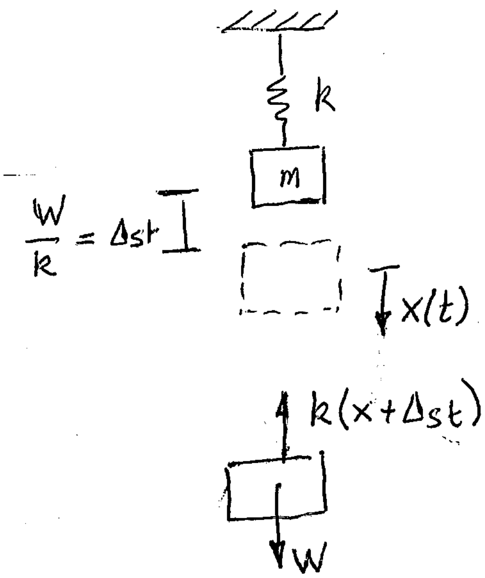
VIBRATION AND TIME RESPONSE

8

8/1 INTRODUCTION

An important and special class of problems in dynamics deals with the linear and angular motions of bodies which oscillate or respond to applied disturbances in the presence of restoring forces. A few examples of this class of dynamics problems are the response of an engineering structure to earthquakes, the vibration of an unbalanced rotating machine, the time response of the plucked string of a musical instrument, the wind-induced vibration of power lines, and the flutter of aircraft wings. In many cases, excessive vibration levels must be reduced to accommodate material limitations or human factors.

In the analysis of every engineering problem, the system under scrutiny must be represented by a physical model. It is often permissible to represent a *continuous* or *distributed-parameter system* (one in which the mass and spring elements are continuously spread over space) by a *discrete* or *lumped-parameter model* (one in which the mass and spring elements are separate and concentrated). Such a modeling scheme is especially desirable when some portions of a continuous system are relatively massive in comparison with other portions. For example, the physical model of a ship propeller shaft is often assumed to be a massless but twistable rod with a disk rigidly attached to each end—one disk representing the turbine and the other representing the propeller. As a second example, we observe that the mass of springs may often be neglected in comparison with that of attached bodies. It should be noted that not every system is reducible to a discrete model. For example, the transverse vibration of a diving board after the departure of the diver is a somewhat difficult problem of distributed-parameter vibration. In this chapter, we shall begin the study of discrete systems, limiting our discussion to those whose configurations may be described with one displacement variable. Such systems are said to possess *one degree of freedom*. For a more detailed study which includes the treatment of two or more degrees of freedom and continuous systems, the student should consult one of the many textbooks devoted solely to the subject of vibrations.



$$+\downarrow \sum F = m\ddot{x}$$

$$W - k(x+\Delta st) = m\ddot{x}$$

$$-kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0$$

$$\ddot{x} + \frac{k}{m}x = 0 \qquad \frac{k}{m} = \omega_n^2 \quad \text{(circular natural frequency)}$$

$$x(t) = A\sin\omega_n t + B\cos\omega_n t$$

$$\dot{x}(t=0) = V_0$$

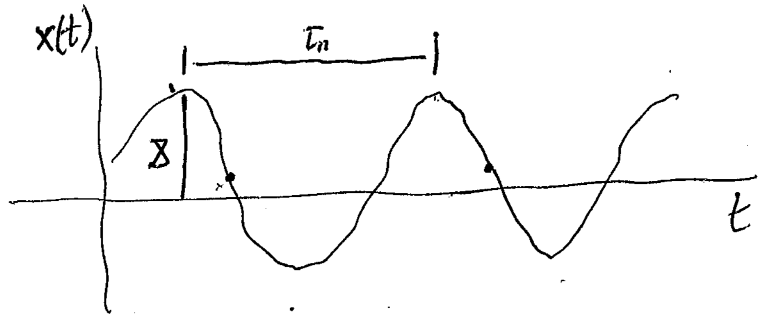
$$x(t=0) = \boxed{x_0 = B}$$

$$\dot{x}(t) = A\omega_n\cos\omega_n t - B\omega_n\sin\omega_n t$$

$$\dot{x}(t=0) = V_0 = A\omega_n \qquad \boxed{A = \frac{V_0}{\omega_n}}$$

$$x(t) = \frac{V_0}{\omega_n}\sin\omega_n t + x_0\cos\omega_n t = \delta\sin(\omega_n t + \phi)$$

$$= \delta\sin\omega_n t\cos\phi + \delta\cos\omega_n t\sin\phi$$



$$\frac{V_0}{\omega_n} = \delta\cos\phi$$

$$x_0 = \delta\sin\phi$$

$$\delta \text{ (amplitude)} = \sqrt{\left(\frac{V_0}{\omega_n}\right)^2 + x_0^2}$$

$$\phi \text{ (lead/lag angle)} = \tan^{-1}\left(\frac{x_0\omega_n}{V_0}\right)$$

ω_n $\frac{\text{rad}}{\text{s}}$

x [m, ft]

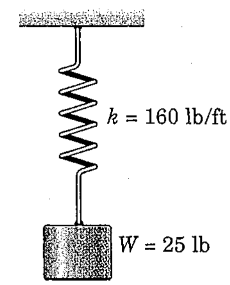
m [kg, slugs = $\frac{\text{lb-sec}^2}{\text{ft}}$]

k [N/m, lb/ft]

Sample Problem 8/1

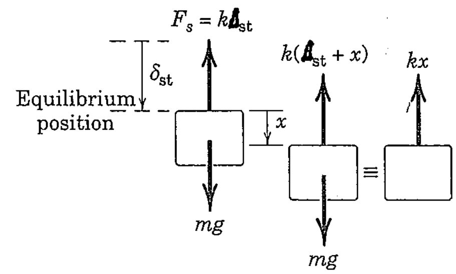
A body weighing 25 lb is suspended from a spring of constant k = 160 lb/ft. At time $t = 0$, it has a downward velocity of 2 ft/sec as it passes through the position of static equilibrium. Determine

(*a*) the static spring deflection δ_{st}
(*b*) the natural frequency of the system in both rad/sec (ω_n) and cycles/sec (f_n)
(*c*) the system period τ
(*d*) the displacement x as a function of time, where x is measured from the position of static equilibrium
(*e*) the maximum velocity v_{max} attained by the mass
(*f*) the maximum acceleration a_{max} attained by the mass.



Solution. **(*a*)** From the spring relationship $F_s = kx$, we see that at equilibrium,

① $$mg = k\delta_{st} \qquad \delta_{st} = \frac{mg}{k} = \frac{25}{160} = 0.1562 \text{ ft or } 1.875 \text{ in.} \quad \textit{Ans.}$$



(*b*) $$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{160}{25/32.2}} = 14.36 \text{ rad/sec} \quad \textit{Ans.}$$

$$f_n = (14.36)\left(\frac{1}{2\pi}\right) = 2.28 \text{ cycles/sec} \quad \textit{Ans.}$$

(*c*) $$\tau = \frac{1}{f_n} = \frac{1}{2.28} = 0.438 \text{ sec} \quad \textit{Ans.}$$

② **(*d*)** From Eq. 8/6:

$$x = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

$$= (0) \cos 14.36t + \frac{2}{14.36} \sin 14.36t$$

$$= 0.1393 \sin 14.36t \quad \textit{Ans.}$$

As an exercise, let us determine x from the alternative Eq. (8/7):

$$x = \sqrt{x_0^2 + (\dot{x}_0/\omega_n)^2} \sin\left[\omega_n t + \tan^{-1}\left(\frac{x_0\omega_n}{\dot{x}_0}\right)\right] = A\sin(\omega_n t + \phi)$$

$$= \sqrt{0^2 + \left(\frac{2}{14.36}\right)^2} \sin\left\{14.36t + \tan^{-1}\left[\frac{(0)(14.36)}{2}\right]\right\}$$

$$= 0.1393 \sin 14.36t$$

(*e*) The velocity is $\dot{x} = 14.36(0.1393) \cos 14.36t = 2 \cos 14.36t$. Since the cosine function cannot be greater than 1 or less than -1, the maximum velocity v_{max} is 2 ft/sec, which, in this case, is the initial velocity. *Ans.*

(*f*) The acceleration is

$$\ddot{x} = -14.36(2) \sin 14.36t = -28.7 \sin 14.36t$$

The maximum acceleration a_{max} is 28.7 ft/sec^2. *Ans.*

① The student should always exercise extreme caution in the matter of units. In the subject of vibrations, it is quite easy to commit errors due to mixing of feet and inches, cycles and radians, and other pairs that frequently enter the calculations.

② Recall that when we refer the motion to the position of static equilibrium, the equation of motion, and therefore its solution, for the present system is identical to that for the horizontally vibrating system.

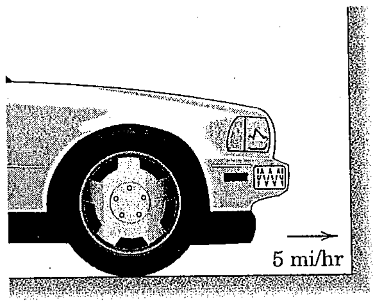
$$\phi = \tan^{-1}\left(\frac{x_0\omega_n}{V_0}\right)$$

$$A = \sqrt{x_0^2 + (V_0/\omega_n)^2}$$

$$V_{max} = A\,\omega_n$$

$$a_{max} = V_{max}\omega_n = A\,\omega_n^2$$

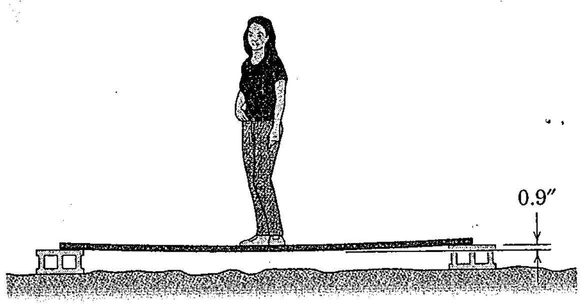
8/16 An energy-absorbing car bumper with its springs initially undeformed has an equivalent spring constant of 3000 lb/in. If the 2500-lb car approaches a massive wall with a speed of 5 mi/hr, determine (*a*) the velocity v of the car as a function of time during contact with the wall, where $t = 0$ is the beginning of the impact, and (*b*) the maximum deflection x_{max} of the bumper.



Problem 8/16

8/17 A 120-lb woman stands in the center of an end-supported board and causes a midspan deflection of 0.9 in. If she flexes her knees slightly in order to cause a vertical vibration, what is the frequency f_n of the motion? Assume elastic response of the board and neglect its relatively small mass.

Ans. f_n = [redacted] Hz

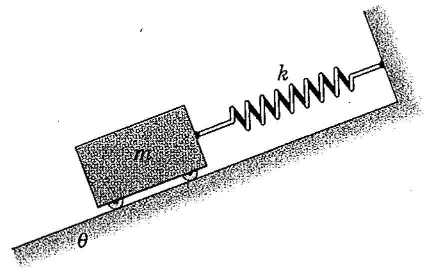


Problem 8/17

$\Delta_{st} = 0.9 \text{ in} = 3/40 \text{ ft}$ $\omega_n = \sqrt{g/\Delta_{st}} =$ [redacted] 20.72 $\frac{rad}{s}$

$f_n = \frac{\omega_n}{2\pi} =$ [redacted] 3.29 Hz

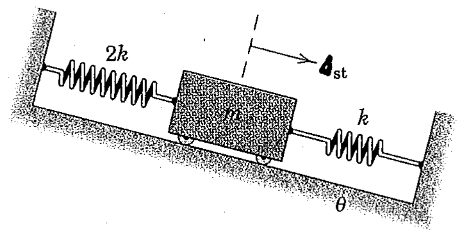
8/18 Prove that the natural frequency f_n of oscillation for the mass m is independent of θ.



Problem 8/18

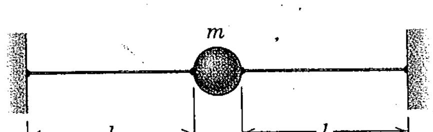
8/19 If both springs are unstretched when the mass is in the central position shown, determine the static deflection Δ_{st} of the mass. What is the period of oscillatory motion about the position of static equilibrium?

Ans. $\Delta_{st} = \dfrac{mg \sin\theta}{3k}$, $\tau_n = 2\pi\sqrt{\dfrac{m}{3k}}$



Problem 8/19

8/20 A small particle of mass m is attached to two highly tensioned wires as shown. Determine the system natural frequency ω_n for small vertical oscillations if the tension T in both wires is assumed to be constant. Is the calculation of the small static deflection of the particle necessary?

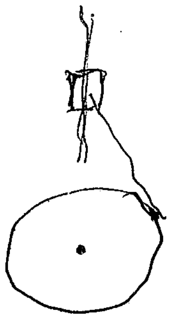


Problem 8/20

natural frequency $f_n = \frac{1}{\tau_n}$ $\quad \frac{\text{cycles}}{\text{sec}}$, Hz

$$\omega_n \tau_n = 2\pi \Rightarrow \frac{1}{\tau_n} = \frac{\omega_n}{2\pi} = f_n \Rightarrow \omega_n = 2\pi f_n$$

$$\left[\frac{rad}{s}\right] \quad \left[\frac{cycles}{sec}\right]$$



$$rpm \times \frac{2\pi}{60} \rightarrow \frac{rad}{sec}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{mg}} = \sqrt{g/\Delta_{st}}$$

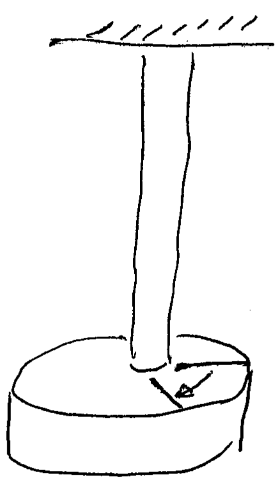
g [m/s^2, ft/s^2] — 9.81, 32.2

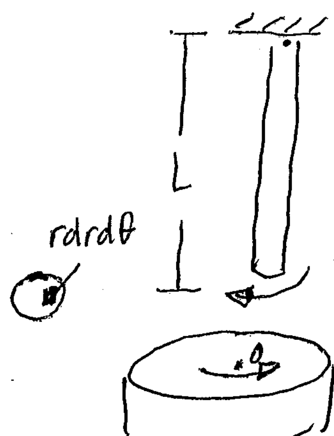
Δ_{st} [m, ft]

$$x(t) = X \sin(\omega_n t + \phi)$$

$$\dot{x}(t) = X\omega_n \cos(\omega_n t + \phi) \qquad v_{max} = X\omega_n$$

$$\ddot{x}(t) = -X\omega_n^2 \sin(\omega_n t + \phi) \qquad a_{max} = X\omega_n^2$$





$$k_t = \frac{JG}{L}$$

$$J = \int r^2 dA$$

$$I_o \ddot{\theta} = \sum T_o = -k_t \theta$$

$$I_o \ddot{\theta} + k_t \theta = 0$$

$$m\ddot{x} + kx = 0$$

$I_o \rightarrow m$

$k_t \rightarrow k$

$\theta \rightarrow x$

$$\omega_n = \sqrt{\frac{k_t}{I_o}}$$

$$\theta(t) = A \sin \omega_n t + B \cos \omega_n t$$

$\theta(t=0) = \theta_0$

$\dot{\theta}(t=0) = \Omega_0$

$$\theta(t) = \frac{\Omega_0}{\omega_n} \sin \omega_n t + \theta_0 \cos \omega_n t = \Theta_0 \sin(\omega_n t + \phi)$$

$$\Theta_0 = \sqrt{\left(\frac{\Omega_0}{\omega_n}\right)^2 + \theta_0^2} \qquad \tan^{-1}\left(\frac{\theta_0 \omega_n}{\Omega_0}\right) = \phi$$

Replacing $\sin\theta$ with θ, which is valid for small oscillations, the equation of motion is

$$\ddot{\theta}+\frac{3}{2}\frac{g}{l}\theta=0$$

This is again similar to equation 2.2, with θ replacing x and $\frac{3g}{2l}$ replacing k/m.

The natural circular frequency is

$$\omega_n=\sqrt{\frac{3}{2}\frac{g}{l}}$$

and the natural frequency, measured in cycles per second, is

$$f_n=\frac{1}{2\pi}\sqrt{\frac{3}{2}\frac{g}{l}}$$

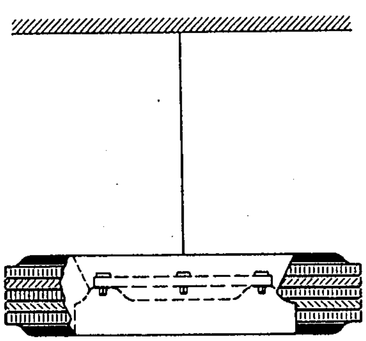
The radius of gyration is

$$I_0=mk_0^2=\frac{1}{3}ml^2$$

$$k_0=\frac{l}{\sqrt{3}}$$

and the distance to the center of percussion is

$$q_0=\frac{k_0^2}{r}=\frac{2}{3}l$$



PROBLEM 2.30 A device designed to determine the moment of inertia of a wheel-tire assembly consists of 2-mm steel suspension wire, 2 m long, and a mounting plate, to which is attached the wheel-tire assembly. The suspension wire is fixed at its upper end and hung vertically. When the system oscillates as a torsional pendulum, the period of oscillation without the wheel-tire assembly is 4 s. With the wheel-tire mounted to the mounting plate, the period of oscillation is 25 s. Determine the moment of inertia of wheel-tire assembly.

Answer: [redacted] kg·m²