Sir Isaac Newton (1642–1727) was an English natural philosopher, a professor of mathematics at Cambridge University, and President of the Royal Society. His *Principia Mathematica* (1687), which deals with the laws and conditions of motion, is considered to be the greatest scientific work ever produced. The definitions of force, mass, and momentum, and his three laws of motion crop up continually in dynamics. Quite fittingly, the unit of force named "Newton" in SI units happens to be the approximate weight of an average apple, which inspired him to study the laws of gravity. (Photo courtesy of David Eugene Smith, *History of Mathematics*, Volume I—General Survey of the History of Elementary Mathematics, Dover Publications, Inc., New York, 1958.)

CHAPTER 2

Free Vibration of Single Degree of Freedom Systems

2.1 Introduction

A system is said to undergo free vibration when it oscillates only under an initial disturbance with no external forces acting after the initial disturbance. The oscillations of the pendulum of a grandfather clock, the vertical oscillatory motion felt by a bicyclist after hitting a road bump, and the motion of a child on a swing under an initial push represent a few examples of free vibration.

Figure 2.1(a) shows a spring-mass system that represents the simplest possible vibratory system. It is called a single degree of freedom system since one coordinate (x) is sufficient to specify the position of the mass at any time. There is no external force applied to the mass; hence the motion resulting from an initial disturbance will be a free vibration. Since there is no element that causes dissipation of energy during the motion of the mass, the amplitude of motion remains constant with time; it is an *undamped* system. In actual practice, except in a vacuum, the amplitude of free vibration diminishes gradually over time, due to the resistance offered by the surrounding medium (such as air). Such vibrations are said to be *damped*. The study

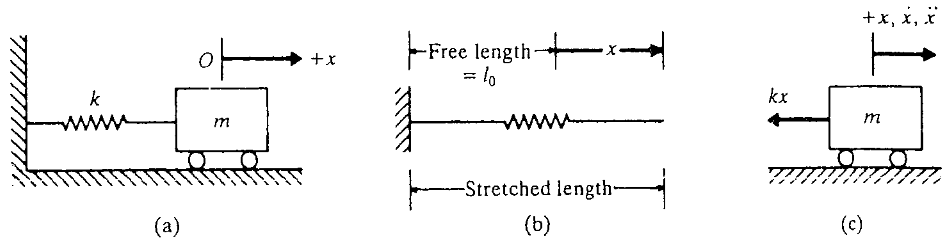


FIGURE 2.1 A spring-mass system in horizontal position.

of the free vibration of undamped and damped single degree of freedom systems is fundamental to the understanding of more advanced topics in vibrations.

Several mechanical and structural systems can be idealized as single degree of freedom systems. In many practical systems, the mass is distributed, but for a simple analysis, it can be approximated by a single point mass. Similarly, the elasticity of the system, which may be distributed throughout the system, can also be idealized by a single spring. For the cam-follower system shown in Fig. 1.32, for example, the various masses were replaced by an equivalent mass (m_{eq}) in Example 1.7. The elements of the follower system (pushrod, rocker arm, valve, and valve spring) are all elastic but can be reduced to a single equivalent spring of stiffness k_{eq}. For a simple analysis, the cam-follower system can thus be idealized as a single degree of freedom spring-mass system, as shown in Fig. 2.2.

Similarly, the structure shown in Fig. 2.3 can be considered a cantilever beam that is fixed at the ground. For the study of transverse vibration, the top mass can be considered a point mass and the supporting structure (beam) can be approximated as a spring to obtain the single degree of freedom model shown in Fig. 2.4. The building frame shown in Fig. 2.5(a) can also be idealized as a spring-mass system, as shown in Fig. 2.5(b). In this case, since the spring constant k is merely the ratio of force to deflection, it can be determined from the geometric and material properties of the columns. The mass of the idealized system is the same as that of the floor if we assume the mass of the columns to be negligible.

2.2 Free Vibration of an Undamped Translational System

2.2.1 Equation of Motion Using Newton's Second Law of Motion

Using Newton's second law of motion, we will consider the derivation of the equation of motion in this section. The procedure we will use can be summarized as follows:

1. Select a suitable coordinate to describe the position of the mass or rigid body in the system. Use a linear coordinate to describe the linear motion of a point mass or the centroid of a rigid body, and an angular coordinate to describe the angular motion of a rigid body.

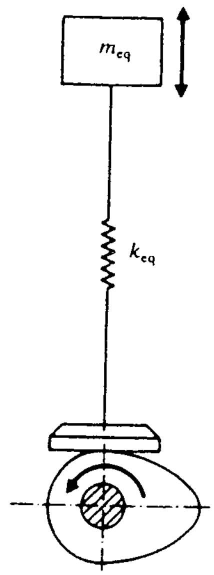
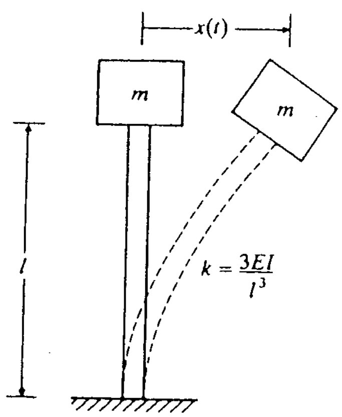
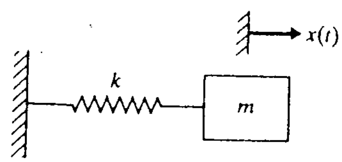


FIGURE 2.2 Equivalent spring-mass system for the cam-follower system of Fig. 1.32.

FIGURE 2.3



(a) Idealization of the tall structure



(b) Equivalent spring-mass system

FIGURE 2.4

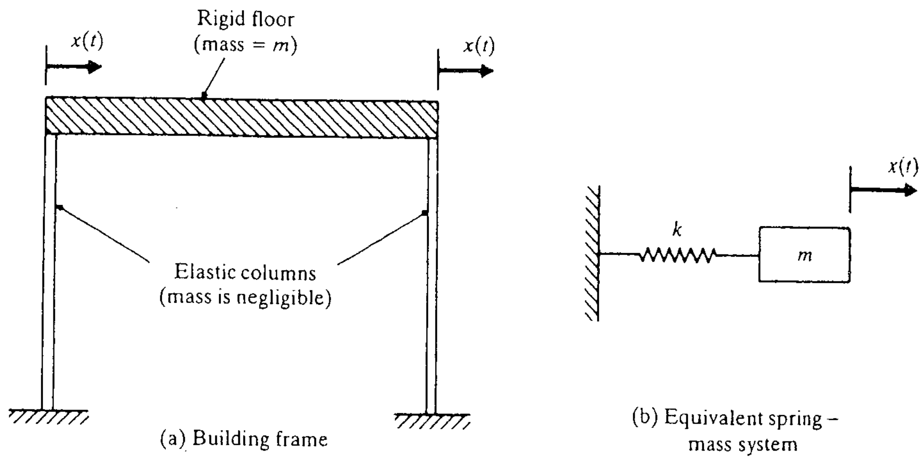


FIGURE 2.5 Idealization of a building frame.

2. Determine the static equilibrium configuration of the system and measure the displacement of the mass or rigid body from its static equilibrium position.
3. Draw the free-body diagram of the mass or rigid body when a positive displacement and velocity are given to it. Indicate all the active and reactive forces acting on the mass or rigid body.
4. Apply Newton's second law of motion to the mass or rigid body shown by the free-body diagram. Newton's second law of motion can be stated as follows:

The rate of change of momentum of a mass is equal to the force acting on it.

Thus, if mass m is displaced a distance $\vec{x}(t)$ when acted upon by a resultant force $\vec{F}(t)$ in the same direction, Newton's second law of motion gives

$$\vec{F}(t) = \frac{d}{dt}\left(m \frac{d\vec{x}(t)}{dt} \right)$$

If mass m is constant, this equation reduces to

$$\vec{F}(t) = m \frac{d^2\vec{x}(t)}{dt^2} = m \ddot{\vec{x}} \tag{2.1}$$

where

$$\ddot{\vec{x}} = \frac{d^2\vec{x}(t)}{dt^2}$$

is the acceleration of the mass. Equation (2.1) can be stated in words as

Resultant force on the mass = mass × acceleration

For a rigid body undergoing rotational motion, Newton's law gives

$$\vec{M}(t) = J\,\ddot{\vec{\theta}} \tag{2.2}$$

where $\vec{M}$ is the resultant moment acting on the body and $\vec{\theta}$ and $\ddot{\vec{\theta}} = \dfrac{d^2\theta(t)}{dt^2}$ are the resulting angular displacement and angular acceleration, respectively. Equation (2.1) or (2.2) represents the equation of motion of the vibrating system.

The procedure is now applied to the undamped single degree of freedom system shown in Fig. 2.1(a). Here the mass is supported on frictionless rollers and can have translatory motion in the horizontal direction. When the mass is displaced a distance $+x$ from its static equilibrium position, the force in the spring is kx and the free-body diagram of the mass can be represented as shown in Fig. 2.1(c). The application of Eq. (2.1) to mass m yields the equation of motion

$$F(t) = -k\,x = m\,\ddot{x}$$

or

$$m\,\ddot{x} + k\,x = 0 \tag{2.3}$$

2.2.2 Equation of Motion Using Other Methods

As stated in Section 1.6, the equations of motion of a vibrating system can be derived using several methods. The applications of D'Alembert's principle, the principle of virtual displacements, and the principle of conservation of energy are considered in this section.

D'Alembert's Principle The equations of motion, Eqs. (2.1) and (2.2), can be rewritten as

$$\vec{F}(t) - m\,\ddot{\vec{x}} = 0 \tag{2.4a}$$

$$\vec{M}(t) - J\,\ddot{\vec{\theta}} = 0 \tag{2.4b}$$

These equations can be considered equilibrium equations provided that $-m\ddot{\vec{x}}$ and $-J\ddot{\vec{\theta}}$ are treated as a force and a moment. This fictitious force (or moment) is known as the inertia force (or inertia moment) and the artificial state of equilibrium implied by Eq. (2.4a) or (2.4b) is known as dynamic equilibrium. This principle, implied in Eq. (2.4a) or (2.4b), is called the D'Alembert's principle. The application of D'Alembert's principle to the system shown in Fig. 2.1(c) yields the equation of motion:

$$-k\,x - m\,\ddot{x} = 0 \qquad \text{or} \qquad m\,\ddot{x} + k\,x = 0 \tag{2.3}$$

Principle of Virtual Displacements The principle of virtual displacements states that "if a system that is in equilibrium under the action of a set of forces is subjected to a virtual displacement, then the total virtual work done by the forces will be zero." Here the virtual displacement is defined as an imaginary infinitesimal displacement given instantaneously. It must be a physically possible displacement that

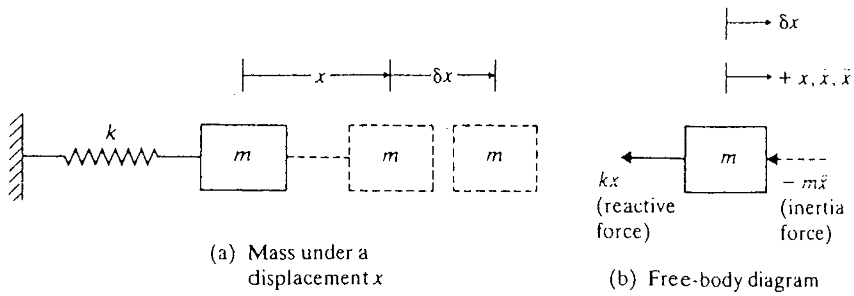


FIGURE 2.6

is compatible with the constraints of the system. The virtual work is defined as the work done by all the forces, including the inertia forces for a dynamic problem, due to a virtual displacement.

Consider a spring-mass system in a displaced position as shown in Fig. 2.6(a), where x denotes the displacement of the mass. Figure 2.6(b) shows the free-body diagram of the mass with the reactive and inertia forces indicated. When the mass is given a virtual displacement δx, as shown in Fig. 2.6(b), the virtual work done by each force can be computed as follows:

$$\text{Virtual work done by the spring force} = \delta W_s = -(k\,x)\,\delta x$$

$$\text{Virtual work done by the inertia force} = \delta W_i = -(m\,\ddot{x})\,\delta x$$

When the total virtual work done by all the forces is set equal to zero, we obtain

$$-m\,\ddot{x}\,\delta x - k\,x\,\delta x = 0 \tag{2.5}$$

Since the virtual displacement can have an arbitrary value, $\delta x \neq 0$, Eq. (2.5) gives the equation of motion of the spring-mass system as

$$m\,\ddot{x} + k\,x = 0 \tag{2.3}$$

Principle of Conservation of Energy A system is said to be conservative if no energy is lost due to friction or energy-dissipating nonelastic members. If no work is done on a conservative system by external forces (other than gravity or other potential forces), then the total energy of the system remains constant. Since the energy of a vibrating system is partly potential and partly kinetic, the sum of these two energies remains constant. The kinetic energy T is stored in the mass by virtue of its velocity, and the potential energy U is stored in the spring by virtue of its elastic deformation. Thus the principle of conservation of energy can be expressed as:

$$T + U = \text{constant}$$

or

$$\frac{d}{dt}(T + U) = 0 \tag{2.6}$$

The kinetic and potential energies are given by

$$T = \tfrac{1}{2}m\dot{x}^2 \tag{2.7}$$

and

$$U = \tfrac{1}{2}kx^2 \tag{2.8}$$

Substitution of Eqs. (2.7) and (2.8) into Eq. (2.6) yields the desired equation

$$m\ddot{x} + kx = 0 \tag{2.3}$$

2.2.3 Equation of Motion of a Spring-Mass System in Vertical Position

Consider the configuration of the spring-mass system shown in Fig. 2.7(a). The mass hangs at the lower end of a spring, which in turn is attached to a rigid support at its upper end. At rest, the mass will hang in a position called the *static equilibrium position,* in which the upward spring force exactly balances the downward gravita-

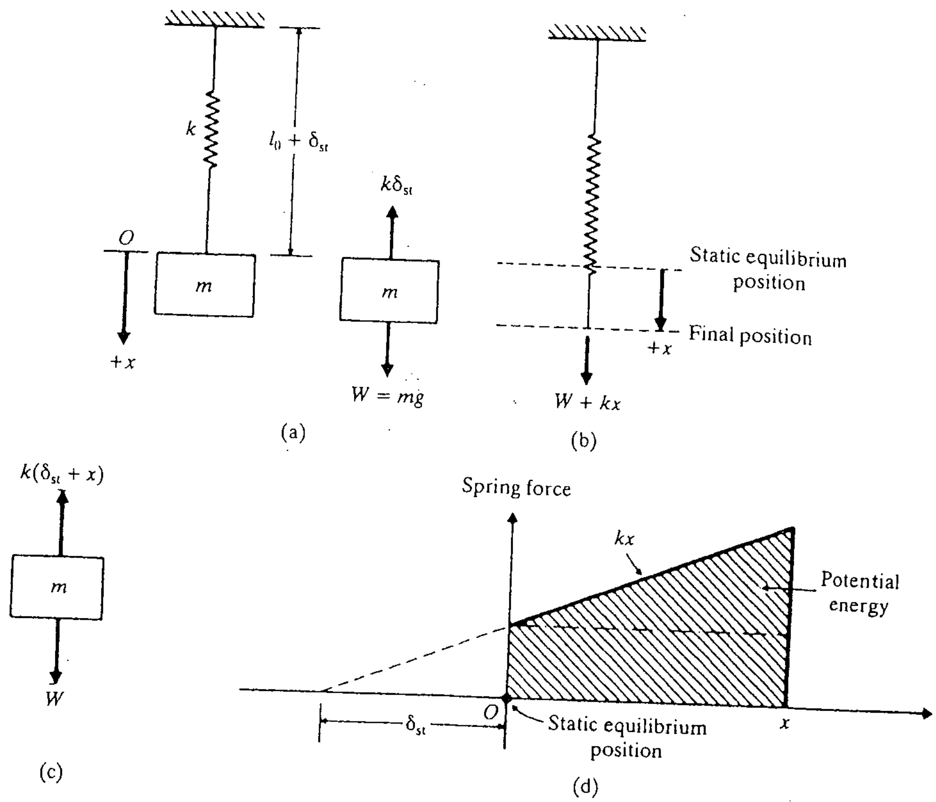


FIGURE 2.7 A spring-mass system in vertical position.

tional force on the mass. In this position the length of the spring is $l_0 + \delta_{st}$, where δ_{st} is the static deflection—the elongation due to the weight W of the mass m. From Fig. 2.7(a), we find that, for static equilibrium,

$$W = mg = k\delta_{st} \tag{2.9}$$

where g is the acceleration due to gravity. Let the mass be deflected a distance $+x$ from its static equilibrium position; then the spring force is $-k(x + \delta_{st})$, as shown in Fig. 2.7(c). The application of Newton's second law of motion to mass m gives

$$m\ddot{x} = -k(x + \delta_{st}) + W$$

and since $k\delta_{st} = W$, we obtain

$$m\ddot{x} + kx = 0 \tag{2.10}$$

Notice that Eqs. (2.3) and (2.10) are identical. This indicates that when a mass moves in a vertical direction, we can ignore its weight, provided we measure x from its static equilibrium position.

Note: Equation (2.10), the equation of motion of the system shown in Fig. 2.7, can also be derived using D'Alembert's principle, the principle of virtual displacements, or the principle of conservation of energy. For example, if the principle of conservation of energy is to be used, we note that the expression for the kinetic energy, T, remains the same as Eq. (2.7). However, the expression for the potential energy, U, is to be derived by considering the weight of the mass. For this we note that the spring force at static equilibrium position ($x = 0$) is mg. When the spring deflects by an amount x, its potential energy is given by (see Fig. 2.7d):

$$mg\,x + \frac{1}{2}kx^2$$

Furthermore, the potential energy of the system due to the change in elevation of the mass (note that $+x$ is downward) is $-mgx$. Thus the net potential energy of the system about the static equilibrium position is given by

$$\begin{aligned} U &= \text{potential energy of the spring} \\ &\quad + \text{change in potential energy due to change in elevation of the mass } m \\ &= mgx + \frac{1}{2}kx^2 - mgx = \frac{1}{2}kx^2 \end{aligned}$$

Since the expressions of T and U remain unchanged, the application of the principle of conservation of energy gives the same equation of motion, Eq. (2.3).

2.2.4 Solution

The solution of Eq. (2,3) can be found by assuming

$$x(t) = Ce^{st} \tag{2.11}$$

where C and s are constants to be determined. Substitution of Eq. (2.11) into Eq. (2.3) gives

$$C(ms^2 + k) = 0$$

Since C cannot be zero, we have

$$ms^2 + k = 0 \tag{2.12}$$

and hence

$$s = \pm\left(-\frac{k}{m}\right)^{1/2} = \pm i\omega_n \tag{2.13}$$

where $i = (-1)^{1/2}$ and

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} \tag{2.14}$$

Equation (2.12) is called the *auxiliary* or the *characteristic* equation corresponding to the differential Eq. (2.3). The two values of s given by Eq. (2.13) are the roots of the characteristic equation, also known as the *eigenvalues* or the *characteristic values* of the problem. Since both values of s satisfy Eq. (2.12), the general solution of Eq. (2.3) can be expressed as

$$x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t} \tag{2.15}$$

where C_1 and C_2 are constants. By using the identities

$$e^{\pm i\alpha t} = \cos \alpha t \pm i \sin \alpha t$$

Eq. (2.15) can be rewritten as

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \tag{2.16}$$

where A_1 and A_2 are new constants. The constants C_1 and C_2 or A_1 and A_2 can be determined from the initial conditions of the system. Two conditions are to be specified to evaluate these constants uniquely. Note that the number of conditions to be specified is the same as the order of the governing differential equation. In the present case, if the values of displacement $x(t)$ and velocity $\dot{x}(t) = (dx/dt)(t)$ are specified as x_0 and $\dot{x}_0$ at $t = 0$, we have, from Eq. (2.16),

$$\begin{aligned} x(t = 0) &= A_1 = x_0 \\ \dot{x}(t = 0) &= \omega_n A_2 = \dot{x}_0 \end{aligned} \tag{2.17}$$

Hence $A_1 = x_0$ and $A_2 = \dot{x}_0/\omega_n$. Thus the solution of Eq. (2.3) subject to the initial conditions of Eq. (2.17) is given by

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \tag{2.18}$$

2.2.5 Harmonic Motion

Equations (2.15), (2.16), and (2.18) are harmonic functions of time. The motion is symmetric about the equilibrium position of the mass m. The velocity is a maximum and the acceleration is zero each time the mass passes through this position. At the extreme displacements, the velocity is zero and the acceleration is a maximum. Since this represents simple harmonic motion (see Section 1.10), the spring-mass system itself is called a *harmonic oscillator*. The quantity ω_n, given by Eq. (2.14), represents the natural frequency of vibration of the system.

Equation (2.16) can be expressed in a different form by introducing the notation

$$A_1 = A \cos \phi$$
$$A_2 = A \sin \phi \tag{2.19}$$

where A and ϕ are the new constants which can be expressed in terms of A_1 and A_2 as

$$A = (A_1^2 + A_2^2)^{1/2} = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} = \text{amplitude}$$
$$\phi = \tan^{-1}\left(\frac{A_2}{A_1}\right) = \tan^{-1}\left(\frac{\dot{x}_0}{x_0 \omega_n}\right) = \text{phase angle} \tag{2.20}$$

Introducing Eq. (2.19) into Eq. (2.16), the solution can be written as

$$x(t) = A \cos(\omega_n t - \phi) \tag{2.21}$$

By using the relations

$$A_1 = A_0 \sin \phi_0$$
$$A_2 = A_0 \cos \phi_0 \tag{2.22}$$

Eq. (2.16) can also be expressed as

$$x(t) = A_0 \sin(\omega_n t + \phi_0) \tag{2.23}$$

where

$$A_0 = A = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} \tag{2.24}$$

and

$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right) \tag{2.25}$$

The nature of harmonic oscillation can be represented graphically as in Fig. 2.8(a). If $\vec{A}$ denotes a vector of magnitude A, which makes an angle $\omega_n t - \phi$ with respect to the vertical (x) axis, then the solution, Eq. (2.21), can be seen to be the projection of the vector $\vec{A}$ on the x-axis. The constants A_1 and A_2 of Eq. (2.16), given by Eq. (2.19), are merely the rectangular components of $\vec{A}$ along two orthogonal axes making angles ϕ and $-(\frac{\pi}{2} - \phi)$ with respect to the vector $\vec{A}$. Since the angle $\omega_n t - \phi$ is a linear function of time, it increases linearly with time; the entire diagram thus rotates anticlockwise at an angular velocity ω_n. As the diagram (Fig. 2.8a) rotates, the projection of $\vec{A}$ onto the x-axis varies harmonically so that the motion repeats itself every time the vector $\vec{A}$ sweeps an angle of 2π. The projection of $\vec{A}$, namely $x(t)$, is shown plotted in Fig. 2.8(b) as a function of time. The phase angle ϕ can also be interpreted as the angle between the origin and the first peak.

Thus, when the mass vibrates in a vertical direction, we can compute the natural frequency and the period of vibration by simply measuring the static deflection δ_{st}. It is not necessary that we know the spring stiffness k and the mass m.

2. From Eq. (2.21), the velocity $\dot{x}(t)$ and the acceleration $\ddot{x}(t)$ of the mass m at time t can be obtained as

$$\dot{x}(t) = \frac{dx}{dt}(t) = -\omega_n A \sin(\omega_n t - \phi) = \omega_n A \cos\left(\omega_n t - \phi + \frac{\pi}{2}\right)$$
$$\ddot{x}(t) = \frac{d^2x}{dt^2}(t) = -\omega_n^2 A \cos(\omega_n t - \phi) = \omega_n^2 A \cos(\omega_n t - \phi + \pi) \tag{2.31}$$

Equation (2.31) shows that the velocity leads the displacement by $\pi/2$ and the acceleration leads the displacement by π.

3. If the initial displacement (x_0) is zero, Eq. (2.21) becomes

$$x(t) = \frac{\dot{x}_0}{\omega_n}\cos\left(\omega_n t - \frac{\pi}{2}\right) = \frac{\dot{x}_0}{\omega_n}\sin \omega_n t \tag{2.32}$$

On the other hand, if the initial velocity ($\dot{x}_0$) is zero, the solution becomes

$$x(t) = x_0 \cos \omega_n t \tag{2.33}$$

4. The response of a single degree of freedom system can be represented in the displacement (x)-velocity ($\dot{x}$) plane, known as the state space or phase plane. For this we consider the displacement given by Eq. (2.21) and the corresponding velocity:

$$x(t) = A \cos(\omega_n t - \phi)$$

or

$$\cos(\omega_n t - \phi) = \frac{x}{A} \tag{2.34}$$

$$\dot{x}(t) = -A\,\omega_n \sin(\omega_n t - \phi)$$

or

$$\sin(\omega_n t - \phi) = -\frac{\dot{x}}{A\,\omega_n} = -\frac{y}{A} \tag{2.35}$$

where $y = \dot{x}/\omega_n$. By squaring and adding Eqs. (2.34) and (2.35), we obtain

$$\cos^2(\omega_n t - \phi) + \sin^2(\omega_n t - \phi) = 1$$

or

$$\frac{x^2}{A^2} + \frac{y^2}{A^2} = 1 \tag{2.36}$$

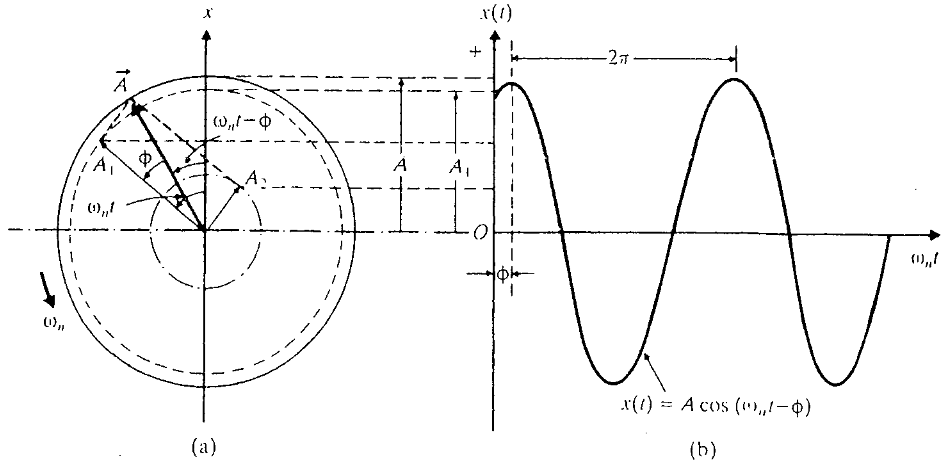


FIGURE 2.8 Graphical representation of the motion of a harmonic oscillator.

Note the following aspects of the spring-mass system:

1. If the spring-mass system is in a vertical position, as shown in Fig. 2.7(a), the circular natural frequency can be expressed as

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} \tag{2.26}$$

The spring constant k can be expressed in terms of the mass m from Eq. (2.9) as

$$k = \frac{W}{\delta_{st}} = \frac{mg}{\delta_{st}} \tag{2.27}$$

Substitution of Eq. (2.27) into Eq. (2.14) yields

$$\omega_n = \left(\frac{g}{\delta_{st}}\right)^{1/2} \tag{2.28}$$

Hence the natural frequency in cycles per second and the natural period are given by

$$f_n = \frac{1}{2\pi}\left(\frac{g}{\delta_{st}}\right)^{1/2} \tag{2.29}$$

$$\tau_n = \frac{1}{f_n} = 2\pi\left(\frac{\delta_{st}}{g}\right)^{1/2} \tag{2.30}$$

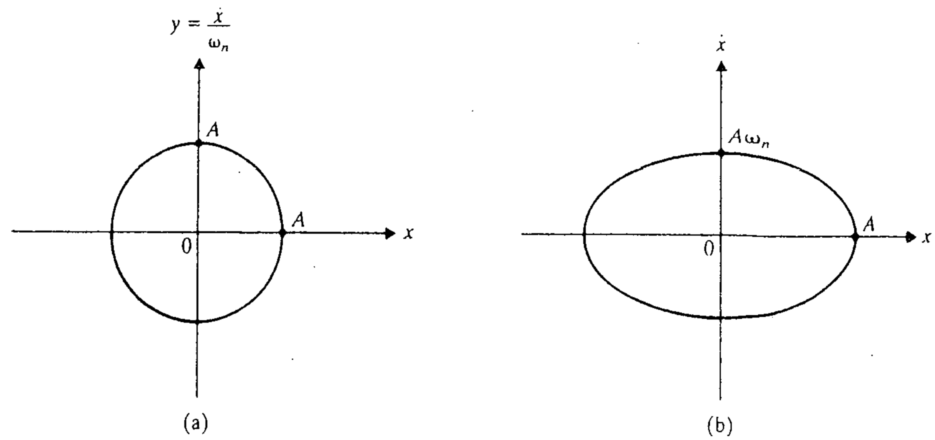


FIGURE 2.9

The graph of Eq. (2.36) in the (x, y)-plane is a circle, as shown in Fig. 2.9(a), and it constitutes the phase plane or state space representation of the undamped system. The radius of the circle, A, is determined by the initial conditions of motion. Note that the graph of Eq. (2.36) in the $(x, \dot{x})$ plane will be an ellipse, as shown in Fig. 2.9(b).

EXAMPLE 2.1 Natural Frequency of a Water Tank

The column of the water tank shown in Fig. 2.10 is 300 ft high and is made of reinforced concrete with a tubular cross section of inner diameter 8 ft and outer diameter 10 ft. The tank weighs 6×10^5 lb with water. Find the natural frequency of transverse vibration of the water tank by neglecting the mass of the column.

Given: Water tank of Fig. 2.10.

Find: Natural frequency of vibration of the tank in transverse direction.

Approach: Find the stiffness of the column and consider the tank as a single degree of freedom system.

Assumptions:

1. Water tank is a point mass.
2. Column has a uniform cross section.
3. Mass of the column is negligible.

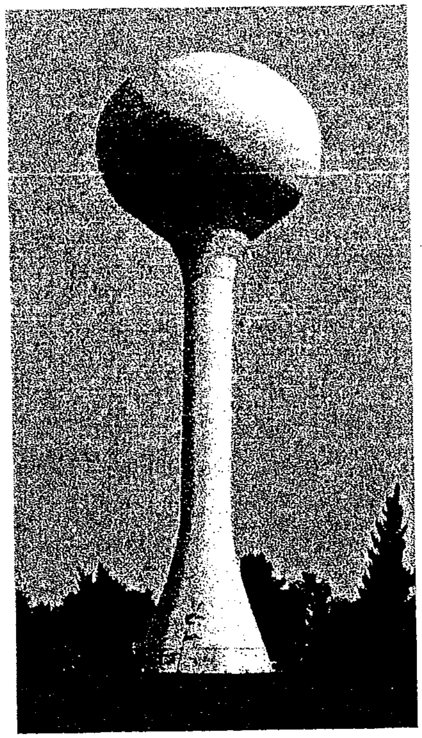


FIGURE 2.10 Elevated tank. (Photo courtesy of West Lafayette Water Company.)

Solution: The water tank can be considered as a cantilever beam with a concentrated load (weight) at the free end. The transverse deflection of the beam, δ, due to a load P is given by $\frac{Pl^3}{3EI}$, where l is the length, E is the Young's modulus and I is the area moment of inertia of the cross section of the beam. The stiffness of the beam (column of the tank) is given by

$$k = \frac{P}{\delta} = \frac{3\,EI}{l^3}$$

In the present case, $l = 3600$ in, $E = 4 \times 10^6$ psi,

$$I = \frac{\pi}{64}(d_0^4 - d_i^4) = \frac{\pi}{64}(120^4 - 96^4) = 600.9554 \times 10^4 \text{ in}^4$$

and hence

$$k = \frac{3\,(4 \times 10^6)\,(600.9554 \times 10^4)}{3600^3} = 1545.6672 \text{ lb/in}$$

The natural frequency of the water tank in transverse direction is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1545.6672 \times 386.4}{6 \times 10^5}} = 0.9977 \text{ rad/sec}$$

■

EXAMPLE 2.2 Natural Frequency of Cockpit of a Firetruck

The cockpit of a firetruck is located at the end of a telescoping boom, as shown in Fig. 2.11(a). The cockpit, along with the fireman, weighs 2000 N. Find the natural frequency of vibration of the cockpit in the vertical direction.

Data: Young's modulus of the material: $E = 2.1 \times 10^{11}$ N/m², Lengths: $l_1 = l_2 = l_3 =$ 3 m, cross-sectional areas: $A_1 = 20$ cm², $A_2 = 10$ cm², $A_3 = 5$ cm².

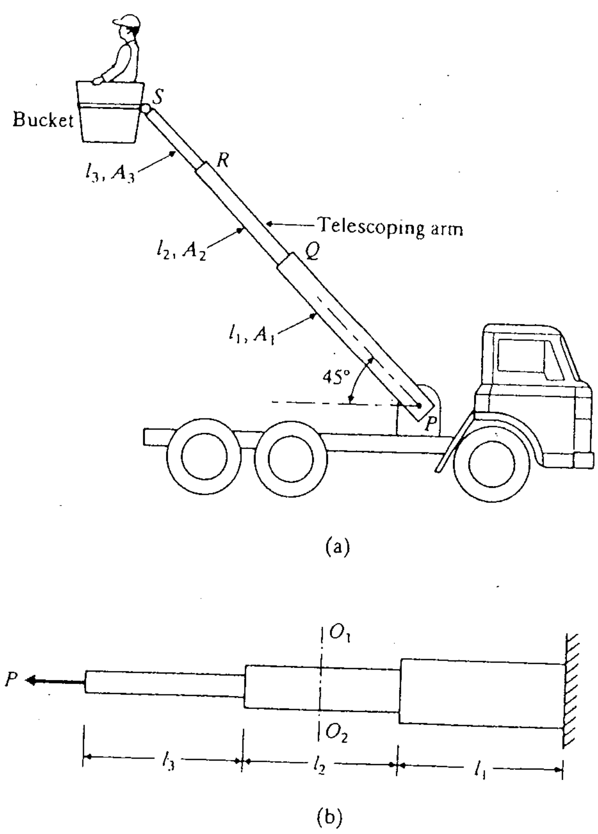


FIGURE 2.11

up by a distance 2 W/k_1, and the center of pulley 2 (point B) moves down by 2 W/k_2. Thus the total movement of the mass m (point O) is

$$2\left(\frac{2W}{k_1} + \frac{2W}{k_2}\right)$$

as the rope on either side of the pulley is free to move the mass downward. If k_{eq} denotes the equivalent spring constant of the system,

$$\frac{\text{Weight of the mass}}{\text{Equivalent spring constant}} = \text{Net displacement of the mass}$$

$$\frac{W}{k_{eq}} = 4W\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = \frac{4W(k_1 + k_2)}{k_1 k_2}$$

$$k_{eq} = \frac{k_1 k_2}{4(k_1 + k_2)} \tag{E.1}$$

By displacing mass m from the static equilibrium position by x, the equation of motion of the mass can be written as

$$m\ddot{x} + k_{eq}x = 0 \tag{E.2}$$

and hence the natural frequency is given by

$$\omega_n = \left(\frac{k_{eq}}{m}\right)^{1/2} = \left[\frac{k_1 k_2}{4m(k_1 + k_2)}\right]^{1/2} \text{ rad/sec} \tag{E.3}$$

or

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{4\pi}\left[\frac{k_1 k_2}{m(k_1 + k_2)}\right]^{1/2} \text{ cycles/sec} \tag{E.4}$$

■

2.3 Free Vibration of an Undamped Torsional System

If a rigid body oscillates about a specific reference axis, the resulting motion is called *torsional vibration.* In this case, the displacement of the body is measured in terms of an angular coordinate. In a torsional vibration problem, the restoring moment may be due to the torsion of an elastic member or to the unbalanced moment of a force or couple.

Figure 2.13 shows a disc, which has a polar mass moment of inertia J_0, mounted at one end of a solid circular shaft, the other end of which is fixed. Let the angular rotation of the disc about the axis of the shaft be θ; θ also represents the angle of twist of the shaft. From the theory of torsion of circular shafts [2.1], we have the relation

$$M_t = \frac{GI_o}{l} \tag{2.37}$$

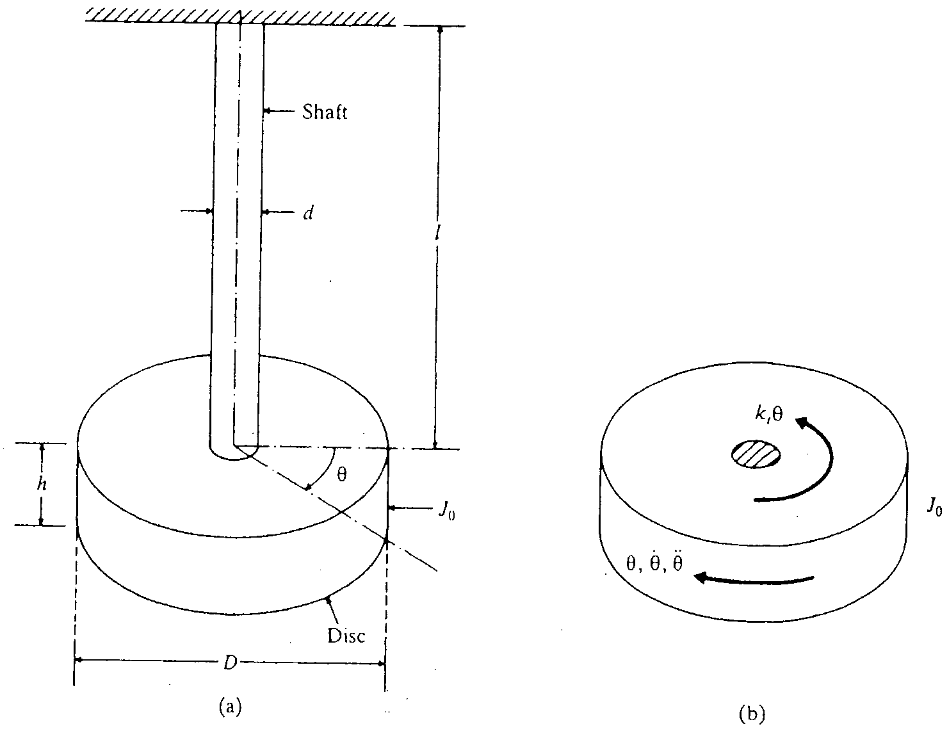


FIGURE 2.13 Torsional vibration of a disc.

where M_t is the torque that produces the twist θ, G is the shear modulus, l is the length of the shaft, I_o is the polar moment of inertia of the cross section of the shaft given by

$$I_o = \frac{\pi d^4}{32} \tag{2.38}$$

and d is the diameter of the shaft. If the disc is displaced by θ from its equilibrium position, the shaft provides a restoring torque of magnitude M_t. Thus the shaft acts as a torsional spring with a torsional spring constant

$$k_t = \frac{M_t}{\theta} = \frac{GI_o}{l} = \frac{\pi G d^4}{32 l} \tag{2.39}$$

2.3.1 Equation of Motion

The equation of the angular motion of the disc about its axis can be derived by using Newton's second law or any of the methods discussed in Section 2.2.2. By considering the free-body diagram of the disc (Fig. 2.13b), we can derive the equation of motion by applying Newton's second law of motion:

$$J_0 \ddot{\theta} + k_t \theta = 0 \tag{2.40}$$

which can be seen to be identical to Eq. (2.3) if the polar mass moment of inertia J_0, the angular displacement θ, and the torsional spring constant k_t are replaced by the mass m, the displacement x, and the linear spring constant k, respectively. Thus the natural circular frequency of the torsional system is

$$\omega_n = \left(\frac{k_t}{J_0}\right)^{1/2} \tag{2.41}$$

and the period and frequency of vibration in cycles per second are

$$\tau_n = 2\pi\left(\frac{J_0}{k_t}\right)^{1/2} \tag{2.42}$$

$$f_n = \frac{1}{2\pi}\left(\frac{k_t}{J_0}\right)^{1/2} \tag{2.43}$$

Note the following aspects of this system:

1. If the cross section of the shaft supporting the disc is not circular, an appropriate torsional spring constant is to be used [2.4, 2.5].
2. The polar mass moment of inertia of a disc is given by

$$J_0 = \frac{\rho h \pi D^4}{32} = \frac{WD^2}{8g}$$

   where ρ is the mass density, h is the thickness, D is the diameter, and W is the weight of the disc.
3. The torsional spring-inertia system shown in Fig. 2.13 is referred to as a *torsional pendulum*. One of the most important applications of a torsional pendulum is in a mechanical clock, where a ratchet and pawl convert the regular oscillation of a small torsional pendulum into the movements of the hands.

2.3.2 Solution

The general solution of Eq. (2.40) can be obtained, as in the case of Eq. (2.3):

$$\theta(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \tag{2.44}$$

where ω_n is given by Eq. (2.41), and A_1 and A_2 can be determined from the initial conditions. If

$$\theta(t = 0) = \theta_0 \quad \text{and} \quad \dot{\theta}(t = 0) = \frac{d\theta}{dt}(t = 0) = \dot{\theta}_0 \tag{2.45}$$

the constants A_1 and A_2 can be found:

$$A_1 = \theta_0$$
$$A_2 = \dot{\theta}_0/\omega_n \tag{2.46}$$

Equation (2.44) can also be seen to represent a simple harmonic motion.

Equation (2.54) shows that $\theta(t)$ increases exponentially with time; hence the motion is unstable. The physical reason for this is that the restoring moment due to the spring ($2kl^2\theta$), which tries to bring the system to equilibrium position, is less than the nonrestoring moment due to gravity $[-W(l/2)\theta]$, which tries to move the mass away from the equilibrium position. Although the stability conditions are illustrated with reference to Fig. 2.17 in this section, similar conditions need to be examined in the vibration analysis of many engineering systems.

2.5 Rayleigh's Energy Method

For a single degree of freedom system, the equation of motion was derived using the energy method in Section 2.2.2. In this section, we shall use the energy method to find the natural frequencies of single degree of freedom systems. The principle of conservation of energy, in the context of an undamped vibrating system, can be restated as

$$T_1 + U_1 = T_2 + U_2 \tag{2.55}$$

where the subscripts 1 and 2 denote two different instants of time. Specifically, we use the subscript 1 to denote the time when the mass is passing through its static equilibrium position and choose $U_1 = 0$ as reference for the potential energy. If we let the subscript 2 indicate the time corresponding to the maximum displacement of the mass, we have $T_2 = 0$. Thus Eq. (2.55) becomes

$$T_1 + 0 = 0 + U_2 \tag{2.56}$$

If the system is undergoing harmonic motion, then T_1 and U_2 denote the maximum values of T and U, respectively, and Eq. (2.56) becomes

$$T_{\max} = U_{\max} \tag{2.57}$$

The application of Eq. (2.57), which is also known as *Rayleigh's energy method*, gives the natural frequency of the system directly, as illustrated in the following examples.

EXAMPLE 2.6 Manometer for Diesel Engine

The exhaust from a single-cylinder four-stroke diesel engine is to be connected to a silencer, and the pressure therein is to be measured with a simple U-tube manometer (see Fig. 2.18). Calculate the minimum length of the manometer tube so that the natural frequency of oscillation of the mercury column will be 3.5 times slower than the frequency of the pressure fluctuations in the silencer at an engine speed of 600 rpm. The frequency of pressure fluctuations in the silencer is equal to

$$\frac{\text{Number of cylinders} \times \text{Speed of the engine}}{2}$$

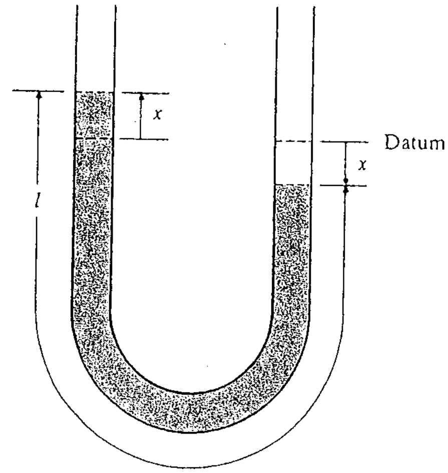


FIGURE 2.18

Given: U-tube manometer, engine speed = 600 rpm, and natural frequency of oscillation = 3.5 times slower than the frequency of pressure fluctuations.

Find: Minimum length of the manometer tube.

Approach: Use energy method to find the natural frequency.

Solution:

1. *Natural frequency of oscillation of the liquid column:* Let the datum in Fig. 2.18 be taken as the equilibrium position of the liquid. If the displacement of the liquid column from the equilibrium position is denoted by x, the change in potential energy is given by

U = potential energy of raised liquid column + potential energy of depressed liquid column

= (weight of mercury raised × displacement of the C.G. of the segment) + (weight of mercury depressed × displacement of the C.G. of the segment)

$$= (Ax\gamma)\frac{x}{2} + (Ax\gamma)\frac{x}{2} = A\gamma x^2 \quad \text{(E.1)}$$

where A is the cross-sectional area of the mercury column and γ is the specific weight of mercury. The change in kinetic energy is given by

$$T = \frac{1}{2}(\text{mass of mercury})(\text{velocity})^2$$

$$= \frac{1}{2}\frac{Al\gamma}{g}(\dot{x})^2 \quad \text{(E.2)}$$

where l is the length of the mercury column. By assuming harmonic motion, we can write

$$x(t) = X \cos \omega_n t \tag{E.3}$$

where X is the maximum displacement and ω_n is the natural frequency. By substituting Eq. (E.3) into Eqs. (E.1) and (E.2), we obtain

$$U = U_{max} \cos^2 \omega_n t \tag{E.4}$$

$$T = T_{max} \sin^2 \omega_n t \tag{E.5}$$

where

$$U_{max} = A \gamma X^2 \tag{E.6}$$

and

$$T_{max} = \frac{1}{2} \frac{A \gamma l \omega_n^2}{g} X^2 \tag{E.7}$$

By equating U_{max} to T_{max}, we obtain the natural frequency:

$$\omega_n = \left(\frac{2g}{l} \right)^{1/2} \tag{E.8}$$

2. *Length of the mercury column:* The frequency of pressure fluctuations in the silencer

$$= \frac{1 \times 600}{2}$$

$$= 300 \text{ rev/min}$$

$$= \frac{300 \times 2\pi}{60} = 10\pi \text{ rad/sec} \tag{E.9}$$

Thus the frequency of oscillations of the liquid column in the manometer is $10\pi/3.5 = 9.0$ rad/sec. By using Eq. (E.8), we obtain

$$\left(\frac{2g}{l} \right)^{1/2} = 9.0 \tag{E.10}$$

or

$$l = \frac{2.0 \times 9.81}{(9.0)^2} = 0.243 \text{ m} \tag{E.11}$$

■

EXAMPLE 2.7 Effect of Mass on ω_n of a Spring

Determine the effect of the mass of the spring on the natural frequency of the spring-mass system shown in Fig. 2.19.

The maximum kinetic energy of the beam itself (T_{max}) is given by

$$T_{max} = \frac{1}{2} \int_0^l \frac{m}{l} \left\{ \dot{y}(x) \right\}^2 dx \tag{E.2}$$

where m is the total mass and (m/l) is the mass per unit length of the beam. Equation (E.1) can be used to express the velocity variation, $\dot{y}(x)$, as

$$\dot{y}(x) = \frac{\dot{y}_{max}}{2l^3}(3\,x^2\,l - x^3) \tag{E.3}$$

and hence Eq. (E.2) becomes

$$T_{max} = \frac{m}{2\,l}\left(\frac{\dot{y}_{max}}{2\,l^3}\right)^2 \int_0^l (3\,x^2\,l - x^3)^2\,dx$$

$$= \frac{1}{2}\,\frac{m}{l}\,\frac{\dot{y}^2_{max}}{4\,l^6}\left(\frac{33}{35}\,l^7\right) = \frac{1}{2}\left(\frac{33}{35}m\right)\dot{y}^2_{max} \tag{E.4}$$

If m_{eq} denotes the equivalent mass of the cantilever (water tank) at the free end, its maximum kinetic energy can be expressed as

$$T_{max} = \frac{1}{2}\,m_{eq}\,\dot{y}^2_{max} \tag{E.5}$$

By equating Eqs. (E.4) and (E.5), we obtain

$$m_{eq} = \frac{33}{35}\,m \tag{E.6}$$

Thus the total effective mass acting at the end of the cantilever beam is given by

$$M_{eff} = M + m_{eq} \tag{E.7}$$

where M is the mass of the water tank. The natural frequency of transverse vibration of the water tank is given by

$$\omega_n = \sqrt{\frac{k}{M_{eff}}} = \sqrt{\frac{k}{M + \frac{33}{35}m}} \tag{E.8}$$

■

2.6 Free Vibration with Viscous Damping

2.6.1 Equation of Motion

As stated in Section 1.9, the viscous damping force F is proportional to the velocity $\dot{x}$ or v and can be expressed as

$$F = -c\dot{x} \tag{2.58}$$

where c is the damping constant or coefficient of viscous damping and the negative sign indicates that the damping force is opposite to the direction of velocity. A single degree of freedom system with a viscous damper is shown in Fig. 2.21. If x is measured from the equilibrium position of the mass m, the application of Newton's law yields the equation of motion:

$$m\ddot{x} = -c\dot{x} - kx$$

or

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.59}$$

2.6.2 Solution

To solve Eq. (2.59), we assume a solution in the form

$$x(t) = Ce^{st} \tag{2.60}$$

where C and s are undetermined constants. Inserting this function into Eq. (2.59) leads to the characteristic equation

$$ms^2 + cs + k = 0 \tag{2.61}$$

the roots of which are

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{2.62}$$

These roots give two solutions to Eq. (2.59):

$$x_1(t) = C_1 e^{s_1 t} \quad \text{and} \quad x_2(t) = C_2 e^{s_2 t} \tag{2.63}$$

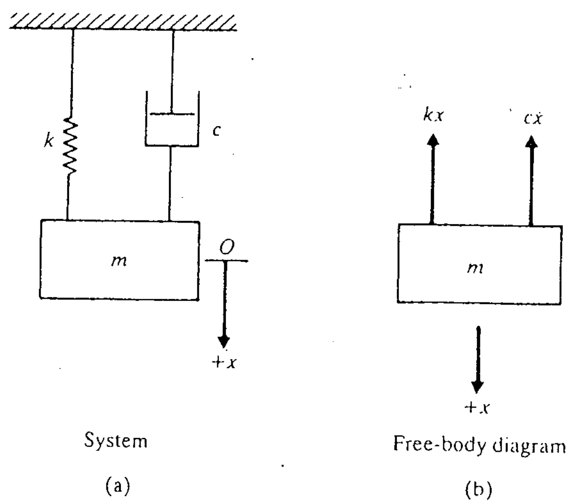


FIGURE 2.21 Single degree of freedom system with viscous damper.

Thus the general solution of Eq. (2.59) is given by a combination of the two solutions $x_1(t)$ and $x_2(t)$:

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$
$$= C_1 e^{\left\{-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\} t} + C_2 e^{\left\{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\} t} \tag{2.64}$$

where C_1 and C_2 are arbitrary constants to be determined from the initial conditions of the system.

Critical Damping Constant and the Damping Ratio. The critical damping c_c is defined as the value of the damping constant c for which the radical in Eq. (2.62) becomes zero:

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

or

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n \tag{2.65}$$

For any damped system, the damping ratio ζ is defined as the ratio of the damping constant to the critical damping constant:

$$\zeta = c/c_c \tag{2.66}$$

Using Eqs. (2.66) and (2.65), we can write

$$\frac{c}{2m} = \frac{c}{c_c}\cdot\frac{c_c}{2m} = \zeta\omega_n \tag{2.67}$$

and hence

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \tag{2.68}$$

Thus the solution, Eq. (2.64), can be written as

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t} \tag{2.69}$$

The nature of the roots s_1 and s_2 and hence the behavior of the solution, Eq. (2.69), depends upon the magnitude of damping. It can be seen that the case $\zeta = 0$ leads to the undamped vibrations discussed in Section 2.2. Hence we assume that $\zeta \neq 0$ and consider the following three cases.

Case 1. *Underdamped system* ($\zeta < 1$ or $c < c_c$ or $c/2m < \sqrt{k/m}$). For this condition, $(\zeta^2 - 1)$ is negative and the roots s_1 and s_2 can be expressed as

$$s_1 = \left(-\zeta + i\sqrt{1 - \zeta^2}\right)\omega_n$$
$$s_2 = \left(-\zeta - i\sqrt{1 - \zeta^2}\right)\omega_n$$

and the solution, Eq. (2.69), can be written in different forms:

$$
\begin{aligned}
x(t) &= C_1 e^{(-\zeta + i\sqrt{1-\zeta^2})\omega_n t} + C_2 e^{(-\zeta - i\sqrt{1-\zeta^2})\omega_n t} \\
&= e^{-\zeta\omega_n t}\left\{ C_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + C_2 e^{-i\sqrt{1-\zeta^2}\omega_n t} \right\} \\
&= e^{-\zeta\omega_n t}\left\{ (C_1 + C_2)\cos\sqrt{1-\zeta^2}\,\omega_n t + i(C_1 - C_2)\sin\sqrt{1-\zeta^2}\,\omega_n t \right\} \\
&= e^{-\zeta\omega_n t}\left\{ C_1'\cos\sqrt{1-\zeta^2}\,\omega_n t + C_2'\sin\sqrt{1-\zeta^2}\,\omega_n t \right\} \\
&= X e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi \right) \\
&= X_0 e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi_0 \right)
\end{aligned}
\tag{2.70}
$$

where (C_1', C_2'), (X, ϕ), and (X_0, ϕ_0) are arbitrary constants to be determined from the initial conditions.

For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, C_1' and C_2' can be found:

$$
C_1' = x_0 \quad \text{and} \quad C_2' = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}
\tag{2.71}
$$

and hence the solution becomes

$$
\begin{aligned}
x(t) = e^{-\zeta\omega_n t}\Bigg\{ & x_0 \cos\sqrt{1-\zeta^2}\,\omega_n t \\
& + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t \Bigg\}
\end{aligned}
\tag{2.72}
$$

The constants (X, ϕ) and (X_0, ϕ_0) can be expressed as

$$
X = X_0 = \sqrt{(C_1')^2 + (C_2')^2}
\tag{2.73}
$$

$$
\phi = \tan^{-1}(C_1'/C_2')
\tag{2.74}
$$

$$
\phi_0 = \tan^{-1}(-C_2'/C_1')
\tag{2.75}
$$

The motion described by Eq. (2.72) is a damped harmonic motion of angular frequency $\sqrt{1-\zeta^2}\,\omega_n$, but because of the factor $e^{-\zeta\omega_n t}$, the amplitude decreases exponentially with time, as shown in Fig. 2.22. The quantity

$$
\omega_d = \sqrt{1-\zeta^2}\,\omega_n
\tag{2.76}
$$

is called the *frequency of damped vibration.* It can be seen that the frequency of damped vibration ω_d is always less than the undamped natural frequency ω_n. The

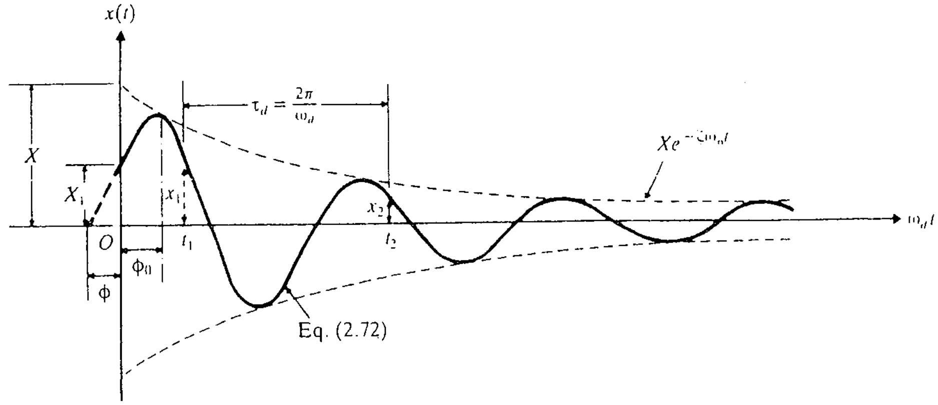


FIGURE 2.22 Underdamped solution.

decrease in the frequency of damped vibration with increasing amount of damping, given by Eq. (2.76), is shown graphically in Fig. 2.23. The underdamped case is very important in the study of mechanical vibrations, as it is the only case which leads to an oscillatory motion [2.10].

Case 2. *Critically damped system* ($\zeta = 1$ or $c = c_c$ or $c/2m = \sqrt{k/m}$). In this case the two roots s_1 and s_2 in Eq. (2.68) are equal:

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n \tag{2.77}$$

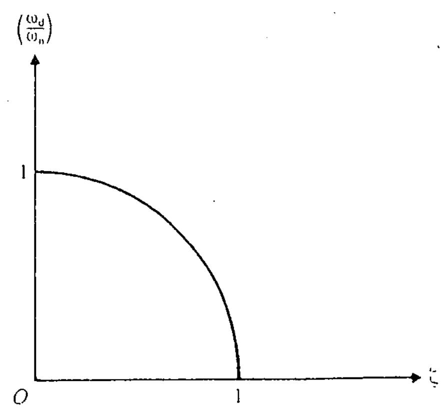


FIGURE 2.23 Variation of ω_d with damping.

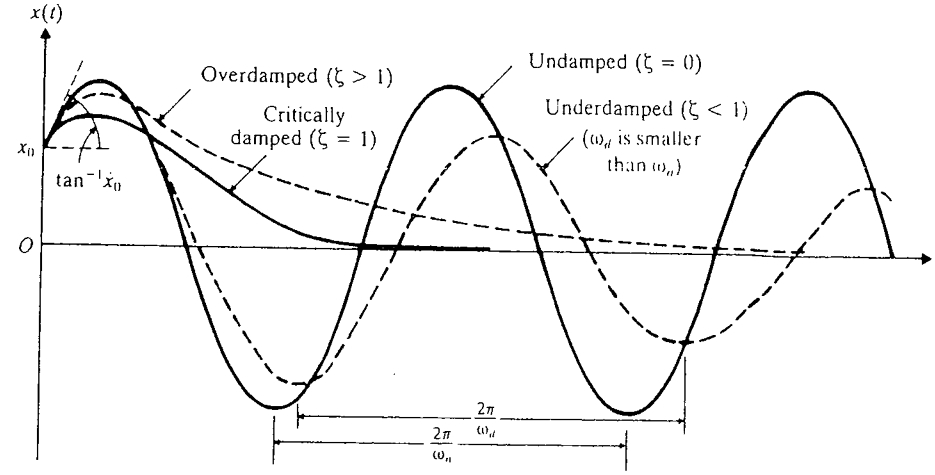


FIGURE 2.24 Comparison of motions with different types of damping.

Because of the repeated roots, the solution of Eq. (2.59) is given by [2.6][1]

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \tag{2.78}$$

The application of the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$ for this case gives

$$\begin{aligned} C_1 &= x_0 \\ C_2 &= \dot{x}_0 + \omega_n x_0 \end{aligned} \tag{2.79}$$

and the solution becomes

$$x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t]e^{-\omega_n t} \tag{2.80}$$

It can be seen that the motion represented by Eq. (2.80) is *aperiodic* (i.e., nonperiodic). Since $e^{-\omega_n t} \to 0$ as $t \to \infty$, the motion will eventually diminish to zero, as indicated in Fig. 2.24.

Case 3. *Overdamped system* ($\zeta > 1$ or $c > c_c$ or $c/2m > \sqrt{k/m}$). As $\sqrt{\zeta^2 - 1} > 0$, Eq. (2.68) shows that the roots s_1 and s_2 are real and distinct and are given by

$$\begin{aligned} s_1 &= (-\zeta + \sqrt{\zeta^2 - 1})\omega_n < 0 \\ s_2 &= (-\zeta - \sqrt{\zeta^2 - 1})\omega_n < 0 \end{aligned}$$

[1]Equation (2.78) can also be obtained by making ζ approach unity in the limit in Eq. (2.72). As $\zeta \to 1$, $\omega_d \to 0$; hence $\cos \omega_d t \to 1$ and $\sin \omega_d t \to \omega_d t$. Thus Eq. (2.72) yields

$$x(t) = e^{-\omega_n t}(C_1' + C_2'\omega_d t) = (C_1 + C_2 t)e^{-\omega_n t}$$

where $C_1 = C_1'$ and $C_2 = C_2'\omega_d$ are new constants.

with $s_2 << s_1$. In this case, the solution, Eq. (2.69), can be expressed as

$$x(t) = C_1 e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + C_2 e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t} \tag{2.81}$$

For the initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$, the constants C_1 and C_2 can be obtained:

$$C_1 = \frac{x_0\omega_n(\zeta + \sqrt{\zeta^2-1}) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2-1}}$$

$$C_2 = \frac{-x_0\omega_n(\zeta - \sqrt{\zeta^2-1}) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2-1}} \tag{2.82}$$

Equation (2.81) shows that the motion is aperiodic regardless of the initial conditions imposed on the system. Since roots s_1 and s_2 are both negative, the motion diminishes exponentially with time, as shown in Fig. 2.24.

Note the following aspects of these systems:

1. The nature of the roots s_1 and s_2 with varying values of damping c or ζ can be shown in a complex plane. In Fig. 2.25, the horizontal and vertical axes are chosen as the real and imaginary axes. The semicircle represents the locus of the roots s_1 and s_2 for different values of ζ in the range $0 < \zeta < 1$. This figure permits us to see instantaneously the effect of the parameter ζ on the behavior of the system. We find that for $\zeta = 0$, we obtain the imaginary roots $s_1 = i\omega_n$ and $s_2 = -i\omega_n$, leading to the solution given in Eq. (2.15). For $0 < \zeta < 1$, the roots s_1 and s_2 are complex conjugate and are located symmetrically about the

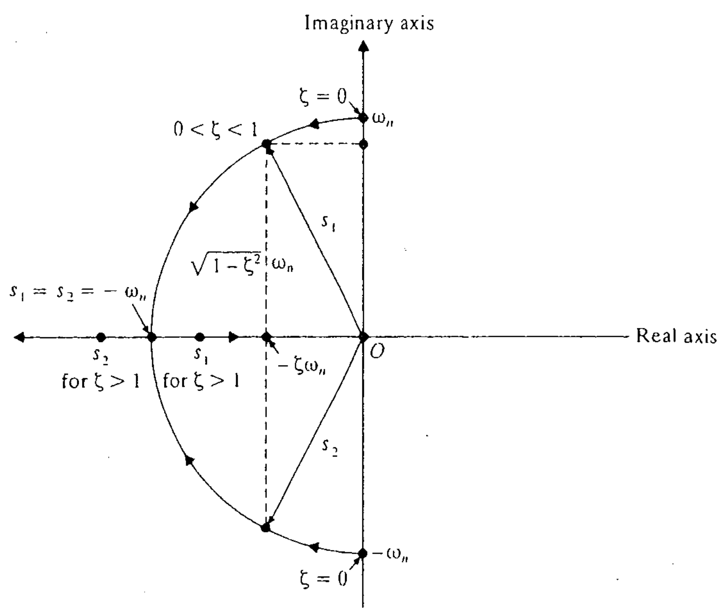


FIGURE 2.25 Locus of s_1 and s_2.

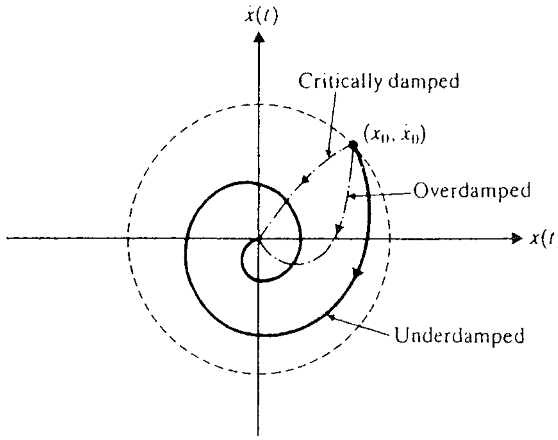


FIGURE 2.26

real axis. As the value of ζ approaches 1, both roots approach the point $-\omega_n$ on the real axis. If $\zeta > 1$, both roots lie on the real axis, one increasing and the other decreasing. In the limit when $\zeta \to \infty$, $s_1 \to 0$ and $s_2 \to -\infty$. The value $\zeta = 1$ can be seen to represent a transition stage, below which both roots are complex and above which both roots are real.

2. A critically damped system will have the smallest damping required for aperiodic motion; hence the mass returns to the position of rest in the shortest possible time without overshooting. The property of critical damping is used in many practical applications. For example, large guns have dashpots with critical damping value, so that they return to their original position after recoil in the minimum time without vibrating. If the damping provided were more than the critical value, some delay would be caused before the next firing.
3. The free damped response of a single degree of freedom system can be represented in phase plane or state space as indicated in Fig. 2.26.

2.6.3 Logarithmic Decrement

The logarithmic decrement represents the rate at which the amplitude of a free damped vibration decreases. It is defined as the natural logarithm of the ratio of any two successive amplitudes. Let t_1 and t_2 denote the times corresponding to two consecutive amplitudes (displacements), measured one cycle apart for an underdamped system, as in Fig. 2.22. Using Eq. (2.70), we can form the ratio

$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2}\cos(\omega_d t_2 - \phi_0)} \tag{2.83}$$

But $t_2 = t_1 + \tau_d$ where $\tau_d = 2\pi/\omega_d$ is the period of damped vibration. Hence $\cos(\omega_d t_2 - \phi_0) = \cos(2\pi + \omega_d t_1 - \phi_0) = \cos(\omega_d t_1 - \phi_0)$, and Eq. (2.83) can be written as

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n (t_1 + \tau_d)}} = e^{\zeta\omega_n \tau_d} \tag{2.84}$$

The logarithmic decrement δ can be obtained from Eq. (2.84):

$$\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\cdot\frac{c}{2m} \tag{2.85}$$

For small damping, Eq. (2.85) can be approximated:

$$\delta \simeq 2\pi\zeta \qquad \text{if} \qquad \zeta << 1 \tag{2.86}$$

Figure 2.27 shows the variation of the logarithmic decrement δ with ζ as given by Eqs. (2.85) and (2.86). It can be noticed that for values up to $\zeta = 0.3$, the two curves are difficult to distinguish.

The logarithmic decrement is dimensionless and is actually another form of the dimensionless damping ratio ζ. Once δ is known, ζ can be found by solving Eq. (2.85):

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \tag{2.87}$$

If we use Eq. (2.86) instead of Eq. (2.85), we have

$$\zeta \simeq \frac{\delta}{2\pi} \tag{2.88}$$

If the damping in the given system is not known, we can determine it experimentally by measuring any two consecutive displacements x_1 and x_2. By taking the natural logarithm of the ratio of x_1 and x_2, we obtain δ. By using Eq. (2.87), we can compute

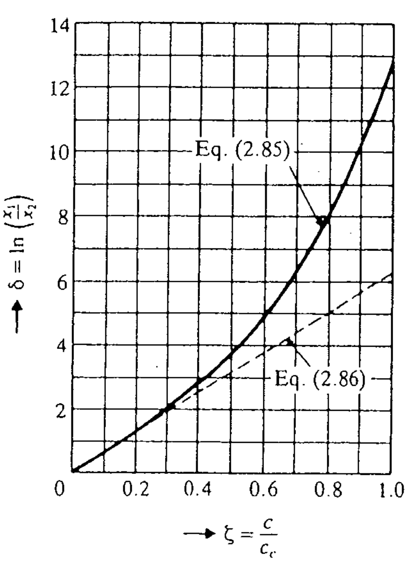


FIGURE 2.27 Variation of logarithmic decrement with damping.

the damping ratio ζ. In fact, the damping ratio ζ can also be found by measuring two displacements separated by any number of complete cycles. If x_1 and x_{m+1} denote the amplitudes corresponding to times t_1 and $t_{m+1} = t_1 + m\tau_d$ where m is an integer, we obtain

$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\frac{x_3}{x_4}\cdots\frac{x_m}{x_{m+1}} \tag{2.89}$$

Since any two successive displacements separated by one cycle satisfy the equation

$$\frac{x_j}{x_{j+1}} = e^{\zeta\omega_n\tau_d} \tag{2.90}$$

Eq. (2.89) becomes

$$\frac{x_1}{x_{m+1}} = (e^{\zeta\omega_n\tau_d})^m = e^{m\zeta\omega_n\tau_d} \tag{2.91}$$

Equations (2.91) and (2.85) yield

$$\delta = \frac{1}{m}\ln\left(\frac{x_1}{x_{m+1}}\right) \tag{2.92}$$

which can be substituted into Eq. (2.87) or Eq. (2.88) to obtain the viscous damping ratio ζ.

2.6.4 Energy Dissipated in Viscous Damping

In a viscously damped system, the rate of change of energy with time (dW/dt) is given by

$$\frac{dW}{dt} = \text{force} \times \text{velocity} = Fv = -cv^2 = -c\left(\frac{dx}{dt}\right)^2 \tag{2.93}$$

using Eq. (2.58). The negative sign in Eq. (2.93) denotes that energy dissipates with time. Assume a simple harmonic motion as $x(t) = X\sin\omega_d t$, where X is the amplitude of motion and the energy dissipated in a complete cycle is given by[2]

$$\Delta W = \int_{t=0}^{(2\pi/\omega_d)} c\left(\frac{dx}{dt}\right)^2 dt = \int_0^{2\pi} cX^2\omega_d\cos^2\omega_d t \cdot d(\omega_d t)$$
$$= \pi c\omega_d X^2 \tag{2.94}$$

[2]In the case of a damped system, simple harmonic motion $x(t) = X\cos\omega_d t$ is possible only when the steady-state response is considered under a harmonic force of frequency ω_d (see Section 3.4). The loss of energy due to the damper is supplied by the excitation under steady-state forced vibration [2.7].

as the maximum kinetic energy ($\frac{1}{2}mv_{\max}^2 = \frac{1}{2}mX^2\omega_d^2$), the two being approximately equal for small values of damping. Thus

$$\frac{\Delta W}{W} = \frac{\pi c \omega_d X^2}{\frac{1}{2}m\omega_d^2 X^2} = 2\left(\frac{2\pi}{\omega_d}\right)\left(\frac{c}{2m}\right) = 2\delta \simeq 4\pi\zeta = \text{constant} \quad (2.99)$$

using Eqs. (2.85) and (2.88). The quantity $\Delta W/W$ is called the *specific damping capacity* and is useful in comparing the damping capacity of engineering materials. Another quantity known as the *loss coefficient* is also used for comparing the damping capacity of engineering materials. The loss coefficient is defined as the ratio of the energy dissipated per radian and the total strain energy:

$$\text{loss coefficient} = \frac{(\Delta W/2\pi)}{W} = \frac{\Delta W}{2\pi W} \quad (2.100)$$

2.6.5 Torsional Systems with Viscous Damping

The methods presented in Sections 2.6.1 through 2.6.4 for linear vibrations with viscous damping can be extended directly to viscously damped torsional (angular) vibrations. For this, consider a single degree of freedom torsional system with a viscous damper, as shown in Fig. 2.29(a). The viscous damping torque is given by (Fig. 2.29b):

$$T = -c_t\dot{\theta} \quad (2.101)$$

where c_t is the torsional viscous damping constant, $\dot{\theta} = d\theta/dt$ is the angular velocity of the disc, and the negative sign denotes that the damping torque is opposite the direction of angular velocity. The equation of motion can be derived as

$$J_0\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0 \quad (2.102)$$

where J_0 = mass moment of inertia of the disc, k_t = spring constant of the system (restoring torque per unit angular displacement), and θ = angular displacement of the disc. The solution of Eq. (2.102) can be found exactly as in the case of linear vibrations. For example, in the underdamped case, the frequency of damped vibration is given by

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \quad (2.103)$$

where

$$\omega_n = \sqrt{\frac{k_t}{J_0}} \quad (2.104)$$

and

$$\zeta = \frac{c_t}{c_{tc}} = \frac{c_t}{2J_0\omega_n} = \frac{c_t}{2\sqrt{k_t J_0}} \quad (2.105)$$

where c_{tc} is the critical torsional damping constant.

i.e.,

$$0.4 = -\left(\frac{v_{a2} - v_{t2}}{0 - 6.26099}\right)$$

i.e.,

$$v_{a2} = v_{t2} + 2.504396 \tag{E.5}$$

The solution of Eqs. (E.3) and (E.5) gives

$$v_{a2} = 1.460898 \text{ m/s} \ ; \ v_{t2} = -1.043498 \text{ m/s}$$

Thus the initial conditions of the anvil are given by

$$x_0 = 0 \ ; \ \dot{x}_0 = 1.460898 \text{ m/s}$$

The damping coefficient is equal to

$$\zeta = \frac{c}{2\sqrt{k\,M}} = \frac{1000}{2\sqrt{(5 \times 10^6)\left(\frac{5000}{9.81}\right)}} = 0.0989949$$

The undamped and damped natural frequencies of the anvil are given by

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{5 \times 10^6}{\left(\frac{5000}{9.81}\right)}} = 98.994949 \text{ rad/s}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 98.994949\sqrt{1 - 0.0989949^2} = 98.024799 \text{ rad/s}$$

The displacement response of the anvil is given by Eq. (2.72):

$$x(t) = e^{-\zeta\omega_n t}\left\{\cos \omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \sin \omega_d t\right\}$$

$$= e^{-9.799995t}\{\cos 98.024799\, t + 0.01490335 \sin 98.024799\, t\} \text{ m}$$ ■

EXAMPLE 2.10 Shock Absorber for a Motorcycle

An underdamped shock absorber is to be designed for a motorcycle of mass 200 kg (Fig. 2.31a). When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in Fig. 2.31(b). Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 sec and the amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e., $x_{1.5} = x_1/4$). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.

Given: Mass = 200 kg; displacement-time curve of the system (Fig. 2.31b); damped period of vibration = 2 sec, $x_{1.5} = x_1/4$; and maximum displacement = 250 mm.

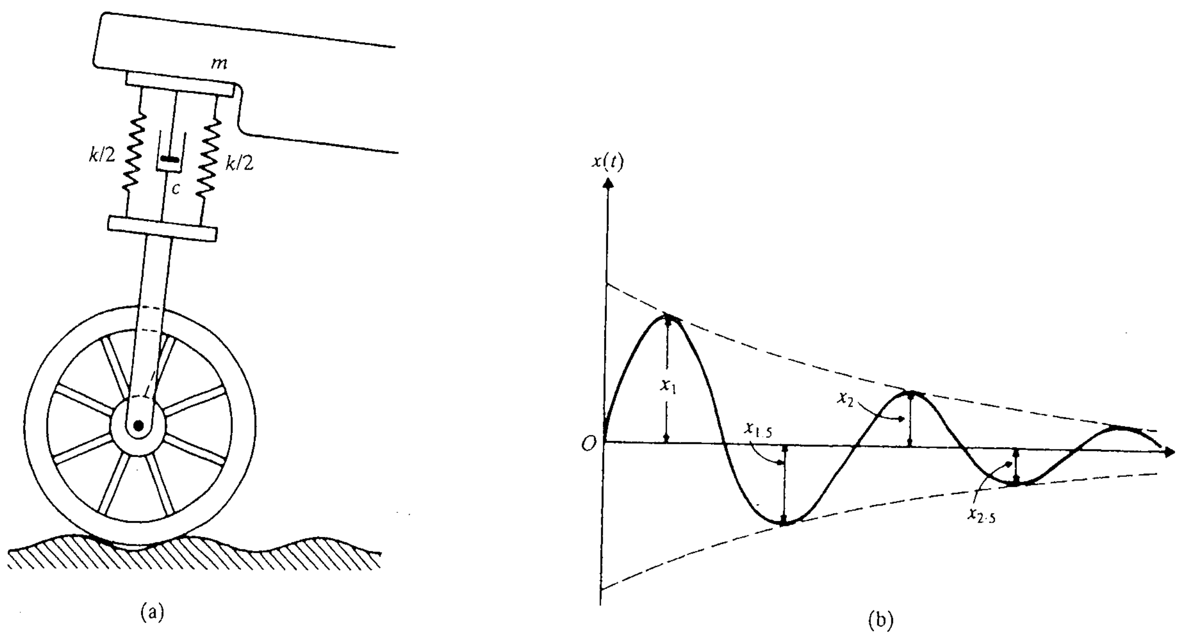


FIGURE 2.31

Find: Stiffness (k), damping constant (c), and initial velocity ($\dot{x}_0$), which results in a maximum displacement of 250 mm.

Approach: Equation for the logarithmic decrement in terms of the damping ratio, equation for the damped period of vibration, time corresponding to maximum displacement for an underdamped system, and envelope passing through the maximum points of an underdamped system.

Solution: Since $x_{1.5} = x_1/4$, $x_2 = x_{1.5}/4 = x_1/16$. Hence the logarithmic decrement becomes

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln(16) = 2.7726 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{E.1}$$

from which the value of ζ can be found as $\zeta = 0.4037$. The damped period of vibration is given to be 2 sec. Hence

$$2 = \tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\omega_n = \frac{2\pi}{2\sqrt{1-(0.4037)^2}} = 3.4338 \text{ rad/sec}$$

The critical damping constant can be obtained:

$$c_c = 2m\omega_n = 2(200)(3.4338) = 1373.54 \text{ N-s/m}$$

Thus the damping constant is given by

$$c = \zeta c_c = (0.4037)(1373.54) = 554.4981 \text{ N-s/m}$$

and the stiffness by

$$k = m\omega_n^2 = (200)(3.4338)^2 = 2358.2652 \text{ N/m}$$

The displacement of the mass will attain its maximum value at time t_1, given by

$$\sin \omega_d t_1 = \sqrt{1 - \zeta^2}$$

(See Problem 2.77.) This gives

$$\sin \omega_d t_1 = \sin \pi t_1 = \sqrt{1 - (0.4037)^2} = 0.9149$$

or

$$t_1 = \frac{\sin^{-1}(0.9149)}{\pi} = 0.3678 \text{ sec}$$

The envelope passing through the maximum points (see Problem 2.77) is given by

$$x = \sqrt{1 - \zeta^2}\, X e^{-\zeta\omega_n t} \tag{E.2}$$

Since $x = 250$ mm, Eq. (E.2) gives at t_1

$$0.25 = \sqrt{1 - (0.4037)^2}\, X e^{-(0.4037)(3.4338)(0.3678)}$$

or

$$X = 0.4550 \text{ m}.$$

The velocity of the mass can be obtained by differentiating the displacement

$$x(t) = X e^{-\zeta\omega_n t} \sin \omega_d t$$

as

$$\dot{x}(t) = X e^{-\zeta\omega_n t}(-\zeta\omega_n \sin \omega_d t + \omega_d \cos \omega_d t) \tag{E.3}$$

When $t = 0$, Eq. (E.3) gives

$$\dot{x}(t = 0) = \dot{x}_0 = X\omega_d = X\omega_n \sqrt{1 - \zeta^2} = (0.4550)(3.4338)(\sqrt{1 - (0.4037)^2})$$
$$= 1.4294 \text{ m/s}$$

■

EXAMPLE 2.11 Analysis of Cannon

The schematic diagram of a large cannon is shown in Fig. 2.32 [2.8]. When the gun is fired, high-pressure gases accelerate the projectile inside the barrel to a very high velocity. The reaction force pushes the gun barrel in the opposite direction of the projectile. Since it is desirable to bring the gun barrel to rest in the shortest time without oscillation, it is made to translate backward against a critically damped spring-damper system called the *recoil mechanism*. In a particular case, the gun barrel and the recoil mechanism have a mass of 500 kg with a recoil spring of stiffness 10,000 N/m. The gun recoils 0.4 m upon firing. Find

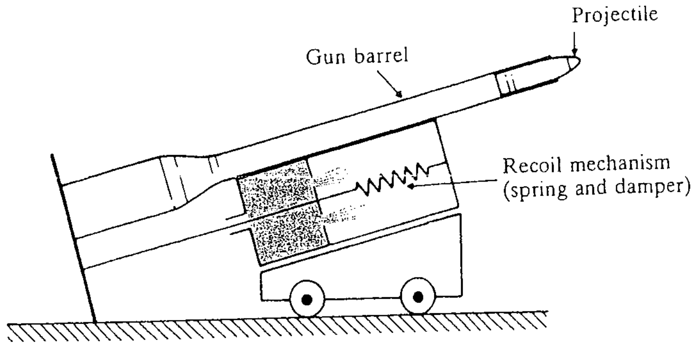


FIGURE 2.32

(1) the critical damping coefficient of the damper, (2) the initial recoil velocity of the gun, and (3) the time taken by the gun to return to a position 0.1 m from its initial position.

Given: Critically damped recoil mechanism with $m = 500$ kg, $k = 10{,}000$ N/m, and recoil distance $= 0.4$ m.

Find: Critical damping coefficient, recoil velocity, and time taken by the gun to return to a position 0.1 m from its initial position.

Approach: Use the response equation of a critically damped system.

Solution

1. The undamped natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10{,}000}{500}} = 4.4721 \text{ rad/sec}$$

and the critical damping coefficient (Eq. 2.65) of the damper is

$$c_c = 2m\omega_n = 2(500)(4.4721) = 4472.1 \text{ N-s/m}$$

2. The response of a critically damped system is given by Eq. (2.78):

$$x(t) = (C_1 + C_2 t)\, e^{-\omega_n t} \tag{E.1}$$

where $C_1 = x_0$ and $C_2 = \dot{x}_0 + \omega_n x_0$. The time t_1 at which $x(t)$ reaches a maximum value can be obtained by setting $\dot{x}(t) = 0$. The differentiation of Eq. (E.1) gives

$$\dot{x}(t) = C_2 e^{-\omega_n t} - \omega_n (C_1 + C_2 t) e^{-\omega_n t}$$

Hence $\dot{x}(t) = 0$ yields

$$t_1 = \left(\frac{1}{\omega_n} - \frac{C_1}{C_2}\right) \tag{E.2}$$

In this case, $x_0 = C_1 = 0$; hence Eq. (E.2) leads to $t_1 = 1/\omega_n$. Since the maximum value of $x(t)$ or the recoil distance is given to be $x_{max} = 0.4$ m, we have

$$x_{max} = x(t = t_1) = C_2 t_1 e^{-\omega_n t_1} = \frac{\dot{x}_0}{\omega_n} e^{-1} = \frac{\dot{x}_0}{e\omega_n}$$

or

$$\dot{x}_0 = x_{max}\,\omega_n e = (0.4)(4.4721)(2.7183) = 4.8626 \text{ m/s}$$

3. If t_2 denotes the time taken by the gun to return to a position 0.1 m from its initial position, we have

$$0.1 = C_2 t_2 e^{-\omega_n t_2} = 4.8626 t_2 e^{-4.4721 t_2} \tag{E.3}$$

The solution of Eq. (E.3) gives $t_2 = 0.8258$ sec. ■

2.7 Free Vibration with Coulomb Damping

In many mechanical systems, *Coulomb* or *dry-friction* dampers are used because of their mechanical simplicity and convenience [2.9]. Also in vibrating structures, whenever the components slide relative to each other, dry-friction damping appears internally. As stated in Section 1.9, Coulomb damping arises when bodies slide on dry surfaces. Coulomb's law of dry friction states that when two bodies are in contact, the force required to produce sliding is proportional to the normal force acting in the plane of contact. Thus the friction force F is given by

$$F = \mu N = \mu W = \mu m g \tag{2.106}$$

where N is the normal force and μ is the coefficient of friction. The friction force acts in a direction opposite to the direction of velocity. Coulomb damping is sometimes called *constant damping,* since the damping force is independent of the displacement and velocity; it depends only on the normal force N between the sliding surfaces.

2.7.1 Equation of Motion

Consider a single degree of freedom system with dry friction as shown in Fig. 2.33(a). Since the friction force varies with the direction of velocity, we need to consider two cases, as indicated in Figs. 2.33(b) and (c).

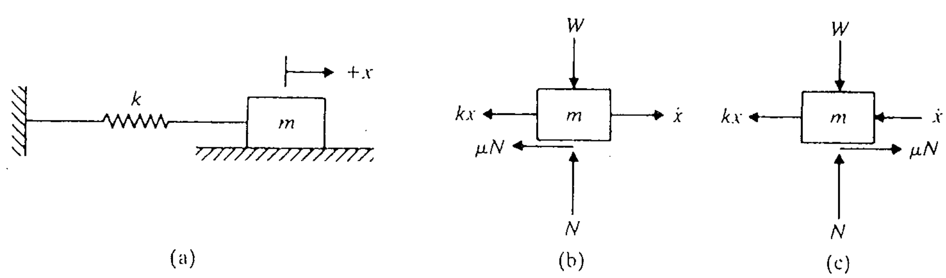


FIGURE 2.33 Spring-mass system with Coulomb damping.

2.11 Is the frequency of a damped free vibration smaller or greater than the natural frequency of the system?

2.12 What is the use of logarithmic decrement?

2.13 Is hysteresis damping a function of the maximum stress?

2.14 What is critical damping and what is its importance?

2.15 What happens to the energy dissipated by damping?

2.16 What is equivalent viscous damping? Is the equivalent viscous damping factor a constant?

2.17 What is the reason for studying the vibration of a single degree of freedom system?

2.18 How can you find the natural frequency of a system by measuring its static deflection?

2.19 Give two practical applications of a torsional pendulum.

2.20 Define these terms: damping ratio, logarithmic decrement, loss coefficient, and specific damping capacity.

2.21 In what ways is the response of a system with Coulomb damping different from that of systems with other types of damping?

2.22 What is complex stiffness?

2.23 Define the hysteresis damping constant.

2.24 Give three practical applications of the concept of center of percussion.

Problems

The problem assignments are organized as follows:

Problems	Section Covered	Topic Covered
2.1–2.50	2.2	Undamped translational systems
2.51–2.64	2.3	Undamped torsional systems
2.65–2.74	2.5	Energy method
2.75–2.97, 2.111	2.6	Systems with viscous damping
2.98–2.107	2.7	Systems with Coulomb damping
2.108–2.110	2.8	Systems with hysteretic damping
2.112–2.115	2.9	Computer program
2.116–2.120	—	Projects

2.1 An industrial press is mounted on a rubber pad to isolate it from its foundation. If the rubber pad is compressed 5 mm by the self-weight of the press, find the natural frequency of the system.

2.2 A spring-mass system has a natural period of 0.21 sec. What will be the new period if the spring constant is (a) increased by 50% and (b) decreased by 50%?

2.3 A spring-mass system has a natural frequency of 10 Hz. When the spring constant is reduced by 800 N/m, the frequency is altered by 45 percent. Find the mass and spring constant of the original system.

2.4 A helical spring, when fixed at one end and loaded at the other, requires a force of 100 N to produce an elongation of 10 mm. The ends of the spring are now rigidly fixed, one end vertically above the other, and a mass of 10 kg is attached at the middle point of its length. Determine the time taken to complete one vibration cycle when the mass is set vibrating in the vertical direction.

2.5 An air-conditioning chiller unit weighing 2000 lb is to be supported by four air springs (Fig. 2.39). Design the air springs such that the natural frequency of vibration of the unit lies between 5 rad/s and 10 rad/s.

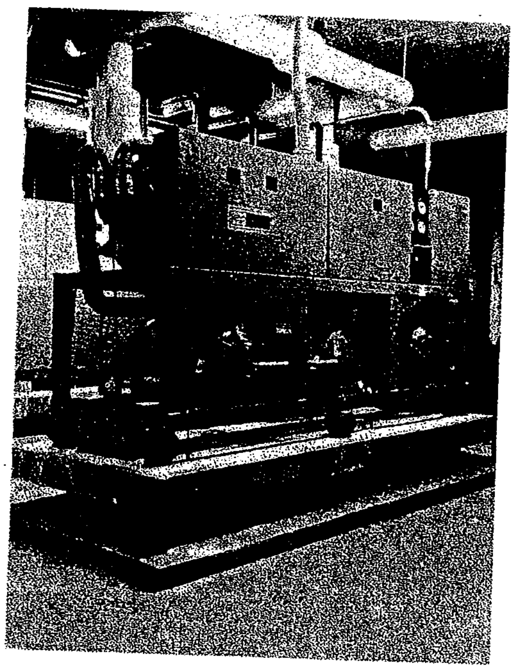


FIGURE 2.39 (Courtesy of *Sound and Vibration*)

2.6 The maximum velocity attained by the mass of a simple harmonic oscillator is 10 cm/sec, and the period of oscillation is 2 sec. If the mass is released with an initial

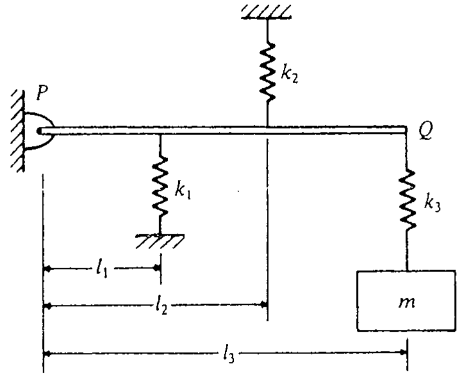


FIGURE 2.40

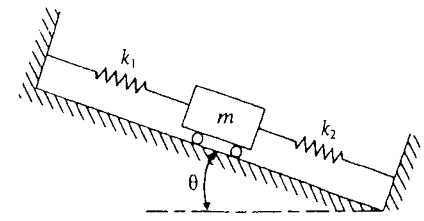


FIGURE 2.41

displacement of 2 cm, find (a) the amplitude, (b) the initial velocity, (c) the maximum acceleration, and (d) the phase angle.

2.7 Three springs and a mass are attached to a rigid, weightless, bar PQ as shown in Fig. 2.40. Find the natural frequency of vibration of the system.

2.8 An automobile having a mass of 2000 kg deflects its suspension springs 0.02 m under static conditions. Determine the natural frequency of the automobile in the vertical direction by assuming damping to be negligible.

2.9 Find the natural frequency of vibration of a spring-mass system arranged on an inclined plane, as shown in Fig. 2.41.

2.10 A loaded mine cart, weighing 5,000 lb, is being lifted by a frictionless pulley and a wire rope, as shown in Fig. 2.42. Find the natural frequency of vibration of the cart in the given position.

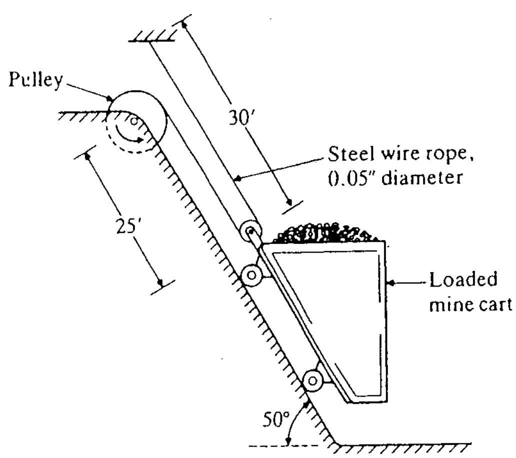


FIGURE 2.42

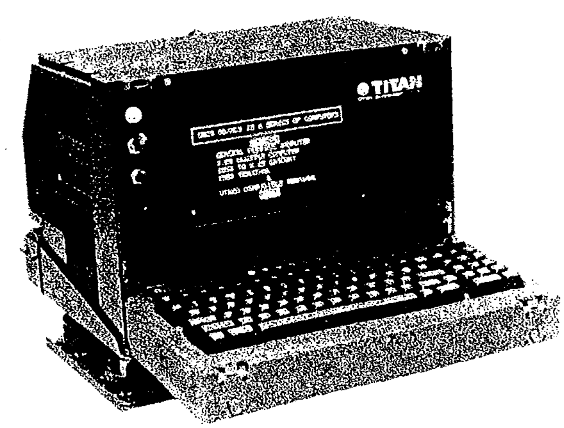


FIGURE 2.43 An electronic chassis mounted on vibration isolators. (Courtesy of Titan SESCO.)

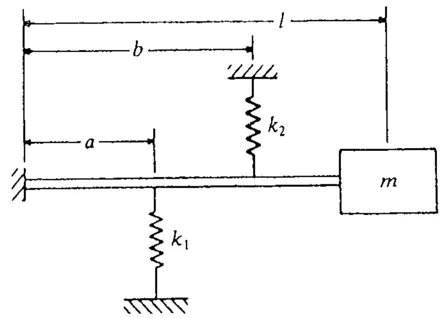


FIGURE 2.44

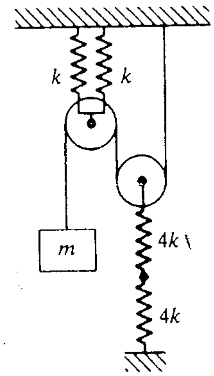


FIGURE 2.45

2.11 An electronic chassis, weighing 500 N, is isolated by supporting it on four helical springs, as shown in Fig. 2.43. Design the springs so that the unit can be used in an environment in which the vibratory frequency ranges from 0 to 5 Hz.

2.12 Find the natural frequency of the system shown in Fig. 2.44 with and without the springs k_1 and k_2 in the middle of the elastic beam.

2.13 Find the natural frequency of the pulley system shown in Fig. 2.45 by neglecting the friction and the masses of the pulleys.

2.14 A weight W is supported by three frictionless and massless pulleys and a spring of stiffness k, as shown in Fig. 2.46. Find the natural frequency of vibration of weight W for small oscillations.

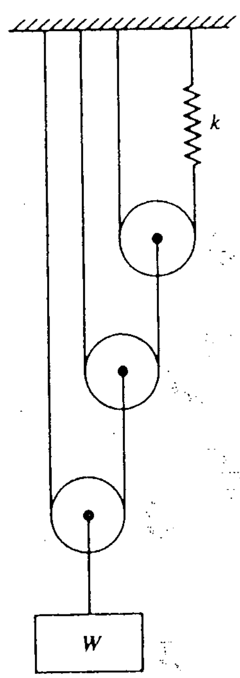


FIGURE 2.46

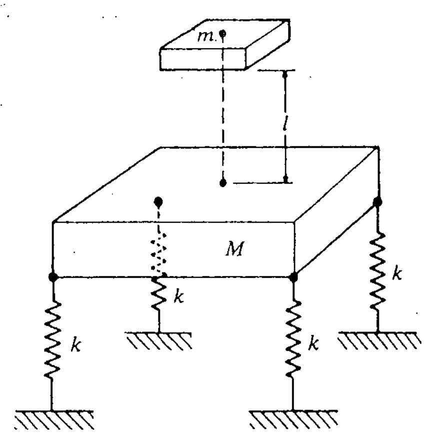


FIGURE 2.47

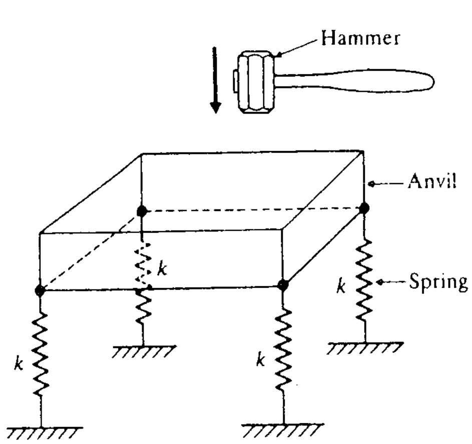


FIGURE 2.48

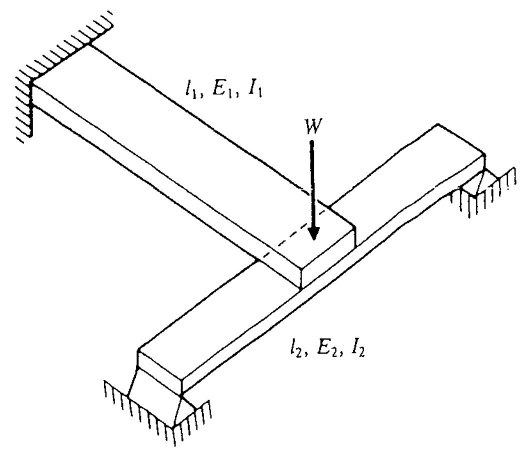


FIGURE 2.49

2.15 A rigid block of mass M is mounted on four elastic supports, as shown in Fig. 2.47. A mass m drops from a height l and adheres to the rigid block without rebounding. If the spring constant of each elastic support is k, find the natural frequency of vibration of the system (a) without the mass m, and (b) with the mass m. Also find the resulting motion of the system in case (b).

2.16 A sledgehammer strikes an anvil with a velocity of 50 ft/sec (Fig. 2.48). The hammer and the anvil weigh 12 lb and 100 lb, respectively. The anvil is supported on four springs, each of stiffness k = 100 lb/in. Find the resulting motion of the anvil (a) if the hammer remains in contact with the anvil, and (b) if the hammer does not remain in contact with the anvil after the initial impact.

2.17 Derive the expression for the natural frequency of the system shown in Fig. 2.49. Note that the load W is applied at the tip of beam 1 and midpoint of beam 2.

2.18 A heavy machine weighing 9810 N is being lowered vertically down by a winch at a uniform velocity of 2 m/sec. The steel cable supporting the machine has a diameter of 0.01 m. The winch is suddenly stopped when the steel cable's length is 20 m. Find the period and amplitude of the ensuing vibration of the machine.

2.19 The natural frequency of a spring-mass system is found to be 2 Hz. When an additional mass of 1 kg is added to the original mass m, the natural frequency is reduced to 1 Hz. Find the spring constant k and the mass m.

2.20 An electrical switchgear is supported by a crane through a steel cable of length 4 m and diameter 0.01 m (Fig. 2.50). If the natural time period of axial vibration of the switchgear is found to be 0.1 s, find the mass of the switchgear.

2.21 Four weightless rigid links and a spring are arranged to support a weight W in two different ways, as shown in Fig. 2.51. Determine the natural frequencies of vibration of the two arrangements.

2.22 A scissors jack is used to lift a load W. The links of the jack are rigid and the collars can slide freely on the shaft against the springs of stiffnesses k_1 and k_2

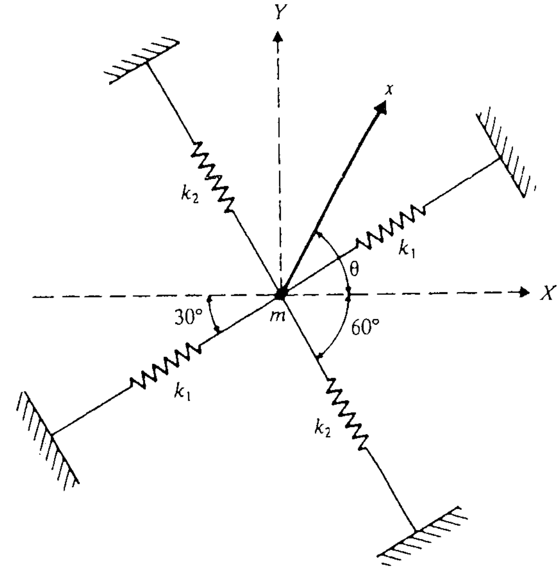


FIGURE 2.55

2.25 A mass m is supported by two sets of springs oriented at 30° and 120° with respect to the X axis, as shown in Fig. 2.55. A third pair of springs, with a stiffness of k_3 each, is to be designed so as to make the system have a constant natural frequency while vibrating in any direction x. Determine the necessary spring stiffness k_3 and the orientation of the springs with respect to the X axis.

2.26 A mass m is attached to a cord that is under a tension T, as shown in Fig. 2.56. Assuming that T remains unchanged when the mass is displaced normal to the cord, (a) write the differential equation of motion for small transverse vibrations, and (b) find the natural frequency of vibration.

2.27 A bungee jumper weighing 160 lb ties one end of an elastic rope of length 200 ft and stiffness 10 lb/in to a bridge and the other end to himself and jumps from the bridge (Fig. 2.57). Assuming the bridge to be rigid, determine the vibratory motion of the jumper about his static equilibrium position.

2.28 An acrobat weighing 120 lb walks on a tightrope, as shown in Fig. 2.58. If the natural frequency of vibration in the given position, in vertical direction, is 10 rad/s, find the tension in the rope.

2.29 The schematic diagram of a centrifugal governor is shown in Fig. 2.59. The length of each rod is l, the mass of each ball is m and the free length of the spring is h. If the shaft speed is ω, determine the equilibrium position and the frequency for small oscillations about this position.

2.30 In the Hartnell governor shown in Fig. 2.60, the stiffness of the spring is 10^4 N/m and the weight of each ball is 25 N. The length of the ball arm is 20 cm and that

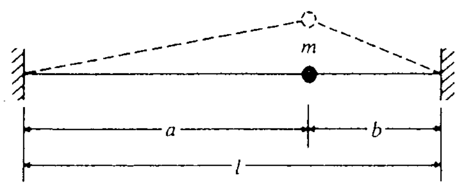


FIGURE 2.56

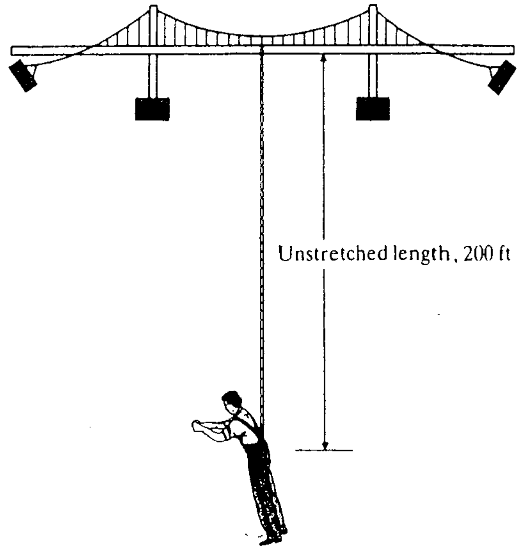


FIGURE 2.57

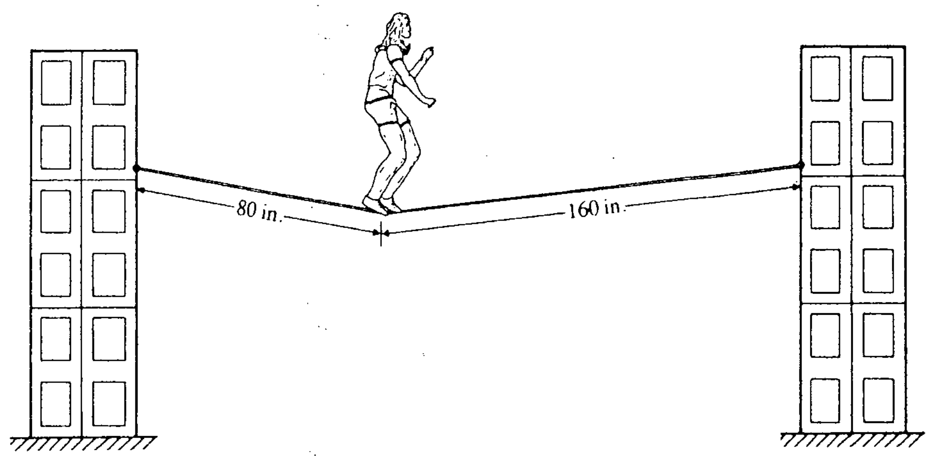


FIGURE 2.58

of the sleeve arm is 12 cm. The distance between the axis of rotation and the pivot of the bell crank lever is 16 cm. The spring is compressed by 1 cm when the ball arm is vertical. Find (a) the speed of the governor at which the ball arm remains vertical, and (b) the natural frequency of vibration for small displacements about the vertical position of the ball arms.

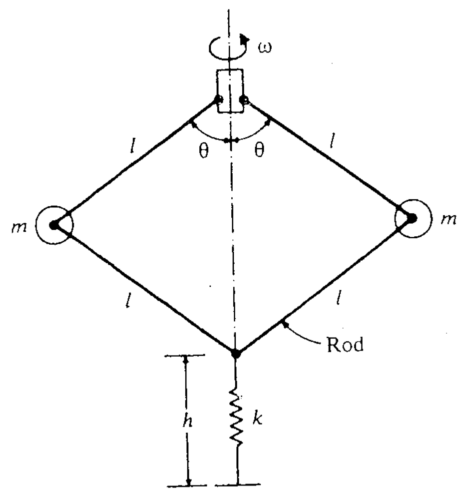


FIGURE 2.59

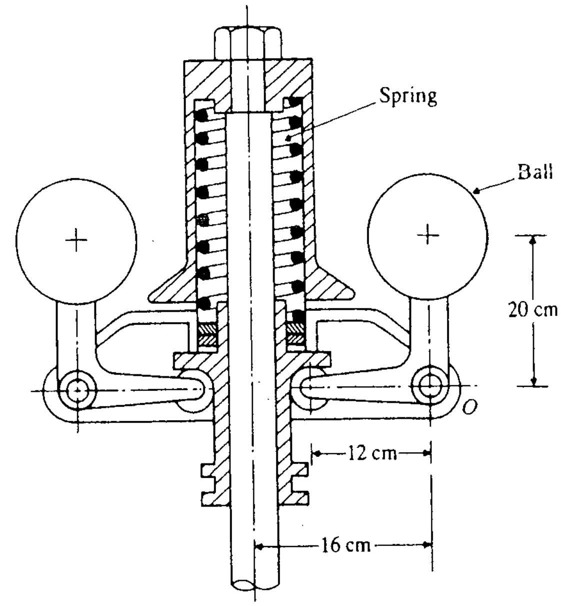


FIGURE 2.60 Hartnell governor.

2.31 A square platform *PQRS* and a car that it is supporting have a combined mass of *M*. The platform is suspended by four elastic wires from a fixed point *O*, as indicated in Fig. 2.61. The vertical distance between the point of suspension *O* and the horizontal equilibrium position of the platform is *h*. If the side of the platform is *a* and the stiffness of each wire is *k*, determine the period of vertical vibration of the platform.

2.32 The inclined manometer, shown in Fig. 2.62, is used to measure pressure. If the total length of mercury in the tube is *L*, find an expression for the natural frequency of oscillation of the mercury.

2.33 The crate, of mass 250 kg, hanging from a helicopter (shown in Fig. 2.63a) can be modeled as shown in Fig. 2.63b. The rotor blades of the helicopter rotate at 300 rpm. Find the diameter of the steel cables so that the natural frequency of vibration of the crate is at least twice the frequency of the rotor blades.

2.34 A pressure vessel head is supported by a set of steel cables of length 2 m as shown in Fig. 2.64. The time period of axial vibration (in vertical direction) is found to vary from 5 s to 4.0825 s when an additional mass of 5,000 kg is added to the pressure vessel head. Determine the equivalent cross-sectional area of the cables and the mass of the pressure vessel head.

2.35 A flywheel is mounted on a vertical shaft, as shown in Fig. 2.65. The shaft has a diameter *d* and length *l* and is fixed at both ends. The flywheel has a weight of *W* and a radius of gyration of *r*. Find the natural frequency of the longitudinal, the transverse, and the torsional vibration of the system.

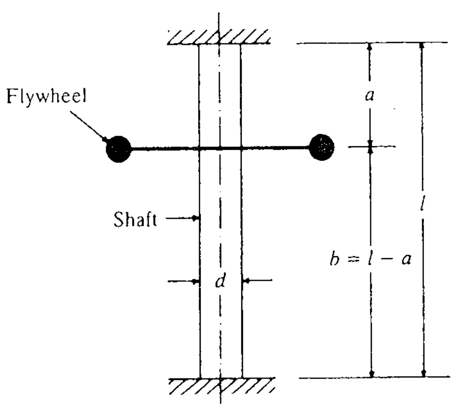


FIGURE 2.65

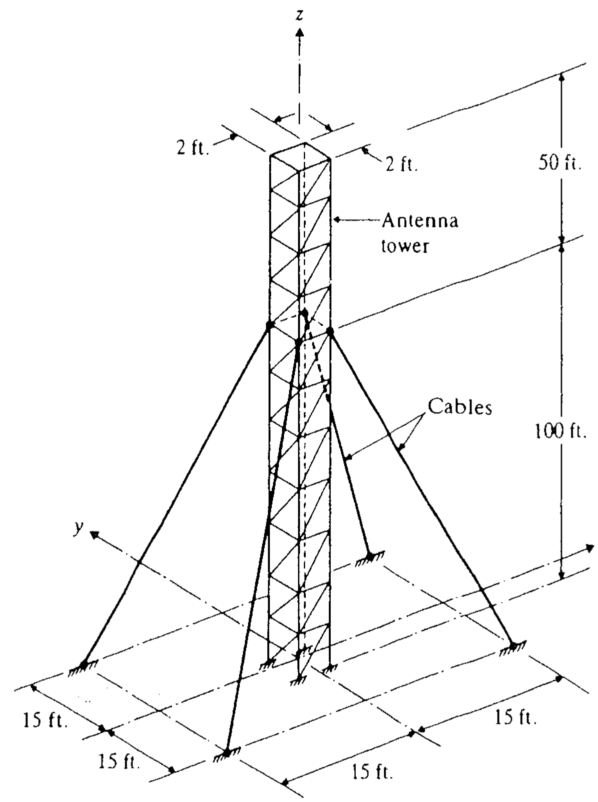


FIGURE 2.66

2.38 A building frame is modeled by four identical steel columns, of weight w each, and a rigid floor of weight W, as shown in Fig. 2.68. The columns are fixed at the ground and have a bending rigidity of EI each. Determine the natural frequency of horizontal vibration of the building frame by assuming the connection between the floor and the columns to be (a) pivoted as shown in Fig. 2.68(a), and (b) fixed against rotation as shown in Fig. 2.68(b). Include the effect of self weights of the columns.

2.39 A pick and place robot arm, shown in Fig. 2.69, carries an object weighing 10 lb. Find the natural frequency of the robot arm in the axial direction for the following data: $l_1 = 12$ in., $l_2 = 10$ in., $l_3 = 8$ in., $E_1 = E_2 = E_3 = 10^7$ psi, $D_1 = 2$ in., $D_2 = 1.5$ in., $D_3 = 1$ in., $d_1 = 1.75$ in., $d_2 = 1.25$ in., $d_3 = 0.75$ in.

2.40 A helical spring of stiffness k is cut into two halves and a mass m is connected to the two halves as shown in Fig. 2.70(a). The natural time period of this system is found to be 0.5 sec. If an identical spring is cut so that one part is 1/4 and the other part 3/4 of the original length, and the mass m is connected to the two parts as shown in Fig. 2.70(b), what would be the natural period of the system?

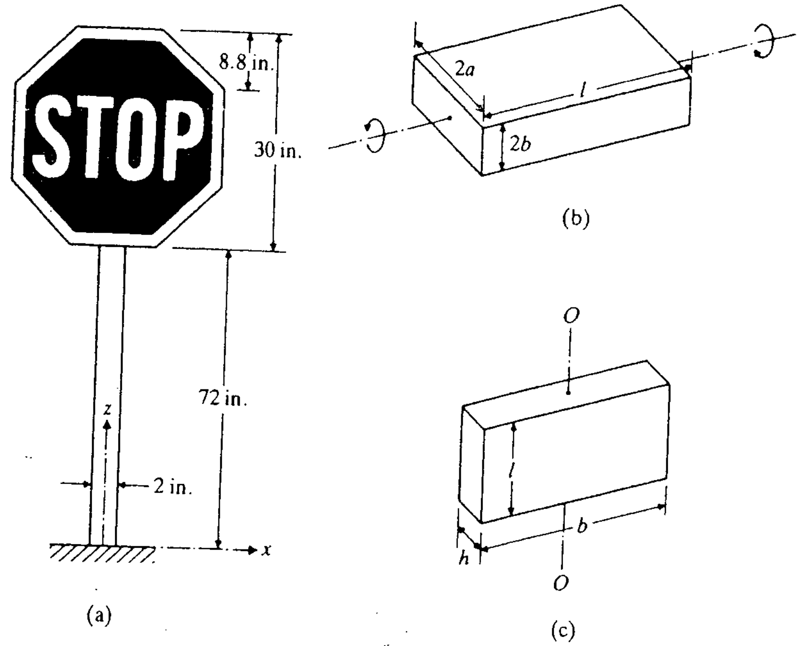


FIGURE 2.67

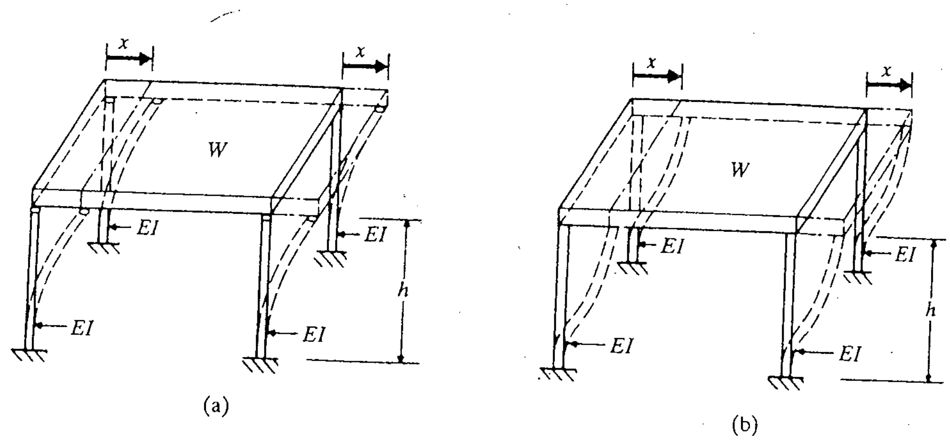


FIGURE 2.68

2.41* Figure 2.71 shows a metal block supported on two identical cylindrical rollers rotating in opposite directions at the same angular speed. When the center of gravity of the block is initially displaced by a distance x, the block will be set into simple harmonic

*The asterisk denotes a design problem or a problem with no unique answer.

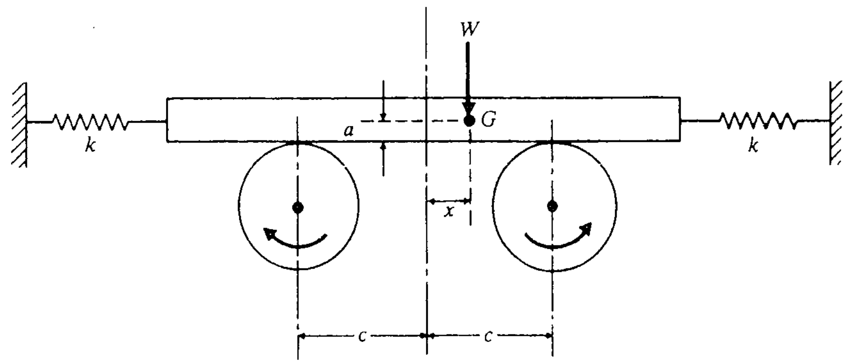


FIGURE 2.72

2.43 An electromagnet weighing 3000 lb is at rest while holding an automobile of weight 2000 lb in a junkyard. The electric current is turned off and the automobile is dropped. Assuming that the crane and the supporting cable have an equivalent spring constant of 10,000 lb/in, find the following: (a) the natural frequency of vibration of the electromagnet; (b) the resulting motion of the electromagnet; and (c) the maximum tension developed in the cable during the motion.

2.44 Derive the equation of motion of the system shown in Fig. 2.73 using the following methods: (a) Newton's second law of motion, (b) D'Alembert's principle, (c) principle of virtual work, and (d) principle of conservation of energy.

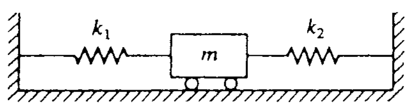


FIGURE 2.73

2.45–
2.46 Draw the free-body diagram and derive the equation of motion using Newton's second law of motion for each of the systems shown in Figs. 2.74 and 2.75.

2.47–
2.48 Derive the equation of motion using the principle of conservation of energy for each of the systems shown in Figs. 2.74 and 2.75.

2.49 A steel beam of length 1 m carries a mass of 50 kg at its free end, as shown in Fig. 2.76. Find the natural frequency of transverse vibration of the mass by modeling it as a single degree of freedom system.

2.50 A steel beam of length 1 m carries a mass of 50 kg at its free end, as shown in Fig. 2.77. Find the natural frequency of transverse vibration of the system by modeling it as a single degree of freedom system.

2.51 A pulley 250 mm in diameter drives a second pulley 1000 mm in diameter by means of a belt (see Fig. 2.78). The moment of inertia of the driven pulley is 0.2 kg-m^2. The belt connecting these pulleys is represented by two springs, each of stiffness k. For what value of k will the natural frequency be 6 Hz?

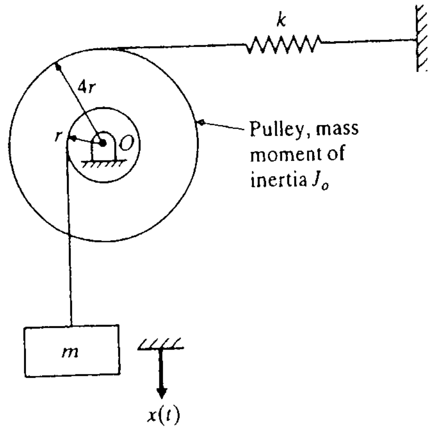


FIGURE 2.74

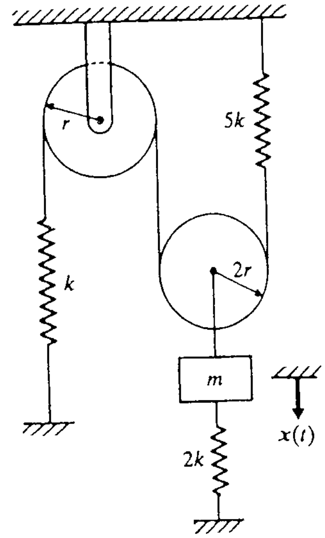


FIGURE 2.75

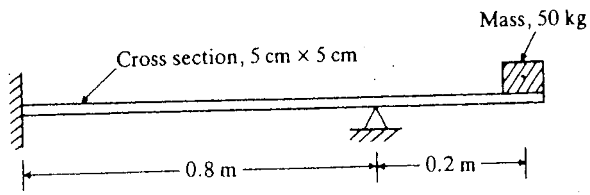


FIGURE 2.76

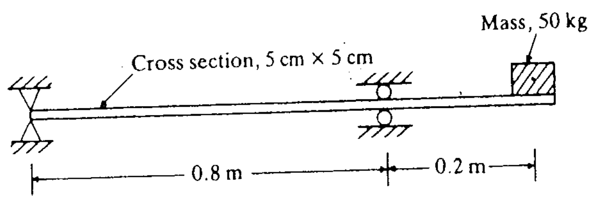


FIGURE 2.77

2.52 Derive an expression for the natural frequency of the simple pendulum shown in Fig. 1.11. Determine the period of oscillation of a simple pendulum having a mass $m = 5$ kg and a length $l = 0.5$ m.

2.53 A mass m is attached at the end of a bar of negligible mass and is made to vibrate in three different configurations, as indicated in Fig. 2.79. Find the configuration corresponding to the highest natural frequency.

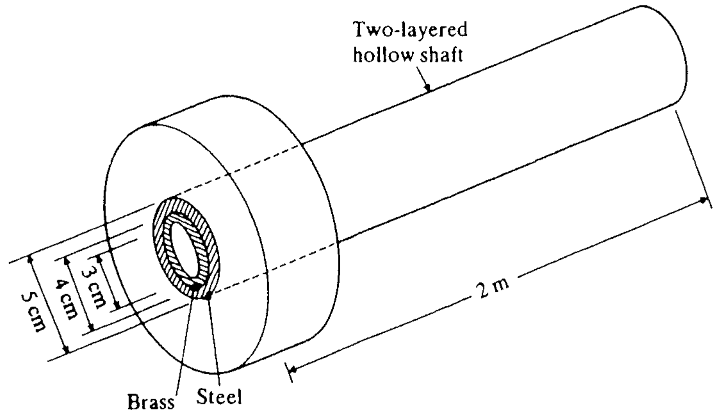


FIGURE 2.82

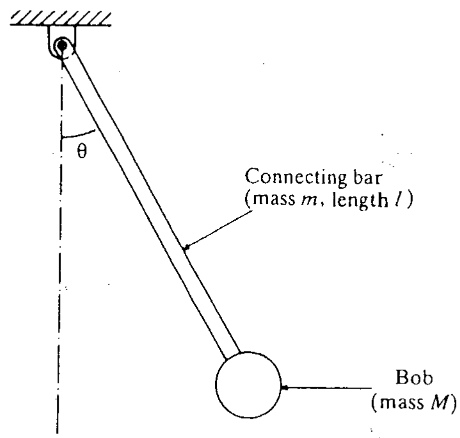


FIGURE 2.83

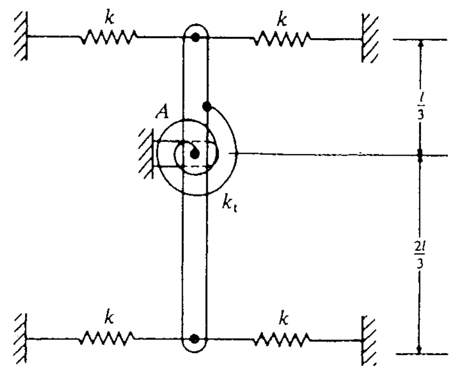


FIGURE 2.84

frequency of the system if $k = 2000$ N/m, $k_t = 1000$ N-m/rad, $m = 10$ kg, and $l = 5$ m.

2.60 A cylinder of mass m and mass moment of inertia J_0 is free to roll without slipping but is restrained by two springs of stiffnesses k_1 and k_2, as shown in Fig. 2.85. Find its natural frequency of vibration. Also find the value of a that maximizes the natural frequency of vibration.

2.61 If the pendulum of Problem 2.52 is placed in a rocket moving vertically with an acceleration of 5 m/s^2, what will be its period of oscillation?

2.62 Find the equation of motion of the uniform rigid bar OA of length l and mass m shown in Fig. 2.86. Also find its natural frequency.

2.63 A uniform circular disc is pivoted at point O, as shown in Fig. 2.87. Find the natural

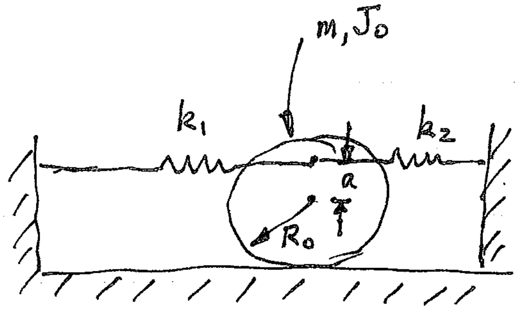


Fig 2.85

2.79 A shock absorber is to be designed to limit its overshoot to 15 percent of its initial displacement when released. Find the damping ratio ζ_0 required. What will be the overshoot if ζ is made equal to (a) $\frac{3}{4}\zeta_0$, and (b) $\frac{5}{4}\zeta_0$?

2.80 The free vibration response of an electric motor of weight 500 N mounted on different types of foundations are shown in Figs. 2.91(a) and (b). Identify the following in each case: (i) the nature of damping provided by the foundation; (ii) the spring constant and damping coefficient of the foundation; and (iii) the undamped and damped natural frequencies of the electric motor.

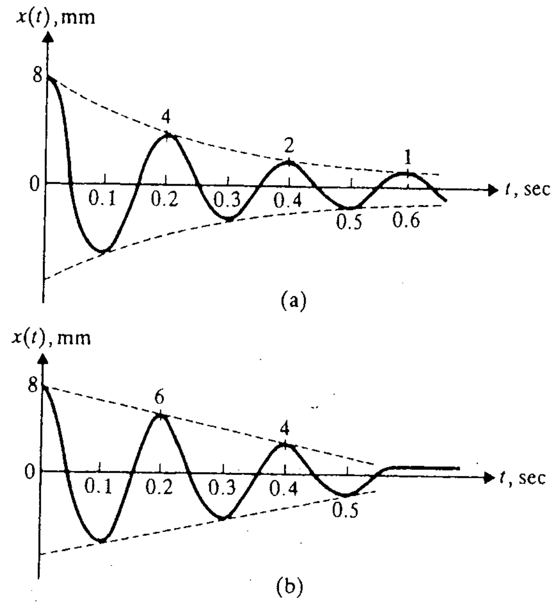


FIGURE 2.91

2.81 For a spring-mass-damper system, $m = 50$ kg and $k = 5000$ N/m. Find the following: (a) critical damping constant c_c; (b) damped natural frequency when $c = c_c/2$; and (c) logarithmic decrement.

2.82 A locomotive car of mass 2000 kg traveling at a velocity $v = 10$ m/sec is stopped at the end of tracks by a spring-damper system, as shown in Fig. 2.92. If the stiffness of the spring is $k = 40$ N/mm and the damping constant is $c = 20$ N-s/mm, determine (a) the maximum displacement of the car after engaging the springs and damper and (b) the time taken to reach the maximum displacement.

2.83 A torsional pendulum has a natural frequency of 200 cycles/min when vibrating in vacuum. The mass moment of inertia of the disc is 0.2 kg-m^2. It is then immersed in oil and its natural frequency is found to be 180 cycles/min. Determine the damping constant. If the disc, when placed in oil, is given an initial displacement of 2°, find its displacement at the end of the first cycle.

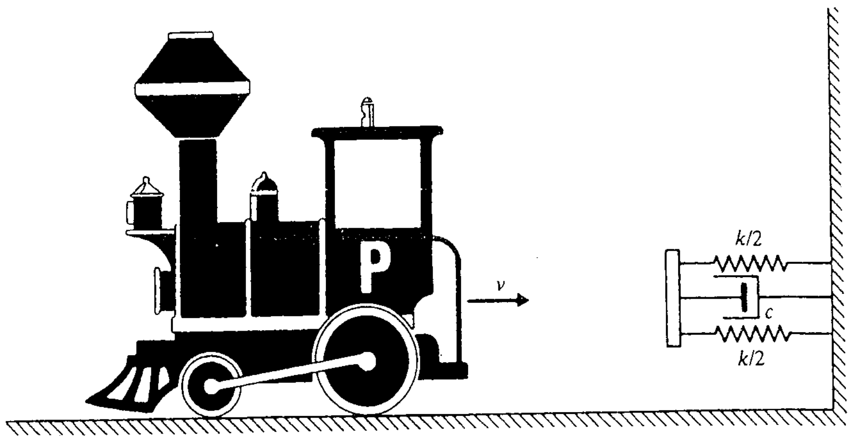


FIGURE 2.92

2.84 A boy riding a bicycle can be modeled as a spring-mass-damper system with an equivalent weight, stiffness and damping constant of 800 N, 50000 N/m, and 1000 N-s/m, respectively. The differential setting of the concrete blocks on the road caused the level surface to decrease suddenly as indicated in Fig. 2.93. If the speed of the bicycle is 5 m/s (18 km/hr), determine the displacement of the boy in the vertical direction. Assume that the bicycle is free of vertical vibration before encountering the step change in the vertical displacement.

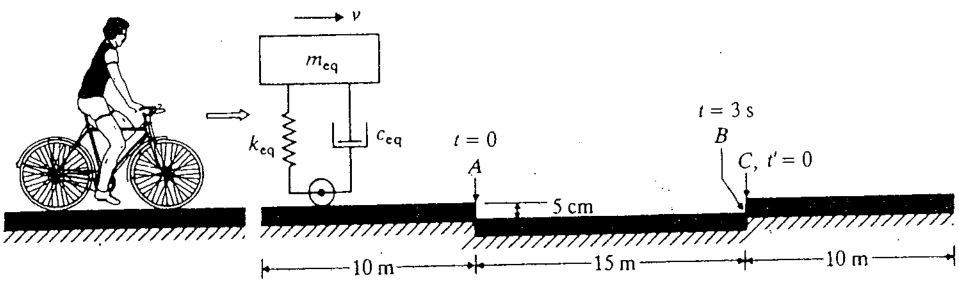


FIGURE 2.93

2.85 A wooden rectangular prism of weight 20 lb, height 3 ft. and cross section 1 ft. × 2 ft. floats and remains vertical in a tub of oil. The frictional resistance of the oil can be assumed to be equivalent to a viscous damping coefficient ζ. When the prism is depressed by a distance of 6 in. from its equilibrium and released, it is found to reach a depth of 5.5 in. at the end of its first cycle of oscillation. Determine the value of the damping coefficient of the oil.

2.86 A body vibrating with viscous damping makes five complete oscillations per second, and in 50 cycles its amplitude diminishes to 10 percent. Determine the logarithmic decrement and the damping ratio. In what proportion will the period of vibration be decreased if damping is removed?

2.87* The maximum permissible recoil distance of a gun is specified as 0.5 m. If the initial recoil velocity is to be between 8 m/sec and 10 m/sec, find the mass of the gun and the spring stiffness of the recoil mechanism. Assume that a critically damped dashpot is used in the recoil mechanism and the mass of the gun has to be at least 500 kg.

2.88 A viscously damped system has a stiffness of 5000 N/m, critical damping constant of 0.2 N-s/mm, and a logarithmic decrement of 2.0. If the system is given an initial velocity of 1 m/sec, determine the maximum displacement of the system.

2.89 Explain why an overdamped system never passes through the static equilibrium position when it is given (a) an initial displacement only and (b) an initial velocity only.

2.90–2.92 Derive the equation of motion and find the natural frequency of vibration of each of the systems shown in Figs. 2.94 to 2.96.

2.93–2.95 Using the principle of virtual work, derive the equation of motion for each of the systems shown in Figs. 2.94 to 2.96.

2.96 A wooden rectangular prism of cross section 40 cm × 60 cm, height 120 cm, and mass 40 kg floats in a fluid, as shown in Fig. 2.90. When disturbed, it is observed to vibrate freely with a natural period of 0.5 s. Determine the density of the fluid.

2.97 The system shown in Fig. 2.97 has a natural frequency of 5 Hz for the following data: $m = 10$ kg, $J_0 = 5$ kg − m^2, $r_1 = 10$ cm, $r_2 = 25$ cm. When the system is disturbed by giving it an initial displacement, the amplitude of free vibration is reduced by 80 percent in 10 cycles. Determine the values of k and c.

2.98 A single degree of freedom system consists of a mass of 20 kg and a spring of stiffness 4000 N/m. The amplitudes of successive cycles are found to be 50, 45, 40, 35, . . . mm. Determine the nature and magnitude of the damping force and the frequency of the damped vibration.

2.99 A mass of 20 kg slides back and forth on a dry surface due to the action of a spring having a stiffness of 10 N/mm. After four complete cycles, the amplitude has been

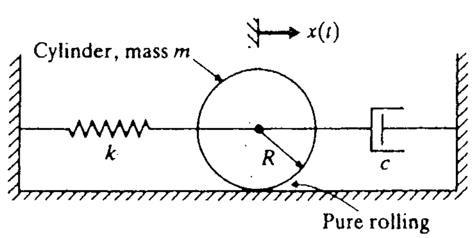


FIGURE 2.94

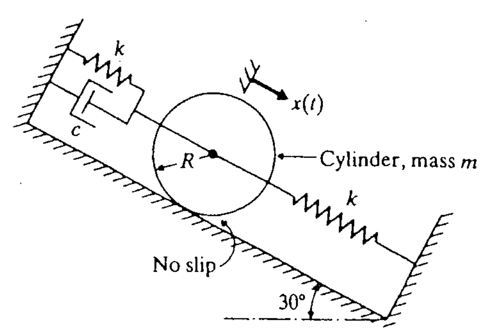


FIGURE 2.95

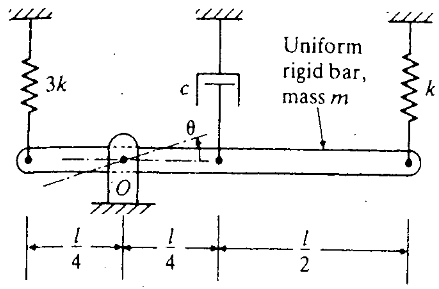


FIGURE 2.96

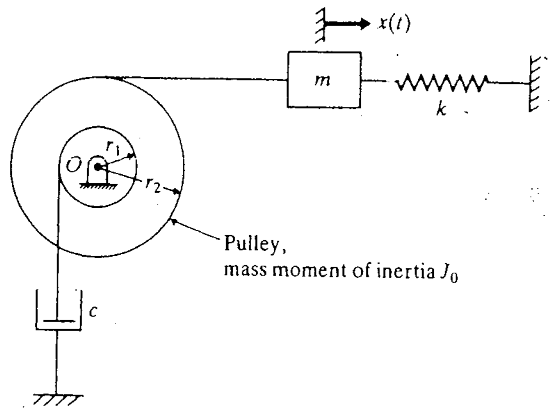


FIGURE 2.97

found to be 100 mm. What is the average coefficient of friction between the two surfaces if the original amplitude was 150 mm? How much time has elapsed during the four cycles?

2.100 A 10-kg mass is connected to a spring of stiffness 3000 N/m and is released after giving an initial displacement of 100 mm. Assuming that the mass moves on a horizontal surface, as shown in Fig. 2.33(a), determine the position at which the mass comes to rest. Assume the coefficient of friction between the mass and the surface to be 0.12.

2.101 A weight of 25 N is suspended from a spring that has a stiffness of 1000 N/m. The weight vibrates in the vertical direction under a constant damping force. When the weight is initially pulled downward a distance of 10 cm from its static equilibrium position and released, it comes to rest after exactly two complete cycles. Find the magnitude of the damping force.

2.102 A mass of 20 kg is suspended from a spring of stiffness 10,000 N/m. The vertical motion of the mass is subject to Coulomb friction of magnitude 50 N. If the spring is initially displaced downward by 5 cm from its static equilibrium position, determine (a) the number of half cycles elapsed before the mass comes to rest, (b) the time elapsed before the mass comes to rest, and (c) the final extension of the spring.

2.103 The Charpy impact test is a dynamic test in which a specimen is struck and broken by a pendulum (or hammer) and the energy absorbed in breaking the specimen is measured. The energy values serve as a useful guide for comparing the impact strengths of different materials. As shown in Fig. 2.98, the pendulum is suspended from a shaft, is released from a particular position, and is allowed to fall and break the specimen. If the pendulum is made to oscillate freely (with no specimen), find (a) an expression for the decrease in the angle of swing for each cycle caused by friction, (b) the solution for $\theta(t)$ if the pendulum is released from an angle θ_0, and (c) the number of cycles after which the motion ceases. Assume the mass of the pendulum as m and the coefficient of friction between the shaft and the bearing of the pendulum as μ.