From
"MECHANICAL VIBRATIONS"
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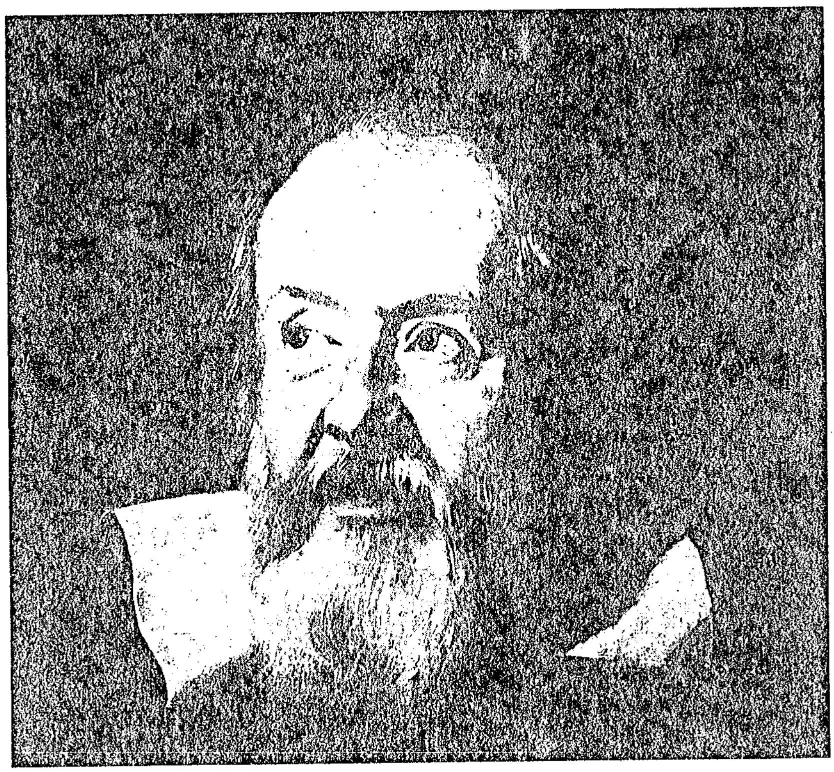
Fundamentals of Vibration

Galileo Galilei (1564–1642), an Italian astronomer, philosopher, and professor of mathematics at the Universities of Pisa and Padua, in 1609 became the first man to point a telescope to the sky. He wrote the first treatise on modern dynamics in 1590. His works on the oscillations of a simple pendulum and the vibration of strings are of fundamental significance in the theory of vibrations.



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1.1 PRELIMINARY REMARKS

This chapter introduces the subject of vibrations in a relatively simple manner. The chapter begins with a brief history of the subject and continues with an examination of its importance. The various steps involved in vibration analysis of an engineering system are outlined, and essential definitions and concepts of vibration are introduced. There follows a presentation of the concept of harmonic analysis, which can be used for the analysis of general periodic motions. No attempt at exhaustive treatment is made in Chapter 1; subsequent chapters will develop many of the ideas in more detail.

1.2 BRIEF HISTORY OF VIBRATION

People became interested in vibration when the first musical instruments, probably whistles or drums, were discovered. Since then, people have applied ingenuity and critical investigation to study the phenomenon of vibration. Galileo discovered the relationship between the length of a pendulum and its frequency and observed the resonance of two bodies that were connected by some energy transfer medium and tuned to the same natural frequency. Further, he observed the interrelationships of the density, tension, length, and frequency of a vibrating string [1.1]. Although it had long been understood that sound was related to the vibration of a mechanical system, it was not clear that pitch is determined by the frequency of vibration until Galileo found the result. At about the same time as Galileo, Hooke showed the relationship between frequency and pitch.

Among mathematicians, Taylor, Bernoulli, d'Alembert, Euler, Lagrange, and Fourier made valuable contributions to the development of vibration theory. Wallis and Sauveur observed, independently, the phenomenon of mode shapes (with stationary points, called nodes) in vibrating strings. They also established that the frequency of the second mode is twice that of the first and the frequency of the third mode three times that of the first. Sauveur is credited with coining the term *fundamental* for the lowest frequency and *harmonics* for the others. Bernoulli first proposed the principle of linear superposition of harmonics: Any general configuration of free vibration is made up of the configurations of individual harmonics, acting independently in varying strengths [1.2].

After the enunciation of Hooke's law of elasticity in 1676, Euler (1744) and Bernoulli (1751) derived the differential equation governing the lateral vibration of prismatic bars and investigated its solution for the case of small deflections. In 1784, Coulomb did both theoretical and experimental studies of the torsional oscillations of a metal cylinder suspended by a wire.

There is an interesting story related to the development of the theory of vibration of plates [1.3]. In 1802, Chladni developed the method of placing sand on a vibrating plate to find its mode shapes and observed the beauty and intricacy of the modal patterns of the vibrating plates. In 1809, the French Academy invited Chladni to give a demonstration of his experiments. Napoleon Bonaparte, who attended the

meeting, was very impressed and presented a sum of 3000 francs to the Academy, to be awarded to the first person to give a satisfactory mathematical theory of the vibration of plates. By the closing date of the competition in October, 1811, only one candidate, Sophie Germain, had entered the contest. But Lagrange, who was one of the judges, noticed an error in the derivation of her differential equation of motion. The Academy opened the competition again, with a new closing date of October, 1813. Sophie Germain again entered the contest, presenting the correct form of the differential equation. However, the Academy did not award the prize to her because the judges wanted physical justification of the assumptions made in her derivation. The competition was opened once more. In her third attempt, Sophie Germain was finally awarded the prize in 1816, although the judges were not completely satisfied with her theory. In fact, it was later found that her differential equation was correct but that the boundary conditions were erroneous. The correct boundary conditions for the vibration of plates were given in 1850 by Kirchoff.

After this, vibration studies were done on a number of practical mechanical and structural systems. In 1877, Lord Rayleigh published his book on the theory of sound [1.4]; it is considered a classic on the subject of vibrations even today. Notable among the many contributions of Rayleigh is the method of finding the fundamental frequency of vibration of a conservative system by making use of the principle of conservation of energy—now known as Rayleigh's method [1.5]. In 1902, Frahm investigated the importance of torsional vibration study in the design of propeller shafts of steamships. The dynamic vibration absorber, which involves the addition of a secondary spring-mass system to eliminate the vibrations of a main system, was also proposed by Frahm in 1909. Among the modern contributors to the theory of vibrations, the names of Stodola, Timoshenko, and Mindlin are notable. Stodola's method of analyzing vibrating beams is also applicable to turbine blades. The works of Timoshenko and Mindlin resulted in improved theories of vibration of beams and plates.

It has long been recognized that many basic problems of mechanics, including those of vibrations, are nonlinear. Although the linear treatments commonly adopted are quite satisfactory for most purposes, they are not adequate in all cases. In nonlinear systems, there often occur phenomena that are theoretically impossible in linear systems. The mathematical theory of nonlinear vibrations began to develop in the works of Poincaré and Lyapunov at the end of the last century. After 1920, studies undertaken by Duffing and van der Pol brought the first definite solutions into the theory of nonlinear vibrations and drew attention to its importance in engineering. In the last 20 years, authors like Minorsky and Stoker have endeavored to collect the main results concerning nonlinear vibrations in the form of monographs [1.6, 1.7].

Random characteristics are present in diverse phenomena such as earthquakes, winds, transportation of goods on wheeled vehicles, and rocket and jet engine noise. It became necessary to devise concepts and methods of vibration analysis for these random effects. Although Einstein considered Brownian movement, a particular type of random vibration, as long ago as 1905, no applications were investigated until 1930. The introduction of the correlation function by Taylor in 1920 and of the

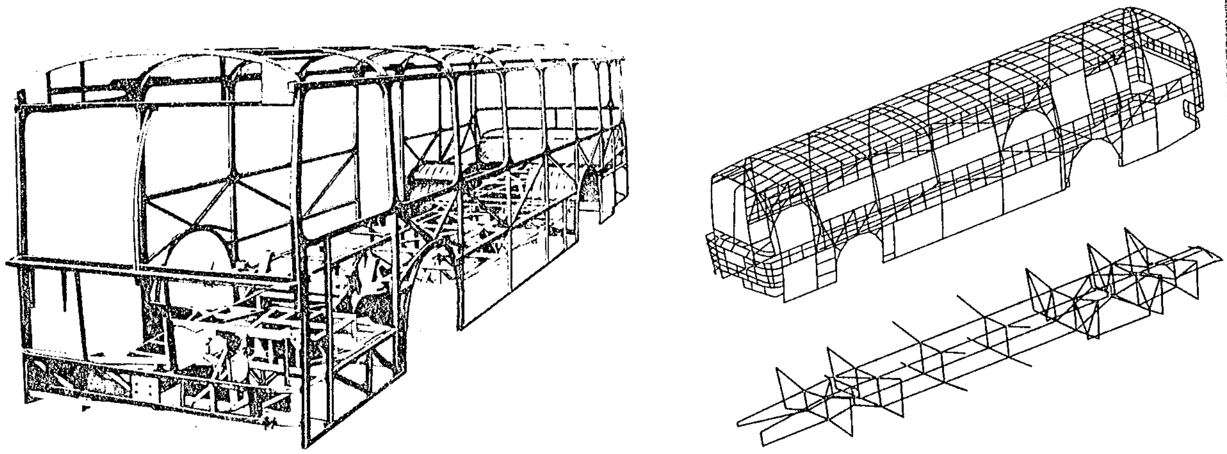


Figure 1.1 Finite element idealization of the body of a bus [1.12]. *(Reprinted with permission © 1974 Society of Automotive Engineers, Inc.)*

spectral density by Wiener and Khinchin in the early 1930s opened new prospects for progress in the theory of random vibrations. Papers by Lin and Rice, published between 1943 and 1945, paved the way for the application of random vibrations to practical engineering problems. The monographs of Crandall and Mark, and Robson systematized the existing knowledge in the theory of random vibrations [1.8, 1.9].

Until about 25 years ago, vibration studies, even those dealing with complex engineering systems, were done by using gross models, with only a few degrees of freedom. However, the advent of high-speed digital computers in the 1950s made it possible to treat moderately complex systems and to generate approximate solutions in semi-closed form, relying on classical solution methods but using numerical evaluation of certain terms that can not be expressed in closed form. The simultaneous development of the finite element method enabled engineers to use digital computers to conduct numerically detailed vibration analysis of complex mechanical, vehicular, and structural systems displaying thousands of degrees of freedom [1.10, 1.11]. Figure 1.1 shows the finite element idealization of the body of a bus [1.12].

1.3 IMPORTANCE OF THE STUDY OF VIBRATION

Early scholars in this field concentrated their efforts on understanding the natural phenomena and developing mathematical theories to describe the vibration of physical systems. In recent times, many investigations have been motivated by the

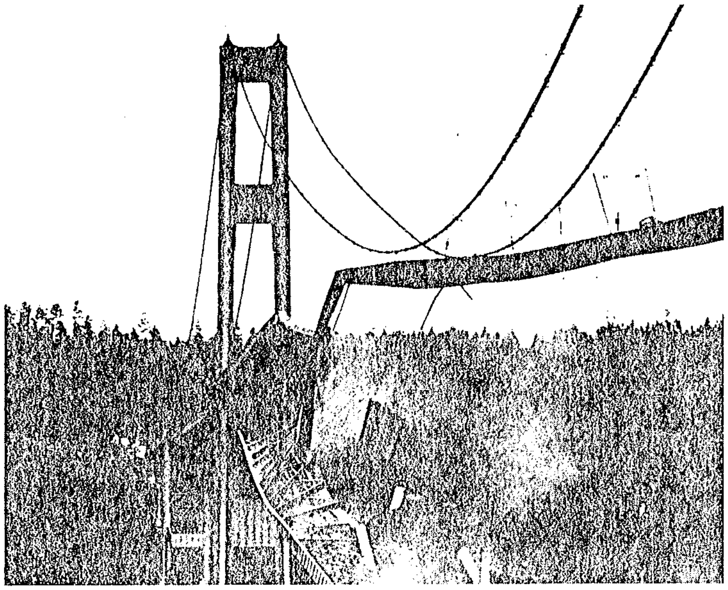


Figure 1.2 Tacoma Narrows bridge during wind-induced vibration. The bridge opened on 1 July 1940 and collapsed on 7 November 1940. *(Farquharson photo, Historical Photography, Collection, University of Washington Libraries.)*

engineering applications of vibration, such as the design of machines, foundations, structures, engines, turbines, and control systems.

Most prime movers have vibrational problems due to the inherent unbalance in the engines. The unbalance may be due to faulty design or poor manufacture. Imbalance in diesel engines, for example, can cause ground waves sufficiently powerful to create a nuisance in urban areas. The wheels of some locomotives can rise more than a centimeter off the track at high speeds due to unbalance. In turbines, vibrations cause spectacular mechanical failures. Engineers have not yet been able to prevent the failures that result from blade and disk vibrations in turbines. Naturally, the structures designed to support heavy centrifugal machines, like motors and turbines, or reciprocating machines, like steam and gas engines and reciprocating pumps, are also subjected to vibration. In all these situations, the structure or machine component subjected to vibration can fail because of material fatigue resulting from the cyclic variation of the induced stress. Furthermore, the vibration causes more rapid wear of machine parts such as bearings and gears and also creates excessive noise.

Whenever the natural frequency of vibration of a machine or structure coincides with the frequency of the external excitation, there occurs a phenomenon known as *resonance*, which leads to excessive deflections and failure. The literature is full of accounts of system failures brought about by resonance and excessive vibration of components and systems (see Fig. 1.2). Because of the devastating effects that vibrations can have on machines and structures, vibration testing [1.13] has become a standard procedure in the design and development of most engineering systems (see Fig. 1.3).

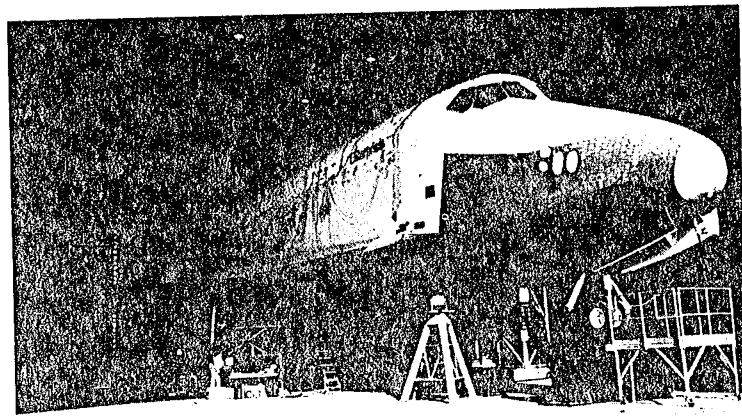
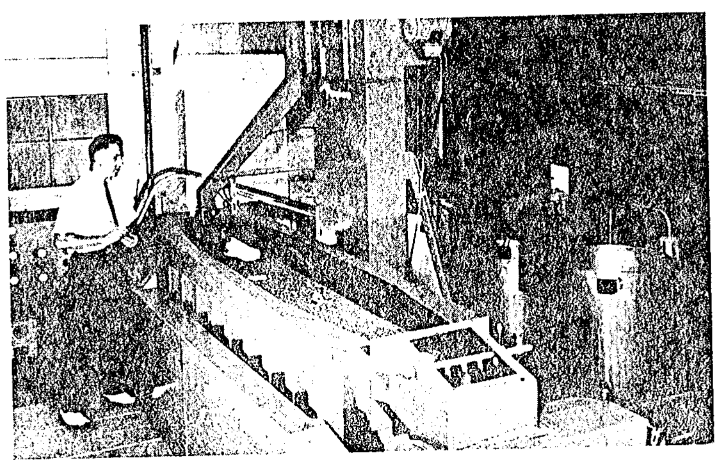
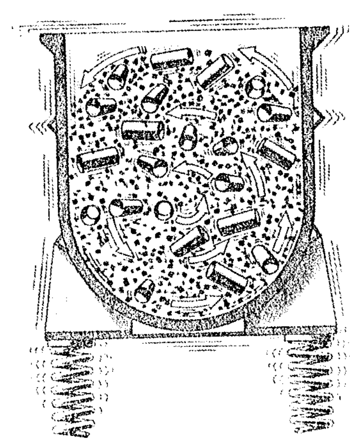


Figure 1.3 Vibration testing of the space shuttle Enterprise. *(Courtesy of NASA.)*

In many engineering systems, a human being acts as an integral part of the system. The transmission of vibration to human beings results in discomfort and loss of efficiency [1.14]. Vibration of instrument panels can cause their malfunction or difficulty in reading the meters [1.15]. Thus one of the important purposes of vibration study is to reduce vibration through proper design of machines and their mountings. In this connection, the mechanical engineer tries to design the engine or

Figure 1.4 Vibratory finishing process. *(Reprinted courtesy of the Society of Manufacturing Engineers © 1964 The Tool and Manufacturing Engineer.)*





machine so as to minimize unbalance, while the structural engineer tries to design the supporting structure so as to ensure that the effect of the imbalance will not be harmful [1.16].

In spite of its detrimental effects, vibration can be utilized profitably in several industrial applications. In fact, the applications of vibratory equipment have increased considerably in recent years [1.17]. For example, vibration is put to work in vibratory conveyors, hoppers, sieves, and compactors. Vibration is also used in pile driving, vibratory testing of materials, vibratory finishing processes, and electronic circuits to filter out the unwanted frequencies (see Fig. 1.4). Vibration has been found to improve the efficiency of certain machining, casting, forging, and welding processes. It is employed to simulate earthquakes for geological research and also to conduct studies in the design of nuclear reactors.

1.4 BASIC CONCEPTS OF VIBRATION

1.4.1 Vibration

Any motion which repeats itself after an interval of time is called *vibration* or *oscillation*. The swinging of a pendulum and the motion of a plucked string are typical examples of vibration. The theory of vibration deals with the study of oscillatory motions of bodies and the forces associated with them.

1.4.2 Elementary Parts of Vibrating Systems

A vibratory system, in general, includes a means for storing potential energy (spring or elasticity), a means for storing kinetic energy (mass or inertia), and a means by which energy is gradually lost (damper).

The vibration of a system involves the transfer of its potential energy to kinetic energy and kinetic energy to potential energy, alternately. If the system is damped, some energy is dissipated in each cycle of vibration and must be replaced by an external source if a state of steady vibration is to be maintained.

As an example, consider the vibration of the simple pendulum shown in Fig. 1.5. Let the bob of mass m be released after giving it an angular displacement θ. At position 1 the velocity of the bob and hence its kinetic energy is zero. But it has a potential energy of magnitude $mgl(1-\cos\theta)$ with respect to the datum position 2. Since the gravitational force mg induces a torque $mgl\sin\theta$ about the point O, the bob starts swinging to the left from position 1. This imparts to the bob a certain angular acceleration in the clockwise direction, and by the time it reaches position 2, all of its potential energy will be converted into kinetic energy. Hence the bob will not stop in position 2, but will continue to swing to position 3. As it crosses the mean position 2, however, a counterclockwise torque starts acting on the bob (again due to gravity) and causes the bob to decelerate. Eventually, the velocity of the bob reduces to zero at the left extreme position. By this time, all the kinetic energy of the bob will be converted to potential energy. Again the gravity torque continues to act and gives the bob a counterclockwise velocity. Hence the bob starts swinging back with progressively increasing velocity and overshoots the mean position again. This sequence of events keeps on repeating, and the pendulum keeps on performing

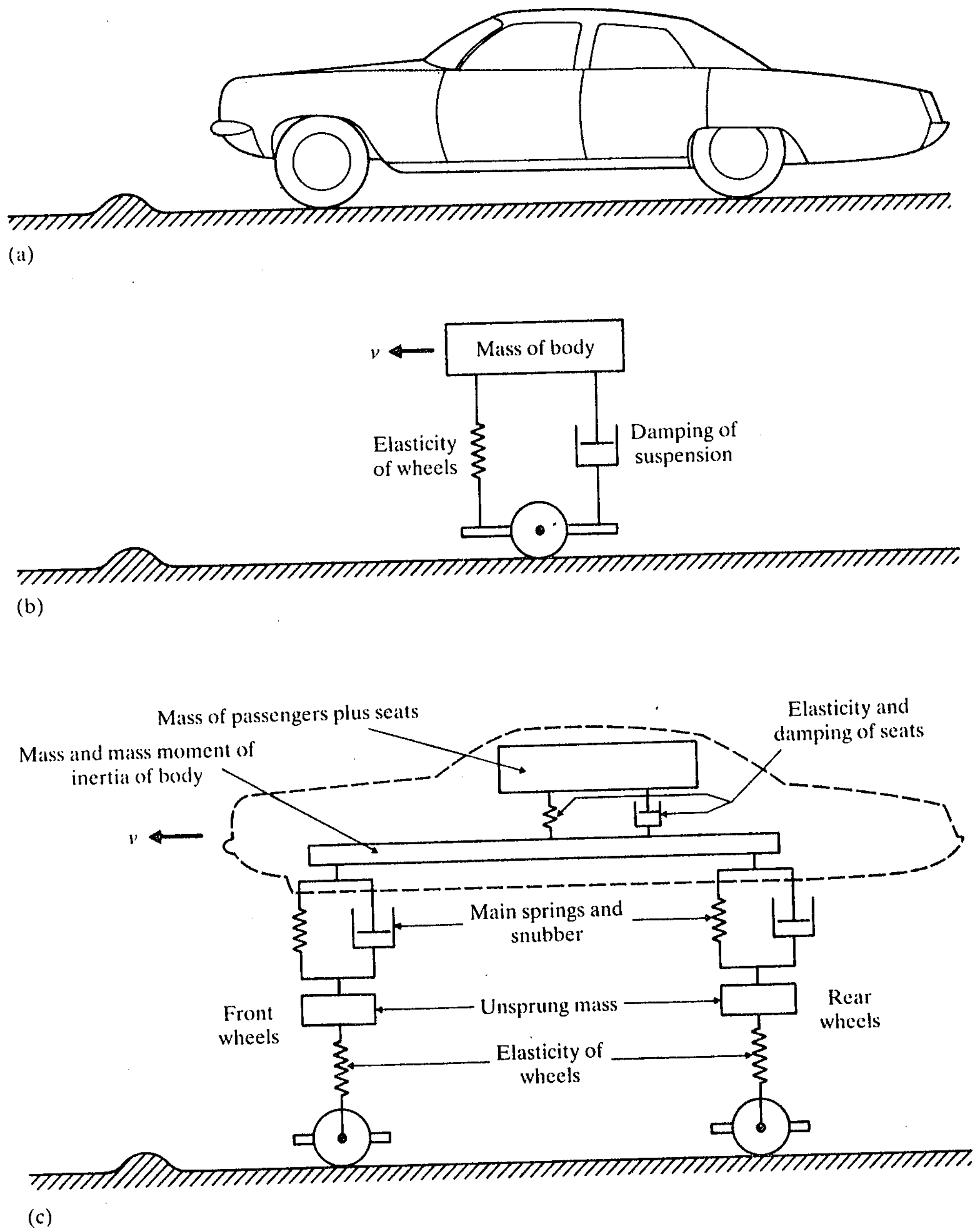


Figure 1.11 Modeling of an automobile.

MECHANICAL VIBRATIONS

SESSION #1

- **PASS OUT COURSE OUTLINE**
- **GO THROUGH HISTORY OF MECH VIB**
- **START ODE REVIEW**

- WANT TO STUDY OSCILLATORY MOTION OF MECHANICAL SYSTEMS

- HISTORY OF VIBS
  - FIRST INTEREST BEGAN W/ MUSICAL INSTRUMENTS (WHISTLES, DRUMS,
  - GALILEO - FOUND RELATIONSHIP BETWEEN LENGTH & FREQ (PENDULUM
    - OBSERVED RESONANCE BETWEEN TWO BODIES CONNECTED BY SPRING.

RESONANCE - WHEN FREQ. OF FORCING FUNCTION = NATURAL FREQ. OF SYSTEM

- OBSERVED RELATIONSHIP OF DENSITY, TENSION, LENGTH & FREQ. OF A VIBRATING STRING.
- PITCH OF SOUND RELATED TO FREQ OF VIBRATION
- HOOKE ALSO FOUND THIS ABOUT THE SAME TIME
- MATHEMATICIANS CONTRIBUTED TO THE UNDERSTANDING OF VIBRATIONS VIA DEVELOPMENT OF DIFFERENTIAL EQUATIONS BOUNDARY CONDITIONS & SOLUTIONS
- early 1700's WALLIS & SAUVEUR OBSERVED VIBRATING STRING MODE SHAPES
  - DEVELOPED RELATIONSHIP BETWEEN FUNDAMENTAL FREQ & HIGHER FREQ.
- SAUVEUR COINED TERMS FUNDAMENTAL = LOWEST FREQ.

  HARMONICS = OTHER FREQ.

* BERNOULLI – PROPOSED LINEAR SUPERPOSITION OF HARMONICS TO DESCRIBE MOTION

* EULER (1744) & BERNOULLI (1751) INVESTIGATED VIB. OF PRISMATIC BARS
* COULOMB (1784) " TORSIONAL " OF METAL CYLINDER
* SOPHIE GERMAIN (1809 – 1816) & KIRCHOFF (1850) INVESTIGATED VIBRATIONS OF PLATES

* LORD RAYLEIGH (1877) ⇒ THEORY OF SOUND
  - USED PRINCIPLE OF CONSERVATION OF ENERGY TO FIND LOWEST FREQ. ; RAYLEIGH'S METHOD.

* FRAHM (1902) STUDIED TORSIONAL VIB IN PROPELLER SHAFTS

* • VIBRATIONS INVOLVING SMALL DEFORMATIONS – LINEAR VIB
* • LARGE DEFORMATIONS – NONLINEAR VIBRATIONS

NONLINEAR VIB. STUDIED BY POINCARÉ & LYAPUNOV LATE 1800'S
1920 – DUFFING & VAN DER POL HELPED BY STUDYING PROBLEMS W/ NONLINEAR RESTORING FORCES & SELF EXCITED OSCILLATIONS

* NON LINEAR VIBRATIONS MONOGRAPHS – STOKER & MINORSKY

* RANDOM VIBRATIONS – (ALSO KNOWN AS NONDETERMINISTIC)
  * EARTHQUAKES
  * BROWNIAN MOTION – STUDIED BY EINSTEIN (1905)
  * MISSILES, SPACE VEHICLES – DUE TO BUFFETING (LAUNCH)
  * PACKAGED ASSEMBLIES – EXPERIENCE BUFFETING (IN SHIPMENT)

* TAYLOR (1920) INTRODUCED CORRELATION FN
* WIENER & KHINCHIN (1930s) INTRODUCED SPECTRAL DENSITY

} STATISTICAL METHODS USED TO AID IN THE DESCRIPTION OF RANDOM FUNCTIONS

- THE ADVENT OF COMPUTERS –

BEFORE – DEALT W/ GROSS MODELS

AFTER – DEALT W/ MORE REFINED MODELS (SEE LAST PAGE OF HANDOUT

— IMPORTANCE OF VIBRATIONS

- IMBALANCE IN ENGINES CAUSE VIBS WHICH CAN BE DEVASTATING
  - LOOSEN PARTS THAT CAN FLY OFF (TURBINE BLADES, DISKS)
  - RESULT IN GROUND WAVES → CAUSE NUISANCE IN URBAN AREAS
  - CAUSE FAILURE THROUGH FATIGUE
  - CAUSE WEAR MORE RAPIDLY
  - EXCESSIVE NOISE
- RESONANCE CAUSE EXCESSIVE DEFLECTIONS (TACOMA BRIDGE)

- MAN/MACHINE INTERFACE CAUSES DISCOMFORT DUE TO VIBS
- CAN CAUSE MALFUNCTION OF INSTRUMENTS

POSITIVE USES – DESIGN OF REACTORS IN NUCLEAR INDUSTRY TO SIM. EARTHQUAKE GROUND MOTIONS

- GEOLOGICAL RESEARCH TO SIMULATE EARTHQUAKE
- CAN IMPROVE EFFICIENCY OF SOME MANUFACTURING PROCESSES

~~BASIC CONCEPTS~~

~~- VIBRATION OR OSCILLATION Any motion which repeats itself after an interval of time~~

~~MOTION MAY BE REGULAR OR DETERMINISTIC~~

~~- IRREGULAR OR NONDETERMINISTIC~~

~~- WE STUDY THE MOTION OF BODIES OR SYSTEMS AS WELL AS THE FORCES THAT ACCOMPANY THIS MOTION OR IS CAUSED BY THIS OSCILLATORY MOTION~~

ODE review

$\boxed{\text{ORDER of DIF EQ. = HIGHEST DERIV. IN EQ.}}$

$y = y(x)$ $\quad y^{IV} + 2y'' = 0$ $\quad$ 4^{th} order

$\boxed{\text{ORDINARY}}$ DIF EQ $\boxed{\text{IF } y = y(x) \text{ only}}$ 1 variable

PARTIAL DIF EQ if $y = y(x,z)$

$$\frac{\partial^2 y}{\partial x^2} + \alpha^2 \frac{\partial^2 y}{\partial z^2} = 0$$

IF COEFF. OF DERIVATIVES – DIF. EQ. w/ CONSTANT COEFF

$$u''(t) + c u'(t) + m u(t) = 0.$$

IF COEFF OF DERIV. ARE NOT CONSTANT

$$a(x) y'' + b(x) y' + c(x) y = 0$$

$\boxed{\text{IF COEFF OF DERIV \& FN ARE FNS OF X only \& power of derivative is 1} \Rightarrow \text{LINEAR}}$

$$a(x) y'' + b(x) y' + c(x) y = 0$$

$$a(x) y'' + b(x)(y')^2 + c(x) y = 0 \qquad \text{NON LINEAR}$$

$$a(x) y'' + b(x) y' y = 0 \qquad \text{NONLINEAR}$$

$$a(x) y'' + b(x) \sin y = 0$$

IF $\boxed{\text{RHS is ZERO}}$ IT IS A $\boxed{\text{HOMOGENEOUS EQUATION}}$

FIRST ORDER DIFF. EQ. – CONST COEFF

$$y' + a y = 0$$

$\frac{dy}{dx} + ay = 0 \Rightarrow \frac{dy}{y} = -a\,dx$

$\ln y = -ax + \ln C$

$y = Ce^{-ax}$

SOLUTION TO THIS is in the form $\boxed{y = Ce^{px}}$

$\Rightarrow y' + ay = Ce^{px}[p + a] = 0 \qquad C \neq 0 \quad e^{px} \neq 0$

$\Rightarrow p = -a$ $\quad \therefore$

$$Ce^{-ax} = y \qquad \text{GENERAL SOLUTION}$$

- TO OBTAIN C NEED initial condition
- FOR EVERY ORDER OF DERIV NEED I.C.

$$y(x=x_0) = y_0$$

$$x = x_0 \quad y = Ce^{-ax_0} \Rightarrow C = y_0 e^{ax_0}$$

$$\therefore \quad y = y_0 e^{-a(x-x_0)}$$

AN ~~FIRST~~ O.D.E. w/ INITIAL CONDITION INITIAL VALUE PROBLEM

WHAT ABOUT IF THE RHS $\neq 0$ (NONHOMOGENEOUS PROBLEM

$$y' + ay = g(x)$$

MULTIPLY BOTH SIDES BY A FN $\mu(x)$ $\ni$. $\mu(x)[y'+ay] = [\mu y]'$

$$\mu' y + \mu y' = \mu(y' + ay) = \mu(x) g(x)$$

$$\mu' y = \mu a y \quad \text{or} \quad \frac{\mu'}{\mu} = a \quad \text{or} \quad (\ln \mu)' = a$$

$$\ln \mu = ax + \ln C \Rightarrow \boxed{\mu = Ce^{ax}}$$

$\mu(x)$ IS AN INTEGRATION FACTOR

$$\therefore \quad [\mu y]' = \mu g(x)$$

$$\mu y = \int^x \mu(t) g(t)\, dt + \text{Const}$$

$$y = \frac{1}{\mu(x)} \int^x \mu(t) g(t)\, dt + \frac{\text{Const}}{\mu(x)}$$

$$\boxed{y(x) = e^{-ax} \int^x e^{at} g(t)\, dt + \text{Const}\, e^{-ax}}$$

SINCE A CONSTANT REMAINS UNKNOWN NEED AN INITIAL CONDITION TO COMPLETELY DEFINE $y(x)$

IF $y(x_0) = y_0$ $\quad y(x) = e^{-ax} \int_{x_0}^x e^{at} g(t)\, dt + y_0 e^{a(x_0) - ax}$

SUPPOSE

$$y' + p(x)\,y = g(x)$$

SUPPOSE $\exists$ a FN $\mu(x)$ $\ni$.

$$\mu y' + \mu p y = [\mu y]' = \mu\, g(x)$$

$$\mu y' + \mu p y = \mu y' + \mu' y$$

$$\Rightarrow \quad \mu p = \mu' \quad \text{or} \quad p = \frac{d\mu/dx}{\mu} \quad \Rightarrow \quad p(x)dx = \frac{d\mu}{\mu}$$

$$\int^x p(t)dt = \int \frac{d\mu}{\mu} = \ln\mu \; \ln C \quad \text{or} \quad \mu(x) = Ce^{\int^x p(t)dt}$$

$$\therefore \quad [\mu y]' = \mu(x) g(x)$$

$$\mu y = \int^x \mu(t)\, g(t)\, dt + \text{Const.}$$

$$y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(t) g(t) dt + \text{const.}\right]$$

w/const

$$y(x) = \frac{1}{\mu(x)}\left[\int_{x_0}^x \mu(t) g(t) dt + \mu(x_0) y_0\right]$$

END LESSON # 1

SESSION #2

EXAMPLE

$$y' + 3y = x. \qquad \Rightarrow \quad y' + ay = g(x)$$

$$\mu(x) = Ce^{ax} \Rightarrow Ce^{3x}$$

$$y(x) = \frac{1}{\cancel{C}e^{3x}} \int^x \cancel{C}e^{3t}\, t\, dt + \text{Const}\, e^{-3x}$$

$$= e^{-3x}\int^x te^{3t} dt + \text{const}\, e^{-3x}$$

$$\left[\begin{array}{ll} u = t & dv = e^{3t}dt \\ du = dt & v = \frac{1}{3}e^{3t} \end{array}\right.$$

$$\int u\,dv = uv - \int v\,du = \frac{1}{3}te^{3t}\Big|^x - \int^x \frac{1}{3}e^{3t}dt$$

$$= \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}$$

$$y(x) = e^{-3x}\left[\frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x}\right] + \text{const}\, e^{-3x}$$

$$= \frac{1}{3}x - \frac{1}{9} + \text{Const}\, e^{-3x}$$

if we assume that at $x=0$ $y=1$ $y(x=0)=1$

$$y(0)=1 = \tfrac{1}{3}(0) - \tfrac{1}{9} + C\cdot \cancel{e^0}^{\,1} = C - \tfrac{1}{9} \qquad C = \tfrac{10}{9}.$$

$$\therefore\ y(x) = \tfrac{1}{3}x - \tfrac{1}{9} + \tfrac{10}{9}e^{-3x}$$

EXAMPLE # 2

$$y' - 2xy = 1 \qquad \text{and IC} \quad y(0)=1$$

$$\Rightarrow\ y' + p(x)\,y = g(x) \qquad p(x) = -2x \qquad g(x)=1$$

$$\mu(x) = C\,e^{\int^x p(t)dt} = C\,e^{\int^x -2t\,dt} = C\,e^{-t^2|^x}$$

$$\mu(x) = C\,e^{-x^2}$$

$$y(x) = \frac{1}{\mu(x)}\left[\int^x \mu(t)\,g(t)\,dt + \text{const.}\right]$$

$$= \frac{1}{Ce^{-x^2}}\left[\int^x \cancel{C}e^{-t^2}\,1\,dt + \text{const}\right]$$

$$= e^{+x^2}\int^x e^{-t^2}dt + \text{const}\,e^{x^2}$$

$$y(0)=1 \qquad 1 = 1\int^0 e^{-t^2}dt + \text{const}\cdot 1$$

$$\text{const} = 1 - \int^0 e^{-t^2}dt$$

$$y(x) = e^{x^2}\int^x e^{-t^2}dt + e^{x^2} - e^{x^2}\int^0 e^{-t^2}dt$$

$$y(x) = e^{x^2}\int_0^x e^{-t^2}dt + e^{x^2} \qquad + e^{x^2}\int_0 e^{-t^2}dt$$

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \qquad \boxed{y(x) = \frac{\sqrt{\pi}}{2}e^{x^2}\,\text{erf}(x) + e^{x^2}}$$

SECOND order ODE

CONCENTRATED ON HOMOGENEOUS EQS w/ CONSTANT COEFF.

$$ay'' + by' + cy = 0 \qquad \text{a sol. is} \qquad Ce^{rx} = y$$

$$\underbrace{(ar^2 + br + c)}_{=0} Ce^{rx} = 0 \qquad C \neq 0 \qquad e^{rx} \neq 0$$

$\boxed{\text{CHARACTERISTIC EQ.}}$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

if $b^2 - 4ac > 0$ real & unequal roots

$b^2 - 4ac = 0$ real & equal roots

if $b^2 - 4ac < 0$ imaginary roots that are complex conjugates

EXAMPLE 1 real roots - uneq

$$y'' + 3y' + 2y = 0 \quad \Rightarrow \quad r^2 + 3r + 2 = 0 \qquad (r+1)(r+2) = 0$$

$r = -1$, $r = -2$

$$y = C_1 e^{-x} + C_2 e^{-2x}$$

Solutions are linearly independent.

$\boxed{\text{since it's 2nd order, have 2 unknown constants.}}$

$\boxed{\text{NEED 2 initial conditions to define } y \text{ uniquely}}$

EXAMPLE 2 real roots - equal

$$y'' + 4y + 4 = 0 \quad \Rightarrow \quad r^2 + 4r + 4 = 0 \qquad (r+2)^2 = 0$$

$$y = C_1 e^{-2x} + 2^{nd} \text{ solution}$$

How do we find the second solution - METHOD OF REDUCTION OF ORDER

METHOD IS GOOD IN CASE OF THE GENERAL EQU.

$$y'' + p(x)y' + q(x)y = 0$$

ASSUME we know one solution $y_1(x)$

ASSUME the second solution is $y = v(x)y_1(x)$

$$y' = v'y_1 + vy_1' \qquad y'' = v''y_1 + 2v'y_1' + vy_1''$$

PUT INTO DE

$$v\underbrace{(y_1'' + py_1' + qy_1)}_{=0} + v'(2y_1' + py_1) + v''y_1 = 0$$

since y_1 satisfies ODE.

THIS IS A FIRST ORDER DE FOR v'

$$v'' + (2y_1'/y_1 + p)v' = 0$$

$$v' = e^{\left[-\int^x (p(t) + 2y_1'/y_1)\, dt\right]}$$

$$= e^{-\int^x p(t)dt} \underbrace{\underbrace{e^{-\int^x 2y_1'/y_1\, dt}}_{e^{-2\ln y_1}}}_{\frac{1}{y_1^2}}$$

$$v' = \frac{}{y_1^2(x)} e^{-\int^x p(t)dt} = c\, u(x).$$

$$v = \int^x u(s)ds + \text{Const.}$$

$$y_2 = y_1 v = y_1(x)\int^x u(s)ds + \cancel{\text{const. } y_1(x)}$$

DONT NEED SINCE y_2 must be indep of y_1

$$\boxed{y_2 = y_1(x)\int^x \frac{1}{y_1^2(s)} e^{-\int^s p(t)dt}\, ds}$$

IN OUR PROB $\quad p(t) = 4 \qquad y_1(s) = e^{-2s}$

$$y_2 = e^{-2x} \int^x \frac{1}{(e^{-2s})^2} e^{-\int^s 4\,dt} \, ds$$

$$= e^{-2x} \int^x \frac{1}{e^{-4s}} e^{-4t\big|^s} \, ds$$

$$e^{-2x} \int^x \frac{\cancel{e^{-4s}}}{\cancel{e^{-4s}}} \, ds = x e^{-2x}$$

$$y_2 = x e^{-2x}$$

$\therefore$ FOR example 2 $\qquad y = C_1 e^{-2x} + C_2 x e^{-2x}$

FOR HOMOG EQ W/ CONSTANT COEFF IF ROOTS are same and one solution is $y_1 = e^{rx}$ $\qquad y_2 = x e^{rx}$

EXAMPLE #3

$$y'' + y' + y = 0 \implies r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1 - 4 \cdot 1 \cdot 1}}{2} = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$y = C_1 e^{\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)x} + C_2 e^{\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)x}$$

$$= e^{-\frac{1}{2}x} \left\{ C_1 e^{i\frac{\sqrt{3}}{2}x} + C_2 e^{-i\frac{\sqrt{3}}{2}x} \right\}$$

EULER FORMULA

~~DE MOIVRE'S THEOREM~~

$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots + \frac{(i\theta)^n}{n!}$$

$$= 1 + \frac{i\theta}{1!} - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots + (-1)^n \frac{\theta^{2n}}{(2n!)} + i(-1)^n \frac{\theta^{2n+1}}{(2n+1)!}$$

$$= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right)$$

$$= \cos\theta + i\sin\theta$$

$$\boxed{c_1 e^{i\theta} + c_2 e^{-i\theta}} = C_1(\cos\theta + i\sin\theta) + C_2(\cos\theta - i\sin\theta)$$

$$= \cos\theta\,[C_1 + C_2] + i\sin\theta\,[C_1 - C_2]$$

$$= \boxed{C_1'\cos\theta + C_2'\sin\theta}$$

C_1 & C_2 are complex no.

C_1' & C_2' are real no.

- note that e^{rx} for r being complex: satisfies $\frac{d}{dx}(e^{rx}) = re^{rx}$ $\cancel{(e^f)' =}$

THUS
$$y(x) = e^{-x/2}\left\{C_1 e^{\frac{i\sqrt{3}x}{2}} + C_2 e^{-\frac{i\sqrt{3}x}{2}}\right\}$$
$$= e^{-x/2}\left\{C_1'\cos\frac{\sqrt{3}x}{2} + C_2'\sin\frac{\sqrt{3}x}{2}\right\}$$

THUS if the roots are complex conjugates $r_1 = \lambda + i\mu$ $r_2 = \lambda - i\mu$ then

$$\boxed{y = e^{\lambda x}\left[C_1\cos\mu x + C_2\sin\mu x\right]}$$

$$\mu = \frac{\sqrt{4ac - b^2}}{2a}$$
$$\lambda = -b/2a$$

Linear Independence of Solutions

For a Second Order Linear Diff Equation - Homogeneous

if $y_1 y_2' - y_1' y_2 \neq 0 = W_{12}(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

at ~~some~~ every pt in the interval of definition of the fn y

then $y = c_1 y_1 + c_2 y_2$

Non homogeneous - Second Order Linear Diff. Equation (SOLDE)

$$y'' + p(x) y' + q(x) y = g(x)$$

Solution

$$y = y_p + y_h$$

where y_h is solution to $y'' + p(x) y' + q(x) y = 0$

y_p is any solution of the nonhomog SOLDE

METHOD OF UNDETERMINED COEFFICIENTS

suppose $g(x) = \sum_{i=1}^{n} g_i(x)$

then $y_p = \sum_{i=1}^{n} y_{p_i}(x)$ when y_{p_i} is solution for $g_i(x)$

EXAMPLE

$$y'' + 4y = 1 + x + \sin x$$

homog solution to $y'' + 4y = 0$ is $y_h = C_1 \sin 2x + C_2 \cos 2x$

let $g_1(x) = 1$ $\quad y_{p_1} = C_1$ $\quad y''_{p_1} + 4y_{p_1} = 4C_1 = g_1(x) = 1$ $\quad C_1 = 1/4$

let $g_2(x) = x$ $\quad y_{p_2} = C_2 x$ $\quad y''_{p_2} + 4y_{p_2} = 4C_2 x = g_2(x) = x$ $\quad C_2 = 1/4$

let $g_3(x) = \sin x$ $\quad y_{p_3} = C_3 \sin x$ $\quad y''_{p_3} + 4y_{p_3} = 3C_3 \sin x = g_3(x) = \sin x$ $\quad C_3 = 1/3$

$$y_p = \sum y_{p_i} = 1/4 + 1/4\, x + 1/3 \sin x$$

$$y = y_p + y_h$$

$y'' + y = 0 \quad (\sin x, \cos x$

$y'' + y = \cos x \qquad Ax\sin x + Cx\cos x = y$

$A\sin x + Ax\cos x - Cx\sin x + C\cos x = y'$

$2A\cos x - Ax\sin x - 2C\sin x - Cx\cos x = y''$

$\therefore 2A\cos x - 2C\sin x = \cos x \quad A = \frac{1}{2} \quad C = 0$

$= \sin x \quad A = 0 \quad C = -\frac{1}{2}$

SESSION #3

SUPPOSE we first look at SOLDE w/ const. coeff.

Suppose $g(x) = \sin nx, \cos nx$ & $r \neq \pm in$

assume $y_p = A \sin nx + B \cos nx$

$g(x) = ax^n$

assume $y_p = Ax^n + Bx^{n-1} + \dots + Ex + F$

$\longrightarrow$ $g(x) = ce^{-px}$ & $y_h = c_1 e^{+rx} + c_2 e^{sx}$

where $r \neq -p \quad s \neq -p$

assume $g(x) = Ae^{-px}$

EXAMPLE $y'' - 3y' - 4y = 2\sin x \Rightarrow \overset{\text{CHAR EQ.}}{(r-4)(r+1) = 0}$; $y_h = c_1 e^{-x} + c_2 e^{4x}$

let $y_p = A\cos x + B \sin x$

put into DE

$$(-A - 3B - 4A)\cos x + (-B + 3A - 4B)\sin x = 2\sin x$$

$$\Rightarrow \quad \underset{0}{\parallel} \qquad \underset{2}{\parallel}$$

$$\Rightarrow A = 3/17 \quad B = -5/17$$

$$y_p = \tfrac{1}{17}(3\cos x - 5\sin x)$$

Suppose $g(x)$ is a linear combination of y_h

$$y'' - 3y' - 4y = 4e^{-x} \qquad y_h = c_2 e^{+4x} + c_1 e^{-x}$$

if $y_p = Ae^{-x}$

$$[A + 3A - 4A]e^{-x} = 4e^{-x}$$ no solution using $y_p = Ae^{-x}$

take $y_p = Axe^{-x} + \quad = (Ax \quad)e^{-x}$

SINCE THE FN $g(x)$ contains a solution of the homog eq. must mult. the homog sol. by a fn whose power is one higher than homog sol.

suppose y_h has solutions $(\sin x, \cos x)$

and $g(x) = \sin x \quad \Rightarrow y_p = (Ax \quad)\sin x + (Bx \quad)\cos x$

$g(x) = x \sin x \quad y_p = (Ax^2 \quad)\sin x + (Cx^2 \quad)\cos x$

METHOD OF VARIATION OF PARAM. (LAGRANGE'S method)

used to solve most general form.

$$y'' + p(x)\,y' + q(x)\,y = g(x)$$

seek a solution of the form

$$y_p = u_1(x)\,y_1(x) + u_2(x)\,y_2(x)$$

where $y_1(x)$ and $y_2(x)$ are linearly indep solutions of the homog equation

$$y_p' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2' \qquad \text{Assume } u_1' y_1 + u_2' y_2 \equiv 0$$

$$y_p'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$$

$$\Rightarrow y_p'' + p y' + q y = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2'' + p\,(u_1 y_1' + u_2 y_2') + q\,(u_1 y_1 + u_2 y_2)$$

$$= u_1 \underbrace{(y_1'' + p y_1' + q y_1)}_{=0} + u_2 \underbrace{(y_2'' + p y_2' + q y_2)}_{=0} + u_1' y_1' + u_2' y_2' = g(x)$$

Thus

$$\left.\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= g(x). \end{aligned}\right\} \qquad \text{first order system for } u_1',\ u_2'$$

solve for $u_1' = \dfrac{-y_2 g}{W(y_1, y_2)}$ $\qquad u_2' = \dfrac{y_1 g}{W(y_1, y_2)}$

since y_1, y_2 are lin indep $W(y_1, y_2) \neq 0$

$$u_1 = -\int^x \frac{y_2(t)\,g(t)\,dt}{W(y_1, y_2)(t)} \qquad u_2 = \int^x \frac{y_1(t)\,g(t)\,dt}{W(y_1, y_2)(t)}$$

and $y_p = u_1 y_1 + u_2 y_2$

EXAMPLE

$$y'' + y = \tan x \qquad \text{let} \quad y_p = u_1(x)\cos x + u_2(x)\sin x$$

$$y_p' = u_1'\cos + u_2'\sin = 0$$

$$-u_1'\sin + u_2'\cos = \tan x$$

$$\begin{pmatrix} \cos & s \\ -\sin & \cos \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \tan x \end{pmatrix}$$

$$-\frac{\tan x \sin}{1} = u_1' \qquad \boxed{u_2' = \frac{\cos \tan}{1} = \sin x}$$

$$-\frac{\sin^2 x}{\cos x} = -\left[\frac{1-\cos^2 x}{\cos x}\right] = -\frac{1}{\cos x} + \cos x = u_1'$$

$$u_2 = -\cos x \qquad u_1 = -\ln|\sec x + \tan x| + \sin x$$

$$\therefore \quad y_p = -\ln| \qquad |\cos x + \cos x \sin x - \sin x \cos x$$

$$y_p = -(\cos x)\ln|\sec x + \tan x|$$

BASIC CONCEPTS

- VIBRATION OR OSCILLATION any motion which repeats itself after an interval of time
  MOTION MAY BE REGULAR OR DETERMINISTIC
- IRREGULAR OR NONDETERMINISTIC (STOCHASTIC)
- WE STUDY THE MOTION OF BODIES OR SYSTEMS AS WELL AS THE FORCES THAT ACCOMPANY THIS MOTION OR IS CAUSED BY THIS OSCILLATORY MOTION

- VIBRATIONS EXIST IN ELECTRIC CIRCUITS
  ACOUSTICS
  CAN BE ELECTROMAGNETIC WAVES
  MECHANICAL SYSTEMS

pluck a string — Ella Fitzgerald & glass

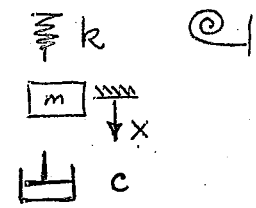
- MECHANICAL VIB CAN CAUSE ACOUSTICAL VIBRATION (VICE VERSA)
- " CAN CAUSE AN ELECTRIC OSCILLATION (")

- BASIC PRINCIPLES ARE THE SAME

- FOR VIBRATIONS TO OCCUR NEED AT LEAST TWO ENERGY STORAGE ELEMENTS

POSSIBILITIES

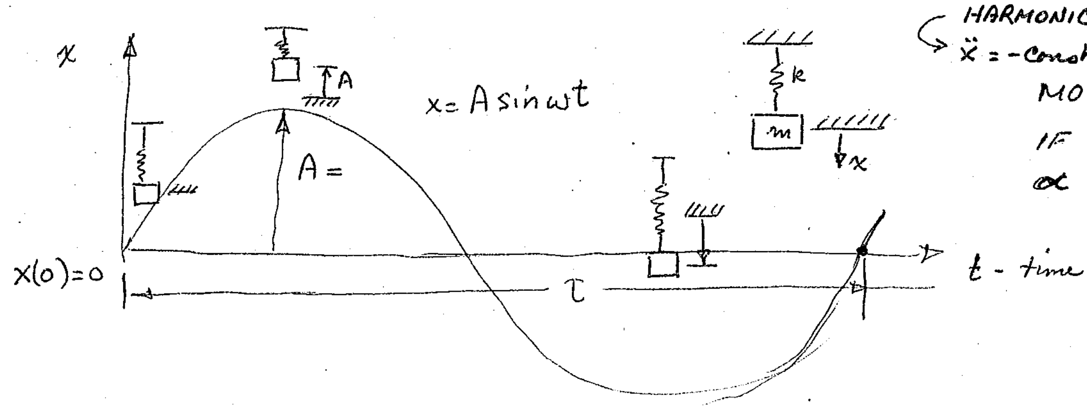
INERTIA or MASS – STORES KINETIC ENERGY
ELASTIC or SPRING MEMBER – " POTENTIAL ENERGY
DAMPER (DASHPOT) – DISSIPATES ENERGY

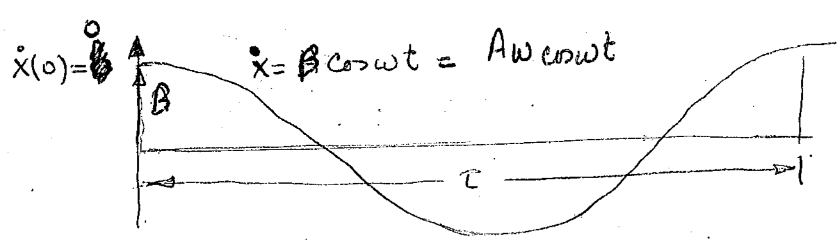


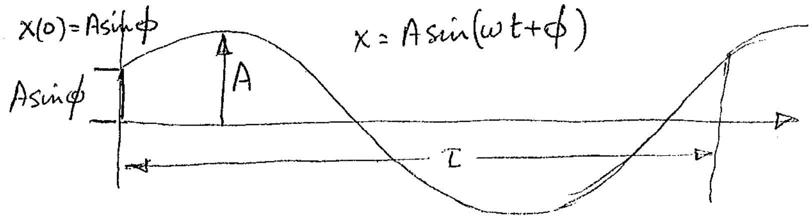
VERTICAL VIBRATIONS OF MASS

- ASSUMPTION – NO HORIZONTAL MOTION OF MASS
- VIBRATION OR OSCILLATION INVOLVES TRANSFER OF ENERGY FROM KE $\rightleftarrows$ PE
- MOTION OF MASS WILL BE CYCLIC OR PERIODIC

PERIODIC HARMONIC MOTION
$\ddot{x} = -\text{Const}\, x$
MOTION OF MASS IF ELASTIC MEMBER $\propto$ TO DEFORMATION





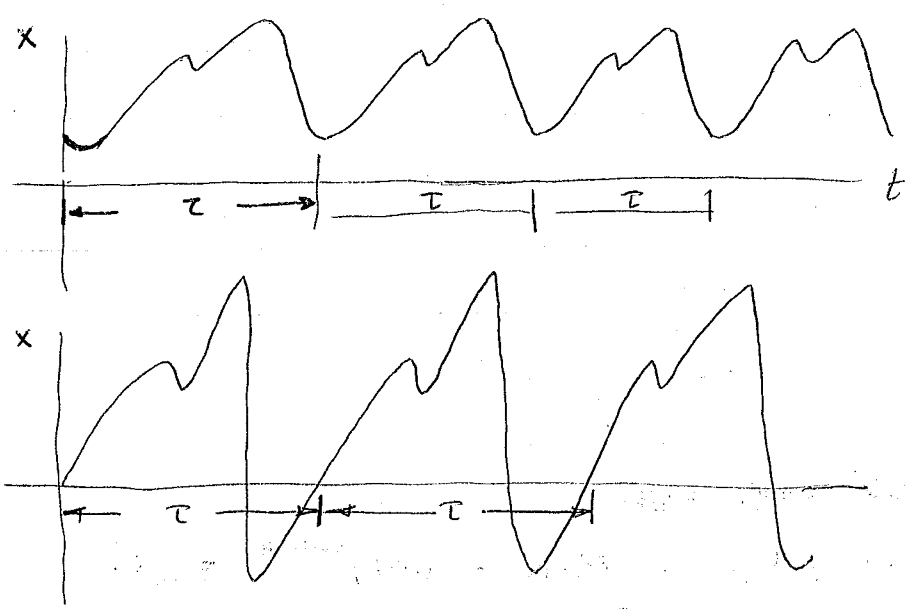


- MOTION IS DEPENDENT ON THE INITIAL CONDITIONS
- A – AMPLITUDE = MAX DISP.

  ϕ – phase ANGLE rad or degrees

  ω – CIRCULAR FREQUENCY rad/sec or degrees/sec

NON HARMONIC PERIODIC MOTION



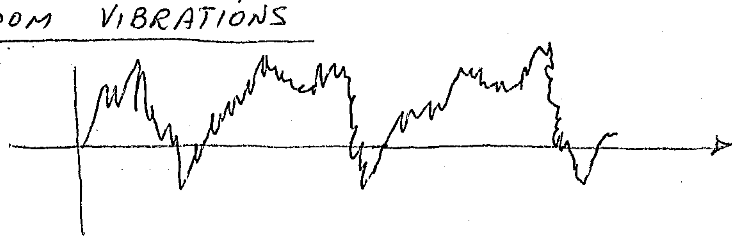
ONE COMPLETE MOVEMENT IS A CYCLE

TIME OF 1 CYCLE IS A PERIOD $= \tau$ (seconds)

FREQUENCY (# OF TIMES) OF CYCLE IN A UNIT TIME $= f$ (cycles/second, Hz)

$\tau f = 1$.

RANDOM VIBRATIONS



PRODUCED BY FORCES THAT ARE IRREGULAR

for a particle 3 DOF's (x, y, z)

for a rigid body of N particles 3N eqns $-$ $(3N-6)^{\text{Constraints}}$ $=$ 6

constraints : $n = 3N - M = 6$ indep variables 3 cartesian + 3 angles.

when constrained to move in a plane : 3 other constraints $\Rightarrow$ 3 DOFs for example $z=0$ $\theta_z = 90^\circ$ & $\theta_x + \theta_y = 90^\circ$

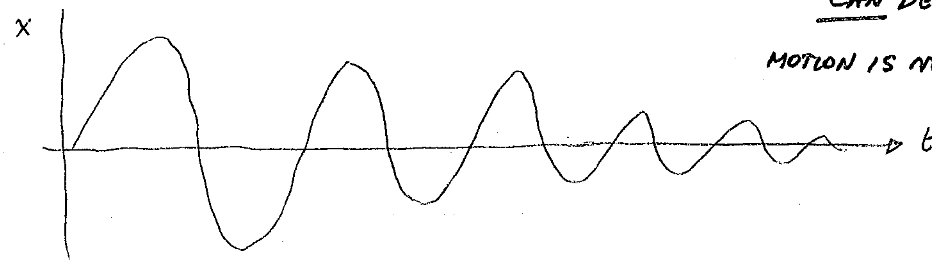
plane + no rot + no lateral : 5 const $\Rightarrow$ 1 DOF

- CAN OCCUR DURING TRANSPORTATION OF GOODS
- IMPACT CONDITIONS

WHEN SYSTEM UNDERGOES RESISTANCE (DAMPING) OSCILLATORY BEHAVIOR DIES OUT

CAN DEFINE A PERIOD

MOTION IS NOT HARMONIC



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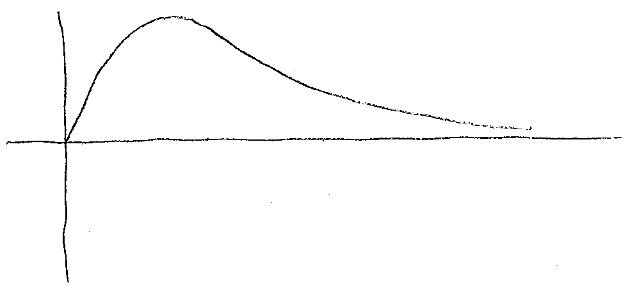


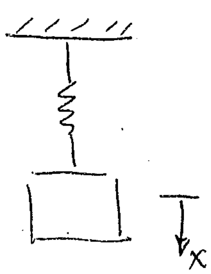
APERIODIC - NONPERIODIC

IF $\exists$ EXTERNAL FORCES CAUSING OSCILLATION - FORCED VIB

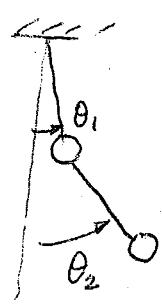
$\nexists$ " " - FREE VIBS

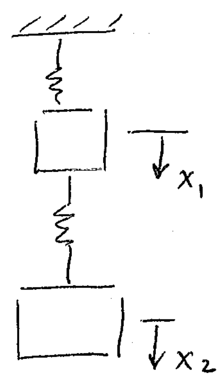
NO. OF DEGREES OF FREEDOM

NO. OF INDEP COORDINATES NEEDED TO FULLY DESCRIBE MOTION

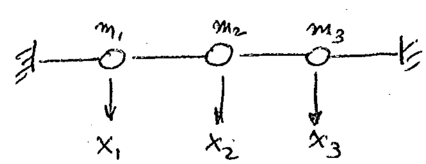


1. DOF





2. DOF



FOR EACH DISCRETE POINT MASS 3 DOF

- IN GENERAL SYSTEMS THAT UNDERGO VIBS ARE COMPLICATED
- CAN BE REPLACED BY SIMPLER SYSTEMS INVOLVING masses, springs, dampers
- THE MODEL IS AN EQUIVALENT SYSTEM
- EACH ELEMENT IS CONSIDERED AS LUMPED
  - SPRING HAS NO MASS, DAMPING
  - MASS IS RIGID - DOES NOT DEFORM, NO DAMPING
  - DAMPER HAS NO MASS, ELASTICITY

SEE PG 12 for car
20 for motorcycle 4th e

STATEMENT OF VIBRATIONS PROBLEM

USE CAR AS EXAMPLE

1. RECOGNIZE THAT VIBS CAN OCCUR
2. DETERMINE WHAT VIBS ARE SIGNIFICANT
3. FORMULATE A SIMPLE MODEL THAT CAPTURES GIST OF PROBLEM
4. * DERIVE GOVERNING EQS.
5. * SOLVE EQS
6. * INTERPRET RESULTS
7. MAKE APPROPRIATE RECOMENDATIONS

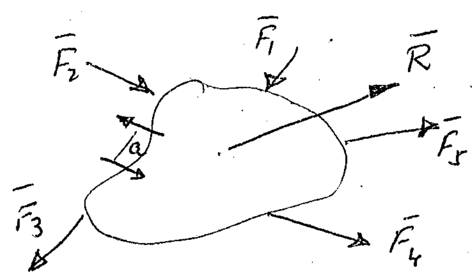
GOVERNING EQS DEPENDENT ON NEWTON'S LAWS

FIRST LAW - BODY REMAINS AT REST OR CONTINUES MOVING A CONST. VEL. UNLESS ACTED UPON BY AN UNBALANCED EXTERNAL FORCE

* SECOND LAW - $\bar{F} = m\bar{a}$ $\quad \bar{a}$ is in direction of $\bar{F}$

THIRD LAW - FOR EVERY ACTION IS AN EQUAL & OPPOSITE REACTION

BODY ACTED UPON BY A SET OF FORCES & COUPLES



$\Sigma \bar{F}_i = \bar{R} = m\bar{a}_G$

where $\bar{a}_G$ is the accel of the mass center

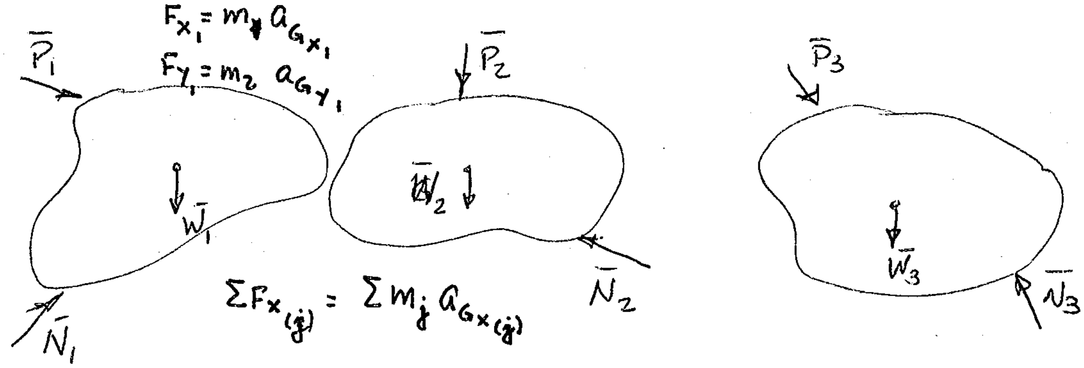
SESSION #4

$\bar{a}_G$ & $\bar{R}$ have same direction but different lines of action

FOR A SYSTEM OF RIGID BODIES UNDER EXTERNAL FORCES

$F_{x_1} = m_1 a_{G_{x_1}}$

$F_{y_1} = m_2 a_{G_{y_1}}$



$\Sigma F_{x_{(j)}} = \Sigma m_j a_{G_{x(j)}}$

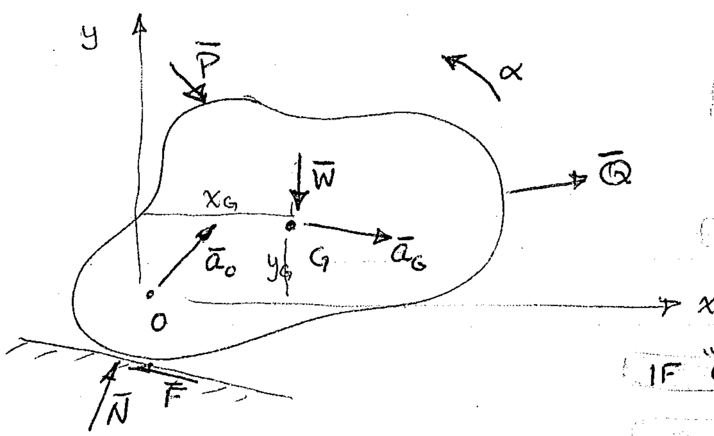
FOR EACH BODY

$\bar{R} = \Sigma m_i \bar{a}_G^{(i)}$

$\bar{R}$ resultant of EXTERNAL FORCES

BODIES MAY OR MAY NOT BE IN CONTACT – CONTACT FORCES CANCEL

PRINCIPLE OF ANGULAR MOTION – PICK AXES THROUGH A PT



$\Sigma T_o = I_o \alpha + m a_{o_y} x_G - m a_{o_x} y_G$

$\bar{r}_{G/o} \times m\bar{a}_o$

$\Sigma T_G = I_G \alpha$ ABOUT G

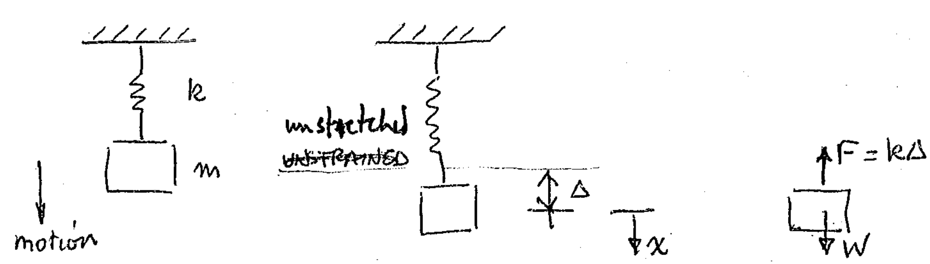
IF "O" IS FIXED $\bar{a}_o = 0$

$\Sigma T_o = I_o \alpha$

DO EQUIV SYSTEMS HERE FIRST (LESSON #5).

UNDAMPED FREE VIBS FOR SDOF SYS

SIMPLEST IS AN ELASTIC MEMBER + MASS

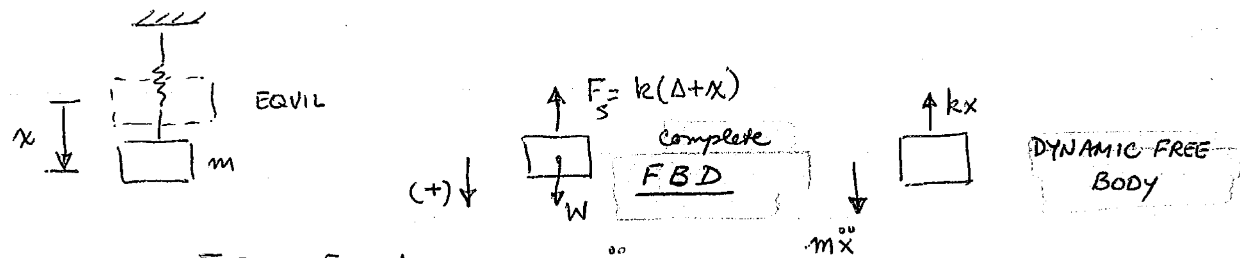


NO EXTERNAL FORCES - FREE VIBS

SDOF - SINCE X IS ONLY VAR. TO DEFINE POSITION OF MASS

BY STATIC EQUILIB $F = k\Delta = W$ $\quad \Delta = \frac{W}{k}$ STATIC DISP. δ_{ST} in RAO

- ASSUME NO HORIZONTAL MOTION OF MASS
-
- SUPPOSE WE DISPLACE MASS & LET GO
- MASS MOVES VERTICALLY ABOUT EQUILIB POSITION



$$\Sigma F = -F_s + W = ma = m\ddot{x}$$
$$= -(\Delta + x)k + W = -kx = m\ddot{x}$$

THIS SHOWS VIB TAKES PLACE ABOUT THE EQUIL POSITION.

LET $\omega^2 = \frac{k}{m}$ $\qquad \ddot{x} + \omega^2 x = 0$ $\qquad \omega_n^2$ in RAO

SOLUTION: $x = A\sin\omega t + B\cos\omega t$

LET $A = C\cos\phi$ $\quad B = C\sin\phi$

$$x = C\left[\cos\phi\sin\omega t + \sin\phi\cos\omega t\right] = C\sin(\omega t + \phi)$$

$C = \sqrt{A^2 + B^2}$ $\quad \phi = \tan^{-1} B/A$

ϕ is the phase angle

ω - CIRCULAR FREQ. $\qquad \omega t = 2\pi$ 1 cycle $\quad \tau = \frac{2\pi}{\omega}$

TO SOLVE COMPLETELY NEED IC

$x(0) = x_0 \implies B = x_0$

$\dot{x}(0) = \dot{x}_0 \implies A = \dot{x}_0/\omega$

$$x = \frac{\dot{x}_0}{\omega}\sin\omega t + x_0\cos\omega t = X\sin(\omega t + \phi)$$

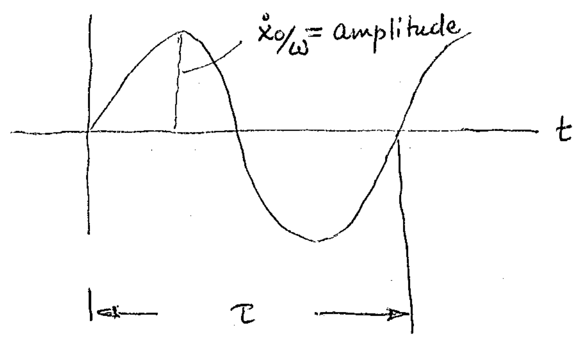
$$X = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega}\right)^2} \qquad \phi = \tan^{-1}\frac{x_0}{\dot{x}_0/\omega}$$

MOTION IS SINUSOIDAL OR HARMONIC

TIME FOR 1 CYCLE $\omega t = 2\pi$ $\quad \frac{2\pi}{\omega} = \tau$ PERIOD

$\tau f = 1 \quad \therefore \quad f = \frac{1}{\tau} = \frac{\omega}{2\pi}$ FREQUENCY

$\omega =$ CIRCULAR FREQ $= 2\pi f$ $= \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{mg/k}} = \sqrt{\frac{g}{W/k}} = \sqrt{\frac{g}{\Delta}}$



FOR $x_0 = 0 \quad \dot{x}_0 \neq 0$

- amplitude - max. travel about equil position
- ω^2 is prop. to g & inversely prop. to static disp

units of ω is radians/sec

τ is seconds

f is cycles/sec or Hertz (Hz)

FOR $\quad x = \frac{\dot{x}_0}{\omega}\sin\omega t + x_0\cos\omega t = X\sin(\omega t + \phi) \qquad X = \sqrt{\left(\frac{\dot{x}_0}{\omega}\right)^2 + x_0^2} \qquad \phi = \tan^{-1}\frac{x_0\omega}{\dot{x}_0}$

$\dot{x} = \dot{x}_0\cos\omega t - x_0\omega\sin\omega t = X\omega\cos(\omega t + \phi) = X\omega\sin(\omega t + \phi + \frac{\pi}{2})$

$= \dot{X}\cos(\omega t + \phi) \qquad \dot{X} = X\omega$ amplitude of veloc.

- velocity is ω times the displacement
- this shows that the velocity is out of phase with the displ by $\pi/2$

$\ddot{x} = -\dot{x}_0\omega\sin\omega t - x_0\omega^2\cos\omega t = -X\omega^2\sin(\omega t + \phi)$

this is governing DE $\quad = -\omega^2 x$

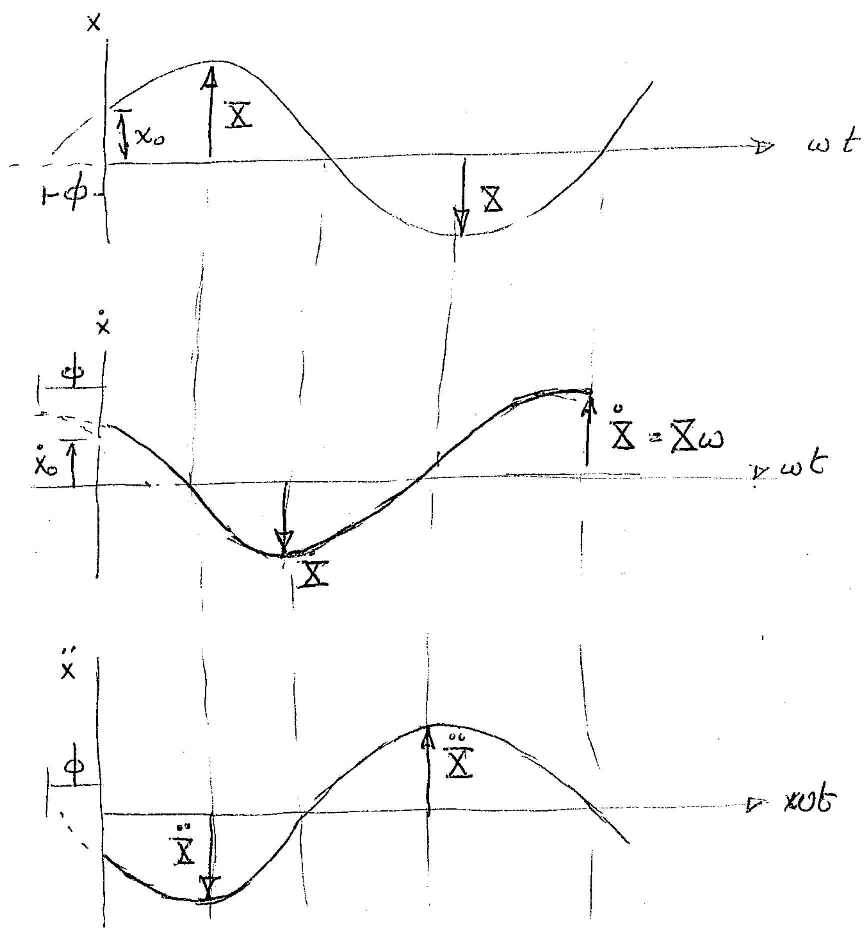
$= -\ddot{X}\sin(\omega t + \phi) \qquad \ddot{X} = X\omega^2 = \dot{X}\omega$ Amplitude of accel

- accel. is negative of displacement curve mult. by ω^2

- since ω is const better to plot x vs. ωt
- ϕ is the shift of the sine curve along horiz axis

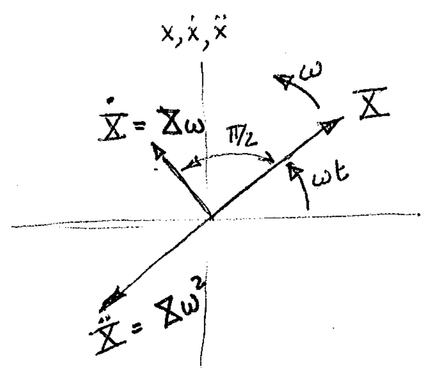
$$\omega t_o + \phi = 0$$

$$t_o = -\phi/\omega$$ defines how much of shift.



if $\dot{x}=0$

GRAPHICAL REPRESENTATION PG. #23 (PG 30 RAO)

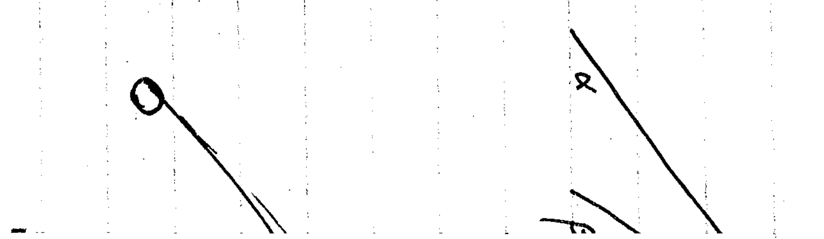


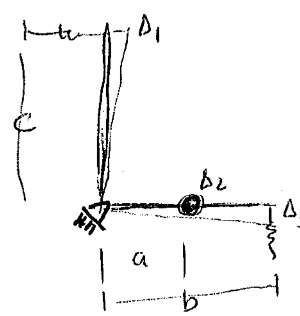
AS VECTOR X moves with angular velocity ω

$\dot{X}$ ~~which~~ leads X by $\pi/2$

$\ddot{X}$ which is $-X$

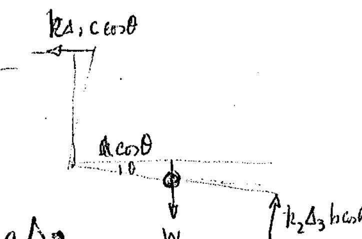
LENGTH OF VECTOR = AMPLITUDE OF $x, \dot{x}, \ddot{x}$





$$PE_f + KE_f = PE_i + KE_i$$

$$KE_f - KE_i + PE_f - PE_i$$



$$PE = 0$$

$$PE_f = \frac{1}{2}k_1\Delta_1^2 + \frac{1}{2}k_2\Delta_3^2 + mg\Delta_2$$

$$I\ddot{\theta} = -k_1c^2\frac{\sin 2\theta}{2} + Wa\cos\theta + k_2 b^2\frac{\sin 2\theta}{2}$$

$$ma^2\ddot{\theta} + (k_1c^2 + k_2b^2)\theta - Wa = 0$$

$$KE_i = 0$$

$$KE_f = \frac{1}{2}m\dot{\Delta}_2^2$$

$$\left(\frac{1}{2}m\dot{\Delta}_2^2\right) + \left(\frac{1}{2}k_1\Delta_1^2 + \frac{1}{2}k_2\Delta_2^2 + mg\Delta_2\right)$$

$\Delta_2 = a\sin\theta$ $\quad \Delta_1 = c\sin\theta$ $\quad \Delta_2 = -a\sin\theta$

$\dot{\Delta}_2 = a\cos\theta\,\dot{\theta}$ $\quad \Delta_2 = -a\sin\theta$ $\quad \dot{\Delta}_2 = -a\cos\theta\,\dot{\theta}$

$$\frac{1}{2}ma^2\cos^2\theta\,\dot{\theta}^2 + \left(\frac{1}{2}k_1c^2\sin^2\theta + \frac{1}{2}k_2b^2\sin^2\theta - mg\,a\sin\theta\right)$$

$$\left\{(k_1c^2 + k_2b^2)\sin^2\theta\right\} - mg\,b\sin\theta\,]$$

$$\frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}\left\{(\quad)2\sin\theta\cos\theta\,\dot{\theta} - mg\,a\cos\theta\,\dot{\theta}\right.$$

$-2\cos\theta\sin\theta\,\dot{\theta}$ $\quad \frac{1}{2}($

$2\cos^2\theta\,\dot{\theta}\,\ddot{\theta}$

$$2\cdot\frac{1}{2}ma^2\dot{\theta}\ddot{\theta} + 2\cdot\frac{1}{2}(k_1c^2+k_2b^2)\sin\theta\cos\theta\,\dot{\theta} - mg\,a\cos\theta\,\dot{\theta}$$

$$\dot{\theta}\left[ma^2\ddot{\theta} + (k_1c^2+k_2b^2)\theta - mga\right] = 0$$

$X = P\sin^2\alpha\, l/AE$ $\quad$ 3EI

$T_1\cos 45 = B\cos\theta$

$T_1\sin 45 + T$

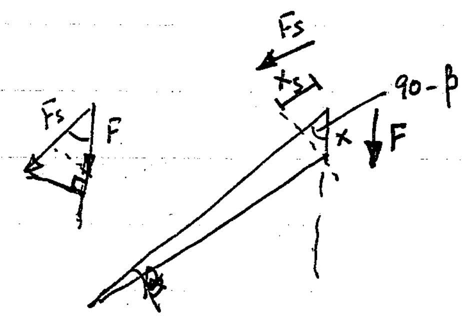
and l_2 $\quad \Delta_2 = \frac{P l_2}{AE}$ $\qquad P = kx$

$k_2 = \frac{AE}{l}$ $\qquad \frac{1}{2} k \Delta^2 = \frac{1}{2} k \Delta^2 = \frac{1}{2} P\Delta = \frac{AE}{l}(x \sin\alpha)^2$

$\therefore \quad \frac{1}{2} k_b \omega^2 + \frac{1}{2} k \Delta^2 + \frac{1}{2} k_2 \Delta_2^2$

$\frac{1}{2}\left[\cancel{\frac{3EI}{l^3 \cos^2\alpha}}\right] x^2 + \frac{1}{2}\left[\cancel{\frac{AE}{l \sin^2\alpha}}\right] x^2 + \frac{1}{2}\left[\frac{AE}{l}\right] x^2$

$k_b = \frac{3EI}{l^3}$ $\quad \omega = \frac{x}{\cos\alpha}$ $\qquad k = \frac{AE}{l}$ $\quad \Delta = \frac{x}{\sin\alpha}$ $\qquad k_2 = \frac{AE}{l}$ $\quad \Delta_2 = x$



$F = k x_q$

$x_s = x \cos(90-\beta)$

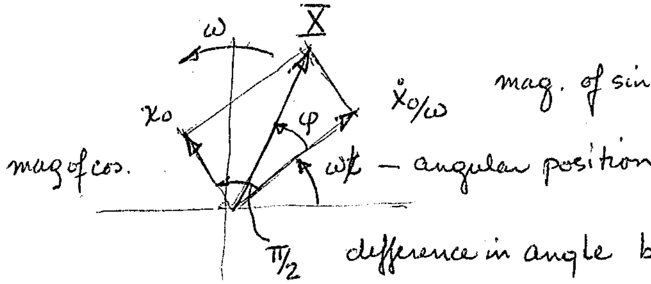
$\cancel{F_s = F\cos(90-\beta)}$ $\qquad F = F_s \cos(90-\beta)$

$F_s = k x_s$ $\qquad \therefore F_s = k x^2 \cos^2(\quad)$

$F_s = k x \cos(\quad)$

$F = F_s \cos(\quad) = \cos(\quad)\left[k x \cos(\quad)\right]$

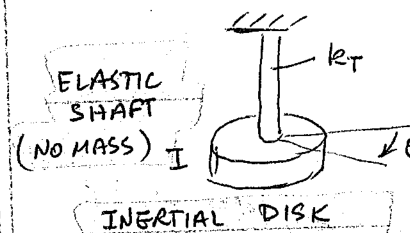
GRAPH. REP. OF EACH PART OF SOLUTION x



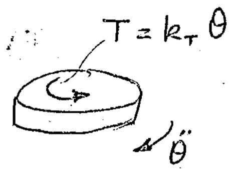
TORSIONAL VIBRATIONS

J polar moment of inertia

$$T = \frac{JG\theta}{L} = k_T\theta \qquad \boxed{k_T = \frac{JG}{L}}$$



k_T $\left[\frac{\text{lb-in}}{\text{rad}}\right]$ $[\text{Nm/rad}]$



I $[\text{lb-in-sec}^2]$ $[\text{kg-m}^2]$ ~~polar~~ mass moment of inertia

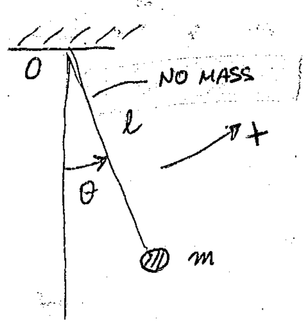
$$\boxed{I\ddot{\theta} = -k_T\theta}$$

$I \Rightarrow J_0$ in Rao

$$\theta = A\sin\omega t + B\cos\omega t \qquad \boxed{\omega = \sqrt{\frac{k_T}{I}}}$$
$$= C\sin(\omega t + \phi)$$

REPLACE θ BY x I BY m k_T by k SAME AS BEFORE

ANGULAR OSCILLATIONS



$I_0 = I_{CG}(\text{of mass}) + m d_\perp^2 = m d_\perp^2$

$I_{CG} = 0$ since mass is assumed at a pt.

$T = mg\cos\theta$ since no accel in radial dir

$mg\sin\theta$

$$I_0\ddot{\theta} = -mg\,l\sin\theta = \Sigma T_0$$

$I_0 = ml^2$ if dimensional character is small wrt l

$KE_i = 0$ $\qquad$ $PE_f = \frac{1}{2}k(a\sin\theta)^2 + mgl\cos\theta$

$KE_f = \frac{1}{2}m(l\dot\theta)^2$ $\qquad$ $PE_i = 0 + mgl$ $\qquad$ $T+P = \text{const}$

$\frac{1}{2}m(l\dot\theta)^2 + \cancel{mgl} = \frac{1}{2}k(a\sin\theta)^2 + mgl\cos\theta - mgl$ $\qquad$ $\Delta KE + \Delta PE = 0$

ΔK_E $\qquad$ ΔPE

$\Delta T + \Delta V$

$\cancel{T-V}$

$\cancel{\Delta T - \Delta V}$

$\cancel{K_f - K_i - (P_f - P_i)}$

$\cancel{(K_f - P_f) - (K_i - P_i)}$

$\Delta KE - \Delta PE = 0$

$K_f - K_i + (P_i - P_f)$

$K_f - K_i + (P_f - P_i) = 0$

$KE + PE = \text{const}$ $\qquad$ — datum at top.

$$\frac{1}{2}m(l\dot\theta)^2 + \frac{1}{2}k(a\sin\theta)^2 - mg(l - l\cos\theta) = \text{const}$$

$$\frac{d}{dt}: \quad ml^2(\dot\theta\ddot\theta) + k(a\sin\theta\, a\cos\theta\,\dot\theta) - mgl\sin\theta\,\dot\theta = 0$$

$$\dot\theta\left[ml^2\ddot\theta + ka^2\sin\theta\cos\theta - mgl\sin\theta\right] = 0$$

$PE = -\text{work}$ $\qquad$ $\text{work} = \int \underline{F}\cdot d\underline{s}$ $\qquad$ $mg\Delta = \text{work}$

$-mg\Delta = PE$

$KE_i + PE_i = KE_f + PE_f$

$(KE_f - KE_i) + (PE_f - PE_i) = 0$

$\left[\frac{1}{2}m(l\dot\theta)^2 - 0\right] + \left[\frac{1}{2}k(a\sin\theta)^2 + mgl\cos\theta - mgl\right] = 0$

$\frac{1}{2}ml^2\dot\theta^2 + \frac{1}{2}ka^2\sin^2\theta - mgl(1-\cos\theta) = 0$

$2\cdot\frac{1}{2}ml^2\dot\theta\ddot\theta + \frac{1}{2}ka^2\cdot 2\sin\theta\cos\theta\,\dot\theta - mgl\sin\theta\,\dot\theta = 0$

$\dot\theta\left[ml^2\ddot\theta - (mgl - ka^2)\theta\right] = 0$ $\qquad$ so $\quad ml^2\ddot\theta + (ka^2 - mgl)\theta = 0$

- if we look at this

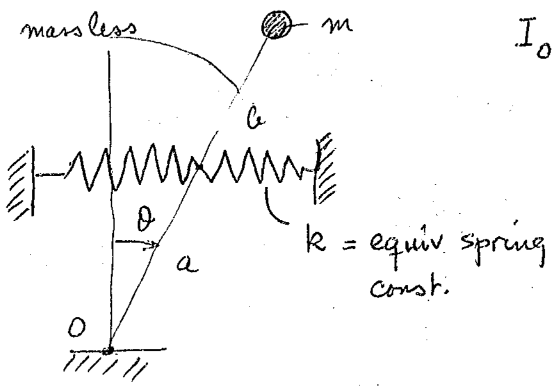
$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \qquad \text{2nd order } \underline{\text{nonlin}} \text{ dif eq.}$$

assume θ is small. $\theta \gg \theta^3/3!$ in sine exp.

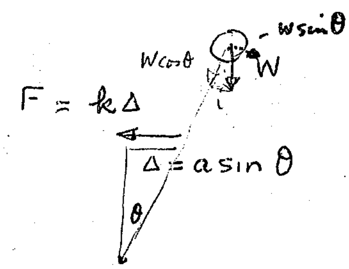
$$ml^2\ddot{\theta} + mgl\left[\theta - \theta^3/3! + \dots\right] = 0$$

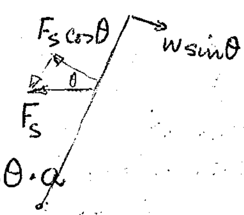
$$ml^2\ddot{\theta} + mgl\theta = 0 \Rightarrow \ddot{\theta} + g/l\,\theta = 0 \qquad \text{let } \omega = \sqrt{g/l}$$

CONSIDER INVERTED PENDULUM - ASSUME SPRING UNSTRAINED WHEN PEND. IS VERTICAL



$I_0 = ml^2$





$$I_0\ddot{\theta} = \sum T_0$$

$$= +W\sin\theta\cdot l - F_s\cos\theta\cdot a \qquad F_s = k\Delta = ka\sin\theta$$

$$= -ka^2\cos\theta\sin\theta + Wl\sin\theta$$

for $\underline{\text{small angles}}$ $\quad I_0\ddot{\theta} = -ka^2\theta + Wl\cancel{\theta} = \theta\left[Wl - ka^2\right]$

$$\text{or} \quad \ddot{\theta} + \left[\frac{ka^2 - Wl}{I_0}\right]\theta = 0$$

if $ka^2 - Wl > 0$ $\qquad \ddot{\theta} + \left[\frac{ka^2 - Wl}{ml^2}\right]\theta = 0 \qquad \omega = \left[\frac{ka^2 - Wl}{ml^2}\right]^{1/2}$

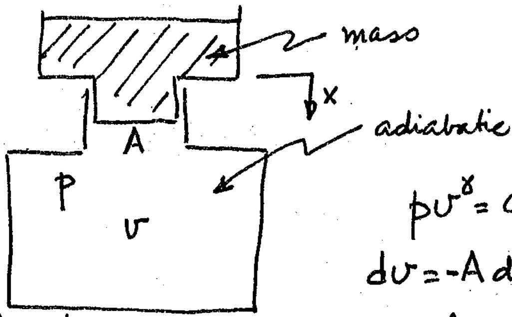
$$\theta = C\sin(\omega t + \varphi) \qquad \text{STABLE OSCIL.}$$

if $ka^2 = Wl$ $\qquad \theta = C_1 t + C_2 \qquad$ CONSTANT-VELOCITY EQUIL.

if $ka^2 - Wl < 0$ $\qquad \theta = C_1\sinh\gamma t + C_2\cosh\gamma t \qquad \gamma = \left[\frac{Wl - ka^2}{ml^2}\right]^{1/2}$

torque (weight) > torque due to spring $\qquad$ UNSTABLE - NON OSCIL

NEGATIVE TERM ON RHS OF D.E. IS A RESTORING TERM



Problem 1.13 #2ed
1-16 in #3, #4

$p v^{\gamma} = \text{const}$

$dv = -A\,dx \qquad \text{since } +dx \Rightarrow -dv$

change from equilib (static) $dp . A = dF$

want $dF = k_{eq}\,dx$

$$dp . A = k_{eq}\,\frac{dv}{-A}$$

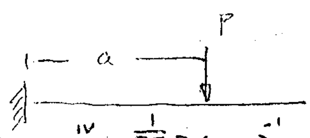
need relation between dp & dv $\Rightarrow$ use $p v^{\gamma} = \text{const}$

$$dp\, v^{\gamma} + p\gamma v^{\gamma-1} dv = 0$$

$$\therefore\ dv = \left[-\frac{\gamma p}{v}\right]^{-1} dp$$

$$\Rightarrow\ dp . A = k_{eq}\left[-\frac{\gamma p}{v}\right]^{-1} dp \,/ -A$$

$$\text{or}\quad k_{eq} = \cancel{\frac{A^2 v}{\gamma p}}\ A^2 \cdot \frac{\gamma p}{v}$$



$w^{IV} = -\frac{1}{EI} P \langle x-a \rangle^{-1}$

$w''' = -\frac{1}{EI} P \langle x-a \rangle^{0} + C_1 \qquad C_1 = \frac{P}{EI}$

$w'' = -\frac{1}{EI} P \langle x-a \rangle^{1} + \frac{P}{EI} x + C_2 \qquad C_2 = -\frac{Pa}{EI}$

$w' = -\frac{1}{EI} P \frac{\langle x-a \rangle^2}{2} + \frac{Px^2}{2EI} - \frac{Pax}{EI} + C_3 \qquad C_3 = 0$

$w = -\frac{1}{EI} P \frac{\langle x-a \rangle^3}{6} + \frac{Px^3}{6EI} - \frac{Pax^2}{2EI} + C_4 \qquad C_4 = 0$

$w(x=L) = -\frac{1}{EI} P \frac{(L-a)^3}{6} + \frac{P}{EI}\frac{L^3}{6} - \frac{3PaL^2}{6EI} = -\frac{2PL^3}{6EI}$ ✓

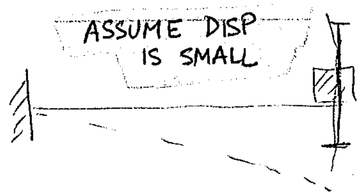
$w =$

SESSION #5

HAND IN HW #2

EQUIVALENT SPRING

- REPLACE ELASTIC MEMBER BY AN EQUIV. SPRING
- REDUCE TO SPRING-MASS SYSTEM.
- EXPRESS FORCE OR MOMENT FOR UNIT DISP AT LOCATION OF MASS OR INERTIAL ELEMENT



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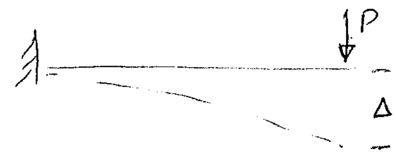
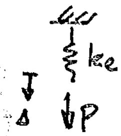


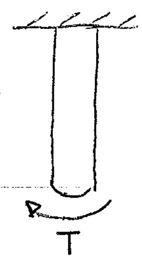
$P = k_E \Delta$

$k_E = P/\Delta$



$\Delta = \dfrac{P\ell^3}{3EI}$ STRENGTH OF MATERIALS

$k_E = \dfrac{3EI}{\ell^3} =$

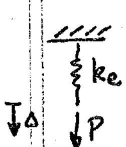


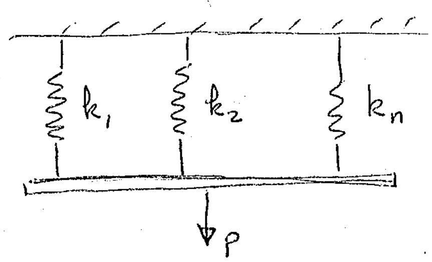
$\theta = T\ell/GJ$ $\quad T = k_T\theta$ $\quad k_T = GJ/\ell$

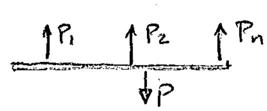
$\frac{P}{i^2}\cdot\frac{i^4}{i} = lb\text{-}in$

- IN A SYSTEM w/ MANY SPRINGS WANT TO FIND EQUIV. SPRING

SPRINGS IN PARALLEL







$\Sigma F = 0 \Rightarrow \Sigma P_i = P$

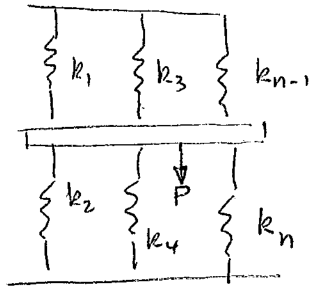
Assume bar remains horiz such that Δ for each is ~~small~~ SAME

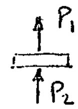
$k_1\Delta = P_1 \; , \; k_2\Delta = P_2 \; \ldots \; k_n\Delta = P_n$

$P_1 + P_2 + \ldots + P_n = P = \Delta(k_1 + k_2 + \ldots + k_n) = \Delta k_E$

$$k_E = \sum_{i=1}^{n} k_i$$

IN PARALLEL NOTE ALL SPRINGS UNDERGOE SAME DEFLECTION

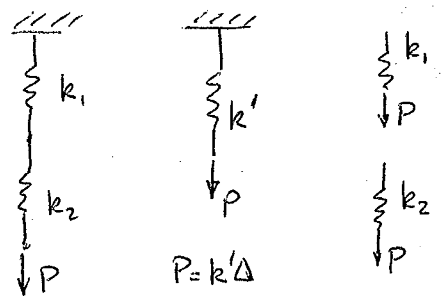




$$P_1 + P_2 = k_1\Delta + k_2\Delta = (k_1 + k_2)\Delta$$

now $\sum P_i = P = \sum k_i \Delta = \Delta \sum k_i = k_E \Delta$

SPRINGS IN SERIES



$P = k_1 \Delta_1$

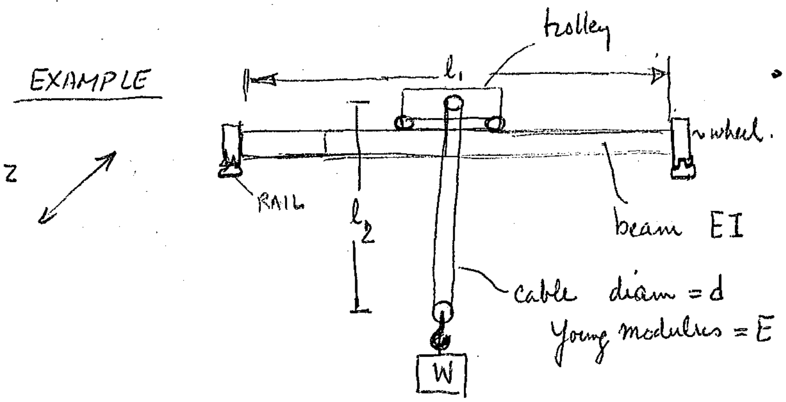
$P = k_2 \Delta_2$

$$\Delta = \Delta_1 + \Delta_2 = \frac{P}{k_1} + \frac{P}{k_2} = \frac{P}{k'}$$

$$\sum \frac{1}{k_i} = \frac{1}{k'}$$

NOTE ALL SPRINGS UNDERGO SAME LOAD

EXAMPLE



- weight W is being lifted overhead by a traveling crane
- mass of trolley, cables, hook small

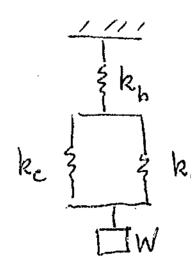
① Assume trolley acts as a point load. at center

$$\Delta = \frac{P\ell^3}{48EI} = \frac{P}{k_b} \qquad k_b = \frac{48EI}{\ell^3}$$

② Each cable holds $\frac{W}{2} = P'$

$$\Delta = \frac{P'\ell_2}{AE} = \frac{P'}{k_c} \Rightarrow k_c = \frac{AE}{\ell_2} = \frac{\pi d^2 E}{4\ell_2}$$

since same disp. cables ⟹ eqiv spring of $2k_c$



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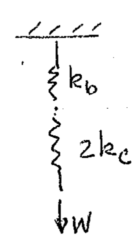


$$\frac{1}{k_E} = \frac{1}{k_b} + \frac{1}{2k_c} = \frac{\ell^3}{48EI} + \frac{\ell_2}{2AE}$$

$$V_1 = \frac{1}{2}k\Delta^2 \qquad T_1 = 0$$

$$T_2 = \frac{1}{2}m\dot{x}^2 \qquad V_2 = \frac{1}{2}k\,(\Delta+x)^2 - W\cdot x$$

$$\frac{1}{2}k\Delta^2 + 0 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k(\Delta+x)^2 - Wx$$

$$0 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k\left[(\Delta+x)^2 - \Delta^2\right] - Wx$$

$$\frac{1}{2}kx(x+2\Delta) - Wx$$

$$\frac{kx^2}{2} + \cancel{\underline{k\Delta x - Wx}} = 0$$

ENERGY METHOD FOR CONSERVATIVE SYSTEMS

- D.E. CAN BE OBTAINED BY USE OF CONSERVATION OF ENERGY

- FOR FREE VIB. SYS. NO ENERGY IS REMOVED $\Rightarrow$ TOTAL ENERGY = CONST

$T = KE \qquad V = PE$

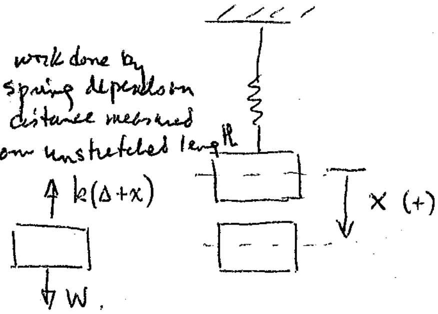
- $$\frac{d}{dt}(T+V) = 0$$

- FOR SMALL VIBS $\quad T = T(\text{veloc}) \qquad V = V(\text{displ.})$

FOR MASS - SPRING OSCILLATOR $\qquad$ work done $= \int_0^x [W - k(\Delta + x')]dx'$

$V = (\text{potential}) = -\text{work} = -\int_0^x [\qquad]dx'$

work done by spring depends on distance measured from unstretched length



$T = \frac{1}{2}m\dot{x}^2 \qquad \dot{x}$ = veloc. of mass.

$$V = \int_x^0 [W - k(\Delta + x')]\,dx' = -\int_x^0 kx'dx' = \frac{kx^2}{2}$$

V: work done to move system from displaced config back to equil. positi

$$T+V = \frac{1}{2}m\dot{x}^2 + \frac{kx^2}{2} = \text{const.} \Rightarrow \frac{d}{dt}(T+V) = \dot{x}(m\ddot{x} + kx) = 0$$

$\dot{x} = 0 \Rightarrow$ no motion (trivial solution)

$m\ddot{x} + kx = 0 \Rightarrow$ non trivial sol.

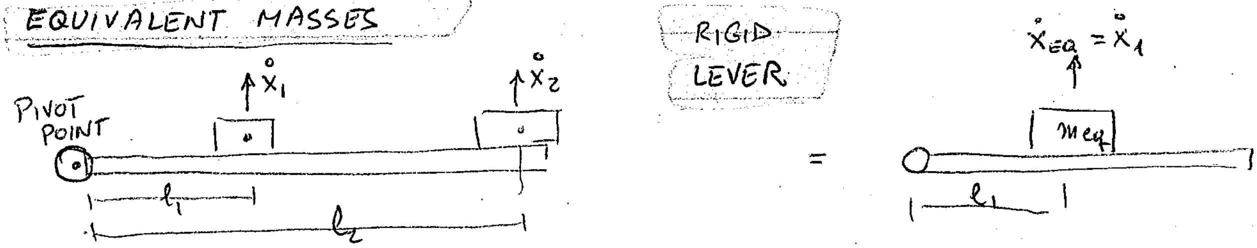
Rayleigh states that $T_{max} = V_{max}$: if solution is harmonic $\quad x = A\sin(\omega t + \phi)$, $\dot{x} = A\omega\cos(\omega t + \phi)$

$T = \frac{1}{2}mA^2\omega^2\cos^2(\quad) \Rightarrow T_{max} = \frac{1}{2}mA^2\omega^2 \quad \Rightarrow \omega^2 = k/m$ or $\omega = \sqrt{k/m}$

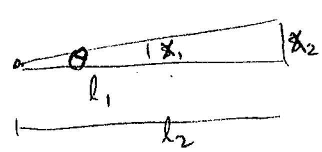
$V = \frac{1}{2}kA^2\sin^2(\quad) \Rightarrow V_{max} = \frac{1}{2}kA^2$

EQUIVALENT MASSES

using equivalent K. energies



FOR SMALL ANGULAR DISPL.



$x_2 = l_2\theta$

$x_1 = l_1\theta$

$\theta = \frac{x_2}{l_2} = \frac{x_1}{l_1}$

$$x_2 = \frac{x_1}{l_1}l_2 \Rightarrow \boxed{\dot{x}_2 = \frac{\dot{x}_1 l_2}{l_1}}$$

1.19 $\frac{1}{2}J_1\dot{\theta}_1^2 + \frac{1}{2}J_2\dot{\theta}_2^2 = KE = \frac{1}{2}J_{eq}\dot{\theta}_1^2$

$n_1 \Delta\theta_1 = n_2 \Delta\theta_2 = 2\pi$

$\therefore \Delta\theta_2 = \frac{n_1}{n_2}\Delta\theta_1 \quad \rightarrow \quad \dot{\theta}_1 = \left(\frac{n_1}{n_2}\right)\dot{\theta}_2$

$J_{eq} = J_1 + J_2\left(\frac{n_1}{n_2}\right)^2 \qquad r_1\Delta\theta_1 = r_2\Delta\theta_2$

EQUATE KE's

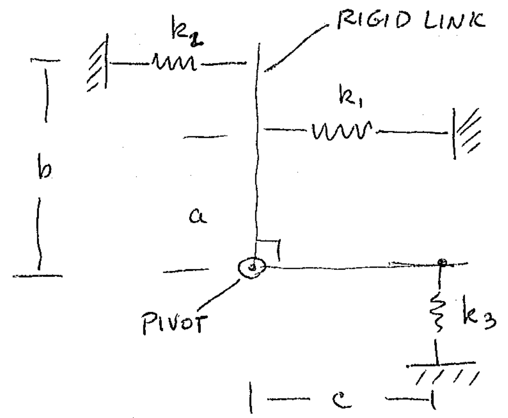
$$\tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 = \tfrac{1}{2} m_{eq} \dot{x}_{EQ}^2 = \tfrac{1}{2} m_{eq} \dot{x}_1^2$$

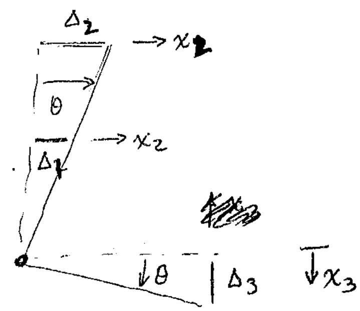
$$\tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \left[\dot{x}_1^2 \frac{l_2^2}{l_1^2} \right] = \tfrac{1}{2} m_{eq} \dot{x}_1^2$$

$$\boxed{m_1 + m_2 \frac{l_2^2}{l_1^2} = m_{EQ}}$$

equating PE's w/ k_{eq} at x_1

EXAMPLE #2





$\Delta_1 = a\theta$

$\Delta_2 = b\theta$ $\qquad \theta = \frac{\Delta_2}{b} = \frac{\Delta_1}{a} \Rightarrow \Delta_2 = \frac{b}{a}\Delta_1$

$\Delta_3 = c\theta$ $\qquad \theta = \frac{\Delta_3}{c} = \frac{\Delta_1}{a} \Rightarrow \Delta_3 = \frac{c}{a}\Delta_1$

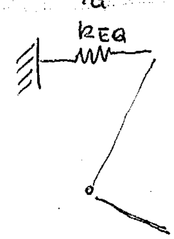
$$U = \tfrac{1}{2} k_1 \Delta_1^2 + \tfrac{1}{2} k_2 \Delta_2^2 + \tfrac{1}{2} k_3 (+\Delta_3)^3 = \tfrac{1}{2} k_{EQ} \Delta_{EQ}^2$$

let $\Delta_{EQ} = \Delta_1$

$$U = \tfrac{1}{2} k_{EQ} \Delta_1^2 = \tfrac{1}{2} k_1 \Delta_1^2 + \tfrac{1}{2} k_2 \left(\frac{b^2}{a^2}\right)\Delta_1^2 + \tfrac{1}{2} k_3 \left(\frac{c^2}{a^2}\right)\Delta_1^2$$

$$= \tfrac{1}{2}\left[k_1 + k_2 b^2/a^2 + k_3 c^2/a^2\right] \Delta_1^2$$

$$\boxed{k_{EQ} = k_1 + k_2\, b^2/a^2 + k_3\, c^2/a^2}$$

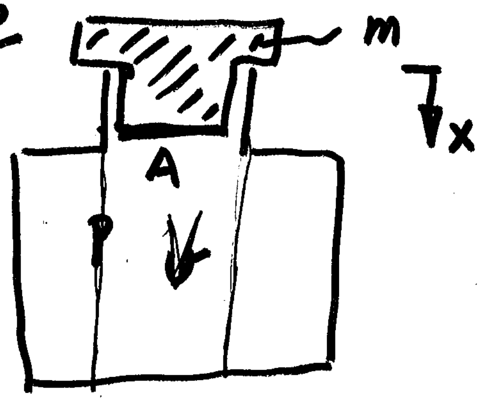


EQUATE PE's.

$$U = \tfrac{1}{2}\left(k_1 a^2 + k_2 b^2 + k_3 c^2\right)\theta^2 = \tfrac{1}{2} K_{eq} \theta^2$$

9/15/04

1.16



$$pV^{\gamma} = \text{constant}$$

$$F = kx$$

$$\Delta V = -A\,\Delta x$$

$$\Delta F = k_{eq}\,\Delta x$$

$$\Delta F = k_{eq} \cdot \frac{\Delta V}{-A}$$

$$\Delta F = A\Delta p$$

$$A\,\Delta p = k_{eq}\,\frac{\Delta V}{-A}$$

$$\Delta p \cdot V^{\gamma} + p\gamma V^{\gamma-1}\Delta V = 0$$

$$\Delta p = \left[-\frac{V}{\gamma p}\right]^{-1}\Delta V$$

$$k_{eq} = A^2\frac{\gamma p}{V}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

$$Q = -Av$$

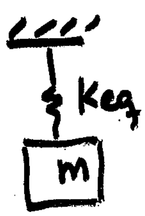
$$\frac{dF}{dt} = k_{eq}\,v$$

$$\frac{dF}{dt} = k_{eq} \cdot \frac{Q}{-A}$$

$$A\frac{\Delta p}{\Delta t} = k_{eq} \cdot \frac{Q}{-A}$$

$$\frac{\Delta p}{\Delta t} = \frac{k_{eq}}{-A^2}\,Q$$

$$Q = \cancel{\frac{k_{eq}}{-A^2}}\ \frac{-A^2}{k_{eq}}\frac{dp}{dt}$$



$$Q = \quad p$$

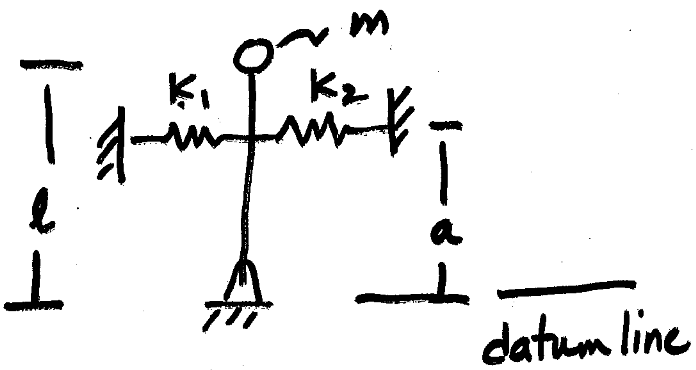
Energy Method

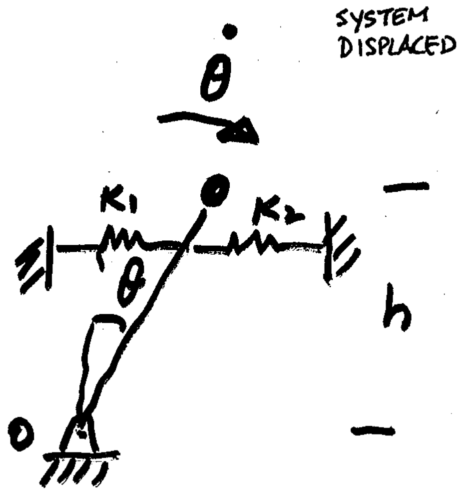
Conservative systems (no damping)

$$(KE + PE) = \text{constant}$$

$$\frac{d}{dt}(KE + PE) = 0 \implies \text{Eqn. of motion}$$

Inverted Pendulum





$$KE_i = 0$$

$$PE_i = \cancel{PE_{springs}} + PE_{mass}$$

$PE_{springs} = 0$ unstretched; $PE_{mass} = mg\ell$

$$KE_f = \frac{1}{2}m(\ell\dot{\theta})^2 = \frac{1}{2}I_0\dot{\theta}^2$$

$$PE_f = PE_{springs} + PE_{mass}$$

$$= \frac{1}{2}k_1(a\sin\theta)^2 + \frac{1}{2}k_2(-a\sin\theta)^2$$

$$+ mg\cdot\ell\cos\theta$$

$$PE_{\text{of mass}} = mgh$$

$$PE_{\text{of springs}} = \frac{1}{2}kx^2$$

$$(KE_f - KE_i) + (PE_f - PE_i) = 0$$

$$KE_f + PE_f = KE_i + PE_i = \text{const}$$

$$\left(\frac{1}{2}m\ell^2\dot{\theta}^2 + 0\right) + \frac{1}{2}(k_1+k_2)a^2\sin^2\theta + mg\ell\cos\theta - \overset{mg\ell}{\cancel{}} = 0$$

$$0 = \frac{1}{\cancel{2}} m l^2 [\cancel{2}\dot{\theta}\ddot{\theta}] + \frac{1}{\cancel{2}}(k_1+k_2)a^2[\cancel{2}\sin\theta\cos\theta\cdot\dot{\theta}] + mgl[-\sin\theta\cdot\dot{\theta}]$$

$$0 = \dot{\theta}\left[ml^2\ddot{\theta} + (k_1+k_2)a^2\sin\theta\cos\theta - mgl\sin\theta\right]$$

$\cancel{\dot{\theta}=0}$ or $[\qquad\qquad] = 0$

after $\frac{d}{dt}$, make small angle assumption

$\cos\theta \sim 1 \quad \sin\theta \sim \theta$

RAYLEIGH METHOD - from energy you can get the natural frequency. **Previous Prob.**

$\frac{1}{2} m l^2 \dot{\theta}^2 =$ KE of the system

$\frac{1}{2}(k_1+k_2)a^2\sin^2\theta - mgl(1-\cos\theta) =$ PE of ~~stem~~ system

$\cos\theta \sim 1 - \theta^2/2$ **ONLY FOR $\cos\theta$ - KEEP 2 TERMS**

$KE_{max} = PE_{max} \Longrightarrow \omega_n^2$

$\frac{1}{2}(k_1+k_2)a^2\theta^2 - mgl\cdot\theta^2/2 =$ PE of the sys. for small θ

if $\theta = A\sin(\omega_n t + \phi)$ for an undamped harmonic system

$\dot{\theta} = A\omega_n\cos(\omega_n t + \phi)$

$$KE = \frac{1}{2} m l^2 \dot{\theta}^2 = \frac{1}{2} m l^2 \left[A\omega_n \cos(\omega_n t + \phi)\right]^2$$

$$= \frac{1}{2} m l^2 A^2 \omega_n^2 \cos^2(\quad)$$

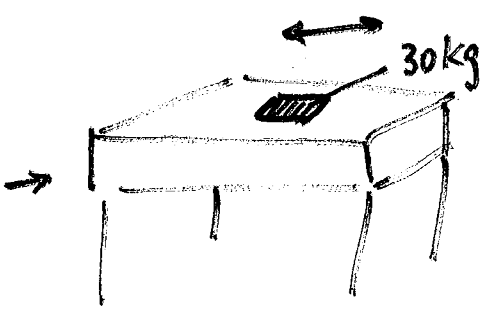
$$\underline{KE_{max}} = \frac{1}{2} m l^2 A^2 \omega_n^2$$

$$PE = \frac{1}{2}\left[(k_1+k_2)a^2 - mgl\right]\theta^2 = \frac{1}{2}\left[(k_1+k_2)a^2 - mgl\right]\left[A\sin(\omega_n t+\phi)\right]^2$$

$$\underline{PE_{max}} = \frac{1}{2}\left[(k_1+k_2)a^2 - mgl\right]A^2$$

$$KE_{max} = \cancel{\frac{1}{2}} m l^2 \cancel{A^2} \omega_n^2 = \cancel{\frac{1}{2}}\left[(k_1+k_2)a^2 - mgl\right]\cancel{A^2} = PE_{max}$$

solve for $$\omega_n^2 = \frac{\left[(k_1+k_2)a^2 - mgl\right]}{ml^2}$$



$\tau_n = 0.4\ sec$

$f\tau = 1$

$\omega_n = 2\pi f$

$2\pi f \tau = 2\pi$

$\omega_n \tau_n = 2\pi$

$$\sqrt{\frac{k}{m}} = \omega_n = \frac{2\pi}{\tau_n} = \frac{2\pi}{.4}$$

$\tau_n' = 0.5\ sec$

Q: m, k ?

$$\sqrt{\frac{k}{m+30}} = \omega_n' = \frac{2\pi}{\tau_n'} = \frac{2\pi}{0.5}$$

$$\sqrt{\frac{K}{m}} = \frac{2\pi}{.4} \implies \frac{K}{m} = \frac{4\pi^2}{.16}$$

$$\sqrt{\frac{K}{m+30}} = \frac{2\pi}{.5} \implies \frac{K}{m+30} = \frac{4\pi^2}{.25}$$

$$\frac{m+30}{K} - \frac{m}{K} = \frac{0.25 - 0.16}{4\pi^2}$$

$$\frac{30}{K} = \frac{0.09}{4\pi^2} \qquad K_{sys} = 13160 \frac{N}{m}$$

$$K_{leg} = \frac{1}{4} K_{sys} = 3290 \frac{N}{m}$$

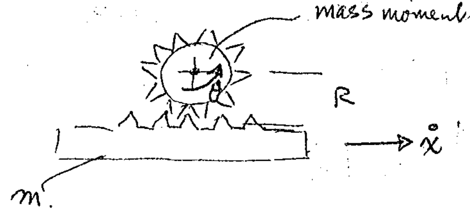
$$\frac{m}{K} = \frac{.16}{4\pi^2} \qquad m = \frac{.16}{4\pi^2} K_{sys} = 53.3 \text{ kg}$$

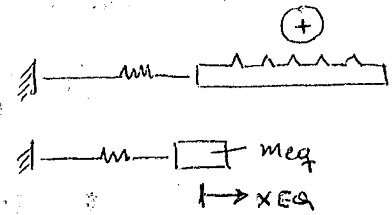
\# 2-4, 6, 7, 13, 15

TRANSLATIONAL & ROTATIONAL MASSES COUPLED TOGETHER

FOR BODIES IN CONTACT

mass moment of inertia I_0





1. EQUIV. TRANSLATING mass.

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I_0 \dot{\theta}^2 = T_{EQ} = \frac{1}{2} \hat{m}_{EQ} \dot{x}_{EQ}^2$$

IF WE CHOOSE $\dot{X}_{EQ} = \dot{X}$ $\qquad R\theta = x \qquad R\dot{\theta} = \dot{x} \qquad \dot{\theta} = \dot{x}/R.$

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I_0 \left(\frac{\dot{x}^2}{R^2}\right) = \frac{1}{2}\left[m + \frac{I_0}{R^2}\right]\dot{x}^2 = \frac{1}{2} m_{EQ}\dot{x}^2$$

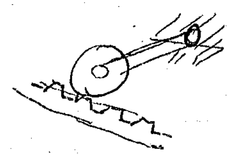
WHAT IS [redacted] ? 0 SINCE WORK BY INTERNAL FORCES = 0

WORK DO TO CONTACT FORCES

$$m_{EQ} = m + \frac{I_0}{R^2}$$

2. EQUIVALENT ROTATING MASS

$$T_{EQ} = \frac{1}{2} J_{EQ} \dot{\theta}_{EQ}^2 = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} I_0 \dot{\theta}^2 = T$$



LET $\dot{\theta}_{EQ} = \dot{\theta} \quad \Rightarrow \quad \dot{x} = R\dot{\theta}$

$$\frac{1}{2} J_{EQ} \dot{\theta}^2 = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} I_0 \dot{\theta}^2 = \frac{1}{2}\left[mR^2 + I_0\right]\dot{\theta}^2$$

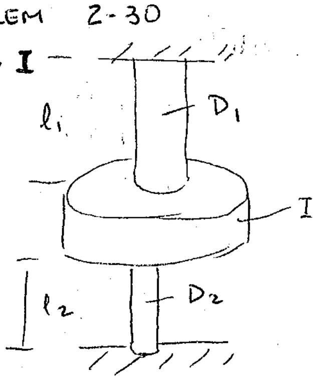
or $$J_{EQ} = I_0 + mR^2$$

HW 2-3, 2-10, 2-13, 2-23 // RAO 2nd ed: 1.7, 1.9, 1.17, 1.16

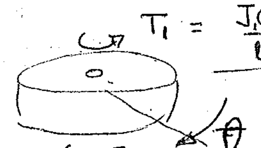
SESSION #6

PROBLEM 2-30

disk - mass moment I about its axis
shafts are fixed
find ODE & freq



UNDERGO SAME ANGULAR DISP.



$$T_1 = \frac{J_1 G}{l_1}\theta \qquad J_i = \frac{\pi D_i^4}{32}$$

$$T_2 = \frac{J_2 G}{l_2}\theta \qquad \Sigma T_i = \text{total torque}$$

$$I\ddot{\theta} = -\left(\frac{J_1}{l_1} + \frac{J_2}{l_2}\right) G\theta$$

$$\omega = \sqrt{\left(\frac{J_1}{l_1} + \frac{J_2}{l_2}\right)\frac{G}{I}} = \sqrt{\frac{G}{I l_1 l_2}\left(J_1 l_2 + J_2 l_1\right)} = \sqrt{\frac{G\pi}{32 I l_1 l_2}\left(D_1^4 l_2 + D_2^4 l_1\right)}$$

a heavy table is supported by flat steel legs. Its $\tau_n = .4$ sec in horiz motion
by adding a 30 kg plate clamped to the table then $\tau_n = .5$ sec
what are the effective k & mass of table

$$\tau_n = \frac{1}{f_n} \qquad f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\Rightarrow \quad \frac{m}{k} = \left(\frac{\tau_n}{2\pi}\right)_1^2 \qquad \frac{m+m_0}{k} = \left(\frac{\tau_n}{2\pi}\right)_2^2$$

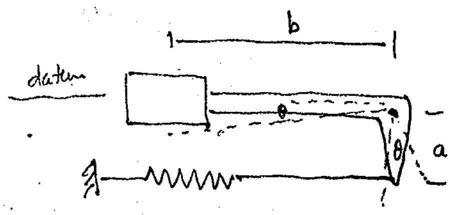
$$\therefore \quad \frac{m_0}{k} = \frac{1}{4\pi^2}\left[\tau_{n_2}^2 - \tau_{n_1}^2\right]$$

$$\text{or} \quad k = \frac{m_0 \cdot 4\pi^2}{\tau_{n_2}^2 - \tau_{n_1}^2} \quad \& \quad m = \frac{k \cdot \tau_{n_1}^2}{4\pi^2}$$

$$= \frac{30 \cdot 4 \cdot 10}{.25 - .16} = \frac{1200}{.09} \cong 13160 \text{ N/m}$$

$$m = \frac{13160 \cdot .16}{4 \cdot \pi^2} = 53.3 \text{ kg}$$

find the natural freq



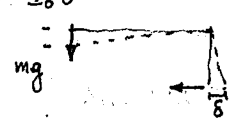
$$k\Delta_{st}\underset{\wedge}{a}^{\cos\theta_{st}} = F_s \cdot a = mg \cdot b \cdot \cos\theta_{st} \qquad \frac{\Delta_{st}}{a} = \frac{y}{b}$$

$$g/\Delta_{st} = \frac{ka}{mb}$$

$$\omega_n = \sqrt{g/\Delta_{st}}$$

$$ka^2 \sin\theta_{st} = mgb$$
$$\sin\theta_{st} = \frac{mgb}{ka^2}$$
$$\theta_{st} = \frac{mgb}{ka^2}$$

$$\Sigma M = I_0 \ddot{\theta}$$



$$\frac{\delta}{a} = \frac{y}{b}$$

θ measured from horizontal

$$mg\,b\cos\theta - k\underbrace{\delta}_{\frac{a^2}{b}\cdot y\cos\theta} a\cos\theta = I_0\ddot{\theta} = mb^2\ddot{\theta}$$

$$mg\,b\cos\theta - ka^2 \sin\theta\cos\theta = mb^2\ddot{\theta}$$

$$mgb = \ddot{\theta}mb^2 + ka^2\theta \qquad \omega_n = \frac{a}{b}\sqrt{\frac{k}{m}}$$

const forcing funct

$$\Rightarrow \quad \ddot{\theta}\frac{mb^2}{ka^2} + (\theta - \theta_{st}) = 0$$

$$\text{or} \quad \ddot{\theta} - \cancel{\ddot{\theta}_{st}} + \frac{ka^2}{mb^2}(\theta - \theta_{st}) = 0$$

if let $\theta - \theta_{st} = \psi$

$$\ddot{\psi} + \frac{ka^2}{mb^2}\psi = 0$$

ψ measured from equil

also $\Delta V = \frac{1}{2}kx^2 - mgy = \frac{1}{2}k\left[a\sin\theta\right]^2 - mg\,b\sin\theta$

$\Delta T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{y}^2 = \frac{1}{2}mb^2\left[\cos\theta\,\dot{\theta}\right]^2$

$$\frac{1}{2}mb^2\dot{\theta}^2 + \frac{1}{2}ka^2\theta^2 - mgb\theta$$

$$\left[mb^2\ddot{\theta} + ka^2\theta - mgb\right]\dot{\theta} = 0 \quad \Rightarrow \quad mb^2\ddot{\theta} + ka^2\theta$$

$$\tau = \frac{1}{f} = \frac{2\pi}{\omega_n} = 2\pi\Big/\sqrt{\frac{G\pi}{32 I l_1 l_2}\left(l_1 D_2^4 + l_2 D_1^4\right)}$$

$$= 2\pi\sqrt{\frac{32 I l_1 l_2}{G\pi}\,\frac{1}{\left(l_1 D_2^4 + l_2 D_1^4\right)}} = 8\sqrt{\frac{2\pi I l_1 l_2}{G\left(l_1 D_2^4 + l_2 D_1^4\right)}}$$

PROBLEM 2-11 Given: mass decreased by 0.4 kg ↑ freq ↑ by 25%

suppose you are told a system represented by a spring-mass had a freq of 2 Hz by ~~adding~~ subtracting 0.4 kg freq increases by 25% what is k, m

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 2\,\text{Hz} \qquad f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m-.4}} = 1.25 f$$

$$(2\pi f)^2 = [2\pi(2)]^2 = \frac{k}{m} \Rightarrow \frac{m}{k} = \frac{1}{16\pi^2}$$

$$(2\pi f')^2 = [2\pi(2.5)]^2 = \frac{k}{m-.4} \Rightarrow \frac{m-.4}{k} = \frac{1}{25\pi^2}$$

$$-\frac{.4}{k} = \frac{1}{25\pi^2} - \frac{m}{k} = \frac{1}{25\pi^2} - \frac{1}{16\pi^2} = \frac{1}{\pi^2}\left[\frac{-9}{16\cdot 25}\right]$$

$$k = \pi^2\frac{(16\cdot 25)}{9}(.4) = 175.46\ N/m.$$

$$m = \frac{k}{16\pi^2} = 1.11\ \text{kg}$$

PROBLEM 2-16

A vibrating SDOF system shows an amplitude of 2cm and a period of 3 sec. Determine the max. velocity & accel

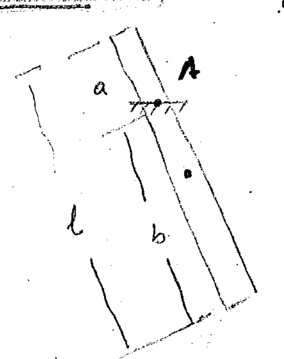
given $X = 2\text{cm}$ $\quad \tau = 3\,\text{sec} = \frac{2\pi}{\omega}$ $\quad \omega = \frac{2}{3}\pi = 2.094$ rad/sec

max veloc $\dot{X} = X\omega = 4.19$ cm/sec

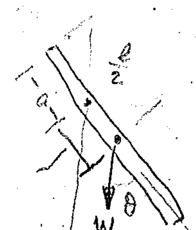
$\ddot{X} = -X\omega^2 = 8.77$ cm/sec^2

PROBLEM 2-21

A slender uniform bar pivots ~~about~~ in a vertical plane about A for small vibs obtain the DE & freq



center of gravity is at $l/2$



$$I_o = I_G + m\left(\tfrac{l}{2} - a\right)^2 \qquad dm = \rho\,dx$$

$$I_G = \int_{-l/2}^{l/2} x^2\,dm = \int_{-l/2}^{l/2} x^2 \rho\,dx = \rho\frac{l^3}{12} = \frac{m l^2}{12}$$

$$I_o = \frac{m\ell^2}{12} + m\left(\frac{\ell^2}{4} - 2\frac{\ell}{2}a + a^2\right) = m\left[\frac{\ell^2}{3} - \ell a + a^2\right]$$

TORQUE OF weight about O $= -W\sin\theta\,(\ell/2 - a)$

$$I_o\ddot{\theta} = -mg\sin\theta(\ell/2 - a) \quad \text{or}$$

$$m\left[\ell^2/3 - \ell a + a^2\right]\ddot{\theta} + mg(\ell/2 - a)\theta = 0 \qquad \text{for small } \theta$$

$$\omega = \sqrt{\frac{mg(\ell/2 - a)}{(\ell^2/3 - \ell a + a^2)m}} = \sqrt{\frac{g(\ell/2 - a)}{\ell^2/3 - \ell a + a^2}}$$

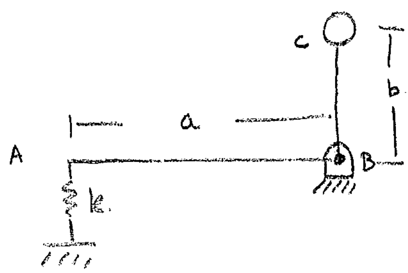
$$f = \frac{1}{2\pi}\omega$$

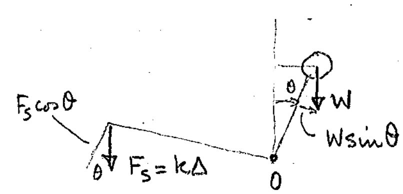
now $\omega_n = \sqrt{g/\ell'}$ for simple pendulum $\ell' = \dfrac{\ell^2/3 - \ell a + a^2}{\ell/2 - a}$

center of percussion

PROBLEM 2-24

WRITE EQ wrt B





$\Delta = a\sin\theta$

$F_s = ka\sin\theta$

Torque due to Weight $W\sin\theta \cdot b$

Torque due to $F_s = (-ka\sin\theta)\cos\theta\, a$

$I_o = mb^2$ if I_G of ball is small

$$I_o\ddot{\theta} = -ka^2\sin\theta\cos\theta + Wb\sin\theta$$

or $$mb^2\ddot{\theta} + (ka^2 - Wb)\theta = 0 \qquad \text{for small } \theta$$

$$\omega = \sqrt{\frac{ka^2 - Wb}{mb^2}} \qquad f = \frac{1}{2\pi}\omega.$$

energy $\frac{1}{2}k_{eq}\Delta_{eq}^2 = \frac{1}{2}k\Delta^2 + mgb(\cos\theta -$

$$\frac{1}{2}k_{eq}\Delta_{eq}^2 = \frac{1}{2}ka^2\theta^2 - \frac{mgb}{2}\theta^2$$

$(a\theta)^2$

$$\therefore k_{eq} = k - \frac{mgb}{a^2}$$

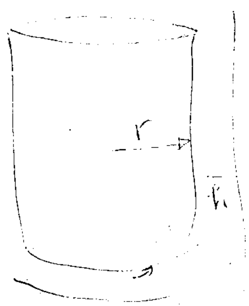
energy $\frac{1}{2}m_{eq}\dot{\Delta}_{eq}^2 = \frac{1}{2}m(b\dot{\theta})^2$

$$m_{eq} = m\left(\frac{b}{a}\right)^2$$

$$\therefore \omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{k - mgb/a^2}{mb^2/a^2}}$$

DO PROBLEMS 2-31, 2-32, 2-33

1.35



Torque $= F \cdot r$

$F = \tau \cdot 2\pi r l = \tau A$

$\tau = \mu \frac{du}{dy} = \mu \frac{V}{h} = \mu \frac{\Omega r}{h}$

$\therefore \quad F = 2\pi r l \cdot \mu \frac{\Omega r}{h} = \frac{2\pi r^2 l \mu}{h} \Omega$

$T = Fr = \frac{2\pi r^3 l \mu}{h} \Omega$

$\therefore \quad T = k_t \Omega$

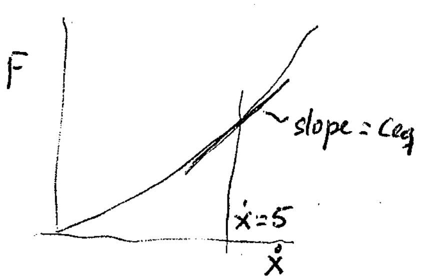
$\tau =$

$T = \int F dr = \int \frac{\mu \Omega r}{h} \cdot 2\pi r dr$

$T = r dF = \mu \frac{\Omega r}{h} \cdot 2\pi r dr$

$\frac{\mu \Omega r^3}{3 h_0} \cdot 2\pi$

1.36 $F = a\dot{x} + b\dot{x}^2$

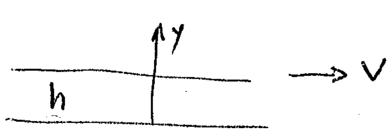


$F = C_{eq} \dot{x} \qquad C_{eq} = \frac{\partial F}{\partial \dot{x}}$

$C_{eq} = a + 2b\dot{x}$

near $\dot{x} = 5 \quad C_{eq} = 5 + 0.2(5) \cdot 2 = 7 \ \frac{N\text{-}s}{m}$

$\therefore \quad F = 7\dot{x}$ near $\dot{x} = 5$



$F = \tau \cdot A = c V$

$\tau = \mu \frac{du}{dy} = \mu \frac{V - 0}{h - 0} = \frac{\mu V}{h}$

$A = l b$

$\therefore \quad F = \left(\frac{\mu}{h} \cdot l b\right) V \qquad C_{eq} = \frac{\mu A}{h}$

To: LEVY @ ENG
From: C. Levy
Subject: for Leah
Date: 9/5/97 Time: 9:49AM

From: SMTP%"mschulma@sdp2.philsch.k12.pa.us" 4-SEP-1997 21:51:03.64
To: LEVYEZ
CC:
Subj: Good News!

Return-Path: <mschulma@phila.k12.pa.us>
Received: from sdp2.philsch.k12.pa.us by servax.fiu.edu with SMTP;
Thu, 4 Sep 1997 21:50:48 -0400 (EDT)
Received: from [170.235.5.12] ([170.235.5.12]) by sdp2.philsch.k12.pa.us (8.6.10) with SMTP id VAA07003 for <levyez@servax.fiu.edu>; Thu, 4 Sep 1997 21:50:41 -0400
X-Sender: mschulma@mail.phila.k12.pa.us
Message-Id: <v01540b00b03515bbed1c@[170.235.5.31]>
Mime-Version: 1.0
Content-Type: text/plain; charset="us-ascii"
Date: Thu, 4 Sep 1997 21:54:27 -0400
To: levyez@servax.fiu.edu
From: mschulma@sdp2.philsch.k12.pa.us (Marcia S. Schulman)
Subject: Good News!

Hi! My father's surgery is over, and hopefully he'll be fine! The doctor cleaned up the bone spurs that had been pressing on his spinal chord in various spot along the column. Now there's enough room for the nerve to move around. No wonder he had been in such pain!

We're so glad today is over! Although he's in alot of post-op pain, and recovery will be slow, there's light at the end of the tunnel.

Hope all is well.

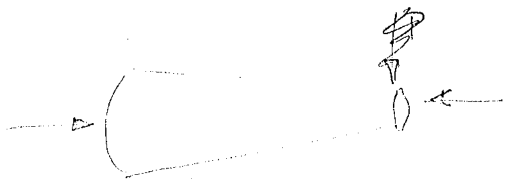
Love,

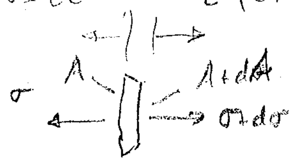
Marcia

DO 1.7, 1.9, 1.17, 1.16

in series   energy   energy

$\sigma = E\epsilon$ $E(\epsilon + d\epsilon) = d\sigma + \sigma$





dx

$$dx \quad \frac{dP}{dA} = \sigma = E\epsilon \qquad dP =$$

$$\sigma A = (\sigma + d\sigma)(A + dA)$$

$$A\,d\sigma = -\sigma\,dA$$

$$\int \frac{d\sigma}{\sigma} = -\frac{dA}{A}$$

$$\ln\sigma = -\ln A + C$$

$$\frac{P}{A} = \sigma = E\epsilon$$

$$\frac{P}{AE}\,dx = du$$

1.11

$$\frac{P}{E}\left[\frac{4\,dx}{\pi\left\{\left(\frac{D-d}{0-l}\right)x + D\right\}^2}\right] = \Delta$$

$$\ln\sigma_D - \ln\sigma_d = \ln A_d - \ln A_D$$

$$\frac{P}{E\pi}\;\frac{\ln Kx + D}{K}\Big|_0^l$$

$$\frac{P}{E\pi} \qquad \frac{4P}{E\pi}\;\frac{-1}{\left(-\frac{D+d}{l}x + D\right)\left(-\frac{D+d}{l}\right)}\Bigg|_0^l = \Delta$$

$$\frac{4P}{E\pi}\left[\frac{-1}{d\left(\frac{-D+d}{l}\right)} + \frac{1}{D\left(\frac{-D+d}{l}\right)}\right]$$

$$\left(\frac{d-D}{l}\right)^{-1}\frac{4P}{E\pi}\left[\frac{-D+d}{dD}\right]$$

$$\frac{4Pl}{E\pi\,dD} = \Delta$$

$$k = \frac{dD\,E\pi}{4l}$$

$$\frac{dD}{4l} = \frac{t(d+t)}{l_2} \qquad \therefore \boxed{l_2 = \frac{4t(d+t)}{dD}\,l}$$

$$A = \frac{\pi(d+2t)^2}{4} - \frac{\pi d^2}{4}$$

$$A = \frac{\pi}{4}\left(2t \cdot 2(d+t)\right)$$

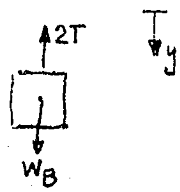
$$= \pi t(d+t)$$

$$\frac{P}{A} = \sigma = E \cdot \frac{\Delta}{l_2}$$

$$\therefore P = \frac{AE}{l_2}\Delta$$

$$P = \frac{E\pi t(d+t)}{l_2}\Delta$$

Isolate block B:



$$m_B V_{B_1} - \overbrace{2T\Delta t}^{\text{Impulse due to 2T}} + \overbrace{W_B \Delta t}^{\text{Impulse due to weight}} = m_B V_{B_2} \qquad \text{here } V_{B_1} = -1 \text{ m/s}$$

by the kinematic eqn $V_{B_2} = -V_{A_2}/2$

With these 3 eqns we can solve for T and V_{A_2} and V_{B_2} when $\Delta t = 2$ sec

$\therefore$ @ $t = 2$ sec $V_{A_2} = 6.02$ m/s $\quad V_{B_2} = -3.01$ m/s or 3.01 m/s $\uparrow$ $\quad T = 16.2$ N

Here we needed to use impulse & momentum to relate the velocities of the blocks to the tension in the cord and kinematics to relate the velocities of the blocks. Remember since by kinematics we picked the directions for $+S_A$ & $+S_B$, we must use same notation for the definition of the directions for the impulse & momentum equations.

$$\tau = \frac{1}{f} = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{G\pi}{32 I l_1 l_2}\left(l_1 D_2^4 + l_2 D_1^4\right)}$$

$$= 2\pi\sqrt{\frac{32 I l_1 l_2}{G\pi}\frac{1}{\left(l_1 D_2^4 + l_2 D_1^4\right)}} = 8\sqrt{\frac{2\pi I l_1 l_2}{G\left(l_1 D_2^4 + l_2 D_1^4\right)}}$$

DAMPED FREE VIBS FOR SDOF

- SO FAR VIBS HAVE BEEN SELF-SUSTAINING
- REAL LIFE VIBS DIE AWAY (VIBS ARE DAMPED)
- THREE TYPES OF DAMPING

1 VISCOUS DAMPING $F_d = -c\dot{x}$ $c = \text{CONST.}$ F_d - DAMPING FORCE

F_d - lbs, N $\quad$ kg/sec, $\frac{N\,sec}{m}$, $\frac{lb\text{-}sec}{in}$

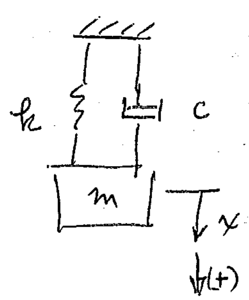
- EXAMPLE - BODY MOVING THRU FLUIDS AT LOW VELOC.
- FORCE OPPOSES MOTION
  - BODIES SLIDE OVER LUBRICATED SURFACE
  - BODIES THROUGH AIR, OIL

SHOCK ABSORBERS, DASHPOTS

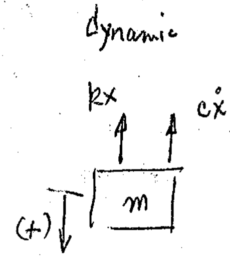
2. COULOMB FRICTION $F = \mu N$ SLIDING OVER DRY SURFACE

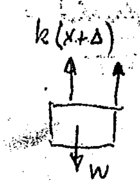
3. HYSTERESIS DAMPING DUE TO INTERNAL FRICTION IN MATERIAL

CONSIDER MASS-SPRING-DASHPOT SYSTEM



dynamic





$c\frac{d}{dt}(x+\Delta) = c\dot{x}$

measure every thing from unstretched length of spring

$\frac{d^2}{dt^2}(x+\Delta) = \ddot{x}$

$$m\ddot{x} = -kx - c\dot{x}$$

$$\ddot{x} + \frac{k}{m}x + \frac{c}{m}\dot{x} = 0$$

$$x = Ce^{st} \Rightarrow \quad s^2 + \frac{k}{m} + \frac{cs}{m} = 0$$

$$s = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}}{2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

$$x = 0 = C_1 e^{-\alpha t} + C_2 e^{-\beta t}$$

$$\frac{C_2}{C_1} = -\frac{e^{-\alpha t}}{e^{-\beta t}} = -e^{(\beta-\alpha)t}$$

$$\frac{\ln\left(\frac{\alpha x_0 + \dot{x}_0}{\beta x_0 + \dot{x}_0}\right)}{2\omega_n\sqrt{\zeta^2 - 1}} = \frac{\ln(-C_2/C_1)}{\beta - \alpha} = \cancel{(\beta-\alpha)}\, t_{cross}$$

SESSION #7

if $\frac{c}{2m} = \sqrt{\frac{k}{m}} = \omega_n$ then $c = c_c$ critical damping cond.

$$(C_1 + C_2 t)e^{-\frac{c_c}{2m}t} = (C_1 + C_2 t)e^{-\omega_n t} \qquad \underline{c_c = 2\sqrt{km} = 2m\omega_n}$$

CAN DEFINE DAMPING FACTOR

$$\frac{c}{c_c} = \zeta \qquad \therefore \quad \frac{c}{2m} = \frac{c}{c_c}\cdot\frac{c_c}{2m} = \zeta(\omega_n) \qquad c = 2m\omega_n\zeta$$

now $\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = \sqrt{\zeta^2\omega_n^2 - \omega_n^2} = \sqrt{\zeta^2 - 1}\,\omega_n$

General Solution $= C_1 e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + C_2 e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}$ for $\zeta \neq 1$

$C_1 e^{s_1 t} + C_2 e^{s_2 t}$

- if $\zeta > 1$ DAMPING CONST > CRIT. DAMP. CONST.

$\zeta > 1$; $\sqrt{\zeta^2 - 1} < \zeta$ $\qquad\therefore\quad -\zeta + \sqrt{\zeta^2 - 1} < 0$

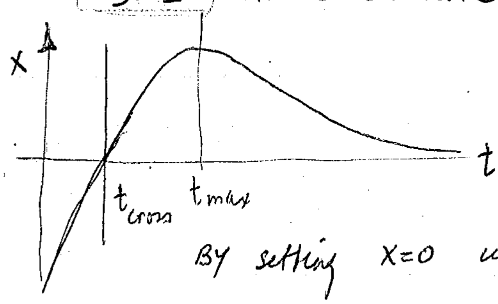
$-\zeta - \sqrt{} < 0$

$x(t) =$ sum of 2 decreasing exponential $= C_1 e^{-\alpha t} + C_2 e^{-\beta t}$

$\alpha = (\zeta - \sqrt{\zeta^2-1})\omega_n \qquad \beta = (\zeta + \sqrt{\zeta^2-1})\omega_n$

$x(t)$ is not periodic or aperiodic

- $\zeta > 1$ ABOVE CRITICAL DAMPING <u>overdamped</u>



$x_0 = C_1 + C_2$ @ $t = 0$

$\dot{x}_0 = -\alpha C_1 - \beta C_2$

$\frac{\alpha x_0 + \dot{x}_0}{\alpha - \beta} = C_2 \qquad \frac{\beta x_0 + \dot{x}_0}{\beta - \alpha} = C_1$

BY setting $x = 0$ we get first crossing pt. $t_{crossing} = \frac{\ln(-C_2/C_1)}{\beta - \alpha}$

$$t_{max} = \frac{\ln(-\beta C_2/\alpha C_1)}{\beta - \alpha} \qquad \beta - \alpha = 2\omega_n\sqrt{\zeta^2 - 1}$$

$\frac{dx}{dt} = 0 \Rightarrow -C_1\alpha e^{-\alpha t} - C_2\beta e^{-\beta t}$

$$x(t=0) = C\sin\phi = x_0$$

$$\dot{x}(t=0) = C\left[-\zeta\omega_n e^{-\zeta\omega_n t}\sin(\quad) + e^{-\zeta\omega_n t}\omega_d\cos(\quad)\right]$$

$$C\left[-\zeta\omega_n \sin\phi + \omega_d\cos\phi\right] = \dot{x}_0$$

$$\left[-\zeta\omega_n x_0 + C\omega_d\cos\phi\right] = \dot{x}_0$$

$$C\cos\phi = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}$$

$$\tan\phi = \frac{x_0\omega_d}{\dot{x}_0 + \zeta\omega_n x_0} \qquad C = \sqrt{x_0^2 + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)^2}$$

$$\text{as } \zeta \to 0 \quad \tan\phi \to \frac{x_0}{\dot{x}_0/\omega_n} \qquad C \to \sqrt{x_0^2 + \frac{\dot{x}_0^2}{\omega_n^2}}$$

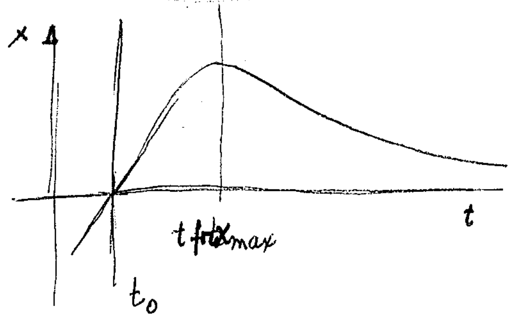
$$x(t) = Ce^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

• IF $\zeta = 1$ $\quad x(t) = (A + Bt)e^{-\omega t}$ APERIODIC FN $\quad X_o = A$

$\dot{X}_o = B - A\omega$

$B = \dot{x}_o + A\omega = \dot{x}_o + x_o\omega$

FOR THE CRITICALLY DAMPED CASE $\zeta = 1$



$x = 0 \Rightarrow A + Bt_o = 0 \quad t_o = -\frac{A}{B}$

t_o is first crossing

$x(t) = (A + Bt)e^{-\omega t} = (x_o[1 + \omega t] + \dot{x}_o t)e^{-\omega t}$

$\dot{x} = \left[(B - A\omega) - B\omega t\right]e^{-\omega t} \quad \dot{x} \Rightarrow 0 \Rightarrow$

$t_{for\ x_{max}} = \frac{1}{\omega} - \frac{A}{B}$

• FOR THE SUBCRITICAL DAMPING CASE $\zeta < 1$ underdamped.

$$x(t) = C_1 e^{(-\zeta + i\sqrt{1-\zeta^2})\omega_n t} + C_2 e^{(-\zeta - i\sqrt{1-\zeta^2})\omega_n t} = \cancel{e^{-\zeta\omega t}\left[C \sin(\omega_d t\right.}$$

$$= C e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ damped circular freq

• FOR $x = 0$

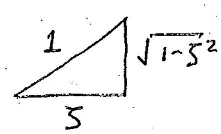
Crossing points $\omega_d t + \phi = n\pi \quad t = (n\pi - \phi)/\omega_d$

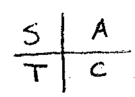
• TO FIND THE MAX OR MIN

$$\dot{x}(t^*) = 0 = C e^{-\zeta\omega t^*}\left[-\zeta\omega \sin(\omega_d t^* + \phi) + \omega_d \cos(\omega_d t^* + \phi)\right]$$

IF $t^* \neq \infty$ $\quad e^{-\zeta\omega t^*}$ and C are not zero

$$\Rightarrow \tan(\omega_d t^* + \phi) = \frac{\omega_d}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$





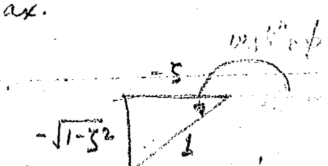
since $\zeta < 1$ $\quad$ tan is (+) in 1^{ST} & 3^{rd} QUAD.

in 1^{ST} QUAD $\quad \sin(\omega_d t_1^* + \phi) = \sqrt{1-\zeta^2} < 1$ $\quad$ MAXS

$$\sin(\omega_d t_1^* + \phi) = \sin(\omega_d t_1^* + \phi + 2\pi) = \dots$$

$$t_1^* = \left[\sin^{-1}\sqrt{1-\zeta^2} - \phi - 2n\pi\right]/\omega_d \quad \text{max.}$$

First max is at $t_1^* = \left[(\sin^{-1}\sqrt{1-\zeta^2}) - \phi\right]/\omega_d$

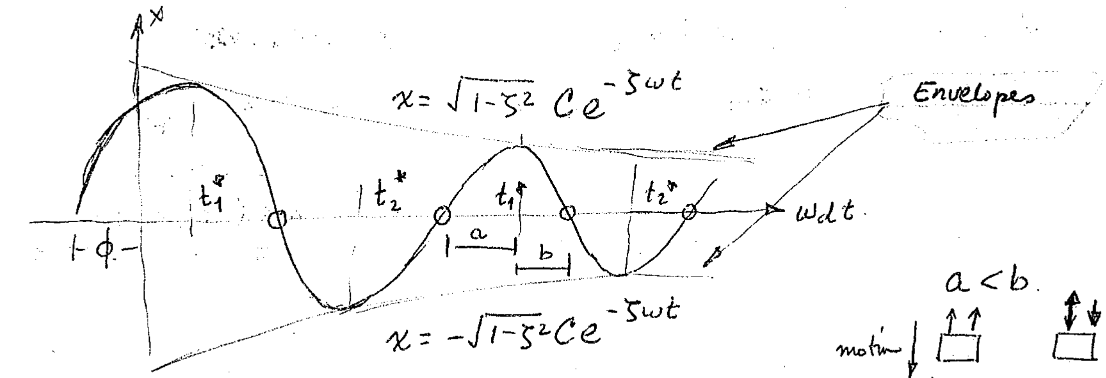


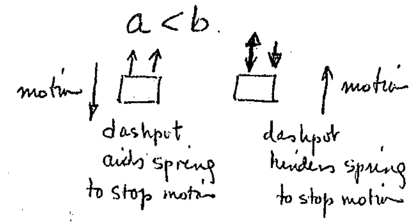
in 3^{rd} QUAD $\quad \sin(\omega_d t_2^* + \phi) = -\sqrt{1-\zeta^2} > -1$ $\quad$ MINS

$$\sin(\omega_d t_2^* + \phi) = \sin(\omega_d t_2^* + \phi + 2\pi) = \dots$$

$$t_2^* = \left[\sin^{-1}(-\sqrt{1-\zeta^2}) - \phi - 2n\pi\right]/\omega_d = \left[\sin^{-1}(\sqrt{1-\zeta^2}) - \phi - (2n-1)\pi\right]/\omega_d$$

$$\omega_d t_2^* = \omega_d t_1^* + \pi$$





- period between maximums. $\omega_d \tau = 2\pi$

$$\tau = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega}.$$

ENVELOPES

- since at max. $\sin(\omega_d t_1^* + \phi) = \sqrt{1-\zeta^2}$

$$x_{ENV.} = C e^{-\zeta \omega t} \sin(\omega_d t_1^* + \phi) = C e^{-\zeta\omega t} \sqrt{1-\zeta^2}.$$

- at min $\sin(\omega_d t_2^* + \phi) = -\sqrt{1-\zeta^2}$

$$x_{ENV.} = C e^{-\zeta\omega t} \sin(\omega_d t_2^* + \phi) = -\sqrt{1-\zeta^2}\, C e^{-\zeta\omega t}$$

EFFECTS OF DAMPING. $\zeta > 0$

- REDUCES THE AMPLITUDE OF THE MOTION
- REDUCES TIME TO FIRST PEAK $\qquad \omega t_1^* = \dfrac{\sin^{-1}\sqrt{1-\zeta^2} - \phi}{\sqrt{1-\zeta^2}} \cong 1 + \dfrac{(1-\zeta^2)}{6} + \dfrac{3}{40}(1-\zeta^2)^2 + \ldots$

$$\omega t_1^* = \pi/2 - \phi \quad \text{vs.} \quad \pi/2 - \zeta - \phi$$

- INCREASES TIME OF MOTION TO RETURN TO NEUTRAL POSITION

$\omega t_{cross} = \pi - \phi$ vs. $\omega t_{cross} = \dfrac{\pi - \phi}{\sqrt{1-\zeta^2}}$ FIRST crossing

$\sin(\omega_n t + \phi) \qquad \sin(\omega_d t + \phi)$

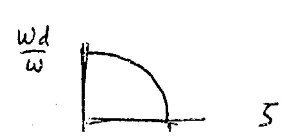
- PERIOD OF DAMPED MOTION > PERIOD OF UNDAMPED MOTION

$\omega_n \tau_n = 2\pi$ vs $\omega_n \tau_d = \dfrac{2\pi}{\sqrt{1-\zeta^2}}$ $\qquad \tau_n = \dfrac{2\pi}{\omega_n}$ & $\tau_d = \dfrac{2\pi}{\omega_d}$ $\qquad$ but $\omega_d < \omega_n \Rightarrow \tau_d > \tau_n$

- PEAKS STARTED TO THE LEFT OF THE UNDAMPED PEAKS
  - SINCE PERIOD INCREASES $\Rightarrow$ PEAKS MOVE TO RIGHT OF UNDAMPED

- DAMPING DUE TO CRITICAL DAMPING CAUSES AMPLITUDE TO DROP TO ZERO MOST QUICKLY FOR SAME I.C.

- FREQ DECREASES $\qquad \omega_d = \sqrt{1-\zeta^2}\,\omega$



2.43 $f_n = .5\ Hz$ $f_d = .45\ Hz = f_n\sqrt{1-\zeta^2}$ $\zeta = \sqrt{.19} = .435$

$c = C_c \cdot \zeta$ $\frac{c_c}{2m} = \omega_n$ $c_c = 2m\omega_n = \frac{c}{\zeta}$

$2m\omega_n\zeta = c$

$2 \cdot 1\ (2\pi)(.5)\sqrt{.19}$

$2 \cdot \pi \quad \cong .435 \cong 3.14\ \frac{N\text{-sec}}{m}$

A simple pendullem is found to vibrate @ a freq of 0.5 Hz in a vacuum & 0.45 Hz in a viscous fluid medium. Find c if the mass of the pendulum bob is 1 kg

```
    .433
    .433
    1299
   12990
  173200
 .187489
```

SESSION #8

LOGARITHMIC DECREMENT

- WANT TO MEASURE THE RATE OF DECAY OF SYSTEM
  - TAKE 2 POINTS IN PHASE (i.e. amplitudes)

$$x_j = C\sqrt{1-\zeta^2}\, e^{-\zeta\omega t_j} \quad \text{or} \quad C e^{-\zeta\omega t_j} \sin(\omega_d t_j + \phi)$$

$$x_{j+1} = C\sqrt{1-\zeta^2}\, e^{-\zeta\omega(t_j+\tau)} \quad \text{or} \quad C e^{-\zeta\omega(t_j+\tau)} \sin[\omega_d(t_j+\tau)+\phi]$$

$$\sin[\omega_d(t_j+\tau)+\phi] = \sin(\omega_d t_j + \phi + 2\pi)$$

$$\frac{x_j}{x_{j+1}} = e^{\zeta\omega_n\tau_d} = \text{constant}; \quad \ln\frac{x_j}{x_{j+1}} = \zeta\omega_n\underbrace{\tau_d}_{2\pi/\omega_d} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

LOGARITHMIC DECREMENT $\boxed{\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}} = \ln\left(\frac{x_j}{x_{j+1}}\right)$ SAME FOR ANY SUCCESSIVE AMPLITUDES

- FOR THE DECAY AFTER n cycles

$$\frac{x_0}{x_n} = \frac{x_0}{x_1}\frac{x_1}{x_2}\frac{x_2}{x_3}\cdots\frac{x_{n-1}}{x_n} = \left(\frac{x_j}{x_{j+1}}\right)^n$$

$$\ln\left(\frac{x_0}{x_n}\right) = n\ln\left(\frac{x_j}{x_{j+1}}\right) = n\delta$$

$$\boxed{\delta = \frac{1}{n}\ln\left(\frac{x_0}{x_n}\right)}$$

or

- DETERMINE NO. OF CYCLES TO CAUSE DECAY FROM x_0 TO x_n

$$\boxed{n = \frac{1}{\delta}\ln\left(\frac{x_0}{x_n}\right) = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta}\ln\left(\frac{x_0}{x_n}\right)}$$

NOT INTEGER

period between cycles

- TIME FOR THIS TO OCCUR IS $n\tau_d = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta}\ln\left(\frac{x_0}{x_n}\right)\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$

$$\boxed{\Delta\tau = n\tau = \frac{1}{\zeta\omega_n}\ln\left(\frac{x_0}{x_n}\right)}$$

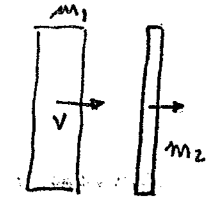
if from exper we can get $\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \boxed{\zeta = \frac{\delta}{\sqrt{(2\pi)^2+\delta^2}}}$

if k & m are known $\boxed{\omega_n = \sqrt{\frac{k}{m}}}$

$$\frac{c}{2m} = \zeta\omega_n \Rightarrow \boxed{c = 2m\zeta\omega_n}$$

A slider of mass $m = 1\ kg$ travels in a cylinder w/ veloc $V = 80\ m/s$ engaging a spring damper system. IF $K = 40\ N/mm$ $c = .2\ N\text{-}s/mm$ find x_{max} & t to x_{max}.

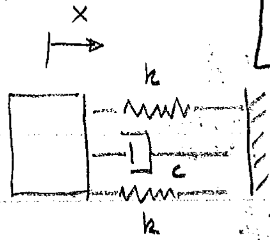
RAO



$m_1 v_1 = (m_1+m_2) v_2$

$v_2 = \frac{m_1}{m_1+m_2} v_1$

2-52



$k_{eq} = 2k =$

c

Given $c = .2\ \frac{N\text{-}s}{mm}$, $k = 40\ N/mm$

$m\ddot{x} + c\dot{x} + k_{eq}x = 0$ w/IC that $x = 0$ @ $t = 0$

$\dot{x} = 80 m/s$ @ $t = 0$

Now $\omega_n = \frac{c_c}{2m} = \sqrt{\frac{k_{eq}}{m}} \Rightarrow c_c = 2\sqrt{\cancel{2}mk_{eq}} = 2\sqrt{\cancel{2}(1)(2 \cdot 40 \cdot 1000)} = \overset{565.686}{\cancel{400}}\ N\text{-}s/m = \overset{.566}{\cancel{.4}}\ \frac{N\text{-}s}{mm}$

$\zeta\omega_n = \frac{c}{2m} = 100$

$c = .2\ N\text{-}s/mm$ $\qquad \zeta = \frac{c}{c_c} = \overset{.3536}{\cancel{.5}} \Rightarrow$ underdamped.

$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{80000}{1}} = 282.84\, rad/s$ $\qquad \omega_d = \omega_n\sqrt{1-\zeta^2} = 282.84(.9354) = 264.57$

initially $x = 0$, $\dot{x} = 80 m/s$

$x = X e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$ $\qquad$ when $t = 0$ $\qquad 0 = X e^{0} \sin\phi$

either $\phi = 0$ or $X = 0$ $\qquad$ if $X = 0 \Rightarrow x = 0\ \forall t$ trivial soln

$\dot{x} = -\zeta\omega X e^{-\zeta\omega_n t}\sin(\omega_d t + \phi) + X\omega_d e^{-\zeta\omega t}\cos(\omega_d t + \phi)$

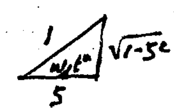
$\dot{x} = 80 = -\zeta\omega_n X e^{0}\sin\phi + X\omega_d e^{0}\cos\phi$ $\qquad$ if $X = 0$ cannot satisfy $\Rightarrow \phi = 0$

$80 = X\omega_d e^{0} \cdot 1 = X\omega_n\sqrt{1-\zeta^2}$ $\qquad \Rightarrow X = \overset{.3023}{\cancel{.3266}}\ m$

$x = .3023\, e^{-(.3536)(282.84)t}\sin\left(\frac{80}{.3023}t\right)$

$\dot{x} = -\zeta\omega_n X e^{-\zeta\omega_n t}\sin\omega_d t + X\omega_n\sqrt{1-\zeta^2}\cos(\omega_d t) = 0$ when $t = t^*$

$\Rightarrow -\zeta\sin\omega_d t + \sqrt{1-\zeta^2}\cos\omega_d t$ $\qquad$ or $\qquad \tan\omega_d t^* = \frac{\sqrt{1-\zeta^2}}{\zeta}$



$t^* = \frac{\sin^{-1}\left(\sqrt{1-\zeta^2}\right)}{\omega_n\sqrt{1-\zeta^2} = \omega_d} = \frac{\frac{69.29 \cdot \pi}{180}}{264.57} = \underline{.00457\ sec}$

$x = .3023\, e^{-\zeta\omega_n t}\sin(\omega_d t)$ $\quad$ w/ $\quad \zeta\omega = 100 \quad \omega_d = 264.57 \quad t = .00457\, sec$

$x = .1790\ m$

Given that 2 successive maxs are diff by a factor of 12

2-49 $\qquad \left(\frac{X_j}{X_{j+1}}\right) = \frac{12}{1} \Rightarrow \ln(12) = \delta = 2.4849 \Rightarrow \zeta = \frac{2.4849}{\sqrt{(2\pi)^2 + \delta^2}} = .36777$

$\zeta_{new} = .73554$

$\delta = \ln\left(\frac{X_j}{X_{j+1}}\right) = \frac{2\pi\zeta_{new}}{\sqrt{1-\zeta_n^2}} = 6.822$

$\frac{X_j}{X_{j+1}} = 917.49$

2.95 in 4th ed.

$$\Delta\tau = n\tau_d = 1 = \frac{1}{5\omega_n}\ln\left(\frac{X_0}{X_5}\right) \Rightarrow \tau_d = .2\text{ sec} = \frac{1}{5}$$

$$\delta = \frac{1}{50}\ln\left(\frac{X_0}{X_{50}}\right) = \frac{1}{50}\ln\left(\frac{X_0}{.1X_0}\right) = .\cancel{00211}\ 04605$$

$$\therefore\ \delta \cong 2\pi\zeta \qquad \zeta = \frac{\delta}{2\pi} = \cancel{.000335}\ .00733$$

$$\omega_d = \frac{2\pi}{\tau_d} = 31.416\text{ rad/s} = \omega_n\sqrt{1-\zeta^2} \Rightarrow \omega_n = 31.415\cancel{9283}\ 677054$$

31.41592654

$\tau_d = \cancel{.199999989}$ .19999463

$\tau_d = .2$

decrease of 2.69×10^{-3} %

Equivalent system, k, c, m

finding ω_n

derive equations of motion

use PE & KE to find equations of motion

stability condition

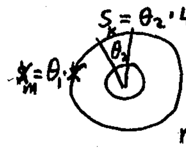
2.47 T = kinetic energy $= T_{mass} + T_{pulley}$

$$= \frac{1}{2}\, m\, \dot{x}^2 + \frac{1}{2}\, J_0\, \dot{\theta}^2 = \frac{1}{2}\,(m\, r^2 + J_0)\, \dot{\theta}^2$$

U = potential energy $= \frac{1}{2}\, k\, x_s^2 \; = \frac{1}{2}\, k\, (4\, r\, \theta)^2 = \frac{1}{2}\, k\, (16\, r^2)\, \theta^2$

Using $\frac{d}{dt}\,(T + U) = 0$ gives

$$(m\, r^2 + J_0)\, \ddot{\theta} + (16\, r^2\, k)\, \theta = 0$$



$\theta_1 = \theta_2$

$x_m = \frac{S_k}{4r}$

$S_k = 4X$

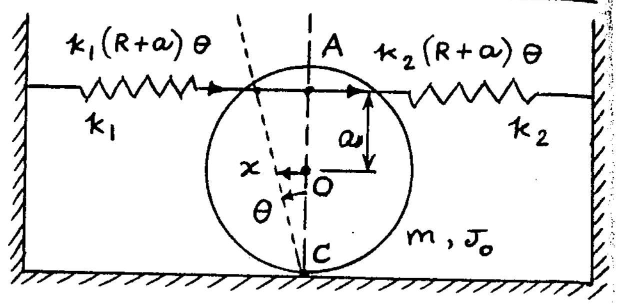
spring moves 4 times distance of mass

2.60 $J_o = \frac{1}{2}\, m\, R^2$, $\quad J_C = \frac{1}{2}\, m\, R^2 + m\, R^2$

Let angular displacement $= \theta$

Equation of motion : $\Sigma T_c = J_c \ddot{\theta}$

$$J_c\, \ddot{\theta} + k_1\, (R+a)^2\, \theta + k_2\, (R+a)^2\, \theta = 0$$



$$\omega_n = \sqrt{\frac{(k_1 + k_2)\,(R+a)^2}{J_c}} = \sqrt{\frac{(k_1 + k_2)(R+a)^2}{1.5\; m\, R^2}} \qquad (E_1)$$

Equation (E_1) shows that ω_n increases with the value of a.

$\therefore$ ω_n will be maximum when $a = R$.

2.83 $\omega_n = 200$ cycles/min $= 20.944$ rad/sec , $\quad \omega_d = 180$ cycles/min $= 18.8496\ \frac{rad}{sec}$

$J_o = 0.2\ kg - m^2$

2.92 in 4th

Since $\omega_d = \sqrt{1 - \zeta^2}\ \omega_n$, $\quad \zeta = \sqrt{1 - \left(\frac{\omega_d}{\omega_n}\right)^2} = \sqrt{1 - \left(\frac{18.8496}{20.944}\right)^2} = 0.4359$

$$= \frac{c_t}{(c_t)_{cri}} = \frac{c_t}{2\, J_o\, \omega_n}$$

$$c_t = 2\, J_o\, \omega_n\, \zeta = 2(0.2)(20.944)(0.4359)$$

$$= 3.6518 \quad N\text{-}m\text{-}s/rad$$

Eq. (2.72) can be used to obtain $\theta(t)$ for $\dot{\theta}_o = 0$, $\theta_o = 2^\circ = 0.03491$ rad and $t = \tau_d = \frac{2\pi}{\omega_d} = 0.3333$ sec ,

$$\theta(t) = e^{-\zeta\,\omega_n t}\ \theta_o \left\{ \cos \omega_d t + \frac{\zeta\, \omega_n}{\omega_d}\ \sin \omega_d t \right\}$$

$$= e^{-(0.4359)(20.944)(0.3333)}\ (0.03491) \left\{ \cos\ 18.8496 \times 0.3333 + \frac{0.4359 \times 20.944}{18.8496}\ \sin\ 18.8496 \times 0.3333 \right\}$$

$$= 0.001665\ rad = 0.09541^\circ$$

or use log decrement since X_o is at a max

10. W.R. Saunders, H.H. Robertshaw and C.A. Rogers, "Experimental studies of structural acoustic control for a shape memory alloy composite beam", *Proceedings of the 31st Structures, Structural Dynamics and Materials Conference, AIAA* Paper No. 90-1090 (1990).

11. C.A. Rogers, C. Liang and C.R. Fuller, "Modeling of shape Memory alloy hybrid composites for structural acoustic control", *Journal of the Acoustical Society of America*, **89(1)**, 210-220 (1991).

12. E.J. Graesser and F.A. Cozzarelli, "Shape memory alloys as new materials for aseismic isolation", ASCE *Journal of Engineering Mechanics*, **117(11)**, 2590-2608 (1991).

13. L.C. Brison, "One dimensional constitutive behavior of shape memory alloys: thermomechanical derivation with non-constant material functions and refined martensite internal variable", *Journal of Intelligent Material Systems and Structures*, **4**, 229-242 (1993).

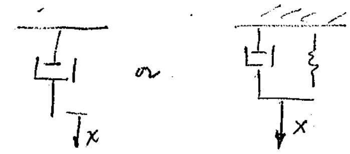
14. K.H. Wu, Y.Q. Liu, M. Maich and H.K. Tseng, "The Mechanical properties of a NiTi_Pd high temperature shape memory alloy", *Proceedings of the SPIE - The International Society of Optical Engineering*, **2189**, 306-313 (1994).

15. Q. Chen and C. Levy, "Smart damping treatment for flexible structure," Mat. Res. Soc. Symp. Proc. **360**, 527-532 (1994).

16. Q. Chen and C. Levy, "Simplified model for combined applications of viscoelastic material and shape memory alloy to vibration control", *Proceedings of SPIE - The International Society of Optical Engineering,* **2443**, 579-587 (1995).

17. Q. Chen and C. Levy, "Active vibration control of elastic beam by means of shape memory alloy layers", *Journal of Smart Materials and Structures*, **5**, 400-406 (1996).

18. D.I.G. Jones, "Temperature-frequency dependence of dynamic properties of damping



$$\text{work} = \bar{F}\cdot d\bar{r} = \overbrace{(\bar{F}\cdot d\bar{v})}^{\text{power}}dt.$$

$$\text{energy} = -\text{work}.$$

ENERGY DISSIPATED

$$\text{Power} = \text{Force} \times \text{velocity} = -cv\cdot v = \frac{d\,\text{work}}{dt}$$

Energy dissipated in one ~~~~ period is $-\int_0^{\tau} \text{power}\, dt = \int_0^{\tau} +c\dot{x}^2 \; dt$

or forced damped syst— forcing fn having ω_d as freq & SS response

but if $x = X \sin \omega_d t$ non damped system with same ω_d as damped system

$$\dot{x} = X\omega_d \cos\omega_d t$$

$$\Delta E = \int_0^{\tau} X^2\omega_d^2 c \;\cos^2\omega_d t \;dt \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}.$$

$$X^2\omega_d c \quad \cos^2\theta \; d\theta$$

$$= c\,\omega_d X^2 \int_0^{2\pi} \frac{1}{2}\, d(\omega_d t) + \frac{\cos 2\omega_d t}{2}\; d(\omega_d t)$$

$$= \left[\frac{\omega_d t}{2} + \frac{\sin 2\omega_d t}{4}\Big|_0^{2\pi}\right] = c\,\omega_d X^2 \pi$$

$$\Delta E \sim X^2$$

$$\tau = \frac{2\pi}{\omega_d},$$

For a non damped system. $KE_{max} = \frac{1}{2}m\dot{x}^2 = \frac{1}{2}m X^2\omega_d^2$

FRACTION OF TOTAL ENERGY THAT IS DISSIPATED IN EACH CYCLE

$$\text{thus}\quad \frac{\Delta E}{E} = \frac{c\,\omega_d X^2 \pi}{\frac{1}{2}m\,\omega_d^2 X^2} = \frac{2c\pi}{m\,\omega_d} \qquad \text{but}\quad \frac{c}{2m} = \zeta\omega_n$$

$$= \frac{4\zeta\omega_n\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{4\zeta\pi}{\sqrt{1-\zeta^2}} = 2\delta$$

non damped & a very lightly damped system have almost the same characteristics $\Rightarrow$ $\zeta \cong 0$ as $\frac{\Delta E}{E} = 4\pi\zeta = \text{const}$ $\quad \zeta < .2$

$$2\delta = \underbrace{4\pi\zeta}_{\text{FOR SMALL }\zeta} = \frac{\Delta E}{E} \qquad \text{SPECIFIC DAMPING CAPACITY}$$

$$\frac{\delta}{\pi} = 2\zeta = \frac{\Delta E}{2\pi E} \qquad \text{LOSS COEFF}$$

} used to ~~relate~~ compare damping capacities of different materials

2^{nd} ed.

2.44

3^{rd} ed 2.76

$$\frac{x_0}{x_1} = 18 \qquad \ln(18) = \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 2.89$$

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = .418$$

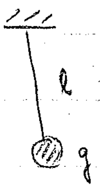
let $\bar{\zeta} = 2\zeta = .836$ $\qquad \frac{4\pi\zeta}{\sqrt{1-4\zeta^2}} = \bar{\delta} = 9.566$ $\qquad \left(\frac{x_0}{x_1}\right)_{new} = e^{\delta} = 14265.4$

2.43

3^{rd} ed 2.75

$$.5\,Hz = \frac{\omega_n}{2\pi} \qquad \omega_n = \pi = \sqrt{g/\ell} \qquad \therefore \ \ell = g/\pi^2 \cong 1m$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad .45 = .5\sqrt{1-\zeta^2} \qquad \zeta = \sqrt{.19} \cong .436$$



$$\overset{I}{(m\ell^2)}\ddot{\theta} + c_t\dot{\theta} + mg\ell\,\theta = 0$$

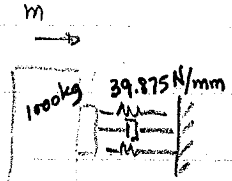
$$\frac{c_{t_c}}{2m\ell^2} = \frac{c_{t_c}}{2I} = \omega_n \qquad c_{t_c} = 2I\omega_n = 2\cdot1\cdot1^2\cdot\pi = 6.28$$

$$\zeta = \frac{c_t}{c_{t_c}} \quad \therefore \quad c_t = c_{t_c}\cdot\zeta \qquad c_t = .43(6.2) \cong 3\ \ 2.7389$$

$$[c_t] \quad \frac{N\text{-}m\text{-}rad}{rad/sec} = \frac{N\text{-}m\text{-}sec}{rad}$$

x(0) = 0
U = 10
m

2.49



$$m\ddot{x} + c\dot{x} + k_{eq}x = 0$$

$$k_{eq} = 2[k/2] \qquad \text{if} \quad x = Ce^{st}$$

$$ms^2 + cs + k_{eq} = 0$$

$$s = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m}$$

$$[20{,}000]^2 - 4[1000][\cancel{40{,}000}\ 79{,}750]$$

$$[1000]^2\{400 - 319\} \cong 81(1000)^2 > 0$$

$$s = \frac{-20{,}000 \pm 9000}{2000} = -\frac{29}{2},\ -\frac{11}{2}$$

$$x = C_1e^{-14.5t} + C_2e^{-5.5t}\ ; \qquad x(0)=0 \quad C_1 + C_2 = 0$$

$$\dot{x}(0) = 10 \quad -14.5C_1 - 5.5C_2 = 10 \qquad -9C_1 = 10 \qquad C_1 = -\frac{10}{9} \qquad C_2 = \frac{10}{9}$$

max disp: $\dot{x} = 0$

$$+\frac{145}{9}e^{-14.5t} - \frac{55}{9}e^{-5.5t} = 0 \quad \text{or} \quad \left[145e^{-9t} - 55\right]\frac{e^{-5.5t}}{9} = 0 \quad \Rightarrow \ln 55/145 = -9t \qquad t = .1077\ sec$$

$$x = C_1e^{-14.5t^*} + C_2e^{-5.5t^*} = .3814\ m$$

HW - DO 3-9, 3-10, 3-17, 3-19

EXAMPLE 3-2

$W = 13.5\ lb \qquad \Delta = .45\ in \qquad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\Delta}} = \frac{c_c}{2m}$

$$c_c = 2m\sqrt{\frac{g}{\Delta}} = 2W\sqrt{\frac{1}{g\Delta}} = 27\ lb\sqrt{\frac{1}{32.2\frac{ft}{sec^2}\cdot\frac{12 in}{ft}\cdot 0.450\ in}}$$

$$= 27\ lb \times .0758 = 2.048\ \frac{lb\text{-}sec}{in}$$

— — — — — —

EXAMPLE 3-7

$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 8\ Hz \qquad m = .20\ lb\ sec^2/in \qquad c = 2.4\ lb\text{-}sec/in$

find $\omega_d = \sqrt{1-\zeta^2}\,\omega$, find $\tau_d = \frac{2\pi}{\omega_d}$

$\omega = 2\pi(8) = 50.2655\ rad/sec$

$c_c = \frac{c_c}{2m} \cdot 2m = \omega \cdot 2m = 50.2655\,(.4) = 20.106\ \frac{lb\,sec}{in}$

$\zeta = \frac{c}{c_c} = .1194 \qquad \sqrt{1-\zeta^2} = .9928 \qquad \omega_d = 49.9059\ \frac{rad}{sec}$

UNDERDAMPED

$f_d = \frac{\omega_d}{2\pi} = 7.943\ Hz \qquad \tau_d = .1259\ sec$

$$x(t) = C_1 e^{-\zeta\omega t}\sin(\omega_d t + \phi)$$
$$= C_1 e^{-6.002t}\sin(49.9059t + \phi)$$

— — — — — —

EXAMPLE 3-12

$x = a\sin\theta$
$\dot{x} = a\cos\theta\,\dot{\theta}$

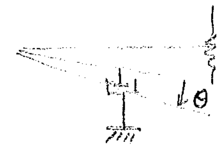
Torque due to spring is $(kl\sin\theta)(\cos\theta)l$

Torque due to damper is $(ca\cos\theta\,\dot{\theta})(\cos\theta)a$

Torque due to weight is $(W\cos\theta)\,l/2$

$I = \int_0^l r^2\,dm = \int_0^l r^2\rho\,dr = r^3/3\,\rho\Big|_0^l = \frac{l^3\rho}{3} = m\frac{l^2}{3} \qquad \rho l = m$

$I_0 \ddot{\theta} = \sum T_0$; for small θ



$$\frac{ml^2}{3}\ddot{\theta} = -kl^2\theta - ca^2\dot{\theta} + \frac{Wl}{2}$$

$$\frac{ml^2}{3}\ddot{\theta} + ca^2\dot{\theta} + kl^2\theta = \frac{Wl}{2}$$ $\frac{W}{2kl} = \theta_{ST}$ Non homog 2^{nd} order LDE

for homog solution $\theta = Ce^{st}$ $\quad \frac{ml^2}{3}s^2 + ca^2 s + kl^2 = 0$ CHAR EQ

$$s = \frac{-ca^2 \pm \sqrt{c^2a^4 - 4kl^2 ml^2/3}}{\frac{2ml^2}{3}}$$

FOR CRITICAL DAMPING $\quad c_c^2 a^4 - 4kl^4 m/3 = 0 \qquad c_c^2 = \frac{4kl^4 m}{3a^4}\cdot\frac{3}{3}$

$$c_c = \frac{2}{3}\frac{l^2}{a^2}\sqrt{3km}$$

Now rewrite ODE to $\quad \ddot{\theta} + \frac{3c}{m}\left(\frac{a}{l}\right)^2\dot{\theta} + \frac{3k}{m}\theta = \frac{3W}{2lm}$

Now $\frac{1}{\frac{2ml^2}{3}}\sqrt{c^2a^4 - \frac{4}{3}kl^4 m} = \sqrt{\frac{9c^2a^4}{4m^2l^4} - \frac{3km\cancel{l^4}}{m^2\cancel{l^4}}} \qquad c_c^2 = \frac{4l^4m^2}{9a^4}\frac{3k}{m}$

$$= \sqrt{\frac{c^2}{c_c^2}\frac{3k}{m} - \frac{3k}{m}} = \sqrt{(\zeta^2-1)3\omega^2}$$

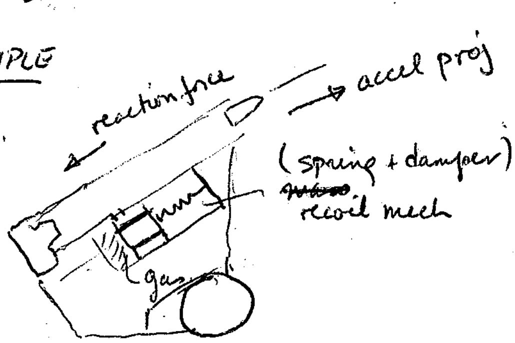
if $\zeta < 1 \qquad \sqrt{3}\sqrt{1-\zeta^2}\,\omega = \sqrt{3}\,\omega_d = \omega_d'$

$$\tau = \frac{1}{f_d'} = \frac{2\pi}{\omega_d'} = \frac{2\pi}{\sqrt{3}\,\omega_d}$$

$$s = \sqrt{3}\left[-\zeta\omega \pm i\omega\sqrt{1-\zeta^2}\right]$$

DID NOT DO — — — — —

EXAMPLE



- want to bring barrel to rest w/o oscill in shortest time
- spring-damper syst is critically

mass = 500 kg $\qquad$ spring stiff = 10000 N/m

gun recoils .4m — $\qquad$ find c_c, $\dot{x}(0)$, t to return to .1m

A gun weighing 500 kg has a spring-dashpot mechanism made up of $k = 10000\,N/m$. The max displ measured when gun fired is 0.4m. Determine time to the amount of time until gun returns to 0.1 m.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{500}} = 4.4721 \text{ rad/sec.}$$

$$c_c = 2m\omega = 2(500)(4.4721) = 4472.1 \text{ N-s/m}$$

FOR CRIT DAMPED $x = (A + Bt)e^{-\omega t}$ $\quad \dot{x} = [B - \omega(A+Bt)]e^{-\omega t}$

max is reached at $t_{max} = \frac{1}{\omega} - \frac{A}{B}$ $\quad$ when $x(0) = 0 \Rightarrow A = 0$

$t_{max} = \frac{1}{\omega}$ $\quad \underline{\dot{x}(t=0) = B}$

recoil velocity

$$x_{max} = Bte^{-\omega t_{max}} = Bt_{max}e^{-1} = \frac{B}{\omega_n}e^{-1} \therefore \quad B = .4\omega e = .4(4.4721)[2.71828]$$

$$= 4.8626 \text{ m/sec}$$

$$x = .1 = Bt_2 e^{-\omega t_2} = (4.8626)\,t_2\, e^{-(4.4721)t_2}$$

solve by iteration.

$t_2 = .8258$ sec.

— — — — — — DID NOT DO

Vierck Problem 3-26 / 3-27

$k = 45$ lb/in, $W = 19.3$ lb and $x_0 = 3$in and $c = 0.057$ lb-sec/in find x_n when $n = 12$

find t when $x_n = \frac{1}{2}x_0$ n

① Given k, $W = mg$ $\Rightarrow \omega$ ② $\omega, m \Rightarrow c_c = 2m\omega$

③ $c_c, c \Rightarrow \zeta = c/c_c$

④ $\zeta, x_0, n \Rightarrow$ $\quad n = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta} \ln\left(\frac{x_0}{x_n}\right)$; $x_0 \exp -\left\{\frac{2\pi\zeta n}{\sqrt{1-\zeta^2}}\right\} = x_n$

HALF LIFE $\quad x_{50\%} = \frac{1}{2}x_0 \quad \therefore \frac{x_0}{x_n} = 2 \quad n = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta}\ln 2$

$$n\tau = \Delta t = \overbrace{\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega}}^{\tau} \cdot n = \text{time to half life}$$

(b) From Eq. (2.116):

$$k = m\,\omega_n^2 = \frac{500}{9.81}\,(31.416)^2 = 5.0304\,(10^4)\ \text{N/m}$$

Using N = W = 500 N,

$$\mu = \frac{0.002\,k}{4\,W} = \frac{(0.002)\,(5.0304\,(10^4))}{4\,(500)} = 0.0503$$

2.84 in 3rd
2.93 in 4th

only for vertical motion

$$c_c = 2\,m\,\omega_n = 2\left(\frac{800}{9.81}\right)(24.7614) = 4038.5566\ \text{N–s/m}$$

$$\zeta = \frac{c}{c_c} = \frac{1000.0}{4038.5566} = 0.2476\ \text{(underdamping)}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 24.7614\sqrt{1-0.2476^2} = 23.9905\ \text{rad/sec}$$

Response of the system:

$$x(t) = X\,e^{-\zeta\omega_n t}\sin(\omega_d\,t + \phi)$$

$$\text{where}\quad X = \left\{x_0^2 + \left(\frac{\dot{x}_0 + \zeta\,\omega_n\,x_0}{\omega_d}\right)^2\right\}^{\frac{1}{2}}$$

for vertical motion $x_0 = 5$cm
$v_0 = 0$

$$= \left\{(0.05)^2 + \left(\frac{(0.2476)\,(24.7614)\,(0.05)}{23.9905}\right)^2\right\}^{\frac{1}{2}} = 0.051607\ \text{m}$$

$$\text{and}\quad \phi = \tan^{-1}\left(\frac{x_0\,\omega_d}{\dot{x}_0 + \zeta\,\omega_n\,x_0}\right) = \tan^{-1}\left(\frac{0.05\,(23.9905)}{0.2476\,(24.7614)\,(0.05)}\right) = 75.6645^\circ$$

Thus the displacement of the boy (positive downward) in vertical direction is given by

$$x(t) = 0.051607\,e^{-6.1309\,t}\sin(23.9905\,t + 75.6645^\circ)\ \text{m}$$

(2.91)
2.96 in 4th ed

Let t_m = time at which $x = x_{max}$ and $\dot{x} = 0$ occur. Here $x_0 = 0$ and $\dot{x}_0$ = initial recoil velocity. By setting $\dot{x}(t) = 0$, Eq. (E_2) gives

$$t_m = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)} = \frac{\dot{x}_0}{\omega_n\,\dot{x}_0} = \frac{1}{\omega_n} \qquad (E_3)$$

With Eq. (E_3) for t_m and $x_0 = 0$, (E_1) gives

$$x_{max} = \dot{x}_0\,t_m\,e^{-\omega_n t_m} = \frac{\dot{x}_0\,e^{-1}}{\omega_n}$$

$$\text{i.e.}\quad \dot{x}_0 = \omega_n\,x_{max}\,e = \omega_n\,(0.5)\,(2.7183) \qquad (E_4)$$

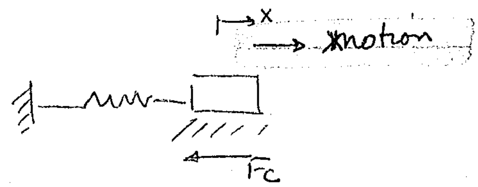
Using $\dot{x}_0 = 10$ m/sec, $\omega_n = 10/(0.5 * 2.7183) = 7.3575\ \frac{\text{rad}}{\text{sec}}$

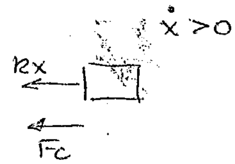
When mass of gun is 500 kg, the stiffness of the spring is

$$k = \omega_n^2 m = (7.3575)^2\,(500) = 27066\ \text{N/m}$$

SESSION #9

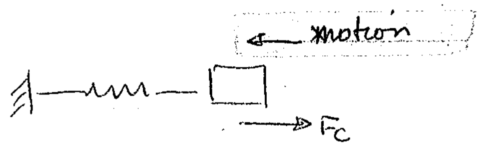
COULOMB FRICTION

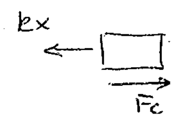




x measured from unstretched length of spring

$\boxed{m\ddot{x} = -kx - F_c}$





$\boxed{m\ddot{x} = -kx + F_c}$

$F_c = \mu N$

$\boxed{\text{ASSUME MOTION IS SMALL} \Rightarrow \text{COULOMB FRICTION } F_c = \text{CONST.}}$

Represent $\boxed{m\ddot{x} = -kx - F_c(\operatorname{sgn}\dot{x})}$ $\qquad \operatorname{sgn}\dot{x} \begin{cases} +1 & \text{if } \dot{x} > 0 \\ -1 & \text{if } \dot{x} < 0 \end{cases}$

Assume motion to the right with $x = x_0$ at $t=0$ $\qquad m\ddot{x} + kx = -F_c$

$\dot{x} = \dot{x}_0$ at $t = 0$ $\qquad \dot{x}_0 > 0$

$\Rightarrow \dot{x}_0 = B\omega \qquad B = \dot{x}_0/\omega$

$\Rightarrow x_0 = A - \frac{F_c}{k} \qquad A = x_0 + \frac{F_c}{k}$

$\Big\}\ \boxed{x(t) = \left(x_0 + \frac{F_c}{k}\right)\cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t - \frac{F_c}{k}}$

MAX MOTION IF $\boxed{\dot{x}(t_{max}) = 0} \Rightarrow -\omega\left(x_0 + \frac{F_c}{k}\right)\sin\omega t_{max} + \dot{x}_0\cos\omega t_{max} = 0$

$\tan\omega t_{max} = \dfrac{\dot{x}_0/\omega}{x_0 + F_c/k} \Rightarrow \boxed{t_{max} = \frac{1}{\omega}\tan^{-1}\left[\frac{\dot{x}_0/\omega}{x_0 + F_c/k}\right]} = \frac{\pi}{\omega}$ if $\dot{x}_0 =$

$\therefore \boxed{x_{max} = x(t_{max}) = X_0} = \left(x_0 + \frac{F_c}{k}\right)\cos\omega t_{max} + \frac{\dot{x}_0}{\omega}\sin\omega t_{max} - \frac{F_c}{k}$

motion to left

$\boxed{x(t) = A\cos\omega t + B\sin\omega t + \frac{F_c}{k}}$

$x(0) = X_0 \qquad \dot{x}(0) = 0 \qquad$ and $\qquad t$ is measured from t_{max}

$x(0) = X_0 = A + \frac{F_c}{k} \qquad A = X_0 - \frac{F_c}{k}$

$\dot{x}(0) = 0 \Rightarrow B = 0$

$\therefore \boxed{x(t) = \left(X_0 - \frac{F_c}{k}\right)\cos\omega t + \frac{F_c}{k}}$

$\boxed{\text{motion is til } \dot{x}(t^*) = 0} \qquad \dot{x}(t^*) = 0 \Rightarrow A\sin\omega t^* = 0 \qquad \boxed{\omega_n t^* = \pi \qquad t^* = \frac{\pi}{\omega_n}}$

$\boxed{t_{TOT} = t_{max} + \frac{\pi}{\omega} = t_{max} + t^*}$

3rd ed. 2.98

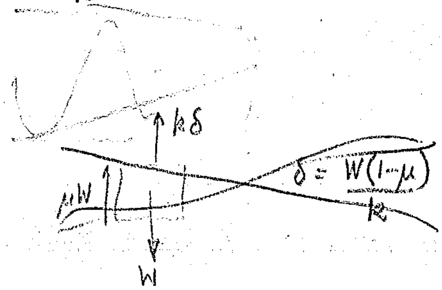
4th ed 2.111

2.55 An SDOF system w/ mass 20 kg & spring of stiffness 4000 N/m. Successive cycles have amplitudes of 50, 45, 40, ... mm. Find type & magnitude of damping force & freq of vibs.

Coulomb $\frac{4F}{k} = .005\,m$ $F = \frac{4000(.005)}{4} = 5N$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{\cancel{4000}}{\cancel{20}}} \cong 14.1$ rad/s

$\Delta_{stop} = \frac{F}{k} = .00125m$ $n = \frac{.050 - .00125}{.005} = \frac{50 - 1.25}{5} = \frac{48.75}{5} = 9.75$ cy



$t = \frac{\pi}{\omega} \Rightarrow \cos\omega t = -1$

$$\therefore \; \boxed{x_{max}} = -\left(X_0 - \frac{F_c}{k}\right) + \frac{F_c}{k} = \boxed{-X_0 + \frac{2F_c}{k}}$$

- motion to right t measured from $t_{max} + \frac{\pi}{\omega} = t_{max} + t^*$

$$x(t) = A\cos\omega t + B\sin\omega t - \frac{F_c}{k}$$

$$x(0) = -X_0 + \frac{2F_c}{k} \qquad \dot{x}(0) = 0$$

$$\dot{x}(0) = 0 \Rightarrow B = 0 \qquad x(0) = -X_0 + \frac{2F_c}{k} = A - \frac{F_c}{k} \Rightarrow A = -X_0 + \frac{3F_c}{k}.$$

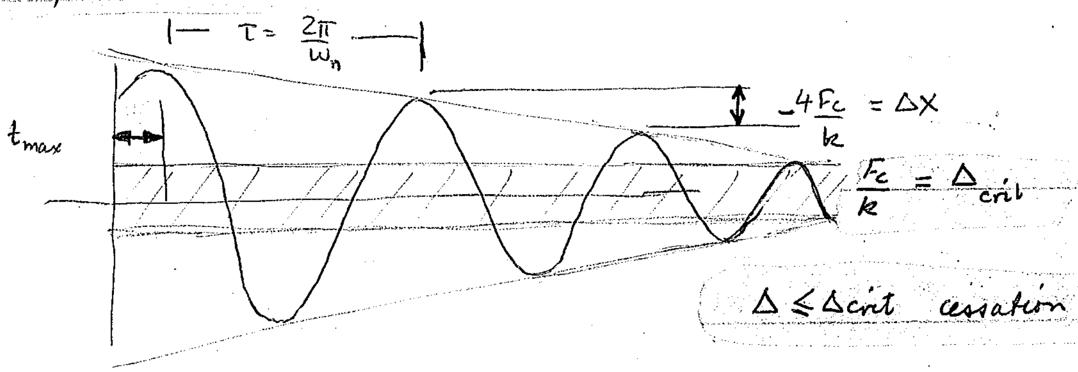
$$x(t) = -\left(X_0 - \frac{3F_c}{k}\right)\cos\omega t - \frac{F_c}{k}$$

cessation of motion is at $\dot{x}(t^{**}_{max}) = 0 \Rightarrow \sin\omega t^{**} = 0 \quad t = \frac{\pi}{\omega} \quad t_{TOT} = t_{max} + \frac{2\pi}{\omega}$

$$x_{max} = x(t_{max}) = \left(X_0 - \frac{3F_c}{k}\right) - \frac{F_c}{k} = X_0 - \frac{4F_c}{k}.$$

period of motion $= \frac{2\pi}{\omega_n}$ ampl. decreases by $\frac{4F_c}{k}$.

- NOTE → ω_n is same as the undamped case
- DECAY IS LINEAR



CESSATION OF motion is when spring force ~~balances~~ is less than the friction force

IF GIVEN X_0 = ampl at frst peak X_n = amp after n cycles

$$\frac{X_0 - X_n}{n} = \Delta X \text{ decay/cycle} = \frac{4F_c}{k}$$

- if given $F_c \Rightarrow k = \frac{4F_c}{\Delta X}$; if given $m \Rightarrow \omega = \sqrt{\frac{k}{m}} \Rightarrow \tau = \frac{2\pi}{\omega}$

  $n\tau$ gives time to go from X_0 to X_n

#4-1

$W = 14.475\ lb \qquad k = 15\ lb/in \qquad P_0 = 24\ lb \qquad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\Delta}} = 20.01\ rad/sec$

find $\delta_{ST} = \frac{W}{k} = \frac{14.475}{15} = .965\ in \qquad X_0 = \frac{P_0}{k} = \frac{24}{15} = 1.6\ in$

given $f_f = 60, 120, 130, 300\ Hz$ find $X = \frac{X_0}{r^2-1} \qquad r = \frac{\omega_f}{\omega}$

$2\pi f_f = \omega_f$

$r = 18.84 \qquad 37.68 \qquad 40.82 \qquad 94.20$

$X = \frac{X_0}{r^2-1} = .0045\ in \qquad .0011\ in \qquad .00096\ in \qquad .00018\ in$

4-4

Given $W = 8\ lb \quad k = 25\ lb/in \quad \omega_f = 2\pi(7\ Hz) = 43.982\ rad/sec \quad P = P_0 \sin\omega_f t$

$x_p = 1.59\ in = X \sin\omega_f t = \frac{X_0}{1-r^2} \sin\omega_f t$ when $\omega_f t = \pi/2$

$r = \frac{\omega_f}{\omega} \qquad \omega = \sqrt{\frac{kg}{W}} = \sqrt{\frac{25 \cdot 32.2 \cdot 12}{8}} = 34.749\ rad/sec$

$r = \frac{43.982}{34.749} = 1.2657 \qquad \therefore \quad x_{p_{max}} = \frac{X_0}{r^2-1} = \frac{X_0}{.602}$

$X_0 = .957\ in = \frac{P_0}{k} \Rightarrow P_0 = 23.93\ lb$

4-15

$\frac{\omega}{2\pi} = 1765\ \frac{cy}{min} \qquad \frac{\omega_f}{2\pi} = 1752\ \frac{cy}{min} \qquad \frac{\omega - \omega_f}{2\pi} = 13\ cy/min = \frac{13\ cy/min}{60\ sec/min} = .\frac{13}{60}\ \frac{cy}{sec}$

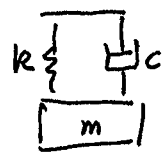
$\tau = \frac{2\pi}{\omega - \omega_f} = \frac{60}{13}\ \frac{sec}{cy} = 4.6154\ sec$

EML 4220 — 9/28/01 TAPED FOR 10/2/01

REVIEW

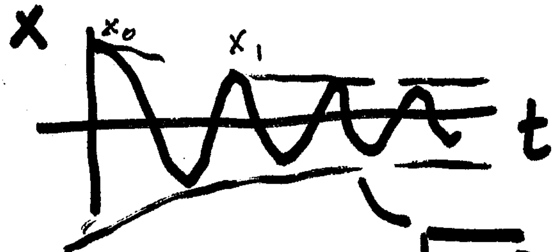
constant damping
independent of {velocity, displacement}

VISCOUS



$$m\ddot{x} + \underline{\underline{c\dot{x}}} + kx = 0$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad \zeta<1$$



$$\frac{\cdot}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}$$

δ

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \ln\left(\frac{x_0}{x_1}\right)$$

$\zeta < .3 \qquad \delta \cong 2\pi\zeta$

LOGARITHMIC DECREMENT

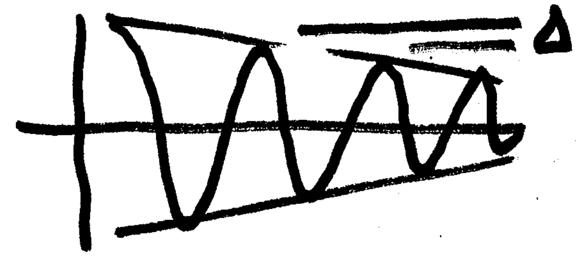
$$\frac{1}{n}\ln\left(x_0/x_n\right) = \delta$$

$t = \infty$ FOR $x = 0$

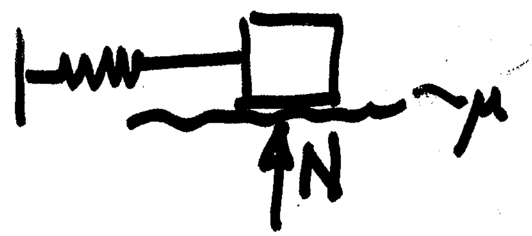
COULOMB

$$m\ddot{x} + kx = \underline{\underline{\pm F}}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$



LINEAR $\Delta = \dfrac{4F}{k} \qquad F = \mu N$



LINEAR DECREMENT

t IS FINITE FOR VIB TO STOP

$m\ddot{x} = -kx - \mu N$

$m\ddot{x} + \mu N = m\ddot{x}$

Solution:

x - positive, $\frac{dx}{dt}$ - positive, or x - negative, dx/dt positive: $x(t) = A_1\cos\omega_n t + A_2\sin\omega_n t - \frac{\mu N}{k}$

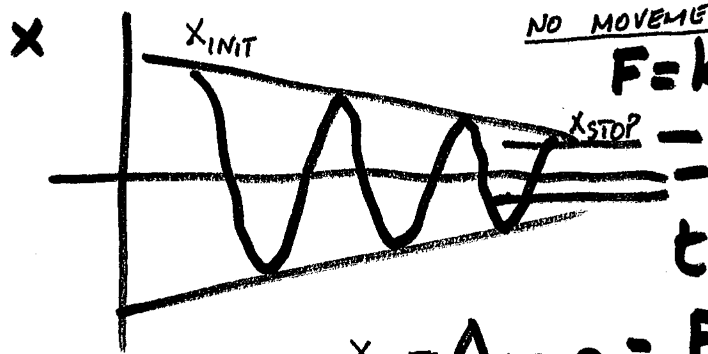
$\omega_n = \sqrt{\frac{k}{m}}$

x - positive, dx/dt ~~positive~~ negative, x - negative, dx/dt negative: $x(t) = A_3\cos\omega_n t + A_4\sin\omega_n t + \frac{\mu N}{k}$

VISCOUS

COULOMB - WHEN FRICTION FORCE IS LARGER THAN SPRING FORCE - NO MOVEMENT

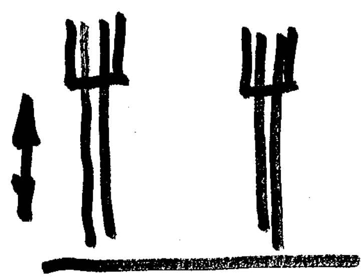
$F = k\Delta$



$X_{STOP} = \Delta_{STOP} = \frac{F}{k}$

$$n \cancel{\delta} = \frac{1}{\delta} \ln\left(\frac{x_0}{x_n}\right)$$

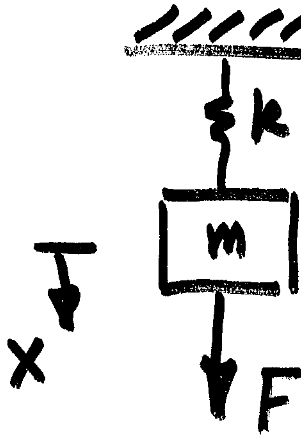
$$\frac{X_{INIT} - X_{STOP}}{4F/k} = n_{cycles}$$



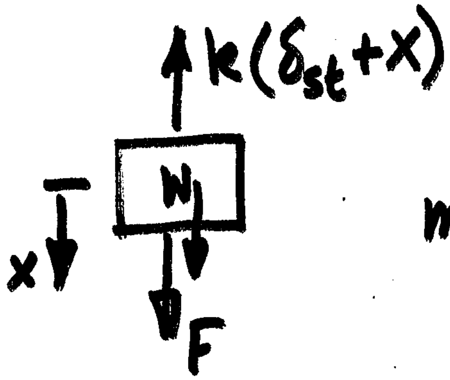
REMINDER: HW DUE TUES. 9~~th~~ OCT

2.83, 2.86, 2.118, 2.100, 2.101

11 Oct



$$F = F(t) = P_0 \sin\omega_f t \qquad \omega_f \neq \omega_n$$



$$m\ddot{x} = +F + \cancel{W} - k(\cancel{\delta_{st}} + x) = F - kx$$

$$m\ddot{x} + kx = P_0 \sin\omega_f t$$

- WANT DYNAMIC RESPONSE DUE TO F

$$x = x_h + x_p$$

x_h: $\quad m\ddot{x} + kx = 0 \Rightarrow x_h = A\cos\omega_n t + B\sin\omega_n t$

x_p: $\quad m\ddot{x} + kx = P_0 \sin\omega_f t \Rightarrow x_p = C\sin\omega_f t$

$$(-m\omega_f^2 + k)\, C\sin\omega_f t = P_0 \sin\omega_f t$$

$$C = \frac{P_0}{k - m\omega_f^2}$$

AMPLITUDE OF VIB DUE TO FORCING FN.

$$C = \frac{P_o/k}{1 - \frac{m}{k}\omega_f^2} = \frac{X_o}{1 - \frac{\omega_f^2}{\omega_n^2}} = \frac{X_o}{1 - r^2}$$

$$r = \frac{\omega_f}{\omega_n}$$ FREQUENCY RATIO

RAO USES $\delta_{st} \Longleftrightarrow X_o$

$$x_p = C \sin \omega_f t = \frac{X_o}{1 - r^2} \sin \omega_f t$$

IF $r < 1$ $F = P_o \sin \omega_f t$ x_p & F IN PHASE

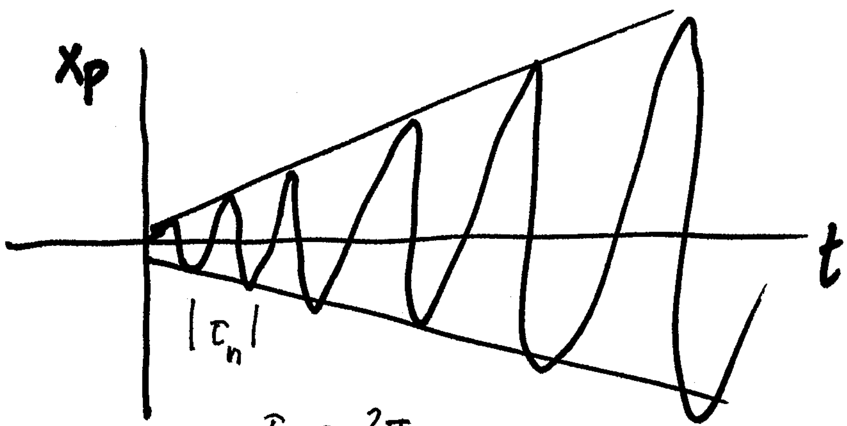
$r > 1$ x_p & F OUT OF PHASE

$r = 1$ (RESONANCE CASE $\omega_f = \omega_n$)

$$m\ddot{x} + kx = P_o \sin \omega_n t$$

$$x_p = \frac{P_o t}{2m\omega_f}\left[-\cos \omega_f t\right]$$

$\sin(\omega_f t - \frac{\pi}{2})$



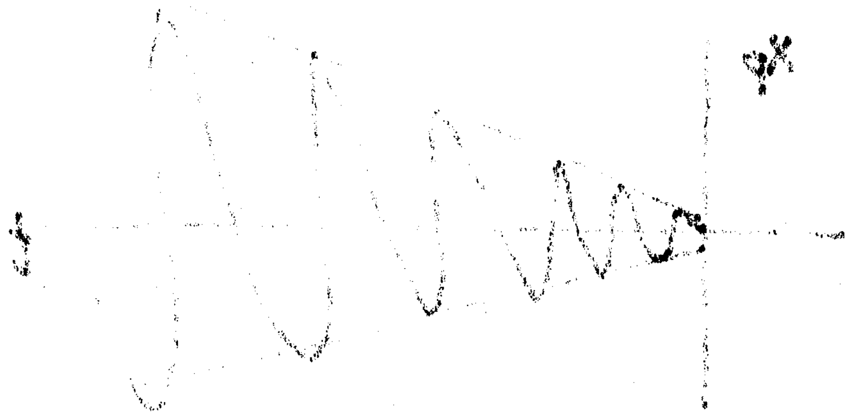
$$\tau_n = \frac{2\pi}{\omega_n}$$

$$x_{TOT} = x_h + x_p$$

$$= A\cos\omega_n t + B\sin\omega_n t + \frac{X_0}{1-r^2}\sin\omega_f t \qquad \omega_f \neq \omega_n$$

$$= \underbrace{A\cos\omega_n t + B\sin\omega_n t}_{\text{FREE VIBS}} + \underbrace{\frac{P_0 t}{2m\omega_f}\sin\left(\omega_n t - \frac{\pi}{2}\right)}_{\text{DUE TO FORCING FUNC.}} \qquad \omega_f = \omega_n$$

$$\frac{P_0 t}{2m\omega_f} = \frac{P_0/k\ t}{2\boxed{m/k\ \omega_f}} = \frac{X_0\omega_f t}{2}$$



$$x_{TOT} = x_h + x_p$$

$$= A\cos\omega_n t + B\sin\omega_n t + \frac{\bar{X}}{1-r^2}\sin\omega_f t \qquad \omega_f \neq \omega_n$$

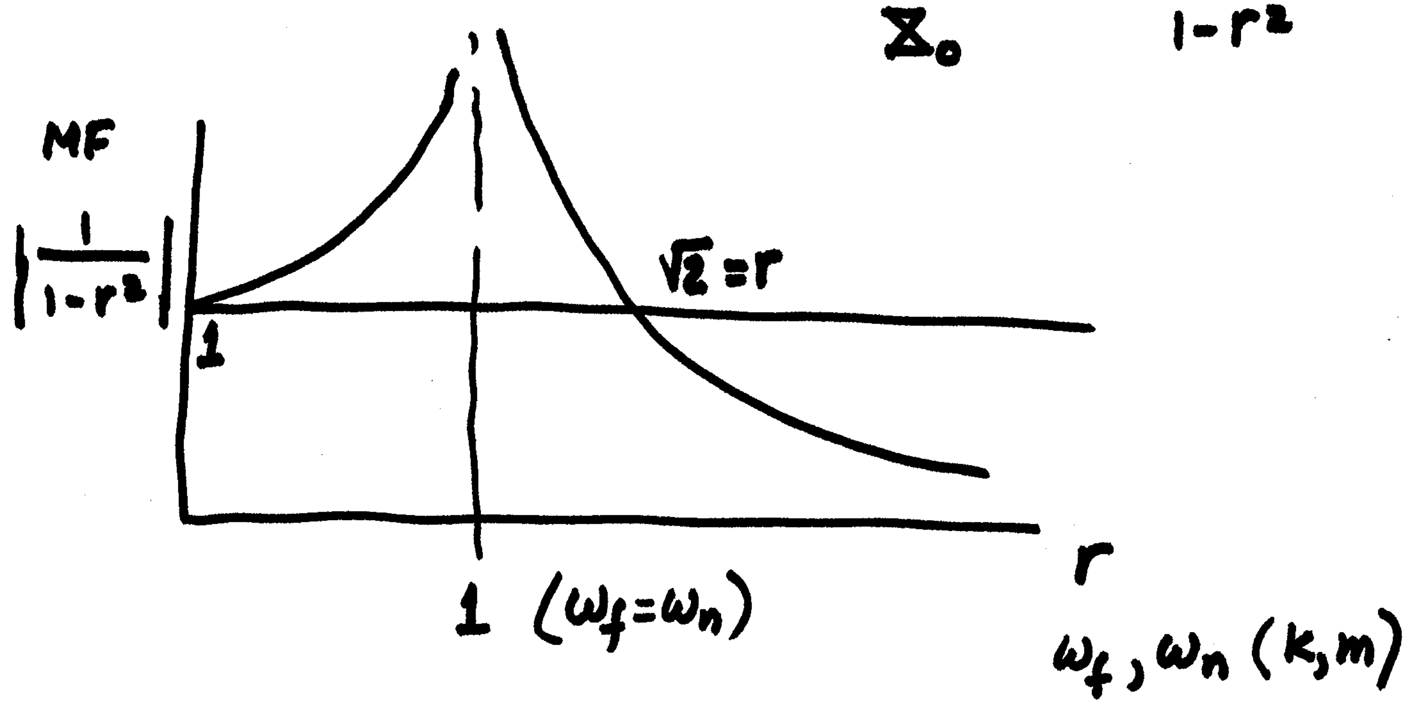
$$= \underbrace{A\cos\omega_n t + B\sin\omega_n t}_{\text{FREE VIBS}} + \underbrace{\frac{F_0 t}{2m\omega_f}\sin\left(\omega_n t - \frac{\pi}{2}\right)}_{\text{DUE TO FORCING FUNC.}} \qquad \omega_f = \omega_n$$

$$\frac{\bar{X}\omega_n t}{2} = \frac{F_0/k \;\; t}{2\boxed{\sqrt{m/k}}} = \frac{F_0 t}{2m\omega_f}$$

$$\frac{X_0}{1-r^2} \Rightarrow$$ AMP. OF DISPL. DUE TO FORCING FN. $\omega_n \neq \omega_f$

$r = \frac{\omega_f}{\omega_n}$ $\quad X_0 = P_0/K$

MF = MAGNIFICATION FACTOR $= \dfrac{\frac{X_0}{1-r^2}}{X_0} = \dfrac{1}{1-r^2}$



THIS PICTURE SHOULD BE USED FOR DESIGNING IN ONLY ONE WAY

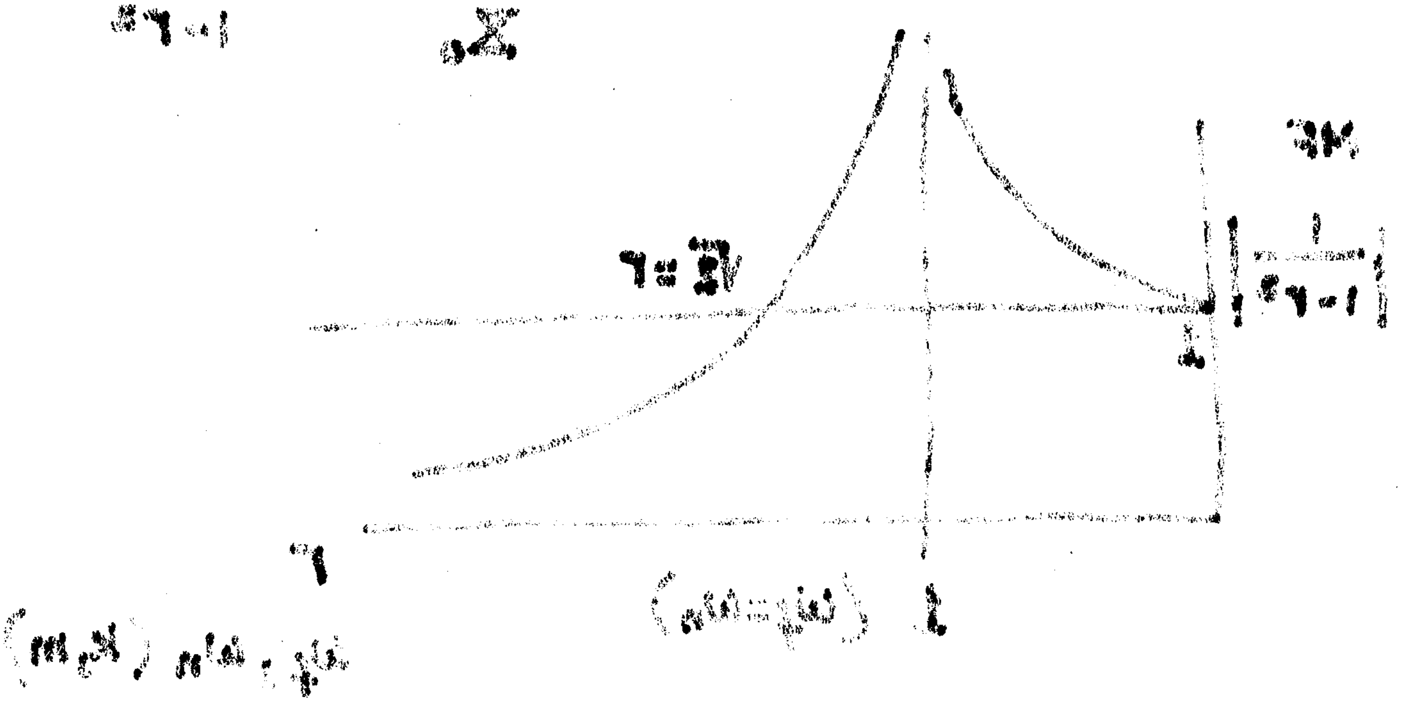
MF VS CHANGES IN ω_f

$$x_{TOT} = x_h + x_p = \underline{A}\cos\omega_n t + \underline{B}\sin\omega_n t + \frac{X_0}{1-r^2}\sin\omega_f t$$

AMP. OF DEFL. DUE TO FORCING FN. $\Leftarrow \frac{X_0}{1-r^2}$

$r = \frac{\omega_f}{\omega_n}$ $\quad X_0 = F_0/k$

MF = MAGNIFICATION FACTOR $= \frac{\frac{X_0}{1-r^2}}{X_0} = \frac{1}{1-r^2}$



THIS PICTURE SHOULD BE USED
FOR DESIGNING IN ONLY ONE WAY

MF VS CHANGES IN ω_f

$x_{tot} = x_h + x_p = A\cos\omega_n t + B\sin\omega_n t + \frac{X_0}{1-r^2}\sin\omega_f t$

TO SOLVE FOR A & B ; GET FULL SOL'N THEN APPLY I.C.s

$$\Rightarrow \quad x(t=0) = x_0$$
$$\dot{x}(t=0) = V_0$$

$$x_0 = A \cdot 1 + B \cdot 0 + \frac{\mathcal{X}_0}{1-r^2} \cdot 0 \Rightarrow A = x_0$$

$$\dot{x} = -A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t + \frac{\mathcal{X}_0}{1-r^2}\,\omega_f \cos\omega_f t$$

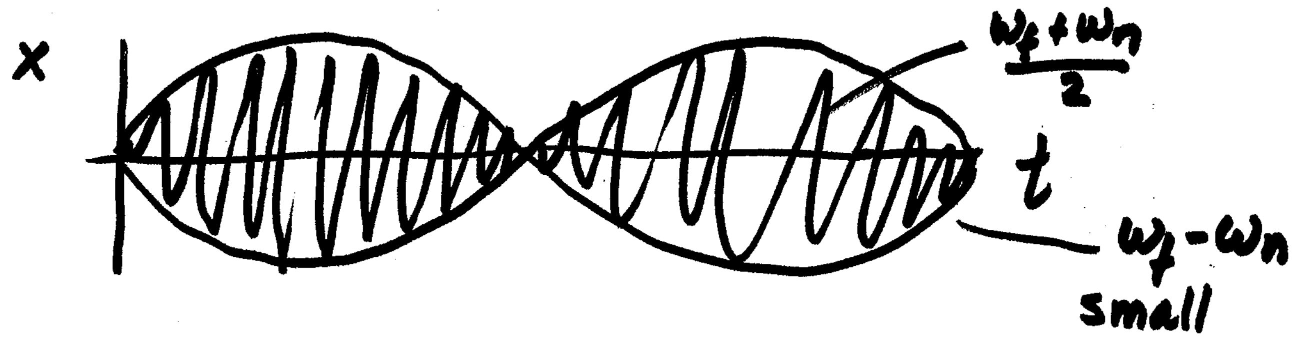
$$\dot{x}(t=0) = B\omega_n + \frac{\mathcal{X}_0}{1-r^2}\,\omega_f = V_0$$

$$B = \left[\frac{V_0}{\omega_n} - \frac{\mathcal{X}_0}{1-r^2} \cdot r\right]$$

$$x = x_0 \cos\omega_n t + \frac{V_0}{\omega_n} \sin\omega_n t + \frac{\mathcal{X}_0}{1-r^2}\left[\sin\omega_f t - r\sin\omega_n t\right]$$

SPECIAL CASE $\quad x(t=0)=0 \quad \dot{x}(t=0)=0$

$$x = \overbrace{\frac{2\mathcal{X}_0}{1-r^2} \sin\left(\frac{\omega_f-\omega_n}{2}\right)t}^{\text{time varying amplitude}} \cdot \cos\left(\frac{\omega_f+\omega_n}{2}\right)t$$



TO SOLVE FOR A & B ; GET FULL SOL'N THEN APPLY I.C.s

$\Rightarrow x(t=0) = x_0$

$\dot{x}(t=0) = V_0$

$$x_0 = A \cdot 1 + B \cdot 0 + \frac{Z_0}{1-r^2} \cdot 0 \Rightarrow A = x_0$$

$$\dot{x} = -A\omega_n \sin\omega_n t + B\omega_n \cos\omega_n t + \frac{Z_0}{1-r^2}\omega_f \cos\omega_f t$$

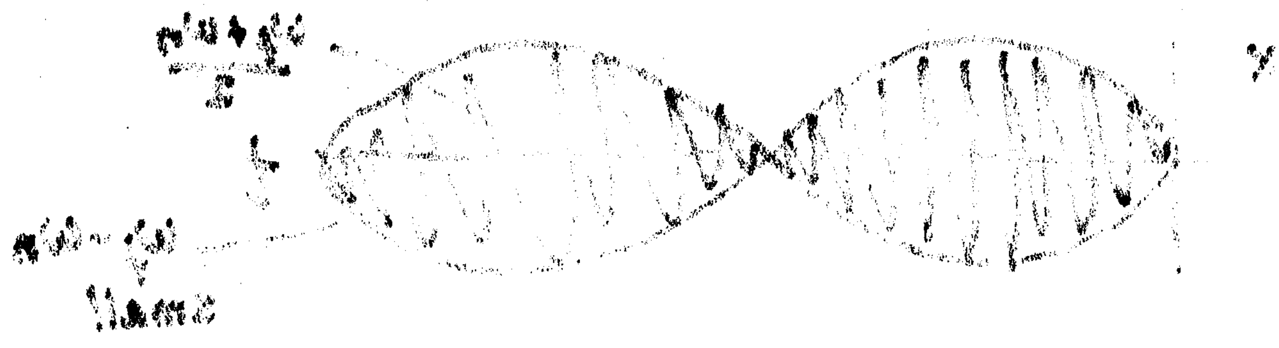
$$\dot{x}(t=0) = B\omega_n + \frac{Z_0}{1-r^2}\omega_f = V_0$$

$$B = \left[\frac{V_0}{\omega_n} - r \cdot \frac{Z_0}{1-r^2}\right]$$

$$x = x_0 \cos\omega_n t + \frac{V_0}{\omega_n}\sin\omega_n t + \frac{Z_0}{1-r^2}\left[\sin\omega_f t - r\sin\omega_n t\right]$$

SPECIAL CASE $x(t=0) = 0$ $\dot{x}(t=0) = 0$

$$x = \frac{2Z_0}{1-r^2} \sin\left(\frac{\omega_f - \omega_n}{2}t\right) \cdot \cos\left(\frac{\omega_f + \omega_n}{2}t\right)$$



$$1-r^2 = (1+r)(1-r) = \underbrace{\left(1+\frac{\omega_f}{\omega_n}\right)}_{\sim 2}\underbrace{\left(1-\frac{\omega_f}{\omega_n}\right)}_{2\Delta\frac{\omega_f}{\omega_n^2}} \cong 4\Delta\frac{\omega_f}{\omega_n^2}$$

IF ω_f IS CLOSE TO ω_n

$$\sin\left(\frac{\omega_f-\omega_n}{2}\right)t \sim \left(\frac{\omega_f-\omega_n}{2}\right)t$$

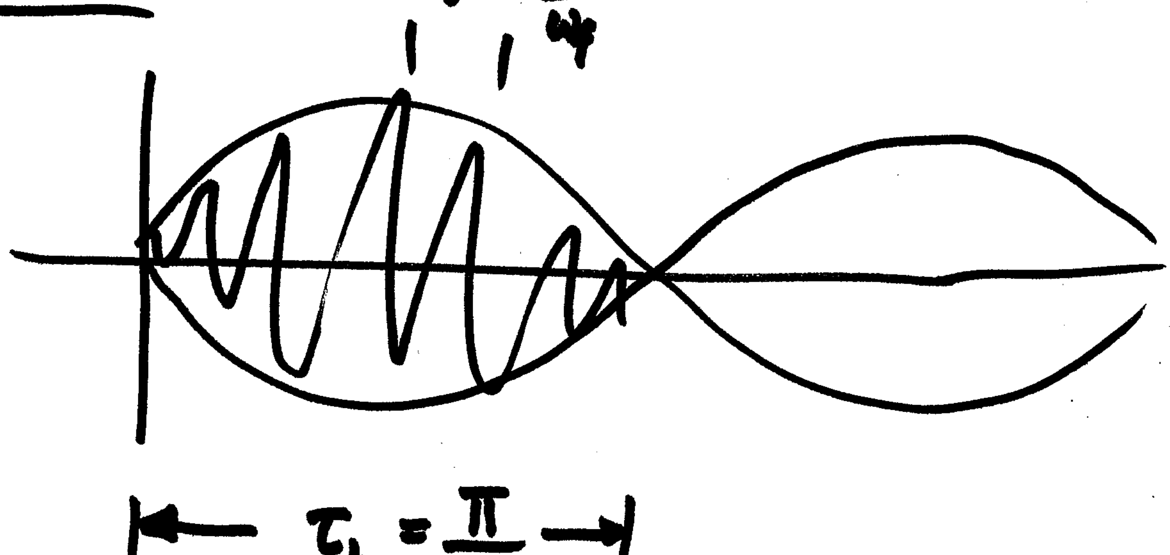
$$\frac{2X_0}{1-r^2}\left(\frac{\omega_f-\omega_n}{2}\right)t \sim -\frac{X_0\,\omega_f t}{2}$$

$$\frac{\omega_n^2-\omega_f^2}{\omega_n^2}\cdot\left(\frac{\omega_f-\omega_n}{2}\right.$$

$$x_p = -\frac{X_0\,\omega_f t}{2}\cos\omega_f t \quad \text{FOR } r=1$$

BEATS

$\tau = \frac{2\pi}{\omega_f}$



$\tau_b = \frac{\pi}{\Delta}$

$$\Delta = \frac{\omega_n-\omega_f}{2}$$

$\Delta m_t \cong \left(\frac{m_t}{m_n}-1\right)\left(\frac{m_t}{m_n}+1\right) = (r-1)(r+1) = r^2-1$

$2\Delta m_t \quad \Delta^2$

If m_t is close to m_n

$\sin\left(\frac{m_t-m_n}{\Delta}\right)\pi \sim \left(\frac{m_t-m_n}{\Delta}\right)\pi$

$\frac{\sin \pi m_t}{\Delta} \sim \left(\frac{m_t-m_n}{\Delta}\right)\pi \quad \frac{2\Delta}{r^2-1}$

For $n=1$ $\quad x_p = -\sum \frac{\sin \pi m_t}{\Delta} \cos \pi m_t$

$\frac{T}{m_t} = T$

BEATS

$\frac{T}{\Delta} = d_p$

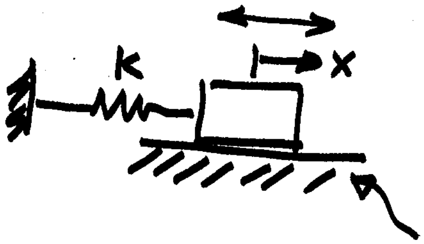
$\frac{m_t-m_n}{\Delta} = \Delta$

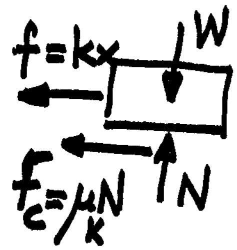
EML 4220

MECHANICAL VIBRATIONS

10/4/05

COULOMB DAMPING   Chpt 2



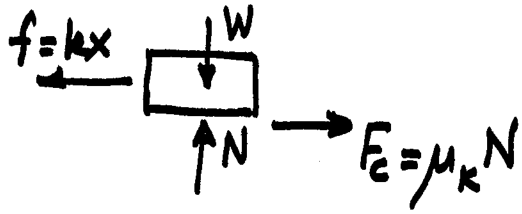


$W=N$

$m\ddot{x}=-kx-F_c$

$m\ddot{x}+kx=-F_c \qquad \dot{x}\neq 0$

motion to left



$\dot{x}\neq 0$

$m\ddot{x}+kx=F_c$

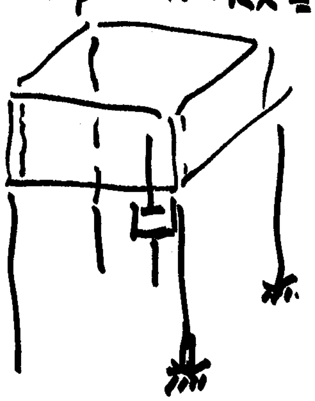
$m\ddot{x}+kx=-F_c$

$x>0$ & motion to right
$x<0$ & motion to the left.

x_h: $m\ddot{x}+kx=0 \Rightarrow x_h(t)=A\cos\omega_n t+B\sin\omega_n t \qquad \omega_n=\sqrt{\frac{k}{m}}$

x_p: $m\ddot{x}+kx=-F_c \Rightarrow x_p=D \Rightarrow kD=-F_c \quad D=-F_c/k$

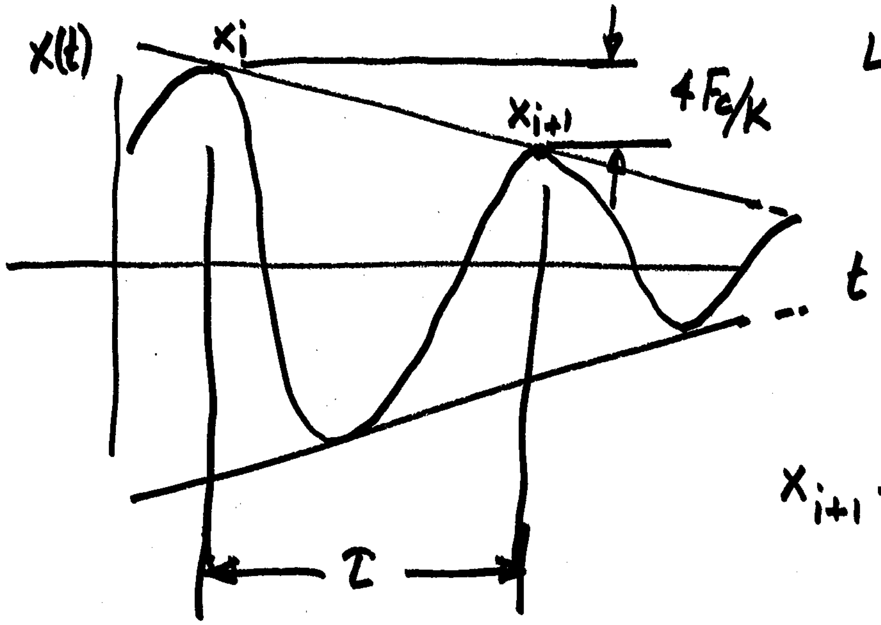
$x_{total}=A\cos\omega_n t+B\sin\omega_n t-F_c/k$



$m\ddot{x}+kx=F_c \Rightarrow x_{total}=\bar{A}\cos\omega_n t+\bar{B}\sin\omega_n t+F_c/k$

$x>0$ & motion to left
$x<0$ & motion to right

as long as t is measured from point when velocity $=0$



LINEAR DECREMEN

$$x_{i+1} - x_i = -4F_c/k$$

$$\frac{x_n - x_0}{n} = -4F_c/k \qquad n = \text{\# of full cycles}$$

Rao $$\frac{x_n - x_0}{n} = -2F_c/k \qquad n = \text{\# of half cycles}$$

Frequency of Vib. for Coulomb Damping is ω_n

ALL WE HAVE WRITTEN APPLIES TO SITUATION WHERE

$$f = kx \geq F_c$$

IF $f < F_c$ THEN SYSTEM STOPS $\Rightarrow$ $x_{STOP} = F_c/k$

$$\frac{x_{STOP} - x_0}{-4F_c/k} = n \quad \text{(to stop)}$$

VISCOUS

ω_d

$t = \infty$ to stop

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{1}{n}\ln\left(\frac{x_0}{x_n}\right)$$

COULOMB

ω_n

t is finite to stop

$$\frac{x_n - x_0}{n} = -\frac{4F_c}{k}$$

Prob 2.111 Pg 213

$m = 20\,kg \quad k = 4000\,N/m$

$x_0 = 50,\ x_1 = 45,\ x_2 = 40 \ldots\ mm$

$F_c = ?\quad$ "ω_d" $= \omega_n$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{20}} = \sqrt{200} \cong 14.1\ rad/s$$

$$x_{i+1} - x_i = 45 - 50 = -5\ mm = -.005\ m = -4F_c/k$$

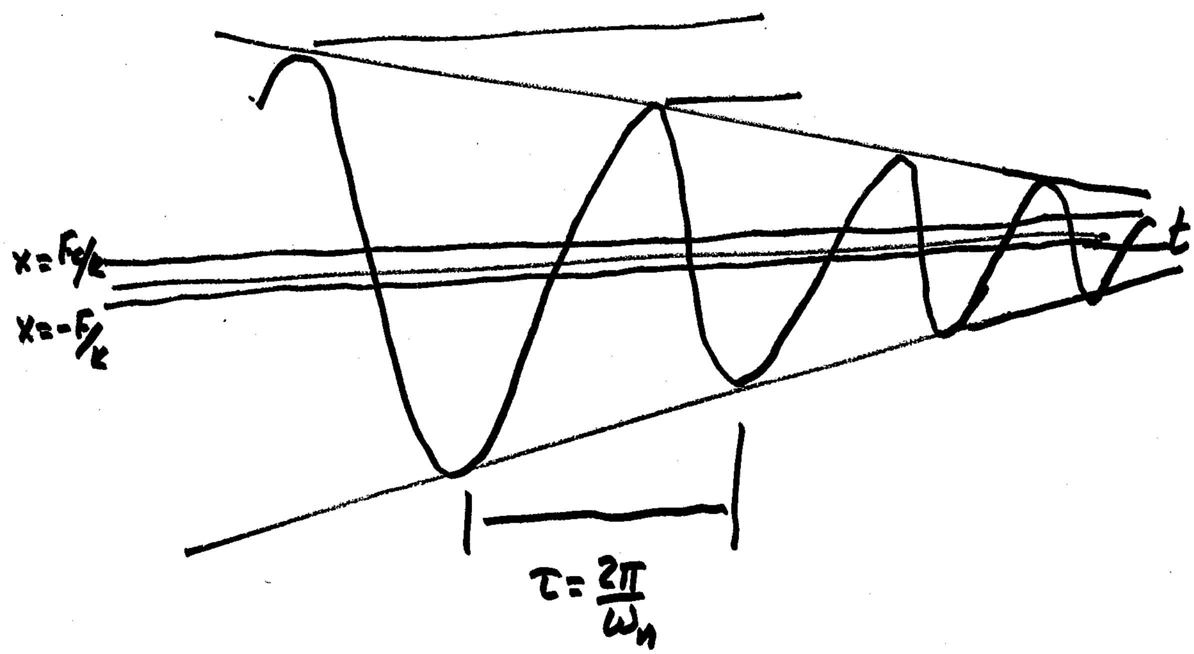
$$F_c = \frac{(-.005)(4000\ N/m)}{-4} = 5N$$

# of cycles to stop

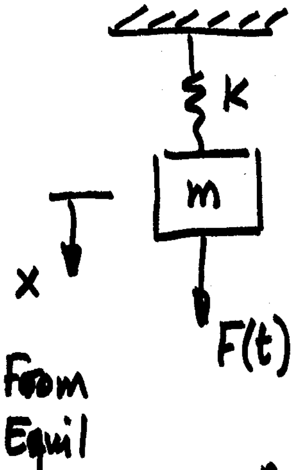
$$\frac{x_{stop} - x_0}{-4F_c/k} = n \qquad x_{stop} = F_c/k = \tfrac{1}{4}(.005) = .00125m$$

$$\frac{.00125 - .05}{-.005} = n \qquad n = 9.75 \text{ full cycles.}$$

$n = 19.5$ half cycles.

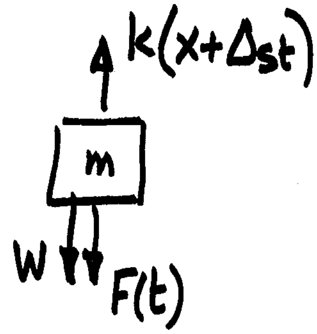


FORCED VIBS OF A SDOF SYSTEM



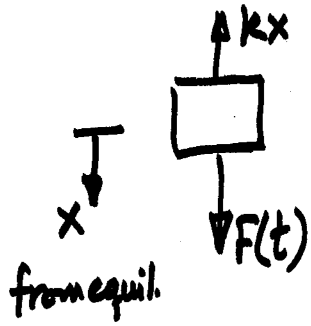
$F(t) = P_0 \sin \omega_f t$

AMPLITUDE OF FORCING FN.



$\Delta_{st} = \dfrac{W}{k}$

FBD



Dynamic FBD

$$m\ddot{x} = \Sigma F = F(t) - kx$$

$$m\ddot{x} + kx = F(t) = P_0 \sin \omega_f t$$

x_h: $m\ddot{x} + kx = 0 \implies x_h(t) = A\cos\omega_n t + B\sin\omega_n t \qquad \omega_n = \sqrt{\dfrac{k}{m}}$

x_p: $m\ddot{x} + kx = P_0 \sin\omega_f t \qquad \omega_f \neq \omega_n \qquad x_p = X \sin\omega_f t$ — DISP. DUE TO FORCING FN.

$$m(-\omega_f^2 X \sin\omega_f t) + k X \sin\omega_f t = P_0 \sin\omega_f t$$

$$X\left[-m\omega_f^2 + k\right] = P_0 \qquad X = \frac{P_0}{k - m\omega_f^2}$$ AMPLITUDE OF FORCED VIB.

$$X_0 = P_0/k \qquad X = \frac{P_0/k}{1 - m\omega_f^2/k} = \frac{X_0}{1 - (\omega_f/\omega_n)^2} = \frac{X_0}{1 - r^2} \qquad r = \omega_f/\omega_n$$

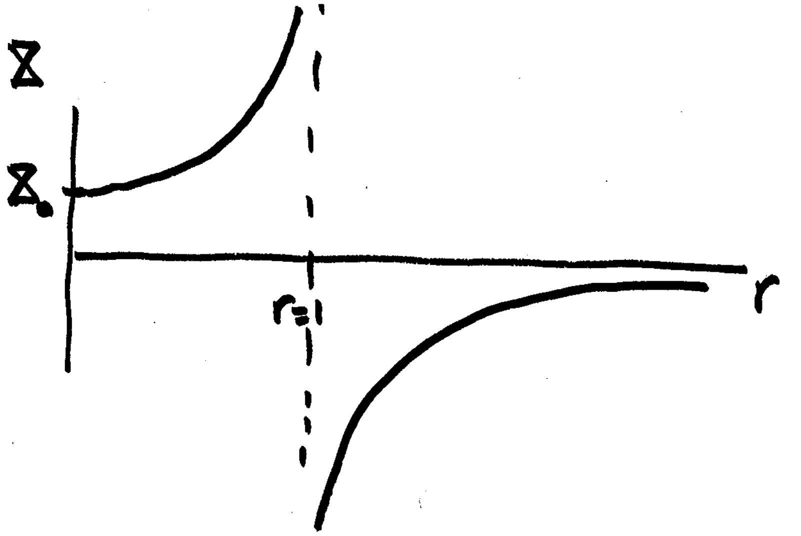
X_0: DISP. OF SPRING DUE TO MAX APPLIED FORCE

r: FREQUENCY RATIO.

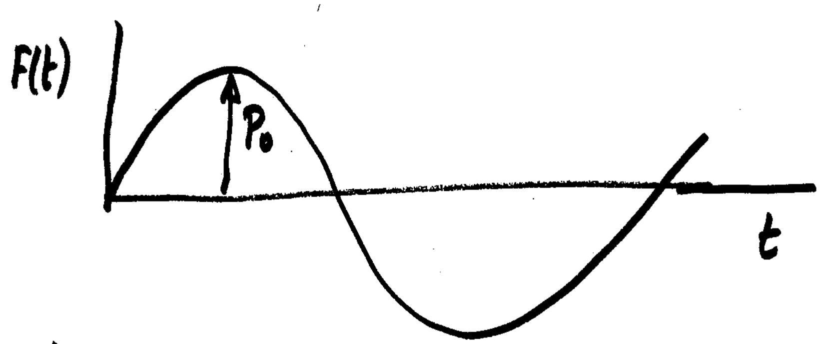
$r = 1$ produces resonance

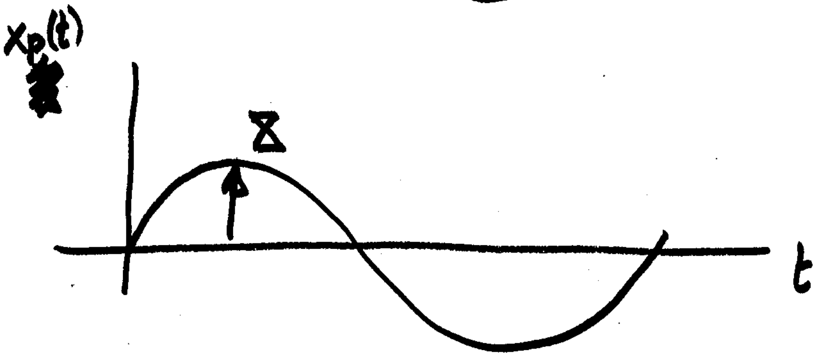
$$x_p = \frac{X_0}{1 - r^2} \sin\omega_f t$$

$$x_{tot} = A\cos\omega_n t + B\sin\omega_n t + \frac{X_0}{1 - r^2}\sin\omega_f t$$

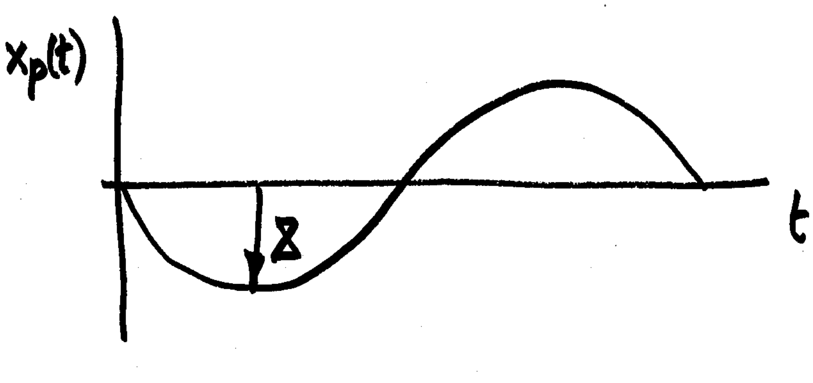


ONLY LOOK AT THIS AS A VARIATION OF ω_f

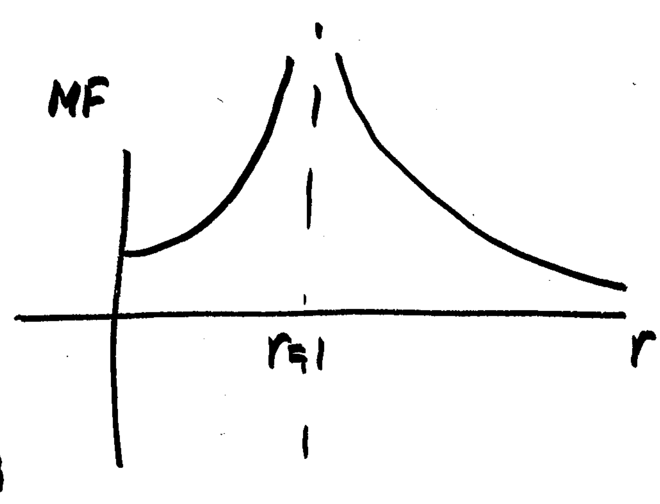




$r<1$



$r>1$



Magnification factor $X/X_0 = \left|\frac{1}{1-r^2}\right|$

3.1, 2, 8, 10, 29, 33, 32, 54, 58 || Reading Chpt 3

HW 3-26, 3-27, 3-30, 3-35

A system is composed of mass & spring subjected to a coulomb force. If initial amplitude is find amplitude after eight cycles & frequency

Problem 3-36 Given FIND X AFTER 8 CYCLES

$m = 57.61\ kg \quad k = 9100\ N/m \quad F_c = 6.825 N \quad X_0 = 4\ cm \quad$ find X_8

$$\Delta X = \frac{4 F_c}{k} = \frac{4(6.825 N)}{9100 N/m} = .003 m = .3 cm$$

at end of 8 cycles decreases by $8(.3 cm) = 2.4 cm \quad \therefore \quad X_8 = X_0 - 8(\Delta X) = 1.6 cm$

- $f = \frac{\omega}{2\pi} \qquad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{9100 N/m}{57.61 kg}} = 12.5682\ rad/sec$

$f = 2\ Hz = \frac{\omega}{2\pi}$

CESSATION OF MOTION

- $X_{max} = \frac{F_c}{k} = .075 cm \qquad \frac{X_0 - .075}{n} = .3$ or $\frac{4 - .075}{.3} = \frac{3.925}{.3} \cong 13.1$ cycles TO CESSATION

TOPIC IV Harmonically Forced Vibs – UNDAMPED CASE

- LOOK AT CASE OF MASS-SPRING SYSTEM INFLUENCED BY A TIME DEPENDENT FORCING FUNCTION

- FORCE MAY BE HARMONIC, OR NONHARMONIC BUT PERIODIC, OR NONPERIODIC OR RANDOM
- WANT DYNAMIC RESPONSE OF SYSTEM CAUSED BY FORCE

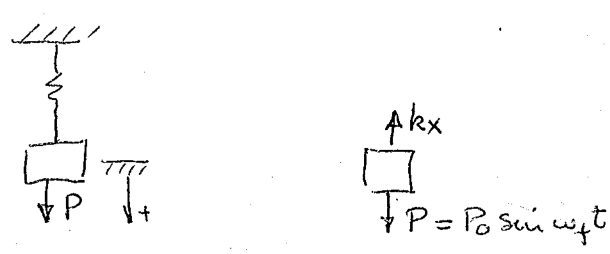
- LOOK AT SDOF W/ HARMONIC FORCING CONDITION OF FORM $P(t) = P_0 \sin(\omega_f t + \psi)$ RAO use $F_0 \cos \omega t$

P_0 is amplitude of $P(t)$

ω_f is the circular freq of FORCING FUNC $\qquad \omega$ in RAO

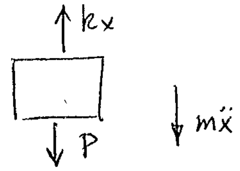
ψ is the phase angle $\qquad \psi$ is DEPENDENT ON I.C. of $P(t)$

LOOK AT CAUSE WHERE $\psi = 0$



SESSION #10

FOR DYNAMIC FREE-BODY DIAG.



$m\ddot{x} = -kx + P$

$m\ddot{x} + kx = P_0 \sin \omega_f t$

$x_p = X$

$m\ddot{x} + kx = 0 \qquad x = A \sin \omega_n t + B \cos \omega_n t \qquad \omega_n = \sqrt{\frac{k}{m}}$

$m\ddot{x} + kx = P_0 \sin \omega_f t$

$\qquad\qquad P_0 \cos \omega_f t$

$x_p = \begin{cases} C \sin \omega_f t \\ C \cos \omega_f t \end{cases} \qquad C = \dfrac{P_0}{k - m\omega_f^2} = \dfrac{P_0}{m} \dfrac{1}{\omega_n^2 - \omega_f^2}$

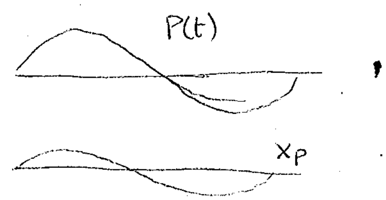
$\dfrac{1}{\left[\left(\frac{\omega_f}{\omega_n}\right)^2\right]} = \dfrac{P_0}{m \cdot k/m} \dfrac{1}{[1 - r^2]} = \dfrac{P_0/k}{1 - r^2} = X = \dfrac{X_0}{1 - r^2}$

ISPL OF SPRING DUE TO P_0 — δ_{ST} in RAO

IS THE FREQ ratio — $\dfrac{\omega}{\omega_n}$ in RAO

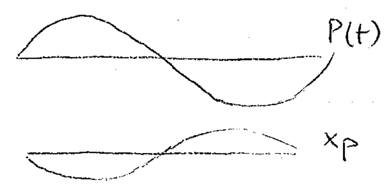
$< 1 - r^2 < 1 \qquad X_p > 0$

p is in phase w/ P(t)



$1 - r^2 < 0 \qquad X_p < 0$

x_p is opposite to P(t)



$1 - r^2 = 0$ and we have resonance

$\dfrac{P_0 t}{2m\omega_f}\begin{bmatrix} -\cos \omega_f t \\ +\sin \omega_f t \end{bmatrix} = \dfrac{P_0/k \; t}{2\omega_f \left(\frac{1}{\omega_f}\right)^2}\begin{bmatrix} -\cos \omega_f t \\ +\sin \omega_f t \end{bmatrix} = \dfrac{X_0 \omega_f t}{2}\begin{bmatrix} -\cos \omega_f t \\ +\sin \omega_f t \end{bmatrix}$

ude of $x_p = \dfrac{X_0 \omega_f t}{2}$ increases linearly w/ time

$\sin(\omega_f t + \pi/2) = -\cos \omega_f t$ — displ lags force by $\pi/2$

$\cos(\omega_f t - \pi/2) = \sin \omega_f t$

$\ln\left(\dfrac{x_0}{x_1}\right) = \dfrac{x_1 + \Delta x}{x_1}$

$= 1 + \dfrac{\Delta x}{x_1}$

$\dfrac{x_0}{x_0 - \Delta x}$

$\dfrac{1}{1 - \epsilon} = 1 + \epsilon + \epsilon^2 + \cdots$

$\delta = \ln(1 + \epsilon + \epsilon^2 + \cdots) = \epsilon$

$\delta \cong \dfrac{\Delta x}{x_0} = \dfrac{x_0 - x_1}{x_0}$

$\delta \cong \dfrac{x_0 - x_1}{x_0}$

$\Delta = \Delta_V + \Delta_f$

$\delta x_0 + \dfrac{4f}{k} = x_0 - x_1$

$\delta + \dfrac{4f}{k x_0} = .001$

$\delta + \dfrac{4f}{k x_{10}} = .002$

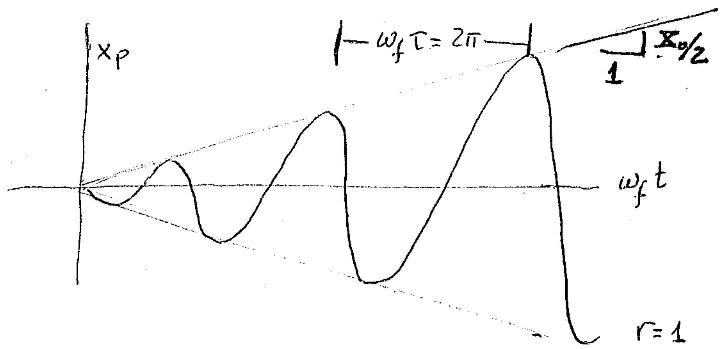
$x_0 \delta - x_{10} \delta = .001 x_0 - .002 x_{10}$

$\delta = \dfrac{.001 x_0 - .002 x_{10}}{x_0 - x_{10}}$

$\dfrac{4f}{k} = (.001 - \delta) x_0$

3. solution to x_h has already been discussed under FREE UNDAMPED motion

4. $$x_p = \frac{X_o \omega_f t}{2} \begin{matrix} \sin(\omega_f t - \pi/2) \\ \cos(\omega_f t - \pi/2) \end{matrix} \quad \text{if} \quad P = \begin{matrix} P_o \sin \omega_f t \\ P_o \cos \omega_f t \end{matrix}$$

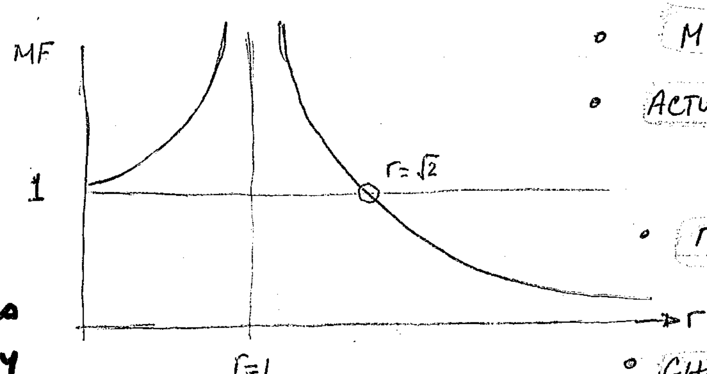


MAGNIFICATION FACTOR

- NORMALLY AMPLITUDE OF FORCED MOTION IS IMPORTANT

- DEFINE MAGNIFICATION RATIO $\boxed{MF = \frac{X}{X_o}} = \frac{X_o/1-r^2}{X_o} = \frac{1}{1-r^2} \quad r<1$

or $\frac{X}{X_o} = \left|\frac{1}{r^2-1}\right| \quad r>1$ RATIO $\frac{\text{DYNAMIC}}{\text{STATIC}}$ AMPLITUDE OF FORCING FN $= \textbf{MF}$

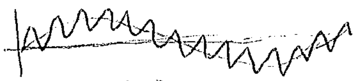


MF varies as r varies only through changes in ω_f

- MF $\to \infty$ as $r \to 1$
- ACTUALLY TIME FOR MF $\to \infty$ IS VERY LARGE
- r CAN BE ALTERED BY CHANGING k, m, ω_f
- CHANGING $k \to X_o$ CHANGES

$r<1$ SEE PG 135

1. IF $\boxed{\begin{matrix}\tau_f > \tau \\ \omega_f < \omega_n\end{matrix}}$ FREE MOTION OSCILLATES ABOUT FORCED VIB
2. $\boxed{\begin{matrix}\omega_f > \omega_n \\ \tau_f < \tau\end{matrix}}$ FORCED MOTION " ABOUT FREE VIB



$r>1$

3. IF $\boxed{\omega=\omega_f \quad r=1}$ for $\boxed{\text{small TIME} \quad \text{FREE VIBS DOMINATES}}$

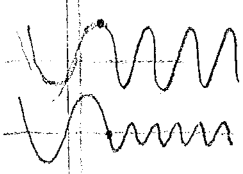
for $\boxed{\text{LARGE TIME} \quad \text{FORCED VIBS DOMINATES}}$

ASSUME $x_c=0$ & $x=x_p$

4. IF FORCED MOTION CEASES – FREE VIBS CONTINUES

DON'T DO

- CONTINUES w/ $x, \dot{x}$ BEING VALUES WHEN FORCED MOTION CEASES
- CHECK OUT SECT 4-5 ON THIS $\boxed{x = x_0 \cos\omega t + \frac{\dot{x}_0}{\omega}\sin\omega t}$



- IF $\boxed{\dot{x}=0 \quad \text{AMPLITUDE REMAINS SAME} \quad \omega \text{ CHANGES}}$
- IF $\boxed{x=0 \quad \text{AMPLITUDE CHANGES}}$

  IF $\boxed{\omega_f<\omega}$ $\boxed{\text{AMPLITUDE} \downarrow}$ $\boxed{\tau_f>\tau}$ $\boxed{\text{CYCLES OCCUR FASTER}}$

  IF $\boxed{\omega_f>\omega}$ $\boxed{\text{"} \quad \uparrow}$ $\boxed{\text{"} \ \tau_f<\tau \ \text{"} \quad \text{SLOWER}}$

- IF $\dot{x}\neq 0$, $x\neq 0$ RESULT IS IN BETWEEN

FOR THE RESONANT CONDITION TO $m\ddot{x}+kx = P_0 \sin\omega_f t$

- $$x_{TOTAL} = x_c + x_p = A\sin\omega_n t + B\cos\omega_n t + \frac{X_0}{1-r^2}\sin\omega_f t$$

- if $x=x_0$ @ $t=0$ & $\dot{x}=\dot{x}_0$ @ $t=0$

$$B = x_0 \quad \text{and} \quad A = \left[\dot{x}_0 - \frac{X_0\omega_f}{1-r^2}\right]\frac{1}{\omega_n} = \frac{\dot{x}_0}{\omega_n} - \frac{X_0 r}{1-r^2}$$

thus $$x = x_0\cos\omega_n t + \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + \frac{X_0}{1-r^2}\left[\sin\omega_f t - r\sin\omega_n t\right]$$

SUPPOSE $x_0 = \dot{x}_0 = 0$; @ resonance $r=1$ $\frac{P_0}{m[\omega_n^2-\omega^2]}$ ~~$x=x_p$ under these conditions~~

$$x = \frac{X_0}{1-r^2}\left[\sin\omega_f t - r\sin\omega_n t\right] = \frac{2X_0}{1-r^2}\sin\left(\frac{\omega_f-\omega_n}{2}\right)t \cdot \cos\left(\frac{\omega_f+\omega_n}{2}\right)t$$

if $\omega_n - \omega_f = 2\Delta \ll 1$ $\qquad \omega_n+\omega_f \cong 2\omega_f$ $\qquad \omega_n^2-\omega_f^2 = 4\omega_f\Delta$

$$1-r^2 \cong 4\Delta/\omega_n \cdot r$$

$$x = -\frac{2X_0}{1-r^2}\sin\Delta t \cdot \cos\omega_f t = \frac{-2X_0}{(4\Delta/\omega)r}\sin\Delta t\cdot\cos\omega_f t = \frac{-X_0}{2}\omega_f t\cos\omega_f t$$

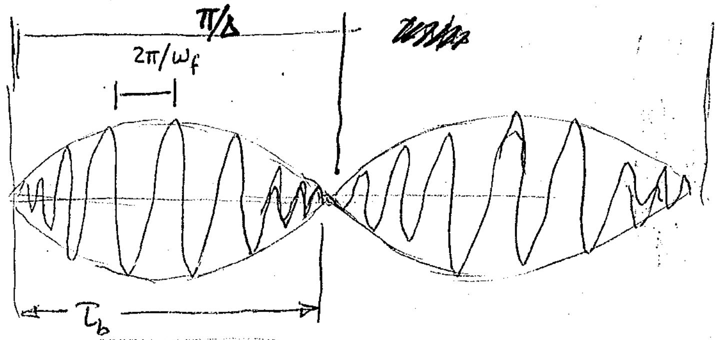
as $r\to 1$

- IF $\omega_f \approx \omega$ but $\neq$ BEATING OCCURS
  - AMPLITUDE ↑ THEN ↓

$r \cong 1$

$$x = -\frac{X_0 \omega}{2\Delta r} \sin \Delta t \cos \omega_f t = \left(-\frac{X_0 \omega}{2\Delta} \cos \omega_f t\right) \sin \Delta t = \left(-\frac{X_0 \omega}{2\Delta} \sin \Delta t\right) \cos \omega_f t$$

- since $\omega_f >> \Delta$, $\tau_f << \tau$ — cosine goes through many cycles before sine goes through one
- $-\frac{X_0 \omega}{2\Delta r} \cos \omega_f t$ is a cyclic varying amplitude
- $\sin \Delta t$ represents envelope of the amplitude

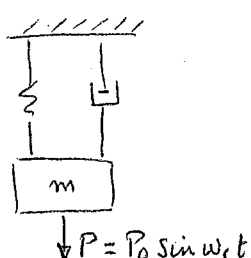


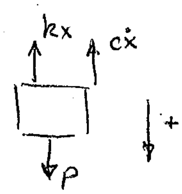
- BEAT PERIOD $\tau_b = \frac{2\pi}{\Delta} = \frac{2\pi}{\omega_n - \omega_f}$ — TIME BETWEEN POINTS OF ZERO MOTION OR BETWEEN MAXIMUM MOTIONS

→ HW 4-2, 6, 13

SESSION #11 + PROBLEMS

FORCED VIBRATIONS w/ VISCOUS DAMPING





DYNAMIC F.B.D.

$m\ddot{x} = -kx - c\dot{x} + P$

$$m\ddot{x} + c\dot{x} + kx = P_0 \sin \omega_f t$$

HAS SOLUTION $x = x_p + x_c$

x_c solves $m\ddot{x} + c\dot{x} + kx = 0$

x_p " $= P_0 \sin \omega_f t$

HAVE STUDIED HOMOGENEOUS – GIVES RISE TO

- UNDERDAMPED, CRITICAL DAMPED, OVERDAMPED
- AFTER SOME TIME – THE SOLUTION OF HOMOGENEOUS DE DIES OUT
- PARTICULAR REMAINS

- SOLUTION TO HOMOG $\longrightarrow$ TRANSIENT
- SOLUTION TO PARTIC $\longrightarrow$ STEADY STATE

- WILL CONSENTRATE ON STEADY STATE – **PARTICULAR SOLN**

- IF $\zeta < 1$ $\quad x_c = Ce^{-\zeta\omega t}\sin(\omega_d t + \phi)$ $\qquad \zeta = \frac{c}{c_c}$; $\omega = \sqrt{\frac{k}{m}} = \frac{c_c}{2m}$

$\omega_d = \sqrt{1-\zeta^2}\,\omega$

- FOR STEADY STATE TAKE $x_p = X\sin(\omega_f t - \psi) \Rightarrow X\cos\psi = A \quad B = -X\sin\psi$

choose $\quad x_p = A\sin\omega_f t + B\cos\omega_f t \qquad$ since DE involves $\ddot{x}, \dot{x}, x$

$$\dot{x}_p = A\omega_f\cos\omega_f t - B\omega_f\sin\omega_f t$$

$$\ddot{x}_p = -A\omega_f^2\sin\omega_f t - B\omega_f^2\cos\omega_f t$$

$$m\ddot{x}_p + c\dot{x}_p + kx_p = P_0\sin\omega_f t.$$

Skip

$$\left[-mA\omega_f^2 - cB\omega_f + kA\right]\sin\omega_f t + \left[-mB\omega_f^2 + cA\omega_f + kB\right]\cos\omega_f t = P_0\sin\omega_f t$$

$$\Rightarrow \quad \left[-m\omega_f^2 + k\right]A - c\omega_f B = P_0 \qquad A = \frac{\begin{vmatrix} P_0 & -c\omega_f \\ 0 & k - m\omega_f^2 \end{vmatrix}}{\text{denom}}$$

$$\left[+c\omega_f\right]A + \left[-m\omega_f^2 + k\right]B = 0$$

$$\text{denominator} = (k - m\omega_f^2)^2 + c^2\omega_f^2 = m^2\left(\left[\omega_n^2 - \omega_f^2\right]^2 + 4\omega_n^2\omega_f^2\zeta^2\right) \qquad B = \frac{\begin{vmatrix} k - m\omega_f^2 & P_0 \\ c\omega_f & 0 \end{vmatrix}}{\text{denom}}$$

$$\boxed{\text{thus } A = \frac{P_0(k - m\omega_f^2)}{\text{denom}} \qquad B = \frac{-c\omega_f P_0}{\text{denom}}, \qquad \text{denom} = (k - m\omega_f^2)^2 + (c\omega_f)^2}$$

Skip this derivation

$$\text{denom} = m^2\omega_n^4\left(\left[1 - r^2\right]^2 + (2r\zeta)^2\right) = \underline{k^2\left[(1 - r^2)^2 + (2r\zeta)^2\right]}$$

just write this

3.5 RESPONSE OF A DAMPED SYSTEM UNDER $F(t) = F_0 e^{i\omega t}$

Let the harmonic forcing function be represented in complex form as $F(t) = F_0 e^{i\omega t}$ so that the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 e^{i\omega t} \tag{3.42}$$

Since the actual excitation is given only by the real part of $F(t)$, the response will also be given only by the real part of $x(t)$ where $x(t)$ is a complex quantity satisfying the differential equation (3.42). F_0 in Eq. (3.42) is, in general, a complex number. By assuming the particular solution $x_p(t)$

$$x_p(t) = X e^{i\omega t} \tag{3.43}$$

we obtain, by substituting Eq. (3.43) into Eq. (3.42),*

$$X = \frac{F_0}{(k - m\omega^2) + ic\omega} \tag{3.44}$$

complex conjugate of $(k-m\omega^2)+ic\omega$

Multiplying the numerator and denominator on the right side of Eq. (3.44) by $[(k - m\omega^2) - ic\omega]$ and separating the real and imaginary parts, we obtain

$$X = F_0\left[\frac{k - m\omega^2}{(k - m\omega^2)^2 + c^2\omega^2} - i\frac{c\omega}{(k - m\omega^2)^2 + c^2\omega^2}\right] \tag{3.45}$$

Using the relation, $x + iy = Ae^{i\phi}$ where $A = \sqrt{x^2 + y^2}$ and $\tan\phi = y/x$, Eq. (3.45) can be expressed as

$$X = \frac{F_0}{\left[(k - m\omega^2)^2 + c^2\omega^2\right]^{1/2}} e^{-i\phi} \tag{3.46}$$

where

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \tag{3.47}$$

Thus the steady-state solution, Eq. (3.43), becomes

$$x_p(t) = \frac{F_0}{\left[(k - m\omega^2)^2 + (c\omega)^2\right]^{1/2}} e^{i(\omega t - \phi)} \tag{3.48}$$

Frequency Response. Equation (3.44) can be rewritten in the form

$$\frac{X}{X_o} = \frac{kX}{F_0} = \frac{1}{1 - r^2 + i2\zeta r} \equiv H(i\omega) \tag{3.49}$$

where $H(i\omega)$ is known as the *complex frequency response* of the system. The absolute

*Equation (3.44) can be written as $Z(i\omega)X = F_0$ where $Z(i\omega) = -m\omega^2 + i\omega c + k$ is called the *mechanical impedance* of the system [3.8].

value of $H(i\omega)$ given by

$$|H(i\omega)| = \left|\frac{kX}{F_0}\right| = \frac{1}{\left[(1-r^2)^2+(2\zeta r)^2\right]^{1/2}} \tag{3.50}$$

denotes the magnification factor defined in Eq. (3.30). Recalling that $e^{i\phi} = \cos\phi + i\sin\phi$, we can show that Eqs. (3.49) and (3.50) are related:

$$H(i\omega) = |H(i\omega)|e^{-i\phi} \tag{3.51}$$

where ϕ is given by Eq. (3.47), which can also be expressed as

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \tag{3.52}$$

Thus Eq. (3.48) can be expressed as

$$x_p(t) = \frac{F_0}{k}|H(i\omega)|e^{i(\omega t-\phi)} \tag{3.53}$$

If $F(t) = F_0 \cos\omega t$, the corresponding steady-state solution is given by the real part of Eq. (3.48):

$$\begin{aligned} x_p(t) &= \frac{F_0}{\left[(k-m\omega^2)^2+(c\omega)^2\right]^{1/2}}\cos(\omega t-\phi) \\ &= \mathrm{Re}\left[\frac{F_0}{k}H(i\omega)e^{i\omega t}\right] = \mathrm{Re}\left[\frac{F_0}{k}|H(i\omega)|e^{i(\omega t-\phi)}\right] \end{aligned} \tag{3.54}$$

which can be seen to be the same as Eq. (3.24). Similarly, if $F(t) = F_0 \sin\omega t$, the corresponding steady-state solution is given by the imaginary part of Eq. (3.48):

$$\begin{aligned} x_p(t) &= \frac{F_0}{\left[(k-m\omega^2)^2+(c\omega)^2\right]^{1/2}}\sin(\omega t-\phi) \\ &= \mathrm{Im}\left[\frac{F_0}{k}|H(i\omega)|e^{i(\omega t-\phi)}\right] \end{aligned} \tag{3.55}$$

Complex Vector Representation of Harmonic Motion. The harmonic excitation and the response of the damped system to that excitation can be represented graphically in the complex plane, and interesting interpretation can be given to the resulting diagram. We first differentiate Eq. (3.53) with respect to time and obtain

$$\begin{aligned} \text{velocity} &= \dot{x}_p(t) = i\omega\frac{F_0}{k}|H(i\omega)|e^{i(\omega t-\phi)} = i\omega x_p(t) \\ \text{acceleration} &= \ddot{x}_p(t) = (i\omega)^2\frac{F_0}{k}|H(i\omega)|e^{i(\omega t-\phi)} = -\omega^2 x_p(t) \end{aligned} \tag{3.56}$$

Because i can be expressed as

$$i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = e^{i\pi/2} \tag{3.57}$$

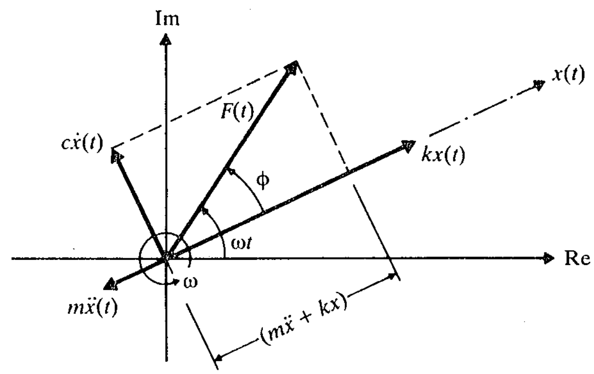


Figure 3.12 Representation of Eq. (3.42) in a complex plane.

we can conclude that the velocity leads the displacement by the phase angle $\pi/2$ and that it is multiplied by ω. Similarly, -1 can be written as

$$-1 = \cos \pi + i \sin \pi = e^{\pi/2} \tag{3.58}$$

Hence the acceleration leads the displacement by the phase angle π, and it is multiplied by ω^2.

Thus the various terms of the equation of motion (3.42) can be represented in the complex plane, as shown in Fig. 3.12. The interpretation of this figure is that the sum of the complex vectors $m\ddot{x}(t)$, $c\dot{x}(t)$, and $kx(t)$ balances $F(t)$, which is precisely what is required to satisfy Eq. (3.42). It is to be noted that the entire diagram rotates with angular velocity ω in the complex plane. If only the real part of the response is to be considered, then the entire diagram must be projected onto the real axis. Similarly, if only the imaginary part of the response is to be considered, then the diagram must be projected onto the imaginary axis. In Fig. 3.12, notice that the force $F(t) = F_0 e^{i\omega t}$ is represented as a vector located at an angle ωt to the real axis. This implies that F_0 is real. If F_0 is also complex, then the force vector $F(t)$ will be located at an angle of $(\omega t + \psi)$, where ψ is some phase angle introduced by F_0. In such a case, all the other vectors, namely, $m\ddot{x}$, $c\dot{x}$, and kx will be shifted by the same angle ψ. This is equivalent to multiplying both sides of Eq. (3.42) by $e^{i\psi}$.

3.6 RESPONSE OF A DAMPED SYSTEM UNDER THE HARMONIC MOTION OF THE BASE

Sometimes the base or support of a spring-mass-damper system undergoes harmonic motion, as shown in Fig. 3.13(a). Let $y(t)$ denote the displacement of the base and $x(t)$ the displacement of the mass from its static equilibrium position at time t. Then the net elongation of the spring is $x - y$ and the relative velocity between the two ends of the damper is $\dot{x} - \dot{y}$. From the free-body diagram shown in Fig. 3.13(b), we

$$X = \sqrt{A^2+B^2} = \frac{P_o}{\sqrt{denom}} = \frac{P_o/k}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} = \frac{X_o}{\sqrt{(1-r^2)^2+(2r\zeta)^2}}$$

$$\tan\psi = -\frac{B}{A} = \frac{c\omega_f}{k - m\omega_f^2} = \frac{2\zeta r}{1-r^2}$$

BOTH X & ψ are fns of ζ & r

$x_p = X \sin(\omega_f t - \psi) \quad \longleftrightarrow \quad P_o \sin \omega_f t = P$ TIME LAG IN RESPONSE $r<1$, LEAD $r>1$

SESSION #12

- SAME MOTION BUT x_p lags $P(t)$ by ψ
- AMPLITUDE IS X
- $\forall\, \zeta > 0$ REDUCES X and for

NOTE IF $\zeta = 0$ (NO DAMPING) $\psi = 0$ & $X = \dfrac{X_o}{|1-r^2|}$ AS BEFORE

- THE LAG TIME t' IS TIME x_p LAGS $P(t)$
  - IE when $t=0$ $\sin\omega_f t = 0$ but $\sin(\omega_f t - \psi) \neq 0$
  - $\sin(\omega_f t' - \psi) = 0$ when $t' = \dfrac{\psi}{\omega_f}$

IF UNDERDAMPED

TOTAL SOL'N is $x = Ce^{-\zeta\omega t}\sin(\omega_d t + \phi) + X \sin(\omega_f t - \psi)$

GIVE HANDOUT

2nd ed. RAO $\quad x = (A+Bt)e^{-\omega_n t} + X \sin(\omega_f t - \psi)$ for critical

129

- LOOK AT PAGE 107 FOR TRANSIENT $\longrightarrow$ STEADY STATE CONVERSION

ASSUMED : $\omega_f < \omega_d \quad = \quad \omega_f < \sqrt{1-\zeta^2}\,\omega_n \Rightarrow r < \sqrt{1-\zeta^2} < 1$

$\tau_f > \tau_d$

- NOTE THAT $X_{DAMPED} < X_{UNDAMPED}$ FOR ALL ω_f
- NOTE WHEN $r = 1$ $\quad X_{RES} = \dfrac{X_o}{2\zeta} \neq \infty$ EXCEPT FOR UNDAMPED

EXTREMUMS OF MF

- $MF = \dfrac{X}{X_o} = \dfrac{1}{\sqrt{(1-r^2)^2 + (2r\zeta)^2}}$ $\qquad MF = MF(r,\zeta)$

take $\dfrac{d(MF)}{dr} = -\dfrac{1}{2}\dfrac{[2(1-r^2)(-2r) + 2\zeta(2r\zeta)]}{[\quad]^{3/2}} = \dfrac{4r[-1+r^2+2\zeta^2]}{2[\quad]^{3/2}} = 0$

$\boxed{dMF/dr = 0}$

when $\boxed{r=0}$

when $\boxed{r \to \infty}$ $\quad d(MF)/dr \to -\dfrac{(-2r^3)}{r^6} = \dfrac{2}{r^3} \to 0 \quad r \to \infty \quad$ ✓

when $\boxed{1-r^2 - 2\zeta^2 = 0}$ $\quad r = \sqrt{1-2\zeta^2} < 1 \quad$ for $0 \le \zeta \le \dfrac{1}{\sqrt{2}} = .707$

for $0 \le \zeta \le .707$ there is a relative max. $\Rightarrow \omega = \omega_n\sqrt{1-2\zeta^2} < \omega_d$.

- THIS DEFINES MAX POINT IN THE RESONANT REGION & since $r<1$ OCCURS TO THE LEFT OF THE RESONANT VALUE $r=1$

Pg 125

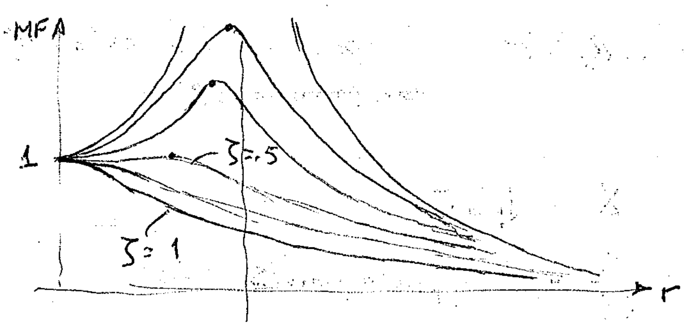
PUT $r = \sqrt{1-2\zeta^2}$ in $MF = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

$$(MF)_{max} = \frac{X_{max}}{X_0} = \frac{1}{\sqrt{(2\zeta^2)^2 + (2\zeta)^2[1-2\zeta^2]}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \text{for } \zeta < 0.707$$

SOL'N TO 3.9

$$\Rightarrow \frac{X_{max}}{X_0} = \frac{X_{RES}}{X_0}\frac{1}{\sqrt{1-\zeta^2}} \Rightarrow \frac{X_{max}}{X_{res}} = \frac{1}{\sqrt{1-\zeta^2}}$$

- FOR $\zeta \geq 0.707$ $\quad (MF)_{max}$ is at $r=0$



3 extremums for $\zeta < .707$
2 extremum for $\zeta > .707$

- $(MF)_{max}$ occurs when $r = \sqrt{1-2\zeta^2}$ or when $\omega_f = \omega_n\sqrt{1-2\zeta^2}$

$$\omega_f = \omega_n\sqrt{1-2\zeta^2} < \underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d} < \omega_n$$

- MF will decrease if r is large
- look at this as ω_f change only when r changes

*** GO TO MASS & VISCOSITY VARIATIONS NEXT – THEN RETURN**

- PHASE ANGLE VARIATION with r

$$\tan\psi = \frac{2\zeta r}{1-r^2} = \frac{c\omega_f}{k - m\omega_f^2}$$

; for ANY const ζ as $r\uparrow$ $\tan\psi\uparrow$ $r<1$

when $r=1$ $\tan\psi = \infty \Rightarrow \psi = \pi/2$

when $r>1$ when $r\uparrow$ $\tan\psi \to -0$

$\Delta E = c\omega_d X^2 \pi$ power $= F \cdot v = cv^2 = c\omega_d^2 X^2 \cos^2 \omega_d t$

max power is $\sim X^2$

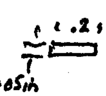
since $\frac{\Delta W}{\text{cycle}} \sim X^2$

max power $\sim X^2$ $X \sim \sqrt{\text{power}}$

points where power drops to $\frac{1}{2}$ its value $\frac{1}{2}$ power $\sim \frac{X^2}{2}$ $\frac{X}{\sqrt{2}} \sim$ half power points $= Q$

Oscill support
3.14

$k = \frac{3EI}{l^3}$ $E = 30 \times 10^6$ psi



$m = \frac{.1 \text{ lb}}{32.2 \text{ ft/s}^2} = 3.106 \times 10^{-3} \frac{\text{lb-s}^2}{\text{ft}}$

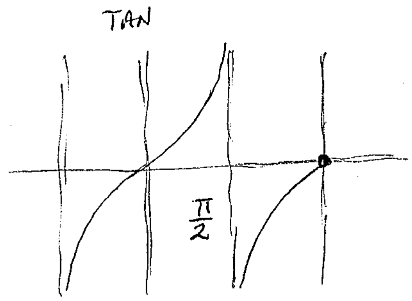
$I = \frac{bh^3}{12} = \frac{(.05)^3(.2)}{12} \text{in}^4 \cdot \frac{\text{ft}^4}{12^4} = 1.005 \times 10^{-10} \text{ ft}^4 = 2.083 \times 10^{-6} \text{ in}^4$

$k_{eq} = \frac{3EI}{l^3} = \frac{3(30 \times 10^6 \frac{\text{lb}}{\text{in}^2})(2.083 \times 10^{-6} \text{ in}^4)}{(10 \text{in})^4} = .1875 \frac{\text{lb}}{\text{in}} \times \frac{12 \text{in}}{\text{ft}}$

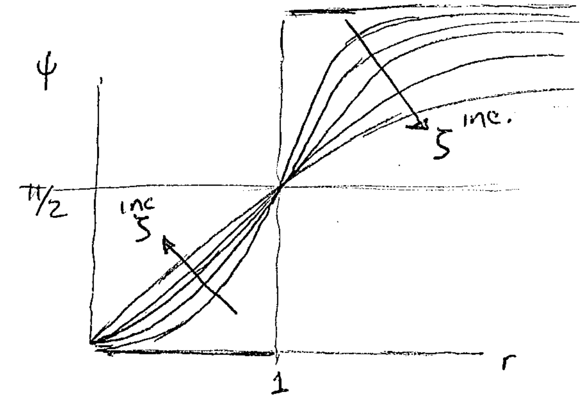
$= 2.25 \frac{\text{lb}}{\text{ft}}$

$m_{bar} = \rho \cdot V = \frac{.283 \frac{\text{lb}}{\text{in}^3}}{32.2 \text{ ft/s}^2} \frac{\text{lb-s}^2}{\text{ft}} \times (10 \times .2 \times .05) \text{ in}^3 = 8.79 \times 10^{-4} \frac{\text{lb-s}^2}{\text{ft}}$

$m_{tot} = m + .23 m_b$ $\omega_n = \sqrt{\frac{k}{m_{tot}}} = 5.65$



$\Rightarrow$



$$\tan\psi = \frac{2\zeta r}{1-r^2}$$

keeping r fixed : as $\zeta\uparrow$ $\tan\psi\uparrow$ for $r<1$

as $\zeta\uparrow$ for $r>1$ $\tan\psi\downarrow$ TOWARDS $-\infty$ $(\psi\to\pi/2)$

FOR $\zeta=0$ and $r<1$ $\tan\psi=0$

$\zeta=0$ and $r>1$ $\tan\psi=0$ but from negative side ie $(\psi\to\pi)$

DISCONTINUITY AT $r=1$

- NOTE $\psi = \psi(\zeta, r) = \psi(c, c_c, \omega_f, \omega) = \psi(c, m, k, \omega_f)$

  BUT NOT P_0

- as $r\to\infty$ $\tan\psi\to -0$ and $\psi\to\pi$ $x_p \to X\sin(\omega_f t-\pi)$
- as $r\to 0$ $\tan\psi\to +0$ and $x_p = X\sin\omega_f t$ $P=P_0\sin\omega_f t$
- x_p will be in phase for $r\ll 1$ but will be out of phase for $r\gg 1$
- $\forall\zeta$ at $r=1$ $x_p = X\sin(\omega_f t-\underline{\underline{\pi/2}}) = \frac{X_0}{2\zeta}\sin(\omega t-\pi/2)$

- for $r<1$ $\omega_f<\omega_n$ $\psi\uparrow$ when $\zeta\uparrow$

  $r>1$ $\omega_f>\omega$ $\psi\uparrow$ when $\zeta\downarrow$

FIRST PART OF SESSION # 13 TIL m & k variations

- QUALITY FACTOR & BANDWIDTH

  REMEMBER $\frac{X_{RES}}{X_0} = \frac{1}{2\zeta}$ $\frac{X_{max}}{X_0} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

  FOR $\zeta\ll 1$ $\frac{X_{max}}{X_0} \cong \boxed{\frac{X_{res}}{X_0} = \frac{1}{2\zeta} = Q}$ (Q factor or quality factor)

  $X/X_0|_{res}$

in EE - when a filter has an max gain at resonance

- DEFINE BANDWIDTH AS DIFFERENCE in FORCED FREQ'S when $MF \doteq \frac{Q}{\sqrt{2}}$

$$MF = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{1}{2\sqrt{2}\zeta} = \frac{Q}{\sqrt{2}}$$

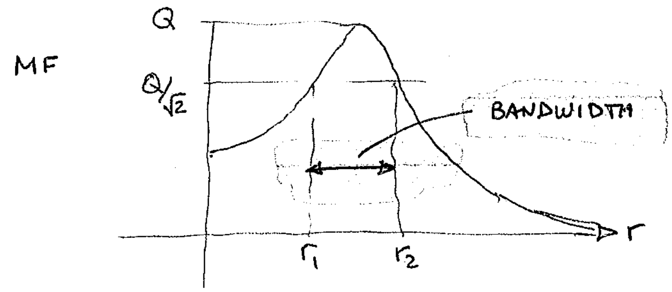
SOLVING FOR $r^2 = (1-2\zeta^2) \pm 2\zeta\sqrt{1+\zeta^2}$

FOR small ζ $\quad r^2 \sim 1 \pm 2\zeta \quad \Rightarrow \quad r \sim 1+\zeta \, , \, 1-\zeta$

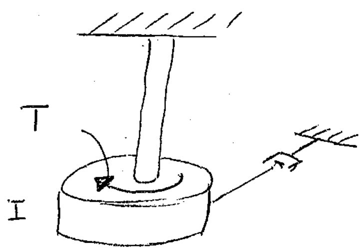
$$r_2 - r_1 = \frac{\omega_{f_2}}{\omega} - \frac{\omega_{f_1}}{\omega} \cong 1+\zeta - (1-\zeta) = 2\zeta$$

or $\quad \omega_{f_2} - \omega_{f_1} = 2\zeta\omega_n \qquad$ BUT $\quad Q = \dfrac{1}{2\zeta} = \dfrac{\omega_n}{\omega_{f_2} - \omega_{f_1}}$

- Q CAN BE USED TO ESTIMATE ζ



r_1, r_2 are the half power pts since power absorbed by the damper is related to the value of X^2 & hence MF



$$I\ddot{\theta} = -k_T\theta - c_T\dot{\theta} + T_0 \sin\omega_f t$$
$$I\ddot{\theta} + c_T\dot{\theta} + k_T\theta = T_0 \sin\omega_f t$$

HW $\quad$ 4-21, 4-28, 4-27, 4-29a $\quad$ FIND BAND-WIDTH, MF_{max}, MF_{RES}

PART OF LESSON # 13 TO HERE

- MASS & SPRING CONST. VARIATIONS
  - when r varies m, k vary. IF k varies X_0 varies
  - when k varies ζ varies $\quad \dfrac{c}{c_c} = \dfrac{c}{2m\omega} = \dfrac{c}{2\sqrt{mk}}$
  - IF m varies ζ varies

  CANNOT USE ORIGINAL GRAPHS → JUMP FROM ONE ζ CURVE TO OTHER

- SPRING k VARIATIONS
  - LOOK AT $\quad X = \dfrac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \qquad \dfrac{dX}{dk} = \dfrac{-(P_0)\,2(k - m\omega_f^2)}{2[\quad]^{3/2}}$
  - $\dfrac{dX}{dk} = 0 \Rightarrow X_{max}$ occurs when $k = m\omega_f^2$ or $X_{max} = \dfrac{P_0}{c\omega_f}$

$$\zeta = \frac{c}{c_c} = \frac{0.60}{3.0} = 0.20$$

$$X_0 = \frac{P_0}{k} = \frac{25}{45} = 0.556 \text{ in.}$$

$$X_{\text{res}} = \frac{X_0}{2\zeta} = \frac{0.556}{2 \times 0.2} = 1.39 \text{ in.}$$

$$X_{\max} = \frac{X_0}{2\zeta\sqrt{1-\zeta^2}} = \frac{0.556}{2 \times 0.2\sqrt{1-(0.2)^2}} = \frac{1.39}{0.980} = 1.42 \text{ in.}$$

4-11. PHASE ANGLE

The value of the phase angle, as defined by Eq. 4-50, depends on the damping factor ζ and the frequency ratio r. This can be studied by plotting ψ against r for various values of ζ. The resulting family of curves is shown in Fig. 4-16. For no damping, $\psi = 0$ from $r = 0$ to $r < 1$, $\psi = 90$ degrees for $r = 1$, and $\psi = 180$ degrees for $r > 1$. This agrees with the analysis and discussion of Section 4-2. For small values of ζ, these same conditions are approximated; that is, the curve approaches the curve for the zero-damping case. All curves go through the point of $\psi = 90$ degrees for $r = 1$. Note that for $\zeta = 0.707$ the ψ curve is approximately linear from $r = 0$ through $r = 1$.

4-12. INFLUENCE OF MASS AND ELASTICITY ON AMPLITUDE

In determining the effect of varying r on the steady-state amplitude, recall that $r = \omega_f\sqrt{m/k}$, and hence r can be varied by changing k or m as well as ω_f. However, if either k or m is changed, this will alter ζ (as $\zeta = c/2\sqrt{mk}$) and distort the interpretation of Fig. 4-15, since a different ζ curve would then have to be used. In addition, altering k will change the reference value X_0. If it is desired to study the effect of varying k, the amplitude relation (Eq. 4-49) can be written in the form

$$X = \frac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \tag{4-71}$$

and X can then be plotted against k for several values of the damping constant c. The resulting family of curves is shown in Fig. 4-17. It should be noted that P_0, m, and ω_f are constant in this consideration. Maximum and minimum points on the curves can be obtained by setting $dX/dk = 0$. From this, it is found that the maximum point occurs for $k = m\omega_f^2$ and is defined by

$$X_{\max} = \frac{P_0}{c\omega_f} \tag{4-72}$$

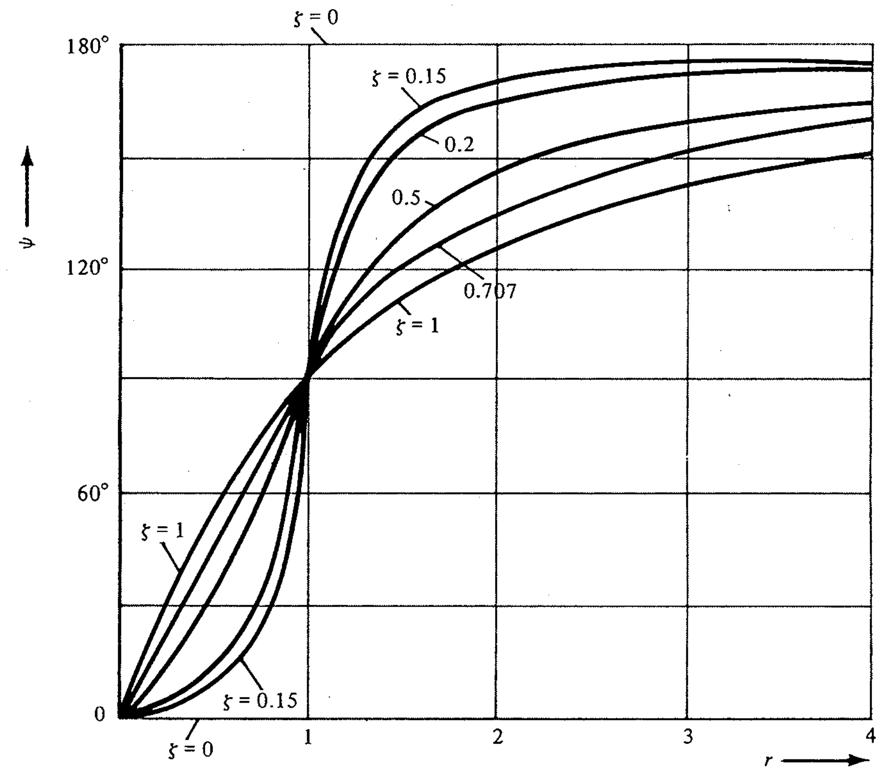


Figure 4-16

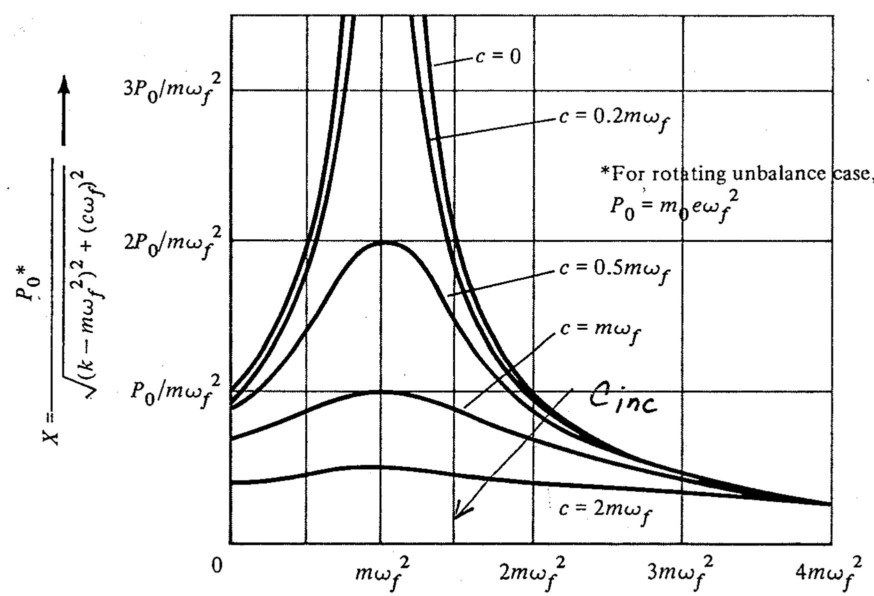


Figure 4-17

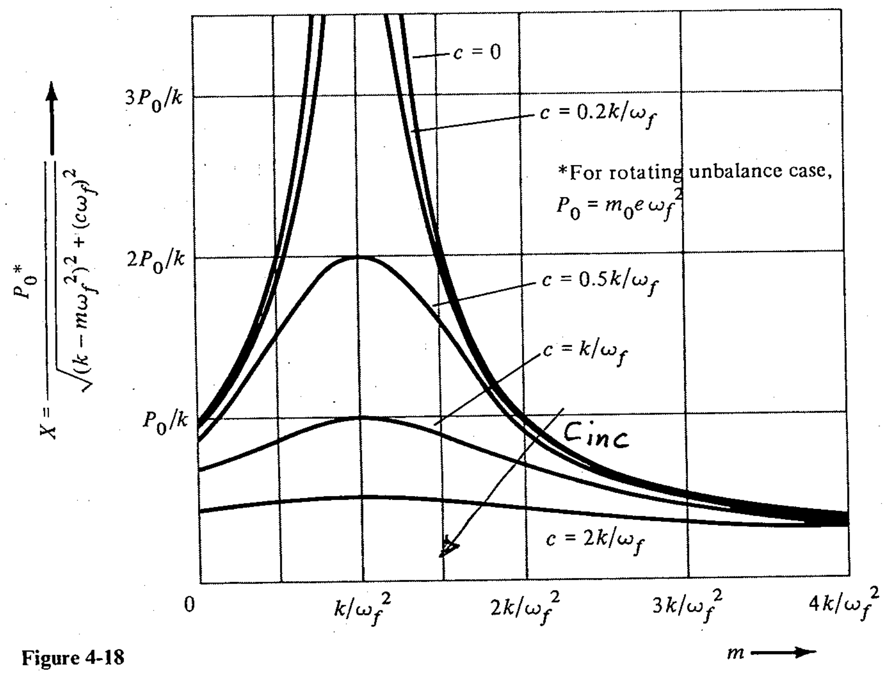


Figure 4-18

In addition, all the curves approach zero as k becomes large. The initial point (for $k = 0$) is given by

$$X = \frac{P_0}{\sqrt{(m\omega_f^2)^2 + (c\omega_f)^2}} \tag{4-73}$$

Reduction in amplitude is attained only as k becomes large. This means, as would be expected, that stiff springs will result in a small amplitude of motion for a given system.

The amplitude relation (Eq. 4-71) can also be used to observe the effect of varying m on the amplitude. In this case X can be plotted against m for various values of c, with P_0, k, and ω_f being taken as constant. The resulting family of curves is shown in Fig. 4-18. Maximum and minimum points on the curves can be determined by setting $dX/dm = 0$. The maximum point occurs at $m = k/\omega_f^2$ and is given by

$$X_{\max} = \frac{P_0}{c\omega_f} \tag{4-74}$$

All the curves approach zero as m becomes large. The initial point (for $m = 0$) is given by

$$X = \frac{P_0}{\sqrt{k^2 + (c\omega_f)^2}} \tag{4-75}$$

As might be anticipated, large values of m result in a reduction in the amplitude.

EXAMPLE 4-4 A machine weighing 19.3 lb is subjected to a harmonic force having a maximum value of 12 lb and a frequency of 300 cycles/min. The clearance for the vibrational movement of the machine is 1 in. Design a lightly damped elastic-support system for the machine, so that the machine does not collide in its movement.

SOLUTION In order to properly limit the movement of the machine, the allowable movement is set at one-half the actual clearance. Thus

$$X = 0.5 \text{ in.}$$

Also,

$$\omega_f = \frac{300}{60} \times 2\pi = 10\pi \text{ rad/sec}$$

For small damping, from Fig. 4-17, the value of c is selected as

$$c = 0.1m\omega_f = 0.1 \times \frac{19.3}{386} \times 10\pi = 0.05\pi \text{ lb sec/in.}$$

$$= 0.157\,08 \text{ lb sec/in.}$$

Then from Eq. 4-71,

$$X = \frac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}$$

$$0.5 = \frac{12}{\sqrt{[k - (19.3/386)(10\pi)^2]^2 + (0.05\pi \times 10\pi)^2}}$$

Expanding and rearranging gives

$$k^2 - 98.6960k + 1883.58 = 0$$

which has the single positive root

$$k = 25.861 \text{ lb/in.}$$

and the design is composed of an elastic support and damping device having the values of k and c determined.

EXAMPLE 4-5 A damped system is driven by the force $P = 0.54 \sin 12t$, where P is in newtons and t is in seconds. The system has a mass of 0.1 kg, and the damping constant is 0.24 N · s/m. (a) Obtain the steady-state amplitude for spring-constant k values of 2, 25, and 90 N/m. (b) Determine the spring constant that will produce the maximum amplitude, and calculate this amplitude.

SOLUTION

$$X = \frac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} = \frac{0.54}{\sqrt{[k - 0.1 \times (12)^2]^2 + (0.24 \times 12)^2}}$$

$$= \frac{0.54}{\sqrt{(k - 14.4)^2 + (2.88)^2}}$$

a. For $k = 2$, $X = 0.04242 \text{ m} = 4.242 \text{ cm}$
For $k = 25$, $X = 4.916 \text{ cm}$
For $k = 90$, $X = 0.7138 \text{ cm}$

b. $$X_{\max} = \frac{P_0}{c\omega_f} = \frac{0.54}{0.24 \times 12} = 0.1875 \text{ m} = 18.75 \text{ cm}$$

for $k = m\omega_f^2 = 0.1 \times (12)^2 = 14.4 \text{ N/m}$

4-13. ROTATING UNBALANCE

A common source of forced vibration is caused by the rotation of a small eccentric mass such as that represented by m_0 in Fig. 4-19(a). This condition results from a setscrew or a key on a rotating shaft, crankshaft rotation, and many other simple but unavoidable situations. Rotating unbalance is inherent in rotating parts, because it is virtually impossible to place the axis of the mass center on the axis of rotation.

For the system shown, the total mass is m and the eccentric mass is m_0, so the mass of the machine body is $(m - m_0)$. The length of the eccentric arm, or the eccentricity of m_0, is represented by e. If the arm rotates with an angular velocity ω_f rad/sec, then the angular position of the arm is defined by $\omega_f t$ with respect to the indicated horizontal reference, where t is time, in seconds. The free-body diagram for this system is shown in Fig. 4-19(b), positive x having been taken as upward. The horizontal motion of $(m - m_0)$ is considered to be prevented by guides. The vertical displacement of m_0 is $(x + e \sin \omega_f t)$. From Eq. 1-8, the differential equation of motion can then be written as

$$(m - m_0)\frac{d^2x}{dt^2} + m_0 \frac{d^2}{dt^2}(x + e \sin \omega_f t) = -kx - c\frac{dx}{dt} \tag{4-76}$$

which can be rearranged in the form

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = m_0 e\omega_f^2 \sin \omega_f t \tag{4-77}$$

Examination of this and comparison to the differential equation (Eq. 4-38) for motion forced by $P = P_0 \sin \omega_f t$ enable the steady-state solution to be set down, from Eq. 4-48, as

$$x = X \sin (\omega_f t - \psi) \tag{4-78}$$

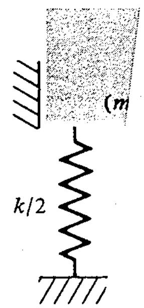


Figure 4-

where

and

Al

F
s
I

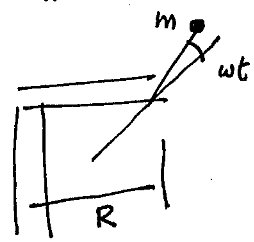
96Hrpm?

$$mv^2/r = mr\omega^2$$

3.8. Rotating unbalanced forces $mr\omega^2$ can be resolved in

$F_y = mr\omega^2 \sin\omega t$ // to y axis

$F_z = mr\omega^2 \cos\omega t$ // to z axis



$$\frac{My}{I} = \sigma_{bend} = \frac{1}{I_z}|M_z|\frac{d_o}{2} = \frac{mr\omega^2 R\left(\frac{d_o}{2}\right)}{\pi/64\,(d_o^4 - d_i^4)}$$

@ center

$$= \frac{(0.1)(0.1)(31.416^2)(0.5)\left(\frac{0.1}{2}\right)}{\frac{\pi}{64}(.1^4 - .08^4)} = 8.5124\times10^4\ N/m^2$$

$$\frac{Tr}{J} = \sigma_{torsion} = \frac{1}{J_y}|M_y|\frac{d_o}{2} = \frac{mr\omega^2 R\left(\frac{d_o}{2}\right)}{\frac{\pi}{32}(d_o^4 - d_i^4)} = 4.2562\times10^4\ N/m^2$$

$$J_y = I_z + I_y$$

3.58 in 4th

\# Do 3.49 important.

3.14 in 4th

3.10

$$\frac{m_o e}{m \cdot 2\zeta} = X\Big|_{r=1} = .55$$

$$\frac{m_o e}{m} = X\Big|_{r\gg1} = .15$$

$\therefore\ \zeta = .1364$

3.55 $P_o = 120$ $\omega_f = 2\pi(2.5173268)$ $k = 2100\ N/m$ $m = 2\ kg$ $\omega_n = \sqrt{\frac{k}{m}}$

$$\frac{P_o/k}{|1-r^2|} = .075$$

3.45 $I = \frac{1}{12}(.5)(.1)^3 = .4167\times10^{-4}\ m^4$ $E = 2.07\times10^{11}\ N/m^2$

$k = \frac{192EI}{\ell^3} = 1.3248\times10^7$

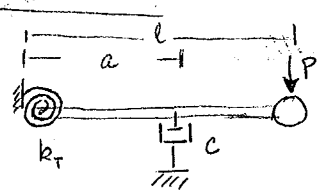
$\omega_n = \sqrt{\frac{K}{m}} = 420.29\ rad/s$ $\omega_f = 2\pi\cdot1200/60 = 125.66$ $r = .3$ $X = .4145\times10^{-3}\ m$

$M = M + .5m$ $\omega_n = \sqrt{\frac{K}{\ }}$ $=$ $m_{beam} = 5\times.5\times.1\times76.5\times10^3/9.81 = 1949.53$

REST OF SESSION #13

EXAMPLE

TORSIONAL SPRING



$T = +k_T\theta$

$T = +c(a\dot{\theta})\,a$ and $P = P_0 \sin\omega_f t$

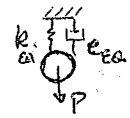
$I = ml^2$ $T = Pl$.

DON'T SPEND SO MUCH TIME

$T_{EQ} = k_{EQ}\,l^2\theta = k_T\theta$

$k_{EQ} = k_T/l^2$

$T_{EQ} = c_{EQ}\,l^2\dot{\theta} = ca^2\dot{\theta} \Rightarrow c_{EQ} = ca^2/l^2$



$m_{EQ}\ddot{x} + c_{EQ}\dot{x} + k_{eq}x = P$

$m\ddot{x} + \frac{ca^2}{l^2}\dot{x} + \frac{k_T}{l^2}x = P_0 \sin\omega_f t$

$ml^2\ddot{\theta} + ca^2\dot{\theta} + k_T\theta = Pl = P_0 l \sin\omega_f t$ now $l\ddot{\theta} = \ddot{x}$, $l\dot{\theta} = \dot{x}$, $l\theta = x$

$\Rightarrow m\ddot{\theta} + ca^2/l^2\,\dot{\theta} + \frac{k_T}{l^2}\theta = \frac{P_0}{l}\sin\omega_f t$

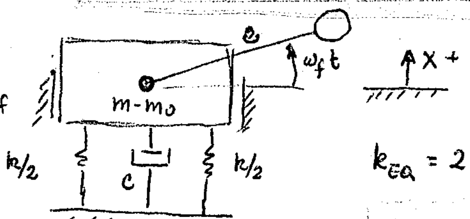
$\Rightarrow \frac{m}{l}\ddot{x} + c\frac{a^2}{l^2}\frac{\dot{x}}{l} + \frac{k_T}{l^2}\frac{x}{l} = \frac{P_0}{l}\sin\omega_f t \Rightarrow m\ddot{x} + \frac{ca^2}{l^2}\dot{x} + \frac{k_T}{l^2}x = P_0 \sin\omega_f t$

now continue with the analysis as normal.

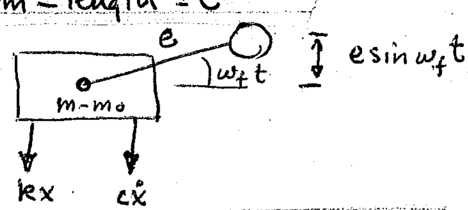
ROTATING UNBALANCE

- COMMON SOURCE OF FORCED VIBRATION DUE TO ROTATION OF ECCENTRIC MASS
- EXAMPLE – CRANKSHAFT ROTATION
- INHERENT PROBLEM – NORMALLY CANNOT CO-LOCATE AXIS OF MASS CENTER & AXIS OF ROTATION
- ASSUME ECCENTRIC MASS IS at end of arm – length = e

ARM ROTATES w/ ANGULAR velocity ω_f



$k_{EQ} = 2(k/2) = k$



$$(m-m_0)\frac{d^2x}{dt^2} + m_0\left[\frac{d^2}{dt^2}(x + e\sin\omega_f t)\right] = -kx - c\dot{x}$$

- measured from static equilib position of total mass $\Delta_{ST} = \underline{mg/k}$

THUS $m\ddot{x} + kx + c\dot{x} = m_0 e\omega_f^2 \sin\omega_f t$ IF we define $P_0 = m_0 e\omega_f^2$

CAN USE RESULTS FROM BEFORE

Rao 2nd 3.24

~~Rao 3rd ed. 3.44~~

Rao 4th ed 3.53

$m = 380\ kg \qquad c = 0 \qquad k = ? \qquad \delta_{st} = \frac{W}{k} = .045 m \qquad k = \frac{380 \cdot 9.81}{.045} \cong 82{,}840 N/m$

$m_o e = .15\ kg\text{-}m \qquad f_f = 1750 rpm \Rightarrow \omega_f = \frac{1750 \cdot 2\pi}{60} = 183.26\ r/s$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}} = \sqrt{\frac{9.81}{.045}} = 14.77\ rad/sec$

$r = \frac{\omega_f}{\omega_n} = \frac{183.26}{14.77} = 12.412$

$$X = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + \cancel{(2\zeta r)^2}^{\,0}}} = \frac{m_o e}{m} \frac{r^2}{|1-r^2|}$$

$$= \frac{.15}{380} \frac{(12.412)^2}{(12.412)^2 - 1} = 3.97 \times 10^{-4} m = .397 mm$$

for $\omega_f \to \infty \qquad X \to \frac{.15}{380} = .0003947$

$$F_{to\ ground} = kX = 82{,}840 \left(\frac{.15}{380}\right) = 32.9 N$$

$$\frac{X}{X_0} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{\frac{1}{1-2\zeta^2}}{\sqrt{\left(1 - \frac{1}{1-2\zeta^2}\right)^2 + \frac{4\zeta^2}{1-2\zeta^2}}}$$

$$\frac{1}{\sqrt{(1-2\zeta^2-1)^2 + 4\zeta^2(1-2\zeta^2)}}$$

$$\frac{1}{\sqrt{4\zeta^4 + 4\zeta^2 - 8\zeta^4}} = \frac{1}{2\zeta\sqrt{1-2\zeta^2}}$$

$$\frac{1}{\sqrt{4\zeta^2 - 4\zeta^4}} =$$

Rao 2nd

~~3.24~~ Rao 3rd ed. 3.44

$m = 380\,kg \quad c = 0 \quad k = ? \quad \delta_{st} = \frac{W}{k} = .045m \quad k = \frac{380 \cdot 9.81}{.045} \cong 82,840\,N/m$

$m_o e = .15\,kg\text{-}m \qquad f_f = 1750\,rpm \Rightarrow \omega_f = \frac{1750 \cdot 2\pi}{60} = 183.26\,rad/s$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{\delta_{st}}} = \sqrt{\frac{9.81}{.045}} = 14.77\,rad/sec$

$r = \frac{\omega_f}{\omega_n} = \frac{183.26}{14.77} = 12.412$

$$X = \frac{m_o e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + \cancel{(2\zeta r)^2}^{\,0}}} = \frac{m_o e}{m} \frac{r^2}{1-r^2}$$

$$= \frac{.15}{380} \frac{(12.412)^2}{(12.412)^2 - 1} = 3.97 \times 10^{-4}\,m = .3973\,mm$$

for $\omega_f \to \infty \qquad X \to \frac{.15}{380} = 0.3947$

$F_{to\ ground} = kX = 82,840\left(\frac{.15}{380}\right) = 32.9\,N$

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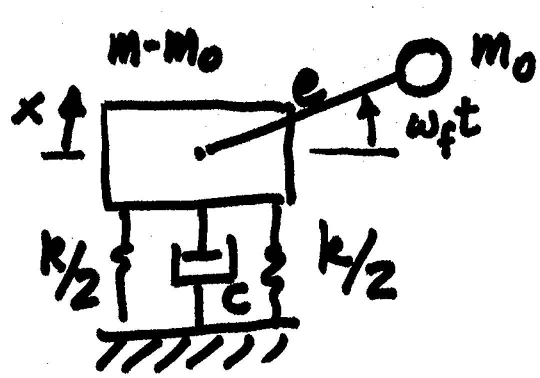
ROTATING UNBALANCE

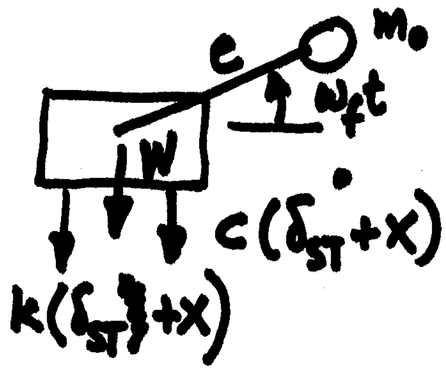


TOTAL SYSTEM MASS = m_o

$$\delta_{st} = \frac{mg}{k}$$

X: IS THE DISPL MEASURED FROM EQUIL OF TOTAL MASS

X: DEFINE FORCED MOTION OF $m - m_o$



$$(m-m_o)\frac{d^2x}{dt^2} + m_o \frac{d^2}{dt^2}(x + e\sin\omega_f t)$$
$$= -k(\cancel{\delta_{st}}+x) - c(\dot{\delta}_{st}+\dot{x}) - \cancel{W}$$

$$m\ddot{x} + c\dot{x} + kx = \underbrace{m_o e \omega_f^2}_{P_o} \sin\omega_f t$$

$$m\ddot{x} + c\dot{x} + kx = P_o \sin\omega_f t$$

$$x_{ss} = x_p = \frac{P_o}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \sin(\omega_f t - \psi)$$

$$\tan\psi = \frac{c\omega_f}{k - m\omega_f^2} = \frac{2\zeta r}{1 - r^2} \qquad \zeta = \frac{c}{c_c} \quad r = \frac{\omega_f}{\omega_n}$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

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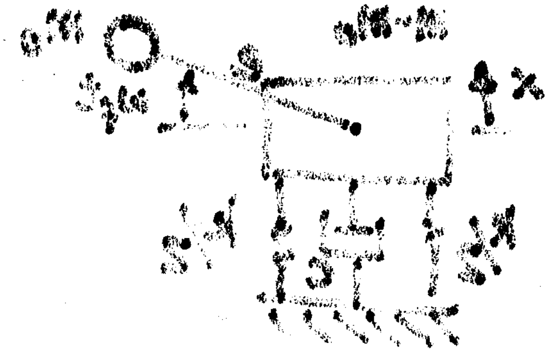
ROTATING UNBALANCE

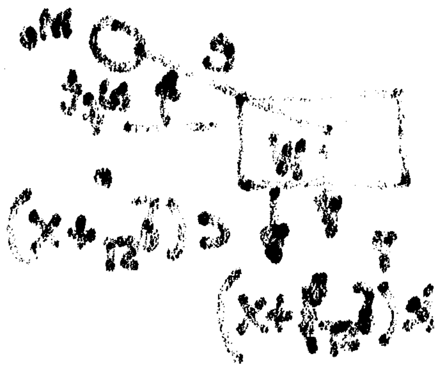
TOTAL SYSTEM MASS = M

$$\delta_{st} = \frac{Mg}{k}$$

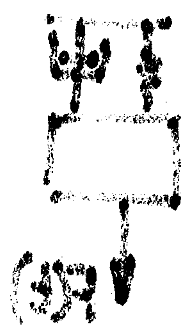


x: IS THE DISPL. MEASURED FROM EQUIL. OF TOTAL MASS

x: DEFINE FORCED MOTION OF $(M-m_0)$



$$(M-m_0)\frac{d^2x}{dt^2} + m_0\frac{d^2}{dt^2}(x + e\sin\omega_f t) = -k(\delta_{st}+x) - c(\dot{\delta}_{st}+\dot{x}) + Mg$$



$$M\ddot{x} + c\dot{x} + kx = \underbrace{m_0 e\omega_f^2}_{P}\sin\omega_f t$$

$$M\ddot{x} + c\dot{x} + kx = P\sin\omega_f t$$

$$X_m = x_p = \frac{P}{\sqrt{(k-M\omega_f^2)^2 + (c\omega_f)^2}}\sin(\omega_f t - \psi)$$

$$\tan\psi = \frac{c\omega_f}{k - M\omega_f^2} = \frac{2\zeta r}{1-r^2}$$

$$r = \frac{\omega_f}{\omega_n} \qquad \omega_n = \sqrt{\frac{k}{M}}$$

FOR ROTATING UNBALANCE

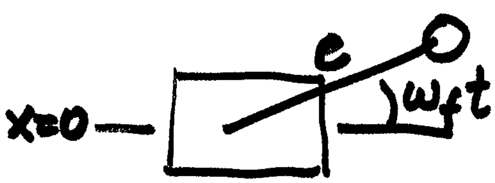
$$x_p = \frac{m_0 e \omega_f^2}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \sin(\omega_f t - \psi)$$

$$\tan\psi = \frac{c\omega_f}{k - m\omega_f^2}$$

TOTAL DISPL = $X_{TRANSIENT} + X_p$

OVERDAMPED   C.D.   UNDERDAMPED

FREE VIBS FOR A DAMPED SYSTEM



WHEN MAIN MASS PASSES THROUGH $X_p = 0 \Rightarrow \psi = \omega_f t$

- THE SAME GRAPHS THAT DEFINE HOW X VARIES WITH m & k FOR  ALSO APPLIES FOR ROTATING UNBALANCE

- VARIATION OF X DUE TO VARIATIONS IN ω_f IS NOT THE SAME

FOR ROTATING UNBALANCE

$$x_p = \frac{m_e \omega_f^2 e}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \sin(\omega_f t - \psi)$$

$$\tan\psi = \frac{c\omega_f}{k - m\omega_f^2}$$

TOTAL DISPL = $x_{TRANSIENT} + x_p$

OVERDAMPED C.D. UNDERDAMPED

FREE VIBS FOR A DAMPED SYSTEM

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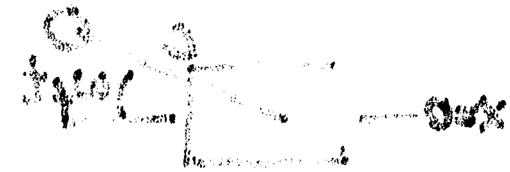


• THE SAME GRAPHS THAT DEFINE HOW X VARIES WITH m & k FOR

ALSO APPLIES FOR ROTATING UNBALANCE

$F(t) = F_0 \sin \omega_f t$

VARIATION OF X DUE TO VARIATIONS IN ω_f IS NOT THE SAME

$$x_p = \frac{m_0 e \omega_f^2}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \sin(\omega_f t - \psi)$$

$$= \frac{m_0 e \omega_f^2 / k \cdot \frac{m}{m}}{\sqrt{\left(1 - \frac{m}{k}\omega_f^2\right)^2 + \left(\frac{c\omega_f}{k}\right)^2}}$$

$$= \underbrace{\boxed{\frac{m_0 e}{m} \bullet \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}}}_{X_{RU}} \sin(\omega_f t - \psi)$$

$$\frac{X_{RU}}{(m_0 e / m)} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\frac{X}{X_0 = P_0/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

$$\frac{m_0 e \omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} \sin(\omega t - \phi)$$

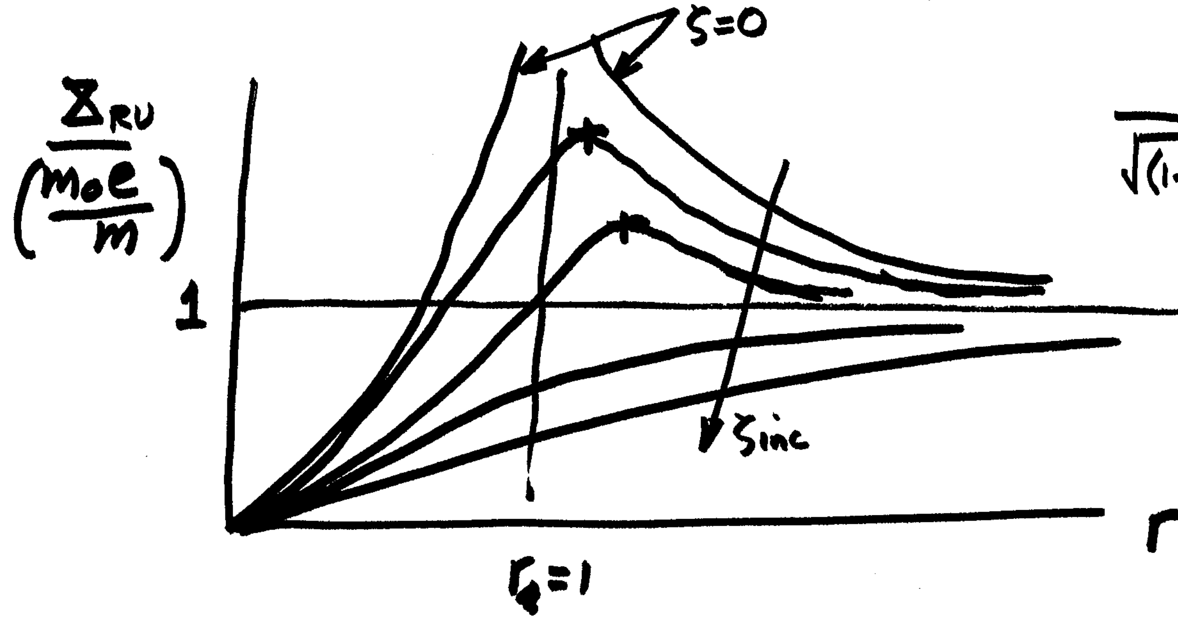
$$\frac{m_0 e \omega^2 / k \cdot \frac{M}{M}}{\sqrt{\left(1 - \frac{M}{k}\omega^2\right)^2 + \left(\frac{c\omega}{k}\right)^2}}$$

$$\boxed{\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \cdot \frac{m_0 e}{M}} \sin(\omega t - \phi)$$

$$Z_{ss}$$

$$\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{Z_{ss}}{\left(m_0 e / M\right)}$$

$$\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{Z}{Z_0 = F_0/k}$$



$$\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

- FOR VERY LARGE ω_f ($r \to \infty$) ALL CURVES TEND TO $\dfrac{X_{RU}}{(m_0 e/m)} = 1$

- $\dfrac{d}{dr}\left(\dfrac{X_{RU}}{m_0 e/m}\right) = 0 \Rightarrow$ max. OCCURS

WHEN $r = \dfrac{1}{\sqrt{1-2\zeta^2}} > 1$

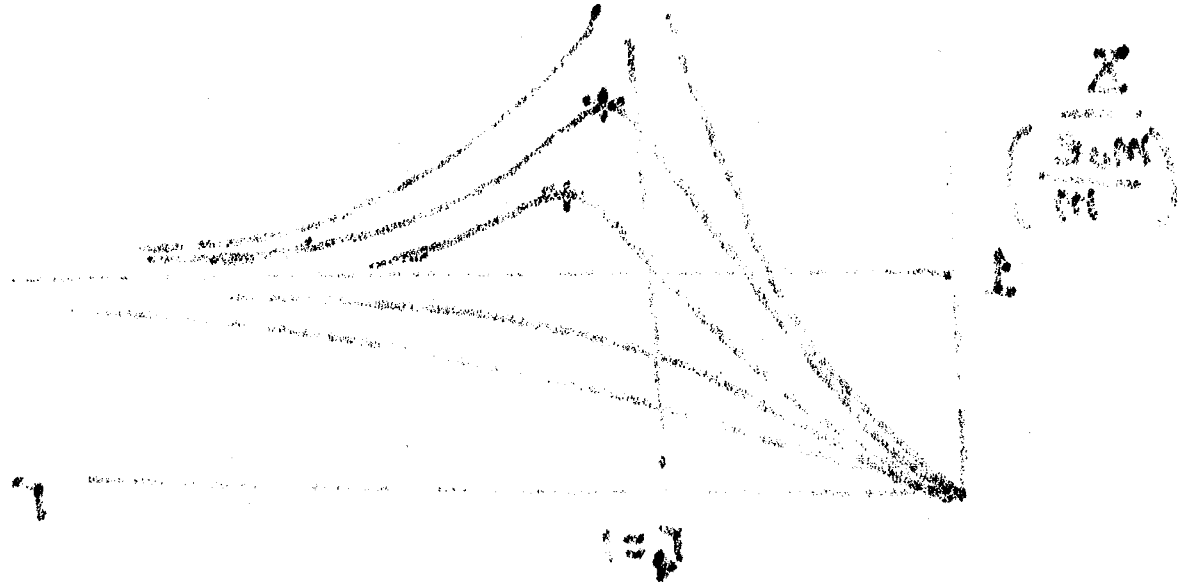
ONLY TRUE UNTIL $\zeta = \dfrac{1}{\sqrt{2}} = 0.707$

r HAS LOCAL MAX WHEN $\zeta \leq \dfrac{1}{\sqrt{2}}$

FOR $\zeta > \dfrac{1}{\sqrt{2}}$ NO <u>LOCAL</u> MAX

$$X_{RU_{max}} = \frac{m_0 e}{m} \cdot \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$ THIS IS FOR LOCAL MAXES $\zeta \leq \dfrac{1}{\sqrt{2}}$

$$X_{RU}\Big|_{r=1} = \frac{m_0 e}{m} \cdot \frac{1}{2\zeta} \; ; \; X_{RU_{max}} = X_{RU}\Big|_{r=1} \cdot \frac{1}{\sqrt{1-\zeta^2}}$$



- THE VERY LARGE γ ($\gamma \to \infty$) ALL CURVES TEND TO $\frac{Z_{max}}{(m_e c^2/m)} = 1$

$0 = \left(\frac{Z_{max}}{m_e c^2/m}\right)\frac{d}{d\xi}$ → MAX. OCCURS

WHEN $\gamma = \frac{1}{\sqrt{1-\xi^2}} > 1$

ONLY TRUE UNTIL $\xi = \frac{1}{\sqrt{3}}$

HAS LOCAL MAX WHEN $\xi \le \frac{1}{\sqrt{3}}$

FOR $\xi > \frac{1}{\sqrt{3}}$ NO LOCAL MAX

THIS IS THE LOCAL MAXIMA $\xi = \frac{1}{\sqrt{3}}$ $\quad \frac{1}{\xi^2\sqrt{1-\xi^2}} \cdot \frac{m_e c^2}{m} = Z_{max}$

$\frac{1}{\xi^2} \cdot \frac{m_e c^2}{m} = Z_{max}$

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the type of mathematical equations employed in describing the system. All types of system models listed in Table 1.1 will be discussed in this book, although distributed models will only be given limited attention.

1.2 System Elements, Their Characteristics, and the Role of Integration

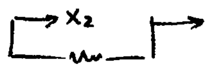
The modeling techniques to be developed in this text will focus initially on the use of a set of simple ideal system elements found in four main types of systems: mechanical, electrical, fluid, and thermal. Transducers, which enable the coupling of these types of system to create mixed system models, will be introduced later.

This set of ideal linear elements is shown in Table 1.2, which also provides their elemental equations and, in the case of energy storage elements, their energy

TABLE 1.2 Ideal System Elements (Linear)

System Type	Mechanical	Electrical	Fluid	Thermal
A-type element	Mass	Capacitor	Fluid capacitor	Thermal capacitor
Elemental equation	$F = m\dfrac{dv}{dt}$	$i = C\dfrac{de}{dt}$	$Q_f = C_f\dfrac{dP}{dt}$	$Q_h = C_t = \dfrac{dT}{dt}$
Energy stored	Kinetic	Electric field	Potential	Thermal
Energy equation	$\mathscr{E}_K = \dfrac{m}{2}v^2$	$\mathscr{E}_E = \dfrac{C}{2}e^2$	$\mathscr{E}_P = \dfrac{C_f}{2}P^2$	$\mathscr{E}_T = \dfrac{C_t}{2}T^2$
T-type element	Spring	Inductor	Inertor	None
Elemental equation	$v = \dfrac{1}{k}\dfrac{dF}{dt}$	$e = L\dfrac{di}{dt}$	$P = I\dfrac{dQ_f}{dt}$	
Energy stored	Potential	Magnetic field	Kinetic	
Energy equation	$\mathscr{E}_P = \dfrac{1}{2k}F^2$	$\mathscr{E}_M = \dfrac{L}{2}i^2$	$\mathscr{E}_K = \dfrac{I}{2}Q_f^2$	
D-type element	Damper	Resistor	Fluid resistor	Thermal resistor
Elemental equation	$F = bv$	$i = \left(\dfrac{1}{R}\right)e$	$Q_f = \left(\dfrac{1}{R_f}\right)P$	$Q_h = \left(\dfrac{1}{R_t}\right)T$
	$v = \dfrac{1}{b}F$	$v = Ri$	$P = R_fQ_f$	$T = R_tQ_h$
Energy dissipation rate	$\dfrac{d\mathscr{E}_D}{dt} = Fv$	$\dfrac{d\mathscr{E}_D}{dt} = ie$	$\dfrac{d\mathscr{E}_D}{dt} = Q_fP$	$\dfrac{d\mathscr{E}_D}{dt} = Q_hT$

Note: Each A-type variable represents a spatial difference across the element.

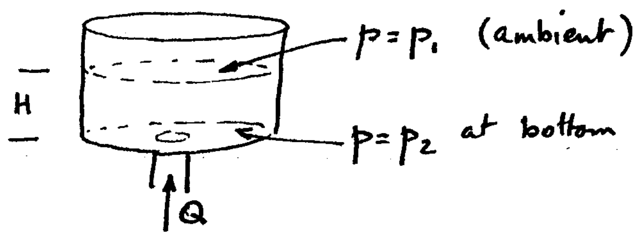
HW's  $x_2 > x_1 \Rightarrow$ compression

if v is continuous $x = \int v\,dt$ is continuous

- FLUID SYSTEMS CAN BE IDEALIZED BY ELEMENTS THAT MODEL FLUID ENERGY STORAGE, DISSIPATION & FLUID TRANSFER
  - FLUID <u>CAPACITANCE</u> STORES ENERGY VIA PRESSURE (ACROSS VAR)
  - FLUID <u>INERTANCE</u> " " VIA FLOW (THROUGH VAR)
  - FLUID <u>RESISTANCE</u> DISSIPATES ENERGY

<u>CAPACITANCE</u>

IN A FLUID RESERVOIR – OPEN TANK IN A GRAVITY FIELD



$\rho Q = \frac{d}{dt}(\rho A H)$ CONSERV. OF MASS

- FOR ρ = const $Q = \frac{d}{dt}(AH)$

- for a reservoir (quasistatic)

$$p_{21} = p_2 - p_1 = \rho g H = g\rho \frac{AH}{A}$$

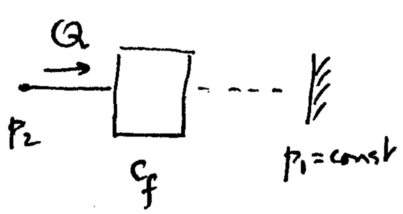
$$\therefore \quad \frac{A}{g\rho}\frac{d}{dt}(p_{21}) = \frac{d}{dt}(AH) = Q$$

if we define $\frac{A}{g\rho} = C_f$ FLUID CAPACITANCE

$$\frac{dV}{dt} = Q = C_f \frac{d}{dt}(p_{21}) \Rightarrow \boxed{\text{volume} = C_f \cdot p_{21}}$$

p = mom. = mass · velocity
q = charge = C · voltage
volume = $C_f \cdot \Delta p$.

- Note that fluid capacitance only has one port through which flow takes place. <u>Pressure is measured with RESPECT TO AMBIENT PRESSURE HERE</u>



SYMBOLIC REPRESENTATION

since $V = C_f\, p_{21}$ & $\frac{dV}{dt} = Q \Rightarrow p_{21} = \frac{1}{C_f}\int Q\,dt + p_{21}^{(t_0)}$

$\left[v_{21} = \frac{1}{m}\int F\,dt + v_{21} \right]$

- also we CAN DEFINE FLUID POTENTIAL ENERGY E_p

or a spring

- FLUID INERTANCE (ANALOGOUS TO electrical [redacted] INDUCTANCE)
  - LOOK AT FLOW THAT IS [redacted] NONSTEADY, FRICTIONLESS, INCOMPRESSIBLE

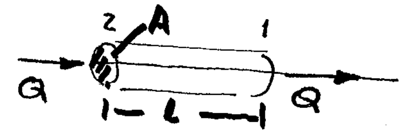
FROM FLUIDS $\frac{\partial \underset{\sim}{v}}{\partial t} + \underset{\sim}{v} \cdot \nabla \underset{\sim}{v} = -\frac{1}{\rho} \nabla p$

and $\frac{\partial \rho}{\partial t} + \underset{\sim}{v} \cdot \nabla \rho = 0$ continuity $\Big\}$ $\frac{\partial (\rho \underset{\sim}{v})}{\partial t} + \underset{\sim}{v} \cdot \nabla(\rho \underset{\sim}{V}) = -\nabla p$

- LOOK AT A NON ACCELERATING PIPE

FOR INCOMPRESSIBLE FLUID $\nabla(\rho \underset{\sim}{V}) = 0 \Rightarrow \rho V_1 A_1 = \rho V_2 A_2 =$ [redacted] $=$ const.

& $\frac{\partial v}{\partial t} = -\frac{1}{\rho}\nabla p = -\frac{1}{\rho l}[p_1 - p_2]$



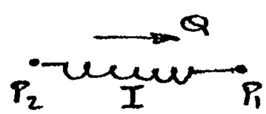
$\therefore \underbrace{\rho l \cdot A}_{\text{fluid mass}} \frac{\partial v}{\partial t} = A p_{21} = (p_2 - p_1) A = \text{force}$; now $Q = Av \Rightarrow A\frac{\partial v}{\partial t} = \frac{\partial Q}{\partial t}$

fluid ~~inertia~~ mass

$\therefore \frac{\rho l}{A} \frac{\partial Q}{\partial t} = P_{21}$ $\qquad \underline{I = \frac{\rho l}{A}}$ = fluid inertance

- define $\frac{d\Gamma_{21}}{dt} = p_{21} \Rightarrow \Gamma_{21} = IQ$ ideal fluid inertance relation

$\underline{p_{21} = I \frac{\partial Q}{\partial t}}$ elemental equation FOR FLUID INERTANCE



in electrical system $\lambda_{21} = Li$ & $v_{21} = L\frac{di}{dt}$

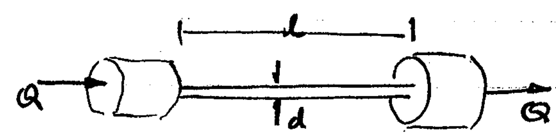
$v_{21} = \frac{1}{K}\, dF/dt$

- energy stored due to fluid inertance is due to kinetic energy of flow

$\underline{\mathcal{E}_k} = \int Q P_{21}\, dt = \int Q\, d\Gamma_{21} = \int Q \cdot I\, dQ = \underline{\tfrac{1}{2} I Q^2} = \frac{1}{2}\frac{\Gamma_{21}^2}{I}$

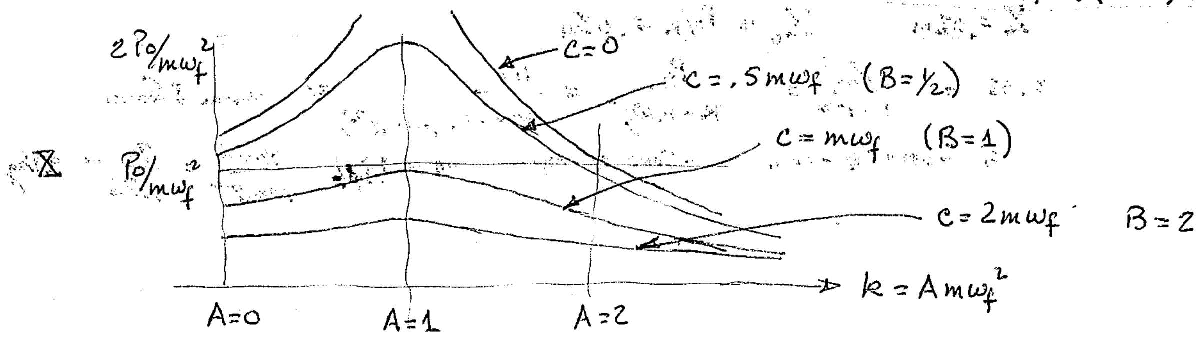
- Γ_{21} is known as the pressure momentum difference $\qquad$ Q is the state variable

- FLOW THRU PIPES NORMAL PRODUCE FRICTION & THERE IS RESISTANCE TO FLOW. ASSUME P_{21} IS SMALL COMPARED TO $P_{Average}$
  - FLOW THROUGH POROUS MEDIUM
  - FLOW THROUGH CAPILLARY TUBE
  - FLOW THROUGH LONG PIPE
  - FLOW THROUGH ORIFICE



$c = B m\omega_f$

let $k = A m\omega_f^2$ $\quad c\omega_f = B m\omega_f^2 \quad \Rightarrow \quad \boxed{X = \frac{P_o}{m\omega_f^2}\frac{1}{\sqrt{(A-1)^2+B^2}}}$



at constant k as $c\uparrow$ $X\downarrow$ as $c\downarrow$ $X\uparrow$

at constant c as $k\uparrow$ $X\uparrow$ TO X_{max} THEN $X\downarrow$

STIFF SPRINGS DECREASE AMPLITUDES

- when $k=0$ $(A=0)$

$$X = \frac{P_o}{\sqrt{(m\omega_f^2)^2+(c\omega_f)^2}}$$

•

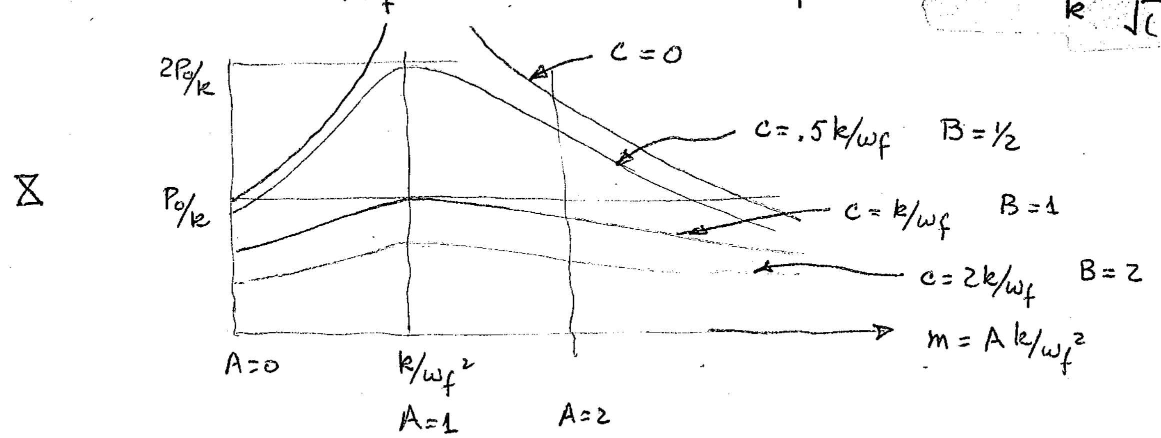
- (mass) VARIATIONS LOOK AT $X = \dfrac{P_o}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}$

$\dfrac{dX}{dm} = 0 \Rightarrow X_{max}$ occurs when $m = \dfrac{k}{\omega_f^2}$

$$\frac{dX}{dm} = \frac{-1}{2}\frac{P_o\cdot 2(k-m\omega_f^2)\omega_f^2}{[\quad]^{3/2}}$$

$$X_{max} = \frac{P_o}{(c\omega_f)} = \frac{P_o/k}{2\zeta r}$$

let $m = Ak/\omega_f^2$ $\quad$ let $c = Bk/\omega_f$ $\quad \Rightarrow \quad \boxed{X = \frac{P_o}{k}\frac{1}{\sqrt{(1-A)^2+(B)^2}}}$



when $m=0$ $\quad X = \dfrac{P_o}{\sqrt{k^2+c^2\omega_f^2}}$ $\qquad$ when $m\to\infty$ $X\to 0$

for constant c as $m\to\infty$ $X\to 0$

for constant m as $c\uparrow$ $X\to 0$

~~3.4~~ 3.8 in 4th ed.

$k = 4000\ N/m \qquad P_0 = 100\ N \qquad 5\ hz = f_f \Rightarrow 10\pi = \omega_f$ damped

$X = .02m \qquad X_0 = P_0/k = .025m$

$$\pm .02 = \frac{X_0}{1-r^2} = \frac{P_0}{k - m\omega_f^2} = \frac{100}{4000 - m(100\pi^2)} \Rightarrow \text{denom} \pm 5000$$

$\Rightarrow +5000 \Rightarrow -\text{mass} \qquad -5000 \Rightarrow +\text{mass} \qquad m \sim \frac{9000}{100\pi^2} = \frac{90}{9.86} \sim 9.1kg$

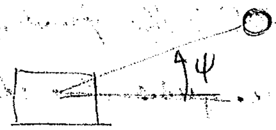
CAVEATS

$\omega = \sqrt{\frac{k}{m}}$

- m includes mass of eccentric mass $(m-m_o)+m_o$
- x measured from equilib position of total mass
- x defines the forced motion of $(m-m_o)$

CAN DEFINE STEADY STATE displ

$x_p = X \sin(\omega_f t - \psi)$



- ψ represents a physical quantity – where arm is wrt horiz when $(\cancel{m-}m_o)$ is at $x=0$

$$\boxed{X} = \frac{P_o}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}} = \frac{m_o e\,\omega_f^2/m\ [m/k]}{\sqrt{[(k-m\omega_f^2)/k]^2+[(c\omega_f)/k]^2}} = \frac{m_o e}{m}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$\tan\psi = \frac{c\omega_f}{k-m\omega_f^2} = \frac{2\zeta r}{1-r^2}$$

- if m_o, e is small we reduce X (importance of reducing eccentricity)
- THE VARIATION OF X versus m, k is same as before if

  $P_o = m_o e \omega_f^2$

  FIG 4-17, 4-18 STILL HOLD variations in k & m still hold

- HOWEVER THE VARIATION OF X vs ω_f IS NOT THE SAME

$$\frac{d}{dr}\left(X/m_o e/m\right) = \frac{2r}{\sqrt{\ }} + -\frac{r^2}{\cancel{2}}\frac{\{\cancel{2}(1-r^2)(-2r)+\cancel{2}(2\zeta r)\cdot 2\zeta\}}{[\ ]^{3/2}} = 0$$

$$\Rightarrow 2r\left(\sqrt{(1-r^2)^2+(2\zeta r)^2}\right)^2 + 2r^2\left\{r(1-r^2) - 2\zeta^2 r\right\} = 0 \quad \Rightarrow r=0 \quad \text{min}$$

also for $\zeta=0$ $r=1$ $\Rightarrow$ absolute maximum

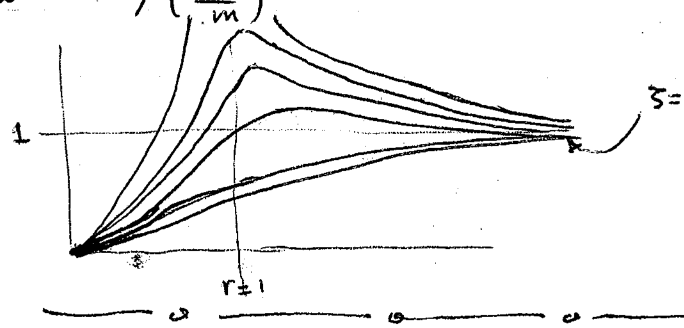
also for $\zeta\neq 0$ $2r\{1+r^2(2\zeta^2-1)\}=0$ $\Rightarrow r=\frac{1}{\sqrt{1-2\zeta^2}} > 1$

MAX. OCCURS TO RIGHT OF $r=1$

$X_{res} = \frac{m_o e}{m}\frac{1}{2\zeta}$

$X_{max} = X_{res}/\sqrt{1-\zeta^2}$

LIM as $r\to\infty$ $X/\left(\frac{m_o e}{m}\right) \to 1$



$\zeta = \frac{1}{\sqrt{2}}$ max occurs at $r=\infty$

4.32 A machine having a total weight of 96.5 lb is mounted on a spring of modulus 900 lb/in and is connected to a dashpot having a damping ratio of 0.25. The machine contains an unbalanced of ($W_0 e$) 5 lb in. If the speed of rotation is 401.1 rpm find (a) the amplitude of steady state motion (b) the max. dynamic force transmitted to the foundation (c) the angular position of the arm when the structure ~~goes~~ thro its neutral position

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MECHANICAL VIBRATIONS

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Problem 3.8 Pg 283 (Problem 3.4 Pg 244 in 3rd ed)

$k = 4000\ N/m$, $P_0 = 100\ N$, $f_f = 5\ Hz$, $X = 20\ mm = .020\ m$

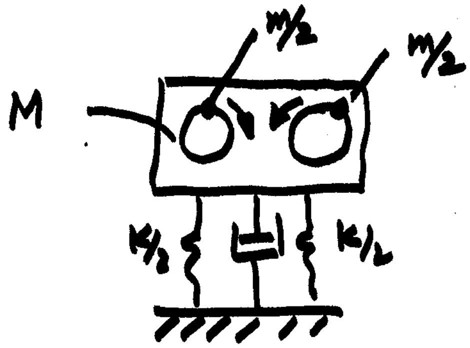
what is m?

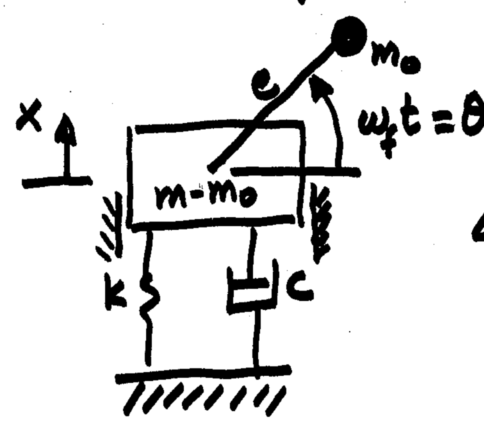
$\omega_f = 2\pi f_f = 10\pi = 31.42\ rad/s$

$$X = \frac{X_0}{1-r^2} = \frac{P_0}{k - m\omega_f^2} = \frac{100}{4000 - m(31.42)^2} = \begin{array}{l}\text{Pick} \\ 0.02 \Rightarrow m<0 \\ -0.02 \Rightarrow m>0\end{array}$$

and $m \cong 9.1\ kg$.

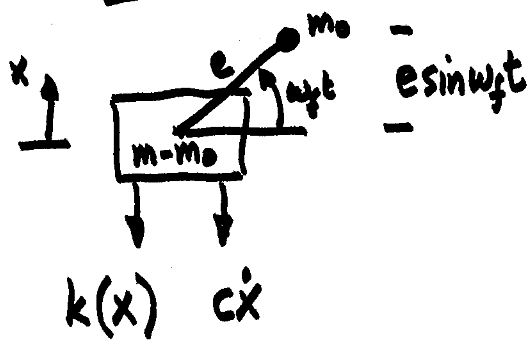
§ 3.7 – Unbalanced Rotating System [Rotating Unbalance]





$$\Delta_{st} = \frac{mg}{k} = \frac{(m-m_0)g + m_0 g}{k}$$

DFBD



m_0's total displ

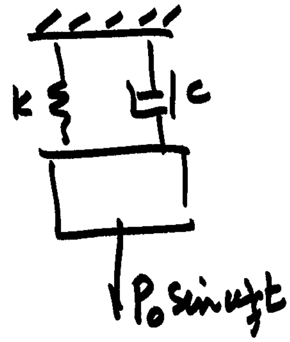
$x + e\sin\omega_f t$

$\Sigma F = \text{mass} \cdot \text{accel}$

$$(-kx) - c\dot{x} = (m-m_0)\frac{d^2x}{dt^2} + m_0 \frac{d^2}{dt^2}\left[x + e\sin\omega_f t\right]$$

$$= m\ddot{x} - m_0 e \omega_f^2 \sin\omega_f t$$

$$\underline{m\ddot{x} + c\dot{x} + kx = m_0 e\, \omega_f^2 \sin\omega_f t}$$



$$m\ddot{x} + c\dot{x} + kx = P_0 \sin \omega_f t$$

$$x_p \text{ (s.s. soln)} = \frac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \sin(\omega_f t - \psi)$$

if $P_0 = m_0 e \omega_f^2$ then we can use solution for forced vib.

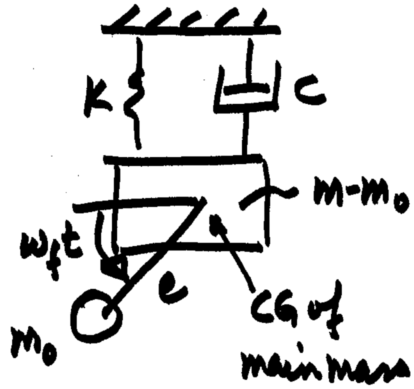
$$m\ddot{x} + c\dot{x} + kx = m_0 e \omega_f^2 \sin \omega_f t$$

$$\underset{\text{(steady state}}{x_{p,RU}} = \underbrace{\frac{m_0 e \omega_f^2}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}}_{X_{RU} = \text{amplitude due to rotating unbalance}} \sin(\omega_f t - \psi) \qquad \tan\psi = \frac{c\omega_f}{k - m\omega_f^2}$$

$$X_{RU} = (m_0 e/m) \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

$$x_{total} = x_{homog} + x_{p,RU}$$

$\hookrightarrow$ $m\ddot{x} + c\dot{x} + kx = 0$ (OD, CD, UD) transients



$$\Psi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \omega_f t$$

when $(m-m_0)$ passes through its equilib. point $(x=0)$

x measures displ of main mass $(m-m_0)$ but static equil. $\Delta_{st} = \frac{W}{k} = \frac{(m-m_0+m_0)g}{k}$

$x_p = X_{RU} \cdot \sin(\omega_f t - \Psi)$

S.S. solution

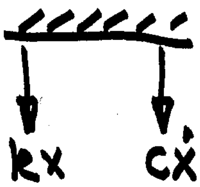
$\underset{\text{S.S.}}{X_{RU}}$ (amplitude due to rotating unbalance)

$$X_{RU} = \frac{m_0 e}{m} \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$= m_0 e \frac{\omega_f^2}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}$$

$$x_{tot} = \underbrace{x_p}_{\text{S.S.}} + \underbrace{x_h \begin{cases} \text{OD} \\ \text{CD} \\ \text{UD} \end{cases}}_{\text{transient}}$$

Forces transmitted to support



$$F_{trans} = kx + c\dot{x} = kX_{RU}\sin(\omega_f t - \Psi) + c\omega_f X_{RU}\cos(\omega_f t - \Psi)$$

$$= \boxed{F_T \sin(\omega_f t - \Psi - \beta)}$$

$$= F_T \sin(\omega_f t - \Psi)\cos\beta - F_T \cos(\omega_f t - \Psi)\sin\beta$$

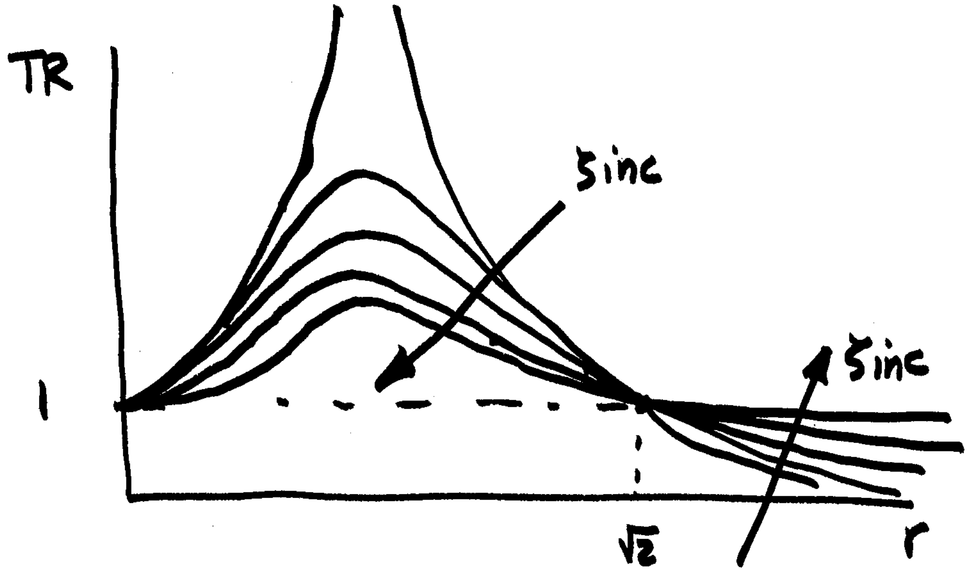
$P_0 = m_0 e \omega_f^2 \sin\omega_f t$

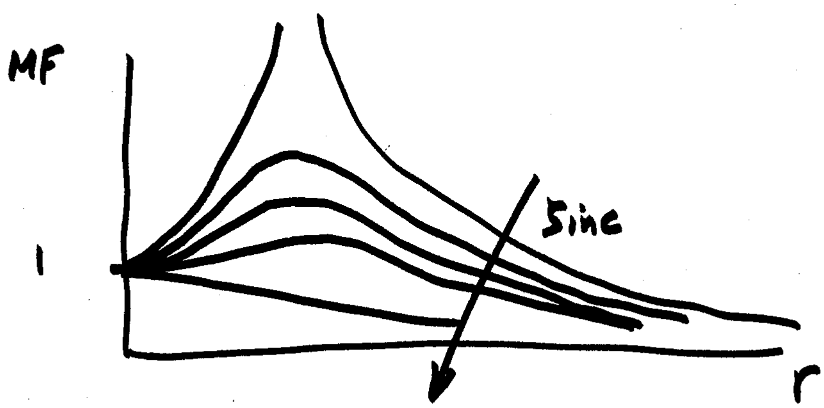
$$\left.\begin{aligned} F_T\cos\beta &= kX_{RU} \\ -F_T\sin\beta &= c\omega_f X_{RU} \end{aligned}\right\} \rightarrow F_T = X_{RU}\sqrt{k^2+(c\omega_f)^2} = kX_{RU}\sqrt{1+(2\zeta r)^2}$$

$$\boxed{\tan\beta = \frac{-c\omega_f}{k} = -2\zeta r}$$

$$\boxed{F_{T,RU} = \frac{\frac{m_0 e}{m}\cdot k r^2 \, \cancel{m_0 e\, k\, \omega_f^2} \sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}} = \boxed{\frac{m_0 e \omega_f^2 \sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(2\zeta r)^2}}}$$

DUE TO ROTATING UNBAL.





in rotating unbalance case

$$F_{T,RU} = \frac{\frac{m_0 e}{m} k r^2 \sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \quad ; \quad \frac{F_{T,RU}}{\frac{m_0 e}{m} k} = \frac{r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

Problem ~~3.43~~ 3.52 Erratum

- The main mass for this problem is the blades drive system + the rotor blades LESS the eccentric mass, i.e., 20 kg − 0.5 kg = 19.5 kg = $(m-m_0)$
- The eccentric mass $m_0 = 0.5$ kg, $e = 0.15$ m
- $\zeta_{TAIL} = 0.15$

The tail section ONLY acts to support the drive system and blades and thus provides the elasticity (spring equivalent) and the damping (viscous equivalent) for the mass (main + eccentric). Tail information is used to get k

$$k_{eq} = \frac{3EI}{l^3} = \frac{3\,(2.5\times10^6\ \text{N-m}^2)}{(4)^3\ \text{m}^3} = 1.172 \times 10^5\ \frac{\text{N}}{\text{m}}$$

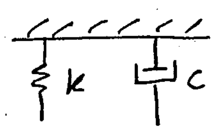
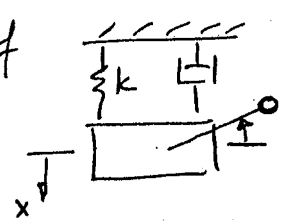
- the tail has weight and so it acts like a spring having weight. This is important for $\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}}$

$$m_{eq} = m_{\substack{drivesys\\+blades}} + 0.25\, m_{tail} = 20 + 0.25(240) = 80\ \text{kg}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{1.172\times10^5}{80}} = 38.27\ \text{rad/sec}$$

$$\omega_f = 1500\,\frac{\text{rev}}{\text{min}} \times \frac{2\pi\,\text{rad}}{\text{rev}} \times \frac{1\,\text{min}}{60\,\text{sec}} = 50\pi\ \text{rad/sec} = 157.08\ \text{rad/sec}$$

$$r = \frac{\omega_f}{\omega_n} = \frac{157.08}{38.27} = 4.1$$

- This work so far is like finding  part of 
- THE MASS OF TAIL DOES NOT PLAY A PART IN x_p
- Now to find the actual forced response, we need to use

$$x_p = \frac{m_0 e}{m} \cdot \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \sin(\omega_f t - \psi)$$

where $m = m_{drivesy+blade} = 20$ kg
$m_0 = 0.5$ kg $\quad r = 4.1$
$e = 0.15$ m $\quad \zeta = 0.15$

$$\left.\right\} \longrightarrow \underline{X_{RU} = 0.003975\,m}$$

when $\omega_f \to \infty \Rightarrow r \to \infty$ $\quad X_{RU} = .00375\,m = \frac{m_0 e}{m}$

$$\psi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = -4.45^\circ$$

WHEN MAIN MASS IS AT X=0, ECCENTRICITY WILL BE AT AN ANGLE OF 4.45° BELOW HORIZONTAL

3.17 $\quad W = 500\,lb \qquad F(t) = 200 \sin 100\pi t \qquad X < .1\,in$

Choose $\quad X = .05\,in$

$$X_{max} = \frac{X_0}{2\zeta\sqrt{1-\zeta^2}} = .05\,in \qquad X_0 = P_0/K$$

Choose $\zeta = 0.01$ Then $X_0 = \dfrac{P_0}{k_{eq}} = \dfrac{200}{k_{eq}}$

$$X_{max} = \frac{X_0}{2(.01)\sqrt{1-(.01)^2}} = \frac{X_0}{.02} = \frac{200}{k_{eq}(.02)} = .05\,in \therefore \; k_{eq} = \frac{2000}{.05(.02)} = 20\times10^4 \frac{lb}{in}$$

$$\frac{k_{eq}}{3} = k = 6.667\times10^4 \frac{lb}{in} \qquad \text{since 3 springs}$$

$$\zeta = \frac{C_{eq}}{C_{cr}} \qquad C_{cr} = 2\sqrt{k_{eq} m} = 2\sqrt{\left(20\times10^4 \frac{lb}{in}\right)\left(\frac{500\,lb}{386.4\frac{in}{sec^2}}\right)} = \cancel{7195}\;719.5 \frac{lb\text{-}sec}{in}$$

32.2×12

$$C_{eq} = \zeta C_c = .01(\cancel{7195}\;719.5) = 7.195 \frac{lb\text{-}sec}{in}$$

$$c = \frac{C_{eq}}{3} = 2.4 \frac{lb\text{-}sec}{in} \qquad \text{since 3 absorbers}$$

Problem 9.29 Pg 733

$m = 500\,kg \quad m_o e = 50\,kg\cdot cm \quad f_f = 300\,rpm \cdot \frac{2\pi}{60} = 31.42\,rad/s$

a) $k \quad c = 0$

b) $\zeta = 0.1 \quad k =$

$\Big\}\ TR,\ \Delta_{st},\ X,\ X_o = P_o/k$

a) For $c=0$ $\quad TR = 0.1$ (Chosen) $\quad \frac{F_T}{P_o} = TR = \frac{1}{|1-r^2|} = \frac{\sqrt{1+\cancel{(2\zeta r)^2}}}{\sqrt{(1-r^2)^2+\cancel{(2\zeta r)^2}}}$

$\Rightarrow r^2 = 11 = \frac{\omega_f^2}{\omega_n^2} = \frac{\omega_f^2 m}{k} \quad \Rightarrow k = \frac{1}{11} m\omega_f^2$

$\hookrightarrow r = 3.32 \longrightarrow \omega_n = \frac{\omega_f}{r} = 9.47\,rad/s \qquad k = \omega_n^2 m = 44863\,N/m$

$\Delta_{st} = \frac{W}{k} = \frac{500\,kg \cdot 9.81}{44863} = 0.1\,m$

$X_o = P_o/k = \frac{m_o e\,\omega_f^2}{k} = \frac{50\times10^{-2}\,kg\text{-}m\,(31.42)^2}{44863} = 0.011\,m$

$X_{RU} = \frac{m_o e}{m}\frac{r^2}{\sqrt{(1-r^2)^2+\cancel{(2\zeta r)^2}}} = \frac{11\,(50\,kg\text{-}cm)}{500} = .0011\,m$

b) Chosen $TR = 0.1 \qquad \zeta = 0.1$

$TR = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = r^2 = 13.37,\ \cancel{-7.-}$

$r = 3.66$

$\omega_n = \omega_f/r = 8.59\,rad/s \qquad k = m\omega_n^2 = 36919\,N/m$

$\Delta_{st} = \frac{W}{k} = 0.125\,m \qquad X_o = P_o/k = 0.013\,m$

$X_{RU} = \frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\ \frac{m_o e}{m} = .001\,m$

QUIZ 10/25 on Ch 3 (General & Rotating Unbalance)

• FOR $\zeta > \frac{1}{\sqrt{2}}$ THERE IS NO RELATIVE MAX BUT as $r \to \infty$ $X / \frac{m_0 e}{m} \to 1$

SESSION # 14

Problem 4-32

Given $m = \frac{W}{g} = \frac{96.5\ lb}{32.2\ ft/sec^2 \cdot 12\frac{in}{ft}} = .2497\ \frac{lb\,sec^2}{in}$; $k = 900\ lb/in$

$\zeta = \frac{c}{c_c} = .25$; given $W_0 e = 5\ lb\text{-}in$ Speed of rotation is 401.1 rpm

since 1 rev = 2π rad. $401.1\ rpm \cdot \frac{2\pi\ rad}{rev} \cdot \frac{1\ min}{60\ sec} = \omega_f = 42.003$ rad/sec

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}} = \sqrt{\frac{900\ lb/in \cdot 32.\ ft/sec^2 \cdot 12\ in/ft}{96.5\ lb}} = 60.041\ rad/sec$$

$r = \frac{\omega_f}{\omega_n} = .7$

$$X = \frac{m_0 e}{m} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \left(\frac{5\ lb\text{-}in}{96.5}\right) \cdot \frac{(.7)^2}{\sqrt{(.51)^2 + [2(.25)(.7)]^2}} = \frac{.49}{\sqrt{.3826}}$$

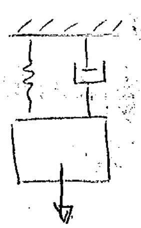
(a) $X = .041$ in

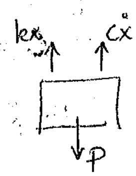
(c) $\tan\psi = \frac{2\zeta r}{1-r^2} = .6863$ $\psi = 34.46°$

HW 4-35, 4-33, 4-41, 4-40

• FORCE TRANSMISSION & ISOLATION

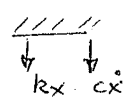
• FORCE TRANSMITTED TO SUPPORT IMPORTANT BESIDES AMPLITUDE X

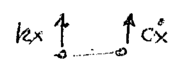




ON THE MASS

ON THE SUPPORT





ON SYSTEM BY SUPPORT

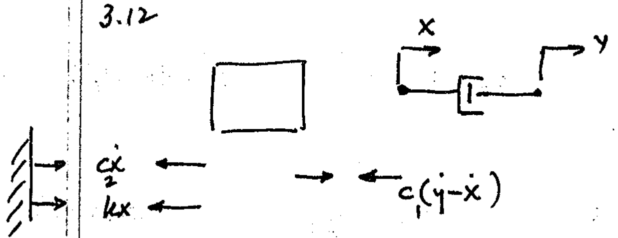
DYNAMIC FORCE $F = kx + c\dot{x}$

since FOR STEADY STATE $x_p = X \sin(\omega_f t - \psi)$, $F = kX \sin(\omega_f t - \psi) + c\omega_f X \cos(\omega_f t - \psi)$

$$X = \frac{P_0/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad \psi = \tan^{-1} \frac{2\zeta r}{1-r^2}$$

Rao 2nd ed

3.12



$$F_T = kx + c_2\dot{x}$$

$$m\ddot{x} = -c_2\dot{x} - kx + c_1(\dot{y}-\dot{x})$$

$$m\ddot{x} + \underset{c_1+c_2}{\cancel{c}}\dot{x} + kx = c_1\dot{y}$$

$$m\ddot{x} + c_{eq}\dot{x} + kx = \underbrace{-\overset{\omega_f}{c_1}\,Y}_{P_0} \sin\omega_f t$$

$$x_p = \frac{-c_1\omega_f Y}{\sqrt{(k-m\omega_f^2)^2 + (c_{eq}\omega_f)^2}} \sin(\omega_f t - \psi) = \frac{-c_1 Y\omega_f/k}{\sqrt{(1-r^2)^2 + (2r[\zeta])^2}} \sin(\omega_f t - \psi)$$

$$\zeta = \frac{c_{eq}}{c_c} = \zeta_1 + \zeta_2$$

$$c_c = 2\sqrt{km}$$

$$\tan\psi = \frac{c_{eq}\omega_f}{k - m\omega_f^2} = \frac{2r(\zeta_1+\zeta_2)}{1-r^2}$$

$$F_T = kX + c_2\dot{X} = \frac{-c_1 Y\omega_f}{\sqrt{\quad}} \sin(\quad) + \frac{c_2\,(-c_1 Y\omega_f/k)\,\omega_f}{\sqrt{\quad}} \cos(\quad)$$

$$= \frac{-c_1 Y\omega_f}{\sqrt{\quad}} \left[\sin(\quad) + c_2/k\,\omega_f \cdot \cos(\quad) \right]$$

$$\frac{-c_1 Y\omega_f \sqrt{1+\left(\frac{c_2\omega_f}{k}\right)^2}}{\sqrt{\quad}} \sin(\omega_f t - \psi - \beta)$$

$$C\cos\beta = 1 \qquad -C\sin\beta = c_2/k\,\omega_f$$

$$C = \sqrt{1+\left(\frac{c_2\omega_f}{k}\right)^2} = \cancel{\quad}$$

$$\tan\beta = -c_2\omega_f/k = -2\zeta_2 r$$

$$\therefore \cancel{-\quad}$$

$$\frac{F_T}{Yk} = \frac{-2\zeta_1 r\sqrt{1+(2\zeta_2 r)^2}}{\sqrt{(1-r^2)^2 + (2r[\zeta_1+\zeta_2])^2}} \sin(\omega_f t - \psi - \beta)$$

the spring due to the maximum impressed force. (c) Obtain the amplitude of the forced motion for the following forced frequencies f_f: 1.5, 3, 3.25, and 7.5 Hz.

4-2. The mass of an undamped mass-spring system is subjected to a harmonic force having a maximum value of 45 lb and a period of 0.25 sec. The body weighs 2.5 lb and the spring constant is 15 lb/in. Determine the amplitude of the forced motion.

4-3. An undamped system is composed of mass $m = 1.1$ kg and spring $k = 4400$ N/m. It is acted on by a harmonic force having maximum value of 440 N and a frequency of 180 cycles/min. Determine the amplitude of the forced motion.

4-4. A weight of 8 lb is suspended from a spring having a constant of 25 lb/in. The system is undamped but the mass is subjected to a harmonic force with a frequency of 7 Hz, resulting in a forced-motion amplitude of 1.59 in. Determine the maximum value of the impressed force.

4-5. The body of an undamped system is driven by a harmonic force having an amplitude of 36 N and a frequency of 450 cycles/min. The body has a mass of 8 kg and exhibits a forced-displacement amplitude of 4.06 mm. Obtain the value of the spring modulus.

4-6. An undamped system consists of an 8.75-kg mass suspended from a spring having a constant of 3500 N/m. A harmonic force acting on the mass and having a maximum value of 187 N causes the system to vibrate with a forced amplitude of 7.6 cm. Determine the frequency of the impressed force.

4-7. A weight W is suspended from a spring having a constant of 25 lb/in. The system is undamped, but the mass is driven by a harmonic force having a frequency of 2 Hz and a maximum value of 10 lb, causing a forced-motion amplitude of 0.5 in. Determine the value of weight W.

4-8. The mass of an undamped system is acted on by force $P = P_0 \cos \omega_f t$, having a maximum value of 10 lb. The spring constant is 3 lb/in. The forced frequency can be any integral multiple of the natural frequency of the system other than 1, thereby avoiding resonance. Only a single forced frequency can exist at a time. Determine the maximum amplitude of the forced motion that could occur.

4-9. Express the relation for the complete motion of an undamped system, excited by a force $P = P_0 \sin \omega_f t$, for the initial conditions of $x = 0$, $\dot{x} = 0$, at $t = 0$. ($r \neq 1$.)

4-10. Write the relation that defines the complete motion of an undamped system, subjected to the harmonic force $P = P_0 \sin \omega_f t$, for the initial conditions of $x = 0$, $\dot{x} = \dot{x}_0$, at $t = 0$. ($r \neq 1$.)

4-11. Express the equation for the complete motion of an undamped system, subjected to the forcing condition $P = P_0 \sin \omega_f t$, for the initial conditions of $x = x_0$, $\dot{x} = 0$, at $t = 0$. ($r \neq 1$.)

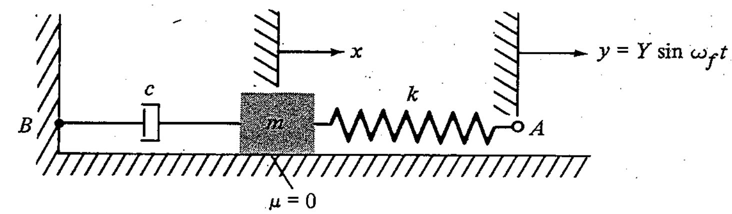
4-12. Write the complete solution for the motion of an undamped system, driven by the force $P = P_0 \sin \omega_f t$, for the initial conditions of $x = x_0$, $\dot{x} = \dot{x}_0$, at $t = 0$. ($r \neq 1$.)

4-13. An undamped system consists of a weight of 19.3 lb and a spring having a modulus of 10 lb/in. The system mass is driven at resonance by harmonic force $P = P_0 \sin \omega_f t$ having a maximum value of 4 lb. Determine the amplitude of the forced motion at the end of (a) $\frac{1}{2}$ cycle, (b) $2\frac{1}{2}$ cycles, (c) $4\frac{1}{2}$ cycles, and (d) $6\frac{1}{2}$ cycles.

4-14. An undamped system is composed of a 4.375-kg mass and a spring having a modulus of 3500 N/m. The mass is driven at resonance by a harmonic force having a maximum value of 14 N. Determine the amplitude of the forced motion at the end of (a) 1 cycle, (b) 5 cycles, and (c) 10 cycles.

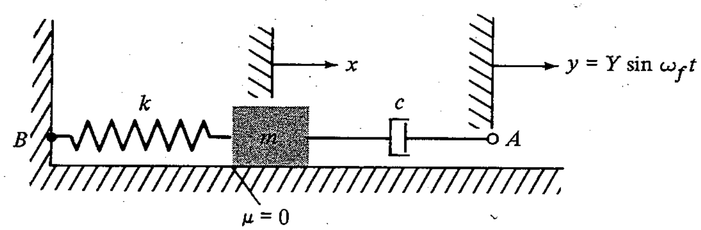
4-15. An undamped system is harmonically forced near resonance, resulting in a beating condition. The natural frequency of the system is 1765 cycles/min, and the forced frequency is 1752 cycles/min. Determine the beat period of the motion.

4-16. For the arrangement shown, x represents the absolute displacement of the mass m, and y is the absolute displacement of the end A of the spring k. Point A is moved according to the relation $y = Y \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation of motion for m. (c) Obtain the solution for the steady-state motion of m. (d) Determine the relation for the impressed force at A. (e) Obtain the expression for the force transmitted to the support at B.



Problem 4-16

4-17. For the arrangement shown, x represents the absolute displacement of the mass m, and y is the absolute displacement of the end A of the dashpot c. The motion of point A is defined by $y = Y \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation of motion for m. (c) Obtain the solution for the steady-state motion of m. (d) Determine the relation for the impressed force at A. (e) Obtain the expression for the force transmitted to the support at B.



Problem 4-17

4-18. The spring k and the dashpot c of the accompanying diagram are fastened together at A; x represents the absolute displacement of m, and y is the absolute displacement of the point A. The motion of A is defined by $y = Y \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation of motion for m. (c) Obtain the solution for the steady-state motion of m. (d) Determine the relation for the impressed force at A.

Undamped system consists of a body weighing 14.475 lb supported by elastic member having $k = 15$ lb/in. Acted on by harmonic force whose max is 24 lb

find $\frac{W}{k}$ P_0/k

his displ. the spring due to the maximum impressed force. (c) Obtain the amplitude of the forced motion for the following forced frequencies f_f: 1.5, 3, 3.25, and 7.5 Hz.

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4-7. A weight W is suspended from a spring having a constant of 25 lb/in. The system is undamped, but the mass is driven by a harmonic force having a frequency of 2 Hz and a maximum value of 10 lb, causing a forced-motion amplitude of 0.5 in. Determine the value of weight W.

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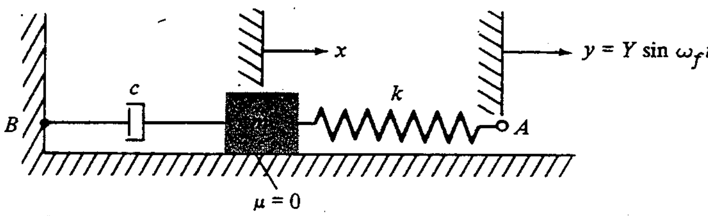

Problem 4-16

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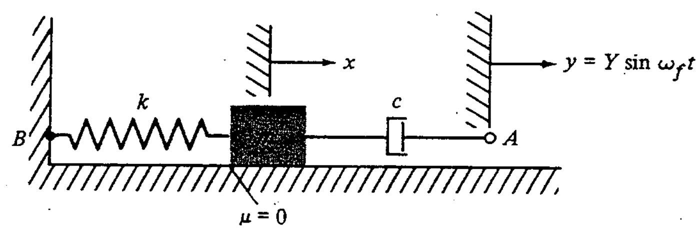


Problem 4-17

4-18. The spring k and the dashpot c of the accompanying diagram are fastened together at A; x represents the absolute displacement of m, and y is the absolute displacement of the point A. The motion of A is defined by $y = Y \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation of motion for m. (c) Obtain the solution for the steady-state motion of m. (d) Determine the relation for the impressed force at A.

HW#8

4-2, 6, 13

4-2

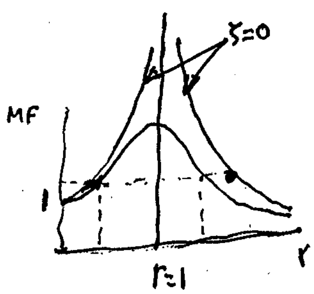
$P = P_o \sin \omega_f t$: $P_o = 45\,lb$ $\quad \tau = .25\,sec = \frac{2\pi}{\omega_f}$; $W = 2.5\,lb$ $\quad k = 15\,lb/in$

find $X = \frac{X_o}{1-r^2}$ $\quad r < 1$

① $\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}} = \sqrt{\frac{15 \cdot 32.2 \cdot 12}{2.5}} = 48.15\,rad/sec$ $\Big\}$ $r = \frac{\omega_f}{\omega} = .522$

② $\omega_f = \frac{2\pi}{\tau} = 8\pi = 25.133\,rad/sec$

$X = \frac{P_o/k}{1-r^2} = \frac{3\,in}{.7275} = 4.123\,in$



4-6 $\quad m = 8.75\,kg$ $\quad k = 3500\,N/m$ $\quad P_o = 187\,N$ $\quad X = 7.6\,cm$

find ω_f (or f_f)

$\omega_n = \sqrt{\frac{k}{m}} = 20\,rad/sec$; $\quad X_o = P_o/k = .0534\,m = 5.34\,cm$ $\quad MF = 1.423 = \frac{X}{X_o}$

Since $MF > 1$ there are 2 solutions. First assume $r < 1$

$X = \frac{X_o}{1-r^2} \Rightarrow 1 - r^2 = \frac{X_o}{X} \Rightarrow r = \sqrt{1 - \frac{X_o}{X}} = .545$

$r = \frac{\omega_f}{\omega_n} \Rightarrow \omega_f = r\omega_n = 10.9\,rad/sec$ $\quad f_f = \frac{\omega_f}{2\pi} = 1.735\,Hz$

Now Assume $r > 1 \Rightarrow X = \frac{X_o}{r^2-1} \Rightarrow r^2 - 1 = \frac{X_o}{X} \Rightarrow r = \sqrt{1 + \frac{X_o}{X}} = 1.305$

$r = \frac{\omega_f}{\omega_n} \Rightarrow \omega_f = r\omega_n = 26.097\,rad/sec$ $\quad f_f = \frac{\omega_f}{2\pi} = 4.153\,Hz$

4-13

$W = 19.3\,lb$, $k = 10\,lb/in$; $P = P_o \sin \omega_f t$ where $P_o = 4\,lb$

system is in resonance $r = 1$. From work in class

$x_p = -\frac{X_o \omega_f t}{2} \cos \omega_f t$ $\quad X_o = P_o/k = .4\,in$

$\omega = \omega_f = \sqrt{\frac{kg}{W}} = 14.149\,rad/sec$

@ $\frac{1}{2}$ cycle $\quad \omega_f t = \pi$ $\quad x_p = -\frac{(.4)}{2}\pi \cos(\pi) = .2\pi = .628\,in$

@ $2\frac{1}{2}$ cycles $\quad \omega_f t = 5\pi$ $\quad x_p = -\frac{(.4)}{2} 5\pi \cos(5\pi) = \pi = 3.142\,in$

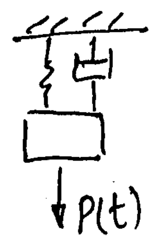
@ $4\frac{1}{2}$ cycles $\quad \omega_f t = 9\pi$ $\quad x_p = -\frac{(.4)}{2} 9\pi \cos(9\pi) = 1.8\pi = 5.655\,in$

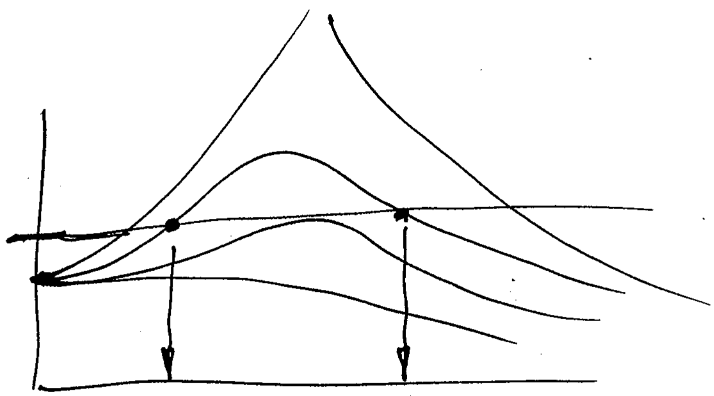
@ $6\frac{1}{2}$ cycles $\quad \omega_f t = 13\pi$ $\quad x_p = -\frac{(.4)}{2} 13\pi \cos(13\pi) = 2.6\pi = 8.168\,in$

EML 4220

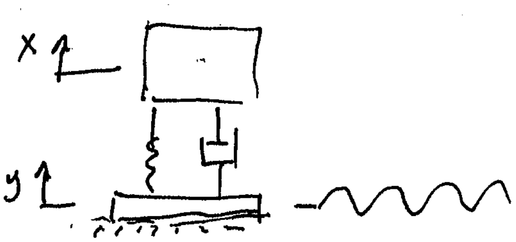
MECH. VIBS - PROBLEM SOLVING

CH. 3, 9, 10 RAO





$$F_{trans} = \frac{P_0 \sqrt{k^2 + (c\omega_f)^2}}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} = X \sqrt{k^2 + (c\omega_f)^2}$$



$y = Y \sin \omega_f t$

Undamped system consists of a body weighing 14.475 lb supported by elastic member having k=15 lb/in, acted on by harmonic force whose max is 24 lb. Find $\frac{W}{k}$ P_0/k

the displ. the spring due to the maximum impressed force. (c) Obtain the amplitude of the forced motion for the following forced frequencies f_f: 1.5, 3, 3.25, and 7.5 Hz.

4-2. The mass of an undamped mass-spring system is subjected to a harmonic force having a maximum value of 45 lb and a period of 0.25 sec. The body weighs 2.5 lb and the spring constant is 15 lb/in. Determine the amplitude of the forced motion.

4-3. An undamped system is composed of mass $m = 1.1$ kg and spring $k = 4400$ N/m. It is acted on by a harmonic force having maximum value of 440 N and a frequency of 180 cycles/min. Determine the amplitude of the forced motion.

4-4. A weight of 8 lb is suspended from a spring having a constant of 25 lb/in. The system is undamped but the mass is subjected to a harmonic force with a frequency of 7 Hz, resulting in a forced-motion amplitude of 1.59 in. Determine the maximum value of the impressed force.

4-5. The body of an undamped system is driven by a harmonic force having an amplitude of 36 N and a frequency of 450 cycles/min. The body has a mass of 8 kg and exhibits a forced-displacement amplitude of 4.06 mm. Obtain the value of the spring modulus.

4-6. An undamped system consists of an 8.75-kg mass suspended from a spring having a constant of 3500 N/m. A harmonic force acting on the mass and having a maximum value of 187 N causes the system to vibrate with a forced amplitude of 7.6 cm. Determine the frequency of the impressed force.

4-7. A weight W is suspended from a spring having a constant of 25 lb/in. The system is undamped, but the mass is driven by a harmonic force having a frequency of 2 Hz and a maximum value of 10 lb, causing a forced-motion amplitude of 0.5 in. Determine the value of weight W.

4-8. The mass of an undamped system is acted on by force $P = P_0 \cos \omega_f t$, having a maximum value of 10 lb. The spring constant is 3 lb/in. The forced frequency can be any integral multiple of the natural frequency of the system other than 1, thereby avoiding resonance. Only a single forced frequency can exist at a time. Determine the maximum amplitude of the forced motion that could occur.

4-9. Express the relation for the complete motion of an undamped system, excited by a force $P = P_0 \sin \omega_f t$, for the initial conditions of $x = 0$, $\dot{x} = 0$, at $t = 0$. $(r \neq 1.)$

4-10. Write the relation that defines the complete motion of an undamped system, subjected to the harmonic force $P = P_0 \sin \omega_f t$, for the initial conditions of $x = 0$, $\dot{x} = \dot{x}_0$, at $t = 0$. $(r \neq 1.)$

4-11. Express the equation for the complete motion of an undamped system, subjected to the forcing condition $P = P_0 \sin \omega_f t$, for the initial conditions of $x = x_0$, $\dot{x} = 0$, at $t = 0$. $(r \neq 1.)$

4-12. Write the complete solution for the motion of an undamped system, driven by the force $P = P_0 \sin \omega_f t$, for the initial conditions of $x = x_0$, $\dot{x} = \dot{x}_0$, at $t = 0$. $(r \neq 1.)$

4-13. An undamped system consists of a weight of 19.3 lb and a spring having a modulus of 10 lb/in. The system mass is driven at resonance by harmonic force $P = P_0 \sin \omega_f t$ having a maximum value of 4 lb. Determine the amplitude of the forced motion at the end of (a) $\frac{1}{2}$ cycle, (b) $2\frac{1}{2}$ cycles, (c) $4\frac{1}{2}$ cycles, and (d) $6\frac{1}{2}$ cycles.

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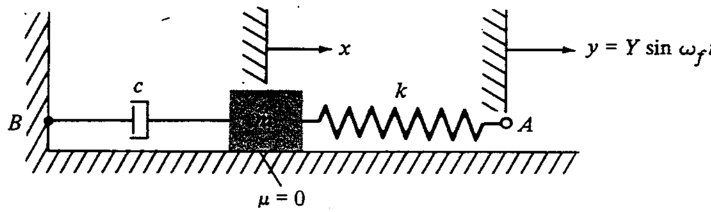

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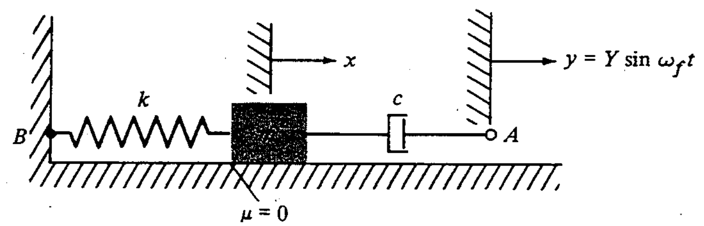

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HW#8

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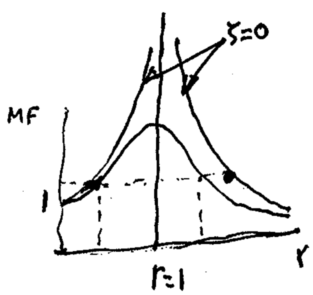

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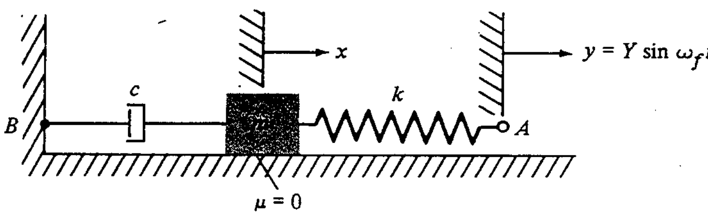
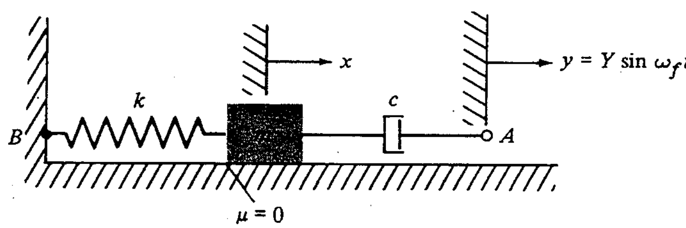


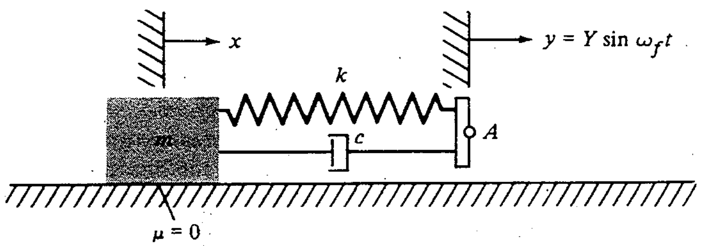
Problem 4-16

4-17. For the arrangement shown, x represents the absolute displacement of the mass m, and y is the absolute displacement of the end A of the dashpot c. The motion of point A is defined by $y = Y \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation of motion for m. (c) Obtain the solution for the steady-state motion of m. (d) Determine the relation for the impressed force at A. (e) Obtain the expression for the force transmitted to the support at B.



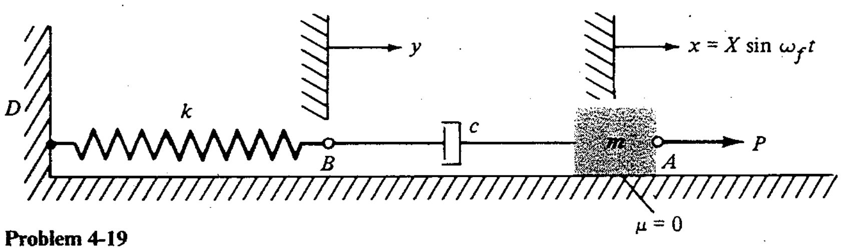
Problem 4-17

4-18. The spring k and the dashpot c of the accompanying diagram are fastened together at A; x represents the absolute displacement of m, and y is the absolute displacement of the point A. The motion of A is defined by $y = Y \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation of motion for m. (c) Obtain the solution for the steady-state motion of m. (d) Determine the relation for the impressed force at A.



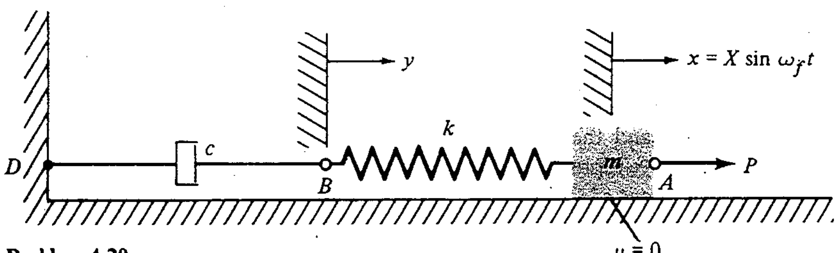
Problem 4-18

4-19. For the arrangement shown, x represents the absolute displacement of the mass m, and y is the absolute displacement of the point B. The impressed force P moves m in accordance with the relation $x = X \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation for the dynamic condition of m. (c) Construct the free-body diagram for the connection point B. (d) Write the differential equation for the connection point B. (e) Obtain the solution for part (d), representing the relation that governs the motion of point B. (f) Determine the relation for the impressed force P. (g) Obtain the expression for the force transmitted to the support at D.



Problem 4-19

4-20. For the arrangement shown, x represents the absolute displacement of the mass m, and y is the absolute displacement of the point B. The impressed force P moves m in accordance with the relation $x = X \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation for the dynamic condition of m. (c) Construct the free-body diagram for the connection point B. (d) Write the differential equation for the connection point B. (e) Obtain the solution for part (d), representing the relation that governs the motion of point B. (f) Determine the relation for the impressed force P. (g) Obtain the expression for the force transmitted to the support at D.



Problem 4-20

4-21. A damped system is composed of a mass of 8.75 kg, a spring having a modulus of 1750 N/m, and a dashpot having a damping constant of 37.13 N · s/m. The mass is acted on by a harmonic force $P = P_0 \sin \omega_f t$ having a maximum value of 220 N and a frequency of 4.50 Hz. Using the symbols of X' and ϕ for the arbitrary constants of the transient, write the complete solution representing the motion of the mass.

4-22. A body having a mass of 100 kg moves along a straight line according to the relation

$$x = 0.12 \sin (3t - \beta) + \frac{0.05 \sin (5.96t + \pi/6)}{e^{0.6t}}$$

where x is the displacement along the line in meters and t is time in seconds. (a) Obtain the physical constants and parameters of the system, including appropriate frequencies, and so on. (b) Obtain the initial displacement and velocity.

4-23. For a damped system excited by the harmonic force $P = P_0 \sin \omega_f t$, plot the magnification-factor curve for $\zeta = 0.3$. Do this from $r = 0$ to $r = 4$, carefully determining the peak value.

4-24. For a damped system driven by the harmonic force $P = P_0 \sin \omega_f t$, plot the phase-angle curve for $\zeta = 0.3$. Do this from $r = 0$ to $r = 4$.

4-25. Develop a relation for the ratio of the maximum amplitude to the resonant amplitude for steady-state motion for the case of a harmonically forced damped system.

4-26. For a damped system excited by the harmonic force $P = P_0 \sin \omega_f t$, plot the curve of the steady-state amplitude against the spring constant k for $c = 0.707 m\omega_f$. Carry this out from $k = 0$ to $k = 4m\omega_f^2$.

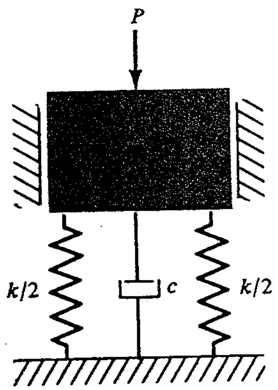
4-27. A damped system has a spring modulus of 24 lb/in. and a damping constant of 0.88 lb sec/in. It is subjected to a harmonic force having a force amplitude of 15 lb. When excited at resonance, the steady-state amplitude is measured as 2.8409 in. Determine (a) the damping factor, (b) the natural undamped frequency of the system, and (c) the damped natural frequency.

4-28. A damped system is subjected to a harmonic force for which the frequency can be adjusted. It is determined experimentally that at twice the resonant frequency, the steady amplitude is one-tenth of that which occurs at resonance. Determine the damping factor for the system.

4-29. A damped torsional system is composed of a shaft having a torsional spring constant $k_T = 60\,000$ lb in./rad, a disk with a mass moment of inertia $I = 24$ lb in. sec^2, and a torsional damping device having a torsional damping constant $c_T = 840$ lb in. sec/rad. A harmonic torque with a maximum value of 2700 lb in., acting on the disk, produces a sustained angular oscillation of 3.368-degree amplitude. (a) Determine the frequency of the impressed torque. (b) Obtain the maximum torque transmitted to the support.

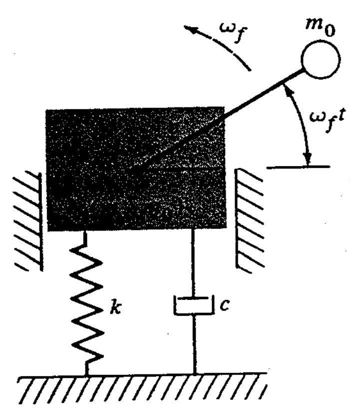
4-30. A torsional system consists of a shaft having a torsional spring constant $k_T = 12\,800$ N · m/rad, a disk with a mass moment of inertia $I = 8$ kg · m^2, and a torsional damping device with a torsional damping constant $c_T = 192$ N · m · s/rad. A harmonic torque with a maximum value of 480 N · m produces a steady angular oscillation of 1.5-degree amplitude. (a) Determine the frequency of the impressed torque. (b) Obtain the maximum torque transmitted to the support.

4-31. A machine having a mass of 70 kg is mounted as shown on springs having a total stiffness of 33 880 N/m. The damping factor is $\zeta = 0.20$. A harmonic force $P = 450 \sin 13.2t$ (where P is in newtons and t is in seconds) acts on the mass. For the sustained or steady-state vibration, determine (a) the amplitude of the motion of the machine, (b) its phase with respect to the exciting force, (c) the transmissibility, (d) the maximum dynamic force transmitted to the foundation, and (e) the maximum velocity of the motion.



Problem 4-31

4-32. A machine having a total weight of 96.5 lb is mounted as shown on a spring having a modulus of 900 lb/in. and is connected to a dashpot with a damping *factor* of 0.25. The machine contains a rotating unbalance ($w_0 e$) of 5 lb in. If the speed of rotation is 401.1 rpm, obtain (a) the amplitude of the steady-state motion, (b) the maximum dynamic force transmitted to the foundation, and (c) the angular position of the arm when the structure goes upward through its neutral position.



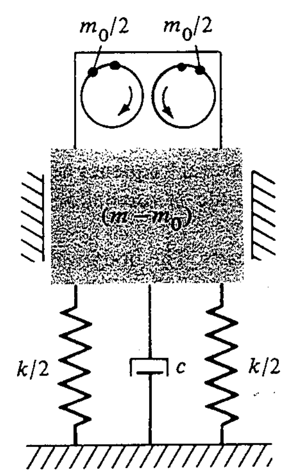
Problem 4-32

4-33. A machine having a total weight of 128.67 lb contains a part that rotates at 859.44 rpm. The machine is supported equally on four identical springs and is

connected to a dashpot for which the damping constant is 10 lb sec/in. In operation the machine has a steady amplitude of 0.116 in. and transmits a maximum dynamic force of 174 lb to the supporting base. (a) Determine the modulus for the springs. (b) Calculate the unbalanced moment ($w_0 e$) for the rotating part.

4-34. A machine with a total mass of 50 kg contains a shaft mechanism that rotates at 1800 rpm. The machine is supported by springs for which the equivalent modulus is 20 000 N/m but the damping constant is unknown. Due to the rotating unbalance, the steady-state amplitude is 1.32 cm and the maximum dynamic force carried by the base is 278 N. Determine (a) the damping constant and (b) the unbalanced moment $m_0 e$.

4-35. A typical vibration exciter is composed of two eccentric masses that rotate oppositely, as represented in the top of the accompanying figure. Thus an oscillation is produced in the vertical direction only, since the horizontal effect cancels. This device is used to determine the vibrational characteristics of the structure to which it is attached. The unbalance ($ew_0/2$) of each exciter wheel is 3 lb in. The total arrangement has a weight $mg = 200$ lb. The exciter speed (of eccentric rotation) was adjusted until a stroboscope showed the structure to be moving upward through its equilibrium position at the instant the eccentric weights were at their top position. The exciter speed then was 840 rpm, and the steady amplitude was 0.75 in. Determine (a) the natural frequency of the structure and the damping factor for the structure. If the speed were changed to 1260 rpm, (b) obtain the steady-state amplitude of the structure and (c) the angular position of the eccentrics as the structure moves upward through its equilibrium position.



Problem 4-35

4-36. A small compressor weighs 69 lb and runs at 875 rpm. It is to be supported by four springs (equally) and no damping is provided. Design the springs, thus determining k, so that only 15% of the shaking force is transmitted to the supporting foundation or structure.

HW # 9

4-21 $m = 8.75$ kg ; $k = 1750$ N/m ; $c = 37.13$ N sec/m. $P = P_0 \sin \omega_f t \Rightarrow P_0 = 220$ N

$f_f = 4.50$ Hz.

$\omega_n = \sqrt{\frac{k}{m}} = 14.142$ rad/sec $\qquad c_c = 2m\omega_n = 247.49 \frac{\text{N-sec}}{m} \qquad \zeta = \frac{c}{c_c} = .15$

since $\zeta < 1$ system is underdamped

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 13.982$ rad/sec

transient $x_c = X' e^{-\zeta\omega_n t} \sin(\omega_d t + \phi) = X' e^{-2.1617t} \sin(13.982t + \phi)$ ✓

$X_0 = P_0/k = .1257$ m ; note X', ϕ obtained from initial conditions

$f_f = 4.50$ Hz $= \frac{\omega_f}{2\pi} \Rightarrow \omega_f = 28.274$ rad/sec $\Rightarrow r = \frac{\omega_f}{\omega_n} = 2.0$

$$X = \frac{X_0}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = .0411 \text{ m} \qquad \tan\psi = \frac{2\zeta r}{1-r^2} = -.2 \qquad \psi = -11.31° = -.1974 \text{ rad}$$

steadystate $x_p = .0411 \sin(28.274t + .1974)$ ✓; $x_{total} = x_c + x_p$

4-27 $k = 24$ lb/in ; $c = .88$ lb·sec/in ; $P_0 = 15$ lb when $r = 1$ $x_p = 2.8409$ in

find ζ ; ω_n ; ω_d note that $x_p|_{r=1} = X_{res}$

$X_{res} = \frac{X_0}{2\zeta} \Rightarrow \underline{X_0 = P_0/k = .625\text{ in}} \Rightarrow \zeta = \frac{1}{2}\frac{X_0}{X_{res}} = \frac{1}{2}\left(\frac{.625}{2.8409}\right) = \underline{.11 = \zeta}$

$\zeta = \frac{c}{c_c}$ $\therefore$ $\underline{c_c = \frac{c}{\zeta} = 8 \text{ lb-sec/in}}$; now $\omega_n^2 = k/m \Rightarrow k = m\omega_n^2$ & $c_c = 2m\omega_n$

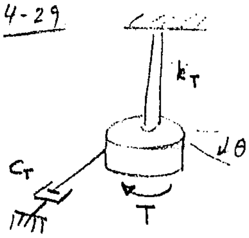
Thus $\frac{k}{c_c} = \frac{\omega_n}{2}$ or $\underline{\omega_n = \frac{2k}{c_c} = 6 \text{ rad/sec}}$ $\qquad f_n = \frac{\omega_n}{2\pi} = .955$ Hz

Now $\underline{\omega_d} = \omega_n\sqrt{1-\zeta^2} = \underline{5.964 \text{ rad/sec}}$ $\quad f_d = .949$ Hz

4-28 gives when $r = 2$ $\quad X_{r=2} = \frac{1}{10} X_{res}$

$$X_{r=2} = \frac{X_0}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{X_0}{\sqrt{(-3)^2 + (4\zeta)^2}} = \frac{1}{10} X_{res} = \frac{1}{10}\frac{X_0}{2\zeta}$$

Solving those terms indicated by (~~~) gives $\zeta = .1531$



<u>4-29</u>

$k_T = 60000$ lb in/rad ; $I = 24$ lb in sec^2/rad, $c_T = 840$ lb in sec/rad

the ODE is $I\ddot{\theta} + c_T\dot{\theta} + k_T\theta = T = T_0 \sin \omega_f t$

also for given $T_0 = 2700$ lb-in results in $\underline{\theta = 3.368^\circ = .0588 \text{ rad}}$

In this case everything we said about the linear case holds here

Thus $MF = \dfrac{\theta}{\theta_0}$ where $\underline{\theta_0 = T_0/k_T = .045 \text{ rad}}$ $\Rightarrow$ $\underline{MF = 1.3067} = \dfrac{0.0588}{0.045}$

now $\sqrt{k_T/I} = \omega_n = 50$ rad/sec ; $\underline{C_{T_{cr}} = 2I\omega_n = 2400 \text{ lb-in sec/rad}}$ $\Rightarrow \underline{\zeta = \dfrac{c}{c_{cr}} = .35}$

$$MF = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 1.3067; \text{ solve for } r \Longrightarrow \quad \underline{r = 1.0722}, \quad \underline{r = .6}.$$

$$r^4 - 1.51r^2 + .414 = 0$$

For $r = 1.0722 = \omega_f/\omega_n \Rightarrow \underline{\omega_f = 1.0722\omega_n = 53.61 \text{ rad/sec}}$ $\quad f_f = \dfrac{\omega}{2\pi} = 8.532$ Hz

$r = .6 = \omega_f/\omega_n \Rightarrow \underline{\omega_f = .6\omega_n = 30 \text{ rad/sec}} \Rightarrow f_f = \dfrac{\omega}{2\pi} = 4.775$ Hz

<u>TO OBTAIN MF_{RES} & MF_{MAX}:</u>

Now $MF_{RES} = \dfrac{\theta_{RES}}{\theta_0} = \dfrac{1}{2\zeta} = 1.429$ @ $r = 1$; $MF_{MAX} = MF_{RES} \cdot \dfrac{1}{\sqrt{1-\zeta^2}} = 1.525$ @ $r = .869$

$= \dfrac{1}{2\zeta} \cdot \dfrac{1}{\sqrt{1-\zeta^2}}$ $\quad = \dfrac{[illegible]}{\sqrt{1-2\zeta^2}}$

<u>TO OBTAIN THE BANDWIDTH:</u>

now $MF_{RES} = Q$ for the bandwidth $\dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \dfrac{1}{2\sqrt{2}\zeta} \Rightarrow r^2 = (1-2\zeta^2) \pm 2\zeta\sqrt{1+\zeta^2}$

$\Rightarrow r_2 = 1.2234$ and $r_1 = .1156 \Rightarrow$ bandwidth is $r_2 - r_1 = 1.1078$

NOTE: Must use full formula for r since ζ <u>is not small</u>.

TO FIND $T_{transmitted}$ $\quad TR = \dfrac{T_{trans}}{T_0} = \dfrac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

when $r = 0.6$ $\quad TR = 1.4169$ $\quad T_{trans} = TR \cdot TO = 3825.5$ lb-in

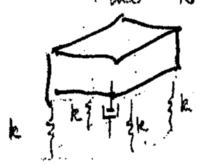
$r = 1.0724$ $\quad TR = 1.6334$ $\quad T_{trans} = TR \cdot T_0 = 4410.1$ lb-in

HW #10

4-33, 35, 40, 41

4-33 Given $\begin{cases} W = 128.67\ lb \quad ; \quad \omega_f = 859.44 \frac{rev}{min} \cdot \frac{2\pi\ rad}{rev} \cdot \frac{1 min}{60 sec} = 90\ rad/sec\ ; \\ c = 10\ lb sec/in \quad ; \quad X = 0.116\ in \quad ; \quad F_T = 174\ lb \end{cases}$

Find k, $W_0 e$



$k_{EQ} = 4k$ now $F_T = X\sqrt{k_{eq}^2 + (c\omega_f)^2}$

$174\ lb = .0116\ in\ \sqrt{k_{EQ}^2 + \left(10 \frac{lb sec}{in} \cdot 90 \frac{rad}{sec}\right)^2}$

$\sqrt{(F_T/X)^2 - (c\omega_f)^2} = k_{EQ} = 1200.00\ lb/in$ $\quad$ $\underline{k = \frac{1}{4}k_{EQ} = 300.0\ lb/in}$

$$X = \frac{P_0}{\sqrt{(k_{EQ} - m\omega_f^2)^2 + (c\omega_f)^2}} \qquad m = \frac{W}{g} = .333\ \frac{lb\text{-}sec^2}{in} \qquad P_0 = m_0 e \omega_f^2$$

$g = 386.4\ in/s^2$

$P_0 = 0.116\ in\sqrt{(k_{EQ} - m\omega_f^2)^2 + (c\omega_f)^2} = 202.92\ lb = m_0 e\omega_f^2 \quad \therefore \quad W_0 e = \frac{(202.92)(32.2)(12)}{(90)^2}\ lb\ in$

$\underline{W_0 e = 9.68\ lb \cdot in}$

as $r \to \infty$ $\quad X \to \frac{m_0 e}{m} = \frac{W_0 e}{W} = \frac{9.68\ lb\text{-}in}{128.67\ lb} \sim .075\ in$

4-35

Given $\begin{cases} eW_0/2 = 3\ lb in \ ; \ W = 200 lb\ ; \text{ when } x = 0 \quad \psi = 90° \Rightarrow r = 1 \text{ from } \tan\psi = \frac{2\zeta r}{1 - r^2} \\ \omega_f = 840 \frac{rev}{min} \cdot \frac{2\pi\ rad}{rev} \cdot \frac{1 min}{60 sec} = 87.965\ rad/sec\ ; \quad X = .75\ in \end{cases}$

Find (a) ω_n (or f_n), ζ

(b) Suppose $\omega_f = 1260 \frac{rev}{min} = 131.95\ rad/sec$ find X and ψ

(a) $$X = \frac{m_0 e}{m}\frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = \frac{W_0 e}{W} \cdot \frac{1}{2\zeta} \qquad \text{since } r = 1$$

$$X = .75\ in = \frac{2(3)}{200}\ \frac{1}{2\zeta}$$

thus $\zeta = \frac{W_0 e}{2WX} = \frac{2(3)}{400(.75)} = \boxed{.02 = \zeta}$

since $r = 1$ $\quad \underline{\omega_n = \omega_f = 87.965\ rad/sec}$ or $\underline{f = 14\ Hz}$

(b) For the system $\omega_f = 131.95\ rad/sec\ ; \ \omega_n = 87.965\ rad/sec \Rightarrow r = 1.5$ also $\zeta = .02$

4-37. Solve Prob. 4-36 considering that damping is also to be included such that the damping factor will be 0.4 for the system. Equal damping at each spring will be provided by a plug or block of energy-absorbing material. Determine the required damping constant as well as the spring constant.

4-38. A refrigerator compressor unit weighs 179 lb and operates at 590 rpm. The maximum dynamic force imposed on the compressor is 20 lb. The unit is to be supported equally by three springs and is to be undamped. (a) Determine the required spring constant if 10% of the dynamic force is to be transmitted to the supporting base. (b) Determine the clearance that must be provided for the unit.

4-39. Solve Prob. 4-38, considering that *each* spring contains a viscous damping absorber having a damping constant of 0.8333 lb sec/in.

4-40. A small machine is known to contain a rotating unbalance on its main shaft. The machine weighs 65 lb and when placed on elastic vibration isolators (which serve as a viscous damping member as well as the elastic support), the static equilibrium displacement is 2.166 in. Also, when the machine is displaced further and released, the subsequent vibration diminishes from an amplitude of 2.350 in. to 0.135 in. in exactly 3 cycles. Operating the machine at resonance produces a sustained amplitude of 0.0193 in. (a) Determine the damping constant for the system. (b) Considering that the shaft diameter is 3 in. so that the eccentric moment arm is 1.5 in., calculate the unbalanced weight. (c) Determine the steady amplitude at an operation speed of twice the resonant speed.

4-41. A damped system with a rotating unbalance is composed of a 96.5-lb body and a spring having a constant of 80 lb/in. but the damping constant is unknown. When operated at resonance, the sustained amplitude is 2.143 in. If the speed of operation is greatly increased, the steady amplitude eventually approaches the value of 0.3572 in. Determine the damping constant for the system.

4-42. The support for a damped system (see Fig. 4-26) is oscillated harmonically with a frequency of 75 cycles/min, causing the mass to vibrate with a steady-state amplitude of 0.6 in. and a maximum dynamic force of 48.75 lb to be transmitted to the support. For the system, the spring modulus is 13 lb/in. and the damping constant is 0.454 lb sec/in. Determine (a) the natural frequency of the system, (b) the mass of the system, and (c) the amplitude of oscillation of the support.

4-43. The support for a damped system is oscillated harmonically. The body weighs 20 lb and the spring constant is 14 lb/in. When the support is oscillated at a rate of 300 cycles/min, the sustained amplitude of the body is 0.793 in. (a) Determine the maximum dynamic force carried by the support. (b) If the support is oscillated at resonance with an amplitude of 1.121 in., the body has a steady amplitude of 3.069 in. Obtain the damping constant for the system.

4-44. A mass-spring system, arranged horizontally, is damped only by Coulomb friction (refer to Fig. 4-27). The system is driven by a harmonic force having a frequency of 54.356 cycles/min and a maximum value of 30 lb, causing the body to oscillate with a steady amplitude of 3.09 in. The spring constant is 12 lb/in. and the body weighs 28.95 lb. Determine the average coefficient of sliding friction for the surfaces involved.

4-45. A machine is designed to produce an abrasive action on a horizontal surface in order to wear and smoothen the surface. It consists of a sliding member weighing 20 lb fixed to an elastic member having a stiffness coefficient of 8 lb/in. The average coefficient of sliding friction of the body against the surface is 0.3. A harmonic force having a maximum value of 15 lb actuates the body at a frequency of 0.9 of the resonant value. Determine the total excursion or stroke of the member.

4-46. A certain mass-spring system exhibits hysteresis damping only. The spring modulus is 310 lb/in. When excited harmonically at resonance, the steady amplitude is 1.540 in. for an energy input of 32 in. lb. When the resonant energy input is increased to 87 in. lb, the sustained amplitude changes to 2.464 in. Determine the hysteresis coefficient β and the exponent γ.

4-47. A small built-up structure shows solid damping characteristics with $\gamma = 2$. A 1000-lb load on the structure causes a static displacement of 2.500 in., and a supported machine that is subjected to a harmonic force having a constant amplitude of 75 lb produces a resonant amplitude of 6.818 in. (a) Determine the hysteresis damping coefficient β and (b) the energy dissipated per cycle by hysteresis at resonance. (c) Calculate the steady amplitude at twice the resonant frequency and (d) at one-half the resonant frequency.

$$\tan\psi = \frac{2\zeta r}{1-r^2} = \frac{2(.02)(1.5)}{1-(2.25)} = -.048 \qquad \underline{\psi = -2.748^\circ}$$

$$X = \frac{m_o e}{m}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{W_o e}{W}\frac{(1.5)^2}{\sqrt{(1.25)^2+(.06)^2}} = \underline{.054 \text{ in}} = X$$

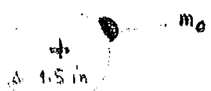
as $r \to \infty \quad X \to \frac{W_o e}{W} = .03$ in

4-40

Given $W = 65$ lb ; $\Delta_{st} = \frac{W}{k_{eq}} = 2.166$ in ; $n = 3 = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta}\ln\left(\frac{2.35}{.135}\right)$

Finally when $r = 1$, $X = .0193$ in; also $e = 1.5$ in

Find: c, m_o (or W_o), if $r = 2$ find X



From $\Delta_{st} \Rightarrow k_{eq} = \frac{W}{\Delta_{st}} = \frac{65 \text{ lb}}{2.166 \text{ in}} = 30.01$ lb/in

but $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{W}} = \underline{13.356 \text{ rad}}/\text{sec}$

thus $c_{cr} = 2m\omega_n = \underline{4.494} \; \frac{\text{lb-sec}}{\text{in}}$; let $\ln\left(\frac{2.35}{.135}\right) = \rho = \ln(17.41) = 2.857$

From $n = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta}\ln\left(\frac{2.35}{.135}\right) \Rightarrow 4\pi^2\zeta^2 n^2 = (1-\zeta^2)\rho^2$ or $\zeta = \frac{\rho}{\sqrt{\rho^2+4\pi^2 n^2}}$

thus $\underline{\zeta = .1499}$. $\underline{c = \zeta c_{cr} = .6734 \frac{\text{lb-sec}}{\text{in}}}$

Now $X = \frac{m_o e}{m}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{W_o e}{W}\frac{1}{2\zeta}$ at $r=1$ $\Rightarrow$ $\underline{W_o = \frac{X W \cdot 2\zeta}{e} = .2507 \text{ lb}}$ (0.0193 in)

Knowing $X = \frac{m_o e}{m}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ then for $r = 2$, $\zeta = .1499$, $W_o = .2507$ lb

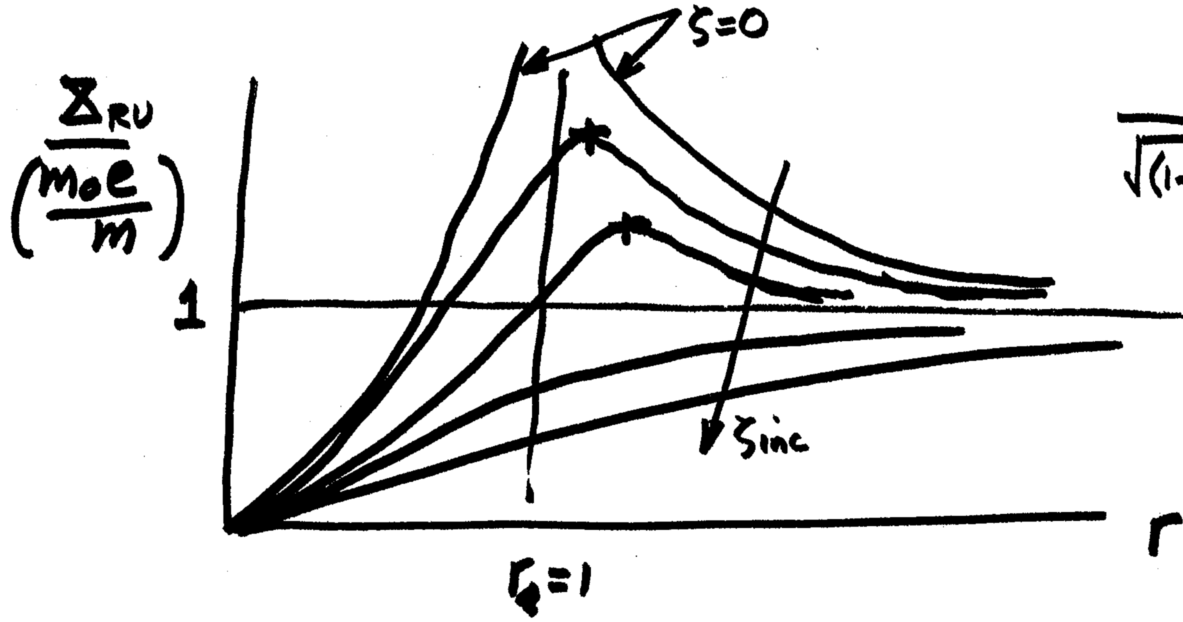
$$\underline{X = .00756 \text{ in}}$$

4-41 Given $W = 96.5$ lb ; $k = 80$ lb/in ; $r = 1 \Rightarrow X = 2.143$ in ; $r \to \infty \Rightarrow X \to .3572$ in

Find c : $X = \frac{m_o e}{m}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$; when $r \to \infty$ $X_{r\to\infty} \to \frac{m_o e}{m} = .3572$ in

now when $r = 1$ $X = \frac{m_o e}{m}\cdot\frac{1}{2\zeta}$ or $\zeta = \frac{X_{r=\infty}}{2X_{r=1}} = \frac{.3572}{2(2.143)} = \underline{.08334 = \zeta}$

$\underline{c_c} = 2m\omega = 2\sqrt{mk} = 2\sqrt{\frac{Wk}{g}} = \underline{8.94 \frac{\text{lb sec}}{\text{in}}}$ $\qquad \underline{c = c_c\zeta = 0.74504 \frac{\text{lb sec}}{\text{in}}}$



$$\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

- FOR VERY LARGE ω_f ($r \to \infty$) ALL CURVES TEND TO $\dfrac{X_{RU}}{(m_o e/m)} = 1$ $\Rightarrow X_{RU} \to \frac{m_o e}{m}$ as $r \to \infty$

- $\dfrac{d}{dr}\left(\dfrac{X_{RU}}{m_o e/m}\right) = 0 \Rightarrow$ MAX. OCCURS WHEN $r = \dfrac{1}{\sqrt{1-2\zeta^2}} > 1$

ONLY TRUE UNTIL $\zeta = \frac{1}{\sqrt{2}} = 0.707$

r HAS LOCAL MAX WHEN $\zeta \le \frac{1}{\sqrt{2}}$

FOR $\zeta > \frac{1}{\sqrt{2}}$ NO <u>LOCAL</u> MAX

$$X_{RU_{max}} = \frac{m_o e}{m} \cdot \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$ THIS IS FOR LOCAL MAXES $\zeta \le \frac{1}{\sqrt{2}}$

$$X_{RU}\Big|_{r=1} = \frac{m_o e}{m} \cdot \frac{1}{2\zeta}\ ; \quad X_{RU_{max}} = X_{RU}\Big|_{r=1} \cdot \frac{1}{\sqrt{1-\zeta^2}}$$

Prob...

4.32 A machine having a total weight of 96.5 lb is mounted on a spring of modulus 900 lb/in and is connected to a dashpot having a damping ratio of 0.25. The machine contains an unbalance of (W_0e) 5 lb in. If the speed of rotation is 401.1 rpm find (a) the amplitude of steady state motion (b) the max. dynamic force transmitted to the foundation (c) the angular position of the arm when the structure goes thru its neutral position

Solution $m = \frac{W}{g} = \frac{96.5}{32.2} = 0.25 \ \frac{\text{lb-sec}^2}{\text{ft}}$; $k = 900 \frac{\text{lb}}{\text{in}} = 10800 \frac{\text{lb}}{\text{ft}}$

$\zeta = \frac{c}{c_{cr}} = 0.25$ given $W_0e = 5$ lb-in $f_f = 401.1$ rpm

$$\left.\begin{aligned} \omega_f &= 401.1 \cdot \frac{2\pi}{60} = 42 \text{ rad/s} \\ \omega_n &= \sqrt{\frac{k}{m}} \cong 60 \text{ rad/s} \end{aligned}\right\} \quad r = \frac{\omega_f}{\omega_n} = 0.7$$

$$X_{RU} = \overset{W_0e/W}{\frac{m_0 e}{m}} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{5 \text{ lb-in}}{96.5 \text{ lb}} \frac{(0.7)^2}{\sqrt{(1-0.7^2)^2 + (2 \cdot 0.25 \cdot 0.7)^2}}$$

$$= 0.041 \text{ in}$$

$$\tan\psi = \frac{2\zeta r}{1-r^2} = 0.6863$$

$\psi = 34.46^\circ = \omega_f t$ when $(m - m_0)$-main mass passes through $x = 0$.

Pall...

4-32 A machine having a total weight of 96.5 lb is mounted on a spring of modulus 900 lb/in and is connected to a dashpot having a damping ratio of 0.25. The machine contains an unbalance of ($W_o e$) 5 lb in. If the speed of rotation is 401.1 rpm find (a) the amplitude of steady state motion (b) the max. dynamic force transmitted to the foundation (c) the angular position of the arm when the structure goes thru its neutral position

Solution $m = \frac{W}{g} = \frac{96.5}{32.2} = 0.25 \ \frac{\text{lb-sec}^2}{\text{ft}}$; $k = 900 \frac{\text{lb}}{\text{in}} = 10800 \frac{\text{lb}}{\text{ft}}$

$\zeta = \frac{c}{c_{cr}} = 0.25$ given $W_o e = 5$ lb-in $f_f = 401.1$ rpm

$$\left.\begin{aligned} \omega_f &= 401.1 \cdot \frac{2\pi}{60} = 42 \text{ rad/s} \\ \omega_n &= \sqrt{\frac{k}{m}} \cong 60 \text{ rad/s} \end{aligned}\right\} \quad r = \frac{\omega_f}{\omega_n} = 0.7$$

$$X_{RU} = \overset{W_o e/W}{\frac{m_o e}{m}} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{5 \text{ lb-in}}{96.5 \text{ lb}} \frac{(0.7)^2}{\sqrt{(1-0.7^2)^2 + (2 \cdot 0.25 \cdot 0.7)^2}}$$

$= 0.041$ in

$$\tan\psi = \frac{2\zeta r}{1-r^2} = 0.6863$$

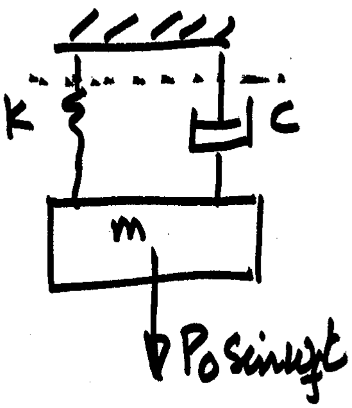
$\psi = 34.46^\circ = \omega_f t$ when $(m - m_o)$-main mass passes through $x = 0$.

(b)

$$F_T = k\,X_{RU}\sqrt{1+(2\zeta r)^2} = 900\,\frac{\text{lb}}{\text{in}}\cdot(0.041\text{ in})\sqrt{1+(2\cdot 0.25\cdot 0.7)^2}$$

$$= \underline{\quad 39.095 \quad}\ \text{lb force.}$$

Force transmitted to support (General Case)



$$m\ddot{x} + c\dot{x} + kx = P_0 \sin\omega_f t$$

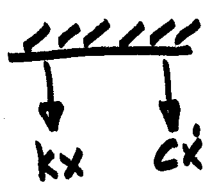
$$x_p = \frac{P_0}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}\sin(\omega_f t-\psi)$$

s.s.

$$= \boxed{\frac{X_0}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}}\sin(\omega_f t-\psi)$$

X – amplitude of forced vibration

$$x_p = X\sin(\omega_f t-\psi)$$



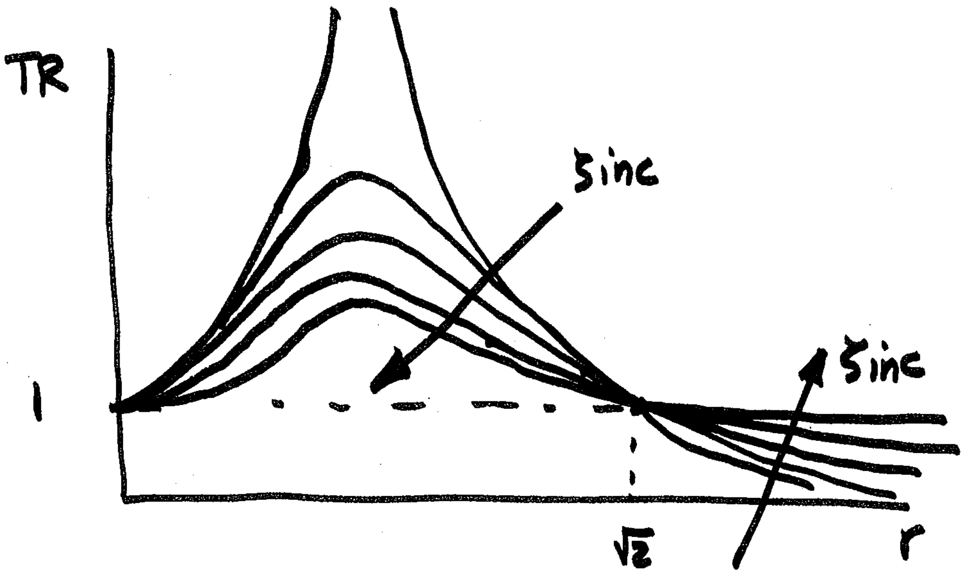
$$= kX\sin(\omega_f t-\psi) + c\omega_f X\cos(\omega_f t-\psi)$$

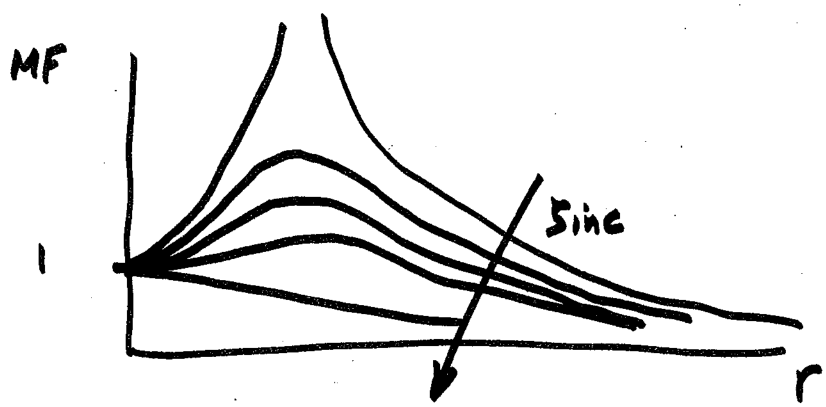
$$x_p = \underbrace{\boxed{X\sqrt{k^2+(c\omega_f)^2}}}_{F_T}\sin(\omega_f t-\psi-\beta)$$

$$\tan\beta = -\frac{c\omega_f}{k}$$

$$F_T = \frac{P_0\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}} = \frac{P_0\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$\underset{\text{transmissibility}}{TR} = F_T/P_0 = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$





in rotating unbalance case

$$F_{T,RU} = \frac{\frac{m_o e}{m} k r^2 \sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \quad ; \quad \frac{F_{T,RU}}{\frac{m_o e}{m} k} = \frac{r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

$$F = X\left[k\sin(\omega_f t-\psi)+c\omega_f\cos(\omega_f t-\psi)\right] \qquad \left.\begin{array}{l}\text{let } C\sin\beta = -c\omega_f\\ C\cos\beta = k\end{array}\right\} C=\sqrt{k^2+(c\omega_f)^2}$$

$$\tan\beta = \frac{-c\omega_f}{k} = -2\zeta r$$

$$F = X\,C\sin(\omega_f t-\psi-\beta)$$

$$= X\,C\sin(\omega_f t-\gamma) \qquad \gamma=\psi+\beta$$

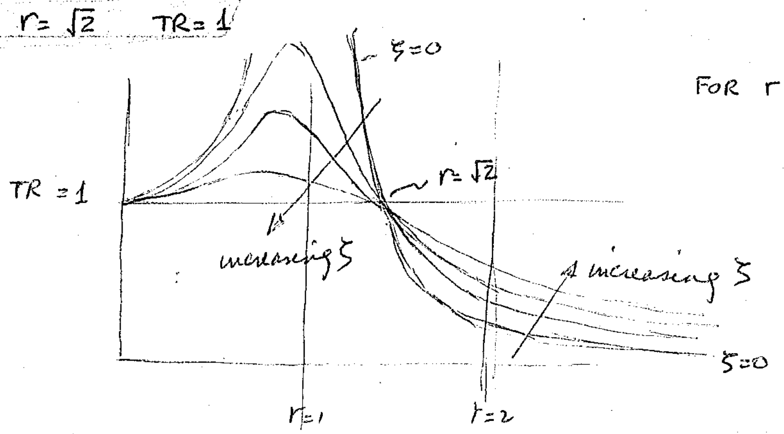
$$= F_{max}\sin(\omega_f t-\gamma) = F_T\sin(\omega_f t-\gamma)$$

$F_{MAX} = XC = X\sqrt{k^2+(c\omega_f)^2}$: max. transmitted force F_T

$$= \frac{X_0\,k\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = P_0\,\frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{P_0\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}$$

$$F_T/P_0 = TR \ (\text{transmissibility}) = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$$

- WHEN $r=0$ $TR=1$ NO FORCING FN. FREQ. – APERIODIC $\forall\zeta$
- WHEN $r=\sqrt{2}$ $TR=1$ $\forall\zeta$



FOR $r \gg 1$ $TR \sim \frac{2\zeta r}{r^2} \sim \frac{2\zeta}{r}$

AS $\zeta\uparrow$ $TR\uparrow$; AS $r\uparrow$ $TR\to 0$

- WHEN $r\to\infty$ $TR\to 0$ $\forall\zeta$ VIBRATIONS OCCUR SO FAST CHANGE IN DISP & VEL $\to 0$

- DAMPING reduces peak force at resonance, but increases F_T for $r>\sqrt{2}$
- NOTE THAT for $r>1$ increase in $\zeta \Rightarrow X\downarrow$

- PEAK OCCURS AT $r=\dfrac{\sqrt{(1+8\zeta^2)^{1/2}-1}}{2\zeta}$ <1 TO LEFT OF RESONANCE

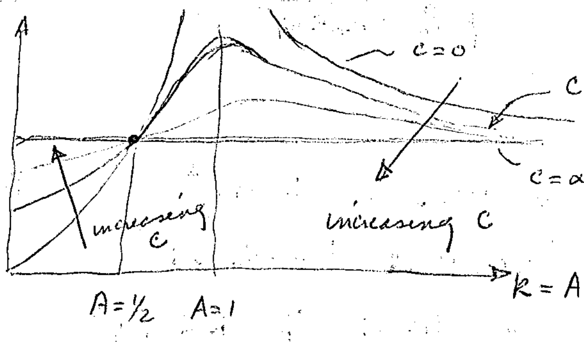
- CAVEAT AGAIN – THAT CHANGE IN $r \Rightarrow$ CHANGE ONLY IN ω_f

- TO LOOK AT CHANGE IN k & m better to use

$$TR = \frac{F_T}{P_0} = \frac{\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}$$

if we take $\boxed{\frac{dTR}{dk} = 0 \Rightarrow k \geqslant m\omega_f^2}$ take $k = Am\omega_f^2$ $c = Bm\omega_f$

then $TR = \frac{F_T}{P_0} = \frac{m\omega_f^2\sqrt{A^2+B^2}}{m\omega_f^2\sqrt{(A-1)^2+B^2}}$ as $k \to \infty \Rightarrow A \to \infty$ $TR \to 1$ $\forall B$ (c)



- for $c \neq 0$ peak value of TR is when $k = m\omega_f^2/2 + \sqrt{(m\omega_f^2/2)^2 + (c\omega_f)^2} > m\omega_f^2$ PEAK TO right of k value at resonance
- TR is reduced only when $k << m\omega_f^2/2$ ie for light damping & small spring const.

EXAM INVOLVES

1. equiv spring equiv mass
2. free vibs w/o damping
3. " w damping
4. forced vibs w/ or w/o damping

- WE CAN DETERMINE mass variations also

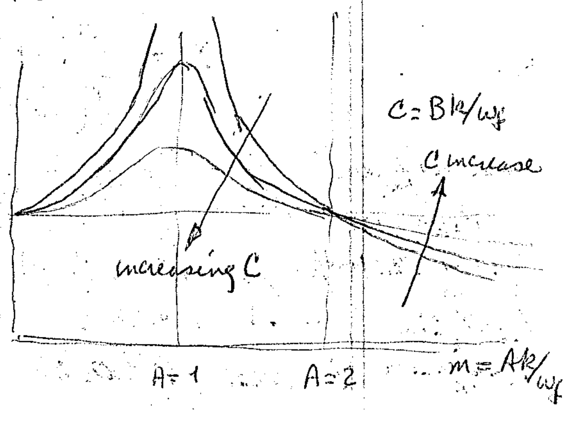
$$\frac{dTR}{dm} = 0 \Rightarrow m = k/\omega_f^2 \quad \text{if} \quad m = Ak/\omega_f^2 \quad c = Bk/\omega_f$$

$$\Rightarrow TR = \frac{F_T}{P_0} = \frac{k\sqrt{1+B^2}}{k\sqrt{(1-A)^2+B^2}} = \frac{\sqrt{1+B^2}}{\sqrt{(1-A)^2+B^2}}$$

when $A = 0 \Rightarrow m = 0$ $TR = 1$ $\forall c$

$A = 2 \Rightarrow m = \frac{2k}{\omega_f^2}$ $TR = 1$ $\forall c$

$A \to \infty$ $TR \to 0$ $\forall c$



ie TR decreases for very large mass $\Rightarrow \delta_{STATIC} = \frac{W}{k} \to \infty$

ALL PEAKS occur at $m = k/\omega_f^2 \Rightarrow A = 1$

$(TR)_{max} = \frac{\sqrt{1+(2\zeta)^2}}{2\zeta} = \sqrt{k^2+(c\omega_f)^2}$

in 4th ed
9.29

Rao 3rd ed
9.27 $\quad m = 500\,kg \quad m_o e = 50\,kg\text{-}cm = .5\,kg\text{-}m \quad f_f = 300\,rpm \Rightarrow 300 \cdot \frac{2\pi}{60} = 31.42\,rad/s$

a) $k \quad c=0$

b) $\zeta = .1 \quad k = small$

$\Big\}\ P_o = m_o e\,\omega_f^2$

4th ed 9.28
9.26
$\omega_f = 62.83\ \frac{rad}{s}$
$TR = 2.5$ @ $r=1$
find curve where ζ has max
$TR = 2.5 \Rightarrow \zeta = .2182$
@ operating speed
$Tr = 0.1$ for $\zeta = 0.2182$
$\Rightarrow$ get $r = 4.985$
$r^4 - 20.86r^2 - 99 = 0 \rightarrow r^2 = 24.84$, $r^2 = 3.986$
$\omega_n r = \omega_f$
$\omega_n = \omega_f / r = 12.61$
$k = m\omega_n^2 = 13000\ N/m$
$c = 2\zeta m\omega_n = 449\ \frac{Ns}{m}$
$TR^2 = \frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}$
$2.5^2 = \frac{1+4\zeta^2}{4\zeta^2} \Rightarrow \zeta$

a) $\Rightarrow$ For $c=0$ TR max occurs at $k = m\omega_f^2$ $\qquad TR = \frac{F_{max}}{P_o}$

note that for k large TR $\downarrow$ but still >1

note that for k small $TR<1$ if $k < m\omega_f^2$

choose $TR = .1 \Rightarrow TR = \frac{1}{|1-r^2|} \Rightarrow r^2 = 11$

$$r^2 = \frac{\omega_f^2 m}{k} = 11 \Rightarrow k = \frac{1}{11} m\omega_f^2 \quad r = 3.32$$

$$\omega_n = \frac{\omega_f}{r} = 9.47\ rad/sec \quad \& \quad k = \omega_n^2 m = 44863\ N/m$$

$$\cancel{\delta_{st} = \frac{W}{k} = \frac{500(9.81)}{44863} \cong .011\,m} \qquad X_o = \frac{P_o}{k} = .011\,m$$

$$X = \frac{r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \cdot \frac{m_o e}{m} = \frac{11 \cdot 1}{10}\left(\frac{.500}{500}\right) = 1.1\,(.001\,m) = .0011\,m$$

case (2)

choose $TR = .1 \quad \zeta = .1$

$$0.1 = \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \Rightarrow r^2 \cong 13.37,\ -7.5 \quad r \cong 3.66\ rad/sec$$

$$\omega_n = \frac{\omega_f}{r} = 8.59\ rad/sec \qquad k = m\omega_n^2 = 36919\ N/m$$

$$\delta_{st} = \frac{W}{k} = \frac{500(9.81)}{36919} \cong .125\,m \qquad X_o = \frac{P_o}{k} = .013\,m$$

$$X = \frac{r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\ \frac{m_o e}{m} = .00108\,m$$

since X shock absorber smaller, N is preferred but since X_k also smaller $\rightarrow$ leads to cheaper design

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Choose $k = \frac{1}{100} m\omega_f^2$
$r^2 = 100 \quad r = 10$
$\omega_n = \frac{\omega_f}{10} = 3.14\ rad/sec$
$\delta_{st} = \frac{W}{k} \cong 1\,m$
$X_o = P_o/k = .1\,m$

$$TR = \frac{\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}}$$

$$\zeta = .1 = \frac{c}{c_c} = \frac{c}{2\omega_n m} = \frac{.1\,m\omega_f}{2m\omega_n} = \frac{1}{20}\,r$$

$$r = 2 \Rightarrow \omega_n = \frac{1}{2}\omega_f = 15.7\ rad/s$$

Choose $.1$ solve for c $\quad .01\,(m^2\omega_f^4 + c^2\omega_f^2) - c^2\omega_f^2 = 0$

$X \cong .00236 \quad TR \cong .236 \qquad \frac{.01}{.99} m^2\omega_f^2 = c^2 \qquad c \cong \frac{1}{10} m\omega_f$