y\cos(\omega_f t - \lambda) \quad \text{with } \lambda = \tan^{-1}\frac{c\omega_f}{k}.$$

4-42

Given $f = 75$ cy/min $\Rightarrow X = .6$ in & $(F_T)_{max} = 48.75$ lb. ; $k = 13$ lb/in

$c = 0.454$ lb sec/in find ω, m and Y

b) now $x = X\sin(\omega_f t - \gamma) = Y\frac{\sqrt{k^2 + (c\omega_f)^2}}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}\sin\omega_f t$ eq (4-100)

need to find m to get Y ; can get m by looking at transmitted force eqn

c) now $(F_T)_{max} = m\omega_f^2 X$ eqn (4-104) $\Rightarrow m = \frac{(F_T)_{max}}{\omega_f^2 X} = \frac{48.75 \text{lb}}{(7.854 \frac{rad}{sec})^2(.6 \text{in})} = 1.317 \frac{\text{lb sec}^2}{\text{in}}$

a) $\omega = \sqrt{\frac{k}{m}} = \pi$ rad/sec $= 3.141592653$ rad/sec

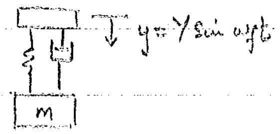
based on the value of m, c, ω_f, k, X obtained we can now find Y

$$Y = X \frac{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}{\sqrt{k^2 + (c\omega_f)^2}} = 3.042 \text{ in}$$

<u>4-43</u>

$W = mg = 20 \text{ lb}$ $\quad k = 14 \text{ lb/in}$; if $f_f = 300 \text{ cy/min} = 5 \text{ Hz} \Rightarrow X = .793 \text{ in}$

a) Find $(F_T)_{max}$; b) if $\omega_f = \omega$ and $Y_{res} = 1.121 \text{ in} \Rightarrow X_{res} = 3.069 \text{ in}$; find c



a). now $F = k(x-y) + c(\dot{x}-\dot{y}) = -m\ddot{x} = m\omega_f^2 X \sin\omega_f t$

$$(F_T)_{max} = m\omega_f^2 X = \frac{W}{g}\omega_f^2 X = \frac{20 \text{ lb}}{386.4 \text{ in/sec}^2}(10\pi)^2(.793 \text{ in}) = 40.51 \text{ lb}.$$

b) when $\omega_f = \omega \Rightarrow r = 1$ $\qquad X_{res} = Y_{res}\left.\frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\right|_{r=1} = \frac{Y_{res}\sqrt{1+4\zeta^2}}{2\zeta}$

now $2\zeta X_{res} = Y_{res}\sqrt{1+4\zeta^2} \Rightarrow 4\zeta(X_{res}^2 - Y_{res}^2) = Y_{res}^2$ or $\zeta = \frac{1}{2}\frac{Y_{res}}{\sqrt{X_{res}^2 - Y_{res}^2}} = .1962.$

now $c = c_c\zeta$ and $c_c = 2m\omega = 2\sqrt{mk}$ $\qquad \frac{W}{g} = m = .0518 \frac{\text{lb sec}^2}{\text{in}}$

$c = .334 \frac{\text{lb sec}}{\text{in}} \Leftarrow c_c = 1.703 \frac{\text{lb sec}}{\text{in}} \Leftarrow$

HW #12

5-6, 5-28

5-6 $\quad P(z) = Bz/\pi \qquad 0 \le z \le \pi$ $\qquad$ over the interval $0 \le z \le 2\pi$

$\qquad -2B + \frac{Bz}{\pi} \qquad \pi \le z \le 2\pi$ $\qquad$ this fn is periodic in z

now if $\quad P = \sum_{n=0}^{\infty} a_n \cos nz + \sum_{n=0}^{\infty} b_n \sin nz$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} P\,dz = \frac{1}{2\pi}\int_0^{\pi} \frac{Bz}{\pi}\,dz + \frac{1}{2\pi}\int_{\pi}^{2\pi}\left(-2+\frac{z}{\pi}\right)B\,dz$$

$$= \frac{B}{2\pi^2}\,\frac{z^2}{2}\Big|_0^{\pi} + \frac{B}{2\pi}\left(-2z+\frac{z^2}{2\pi}\right)\Big|_{\pi}^{2\pi} = \frac{B}{4} + \frac{B}{2\pi}\left(-2\pi + 2\pi - \frac{\pi}{2}\right) = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} P\cos nz\,dz = \frac{B}{\pi^2}\int_0^{\pi} z\cos nz\,dz + \frac{B}{\pi}\int_{\pi}^{2\pi}\left(-2+\frac{z}{\pi}\right)\cos nz\,dz$$

$$= \frac{B}{\pi^2}\left[\frac{z\sin nz}{n} - \frac{\cos nz}{n^2}\right]_0^{\pi} - \frac{2B}{\pi}\,\frac{\sin nz}{n}\Big]_{\pi}^{2\pi} + \frac{B}{\pi^2}\left[\frac{z\sin nz}{n} - \frac{\cos nz}{n^2}\right]_{\pi}^{2\pi}$$

$$= \frac{B}{\pi^2}\left[-\frac{(-1)^n}{n^2} + \frac{1}{n^2}\right] + \frac{B}{\pi}\left[-\frac{1}{n^2} + \frac{(-1)^n}{n^2}\right] = 0$$

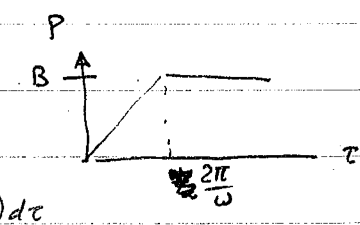
$$b_n = \frac{1}{\pi}\int_0^{2\pi} P\sin nz\,dz = \frac{B}{\pi^2}\int_0^{\pi} z\sin nz\,dz + \frac{B}{\pi}\int_{\pi}^{2\pi}\left(-2+\frac{z}{\pi}\right)\sin nz\,dz$$

$$= \frac{B}{\pi^2}\left[-\frac{z\cos nz}{n} + \frac{\sin nz}{n^2}\right]_0^{\pi} + \frac{2B}{\pi}\,\frac{\cos nz}{n}\Big]_{\pi}^{2\pi} + \frac{B}{\pi^2}\left[-\frac{z\cos nz}{n} + \frac{\sin nz}{n^2}\right]_{\pi}^{2\pi}$$

$$= \frac{B}{\pi^2}\left[-\frac{\pi(-1)^n}{n}\right] + \frac{2B}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n}\right] + \frac{B}{\pi^2}\left[-\frac{2\pi}{n} + \frac{\pi}{n}(-1)^n\right]$$

$$= \frac{2B}{\pi}\,\frac{(-1)^{n+1}}{n}$$

Thus $\quad P(z) = \frac{2B}{\pi}\left\{\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nz\right\}$



5-28 $\quad$ for the undamped system

$$x = \frac{1}{\sqrt{mk}}\int_0^t P(\tau)\sin\omega(t-\tau)\,d\tau$$

where

$$P(\tau) = \frac{B\tau\omega}{2\pi} \qquad \text{for} \quad 0 \le \tau \le \frac{2\pi}{\omega}$$

$$= B \qquad \text{for} \quad \tau \ge \frac{2\pi}{\omega}$$

thus

$$x = \frac{1}{\sqrt{mk}}\begin{cases} \frac{B}{2\pi}\int_0^t \omega\tau\sin\omega(t-\tau)\,d\tau & \text{for } 0 \le t \le \frac{2\pi}{\omega} \\ \frac{B}{2\pi}\int_0^{2\pi/\omega} \omega\tau\sin\omega(t-\tau)\,d\tau & \text{for } t \ge \frac{2\pi}{\omega} \end{cases}$$

t

for $0 \le t \le \frac{2\pi}{\omega}$

$$x = \frac{B}{2\pi\omega}\,\frac{1}{\sqrt{mk}}\left\{\omega\tau\cos(\omega t-\omega\tau)+\sin(\omega t-\omega\tau)\right\}_0^t$$

$$x = \frac{B}{2\pi\,\omega}\cdot\frac{1}{\sqrt{mk}}\cdot\left\{\omega t-\sin\omega t\right\}$$

for $t \geqslant \frac{2\pi}{\omega}$

$$x = \frac{B}{2\pi\omega}\,\frac{1}{\sqrt{mk}}\left\{\omega\tau\cos(\omega t-\omega\tau)+\sin(\omega t-\omega\tau)\right\}_0^{\tau=\frac{2\pi}{\omega}}$$

$$+\frac{B}{\omega}\,\frac{1}{\sqrt{mk}}\left\{\cos(\omega t-\omega\tau)\right\}_{\tau=\frac{2\pi}{\omega}}^{\tau=t}$$

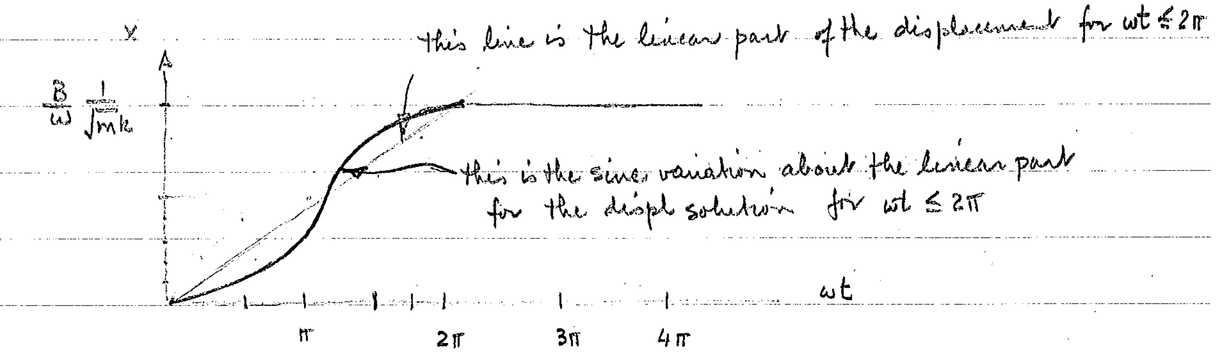
or

$$x = \frac{B}{2\pi\omega}\,\frac{1}{\sqrt{mk}}\left\{2\pi\overbrace{\cos(\omega t-2\pi)}^{\cos\omega t}+\overbrace{\sin(\omega t-2\pi)}^{\sin\omega t}-\sin(\omega t)\right\}$$

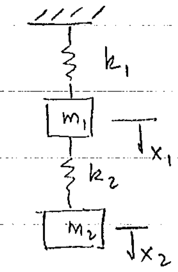
$$+\frac{B}{\omega}\,\frac{1}{\sqrt{mk}}\left\{1-\overbrace{\cos(\omega t-2\pi)}^{\cos\omega t}\right\}$$

for $t \geqslant \frac{2\pi}{\omega}$

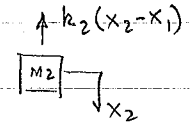
$$x = \frac{B}{\omega}\,\frac{1}{\sqrt{mk}}$$

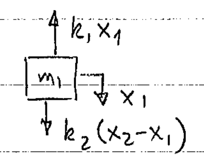


HW # 13 - 7.1, 4, 7



a)





b) for m_2 : $m_2\ddot{x}_2 = -k_2(x_2-x_1)$ or $m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0$ (1)

for m_1 : $m_1\ddot{x}_1 = k_2(x_2-x_1) - k_1x_1$ or $m_1\ddot{x}_1 + (k_2+k_1)x_1 - k_2x_2 = 0$ (2)

c) let $m_2 = m_1 = m$ $k_2 = k_1 = k$

let $x_2 = A_2\sin(\omega t+\varphi)$; $x_1 = A_1 \sin(\omega t - \varphi) \Rightarrow$ put into (1), (2) to get

$$\left.\begin{aligned} &(3) \quad -mA_2\omega^2 + kA_2 - kA_1 = 0 \\ &(4) \quad -mA_1\omega^2 + (2k)A_1 - kA_2 = 0 \end{aligned}\right\} = \begin{bmatrix} -k & k-m\omega^2 \\ 2k-m\omega^2 & -k \end{bmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

if $\sin(\omega t+\varphi) \neq 0$.

Now let det $= 0 \Rightarrow k^2 - (k-m\omega^2)(2k-m\omega^2) = 0 \Rightarrow -k^2 + 3km\omega^2 - m^2\omega^4 = 0$

$$\omega^2 = \frac{-3km \pm \sqrt{9k^2m^2 - 4k^2m^2}}{-2m^2} = \frac{-3km \pm \sqrt{5}\,km}{-2m^2} = \frac{3\pm\sqrt{5}}{2}\,\frac{k}{m}$$

$\omega_2 = 1.618\sqrt{\frac{k}{m}}$ $\boxed{\omega_1 = .618\sqrt{\frac{k}{m}} \quad \text{fundamental freq}}$

for $\underline{\omega_1 = .618\sqrt{\frac{k}{m}}}$ and eq. (3): $-kA_1 + \left(k - \cancel{m}\left(\frac{3-\sqrt{5}}{2}\right)\frac{k}{\cancel{m}}\right)A_2 = 0$ or $-\cancel{k}A_1 + \overbrace{\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)}^{.618}\cancel{k}A_2 = 0$

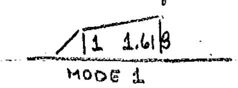
$\Rightarrow A_1 = .618\,A_2$ or $A_2 = 1.618\,A_1$

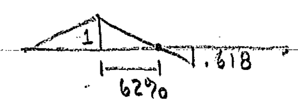
- normally x_1 is normally normalized so that $(x_1)_{max} = 1 \Rightarrow A_1 = 1 \Rightarrow A_2 = 1.618$

from $\omega_2 = 1.618\sqrt{\frac{k}{m}}$ and eq. (3): $-kA_1 + \left(k - \cancel{m}\left(\frac{3+\sqrt{5}}{2}\right)\frac{k}{\cancel{m}}\right)A_2 = 0$ or $-A_1 - \left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right)A_2 = 0$

$\Rightarrow A_1 = -1.618\,A_2$ or $A_2 = -.618\,A_1$

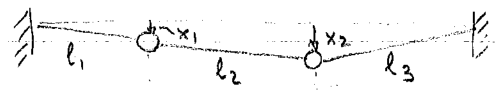
- since x_1 is normalized to $(x_1)_{max} = 1 \Rightarrow A_1 = 1 \Rightarrow A_2 = -.618$



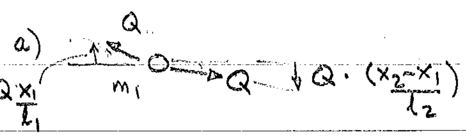


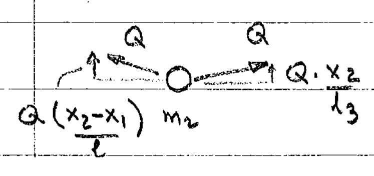
NODE IS ~ .62% of distance from equil. position of mass m_1

7.4



a)





what is shown here are the components of the force in the direction of motion (x_1, x_2 are measured ↓)

b. $m_1\ddot{x}_1 = \dfrac{Q(x_2-x_1)}{l_2} - Q\left(\dfrac{x_1}{l_1}\right)$

$m_2\ddot{x}_2 = -\dfrac{Q(x_2-x_1)}{l_2} - Q\left(x_2/l_3\right)$

c) if $m_1 = m_2 = m$ and $l_1 = l_2 = l_3$

$m\ddot{x}_1 = \dfrac{Q(x_2-x_1)}{l} - Q\left(\dfrac{x_1}{l}\right) \Rightarrow \ddot{x}_1 + \dfrac{2Q}{ml}x_1 - \dfrac{Qx_2}{ml} = 0$ (1)

$m\ddot{x}_2 = -\dfrac{Q(x_2-x_1)}{l} - Q\left(\dfrac{x_2}{l}\right) \Rightarrow \ddot{x}_2 + \dfrac{2Q}{ml}x_2 - \dfrac{Qx_1}{ml} = 0$ (2)

let $x_1 = A_1\sin(\omega t+\varphi)$ & $x_2 = A_2\sin(\omega t+\varphi)$ and put into (1), (2)

$\Rightarrow \left(-\omega^2 + \dfrac{2Q}{ml}\right)A_1 - \dfrac{Q}{ml}A_2 = 0$ (3) if $\sin(\omega t+\varphi) \neq 0$

$-\dfrac{Q}{ml}A_1 + \left(-\omega^2 + \dfrac{2Q}{ml}\right)A_2 = 0$ (4)

if $A_1 \neq A_2 \neq 0 \Rightarrow$ determinant of the coeffs of A_1 & $A_2 = 0$ ie

$$\begin{vmatrix} -\omega^2 + \dfrac{2Q}{ml} & -\dfrac{Q}{ml} \\ -\dfrac{Q}{ml} & -\omega^2 + \dfrac{2Q}{ml} \end{vmatrix} = 0 \Rightarrow \omega^4 - \frac{4Q}{ml}\omega^2 + \frac{3Q^2}{(ml)^2} = 0$$

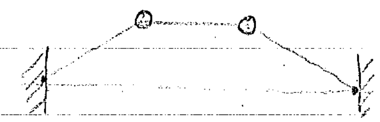
or $\left(\omega^2 - \dfrac{3Q}{ml}\right)\left(\omega^2 - \dfrac{Q}{ml}\right) = 0$

$\Rightarrow \omega_1 = \sqrt{\dfrac{Q}{ml}}$ $\quad \omega_2 = \sqrt{\dfrac{3Q}{ml}}$

To find the fundamental modes: put ω_1 into (3)

$\Rightarrow \left(-\dfrac{Q}{ml} + \dfrac{2Q}{ml}\right)A_1 - \dfrac{Q}{ml}A_2 = 0 \Rightarrow A_1 = A_2$; if $A_1 = 1 \Rightarrow A_2 = 1$

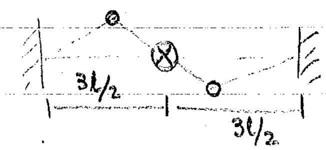
when x_1 is normalized to $(x_1)_{max} = 1$.



mode shape for fundamental freq.

for the 2nd mode : put ω_2 into (3) $\Rightarrow \left(-\dfrac{3Q}{ml} + \dfrac{2Q}{ml}\right)A_1 - \dfrac{Q}{ml}A_2 = 0 \Rightarrow A_1 = -A_2$;

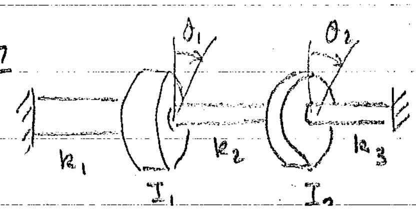
if $A_1 = 1 \Rightarrow A_2 = -1$ when x_1 is normalized to $(x_1)_{max} = +1$.



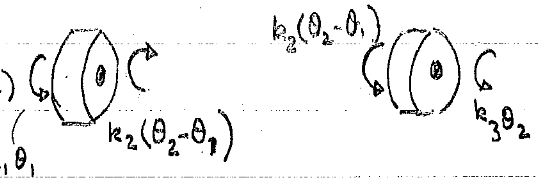
mode shape for 2nd freq.

⊗ represents the node at a distance of $3l/2$ from either end.

7-7



a)



b) $I_1 \ddot{\theta}_1 = -k_1\theta_1 + k_2(\theta_2 - \theta_1)$ $\qquad$ $I_2\ddot{\theta}_2 = -k_3\theta_2 - k_2(\theta_2 - \theta_1)$

or $I_1\ddot{\theta}_1 + (k_1+k_2)\theta_1 - k_2\theta_2 = 0 \quad (1)$ $\qquad$ $I_2\ddot{\theta}_2 + (k_2+k_3)\theta_2 - k_2\theta_1 = 0 \quad (2)$

c) let $I_1 = I$ $\quad I_2 = 2I$ and $k_1 = k_2 = k_3 = k$; and let $\theta_1 = A_1 \sin(\omega t + \varphi)$

$\theta_2 = A_2 \sin(\omega t + \varphi)$; put into these into (1,2) to get

$$\left.\begin{aligned} (-\omega^2 I + 2k)A_1 - kA_2 &= 0 \quad (3) \\ -kA_1 + (-2I\omega^2 + 2k)A_2 &= 0 \end{aligned}\right\} \text{ if } \sin(\omega t+\varphi) \neq 0$$

Now if $A_1 \neq A_2 \neq 0 \Rightarrow$ determinant of coeffs $= 0$ or

$$\begin{vmatrix} -\omega^2 I + 2k & -k \\ -k & -2I\omega^2 + 2k \end{vmatrix} = 0 \quad \text{or} \quad 2I^2\omega^4 - 6I\omega^2 k + 3k^2 = 0$$

$$\omega^2 = \frac{6Ik \pm \sqrt{36I^2k^2 - 4(2I^2)(3k^2)}}{4I^2} = \frac{3}{2}\frac{k}{I} \pm \sqrt{\frac{3}{4}}\,\frac{k}{I} = 2.366\frac{k}{I},\ .634\frac{k}{I}$$

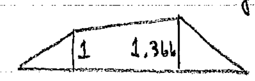
$\omega_1 = .7962\sqrt{\frac{k}{I}} \qquad \omega_2 = 1.538\sqrt{\frac{k}{I}}$

For the fundamental freq ω_1 go to (3) $\Rightarrow \left(-\left(\frac{3}{2} - \sqrt{\frac{3}{4}}\right) + 2\right)kA_1 - kA_2 = 0$ or

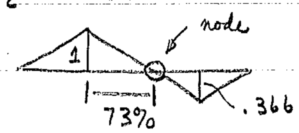
$A_2 = 1.366 A_1 \Rightarrow$ if $A_1 = 1 \Rightarrow A_2 = 1.366$

For ω_2 got to (3) $\Rightarrow \left(-\left(3/2 - \sqrt{3/4}\right) + 2\right)kA_1 - kA_2 = 0 \Rightarrow A_2 = -.366 A_1$

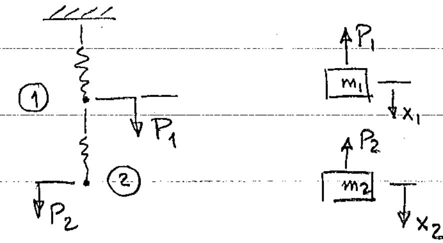
if $A_1 = 1 \Rightarrow A_2 = -.366$



mode 1



7-2 Look at



Here P_1 and P_2 represent the total external force on each mass

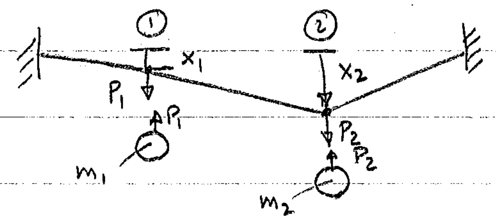
look at pt 1 ; if $P_2 = 0 \Rightarrow x_1 = a_{11} P_1$. When $P_2 = 0$, due to force P_1 $P_1 = k_1 x_1$ only since spring 2 just goes along for the ride & doesn't extend. Since $x_1 = a_{11} P_1$ (or $x_1 \cdot \frac{1}{a_{11}} = P_1$) and $P_1 = k_1 x_1 \Rightarrow \underline{1/k_1 = a_{11}}$

now Since k_2 doesn't extend $\Rightarrow x_2 = x_1$; but $x_1 = a_{11} P_1$; $\Rightarrow x_2 = a_{11} P_1$. but by definition $x_2 = a_{21} P_1$ when $P_2 = 0 \Rightarrow \underline{a_{11} = a_{21} = 1/k_1}$. By maxwell-betty theorem $\underline{a_{12} = a_{21} = 1/k_1}$

Now look at pt. 2 ; if $P_1 = 0 \Rightarrow x_2 = a_{22} P_2$; but for the springs k_1, k_2 they are in series $\Rightarrow 1/k_{eq} = \frac{1}{k_1} + \frac{1}{k_2}$ or $k_{eq} = \frac{k_1 k_2}{k_1 + k_2}$. Now $P_2 = k_{eq} x_2$ also $\Rightarrow$ $\underline{a_{22} = \frac{1}{k_{eq}} = \frac{k_1 + k_2}{k_1 k_2}}$

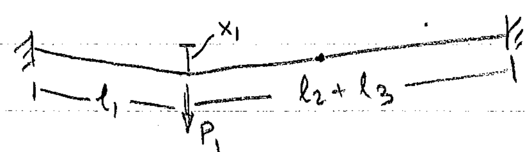
Now by setting $k_3 = 0$ in the formulas derived in class we get the above.

7-5 Look at

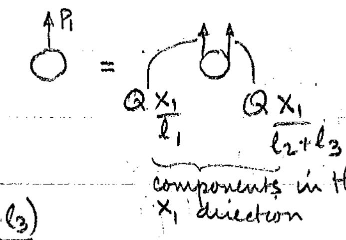


again P_1 and P_2 represent the total external forces acting on each mass

look at pt. 1 ; if $P_2 = 0 \Rightarrow x_1 = a_{11} P_1$ and the system will look like this

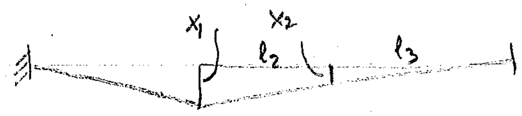


and



$$\therefore\ P_1 = \underbrace{\left(\frac{Q}{\ell_1} + \frac{Q}{\ell_2 + \ell_3}\right)}_{k_{eq}} x_1 = \frac{1}{a_{11}} P_1 \Rightarrow \underline{a_{11} = \frac{\ell_1(\ell_2 + \ell_3)}{Q \ell_0}}$$

where $\ell_0 = \Sigma \ell_i$

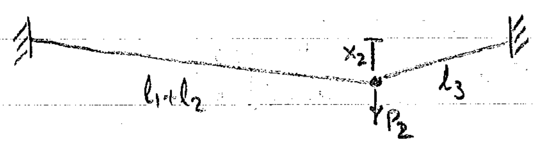


by similar Δ's $\quad \dfrac{x_1}{l_2+l_3} = \dfrac{x_2}{l_3}$

$\therefore \quad x_2 = \dfrac{x_1 l_3}{l_2+l_3}$; but $x_1 = a_{11}P_1 \quad \therefore \quad x_2 = a_{11}\dfrac{P_1 l_3}{l_2+l_3} = \dfrac{l_1 l_3}{Q l_0}P_1$

but when $P_2 = 0 \quad x_2 = a_{21}P_1 \implies \underline{a_{21} = \dfrac{l_1 l_3}{Q(l_0)}} = a_{12}$ by Maxwell Theorem

Now look at pt 2 on original diagram ; if $P_1 = 0 \implies x_2 = a_{22}P_2$
and the system looks like this



and $\quad \overset{\uparrow P_2}{\bigcirc} = \underbrace{\dfrac{Qx_2}{l_1+l_2} \quad \dfrac{Qx_2}{l_3}}_{\text{components of the force in the } x_2 \text{ direction}}$

$\therefore \quad P_2 = \left(\dfrac{Q}{l_1+l_2} + \dfrac{Q}{l_3}\right)x_2 = \dfrac{1}{a_{22}}x_2 \implies \underline{a_{22} = \dfrac{l_3(l_1+l_2)}{Q l_0}}$

when $l_0 = \Sigma l_i$

NOTE: Since it is assumed that Q doesn't change appreciably when the masses are displaced, then the masses are in equilibrium in the horizontal direction, hence no motion in the horizontal direction.

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{\theta} \end{pmatrix} + \begin{bmatrix} k_1+k_2 & k_2b_2-k_1b_1 \\ k_2b_2-k_1b_1 & k_1b_1^2+k_2b_2^2 \end{bmatrix} \begin{pmatrix} x \\ \theta \end{pmatrix}$$

$$\omega_n^2 = \frac{\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} 11k & -8ak \\ -8ak & 14ka^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}}{\begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} m & 0 \\ 0 & 100ma^2 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix}}$$

$$A = \frac{mg}{11k}$$

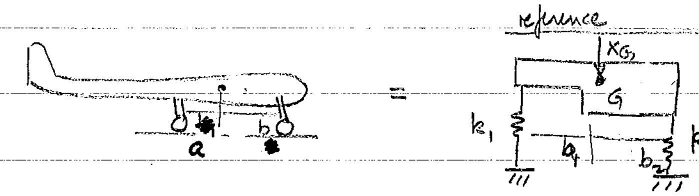
$$B \Rightarrow W \cdot b = k_1 \cdot (b+a)^2 \theta$$

$$\frac{mg \cdot 2a}{10k(3a)^2} = \frac{mg}{45ka}$$

$$\frac{B}{A} = \frac{4\,\cancel{\frac{mg}{45ka}}}{\cancel{\frac{mg}{11k}}} = \frac{1}{4a}$$

HW #15

7-25



$b = 2a$

$k_1 = 10k \quad k_2 = k$

$J_G = 100\,ma^2$

This is basically the problem studied in class; therefore if we decide the measure the displacement of the center of gravity (G) from the reference position, and consider rotations measured from the reference line, then our equations of motion will be

$$m\ddot{x}_G + (k_1+k_2)x_G + (k_2b_2 - k_1b_1)\theta = m\ddot{x}_G + (11k)x_G - 8ka\theta = 0 \qquad (1)$$

$$J_G\ddot{\theta} + (k_2b_2 - k_1b_1)x_G + (k_1b_1^2 + k_2b_2^2)\theta = -100ma^2\ddot{\theta} - 8ka\,x_G + 14ka^2\theta = 0 \quad (2)$$

Now let: $x = A\sin(\omega t+\varphi)$, $\theta = B\sin(\omega t+\varphi)$. Put into (1) and (2) to get

$$\left.\begin{array}{ll}(3) & (-m\omega^2 + 11k)A + (-8ak)B = 0\\ (4) & (-8ka)A + (-100ma^2\omega^2 + 14ka^2)B = 0.\end{array}\right\}\text{ amplitude eqns.}$$

for non trivial solutions

$$\begin{vmatrix} -m\omega^2+11k & -8ak \\ -8ka & -100ma^2\omega^2+14ka^2 \end{vmatrix} = 0 \Rightarrow \begin{array}{l} 100m^2a^2\omega^4 - 1114mka^2\omega^2 + 90k^2a^2 = 0 \\ \text{or } \omega^4 - 11.14\frac{k}{m}\omega^2 + .9\,k^2/m^2 = 0 \end{array}$$

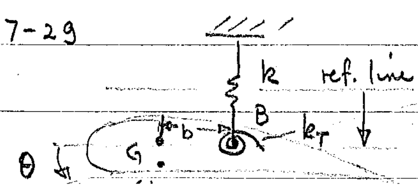
Solve for $\omega = \sqrt{\dfrac{11.14 \pm \sqrt{(11.14)^2 - 4(.9)}}{2}}$ $\qquad \omega_1 = .2853\sqrt{\frac{k}{m}} \qquad \omega_2 = 3.3254\sqrt{\frac{k}{m}}$

Now take ω_1 and put into (3) to get $B = 1.365\ A/a$

Now take ω_2 and put into (3) to get $B = -.00733\ \frac{A}{a}$.

note that since θ is in radians B must also be in radians, and A and a must have the same units of length.

7-29



Since the springs are connected to point B, need to use the second method discussed in class for the unbalanced vibrating mass (shown above)

when point G moves down and rotates to point G' due to vibration then, from the ref. line, the displ of point G is $y+b\theta$ and the acceleration $\ddot{y}+b\ddot{\theta}$.

The external force acting of the body is due to the linear spring and is equal to $-ky$. Thus $\underline{\underline{m(\ddot{y}+b\ddot{\theta}) = -ky}}$ is the equation of linear moti

• Similarly using the equation (1-10) for motion about a point that is not fixed or is not the center of gravity, we need to define the torques about point B. Thus Σ Torques $= -k_T\theta$. Also as in the problem solved in class, since B is not the mass center, Σ Torques $= \ddot{\theta} I_B + m\ddot{y} b \cancel{\cos\theta}^{1}$

Thus $\underline{\underline{\ddot{\theta} I_B + m\ddot{y} b = -k_T\theta}}$.

Let $y = A\sin(\omega t+\varphi)$ $\theta = B\sin(\omega t+\varphi)$ and put in the underlined equation

Thus

$$\left.\begin{aligned} -m\omega^2 A + kA - m\omega^2 bB &= 0 \\ -I_B\omega^2 B + k_T B - m\omega^2 bA &= 0 \end{aligned}\right\} \text{ if } \sin(\omega t+\varphi) \neq 0.$$

These are the Amplitude Equations,

To find the characteristic equation, look at the determinant of the coefficient

Thus
$$\begin{vmatrix} -m\omega^2+k & -m\omega^2 b \\ -m\omega^2 b & -I_B\omega^2+k_T \end{vmatrix} = 0$$

This leads to the characteristic equation (or the frequency equation)

$$mI_B\omega^4 - mk_T\omega^2 - I_B k\omega^2 - m^2\omega^4 b^2 + kk_T = 0$$

or $$\omega^4\left[mI_B - m^2b^2\right] - \left[mk_T + I_B k\right]\omega^2 + kk_T = 0 \qquad (*)$$

Now I_G (moment of inertia about the mass center) can related to I_B by the parallel axis theorem, ie $I_B = I_G + mb^2$. Thus $mI_B - m^2b^2 = mI_G$ and $I_B k = I_G k + kmb^2$ and $(*)$ becomes

$$\omega^4\left[mI_G\right] - \omega^2\left[I_G k + m(k_T + kb^2)\right] + kk_T = 0$$