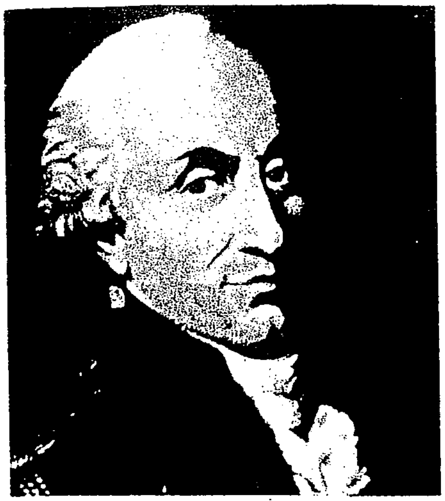


Charles Augustin de Coulomb (1736–1806) was a French military engineer and physicist. His early work on statics and mechanics was presented in 1779 in his great memoir *The Theory of Simple Machines,* which describes the effect of resistance and the so-called "Coulomb's law of proportionality" between friction and normal pressure. In 1784, he obtained the correct solution to the problem of the small oscillations of a body subjected to torsion. He is well known for his laws of force for electrostatic and magnetic charges. His name is remembered through the unit of electric charge. (Courtesy of *Applied Mechanics Reviews*).

CHAPTER 3

Harmonically Excited Vibration

3.1 Introduction

A mechanical or structural system is said to undergo forced vibration whenever external energy is supplied to the system during vibration. External energy can be supplied to the system through either an applied force or an imposed displacement excitation. The applied force or displacement excitation may be harmonic, nonharmonic but periodic, nonperiodic, or random in nature. The response of a system to a harmonic excitation is called *harmonic response.* The nonperiodic excitation may have a long or short duration. The response of a dynamic system to suddenly applied nonperiodic excitations is called *transient response.*

In this chapter, we shall consider the dynamic response of a single degree of freedom system under harmonic excitations of the form $F(t) = F_0 e^{i(\omega t + \phi)}$ or $F(t) = F_0 \cos(\omega t + \phi)$ or $F(t) = F_0 \sin(\omega t + \phi)$, where F_0 is the amplitude, ω is the frequency, and ϕ is the phase angle of the harmonic excitation. The value of ϕ depends on the value of $F(t)$ at $t = 0$ and is usually taken to be zero. Under a harmonic excitation, the response of the system will also be harmonic. If the frequency of excitation coincides with the natural frequency of the system, the response of the system will be very large. This condition, known as resonance, is to be avoided to prevent failure of the system. The vibration produced by an unbalanced rotating machine, the oscillations of a tall chimney due to vortex shedding in a steady wind, and the vertical motion of an automobile on a sinusoidal road surface are examples of harmonically excited vibration.

3.2 Equation of Motion

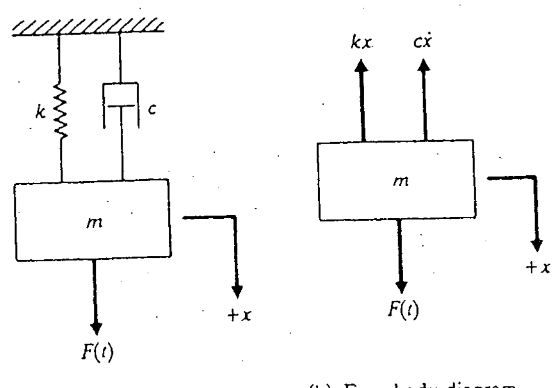
If a force $F(t)$ acts on a viscously damped spring-mass system as shown in Fig. 3.1, the equation of motion can be obtained using Newton's second law:

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{3.1}$$

Since this equation is nonhomogeneous, its general solution $x(t)$ is given by the sum of the homogeneous solution, $x_h(t)$, and the particular solution, $x_p(t)$. The homogeneous solution, which is the solution of the homogeneous equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{3.2}$$

represents the free vibration of the system and was discussed in Chapter 2. As seen in Section 2.6.2, this free vibration dies out with time under each of the three possible conditions of damping (underdamping, critical damping, and overdamping) and under all possible initial conditions. Thus the general solution of Eq. (3.1) eventually reduces to the particular solution $x_p(t)$, which represents the steady-state vibration. The steady-state motion is present as long as the forcing function is present. The variations of homogeneous, particular, and general solutions with time for a typical case are shown in Fig. 3.2. It can be seen that $x_h(t)$ dies out and $x(t)$ becomes $x_p(t)$ after some time (τ in Fig. 3.2). The part of the motion that dies out due to damping (the free vibration part) is called *transient*. The rate at which the transient motion decays depends on the values of the system parameters k, c, and m. In this chapter, except in Section 3.3, we ignore the transient motion and derive only the particular solution of Eq. (3.1), which represents the steady-state response, under harmonic forcing functions.



(a) (b) Free-body diagram

FIGURE 3.1 A spring-mass-damper system.

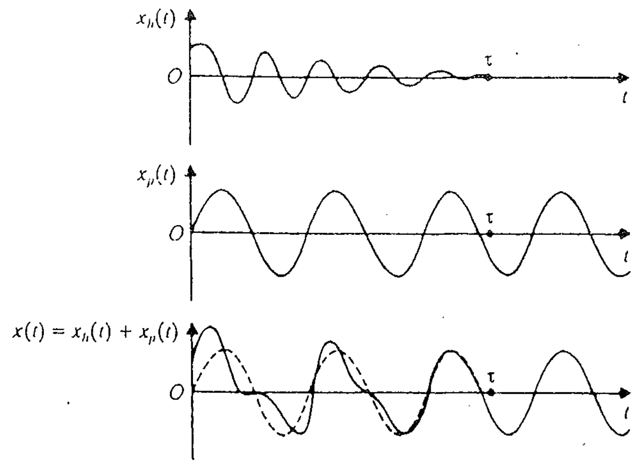


FIGURE 3.2 Homogenous, particular, and general solutions of Eq. (3.1) for an underdamped case.

3.3 Response of an Undamped System under Harmonic Force

Before studying the response of a damped system, we consider an undamped system subjected to a harmonic force, for the sake of simplicity. If a force $F(t) = F_0 \cos \omega t$ acts on the mass m of an undamped system, the equation of motion, Eq. (3.1), reduces to

$$m\ddot{x} + kx = F_0 \cos \omega t \tag{3.3}$$

The homogeneous solution of this equation is given by

$$x_h(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t \tag{3.4}$$

where $\omega_n = (k/m)^{1/2}$ is the natural frequency of the system. Because the exciting force $F(t)$ is harmonic, the particular solution $x_p(t)$ is also harmonic and has the same frequency ω. Thus we assume a solution in the form

$$x_p(t) = X \cos \omega t \tag{3.5}$$

where X is a constant that denotes the maximum amplitude of $x_p(t)$. By substituting Eq. (3.5) into Eq. (3.3) and solving for X, we obtain

$$X = \frac{F_0}{k - m\omega^2} \tag{3.6}$$

Thus the total solution of Eq. (3.3) is

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t \tag{3.7}$$

Using the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, we find that

$$C_1 = x_0 - \frac{F_0}{k - m\omega^2}, \qquad C_2 = \frac{\dot{x}_0}{\omega_n} \tag{3.8}$$

and hence

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right)\sin\omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\cos\omega t \tag{3.9}$$

The maximum amplitude X in Eq. (3.6) can also be expressed as

$$\frac{X}{\delta_{st}} = \frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2} \tag{3.10}$$

where $\delta_{st} = F_0/k$ denotes the deflection of the mass under a force F_0 and is sometimes called "static deflection" since F_0 is a constant (static) force. The quantity X/δ_{st} represents the ratio of the dynamic to the static amplitude of motion and is called the *magnification factor, amplification factor,* or *amplitude ratio*. The variation of the amplitude ratio, X/δ_{st}, with the frequency ratio $r = \omega/\omega_n$ (Eq. 3.10) is shown in Fig. 3.3. From this figure, the response of the system can be identified to be of three types.

Case 1. When $0 < \omega/\omega_n < 1$, the denominator in Eq. (3.10) is positive and the response is given by Eq. (3.5) without change. The harmonic response of the system $x_p(t)$ is said to be in phase with the external force as shown in Fig. 3.4.

Case 2. When $\omega/\omega_n > 1$, the denominator in Eq. (3.10) is negative, and the steady-state solution can be expressed as

$$x_p(t) = -X\cos\omega t \tag{3.11}$$

where the amplitude of motion X is redefined to be a positive quantity as

$$X = \frac{\delta_{st}}{\left(\dfrac{\omega}{\omega_n}\right)^2 - 1} \tag{3.12}$$

The variations of $F(t)$ and $x_p(t)$ with time are shown in Fig. 3.5. Since $x_p(t)$ and $F(t)$ have opposite signs, the response is said to be 180° out of phase with the

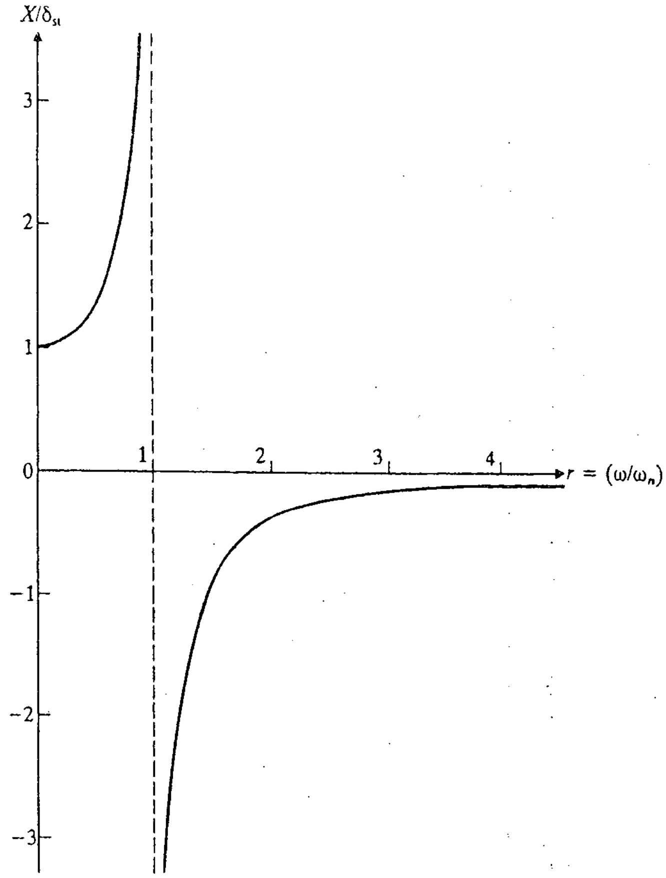


FIGURE 3.3

external force. Further, as $\omega/\omega_n \to \infty$, $X \to 0$. Thus the response of the system to a harmonic force of very high frequency is close to zero.

Case 3. When $\omega/\omega_n = 1$, the amplitude X given by Eq. (3.10) or (3.12) becomes infinite. This condition, for which the forcing frequency ω is equal to the natural frequency of the system ω_n, is called *resonance*. To find the response for this condition, we rewrite Eq. (3.9) as

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st} \left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\dfrac{\omega}{\omega_n} \right)^2} \right] \tag{3.13}$$

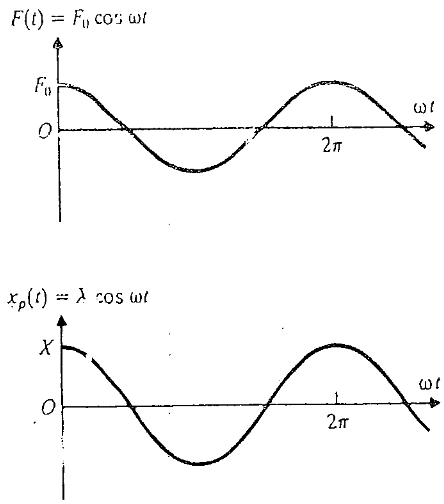


FIGURE 3.4

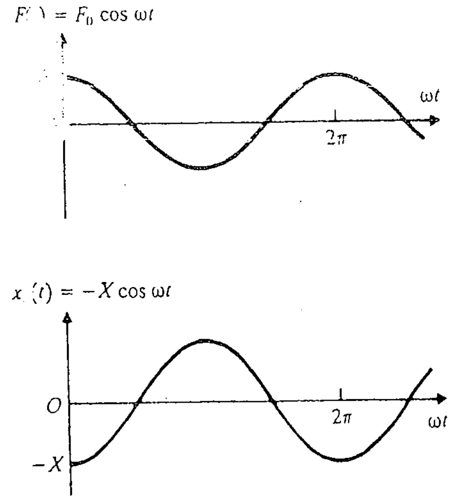


FIGURE 3.5

Since the last term of this equation takes an indefinite form for $\omega = \omega_n$, we apply L'Hospital's rule [3.1] to evaluate the limit of this term:

$$\lim_{\omega \to \omega_n} \left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\dfrac{\omega}{\omega_n} \right)^2} \right] = \lim_{\omega \to \omega_n} \left[\frac{\dfrac{d}{d\omega}(\cos \omega t - \cos \omega_n t)}{\dfrac{d}{d\omega}\left(1 - \dfrac{\omega^2}{\omega_n^2} \right)} \right]$$

$$= \lim_{\omega \to \omega_n} \left[\frac{t \sin \omega t}{2 \dfrac{\omega}{\omega_n^2}} \right] = \frac{\omega_n t}{2} \sin \omega_n t. \tag{3.14}$$

Thus the response of the system at resonance becomes

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st} \omega_n t}{2} \sin \omega_n t \tag{3.15}$$

It can be seen from Eq. (3.15) that at resonance, $x(t)$ increases indefinitely. The last term of Eq. (3.15) is shown in Fig. 3.6, from which the amplitude of the response can be seen to increase linearly with time.

3.3.1 Total Response

The total response of the system, Eq. (3.7) or Eq. (3.9), can also be expressed as

$$x(t) = A \cos(\omega_n t - \phi) + \frac{\delta_{st}}{1 - \left(\dfrac{\omega}{\omega_n} \right)^2} \cos \omega t; \qquad \text{for } \frac{\omega}{\omega_n} < 1 \tag{3.16}$$

$$x(t) = A \cos(\omega_n t - \phi) - \frac{\delta_{st}}{1 - \left(\dfrac{\omega}{\omega_n} \right)^2} \cos \omega t; \qquad \text{for } \frac{\omega}{\omega_n} > 1 \tag{3.17}$$

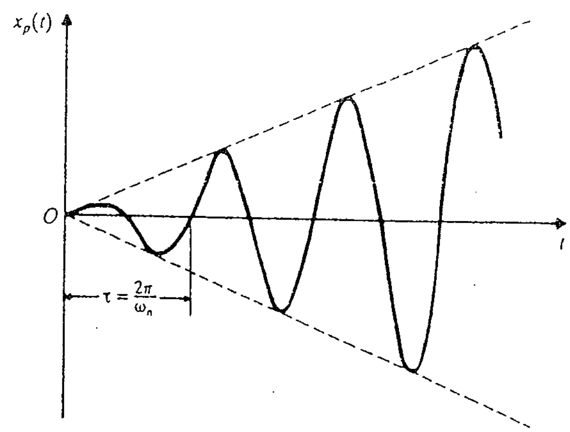


FIGURE 3.6

where A and ϕ can be determined as in the case of Eq. (2.21). Thus the complete motion can be expressed as the sum of two cosine curves of different frequencies. In Eq. (3.16), the forcing frequency ω is smaller than the natural frequency, and the total response is shown in Fig. 3.7(a). In Eq. (3.17), the forcing frequency is greater than the natural frequency, and the total response appears as shown in Fig. 3.7(b).

3.3.2 Beating Phenomenon

If the forcing frequency is close to, but not exactly equal to, the natural frequency of the system, a phenomenon known as *beating* may occur. In this kind of vibration, the amplitude builds up and then diminishes in a regular pattern. The phenomenon of beating can be explained by considering the solution given by Eq. (3.9). If the initial conditions are taken as $x_0 = \dot{x}_0 = 0$, Eq. (3.9) reduces to

$$x(t) = \frac{(F_0/m)}{\omega_n^2 - \omega^2}(\cos \omega t - \cos \omega_n t)$$

$$= \frac{(F_0/m)}{\omega_n^2 - \omega^2}\left[2 \sin \frac{\omega + \omega_n}{2}t \cdot \sin \frac{\omega_n - \omega}{2}t\right] \tag{3.18}$$

Let the forcing frequency ω be slightly less than the natural frequency:

$$\omega_n - \omega = 2\varepsilon \tag{3.19}$$

where ε is a small positive quantity. Then $\omega_n \approx \omega$ and

$$\omega + \omega_n \simeq 2\omega \tag{3.20}$$

Multiplication of Eqs. (3.19) and (3.20) gives

$$\omega_n^2 - \omega^2 = 4\varepsilon\omega \tag{3.21}$$

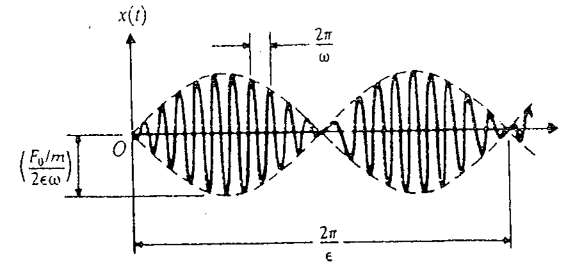


FIGURE 3.8

or the points of maximum amplitude is called the *period of beating* (τ_b) and is given by

$$\tau_b = \frac{2\pi}{2\varepsilon} = \frac{2\pi}{\omega_n - \omega} \tag{3.23}$$

with the frequency of beating defined as

$$\omega_b = 2\varepsilon = \omega_n - \omega$$

(3.22)

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EXAMPLE 3.1 Plate Supporting a Pump

A reciprocating pump, weighing 150 lb, is mounted at the middle of a steel plate of thickness 0.5 in., width 20 in., and length 100 in., clamped along two edges as shown in Fig. 3.9. During operation of the pump, the plate is subjected to a harmonic force, $F(t) = 50 \cos 62.832\, t$ lb. Find the amplitude of vibration of the plate.

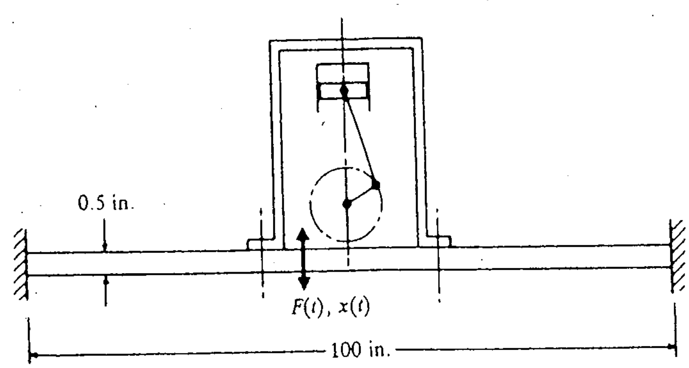


FIGURE 3.9

Given: Pump weight = 150 lb; plate dimensions: thickness (t) = 0.5 in., width (w) = 20 in., and length (l) = 100 in.; and harmonic force: $F(t)$ = 50 cos 62.832 t lb.

Find: Amplitude of vibration of the plate, X.

Approach: Find the stiffness of the plate by modeling it as a clamped beam. Use the equation for the response under harmonic excitation.

Solution: The plate can be modeled as a fixed-fixed beam having Young's modulus (E) = 30×10^6 psi, length (l) = 100 in., and area moment of inertia (I) = $\frac{1}{12}(20)(0.5)^3$ = 0.2083 in^4. The bending stiffness of the beam is given by

$$k = \frac{192EI}{l^3} = \frac{192(30 \times 10^6)(0.2083)}{(100)^3} = 1200.0 \text{ lb/in.} \tag{E.1}$$

The amplitude of harmonic response is given by Eq. (3.6) with F_0 = 50 lb, m = 150/386.4 lb-sec^2/in. (neglecting the weight of the steel plate), k = 1200.0 lb/in., and ω = 62.832 rad/sec. Thus Eq. (3.6) gives

$$X = \frac{F_0}{k - m\omega^2} = \frac{50}{1200.0 - (150/386.4)(62.832)^2} = -0.1504 \text{ in.} \tag{E.2}$$

The negative sign indicates that the response $x(t)$ of the plate is out of phase with the excitation $F(t)$. ■

3.4 Response of a Damped System under Harmonic Force

If the forcing function is given by $F(t) = F_0 \cos \omega t$, the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \tag{3.24}$$

The particular solution of Eq. (3.24) is also expected to be harmonic; we assume it in the form[1]

$$x_p(t) = X \cos(\omega t - \phi) \tag{3.25}$$

where X and ϕ are constants to be determined. X and ϕ denote the amplitude and phase angle of the response, respectively. By substituting Eq. (3.25) into Eq. (3.24), we arrive at

$$X[(k - m\omega^2)\cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)] = F_0 \cos \omega t \tag{3.26}$$

Using the trigonometric relations

$$\cos(\omega t - \phi) = \cos \omega t \cos \phi + \sin \omega t \sin \phi$$

$$\sin(\omega t - \phi) = \sin \omega t \cos \phi - \cos \omega t \sin \phi$$

[1]Alternatively, we can assume $x_p(t)$ to be of the form $x_p(t) = C_1 \cos \omega t + C_2 \sin \omega t$, which also involves two constants C_1 and C_2. But the final result will be the same in both the cases.

in Eq. (3.26) and equating the coefficients of $\cos \omega t$ and $\sin \omega t$ on both sides of the resulting equation, we obtain

$$X\left[(k - m\omega^2)\cos\phi + c\omega\sin\phi\right] = F_0$$
$$X\left[(k - m\omega^2)\sin\phi - c\omega\cos\phi\right] = 0 \tag{3.27}$$

Solution of Eqs. (3.27) gives

$$X = \frac{F_0}{\left[(k - m\omega^2)^2 + c^2\omega^2\right]^{1/2}} \tag{3.28}$$

and

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \tag{3.29}$$

By inserting the expressions of X and ϕ from Eqs. (3.28) and (3.29) into Eq. (3.25) we obtain the particular solution of Eq. (3.24). Figure 3.10 shows typical plots of the forcing function and (steady-state) response. Dividing both the numerator and denominator of Eq. (3.28) by k and making the following substitutions

$$\omega_n = \sqrt{\frac{k}{m}} = \text{undamped natural frequency,}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}}; \quad \frac{c}{m} = 2\zeta\omega_n,$$

$$\delta_{st} = \frac{F_0}{k} = \text{deflection under the static force } F_0, \text{ and}$$

$$r = \frac{\omega}{\omega_n} = \text{frequency ratio}$$

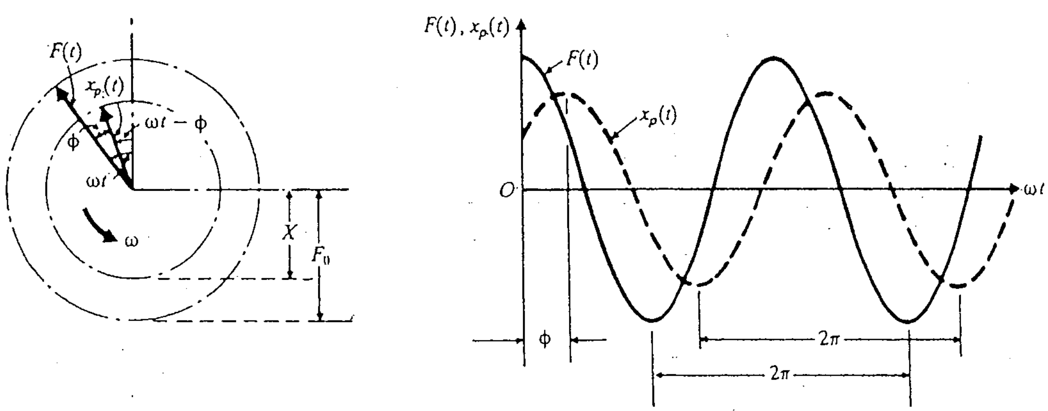


FIGURE 3.10 Graphical representation of forcing function and response.

we obtain

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{3.30}$$

and

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \tag{3.31}$$

As stated in Section 3.3, the quantity $M = X/\delta_{st}$ is called the *magnification factor, amplification factor,* or *amplitude ratio.* The variations of X/δ_{st} and ϕ with the frequency ratio r and the damping ratio ζ are shown in Fig. 3.11.

The following characteristics of the magnification factor (M) can be noted from Eq. (3.30) and Fig. 3.11(a):

1. For an undamped system ($\zeta = 0$), Eq. (3.30) reduces to Eq. (3.10), and $M \to \infty$ as $r \to 1$.

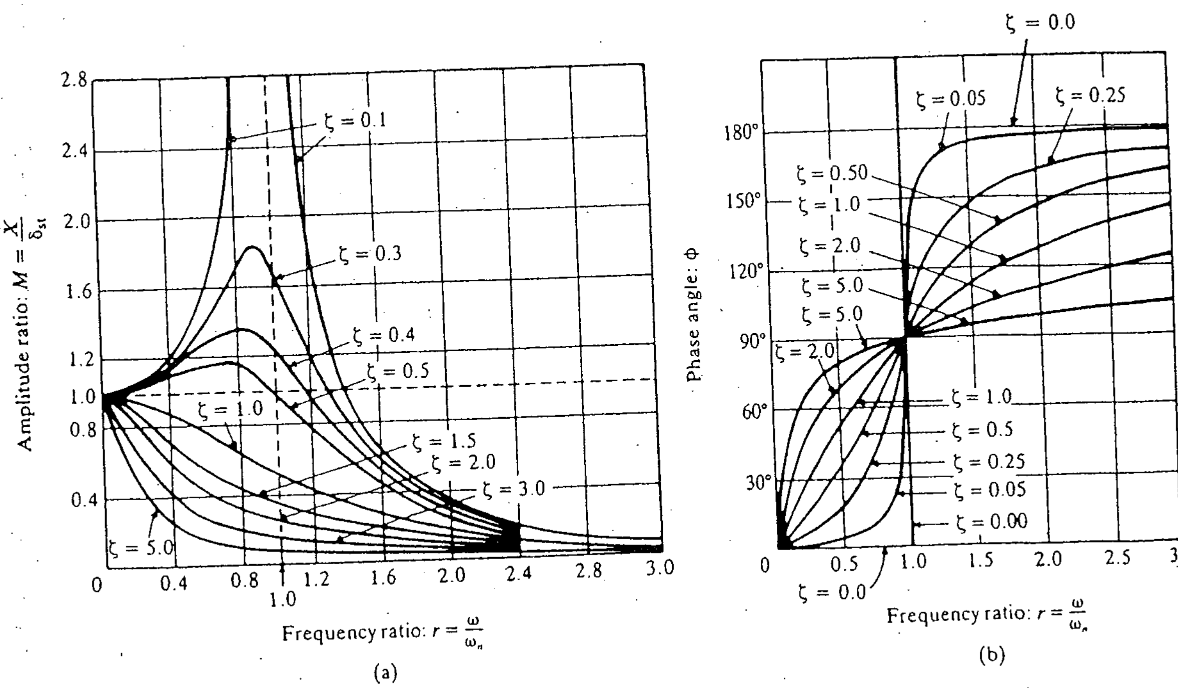


FIGURE 3.11 Variation of X and ϕ with frequency ratio r.

2. Any amount of damping ($\zeta > 0$) reduces the magnification factor (M) for all values of the forcing frequency.
3. For any specified value of r, a higher value of damping reduces the value of M.
4. In the degenerate case of a constant force (when $r = 0$), the value of $M = 1$.
5. The reduction in M in the presence of damping is very significant at or near resonance.
6. The amplitude of forced vibration becomes smaller with increasing values of the forcing frequency (that is, $M \to 0$ as $r \to \infty$).
7. For $0 < \zeta < \dfrac{1}{\sqrt{2}}$, the maximum value of M occurs when (see Problem 3.19)

$$r = \sqrt{1 - 2\zeta^2} \quad \text{or} \quad \omega = \omega_n \sqrt{1 - 2\zeta^2} \tag{3.32}$$

which can be seen to be lower than the undamped natural frequency ω_n and the damped natural frequency $\omega_d = \omega_n \sqrt{1 - \zeta^2}$.

8. The maximum value of X (when $r = \sqrt{1 - 2\zeta^2}$) is given by

$$\left(\frac{X}{\delta_{st}}\right)_{\max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{3.33}$$

and the value of X at $\omega = \omega_n$ by

$$\left(\frac{X}{\delta_{st}}\right)_{\omega = \omega_n} = \frac{1}{2\zeta} \tag{3.34}$$

Equation (3.33) can be used for the experimental determination of the measure of damping present in the system. In a vibration test, if the maximum amplitude of the response $(X)_{\max}$ is measured, the damping ratio of the system can be found using Eq. (3.33). Conversely, if the amount of damping is known, one can make an estimate of the maximum amplitude of vibration.

9. For $\zeta = \dfrac{1}{\sqrt{2}}$, $\dfrac{dM}{dr} = 0$ when $r = 0$. For $\zeta > \dfrac{1}{\sqrt{2}}$, the graph of M monotonically decreases with increasing values of r.

The following characteristics of the phase angle can be observed from Eq. (3.31) and Fig. 3.11(b):

1. For an undamped system ($\zeta = 0$), Eq. (3.31) shows that the phase angle is 0 for $0 < r < 1$ and 180° for $r > 1$. This implies that the excitation and response are in phase for $0 < r < 1$ and out of phase for $r > 1$ when $\zeta = 0$.
2. For $\zeta > 0$ and $0 < r < 1$, the phase angle is given by $0 < \phi < 90°$, implying that the response lags the excitation.
3. For $\zeta > 0$ and $r > 1$, the phase angle is given by $90° < \phi < 180°$, implying that the response leads the excitation.
4. For $\zeta > 0$ and $r = 1$, the phase angle is given by $\phi = 90°$, implying that the phase difference between the excitation and the response is 90°.

5. For $\zeta > 0$ and large values of r, the phase angle approaches 180°, implying that the response and the excitation are out of phase.

3.4.1 Total Response

The complete solution is given by $x(t) = x_h(t) + x_p(t)$ where $x_h(t)$ is given by Eq. (2.64). Thus

$$x(t) = X_0 e^{-\zeta\omega_n t}\cos(\omega_d t - \phi_0) + X\cos(\omega t - \phi) \tag{3.35}$$

where

$$\omega_d = \sqrt{1 - \zeta^2}\cdot\omega_n \tag{3.36}$$

$$r = \frac{\omega}{\omega_n} \tag{3.37}$$

X and ϕ are given by Eqs. (3.30) and (3.31), respectively, and X_0 and ϕ_0 can be determined from the initial conditions.

3.4.2 Quality Factor and Bandwidth

For small values of damping ($\zeta < 0.05$), we can take

$$\left(\frac{X}{\delta_{st}}\right)_{max} \simeq \left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta} = Q \tag{3.38}$$

The value of the amplitude ratio at resonance is also called *Q factor* or *quality factor* of the system, in analogy with some electrical-engineering applications, such as the tuning circuit of a radio, where the interest lies in an amplitude at resonance that is as large as possible [3.2]. The points R_1 and R_2, where the amplification factor falls to $Q/\sqrt{2}$, are called *half power points* because the power absorbed (ΔW) by the damper (or by the resistor in an electrical circuit), responding harmonically at a given frequency, is proportional to the square of the amplitude (see Eq. 2.94):

$$\Delta W = \pi c\omega X^2 \tag{3.39}$$

The difference between the frequencies associated with the half power points R_1 and R_2 is called the *bandwidth* of the system (see Fig. 3.12). To find the values of R_1 and R_2, we set $X/\delta_{st} = Q/\sqrt{2}$ in Eq. (3.30) so that

$$\frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = \frac{Q}{\sqrt{2}} = \frac{1}{2\sqrt{2}\zeta}$$

or

$$r^4 - r^2(2 - 4\zeta^2) + (1 - 8\zeta^2) = 0 \tag{3.40}$$

The solution of Eq. (3.40) gives

$$r_1^2 = 1 - 2\zeta^2 - 2\zeta\sqrt{1 + \zeta^2}, \qquad r_2^2 = 1 - 2\zeta^2 + 2\zeta\sqrt{1 + \zeta^2} \tag{3.41}$$

3.13 How does the force transmitted to the base change as the speed of the machine increases?

3.14 If a vehicle vibrates badly while moving on a uniformly bumpy road, will a change in the speed improve the condition?

3.15 Is it possible to find the maximum amplitude of a damped forced vibration for any value of r by equating the energy dissipated by damping to the work done by the external force?

3.16 What assumptions are made about the motion of a forced vibration with nonviscous damping in finding the amplitude?

3.17 Is it possible to find the approximate value of the amplitude of a damped forced vibration without considering damping at all? If so, under what circumstances?

3.18 Is dry friction effective in limiting the reasonant amplitude?

3.19 How do you find the response of a viscously damped system under rotating unbalance?

3.20 What is the frequency of the response of a viscously damped system when the external force is $F_0 \sin \omega t$? Is this response harmonic?

3.21 What is the difference between the peak amplitude and the resonant amplitude?

3.22 Why is viscous damping used in most cases rather than other types of damping?

3.23 What is self-excited vibration?

Problems

The problem assignments are organized as follows:

Problems	Section Covered	Topic Covered
3.1–3.16	3.3	Undamped systems
3.17–3.32	3.4	Damped systems
3.33–3.41	3.6	Base excitation
3.42–3.52	3.7	Rotating unbalance
3.53–3.55	3.8	Response under Coulomb damping
3.56–3.57	3.9	Response under hysteresis damping
3.58–3.61	3.10	Response under other types of damping
3.62–3.65	3.11	Self excitation and stability
3.66–3.69	3.12	Computer program
3.70–3.71	—	Projects

3.1 A weight of 50 N is suspended from a spring of stiffness 4000 N/m and is subjected to a harmonic force of amplitude 60 N and frequency 6 Hz. Find (a) the extension of the spring due to the suspended weight, (b) the static displacement of the spring due to the maximum applied force, and (c) the amplitude of forced motion of the weight.

3.2 A spring-mass system is subjected to a harmonic force whose frequency is close to the natural frequency of the system. If the forcing frequency is 39.8 Hz and the natural frequency is 40.0 Hz, determine the period of beating.

3.3 A spring-mass system consists of a mass weighing 100 N and a spring with a stiffness of 2000 N/m. The mass is subjected to resonance by a harmonic force $F(t) = 25 \cos \omega t$ N. Find the amplitude of the forced motion at the end of (a) $\frac{1}{4}$ cycle, (b) $2\frac{1}{2}$ cycles, and (c) $5\frac{3}{4}$ cycles.

3.4 A mass m is suspended from a spring of stiffness 4000 N/m and is subjected to a harmonic force having an amplitude of 100 N and a frequency of 5 Hz. The amplitude of the forced motion of the mass is observed to be 20 mm. Find the value of m.

3.5 A spring-mass system with $m = 10$ kg and $k = 5000$ N/m is subjected to a harmonic force of amplitude 250 N and frequency ω. If the maximum amplitude of the mass is observed to be 100 mm, find the value of ω.

3.6 In Fig. 3.1(a), a periodic force $F(t) = F_0 \cos \omega t$ is applied at a point on the spring that is located at a distance of 25 percent of its length from the fixed support. Assuming that $c = 0$, find the steady-state response of the mass m.

3.7 An aircraft engine has a rotating unbalanced mass m at radius r. If the wing can be modeled as a cantilever beam of uniform cross section $a \times b$, as shown in Fig. 3.34(b), determine the maximum deflection of the wing at an engine speed of N rpm. Assume damping to be negligible.

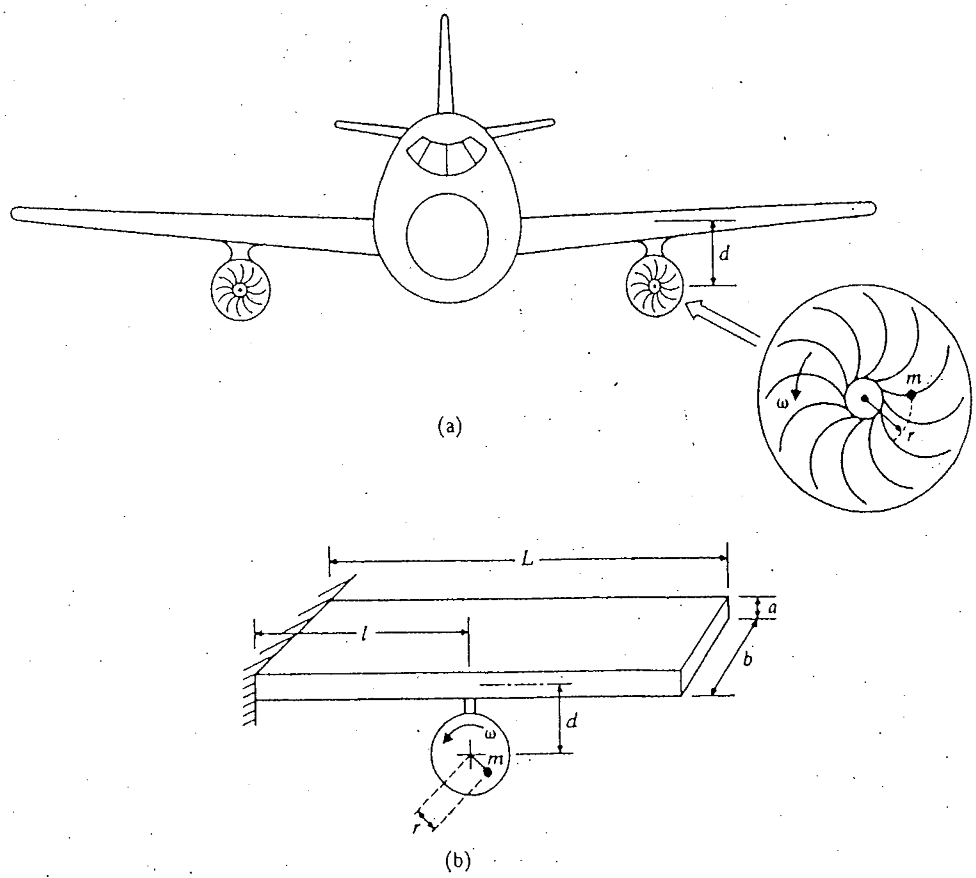
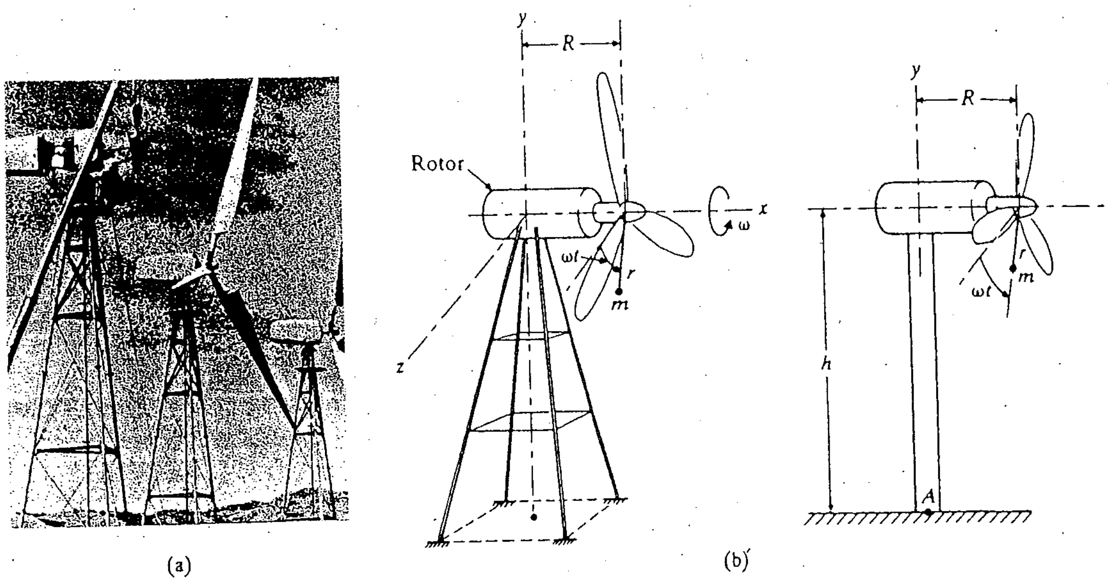


FIGURE 3.34

3.8 A three-bladed wind turbine (Fig. 3.35a) has a small unbalanced mass m located at a radius r in the plane of the blades. The blades are located from the central vertical (y) axis at a distance R and rotate at an angular velocity of ω. If the supporting truss can be modeled as a hollow steel shaft of outer diameter 0.1 m and inner diameter 0.08 m, determine the maximum stresses developed at the base of the support (point A). The mass moment of inertia of the turbine system about the vertical (y) axis is J_0. Assume R = 0.5 m, m = 0.1 kg, r = 0.1 m, J_0 = 100 kg-m^2, h = 8 m, and ω = 31.416 rad/sec.

3.9 An electromagnetic fatigue testing machine is shown in Fig. 3.36 in which an alternating force is applied to the specimen by passing an alternating current of frequency f through the armature. If the weight of the armature is 40 lb, the stiffness of the spring (k_1) is 10,217.0296 lb/in and the stiffness of the steel specimen is 75 × 10^4 lb/in, determine the frequency of the a.c. current that induces a stress in the specimen that is twice the amount generated by the magnets.

• 3.10 The spring actuator shown in Fig. 3.37 operates by using the air pressure from a pneumatic controller (p) as input and providing an output displacement to a valve (x) proportional to the input air pressure. The diaphragm, made of a fabric-base rubber, has an area A and deflects under the input air pressure against a spring of stiffness k. Find the response of the valve under a harmonically fluctuating input air pressure $p(t) = p_0 \sin \omega t$ for the following data: p_0 = 10 psi, ω = 8 rad/s, A = 100 in^2, k = 400 lb/in, weight of spring = 15 lb, and weight of valve and valve rod = 20 lb.



(a) (b)

FIGURE 3.35 (Photo courtesy of *Power Transmission Design*)

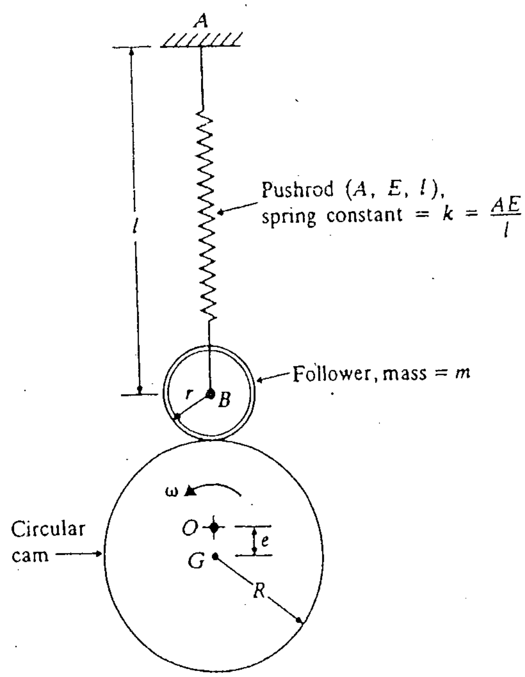


FIGURE 3.38

3.11 In the cam-follower system shown in Fig. 3.38, the rotation of the cam imparts a vertical motion to the follower. The pushrod, which acts as a spring, has been compressed by an amount x_0 before assembly. Determine the following: (a) equation of motion of the follower, including the gravitational force; (b) force exerted on the follower by the cam; and (c) conditions under which the follower loses contact with the cam.

3.12* Design a solid steel shaft supported in bearings which carries the rotor of a turbine at the middle. The rotor weighs 500 lb and delivers a power of 200 hp at 3000 rpm. In order to keep the stress due to the unbalance in the rotor small, the critical speed of the shaft is to be made one-fifth of the operating speed of the rotor. The length of the shaft is to be made equal to at least 30 times its diameter.

3.13 A hollow steel shaft, of length 100 in., outer diameter 4 in. and inner diameter 3.5 in., carries the rotor of a turbine, weighing 500 lb, at the middle and is supported at the ends in bearings. The clearance between the rotor and the stator is 0.5 in. The rotor has an eccentricity equivalent to a weight of 0.5 lb at a radius of 2 in. A limit switch is installed to stop the rotor whenever the rotor touches the stator. If the rotor operates at resonance, how long will it take to activate the limit switch? Assume the initial displacement and velocity of the rotor perpendicular to the shaft to be zero.

3.14 A steel cantilever beam, carrying a weight of 0.1 lb at the free end, is used as a frequency meter.[6] The beam has a length of 10 in., width of 0.2 in., and thickness of

*The asterisk denotes a design type problem or a problem with no unique answer.

[6]The use of cantilever beams as frequency meters is discussed in detail in Section 10.4.

0.05 in. The internal friction is equivalent to a damping ratio of 0.01. When the fixed end of the beam is subjected to a harmonic displacement $y(t) = 0.05 \cos \omega t$, the maximum tip displacement has been observed to be 2.5 in. Find the forcing frequency.

3.15 Derive the equation of motion and find the steady-state response of the system shown in Fig. 3.39 for rotational motion about the hinge O for the following data: $k_1 = k_2 = 5000$ N/m, $a = 0.25$ m, $b = 0.5$ m, $l = 1$ m, $M = 50$ kg, $m = 10$ kg, $F_0 = 500$ N, $\omega = 1000$ rpm.

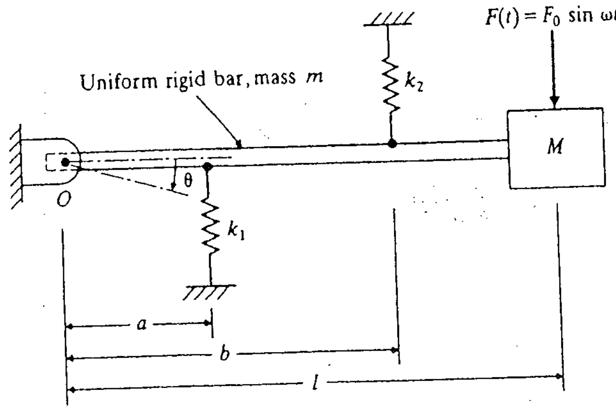


FIGURE 3.39

3.16 Derive the equation of motion and find the steady-state solution of the system shown in Fig. 3.40 for rotational motion about the hinge O for the following data: $k = 5000$ N/m, $l = 1$ m, $m = 10$ kg, $M_0 = 100$ N-m, $\omega = 1000$ rpm.

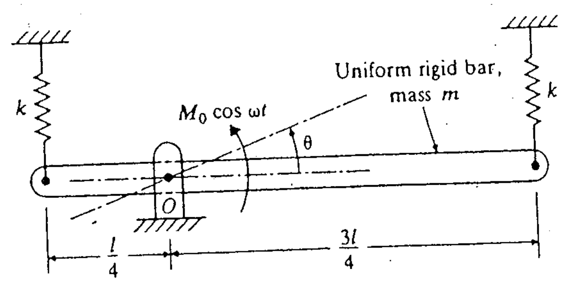


FIGURE 3.40

3.17 A four-cylinder automobile engine is to be supported on three shock mounts, as indicated in Fig. 3.41. The engine block assembly weighs 500 lb. If the unbalanced force generated by the engine is given by $200 \sin 100\pi t$ lb, design the three shock mounts (each of stiffness k and viscous damping constant c) such that the amplitude of vibration is less than 0.1 in.

3.18 The propeller of a ship, of weight 10^5 N and polar mass moment of inertia 10,000 kg-m^2, is connected to the engine through a hollow stepped steel propeller shaft, as shown in Fig. 3.42. Assuming that water provides a viscous damping ratio of 0.1,

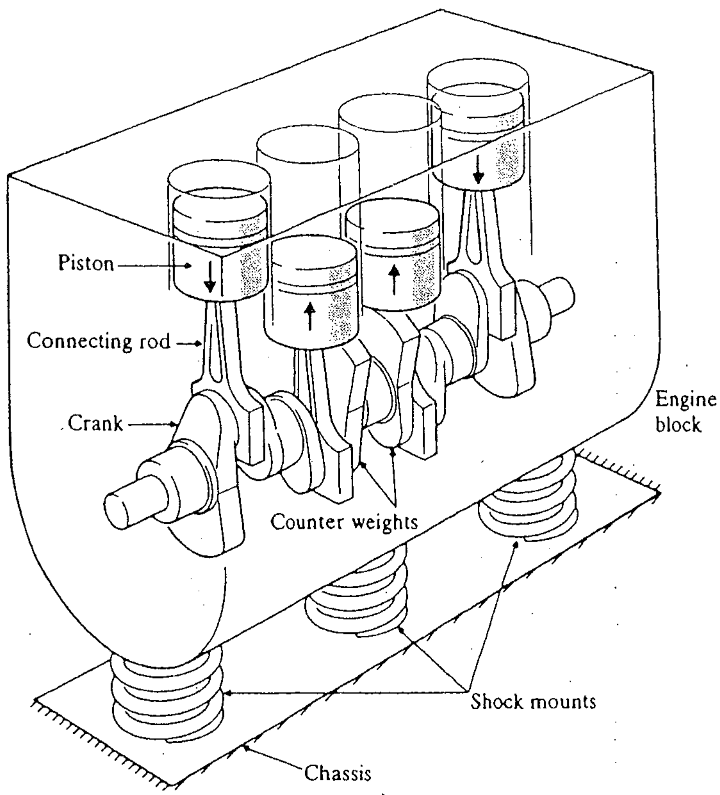


FIGURE 3.41

determine the torsional vibratory response of the propeller when the engine induces a harmonic angular displacement of 0.05 sin 314.16 t rad at the base (point A) of the propeller shaft.

3.19 Find the frequency ratio $r = \omega/\omega_n$ at which the amplitude of a single degree of freedom damped system attains the maximum value. Also find the value of the maximum amplitude.

3.20 Figure 3.43 shows a permanent-magnet moving coil ammeter. When current (I) flows through the coil wound on the core, the core rotates by an angle proportional to the magnitude of the current that is indicated by the pointer on a scale. The core, with the coil, has a mass moment of inertia J_0, the torsional spring constant of k_t, and the torsional damper has a damping constant of c_t. The scale of the ammeter is calibrated such that when a d.c. current of magnitude 1 ampere is passed through the coil, the pointer indicates a current of 1 ampere. The meter has to be recalibrated for measuring the magnitude of a.c. current. Determine the steady-state value of the current indicated by the pointer when an a.c. current of magnitude 5 amperes and frequency 50 Hz is passed through the coil. Assume $J_0 = 0.001$ N-m^2, $k_t = 62.5$ N-m/rad and $c_t = 0.5$ N-m-s/rad.

3.21 A spring-mass-damper system is subjected to a harmonic force. The amplitude is found to be 20 mm at resonance and 10 mm at a frequency 0.75 times the resonant frequency. Find the damping ratio of the system.

3.22 For the system shown in Fig. 3.44, x and y denote, respectively, the absolute displacements of the mass m and the end Q of the dashpot c_1. (a) Derive the equation of motion of the mass m, (b) find the steady state displacement of the mass m, and (c) find the force transmitted to the support at P, when the end Q is subjected to the harmonic motion $y(t) = Y \cos \omega t$.

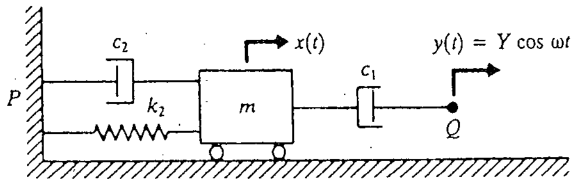


FIGURE 3.44

3.23 Show that, for small values of damping, the damping ratio ζ can be expressed as

$$\zeta = \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1}$$

where ω_1 and ω_2 are the frequencies corresponding to the half power points.

3.24 A torsional system consists of a disc of mass moment of inertia $J_0 = 10$ kg-m^2, a torsional damper of damping constant $c_t = 300$ N-m-s/rad, and a steel shaft of diameter 4 cm and length 1 m (fixed at one end and attached to the disc at the other end). A steady angular oscillation of amplitude 2° is observed when a harmonic torque of magnitude 1000 N-m is applied to the disc. (a) Find the frequency of the applied torque, and (b) find the maximum torque transmitted to the support.

3.25 For a vibrating system, $m = 10$ kg, $k = 2500$ N/m, and $c = 45$ N-s/m. A harmonic force of amplitude 180 N and frequency 3.5 Hz acts on the mass. If the initial displacement and velocity of the mass are 15 mm and 5 m/s, find the complete solution representing the motion of the mass.

3.26 The peak amplitude of a single degree of freedom system, under a harmonic excitation, is observed to be 0.2 in. If the undamped natural frequency of the system is 5 Hz, and the static deflection of the mass under the maximum force is 0.1 in., (a) estimate the damping ratio of the system, and (b) find the frequencies corresponding to the amplitudes at half power.

3.27 The landing gear of an airplane can be idealized as the spring-mass-damper system shown in Fig. 3.45. If the runway surface is described $y(t) = y_0 \cos \omega t$, determine the values of k and c that limit the amplitude of vibration of the airplane (x) to 0.1 m. Assume $m = 2000$ kg, $y_0 = 0.2$ m and $\omega = 157.08$ rad/s.

3.28 A precision grinding machine (Fig. 3.46) is supported on an isolator that has a stiffness of 1 MN/m and a viscous damping constant of 1 kN-s/m. The floor on which the machine is mounted is subjected to a harmonic disturbance due to the operation of an unbalanced engine in the vicinity of the grinding machine. Find the maximum acceptable displacement amplitude of the floor if the resulting amplitude of vibration

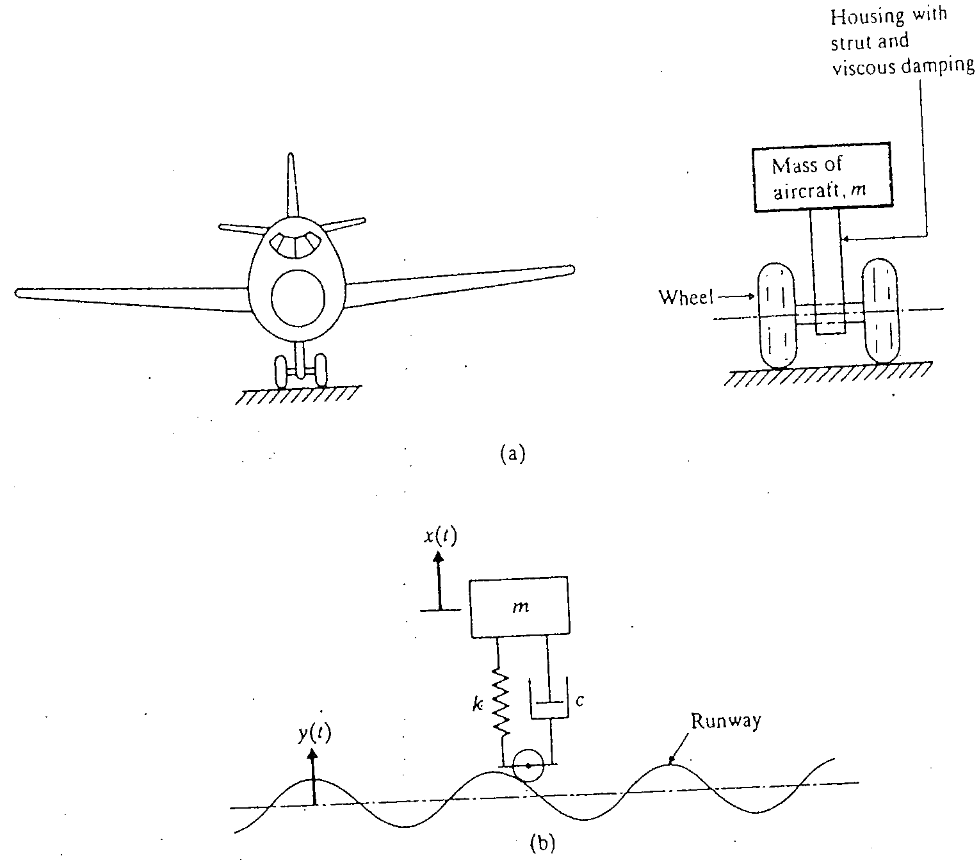


FIGURE 3.45

of the grinding wheel is to be restricted to 10^{-6} m. Assume that the grinding machine and the wheel are a rigid body of weight 5,000 N.

3.29 Derive the equation of motion and find the steady-state response of the system shown in Fig. 3.47 for rotational motion about the hinge O for the following data: k = 5000 N/m, l = 1 m, c = 1000 N-s/m, m = 10 kg, M_0 = 100 N-m, ω = 1000 rpm.

3.30 An air compressor of mass 100 kg is mounted on an elastic foundation. It has been observed that, when a harmonic force of amplitude 100 N is applied to the compressor, the maximum steady-state displacement of 5 mm occurred at a frequency of 300 rpm. Determine the equivalent stiffness and damping constant of the foundation.

3.31 Find the steady-state response of the system shown in Fig. 3.48 for the following data: k_1 = 1000 N/m, k_2 = 500 N/m, c = 500 N-s/m, m = 10 kg, r = 5 cm, J_0 = 1 kg-m^2, F_0 = 50 N, ω = 20 rad/s.

3.32 A uniform slender bar of mass m may be supported in one of two ways indicated in Figs. 3.49(a) and (b). Determine the arrangement that results in a reduced steady-state response of the bar under a harmonic force, $F_0 \sin \omega t$, applied at the middle of the bar, as shown in the figure.

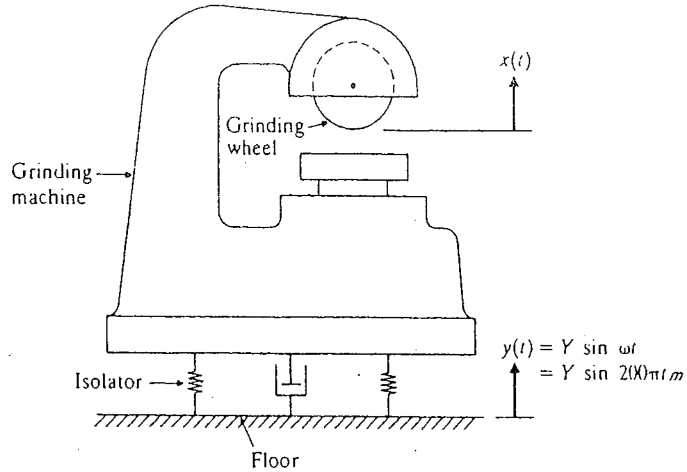


FIGURE 3.46

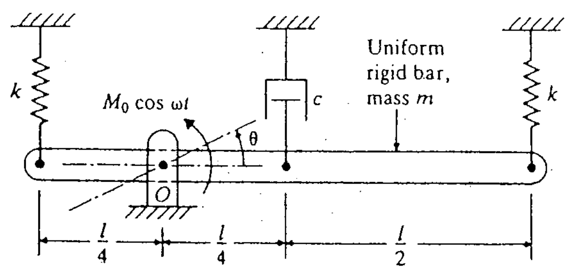


FIGURE 3.47

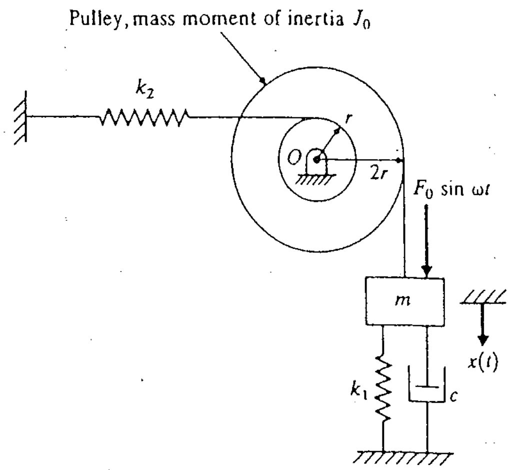


FIGURE 3.48

3.33 A single story building frame is subjected to a harmonic ground acceleration as shown in Fig. 3.50. Find the steady-state motion of the floor (mass m).

3.34 Find the horizontal displacement of the floor (mass m) of the building frame shown in Fig. 3.50 when the ground acceleration is given by $\ddot{x}_g = 100 \sin \omega t$ mm/sec^2. Assume $m = 2000$ kg, $k = 0.1$ MN/m, $\omega = 25$ rad/sec, and $x_g(t = 0) = \dot{x}_g(t = 0) = x(t = 0) = \dot{x}(t = 0) = 0$.

Jean Baptiste Joseph Fourier (1768–1830) was a French mathematician and a professor at the Ecole Polytechnique in Paris. His works on heat flow, published in 1822, and on trigonometric series are well known. The expansion of a periodic function in terms of harmonic functions has been named after him as the "Fourier series." (Reprinted with permission from *Applied Mechanics Reviews*).

CHAPTER 4

Vibration Under General Forcing Conditions

4.1 Introduction

The response of a single degree of freedom system under general, nonharmonic, forcing functions is considered in this chapter. A general forcing function may be periodic (nonharmonic) or nonperiodic. A nonperiodic forcing function may be acting for a short, long, or infinite duration. If the duration of the forcing function or excitation is small compared to the natural time period of the system, the forcing function or excitation is called a shock. The motion imparted by a cam to the follower, the vibration felt by an instrument when its package is dropped from a height, the force applied to the foundation of a forging press, the motion of an automobile when it hits a pothole, and the ground vibration of a building frame during an earthquake are examples of general forcing functions.

If the forcing function is periodic but not harmonic, it can be replaced by a sum of harmonic functions using the harmonic analysis procedure discussed in Section 1.11. Using the principle of superposition, the response of the system can then be determined by superposing the responses due to the individual harmonic forcing functions. On the other hand, if the system is subjected to a suddenly applied nonperiodic force, the response will involve transient vibration. The transient response of a system can be found by using what is known as the *convolution integral.*

4.2 Response Under a General Periodic Force

When the external force $F(t)$ is periodic with period $\tau = 2\pi/\omega$, it can be expanded in a Fourier series (see Section 1.11):

$$F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t \tag{4.1}$$

where

$$a_j = \frac{2}{\tau}\int_0^{\tau} F(t) \cos j\omega t \, dt, \qquad j = 0, 1, 2, \ldots \tag{4.2}$$

and

$$b_j = \frac{2}{\tau}\int_0^{\tau} F(t) \sin j\omega t \, dt, \qquad j = 1, 2, \ldots \tag{4.3}$$

The equation of motion of the system can be expressed as

$$m\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t \tag{4.4}$$

The right-hand side of this equation is a constant plus a sum of harmonic functions. Using the principle of superposition, the steady-state solution of Eq. (4.4) is the sum of the steady-state solutions of the following equations:

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} \tag{4.5}$$

$$m\ddot{x} + c\dot{x} + kx = a_j \cos j\omega t \tag{4.6}$$

$$m\ddot{x} + c\dot{x} + kx = b_j \sin j\omega t \tag{4.7}$$

Noting that the solution of Eq. (4.5) is given by

$$x_p(t) = \frac{a_0}{2k} \tag{4.8}$$

and using the results of Section 3.4, we can express the solutions of Eqs. (4.6) and (4.7), respectively, as

$$x_p(t) = \frac{(a_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta j r)^2}} \cos(j\omega t - \phi_j) \tag{4.9}$$

$$x_p(t) = \frac{(b_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta j r)^2}} \sin(j\omega t - \phi_j) \tag{4.10}$$

where

$$\phi_j = \tan^{-1}\left(\frac{2\zeta j r}{1 - j^2 r^2}\right) \tag{4.11}$$

and

$$r = \frac{\omega}{\omega_n} \tag{4.12}$$

Thus the complete steady-state solution of Eq. (4.4) is given by

$$x_p(t) = \frac{a_0}{2k} + \sum_{j=1}^{\infty} \frac{(a_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta j r)^2}} \cos(j\omega t - \phi_j) + \sum_{j=1}^{\infty} \frac{(b_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta j r)^2}} \sin(j\omega t - \phi_j) \tag{4.13}$$

It can be seen from the solution, Eq. (4.13), that the amplitude and phase shift corresponding to the jth term depend on j. If $j\omega = \omega_n$, for any j, the amplitude of the corresponding harmonic will be comparatively large. This will be particularly true for small values of j and ζ. Further, as j becomes larger, the amplitude becomes smaller and the corresponding terms tend to zero. Thus the first few terms are usually sufficient to obtain the response with reasonable accuracy.

The solution given by Eq. (4.13) denotes the steady-state response of the system. The transient part of the solution arising from the initial conditions can also be included to find the complete solution. To find the complete solution, we need to evaluate the arbitrary constants by setting the value of the complete solution and its derivative to the specified values of initial displacement $x(0)$ and the initial velocity $\dot{x}(0)$. This results in a complicated expression for the transient part of the total solution.

EXAMPLE 4.1 Periodic Vibration of a Hydraulic Valve

In the study of vibrations of valves used in hydraulic control systems, the valve and its elastic stem are modeled as a damped spring-mass system as shown in Fig. 4.1(a). In addition to the spring force and damping force, there is a fluid pressure force on the valve that changes with the amount of opening or closing of the valve. Find the steady-state response of the valve when the pressure in the chamber varies as indicated in Fig. 4.1(b). Assume $k = 2500$ N/m, $c = 10$ N-s/m, and $m = 0.25$ kg.

Given: Hydraulic control valve with $m = 0.25$ kg, $k = 2500$ N/m, and $c = 10$ N-s/m and pressure on the valve as given in Fig. 4.1(b).

The natural frequency of the valve is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{0.25}} = 100 \text{ rad/sec} \tag{E.14}$$

and the forcing frequency ω by

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{2} = \pi \text{ rad/sec} \tag{E.15}$$

Thus the frequency ratio can be obtained:

$$r = \frac{\omega}{\omega_n} = \frac{\pi}{100} = 0.031416 \tag{E.16}$$

and the damping ratio:

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{10.0}{2(0.25)(100)} = 0.2 \tag{E.17}$$

The phase angles ϕ_1 and ϕ_3 can be computed as follows:

$$\begin{aligned}\phi_1 &= \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)\\ &= \tan^{-1}\left(\frac{2 \times 0.2 \times 0.031416}{1 - 0.031416^2}\right) = 0.0125664 \text{ rad}\end{aligned} \tag{E.18}$$

and

$$\begin{aligned}\phi_3 &= \tan^{-1}\left(\frac{6\zeta r}{1 - 9r^2}\right)\\ &= \tan^{-1}\left(\frac{6 \times 0.2 \times 0.031416}{1 - 9(0.031416)^2}\right) = 0.0380483 \text{ rad}\end{aligned} \tag{E.19}$$

In view of Eqs. (E.2) and (E.14) to (E.19), the solution can be written as

$$\begin{aligned}x_p(t) = {} & 0.019635 - 0.015930 \cos(\pi t - 0.0125664)\\ & - 0.0017828 \cos(3\pi t - 0.0380483) \text{ m}\end{aligned} \tag{E.20}$$

■

4.3 Response Under a Periodic Force of Irregular Form

In some cases, the force acting on a system may be quite irregular and may be determined only experimentally. Examples of such forces include wind-and earthquake-induced forces. In such cases, the forces will be available in graphical form and no analytical expression can be found to describe $F(t)$. Sometimes, the value of $F(t)$ may be available only at a number of discrete points $t_1, t_2, \ldots, t_N$. In all these cases, it is possible to find the Fourier coefficients by using a numerical

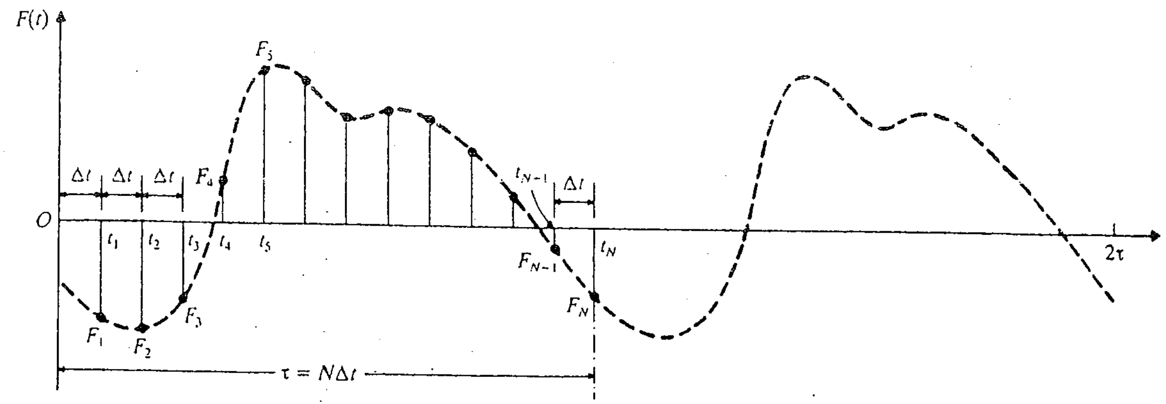


FIGURE 4.2

integration procedure, as described in Section 1.11. If $F_1, F_2, \ldots, F_N$ denote the values of $F(t)$ at $t_1, t_2, \ldots, t_N$, respectively, where N denotes an even number of equidistant points in one time period $\tau(\tau = N\Delta t)$, as shown in Fig. 4.2, the application of trapezoidal rule [4.1] gives

$$a_0 = \frac{2}{N}\sum_{i=1}^{N} F_i \tag{4.14}$$

$$a_j = \frac{2}{N}\sum_{i=1}^{N} F_i \cos\frac{2j\pi t_i}{\tau}, \qquad j = 1, 2, \ldots \tag{4.15}$$

$$b_j = \frac{2}{N}\sum_{i=1}^{N} F_i \sin\frac{2j\pi t_i}{\tau}, \qquad j = 1, 2, \ldots \tag{4.16}$$

Once the Fourier coefficients a_0, a_j, and b_j are known, the steady-state response of the system can be found using Eq. (4.13) with

$$r = \left(\frac{2\pi}{\tau\omega_n}\right)$$

EXAMPLE 4.2 Steady-State Vibration of a Hydraulic Valve

Find the steady-state response of the valve in Example 4.1 if the pressure fluctuations in the chamber are found to be periodic. The values of pressure measured at 0.01 second intervals in one cycle are given below.

Time, t_i (seconds)	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12
$p_i = p(t_i)$ (kN/m^2)	0	20	34	42	49	53	70	60	36	22	16	7	0

Given: Arbitrary pressure fluctuations on the valve, shown in Fig. 4.1(a).

Find: Steady-state response of the valve.

Approach: Find Fourier series expansion of the pressure acting on the valve using numerical procedure. Add the responses due to individual harmonic force components.

Solution: The Fourier analysis of the pressure fluctuations (see Example 1.13) gives the result

$$p(t) = 34083.3 - 26996.0 \cos 52.36t + 8307.7 \sin 52.36t + 1416.7 \cos 104.72t + 3608.3 \sin 104.72t - 5833.3 \cos 157.08t + 2333.3 \sin 157.08t + \ldots \text{ N/m}^2 \qquad \text{(E.1)}$$

Other quantities needed for the computation are

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{0.12} = 52.36 \text{ rad/sec}$$

$$\omega_n = 100 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = 0.5236$$

$$\zeta = 0.2$$

$$A = 0.000625\pi \text{ m}^2$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) = \tan^{-1}\left(\frac{2 \times 0.2 \times 0.5236}{1 - 0.5236^2}\right) = 16.1°$$

$$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1 - 4r^2}\right) = \tan^{-1}\left(\frac{4 \times 0.2 \times 0.5236}{1 - 4 \times 0.5236^2}\right) = -77.01°$$

$$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1 - 9r^2}\right) = \tan^{-1}\left(\frac{6 \times 0.2 \times 0.5236}{1 - 9 \times 0.5236^2}\right) = -23.18°$$

The steady-state response of the valve can be expressed, using Eq. (4.13), as

$$x_p(t) = \frac{34083.3A}{k} - \frac{(26996.0A/k)}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \cos(52.36t - \phi_1) + \frac{(8309.7A/k)}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \sin(52.36t - \phi_1)$$

$$+ \frac{(1416.7A/k)}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}} \cos(104.72t - \phi_2)$$

$$+ \frac{(3608.3A/k)}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}} \sin(104.72t - \phi_2)$$

$$- \frac{(5833.3A/k)}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}} \cos(157.08t - \phi_3)$$

$$+ \frac{(2333.3A/k)}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}} \sin(157.08t - \phi_3)$$

■

4.4 Response Under a Nonperiodic Force

We have seen that periodic forces of any general wave form can be represented by Fourier series as a superposition of harmonic components of various frequencies. The response of a linear system is then found by superposing the harmonic response to each of the exciting forces. When the exciting force $F(t)$ is nonperiodic, such as that due to the blast from an explosion, a different method of calculating the response is required. Various methods can be used to find the response of the system to an arbitrary excitation. Some of these methods are as follows:

1. Representing the excitation by a Fourier integral
2. Using the method of convolution integral
3. Using the method of Laplace transformation
4. First approximating $F(t)$ by a suitable interpolation model and then using a numerical procedure
5. Numerically integrating the equations of motion

We shall discuss Methods 2, 3, and 4 in the following sections and Method 5 in Chapter 11.

4.5 Convolution Integral

A nonperiodic exciting force usually has a magnitude that varies with time; it acts for a specified period of time and then stops. The simplest form of such a force is the impulsive force. An impulsive force is one that has a large magnitude F and acts for a very short period of time Δt. From dynamics we know that impulse can be measured by finding the change in momentum of the system caused by it [4.2].

If $\dot{x}_1$ and $\dot{x}_2$ denote the velocities of the mass m before and after the application of the impulse, we have

$$\text{Impulse} = F\Delta t = m\dot{x}_2 - m\dot{x}_1 \tag{4.17}$$

By designating the magnitude of the impulse $F\Delta t$ by $\underset{\sim}{F}$, we can write, in general,

$$\underset{\sim}{F} = \int_t^{t+\Delta t} F\,dt \tag{4.18}$$

A unit impulse ($\underset{\sim}{f}$) is defined as

$$\underset{\sim}{f} = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} F\,dt = F\,dt = 1 \tag{4.19}$$

It can be seen that in order for $F\,dt$ to have a finite value, F tends to infinity (since dt tends to zero). Although the unit impulse function has no physical meaning, it is a convenient tool in our present analysis.

4.5.1 Response to an Impulse

We first consider the response of a single degree of freedom system to an impulse excitation; this case is important in studying the response under more general excitations. Consider a viscously damped spring-mass system subjected to a unit impulse at $t = 0$, as shown in Figs. 4.3(a) and (b). For an underdamped system, the solution of the equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{4.20}$$

is given by Eq. (2.72) as follows:

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\} \tag{4.21}$$

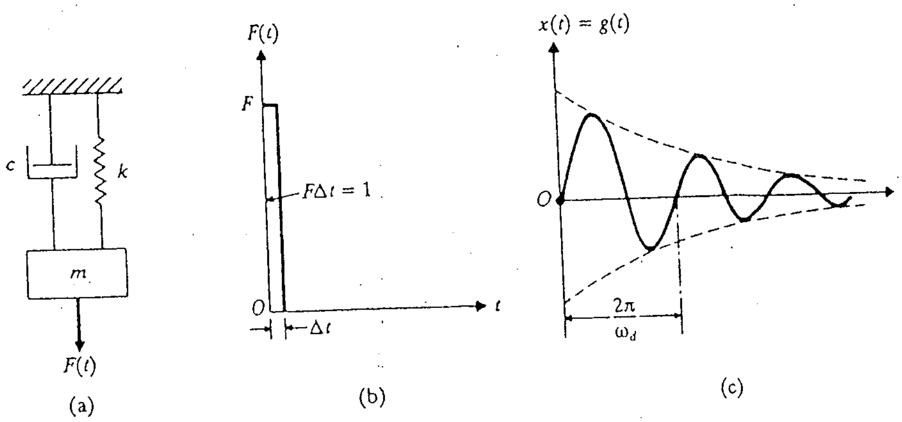


FIGURE 4.3

where

$$\zeta = \frac{c}{2m\omega_n} \tag{4.22}$$

$$\omega_d = \omega_n\sqrt{1 - \zeta^2} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \tag{4.23}$$

$$\omega_n = \sqrt{\frac{k}{m}} \tag{4.24}$$

If the mass is at rest before the unit impulse is applied ($x = \dot{x} = 0$ for $t < 0$ or at $t = 0^-$), we obtain, from the impulse-momentum relation,

$$\text{Impulse} = f = 1 = m\dot{x}\,(t = 0) - m\dot{x}\,(t = 0^-) = m\dot{x}_0 \tag{4.25}$$

Thus the initial conditions are given by

$$x(t = 0) = x_0 = 0$$

$$\dot{x}(t = 0) = \dot{x}_0 = \frac{1}{m} \tag{4.26}$$

In view of Eq. (4.26), Eq. (4.21) reduces to

$$x(t) = g(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t \tag{4.27}$$

Equation (4.27) gives the response of a single degree of freedom system to a unit impulse, which is also known as the *impulse response function*, denoted by $g(t)$. The function $g(t)$, Eq. (4.27), is shown in Fig. 4.3(c).

If the magnitude of the impulse is F instead of unity, the initial velocity $\dot{x}_0$ is F/m and the response of the system becomes

$$x(t) = \frac{Fe^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t = Fg(t) \tag{4.28}$$

If the impulse F is applied at an arbitrary time $t = \tau$, as shown in Fig. 4.4(a), it will change the velocity at $t = \tau$ by an amount F/m. Assuming that $x = 0$ until the impulse is applied, the displacement x at any subsequent time t, caused by a change in the velocity at time τ, is given by Eq. (4.28) with t replaced by the time elapsed after the application of the impulse, that is, $t - \tau$. Thus we obtain

$$x(t) = Fg(t - \tau) \tag{4.29}$$

This is shown in Fig. 4.4(b).

4.5.2 Response to General Forcing Condition

Now we consider the response of the system under an arbitrary external force $F(t)$, shown in Fig. 4.5. This force may be assumed to be made up of a series of impulses of varying magnitude. Assuming that at time τ, the force $F(\tau)$ acts on the system for a short period of time $\Delta\tau$, the impulse acting at $t = \tau$ is given by $F(\tau)\,\Delta\tau$. At

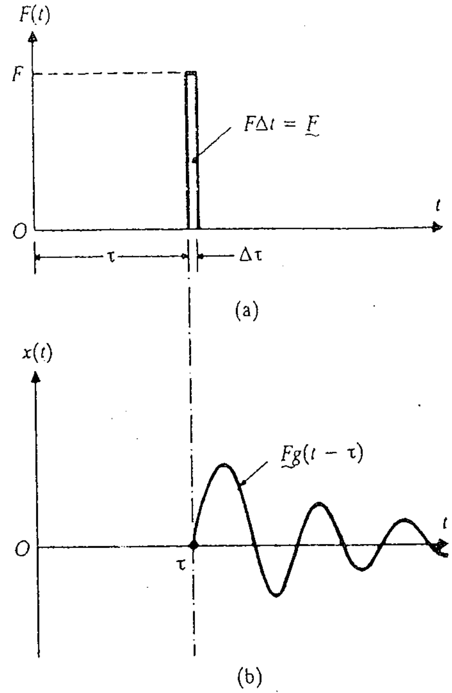


FIGURE 4.4

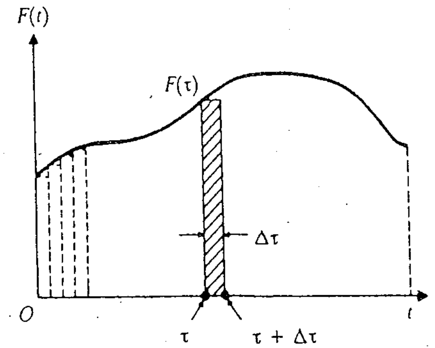


FIGURE 4.5 An arbitrary (nonperiodic) forcing function.

any time t, the elapsed time since the impulse is $t - \tau$, so the response of the system at t due to this impulse alone is given by Eq. (4.29) with $\underset{\sim}{F} = F(\tau)\,\Delta\tau$:

$$\Delta x(t) = F(\tau)\,\Delta\tau g\,(t - \tau) \tag{4.30}$$

The total response at time t can be found by summing all the responses due to the elementary impulses acting at all times τ:

$$x(t) \simeq \Sigma F(\tau) g(t - \tau)\,\Delta\tau \tag{4.31}$$

Letting $\Delta\tau \to 0$ and replacing the summation by integration, we obtain

$$x(t) = \int_0^t F(\tau) g(t - \tau)\,d\tau \tag{4.32}$$

By substituting Eq. (4.27) into Eq. (4.32), we obtain

$$x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau) e^{-\zeta\omega_n(t-\tau)} \sin \omega_d\,(t - \tau)\,d\tau \tag{4.33}$$

which represents the response of an underdamped single degree of freedom system to the arbitrary excitation $F(t)$. Note that Eq. (4.33) does not consider the effect of initial conditions of the system. The integral in Eq. (4.32) or Eq. (4.33) is called the *convolution* or *Duhamel integral*. In many cases the function $F(t)$ has a form that permits an explicit integration of Eq. (4.33). In case such integration is not

4.3 What is the Duhamel integral? What is its use?

4.4 How are the initial conditions determined for a single degree of freedom system subjected to an impulse at $t = 0$?

4.5 Derive the equation of motion of a system subjected to base excitation.

4.6 What is a response spectrum?

4.7 What are the advantages of the Laplace transformation method?

4.8 What is the use of the pseudo spectrum?

4.9 How is the Laplace transform of a function $x(t)$ defined?

4.10 Define these terms: generalized impedance and admittance of a system.

4.11 State the interpolation models that can be used for approximating an arbitrary forcing function.

4.12 How many resonant conditions are there when the external force is not harmonic?

4.13 How do you compute the frequency of the first harmonic of a periodic force?

4.14 What is the relation between the frequencies of higher harmonics and the frequency of the first harmonic for a periodic excitation?

Problems

The problem assignments are organized as follows:

Problems	Section Covered	Topic Covered
4.1–4.10	4.2	Response under general periodic force
4.11–4.13	4.3	Periodic force of irregular form
4.14–4.34	4.5	Convolution integral
4.35–4.44	4.6	Response spectrum
4.45–4.47	4.7	Laplace transformation
4.48–4.51	4.8	Irregular forcing conditions using numerical methods
4.52–4.57	4.9	Computer program
4.58–4.60	—	Projects

4.1–4.4 Find the steady-state response of the hydraulic control valve shown in Fig. 4.1(a) to the forcing functions obtained by replacing $x(t)$ with $F(t)$ and A with F_0 in Figs. 1.87–1.90.

4.5 Find the steady-state response of a viscously damped system to the forcing function obtained by replacing $x(t)$ and A with $F(t)$ and F_0, respectively, in Fig. 1.46(a).

4.6 The torsional vibrations of a driven gear mounted on a shaft (see Fig. 4.29) under steady conditions are governed by the equation:

$$J_0\ddot{\theta} + k_t\theta = M_t$$

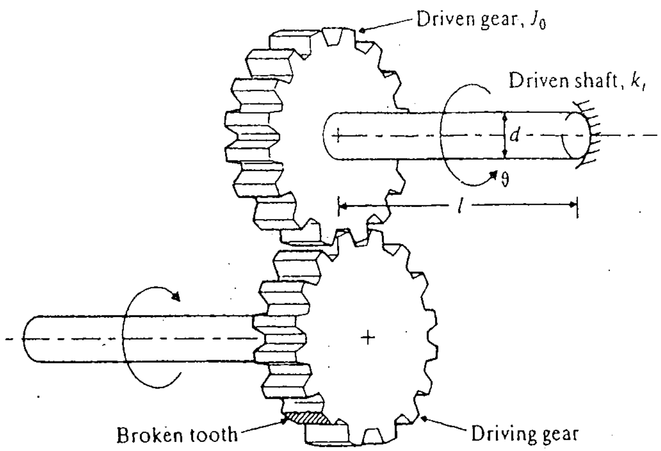


FIGURE 4.29

where k_t is the torsional stiffness of the driven shaft, M_t is the torque transmitted, J_0 is the mass moment of inertia, and θ is the angular deflection of the driven gear. If one of the 16 teeth on the driving gear breaks, determine the resulting torsional vibration of the driven gear for the following data.

Driven gear: $J_0 = 0.1$ N-m-s^2, speed = 1000 rpm, driven shaft: material - steel, solid circular section with diameter 5 cm and length 1 m, $M_{t0} = 1000$ N-m.

4.7 A slider crank mechanism is used to impart motion to the base of a spring-mass-damper system, as shown in Fig. 4.30. Approximating the base motion $y(t)$ as a series of harmonic functions, find the response of the mass for $m = 1$ kg, $c = 10$ N-s/m, $k = 100$ N/m, $r = 10$ cm, $l = 1$ m, and $\omega = 100$ rad/s.

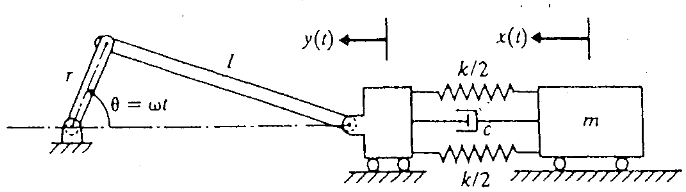


FIGURE 4.30

4.8 The base of a spring-mass-damper system is subjected to the periodic displacement shown in Fig. 4.31. Determine the response of the mass using the principle of superposition.

4.9 The base of a spring-mass system, with Coulomb damping, is connected to the slider crank mechanism shown in Fig. 4.32. Determine the response of the system for a coefficient of friction μ between the mass and the surface by approximating the motion $y(t)$ as a series of harmonic functions for $m = 1$ kg, $k = 100$ N/m, $r = 10$ cm, $l = 1$ m, $\mu = 0.1$, and $\omega = 100$ rad/s. Discuss the limitations of your solution.

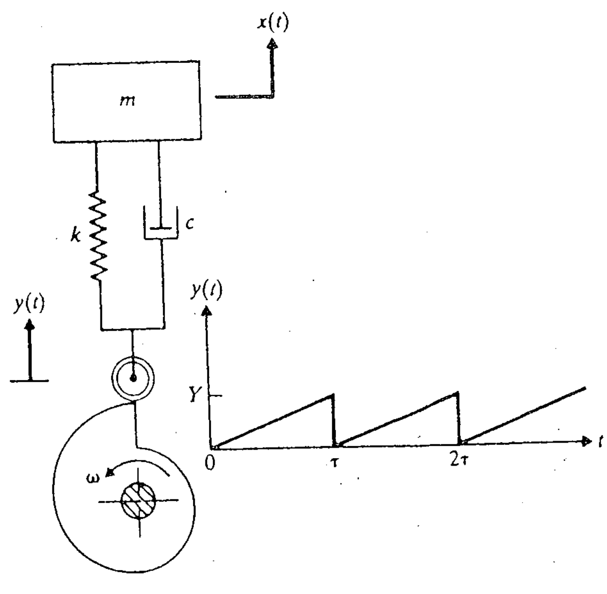


FIGURE 4.31

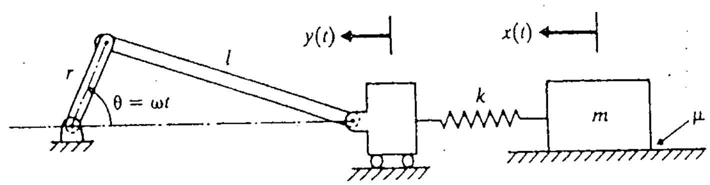


FIGURE 4.32

4.10 A roller cam is used to impart a periodic motion to the base of the spring-mass system shown in Fig. 4.33. If the coefficient of friction between the mass and the surface is μ, find the response of the system using the principle of superposition. Discuss the validity of the result.

4.11 Find the response of a damped system with $m = 1$ kg, $k = 15$ kN/m. and $\zeta = 0.1$ under the action of a periodic forcing function, as shown in Fig. 1.92.

4.12 Find the response of a viscously damped system under the periodic force whose values are given in Problem 1.69. Assume that M_t denotes the value of the force in Newtons at time t_i seconds. Use $m = 0.5$ kg, $k = 8000$ N/m, and $\zeta = 0.06$.

4.13 Find the displacement of the water tank shown in Fig. 4.34(a) under the periodic force shown in Fig. 4.34(b) by treating it as an undamped single degree of freedom system. Use the numerical procedure described in Section 4.3.

4.14 Sandblasting is a process in which an abrasive material, entrained in a jet, is directed onto the surface of a casting to clean its surface. In a particular setup for sandblasting, the casting of mass m is placed on a flexible support of stiffness k as shown in Fig.

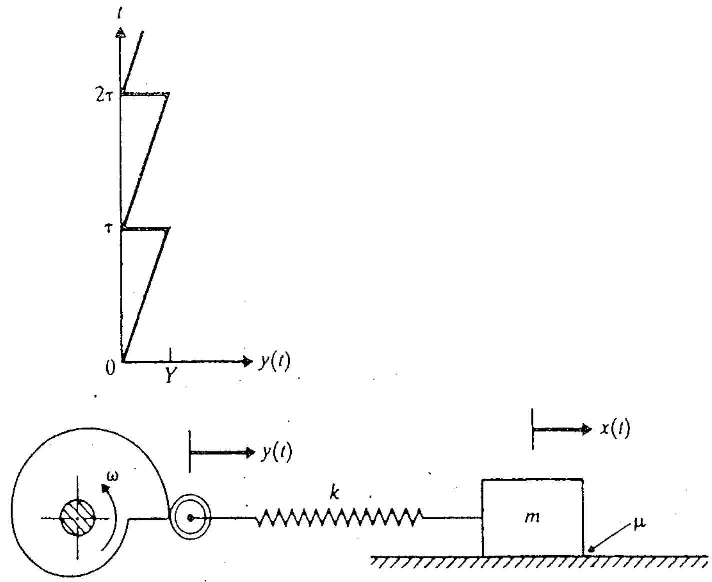


FIGURE 4.33

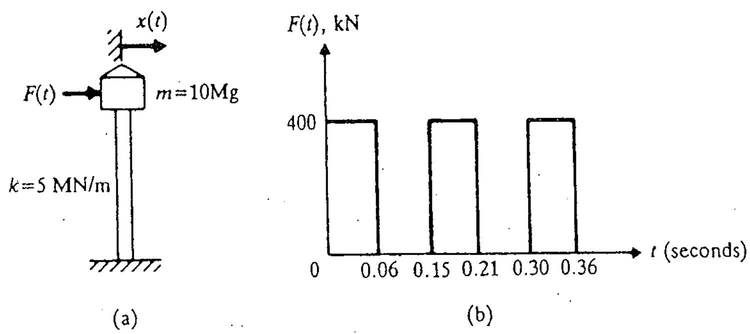


FIGURE 4.34

4.35(a). If the force exerted on the casting due to the sandblasting operation varies as shown in Fig. 4.35(b), find the response of the casting.

4.15 The frame, anvil, and the base of the forging hammer, shown in Fig. 4.36(a), have a total mass of m. The support elastic pad has a stiffness of k. If the force applied by the hammer is given by Fig. 4.36(b), find the response of the anvil.

4.16 Find the displacement of a damped single degree of freedom system under the forcing function $F(t) = F_0 e^{-\alpha t}$ where α is a constant.

4.17 A compressed air cylinder is connected to the spring-mass system shown in Fig. 4.37(a). Due to a small leak in the valve, the pressure on the piston, $p(t)$, builds up

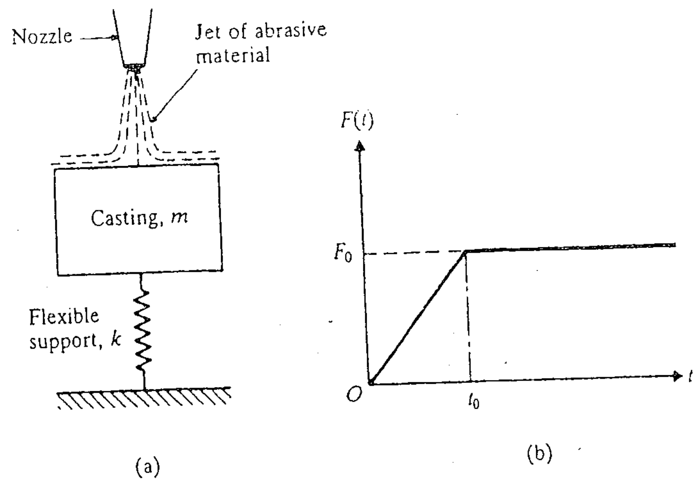


FIGURE 4.35

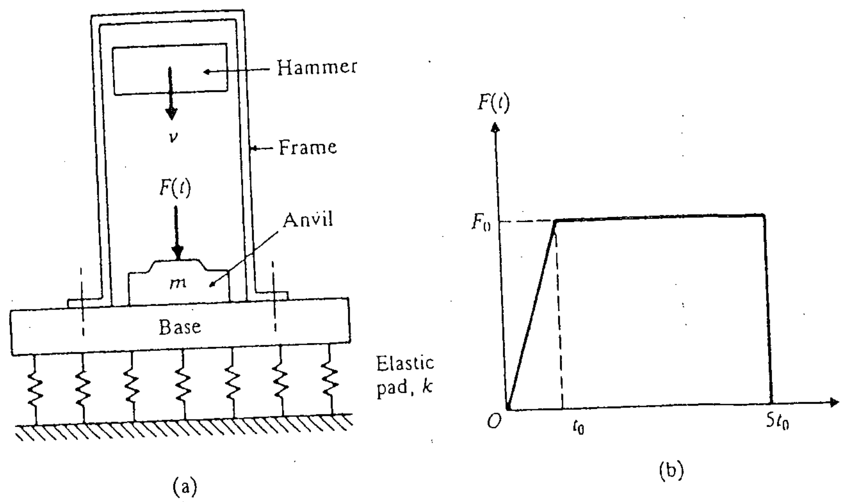


FIGURE 4.36

as indicated in Fig. 4.37(b). Find the response of the piston for the following data: $m = 10$ kg, $k = 1000$ N/m, and $d = 0.1$ m.

4.18 Find the transient response of an undamped spring-mass system for $t > \pi/\omega$ when the mass is subjected to a force

$$F(t) = \begin{cases} \dfrac{F_0}{2}(1 - \cos \omega t) & \text{for } 0 \leq t \leq \dfrac{\pi}{\omega} \\ F_0 & \text{for } t > \dfrac{\pi}{\omega} \end{cases}$$

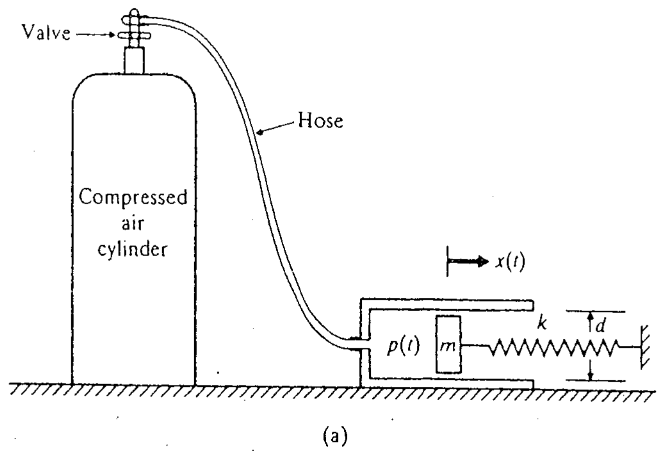


FIGURE 4.37

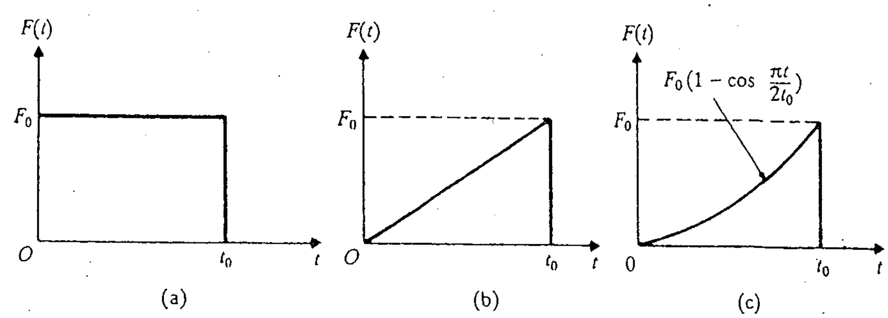


FIGURE 4.38

Assume that the displacement and velocity of the mass are zero at $t = 0$.

4.19–4.21 Use the Dahamel integral method to derive expressions for the response of an undamped system subjected to the forcing functions shown in Figs. 4.38(a) to (c).

4.22 Figure 4.39 shows a one degree of freedom model of a motor vehicle traveling in the horizontal direction. Find the relative displacement of the vehicle as it travels over a road bump of the form $y(s) = Y \sin \pi s/\delta$.

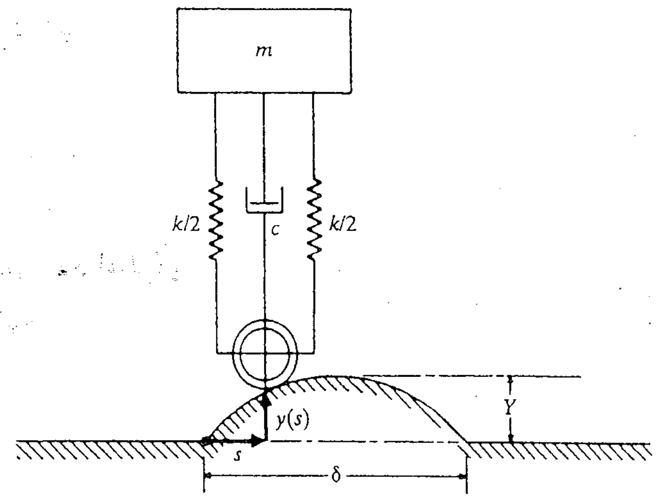


FIGURE 4.39

4.23 A vehicle traveling at a constant speed v in the horizontal direction encounters a triangular road bump, as shown in Fig. 4.40. Treating the vehicle as an undamped spring-mass system, determine the response of the vehicle in the vertical direction.

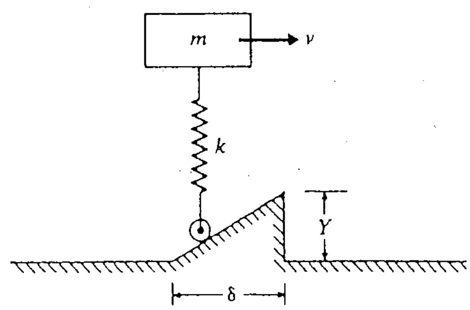


FIGURE 4.40

4.24 An automobile, having a mass of 1000 kg, runs over a road bump of the shape shown in Fig. 4.41. The speed of the automobile is 50 km/hr. If the undamped natural period of vibration in the vertical direction is 1.0 second, find the response of the car by assuming it as a single degree of freedom undamped system vibrating in the vertical direction.

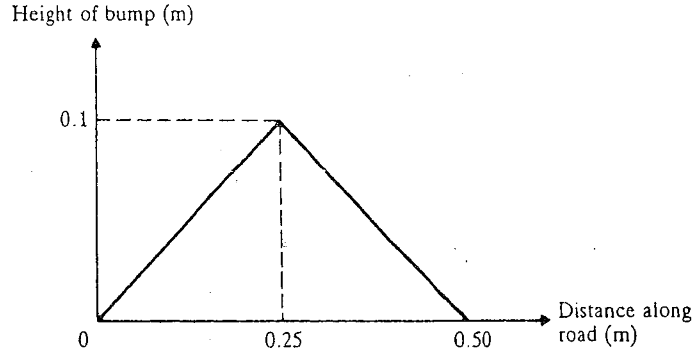


FIGURE 4.41

4.25 A camcorder of mass m is packed in a container using a flexible packing material. The stiffness and damping constant of the packing material are given by k and c, respectively, and the mass of the container is negligible. If the container is dropped accidentally from a height of h onto a rigid floor (see Fig. 4.42), find the motion of the camcorder.

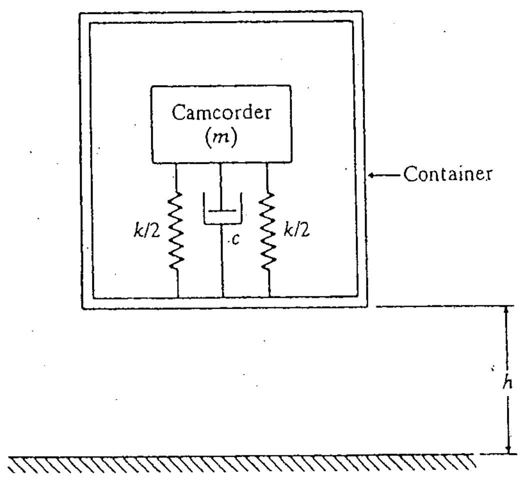


FIGURE 4.42

4.27 in new pg 369

4.26 An airplane, taxiing on a runway, encounters a bump. As a result, the root of the wing is subjected to a displacement that can be expressed as

$$y(t) = \begin{cases} Y(t^2/t_0^2), & 0 \le t \le t_0 \\ 0, & t > t_0 \end{cases}$$

Find the response of the mass located at the tip of the wing if the stiffness of the wing is k (see Fig. 4.43).

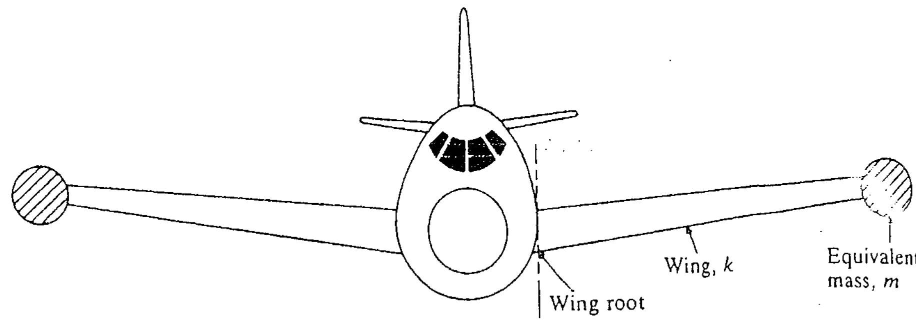


FIGURE 4.43

4.27 Derive Eq. (E.1) of Example 4.6.

4.28 In a static firing test of a rocket, the rocket is anchored to a rigid wall by a spring-damper system, as shown in Fig. 4.44(a). The thrust acting on the rocket reaches its maximum value F in a negligibly short time and remains constant until the burnout time t_0, as indicated in Fig. 4.44(b). The thrust acting on the rocket is given by $F = m_0 v$ where m_0 is the constant rate at which fuel is burnt and v is the velocity of the jet stream. The initial mass of the rocket is M, so that its mass at any time t is given by $m = M - m_0 t$, $0 \leq t \leq t_0$. If the data are $k = 7.5 \times 10^6$ N/m, $c = 0.1 \times 10^6$ N-s/m, $m_0 = 10$ kg/s, $v = 2000$ m/s, $M = 2000$ kg, and $t_0 = 100$ s, (1) derive the equation of motion of the rocket, and (2) find the maximum steady-state displacement of the rocket by assuming an average (constant) mass of $(M - \frac{1}{2}m_0 t_0)$.

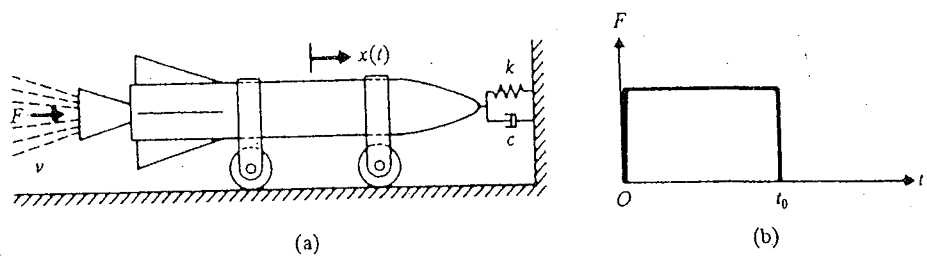


FIGURE 4.44

4.29 Show that the response to a unit step function $h(t)$ ($F_0 = 1$ in Fig. 4.6b) is related to the impulse response function $g(t)$, Eq. (4.27), as follows:

$$g(t) = \frac{dh(t)}{dt}$$

4.30 Show that the convolution integral, Eq. (4.33), can also be expressed in terms of the response to a unit step function $h(t)$ as

$$x(t) = F(0)\, h(t) + \int_0^t \frac{dF(\tau)}{d\tau} h(t - \tau)\, d\tau$$

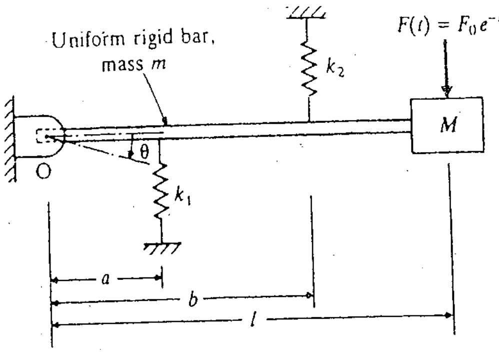


FIGURE 4.45

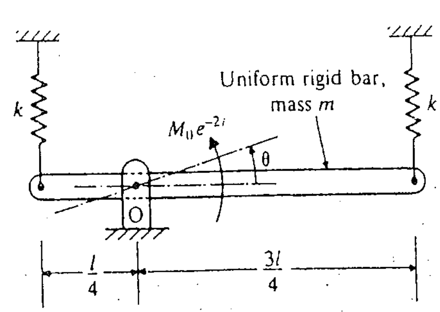


FIGURE 4.46

4.31 Find the response of the rigid bar shown in Fig. 4.45 using convolution integral for the following data: $k_1 = k_2 = 5000$ N/m, $a = 0.25$ m, $b = 0.5$ m, $l = 1.0$ m, $M = 50$ kg, $m = 10$ kg, $F_0 = 500$ N.

4.32 Find the response of the rigid bar shown in Fig. 4.46 using convolution integral for the following data: $k = 5000$ N/m, $l = 1$ m, $m = 10$ kg, $M_0 = 100$ N-m.

4.33 Find the response of the rigid bar shown in Fig. 4.47 using convolution integral when the end P of the spring PQ is subjected to the displacement, $x(t) = x_0\, e^{-t}$. Data: $k = 5000$ N/m, $l = 1$ m, $m = 10$ kg, $x_0 = 1$ cm.

4.34 Find the response of the mass shown in Fig. 4.48 under the force $F(T) = F_0\, e^{-t}$ using convolution integral. Data: $k_1 = 1000$ N/m, $k_2 = 500$ N/m, $r = 5$ cm, $m = 10$ kg, $J_0 = 1$ kg-m^2, $F_0 = 50$ N.

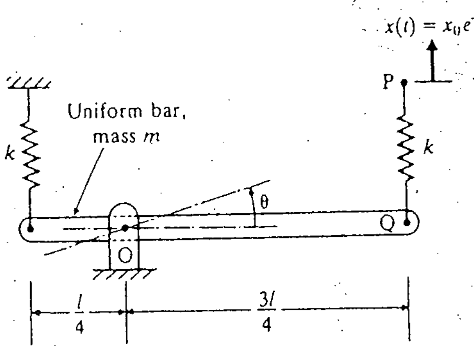


FIGURE 4.47

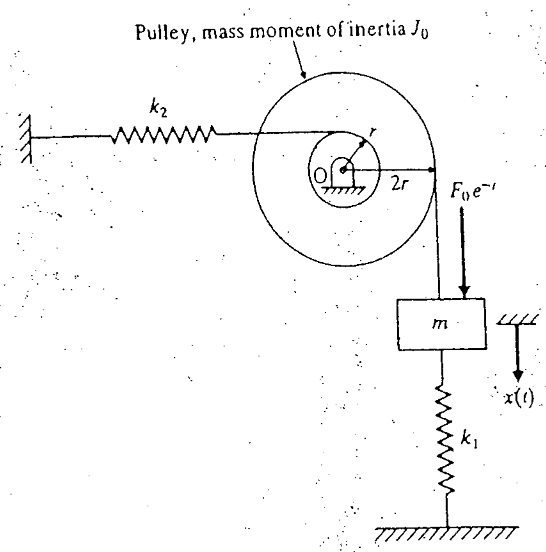


FIGURE 4.48

The damping ratios obtainable with different types of construction/arrangement are indicated below:

Type of Construction/Arrangement	Equivalent Viscous Damping Ratio (%)
Welded construction	1–4
Bolted construction	3–10
Steel frame	5–6
Unconstrained viscoelastic layer on steel-concrete girder	4–5
Constrained viscoelastic layer on steel-concrete girder	5–8

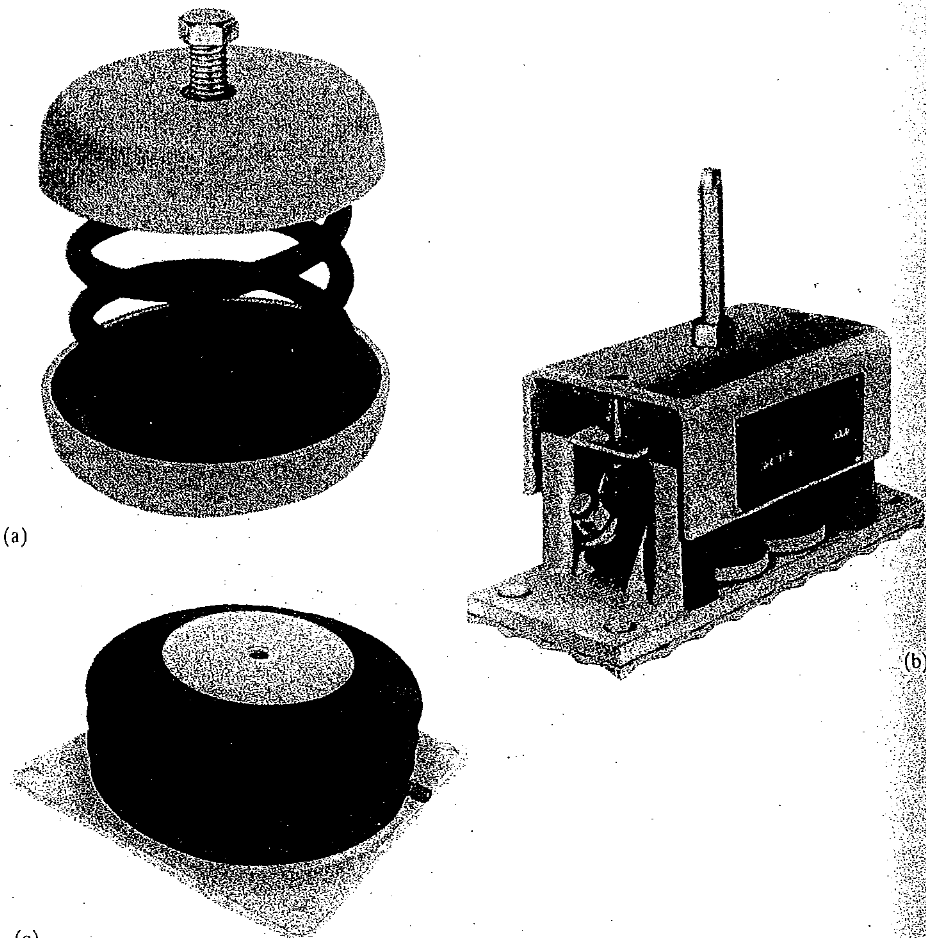
9.9 Vibration Isolation

Vibration isolation is a procedure by which the undesirable effects of vibration are reduced [9.21–9.24]. Basically, it involves the insertion of a resilient member (or isolator) between the vibrating mass (or equipment or payload) and the source of vibration so that a reduction in the dynamic response of the system is achieved under specified conditions of vibration excitation. An isolation system is said to be active or passive depending on whether or not external power is required for the isolator to perform its function. A passive isolator consists of a resilient member (stiffness) and an energy dissipator (damping). Examples of passive isolators include metal springs, cork, felt, pneumatic springs, and elastomer (rubber) springs. Figure 9.16 shows typical spring and pneumatic mounts that can be used as passive isolators, and Fig. 9.17 illustrates the use of passive isolators in the mounting of a high-speed punch press [9.25]. The optimal synthesis of vibration isolators is presented in Refs. [9.26–9.30].

An active isolator is comprised of a servomechanism with a sensor, signal processor, and an actuator. The effectiveness of an isolator is stated in terms of its transmissibility. The transmissibility (T_r) is defined as the ratio of the amplitude of the force transmitted to that of the exciting force.

Vibration isolation can be used in two types of situations. In the first type, the foundation or base of a vibrating machine is protected against large unbalanced forces (as in the case of reciprocating and rotating machines) or impulsive forces (as in the case of forging and stamping presses). In these cases, if the system is modeled as a single degree of freedom system as shown in Fig. 9.18(a), the force is transmitted to the foundation through the spring and the damper. The force transmitted to the base (F_t) is given by

$$F_t(t) = kx(t) + c\dot{x}(t) \tag{9.79}$$



(a) (b) (c)

FIGURE 9.16 (a) Undamped spring mount; (b) damped spring mount; (c) pneumatic rubber mount. *(Courtesy of* Sound and Vibration.*)*

If the force transmitted to the base $F_t(t)$ varies harmonically, as in the case of unbalanced reciprocating and rotating machines, the resulting stresses in the foundation bolts also vary harmonically, which might lead to fatigue failure. Even if the force transmitted is not harmonic, its magnitude is to be limited to safe permissible values.

In the second type, the system is protected against the motion of its foundation or base (as in the case of protection of a delicate instrument or equipment from the motion of its container). If the delicate instrument is modeled as a single degree of freedom system, as shown in Fig. 9.18(b), the force transmitted to the instrument (mass m in Fig. 9.18b) is given by

$$F_t(t) = m\ddot{x}(t) \equiv k[x(t) - y(t)] + c[\dot{x}(t) - \dot{y}(t)] \tag{9.80}$$

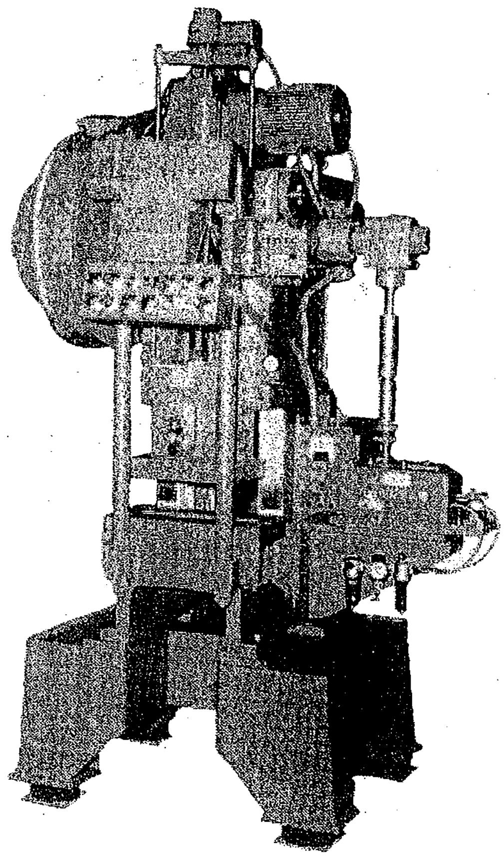


FIGURE 9.17 High-speed punch press mounted on pneumatic rubber mounts. (*Courtesy of* Sound and Vibration.)

where $(x - y)$ and $(\dot{x} - \dot{y})$ denote the relative displacement and relative velocity of the spring and the damper, respectively. In many practical problems, the package is to be designed properly to avoid transmission of large forces to the delicate instrument to avoid damage.

9.9.1 Vibration Isolation System with Rigid Foundation

Reduction of the Force Transmitted to Foundation. When a machine is bolted directly to a rigid foundation or floor, the foundation will be subjected to a harmonic load due to the unbalance in the machine in addition to the static load due to the weight of the machine. Hence an elastic or resilient member is placed between the machine and the rigid foundation to reduce the force transmitted to the foundation. The system can then be idealized as a single degree of freedom system, as shown

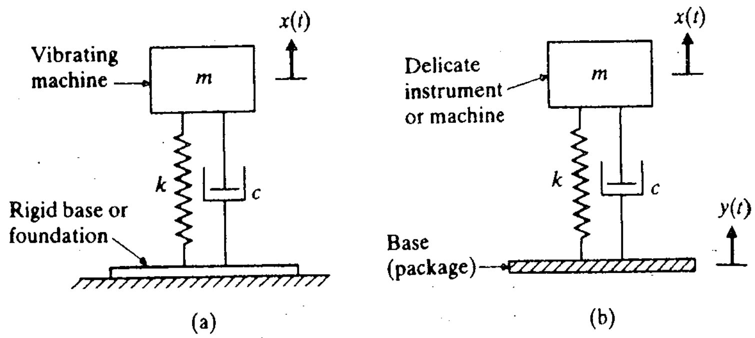


FIGURE 9.18

in Fig. 9.19(a). The resilient member is assumed to have both elasticity and damping and is modeled as a spring k and a dashpot c, as shown in Fig. 9.19(b). It is assumed that the operation of the machine gives rise to a harmonically varying force $F(t) = F_0 \cos \omega t$. The equation of motion of the machine (of mass m) is given by

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \tag{9.81}$$

Since the transient solution dies out after some time, only the steady-state solution will be left. The steady-state solution of Eq. (9.81) is given by (see Eq. 3.25)

$$x(t) = X \cos (\omega t - \phi) \tag{9.82}$$

where

$$X = \frac{F_0}{[(k - m\omega^2)^2 + \omega^2 c^2]^{1/2}} \tag{9.83}$$

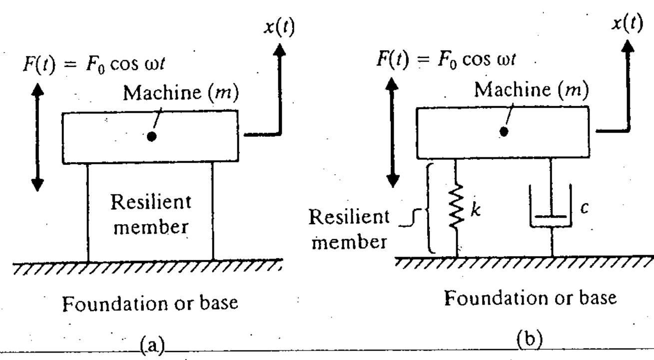


FIGURE 9.19 Machine and resilient member on rigid foundation.

and

$$\phi = \tan^{-1}\left(\frac{\omega c}{k - m\omega^2}\right) \tag{9.84}$$

The force transmitted to the foundation through the spring and the dashpot, $F_t(t)$, is given by

$$F_t(t) = kx(t) + c\dot{x}(t) = kX\cos(\omega t - \phi) - c\omega X\sin(\omega t - \phi) \tag{9.85}$$

The magnitude of the total transmitted force (F_T) is given by

$$\begin{aligned} F_T &= [(kx)^2 + (c\dot{x})^2]^{1/2} = X\sqrt{k^2 + \omega^2 c^2} \\ &= \frac{F_0(k^2 + \omega^2 c^2)^{1/2}}{[(k - m\omega^2)^2 + \omega^2 c^2]^{1/2}} \end{aligned} \tag{9.86}$$

The transmissibility or transmission ratio of the isolator (T_r) is defined as the ratio of the magnitude of the force transmitted to that of the exciting force:

$$\begin{aligned} T_r = \frac{F_T}{F_0} &= \left\{\frac{k^2 + \omega^2 c^2}{(k - m\omega^2)^2 + \omega^2 c^2}\right\}^{1/2} \\ &= \left\{\frac{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right\}^{1/2} \end{aligned} \tag{9.87}$$

where $r = \dfrac{\omega}{\omega_n}$ is the frequency ratio. The variation of T_r with the frequency ratio $r = \dfrac{\omega}{\omega_n}$ is shown in Fig. 9.20. In order to achieve isolation, the force transmitted to the foundation needs to be less than the excitation force. It can be seen, from Fig. 9.20, that the forcing frequency has to be greater than $\sqrt{2}$ times the natural frequency of the system in order to achieve isolation of vibration.

Notes

1. The magnitude of the force transmitted to the foundation can be reduced by decreasing the natural frequency of the system (ω_n).
2. The force transmitted to the foundation can also be reduced by decreasing the damping ratio. However, since vibration isolation requires $r > \sqrt{2}$, the machine should pass through resonance during start-up and stopping. Hence, some damping is essential to avoid infinitely large amplitudes at resonance.
3. Although damping reduces the amplitude of the mass (X) for all frequencies, it reduces the maximum force transmitted to the foundation (F_t) only if $r < \sqrt{2}$. Above that value, the addition of damping increases the force transmitted.
4. If the speed of the machine (forcing frequency) varies, we must compromise in choosing the amount of damping to minimize the force transmitted. The

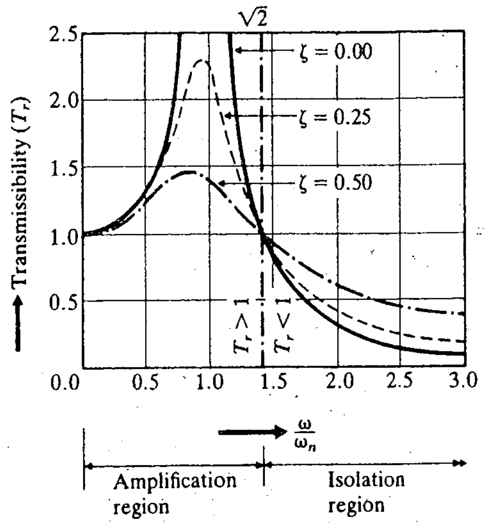


FIGURE 9.20 Variation of transmission ratio (T_r) with ω.

amount of damping should be sufficient to limit the amplitude X and the force transmitted F_t while passing through the resonance, but not so much to increase unnecessarily the force transmitted at the operating speed.

Reduction of the Force Transmitted to the Mass. If a sensitive instrument or machine of mass m is to be isolated from the unwanted harmonic motion of its base, the governing equation is given by Eq. (3.75):

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{9.88}$$

where $z = x - y$ denotes the displacement of the mass relative to the base. If the base motion is harmonic, then the motion of the mass will also be harmonic. Hence the displacement transmissibility, $T_d = \frac{X}{Y}$, is given by Eq. (3.68)

$$T_d = \frac{X}{Y} = \left\{ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right\}^{\frac{1}{2}} \tag{9.89}$$

where the right-hand side expression in Eq. (9.89) can be identified to be same as that in Eq. (9.87). Note that Eq. (9.89) is also equal to the ratio of the maximum steady-state accelerations of the mass and the base.

Reduction of the Force Transmitted to the Foundation Due to Rotating Unbalance. The excitation force caused by a rotating unbalance is given by

$$F(t) = F_0 \sin \omega t \equiv me\omega^2 \sin \omega t \tag{9.90}$$

The natural frequencies of the system are given by the roots of the equation

$$\begin{vmatrix} (k - m_1\omega^2) & -k \\ -k & (k - m_2\omega^2) \end{vmatrix} = 0 \tag{9.97}$$

The roots of Eq. (9.97) are given by

$$\omega_1^2 = 0, \qquad \omega_2^2 = \frac{(m_1 + m_2)\,k}{m_1 m_2} \tag{9.98}$$

The value $\omega_1 = 0$ corresponds to rigid-body motion since the system is unconstrained. In the steady state, the amplitudes of m_1 and m_2 are governed by Eq. (9.96), whose solution yields

$$X_1 = \frac{(k - m_2\omega^2)\,F_0}{[(k - m_1\omega^2)(k - m_2\omega^2) - k^2]} \tag{9.99}$$

$$X_2 = \frac{kF_0}{[(k - m_1\omega^2)(k - m_2\omega^2) - k^2]} \tag{9.100}$$

The force transmitted to the supporting structure (F_t) is given by the amplitude of $m_2\ddot{x}_2$:

$$F_t = -m_2\omega^2 X_2 = \frac{-m_2 k\omega^2 F_0}{[(k - m_1\omega^2)(k - m_2\omega^2) - k^2]} \tag{9.101}$$

The transmissibility of the isolator (T_r) is given by

$$T_r = \frac{F_t}{F_0} = \frac{-m_2 k\omega^2}{[(k - m_1\omega^2)(k - m_2\omega^2) - k^2]}$$

$$= \frac{1}{\left(\dfrac{m_1 + m_2}{m_2} - \dfrac{m_1\omega^2}{k}\right)} = \frac{m_2}{(m_1 + m_2)}\left(\frac{1}{1 - \dfrac{\omega^2}{\omega_2^2}}\right) \tag{9.102}$$

where ω_2 is the natural frequency of the system given by Eq. (9.98). Equation (9.102) shows, as in the case of an isolator on a rigid base, that the force transmitted to the foundation becomes less as the natural frequency of the system ω_2 is reduced.

EXAMPLE 9.3 Spring Support for Exhaust Fan

An exhaust fan, rotating at 1000 rpm, is to be supported by four springs, each having a stiffness of K. If only 10 percent of the unbalanced force of the fan is to be transmitted to the base, what should be the value of K? Assume the mass of the exhaust fan to be 40 kg.

Given: Exhaust fan with mass = 40 kg, rotational speed = 1000 rpm, and permissible shaking force to be transmitted to base = 10 percent.

Find: Stiffness (K) of each of the four supporting springs.

Approach: Use transmissibility equation.

Solution: Since the transmissibility has to be 0.1, we have, from Eq. (9.87),

$$0.1 = \left[\frac{1 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2}{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\zeta \frac{\omega}{\omega_n}\right)^2} \right]^{1/2} \tag{E.1}$$

where the forcing frequency is given by

$$\omega = \frac{1000 \times 2\pi}{60} = 104.72 \text{ rad/sec} \tag{E.2}$$

and the natural frequency of the system by

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} = \left(\frac{4K}{40}\right)^{1/2} = \frac{\sqrt{K}}{3.1623} \tag{E.3}$$

By assuming the damping ratio to be $\zeta = 0$, we obtain from Eq. (E.1),

$$0.1 = \frac{\pm 1}{\left\{1 - \left(\frac{104.72 \times 3.1623}{\sqrt{K}}\right)^2\right\}} \tag{E.4}$$

To avoid imaginary values, we need to consider the negative sign on the right-hand side of Eq. (E.4). This leads to

$$\frac{331.1561}{\sqrt{K}} = 3.3166$$

or

$$K = 9969.6365 \text{ N/m}$$ ■

EXAMPLE 9.4 Isolation of Vibrating System

A vibrating system is to be isolated from its supporting base. Find the required damping ratio that must be achieved by the isolator to limit the transmissibility at resonance to T_r = 4. Assume the system to have a single degree of freedom.

Given: Transmissibility at resonance = 4.

Find: Damping ratio of the isolator.

Approach: Find the equation for the transmissibility at resonance.

Solution: By setting $\omega = \omega_n$, Eq. (9.87) gives

$$T_r = \frac{\sqrt{1 + (2\zeta)^2}}{2\zeta}$$

or

$$\zeta = \frac{1}{2\sqrt{T_r^2 - 1}} = \frac{1}{2\sqrt{15}} = 0.1291$$

9.9.3 Vibration Isolation System with Partially Flexible Foundation

Figure 9.22 shows a more realistic situation in which the base of the isolator, instead of being completely rigid or completely flexible, is partially flexible [9.34]. We can define the mechanical impedance of the base structure, $Z(\omega)$, as the force at frequency ω required to produce a unit displacement of the base (as in Section 3.5):

$$Z(\omega) = \frac{\text{Applied force of frequency } \omega}{\text{Displacement}}$$

The equations of motion are given by[6]

$$m_1 \ddot{x}_1 + k\,(x_1 - x_2) = F_0 \cos \omega t \tag{9.103}$$
$$k\,(x_2 - x_1) = -x_2 Z(\omega) \tag{9.104}$$

By substituting the harmonic solution

$$x_j(t) = X_j \cos \omega t, \qquad j = 1, 2 \tag{9.105}$$

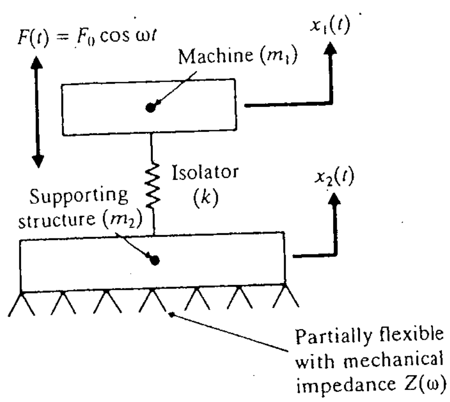


FIGURE 9.22 Machine with isolator on a partially flexible foundation.

[6]If the base is completely flexible with an unconstrained mass of m_2, $Z(\omega) = -\omega^2 m_2$, and Eqs. (9.104) and (9.105) lead to Eq. (9.94).

points can be determined by setting

$$\frac{d}{dr}\left(\frac{X}{m_0 e/m}\right) = 0$$

This results in defining

$$\frac{X}{m_0 e/m} = 0$$

as the initial minimum point of all curves. Also, all curves approach unity as r becomes large. Finally, the maximum point occurs at

$$r = \frac{1}{\sqrt{1 - 2\zeta^2}} > 1 \tag{4-83}$$

Accordingly, the peaks occur to the right of the resonance value of $r = 1$. For $\zeta = 0.707$ the curve rises through its entirety, with the maximum equal to 1 as r approaches infinity.

Figure 4-20 is adequate, provided that the variation in r is limited to changing ω_f. Note that small amplitude occurs only at low operating frequencies, as would be expected.

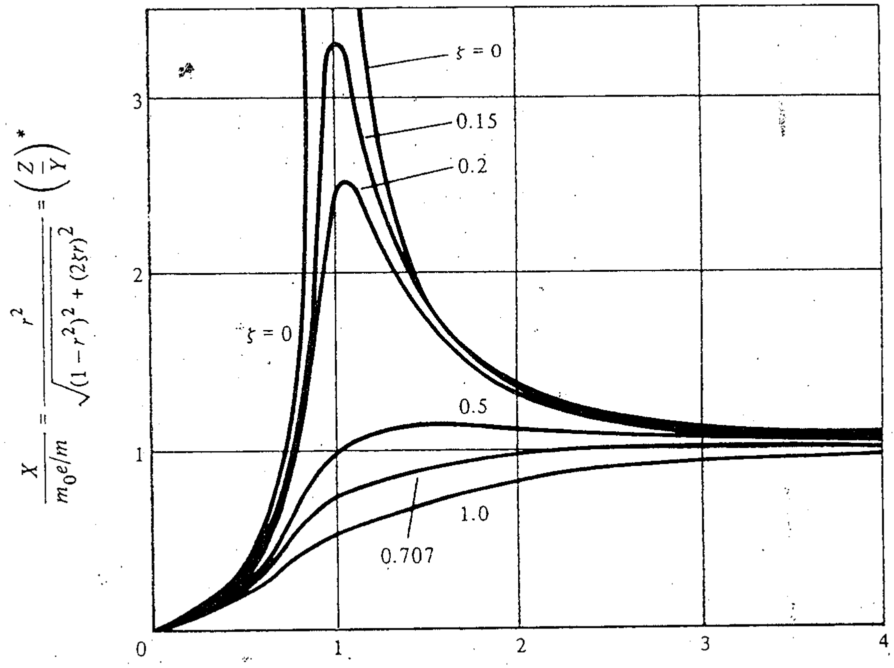
Since ζ (as well as r) is dependent on k and m, Fig. 4-20 does not properly show the effect of varying k or m. Also, the reference $m_0 e/m$ is affected by altering m. The effect of varying k or m can be observed by writing the amplitude relation here in the form given by Eq. 4-79. The amplitude X can then be plotted against either k or m for various values of the damping constant c. The resulting families of curves will be identical to those of Figs. 4-17 and 4-18, provided P_0 is replaced by $m_0 e\omega_f^2$.

In all the preceding discussion, it should be noted that amplitude is dependent on the quantity $m_0 e$ and that if either m_0 or e is small, the amplitude will become small. This merely emphasizes the importance of reducing the eccentric condition insofar as may be possible.

EXAMPLE 4-6 A machine with a rotating shaft has a total weight of 200 lb and is supported by springs. The damping constant for the system is found to be 3 lb sec/in. The resonant speed is determined experimentally to be 1200 rpm, and the corresponding amplitude of the main mass of the machine is 0.50 in. Determine the amplitude for a speed of 2400 rpm. Also determine the fixed value the amplitude will eventually approach at high speed.

SOLUTION Since the machine oscillates when in operation, the rotating part must contain an eccentric mass. At resonance, Eq. 4-79 becomes

$$X = \frac{m_0 e\omega_f}{c}$$



*For oscillating support case.

Figure 4-20

so that

$$m_0 e = \frac{Xc}{\omega_f} = \frac{0.5 \times 3}{40\pi} = 0.011\,93 \text{ lb sec}^2$$

$$k = m\omega_f^2 = \frac{200}{386} \times (40\pi)^2 = 8180 \text{ lb/in.}$$

Then, at 2400 rpm,

$$X = \frac{m_0 e\omega_f^2}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}$$

$$= \frac{0.011\,93 \times (80\pi)^2}{\sqrt{[8180 - \frac{200}{386} \times (80\pi)^2]^2 + (3 \times 80\pi)^2}}$$

$$= 0.0306 \text{ in.}$$

For further increase in speed, the amplitude approaches the value defined by

$$X = \frac{m_0 e}{m} = \frac{0.011\,93 \times 386}{200} = 0.0230 \text{ in.}$$

SOLUTION

$$X = \frac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} = \frac{0.54}{\sqrt{[k - 0.1 \times (12)^2]^2 + (0.24 \times 12)^2}}$$

$$= \frac{0.54}{\sqrt{(k - 14.4)^2 + (2.88)^2}}$$

a. For $k = 2$, $X = 0.04242$ m $= 4.242$ cm
For $k = 25$, $X = 4.916$ cm
For $k = 90$, $X = 0.7138$ cm

b. $X_{max} = \dfrac{P_0}{c\omega_f} = \dfrac{0.54}{0.24 \times 12} = 0.1875$ m $= 18.75$ cm

for $k = m\omega_f^2 = 0.1 \times (12)^2 = 14.4$ N/m

4-13. ROTATING UNBALANCE

A common source of forced vibration is caused by the rotation of a small eccentric mass such as that represented by m_0 in Fig. 4-19(a). This condition results from a setscrew or a key on a rotating shaft, crankshaft rotation, and many other simple but unavoidable situations. Rotating unbalance is inherent in rotating parts, because it is virtually impossible to place the axis of the mass center on the axis of rotation.

For the system shown, the total mass is m and the eccentric mass is m_0, so the mass of the machine body is $(m - m_0)$. The length of the eccentric arm, or the eccentricity of m_0, is represented by e. If the arm rotates with an angular velocity ω_f rad/sec, then the angular position of the arm is defined by $\omega_f t$ with respect to the indicated horizontal reference, where t is time, in seconds. The free-body diagram for this system is shown in Fig. 4-19(b), positive x having been taken as upward. The horizontal motion of $(m - m_0)$ is considered to be prevented by guides. The vertical displacement of m_0 is $(x + e \sin \omega_f t)$. From Eq. 1-8, the differential equation of motion can then be written as

$$(m - m_0)\frac{d^2x}{dt^2} + m_0 \frac{d^2}{dt^2}(x + e \sin \omega_f t) = -kx - c\frac{dx}{dt} \tag{4-76}$$

which can be rearranged in the form

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = m_0 e\omega_f^2 \sin \omega_f t \tag{4-77}$$

Examination of this and comparison to the differential equation (Eq. 4-38) for motion forced by $P = P_0 \sin \omega_f t$ enable the steady-state solution to be set down, from Eq. 4-48, as

$$x = X \sin(\omega_f t - \psi) \tag{4-78}$$

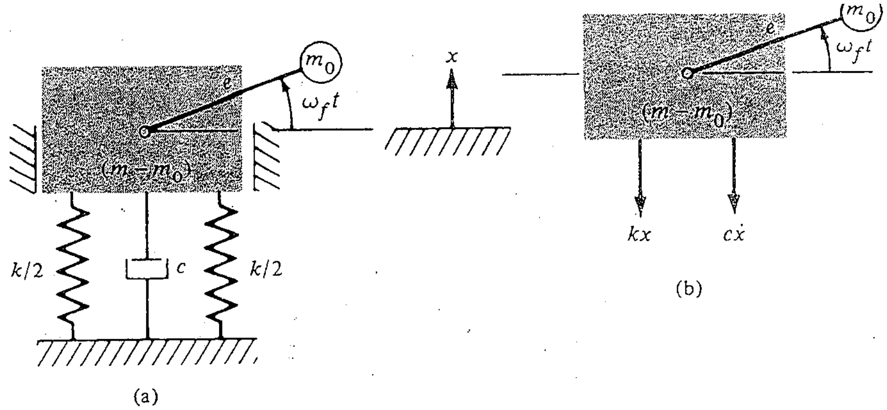


Figure 4-19

where

$$X = \frac{m_0 e\omega_f^2}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \tag{4-79}$$

$$= \frac{(m_0 e/m)\omega_f^2(m/k)}{\sqrt{[(k - m\omega_f^2)/k]^2 + (c\omega_f/k)^2}} = \frac{m_0 e}{m}\frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4-80}$$

and

$$\frac{X}{m_0 e/m} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4-81}$$

Also

$$\tan \psi = \frac{2\zeta r}{1 - r^2} \tag{4-82}$$

Here $\omega = \sqrt{k/m}$ represents the natural circular frequency of the undamped system (including the mass m_0), but x defines the forced motion of the main mass $(m - m_0)$. It should be noted that for this case ψ will be represented physically by the angle of the eccentric arm relative to the horizontal reference of $\omega_f t$. Thus for a value of ψ determined by Eq. 4-82, the arm would be at this angle when the main body is at its neutral position, moving upward. (Since the motion lags the forcing condition, the arm then leads the motion by the angle ψ determined.) The steady-state amplitude is generally significant, and this can be studied by plotting

$$\frac{X}{m_0 e/m}$$

against the frequency ratio r for various values of the damping factor ζ, resulting in the family of curves shown in Fig. 4-20. Maximum and minimum

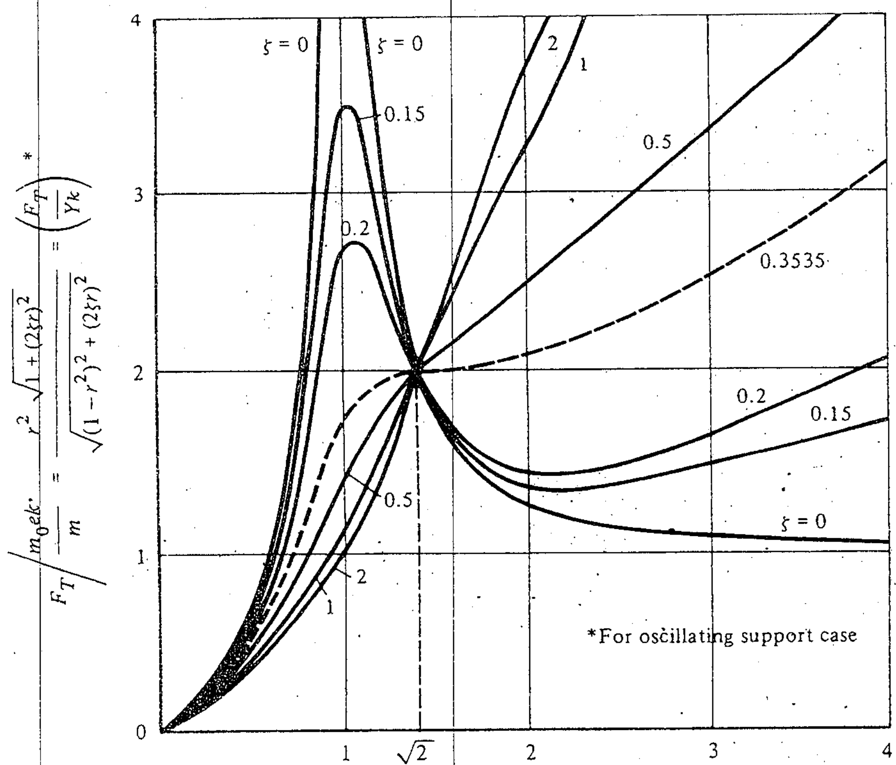


Figure 4-25

$m_0 e\omega_f^2$ for P_0 in Eqs. 4-88 and 4-89. Then

$$F_T = m_0 e\omega_f^2 \frac{\sqrt{k^2 + (c\omega_f)^2}}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} = m_0 e\omega_f^2 \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4-93}$$

Multiplying the numerator and denominator by k/m and rearranging gives

$$F_T = \frac{m_0 ek}{m} \cdot \frac{r^2\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4-94}$$

whence

$$\frac{F_T}{m_0 ek/m} = \frac{r^2\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4-95}$$

The effect of varying ω_f on the transmitted force can be shown by plotting $F_T/(m_0 ek/m)$ against r for various values of the damping factor ζ. In so doing, k and m are taken as constant. The reference $m_0 ek/m$ is then fixed. The resulting family of curves is shown in Fig. 4-25. Damping serves to limit the transmitted force in the region of resonance. A crossover point occurs at $r = \sqrt{2}$, for which $F_T/(m_0 ek/m)$ has a value of 2. For no damping, the curve

approaches a value of 1 as r approaches infinity. When damping is present, the force becomes very large as r increases, and the greater the damping, the more rapidly this occurs. Even for small damping, the increase in the transmitted force is significant. Since frequency ratios of 10 or more are common in practice, the seriousness of damping is evident.

The maximum and minimum points for the family of curves here can be determined by setting

$$\frac{d}{dr}\left(\frac{F_T}{m_0 ek/m}\right) = 0$$

The resulting expression is satisfied by the following conditions:

1. For $r = 0$. This defines the initial point of $F_T/(m_0 ek/m) = 0$ for all curves.
2. By the roots of the relation

$$2\zeta^2 r^6 + (16\zeta^4 - 8\zeta^2)r^4 + (8\zeta^2 - 1)r^2 + 1 = 0$$

If $0 < \zeta < \sqrt{2}/4$, there are two positive real roots of this relation. One of these will be between $r = 0$ and $r = \sqrt{2}$, and will define a maximum point on the curve. The other will be $r > \sqrt{2}$ and will define a minimum point on the curve. If $\zeta > \sqrt{2}/4$, there is no maximum point on the curve.

If it is desired to determine the effect of varying k and m on the transmitted force, this can be done by using Eq. 4-93 and arranging it as

$$\frac{F_T}{m_0 e\omega_f^2} = \frac{\sqrt{k^2 + (c\omega_f)^2}}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \tag{4-96}$$

Since the forcing frequency ω_f, the mass m_0, and the eccentricity e are to be held constant, this case becomes identical to those shown in Figs. 4-23 and 4-24, and no further analysis is needed here.

EXAMPLE 4-7 For the rotating eccentric of Example 4-6, calculate the maximum dynamic force transmitted to the foundation at the resonant speed. Also obtain F_T for the crossover point of Fig. 4-25.

SOLUTION At resonance, Eq. 4-93 reduces to

$$F_T = \frac{m_0 e\omega_f}{c}\sqrt{k^2 + (c\omega_f)^2} = X_{res}\sqrt{k^2 + (c\omega_f)^2}$$

$$= 0.5\sqrt{(8180)^2 + (3 \times 40\pi)^2} = 4094 \text{ lb}$$

For the crossover,

$$F_T = \frac{2m_0 e}{m}k = 2 \times 0.0230 \times 8180 = 376 \text{ lb}$$