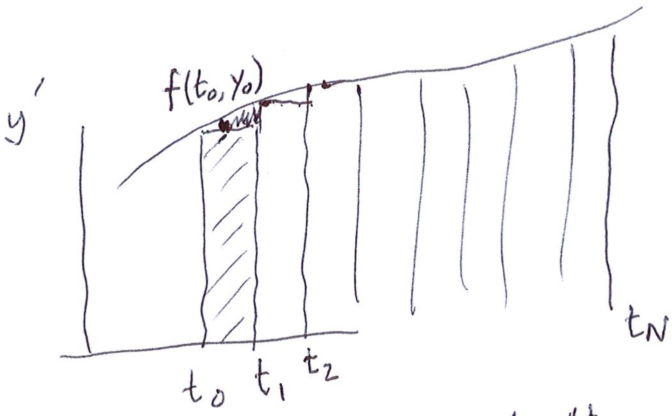
Solve S.V. equation

$\frac{dy}{dt} = f(t,y) \quad y = y(t)$

given $y(t=t_0) = Y_0$

$\int_{Y_0}^{y} d\bar{y} \Leftarrow \int \frac{dy}{dt} \cdot dt = \int_{t_0}^{t} f(\bar{t}, y) d\bar{t}$



$t_1 = t_0 + \Delta t, \quad t_2 = t_1 + \Delta t$

$$Y - Y_0 = \bar{Y}\Big|_{Y_0}^{Y} = \int_{y_0}^{y} d\bar{y} = \int_{t_0}^{t} f(\bar{t}, y) dt \cong f(t_0, Y_0)\Delta t$$

$$y(t_0 + \Delta t) = y(t_1) \cong Y_1 = Y_0 + f(t_0, Y_0)\Delta t \qquad \text{Euler Method.}$$

Given $t_1, Y_1 \Longrightarrow$ find Y_2 using same method.

$$Y_2 = Y_1 + f(t_1, Y_1)\Delta t$$

algorithm

GIVEN $t_0, Y_0, f(t,y), \Delta t$

DO $I = 1, 2, \ldots, N \Longrightarrow \frac{t_N - t_0}{\Delta t} = N$

$Y_I = Y_{I-1} + f(t_{I-1}, Y_{I-1})\Delta t$

$t_I = t_{I-1} + \Delta t$

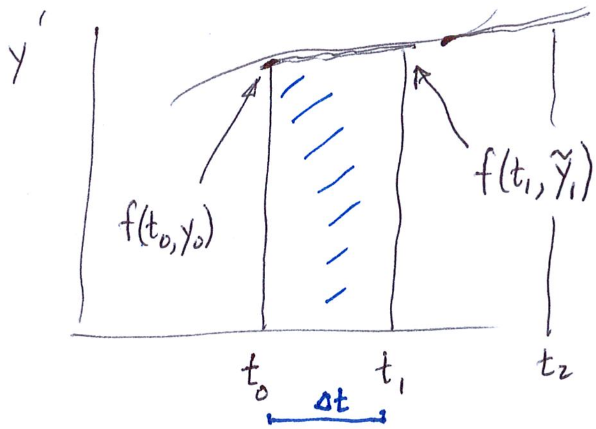
print t_I, Y_I

$$y(t_0 + \Delta t) = y(t_0) + \frac{dy}{dt}\Big|_{t=t_0} \cdot \Delta t + \frac{d^2y}{dt^2}\Big|_{t=t_0} \frac{\Delta t^2}{2!} + \frac{d^3y}{dt^3}\Big|_{t=t_0} \frac{\Delta t^3}{3!}$$

$$= y(t_0) + f(t_0, y_0)\Delta t \quad \Big\| \longrightarrow \text{local error} \quad C\Delta t^2$$

per step error

for many steps error accumulates $\Rightarrow$ global $D\Delta t$



area under graph using trapezoidal rule

$$y(t=t_1) \cong y_1 = y_0 + \frac{f(t_0,y_0) + f(t_1,\tilde{y}_1)}{2}\Delta t$$

$$\tilde{y}_1 = y_0 + f(t_0,y_0)\Delta t$$

$$t_1 = t_0 + \Delta t$$

MOD EULER

GIVEN $t_0, y_0, f(t,y), \Delta t$

DO $I = 1, 2, \dots, N$

$$y_{IP} = y_{I-1} + f(t_{I-1}, y_{I-1})\Delta t$$

$$t_I = t_{I-1} + \Delta t$$

$$y_I = y_{I-1} + \left[f(t_{I-1}, y_{I-1}) + f(t_I, y_{IP})\right]\frac{\Delta t}{2}$$

print t_I, y_I

local error $C\Delta t^3$ (per step)

global error $D\Delta t^2$ (over many steps)

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for 2 s.v. eqs.

$$\frac{dy}{dt} = f(t,y,z) \qquad @\ t = t_0 \quad y = y_0 \quad z = z_0 \qquad (x = x_0)$$

$$\frac{dz}{dt} = g(t,y,z) \qquad \frac{dx}{dt} = h(t,y,z,x)$$

$$y_1 = y_0 + f(t_0,y_0,z_0)\Delta t$$

$$z_1 = z_0 + g(t_0,y_0,z_0)\Delta t$$

3 state variables

$$x_1 = x_0 + h(t_0,y_0,z_0,x_0)\Delta t$$

$$t_1 = t_0 + \Delta t$$

GIVEN: $t_0, y_0, z_0,\ f(t,y,z),\ g(t,y,z),\ \Delta t$

DO $I = 1, 2, \ldots, N$ $\qquad N = \dfrac{t_N - t_0}{\Delta t}$

$y_I = y_{I-1} + f(t_{I-1}, y_{I-1}, z_{I-1})\Delta t$

$z_I = z_{I-1} + g(t_{I-1}, y_{I-1}, z_{I-1})\Delta t$

$t_I = t_{I-1} + \Delta t$

print t_I, y_I, z_I

Euler Method for 2 state variable eqs.

$x_I = x_{I-1} + h(t_{I-1}, y_{I-1}, z_{I-1}, x_{I-1})\Delta t$ For 3 state variables

for example

$f(t,y,z) = e^t + \sin y + 2z$

$g(t,y,z) = t^2 + 4yz$

$f(t_{I-1}, y_{I-1}, z_{I-1}) = e^{t_{I-1}} + \sin y_{I-1} + 2z_{I-1}$

$g(t_{I-1}, y_{I-1}, z_{I-1}) = t_{I-1}^2 + 4y_{I-1} \cdot z_{I-1}$

MOD EULER 2 STATE VARIABLES

Given $t_0, y_0, z_0,\ f(t,y,z),\ g(t,y,z),\ \Delta t$

DO $I = 1, 2, \ldots, N$

$y_{IP} = y_{I-1} + f(t_{I-1}, y_{I-1}, z_{I-1})\Delta t$

$z_{IP} = z_{I-1} + g(t_{I-1}, y_{I-1}, z_{I-1})\Delta t$

3 S.V. EQS.

$x_{IP} = x_I + h(t_{I-1}, y_{I-1}, z_{I-1}, x_{I-1})\Delta t$

$t_I = t_{I-1} + \Delta t$

$y_I = y_{I-1} + \left[f(t_{I-1}, y_{I-1}, z_{I-1}) + f(t_I, y_{IP}, z_{IP})\right]\dfrac{\Delta t}{2}$

$z_I = z_{I-1} + \left[g(t_{I-1}, y_{I-1}, z_{I-1}) + g(t_I, y_{IP}, z_{IP})\right]\dfrac{\Delta t}{2}$

$x_I = x_{I-1} + \left[h(\_\_\_) + h(\_\_\_)\right]\dfrac{\Delta t}{2}$

print t_I, y_I, z_I

Example: given $y' = f(t,y) = 5y^2t + 3t$ with $y=1$ at $t=0$, use Euler & Mod Euler

euler

$$\frac{dy}{dt} = 5y^2t + 3t = f(t,y)$$

Given: $y_0 = 1$ when $t_0 = 0$

$f(t_0, y_0) = 0$ let $\Delta t = 0.1$ sec (our choice of step)

t	y
0	1
.1	1
.2	1.08
.3	1.2566

$y_1 = y_0 + f(t_0,y_0)\Delta t = 1 + 0(.1) = \underline{1}$

$t_1 = t_0 + \Delta t = \underline{0.1}$

$f(t_1,y_1) = 5(1)^2(0.1) + 3(.1) = .8$

$y_2 = y_1 + f(t_1,y_1)\Delta t = 1 + 0.8(.1) = \underline{1.08}$

$t_2 = t_1 + \Delta t = .1 + .1 = \underline{.2}$

$f(t_2,y_2) = 5(1.08)^2(.2) + 3(.2) = 1.7664$

$y_3 = y_2 + f(t_2,y_2)\Delta t = 1.08 + 1.7664(.1) = \underline{1.25664}$

$t_3 = t_2 + \Delta t = 0.2 + 0.1 = \underline{.3}$

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mod euler

$$\frac{dy}{dt} = 5y^2t + 3t = f(t,y)$$

let $y_0 = 1$ when $t_0 = 0$

t	y
0	1
.1	1.04
.2	1.1752

$f(t_0,y_0) = 5(1)^2(0) + 3(0) = 0$ let $\Delta t = 0.1$ sec

predicted $\tilde{y}_1 = y_0 + f(t_0,y_0)\Delta t = 1 + 0(.1) = 1$

$t_1 = t_0 + \Delta t = 0 + 0.1 = 0.1$

$f(t_1,\tilde{y}_1) = 5\tilde{y}_1^2 t_1 + 3t_1 = 5(1)^2(.1) + 3(.1) = 0.8$

corrected $y_1 = y_0 + [f(t_0,y_0) + f(t_1,\tilde{y}_1)]\frac{\Delta t}{2} = 1 + [0 + 0.8]\frac{0.1}{2} = \underline{1.04}$

$t_1 = t_0 + \Delta t = \underline{0.1}$

$f(t_1,y_1) = 5(1.04)^2(.1) + 3(.1) = .8408$

predicted $\tilde{y}_2 = y_1 + f(t_1,y_1)\Delta t = 1.04 + .8408(.1) = 1.1241$

$t_2 = t_1 + \Delta t = 0.1 + 0.1 = 0.2$

$f(t_2,\tilde{y}_2) = 5\tilde{y}_2^2 t_2 + 3t_2 = 5(1.1241)^2(.2) + 3(.2) = 1.8636$

corrected $y_2 = y_1 + [f(t_1,y_1) + f(t_2,\tilde{y}_2)]\frac{\Delta t}{2} = 1.04 + [.8408 + 1.8636]\frac{.1}{2} = \underline{1.1752}$

$t_2 = t_1 + \Delta t = \underline{0.2}$