Applied Numerical Methods
by James, Smith, Wolford
International Textbook Co.
1967

T	X	Y	VX	VY	V
5.0	1044.1	272.5	109.76	97.66	146.92
5.5	1096.6	323.0	100.16	104.49	144.74
6.0	1144.5	376.9	91.48	110.82	143.70
6.4	1179.8	422.1	85.11	115.52	143.49
6.5	1188.2	433.7	83.59	116.64	143.50
7.0	1228.2	493.4	76.38	121.95	143.90
7.5	1264.7	555.6	69.76	126.76	144.69
8.0	1298.0	620.1	63.67	131.08	145.73
8.5	1328.4	686.6	58.08	134.93	146.90
9.0	1356.2	755.0	52.94	138.35	148.13
9.5	1381.5	824.9	48.21	141.36	149.36
10.0	1404.5	896.3	43.88	144.00	150.54
10.5	1425.4	968.9	39.90	146.30	151.64
11.0	1444.4	1042.5	36.26	148.30	152.67
11.5	1461.7	1117.1	32.94	150.03	153.60
12.0	1477.4	1192.5	29.90	151.52	154.44
12.5	1491.7	1268.6	27.13	152.80	155.19
13.0	1504.6	1345.3	24.60	153.90	155.85
13.5	1516.3	1422.5	22.30	154.83	156.43
14.0	1526.9	1500.1	20.21	155.63	156.94
14.5	1536.6	1578.1	18.31	156.31	157.38
15.0	1545.3	1656.4	16.58	156.89	157.77
15.5	1553.2	1735.0	15.01	157.38	158.10
16.0	1560.3	1813.8	13.59	157.80	158.38
16.5	1566.8	1892.8	12.30	158.15	158.63
17.0	1572.7	1971.9	11.14	158.44	158.83
17.5	1578.0	2051.2	10.08	158.69	159.01
18.0	1582.8	2130.6	9.12	158.90	159.16
18.5	1587.1	2210.1	8.25	159.08	159.29
19.0	1591.0	2289.7	7.46	159.23	159.40
19.5	1594.6	2369.3	6.75	159.35	159.50
20.0	1597.8	2449.0	6.11	159.46	159.58

STOP 0000

The computer results show that a minimum velocity of 143.49 ft/sec is reached when $t = 6.4$ sec, which indicates the optimum time to open the parachute. The velocity data are plotted in Fig. 6-17.

The Non-self-starting Modification. For purposes of easily estimating the per-step truncation error in the corrector equation,

$$P(y_{i+1}) = y_{i-1} + 2y_i'(\Delta x) \qquad \text{6-23(a')}$$

is sometimes used as a predictor equation in place of Eq. 6-23(a). (See Sec. 6-9 for a discussion of error estimation.) However, when this substi-

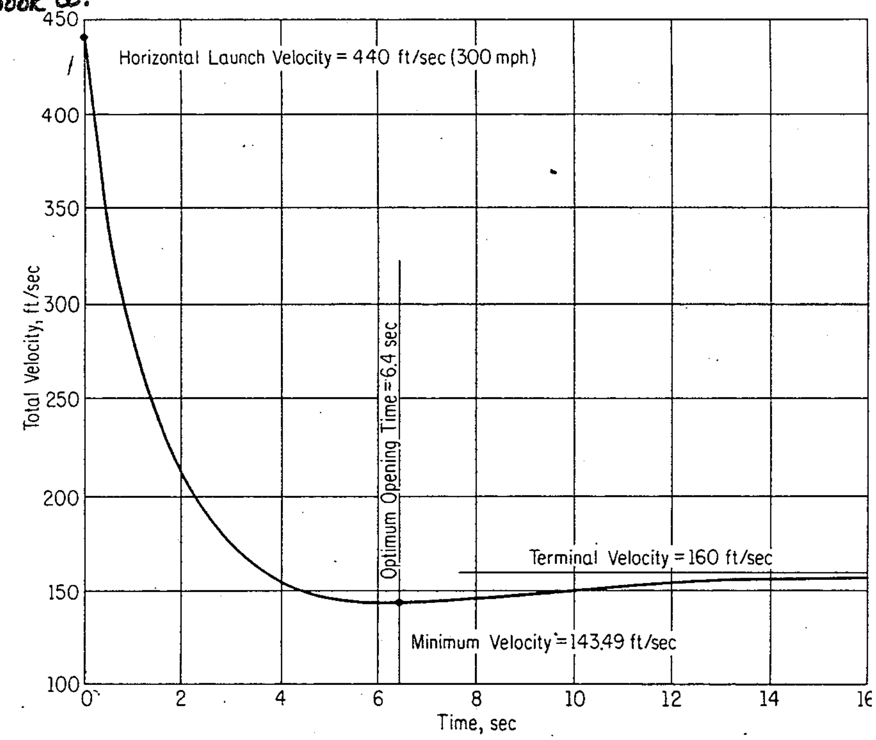

FIG. 6-17. Total velocity of paratrooper.

tution is made, the convenient self-starting feature of the previousl described modification of Euler's method is lost. In addition to the usu initial value of y required for solving a first order differential equatio one additional value of y must be known to get the method started. Int gration by this method is usually started by applying Eq. 6-23(a) and (t for the first step only and then using Eq. 6-23(a') in place of Eq. 6-23 fo subsequent steps.

Equation 6-23(a') approximates the area in two strips under the curve, each of width Δx, with a rectangle of width $2(\Delta x)$ and height y_i'.

6-5. Runge-Kutta Methods

A Runge-Kutta method is one which employs a recurrence formul of the form

$$y_{i+1} = y_i + a_1k_1 + a_2k_2 + a_3k_3 + \cdots + a_nk_n \qquad (6\text{-}4$$

to calculate successive values of the dependent variable y of the diffe ential equation

$$\frac{dy}{dx} = y' = f(x,y) \qquad (6\text{-}42$$

where

$$\begin{aligned} k_1 &= (\Delta x) f(x_i, y_i) \\ k_2 &= (\Delta x) f(x_i + p_1\,\Delta x, y_i + q_{11} k_1) \\ k_3 &= (\Delta x) f(x_i + p_2\,\Delta x, y_i + q_{21} k_1 + q_{22} k_2) \\ &\cdots\cdots \\ k_n &= (\Delta x) f(x_i + p_{n-1}\,\Delta x, y_i + q_{n-1,1} k_1 \\ &\qquad + q_{n-1,2} k_2 + \cdots + q_{n-1,n-1} k_{n-1}) \end{aligned} \tag{6-43}$$

The a's, p's, and q's must assume values such that Eq. 6-41 accurately yields successive values of y. As will be shown later, these values are determined by making Eq. 6-41 equivalent to a certain specified number of terms of a Taylor-series expansion of y about x_i.

Runge-Kutta methods are self-starting, and it is theoretically possible to develop one having any desired degree of accuracy. These were among the earliest employed in the numerical solution of differential equations, and they are still widely used. As with any method, they possess certain advantages and disadvantages which must be weighed in considering their suitability for a particular application. The principal advantage of the Runge-Kutta methods is their self-starting feature and resulting ease of programming, and one disadvantage is the requirement that the function $f(x, y)$ must be evaluated for several slightly different values of x and y in every step of the solution (in every incrementation of x by Δx). This repeated determination of $f(x, y)$ usually results in a less efficient method, with respect to computing time, than do other methods, of comparable accuracy, in which previously determined values of the dependent variable are used in subsequent steps. (Hamming's method and Milne's method are examples of the latter type. They are discussed in later portions of this chapter.) Another disadvantage is that it is more difficult to estimate the per-step error for higher-order Runge-Kutta solutions than for solutions obtained by some other commonly used methods.

Before illustrating the general procedure for developing a Runge-Kutta method, let us review 1) the definition of a *total differential* and 2) the expansion of a function of 2 variables in a *Taylor series*, since both will be pertinent to the development.

1. If $y' = f(x, y)$, the total differential dy' is

$$dy' = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

Upon dividing by dx, we obtain

$$\frac{dy'}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \tag{6-44}$$

Differentiating again with respect to x yields

$$\frac{d^2y'}{dx^2} = \frac{\partial\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right)}{\partial x} + \frac{\partial\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}\right)}{\partial y}\left(\frac{dy}{dx}\right) \tag{6-45}$$

Higher derivatives of y' may be determined in a similar manner.

2. Consider some general function of 2 variables such as $z = f(x, y)$ Such a function may be expanded about a point x_i, y_i in a Taylor serie for functions of 2 variables, as follows:

$$z(x_i + h, y_i + j) = f(x_i, y_i) + h\left[\frac{\partial f}{\partial x}(x_i, y_i)\right] + j\left[\frac{\partial f}{\partial y}(x_i, y_i)\right] + \frac{1}{2!}\left\{h^2\left[\frac{\partial^2 f}{\partial x^2}(x_i, y_i)\right] + 2hj\left[\frac{\partial^2 f}{\partial x\,\partial y}(x_i, y_i)\right] + j^2\left[\frac{\partial^2 f}{\partial y^2}(x_i, y_i)\right]\right\} + \cdots \tag{6-46}$$

where h and j are increments of x and y, respectively.

We are now ready to proceed with the development of a *second-orde* Runge-Kutta method ($n = 2$ in Eq. 6-41). Equation 6-41 becomes

$$y_{i+1} = y_i + a_1 k_1 + a_2 k_2 \tag{6-47}$$

in which

$$\left.\begin{aligned} k_1 &= (\Delta x) f(x_i, y_i) \\ k_2 &= (\Delta x) f(x_i + p_1\,\Delta x, y_i + q_{11} k_1) \end{aligned}\right\} \tag{6-48}$$

Our problem is to determine values for a_1, a_2, p_1, and q_{11}, so that Eq 6-47 yields an accurate value of y_{i+1}. A graphical interpretation of th k functions is illustrated in Fig. 6-18. The crosshatched area represent k_1, and k_2 is represented by the shaded area. (There would be simila rectangular areas for each k function of Eq. 6-41 for higher-order Runge Kutta equations.) For the second-order method being discussed, Eq. 6-4 could be written as

$$y_{i+1} = y_i + a_1\,(\text{crosshatched area}) + a_2\,(\text{shaded area})$$

It should be evident that the size of the shaded area representing k depends upon the values determined for p_1 and q_{11}.

We shall determine values for a_1, a_2, p_1, and q_{11} by making Eq. 6-4 equivalent to a truncated Taylor-series expansion of y about x_i. As th first step, let us expand y_{i+1} about x_i. We obtain

$$y_{i+1} = y_i + (\Delta x)\,y_i' + \frac{(\Delta x)^2}{2!}\,y_i'' + \cdots \tag{6-49}$$

Remembering that the given differential equation is $y' = f(x, y)$, we writ

$$y_i' = f(x_i, y_i) \tag{6-50}$$

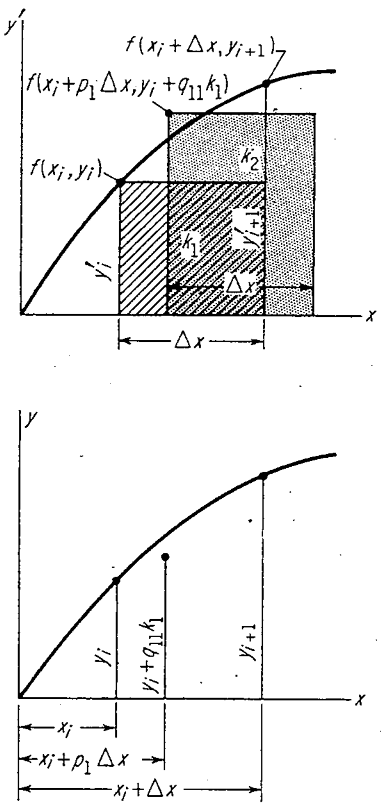

FIG. 6-18. Graphical interpretation of k functions.

From Eq. 6-44 it can be seen that

$$y_i'' = \frac{\partial f}{\partial x}(x_i, y_i) + \left[\frac{\partial f}{\partial y}(x_i, y_i)\right]\left[f(x_i, y_i)\right] \tag{6-51}$$

Substituting Eqs. 6-50 and 6-51 into Eq. 6-49, we obtain the expansion in the following form:

$$y_{i+1} = y_i + (\Delta x) f(x_i, y_i)$$
$$+ \frac{(\Delta x)^2}{2!}\left\{\frac{\partial f}{\partial x}(x_i, y_i) + \left[\frac{\partial f}{\partial y}(x_i, y_i)\right]\left[f(x_i, y_i)\right]\right\} + \frac{(\Delta x)^3}{3!}\left\{\cdots\right\} + \cdots \tag{6-52}$$

Looking at Eqs. 6-47, 6-48, and 6-52, we see that k_2 must be expressed in terms of $f(x_i, y_i)$, $\partial f/\partial x(x_i, y_i)$, and $\partial f/\partial y(x_i, y_i)$ if Eqs. 6-47 and 6-52 are to contain similar terms. This can be accomplished by expanding k_2

in a Taylor series for functions of 2 variables about x_i, y_i. Using the first 3 terms of Eq. 6-46 and realizing that $h = p_1 \Delta x$ and $j = q_{11} k_1$ (see Eq. 6-48), we may write

$$k_2 = \Delta x\left\{f(x_i, y_i) + p_1 \Delta x\left[\frac{\partial f}{\partial x}(x_i, y_i)\right] + q_{11}k_1\left[\frac{\partial f}{\partial y}(x_i, y_i)\right]\right\} \tag{6-53}$$

Substituting the first of Eq. 6-48 and Eq. 6-53 into Eq. 6-47, we obtain

$$y_{i+1} = y_i + a_1(\Delta x) f(x_i, y_i) + a_2(\Delta x) f(x_i, y_i)$$
$$+ a_2(\Delta x)^2\left\{p_1 \frac{\partial f}{\partial x}(x_i, y_i) + q_{11}\left[\frac{\partial f}{\partial y}(x_i, y_i)\right]\left[f(x_i, y_i)\right]\right\} \tag{6-54}$$

Equating coefficients of similar terms in Eqs. 6-52 and 6-54, we obtain the following 3 independent equations:

$$\begin{aligned} a_1 + a_2 &= 1 \\ a_2 p_1 &= 1/2 \\ a_2 q_{11} &= 1/2 \end{aligned} \tag{6-55}$$

which contain 4 unknowns. By arbitrarily assigning a value to 1 unknown and then solving for the other 3, we can obtain as many different sets of values as we desire and, in turn, as many different sets of Eqs. 6-47 and 6-48 as desired.

A solution obtained by the use of Eq. 6-47 in a step-by-step integration will have a per-step truncation error of order $(\Delta x)^3$, since terms containing $(\Delta x)^3$ and higher powers of Δx were neglected in the development. Thus, this is known as a *second-order* Runge-Kutta method.

If a particular second-order method is defined by letting $a_1 = 1/2$ in Eq. 6-55, then

$$\begin{aligned} a_2 &= 1/2 \\ p_1 &= 1 \\ q_{11} &= 1 \end{aligned}$$

Equations 6-47 and 6-48 then become

$$y_{i+1} = y_i + 1/2(k_1 + k_2) \qquad \text{(a)}$$

with

$$\begin{aligned} k_1 &= (\Delta x) f(x_i, y_i) \\ k_2 &= (\Delta x) f(x_i + \Delta x, y_i + k_1) \end{aligned} \qquad \text{(b)} \tag{6-56}$$

This set of equations may be used to solve first-order differential equations with an accuracy comparable to that of Euler's modified methods. In fact, if Eq. 6-56b is substituted into Eq. 6-56a, the resulting equation

$$y_{i+1} = y_i + 1/2\{(\Delta x) f(x_i, y_i)$$
$$+ (\Delta x) f[x_i + \Delta x, y_i + (\Delta x) f(x_i, y_i)]\} \tag{6-57}$$

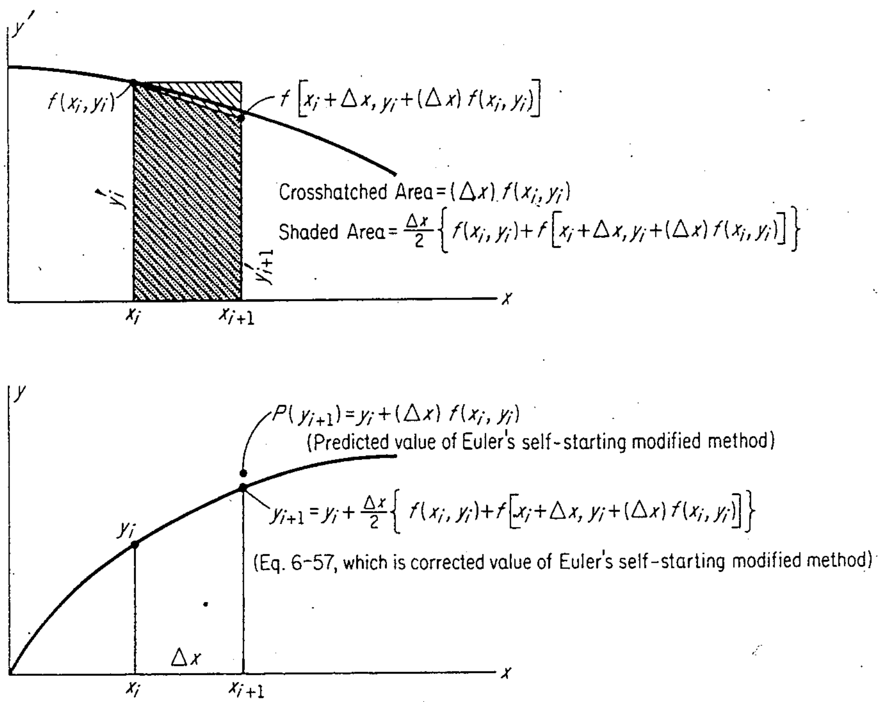

FIG. 6-19. Graphical representation of a second-order Runge-Kutta method.

is exactly equivalent to Euler's self-starting modified method with the iteration at each step omitted. This may be verified by examining the graphical representation of the equation in Fig. 6-19.

Because the second-order Runge Kutta method has only the order of accuracy of Euler's modified methods, let us consider Runge-Kutta methods with higher orders of accuracy. Since the development of such methods rapidly increases in complexity with an increase in accuracy, and since the general procedure has just been demonstrated for a second-order method, the developments will not be given for higher orders. For example, in developing a third-order method, all Taylor-series terms containing $(\Delta x)^3$ must be retained; in developing a fourth-order method, all $(\Delta x)^4$ terms must be retained, and so forth.

If $n = 3$, in Eq. 6-41, and a procedure similar to the one previously outlined is followed, the equating of coefficients of terms containing Δx, $(\Delta x)^2$, and $(\Delta x)^3$ will yield a set of equations with 6 constraints on the parameters. With 8 parameters involved, 2 of them can be assigned values arbitrarily. With a particular choice of values for 2 such parameters, the following *third-order* Runge-Kutta formula can be obtained:

$$\underline{y_{i+1} = y_i + \tfrac{1}{6}(k_1 + 4k_2 + k_3)} \tag{6-58}$$

in which

$$\left.\begin{aligned} k_1 &= (\Delta x) f(x_i, y_i) \\ k_2 &= (\Delta x) f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_1}{2}\right) \\ k_3 &= (\Delta x) f(x_i + \Delta x, y_i - k_1 + 2k_2) \end{aligned}\right\} \tag{6-59}$$

Since terms containing $(\Delta x)^4$ are neglected, the error is said to be of order $(\Delta x)^4$.

A well-known *fourth-order* Runge-Kutta method, resulting from taking n equal to 4 in Eq. 6-41, equating terms through and including those containing $(\Delta x)^4$, and selecting a particular set of 2 arbitrary parameter values, is expressed as follows:[3]

$$\underline{y_{i+1} = y_i + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)} \tag{6-60}$$

in which

$$\left.\begin{aligned} k_1 &= (\Delta x) f(x_i, y_i) \\ k_2 &= (\Delta x) f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_1}{2}\right) \\ k_3 &= (\Delta x) f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_2}{2}\right) \\ k_4 &= (\Delta x) f(x_i + \Delta x, y_i + k_3) \end{aligned}\right\} \tag{6-61}$$

In this method the per-step error is of order $(\Delta x)^5$.

Having discussed the general method of developing Runge-Kutta formulas and having shown several specific different-order Runge-Kutta equations, let us illustrate the use of the third-order method in solving a first-order differential equation.

EXAMPLE 6-3

The circuit shown in Fig. 6-20 contains a source of emf, an inductance, and a resistor, the magnitude of the latter varying with its tempera-

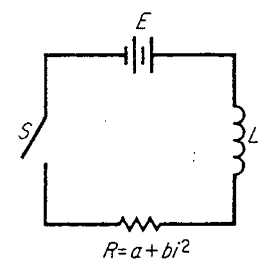

FIG. 6-20. Circuit diagram for Example 6-3.

[3]The interested reader will find a complete derivation in *Mathematical Methods for Digital Computers*. Anthony Ralston and Herbert S. Wilf, eds. New York: John Wiley & Sons, Inc., 1960.

6-6. Solution of Simultaneous Ordinary Differential Equations by Runge-Kutta Methods

It is frequently necessary to solve sets of simultaneous first-order differential equations in analyzing engineering systems. Such equations occur most frequently, perhaps, in obtaining solutions of higher-order differential equations which are transformed to sets of first-order differential equations as part of the solution process. Runge-Kutta methods are well suited for the solution of such equations, and their application will be discussed here.

Let us first consider the solution of 2 simultaneous first-order differential equations of the form

$$\left.\begin{aligned} \frac{dy}{dx} &= f[x, y(x), u(x)] \\ \frac{du}{dx} &= F[x, y(x), u(x)] \end{aligned}\right. \tag{6-64}$$

where the initial values ($y = y_0$, $u = u_0$, when $x = x_0$) are known. Runge-Kutta formulas can be used to solve Eq. 6-64. Using the fourth-order method already described (Eqs. 6-60 and 6-61), for example, the following *sets* of equations would be used:

$$y_{i+1} = y_i + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{6-65}$$

where

$$\left.\begin{aligned} k_1 &= (\Delta x) f(x_i, y_i, u_i) \\ k_2 &= (\Delta x) f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_1}{2}, u_i + \frac{q_1}{2}\right) \\ k_3 &= (\Delta x) f\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_2}{2}, u_i + \frac{q_2}{2}\right) \\ k_4 &= (\Delta x) f(x_i + \Delta x, y_i + k_3, u_i + q_3) \end{aligned}\right\} \tag{6-66}$$

and

$$u_{i+1} = u_i + \tfrac{1}{6}(q_1 + 2q_2 + 2q_3 + q_4) \tag{6-67}$$

where

$$\left.\begin{aligned} q_1 &= (\Delta x) F(x_i, y_i, u_i) \\ q_2 &= (\Delta x) F\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_1}{2}, u_i + \frac{q_1}{2}\right) \\ q_3 &= (\Delta x) F\left(x_i + \frac{\Delta x}{2}, y_i + \frac{k_2}{2}, u_i + \frac{q_2}{2}\right) \\ q_4 &= (\Delta x) F(x_i + \Delta x, y_i + k_3, u_i + q_3) \end{aligned}\right\} \tag{6-68}$$

The solution begins by substituting the initial values of y and u, which

must be known, into the given differential equations to obtain initial values of the functions f and F. Values of k_1 and q_1 are next obtained by multiplying the initial values of f and F, respectively, by Δx, as indicated in Eqs. 6-66 and 6-68. With values of k_1 and q_1 known, k_2 and q_2 are next evaluated, then k_3 and q_3, and finally k_4 and q_4. Then the recurrence formulas (Eqs. 6-65 and 6-67) are used to obtain values of y and u at $x = x_i + \Delta x$ (y_{i+1} and u_{i+1}). These new values of y and u are then used as beginning values in starting the procedure (just described) over again, to obtain values of y_{i+2} and u_{i+2} at $x = x_i + 2\Delta x$, and so on, until the desired range of integration has been covered.

Any number of simultaneous first-order differential equations may be solved by merely using a set of equations such as those shown for each dependent variable appearing in the set of simultaneous differential equations.

An nth-order differential equation can be solved by transforming the equation to a set of n simultaneous first-order differential equations and applying n Runge-Kutta formulas, as discussed in the preceding paragraphs. The procedure can be simplified by combining equations, as will be shown in the following paragraphs.

Consider the second-order differential equation

$$\frac{d^2x}{dt^2} = f\left(t, x, \frac{dx}{dt}\right) \tag{6-69}$$

Letting $v = dx/dt$, Eq. 6-69 can be transformed to the 2 first-order differential equations

$$\left.\begin{aligned} \frac{dv}{dt} &= f(t, x, v) \\ \frac{dx}{dt} &= v \end{aligned}\right\} \tag{6-70}$$

Referring to Eqs. 6-65 through 6-68, the following 2 fourth-order Runge-Kutta formulas could be used to solve Eq. 6-70:

$$v_{i+1} = v_i + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \tag{6-71}$$

where

$$\left.\begin{aligned} k_1 &= (\Delta t) f(t_i, x_i, v_i) \\ k_2 &= (\Delta t) f\left(t_i + \frac{\Delta t}{2}, x_i + \frac{q_1}{2}, v_i + \frac{k_1}{2}\right) \\ k_3 &= (\Delta t) f\left(t_i + \frac{\Delta t}{2}, x_i + \frac{q_2}{2}, v_i + \frac{k_2}{2}\right) \\ k_4 &= (\Delta t) f(t_i + \Delta t, x_i + q_3, v_i + k_3) \end{aligned}\right\} \tag{6-72}$$

and

$$x_{i+1} = x_i + \tfrac{1}{6}(q_1 + 2q_2 + 2q_3 + q_4) \tag{6-73}$$

where

$$\left.\begin{aligned} q_1 &= (\Delta t)F(v_i) &&= (\Delta t)(v_i) \\ q_2 &= (\Delta t)F\left(v_i + \frac{k_1}{2}\right) &&= (\Delta t)\left(v_i + \frac{k_1}{2}\right) \\ q_3 &= (\Delta t)F\left(v_i + \frac{k_2}{2}\right) &&= (\Delta t)\left(v_i + \frac{k_2}{2}\right) \\ q_4 &= (\Delta t)F(v_i + k_3) &&= (\Delta t)(v_i + k_3) \end{aligned}\right\} \tag{6-74}$$

However, we may obtain a more compact set of equations by substituting the expressions for the q's into both the expressions for the k's and the recurrence formula of Eq. 6-73. Performing these substitutions yields

$$\begin{aligned} x_{i+1} &= x_i + (\Delta t)v_i + \frac{\Delta t}{6}(k_1 + k_2 + k_3) \\ v_{i+1} &= v_i + {}^1/_6(k_1 + 2k_2 + 2k_3 + k_4) \end{aligned} \tag{6-75}$$

where

$$\left.\begin{aligned} k_1 &= (\Delta t)f(t_i, x_i, v_i) \\ k_2 &= (\Delta t)f\left[t_i + \frac{(\Delta t)}{2}, x_i + \frac{(\Delta t)}{2}v_i, v_i + \frac{k_1}{2}\right] \\ k_3 &= (\Delta t)f\left[t_i + \frac{(\Delta t)}{2}, x_i + \frac{(\Delta t)}{2}v_i + \frac{(\Delta t)}{4}k_1, v_i + \frac{k_2}{2}\right] \\ k_4 &= (\Delta t)f\left[t_i + \Delta t, x_i + (\Delta t)v_i + \frac{(\Delta t)}{2}k_2, v_i + k_3\right] \end{aligned}\right\} \tag{6-76}$$

Example 6-4

To illustrate the application of Eqs. 6-75 and 6-76 in solving a second-order differential equation, consider the vibrating system shown in Fig. 6-23a. A unit mass is attached to a spring having a spring constant of

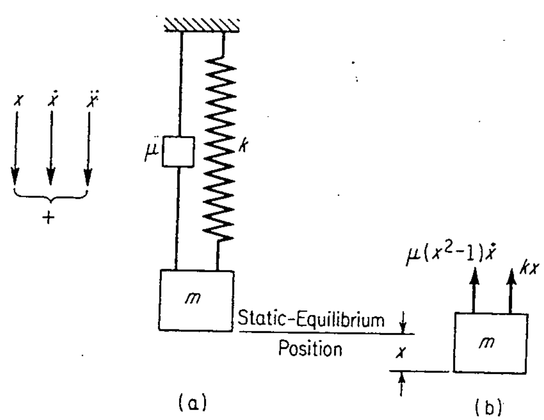

FIG. 6-23. System of Example 6-4.

1 lb/ft, and a damping device that exerts a force on the mass which depends upon both the displacement and the velocity of the mass. The damping force may be expressed mathematically as

$$F_d = \mu(x^2 - 1)\dot{x} \tag{6-77}$$

where μ is a damping coefficient. It is desired to obtain x-t and $\dot{x}$-t curves for studying the displacement and velocity characteristics of the system.

A free-body diagram of the mass is shown in Fig. 6-23b. The gravity force acting on such a system is customarily not shown on the free body, since the amplitude of vibration x is measured from the static-equilibrium position of the mass, in which position the gravity force is equal to the spring force, and these forces cancel from the resulting differential equation.

Noting the positive direction assumed for x, $\dot{x}$, and $\ddot{x}$, as shown in the figure, and the positive displacement used in determining the free-body diagram, the application of Newton's second law yields

$$-kx - \mu(x^2 - 1)\dot{x} = m\ddot{x}$$

or, more simply,

$$\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0 \tag{6-78}$$

since $m = k = 1$.

Equation 6-78 is a classical nonlinear differential equation known as *Van der Pol's* equation. Although the mechanical system from which Eq. 6-78 was derived is a hypothetical one, this equation does actually represent the behavior of a certain type of electronic oscillator.[6]

The reader who is familiar with vibration theory will recognize that if $|x| > 1$, the second term of Eq. 6-78 is a *positive-damping* term which represents a loss of energy from the system. Conversely, if $|x| < 1$ this term is a *negative-damping* term which represents an addition of energy to the system. Thus, if we were initially to displace the mass less than 1 ft and then release it, the resulting vibrations would increase in amplitude until they reached some magnitude exceeding 1 ft, at which time energy would be removed from the system during the portion of each cycle in which $|x| > 1$. The buildup in amplitude would thus be limited, and it would seem reasonable to expect the amplitude of vibration to reach some stable state after some interval of time.

To program Eq. 6-78 for solution on the computer, we note that

$$\left.\begin{aligned} \dot{x} &= \frac{dx}{dt} \\ \ddot{x} &= \frac{d\dot{x}}{dt} \end{aligned}\right\} \tag{6-79}$$

so that

[6]Minorsky, N. *Introduction to Nonlinear Mechanics*. Ann Arbor, Mich.: J. W. Edwards, Publisher, Inc., 1947, p. 293.