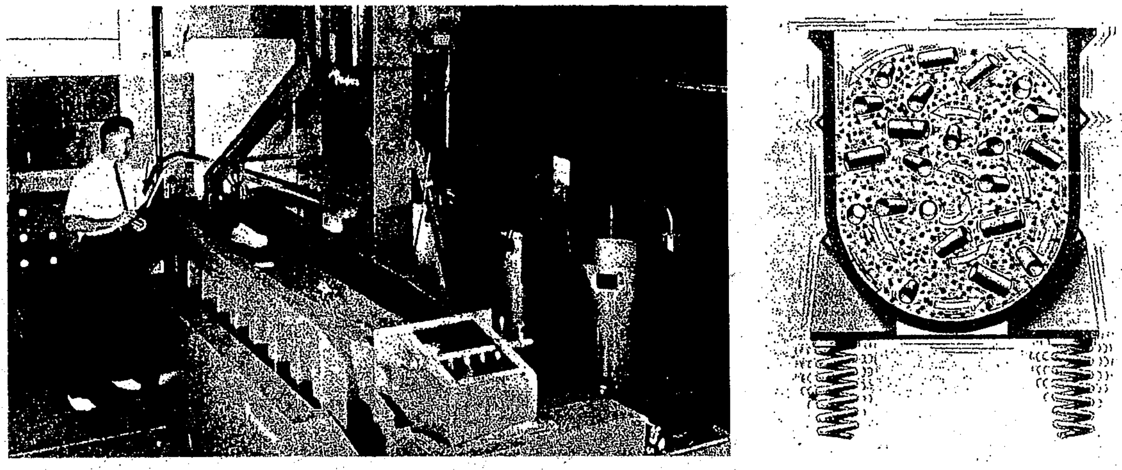


FIGURE 1.10 Vibratory finishing process. (Reprinted courtesy of the Society of Manufacturing Engineers, © 1964 The Tool and Manufacturing Engineer.)

have increased considerably in recent years [1.21]. For example, vibration is put to work in vibratory conveyors, hoppers, sieves, compactors, washing machines, electric toothbrushes, dentist's drills, clocks, and electric massaging units. Vibration is also used in pile driving, vibratory testing of materials, vibratory finishing processes, and electronic circuits to filter out the unwanted frequencies (see Fig. 1.10). Vibration has been found to improve the efficiency of certain machining, casting, forging, and welding processes. It is employed to simulate earthquakes for geological research and also to conduct studies in the design of nuclear reactors.

1.4 Basic Concepts of Vibration

1.4.1 Vibration

Any motion that repeats itself after an interval of time is called *vibration* or *oscillation*. The swinging of a pendulum and the motion of a plucked string are typical examples of vibration. The theory of vibration deals with the study of oscillatory motions of bodies and the forces associated with them.

1.4.2 Elementary Parts of Vibrating Systems

A vibratory system, in general, includes a means for storing potential energy (spring or elasticity), a means for storing kinetic energy (mass or inertia), and a means by which energy is gradually lost (damper).

The vibration of a system involves the transfer of its potential energy to kinetic energy and kinetic energy to potential energy, alternately. If the system is damped, some energy is dissipated in each cycle of vibration and must be replaced by an external source if a state of steady vibration is to be maintained.

As an example, consider the vibration of the simple pendulum shown in Fig. 1.11. Let the bob of mass m be released after giving it an angular displacement θ.

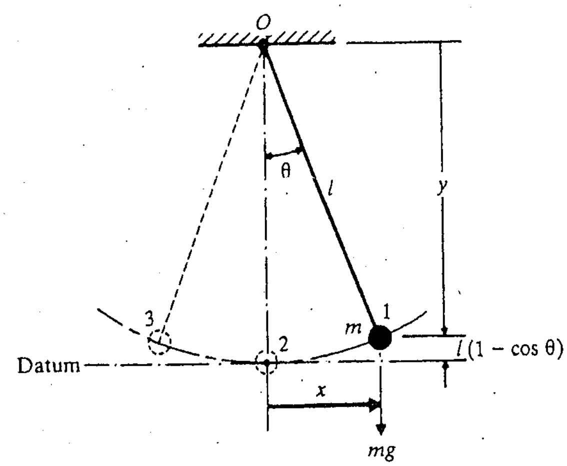


FIGURE 1.11 A simple pendulum.

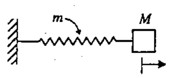
At position 1 the velocity of the bob and hence its kinetic energy is zero. But it has a potential energy of magnitude $mgl(1 - \cos\theta)$ with respect to the datum position 2. Since the gravitational force mg induces a torque $mgl \sin\theta$ about the point O, the bob starts swinging to the left from position 1. This gives the bob certain angular acceleration in the clockwise direction, and by the time it reaches position 2, all of its potential energy will be converted into kinetic energy. Hence the bob will not stop in position 2, but will continue to swing to position 3. However, as it passes the mean position 2, a counterclockwise torque starts acting on the bob due to gravity and causes the bob to decelerate. The velocity of the bob reduces to zero at the left extreme position. By this time, all the kinetic energy of the bob will be converted to potential energy. Again due to the gravity torque, the bob continues to attain a counterclockwise velocity. Hence the bob starts swinging back with progressively increasing velocity and passes the mean position again. This process keeps repeating, and the pendulum will have oscillatory motion. However, in practice, the magnitude of oscillation (θ) gradually decreases and the pendulum ultimately stops due to the resistance (damping) offered by the surrounding medium (air). This means that some energy is dissipated in each cycle of vibration due to damping by the air.

1.4.3 Degree of Freedom

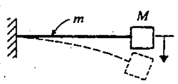
The minimum number of independent coordinates required to determine completely the positions of all parts of a system at any instant of time defines the degree of freedom of the system. The simple pendulum shown in Fig. 1.11, as well as each of the systems shown in Fig. 1.12, represents a single degree of freedom system. For example, the motion of the simple pendulum (Fig. 1.11) can be stated either in terms of the angle θ or in terms of the Cartesian coordinates x and y. If the coordinates x and y are used to describe the motion, it must be recognized that these coordinates are not independent. They are related to each other through the relation $x^2 + y^2 =$

Equivalent Masses, Springs and Dampers

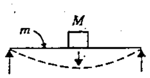
Equivalent masses



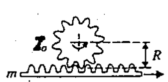
Mass (M) attached at end of spring of mass m — $m_{eq} = M + \frac{m}{3}$



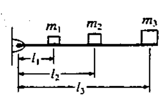
Cantilever beam of mass m carrying an end mass M — $m_{eq} = M + 0.23\,m$



Simply supported beam of mass m carrying a mass M at the middle — $m_{eq} = M + 0.5\,m$

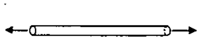


Coupled translational and rotational masses — $m_{eq} = m + \frac{J_0}{R^2}$; $J_{eq} = J_0 + mR^2$

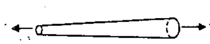


Masses on a hinged bar — $m_{eq_1} = m_1 + \left(\frac{l_2}{l_1}\right)^2 m_2 + \left(\frac{l_3}{l_1}\right)^2 m_3$

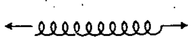
Equivalent springs



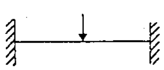
Rod under axial load (l = length, A = cross sectional area) — $k_{eq} = \frac{EA}{l}$



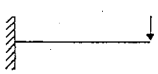
Tapered rod under axial load (D, d = end diameters) — $k_{eq} = \frac{\pi EDd}{4l}$



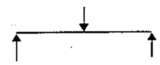
Helical spring under axial load (d = wire diameter, D = mean coil diameter, n = number of active turns) — $k_{eq} = \frac{Gd^4}{8nD^3}$



Fixed-fixed beam with load at the middle — $k_{eq} = \frac{192\,EI}{l^3}$

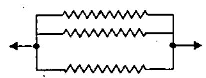


Cantilever beam with end load — $k_{eq} = \frac{3EI}{l^3}$

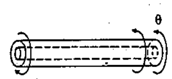


Simply supported beam with load at the middle — $k_{eq} = \frac{48EI}{l^3}$

Springs in series — $\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n}$

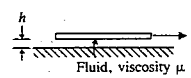


Springs in parallel — $k_{eq} = k_1 + k_2 + \cdots + k_n$

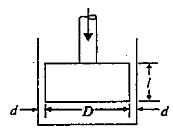


Hollow shaft under torsion (l = length, D = outer diameter, d = inner diameter) — $k_{eq} = \frac{\pi G}{32l}(D^4 - d^4)$

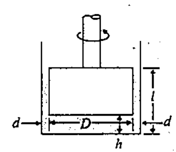
Equivalent viscous dampers



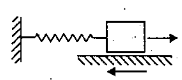
Relative motion between parallel surfaces (A = area of smaller plate) — $c_{eq} = \frac{\mu A}{h}$



Dashpot (axial motion of a piston in a cylinder) — $c_{eq} = \mu \frac{3\pi D^3 l}{4d^3}\left(1 + \frac{2d}{D}\right)$



Torsional damper — $c_{eq} = \frac{\pi\mu D^2(l-h)}{2d} + \frac{\pi\mu D^3}{32h}$



Dry friction (Coulomb damping) (fN = friction force, ω = frequency, X = amplitude of vibration) — $c_{eq} = \frac{4fN}{\pi\omega X}$

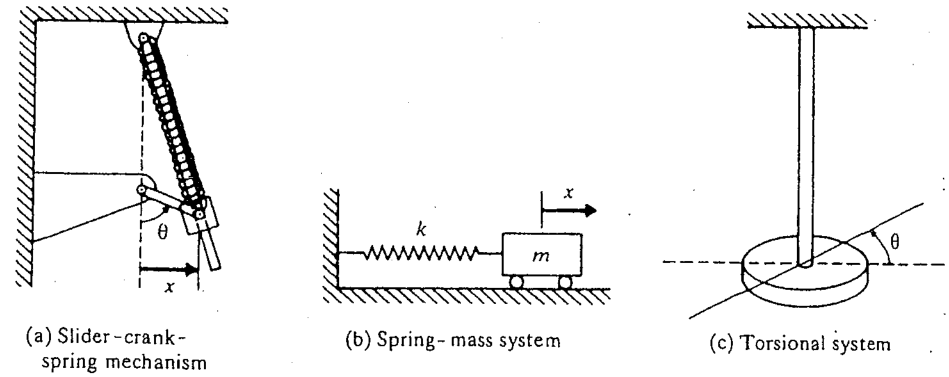


FIGURE 1.12 Single degree of freedom systems.

l^2, where l is the constant length of the pendulum. Thus any one coordinate can describe the motion of the pendulum. In this example, we find that the choice of θ as the independent coordinate will be more convenient than the choice of x or y. For the slider shown in Fig. 1.12(a), either the angular coordinate θ or the coordinate x can be used to describe the motion. In Fig. 1.12(b), the linear coordinate x can be used to specify the motion. For the torsional system (long bar with a heavy disk at the end) shown in Fig. 1.12(c), the angular coordinate θ can be used to describe the motion.

Some examples of two and three degree of freedom systems are shown in Figs. 1.13 and 1.14, respectively. Figure 1.13(a) shows a two mass–two spring system that is described by the two linear coordinates x_1 and x_2. Figure 1.13(b) denotes a two rotor system whose motion can be specified in terms of θ_1 and θ_2. The motion of the system shown in Fig. 1.13(c) can be described completely either by X and θ or by x, y, and X. In the latter case, x and y are constrained as $x^2 + y^2 = l^2$ where l is a constant.

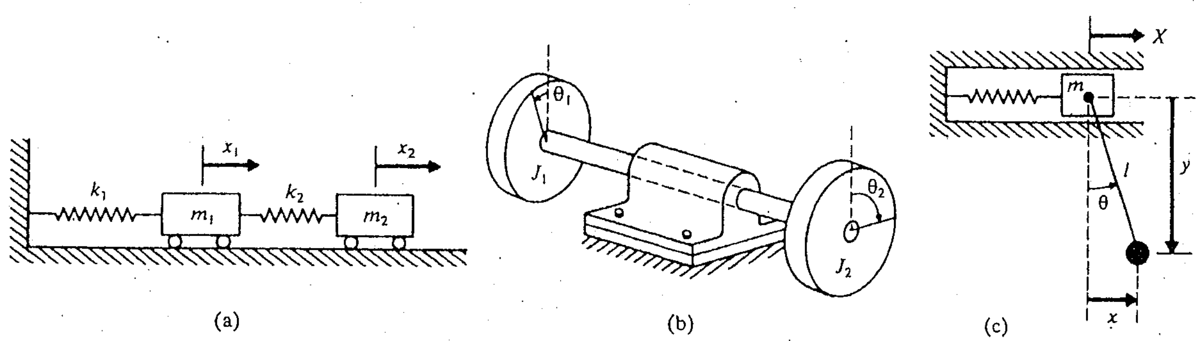


FIGURE 1.13 Two degree of freedom systems.

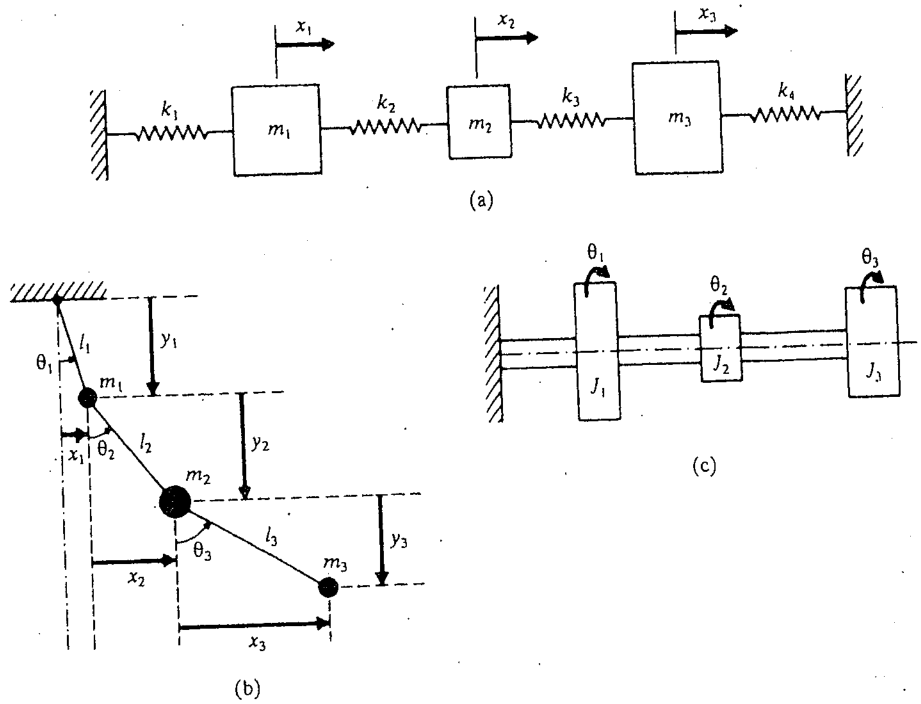


FIGURE 1.14 Three degree of freedom systems.

For the systems shown in Figs. 1.14(a) and 1.14(c), the coordinates x_i ($i = 1, 2, 3$) and θ_i ($i = 1, 2, 3$) can be used, respectively, to describe the motion. In the case of the system shown in Fig. 1.14(b), θ_i ($i = 1, 2, 3$) specifies the positions of the masses m_i ($i = 1, 2, 3$). An alternate method of describing this system is in terms of x_i and y_i ($i = 1, 2, 3$); but in this case the constraints $x_i^2 + y_i^2 = l_i^2$ ($i = 1, 2, 3$) have to be considered.

The coordinates necessary to describe the motion of a system constitute a set of *generalized coordinates*. The generalized coordinates are usually denoted as q_1, q_2, . . . and may represent Cartesian and/or non-Cartesian coordinates.

1.4.4 Discrete and Continuous Systems

A large number of practical systems can be described using a finite number of degrees of freedom, such as the simple systems shown in Figs. 1.11 to 1.14. Some systems, especially those involving continuous elastic members, have an infinite number of degrees of freedom. As a simple example, consider the cantilever beam shown in Fig. 1.15. Since the beam has an infinite number of mass points, we need an infinite number of coordinates to specify its deflected configuration. The infinite number of coordinates defines its elastic deflection curve. Thus the cantilever beam has an infinite number of degrees of freedom. Most structural and machine systems have deformable (elastic) members and therefore have an infinite number of degrees of freedom.

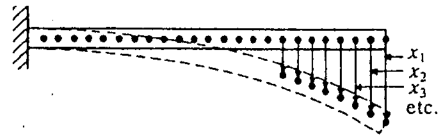


FIGURE 1.15 A cantilever beam (an infinite number of degrees of freedom system).

Systems with a finite number of degrees of freedom are called *discrete* or *lumped parameter* systems, and those with an infinite number of degrees of freedom are called *continuous* or *distributed* systems.

Most of the time, continuous systems are approximated as discrete systems, and solutions are obtained in a simpler manner. Although treatment of a system as continuous gives exact results, the analysis methods available for dealing with continuous systems are limited to a narrow selection of problems, such as uniform beams, slender rods, and thin plates. Hence most of the practical systems are studied by treating them as finite lumped masses, springs, and dampers. In general, more accurate results are obtained by increasing the number of masses, springs, and dampers—that is, by increasing the number of degrees of freedom.

1.5 Classification of Vibration

Vibration can be classified in several ways. Some of the important classifications are as follows.

1.5.1 Free and Forced Vibration

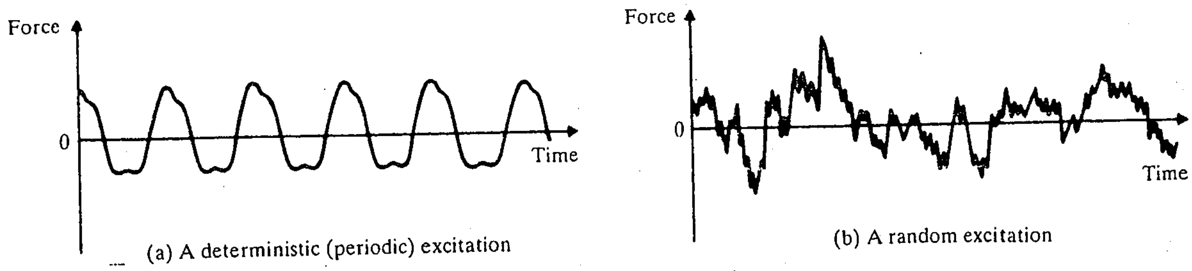
Free Vibration. If a system, after an initial disturbance, is left to vibrate on its own, the ensuing vibration is known as free vibration. No external force acts on the system. The oscillation of a simple pendulum is an example of free vibration.

Forced Vibration. If a system is subjected to an external force (often, a repeating type of force), the resulting vibration is known as *forced vibration*. The oscillation that arises in machines such as diesel engines is an example of forced vibration.

If the frequency of the external force coincides with one of the natural frequencies of the system, a condition known as *resonance* occurs, and the system undergoes dangerously large oscillations. Failures of such structures as buildings, bridges, turbines, and airplane wings have been associated with the occurrence of resonance.

1.5.2 Undamped and Damped Vibration

If no energy is lost or dissipated in friction or other resistance during oscillation, the vibration is known as *undamped vibration*. If any energy is lost in this way, on the other hand, it is called *damped vibration*. In many physical systems, the amount of damping is so small that it can be disregarded for most engineering purposes. However, consideration of damping becomes extremely important in analyzing vibratory systems near resonance.



(a) A deterministic (periodic) excitation

(b) A random excitation

FIGURE 1.16

1.5.3 Linear and Nonlinear Vibration

If all the basic components of a vibratory system—the spring, the mass, and the damper—behave linearly, the resulting vibration is known as *linear vibration*. On the other hand, if any of the basic components behave nonlinearly, the vibration is called *nonlinear vibration*. The differential equations that govern the behavior of linear and nonlinear vibratory systems are linear and nonlinear, respectively. If the vibration is linear, the principle of superposition holds, and the mathematical techniques of analysis are well developed. For nonlinear vibration, the superposition principle is not valid, and techniques of analysis are less well known. Since all vibratory systems tend to behave nonlinearly with increasing amplitude of oscillation, a knowledge of nonlinear vibration is desirable in dealing with practical vibratory systems.

1.5.4 Deterministic and Random Vibration

If the value or magnitude of the excitation (force or motion) acting on a vibratory system is known at any given time, the excitation is called *deterministic*. The resulting vibration is known as *deterministic vibration*.

In some cases, the excitation is *nondeterministic* or *random*; the value of the excitation at a given time cannot be predicted. In these cases, a large collection of records of the excitation may exhibit some statistical regularity. It is possible to estimate averages such as the mean and mean square values of the excitation. Examples of random excitations are wind velocity, road roughness, and ground motion during earthquakes. If the excitation is random, the resulting vibration is called *random vibration*. In the case of random vibration, the vibratory response of the system is also random; it can be described only in terms of statistical quantities. Figure 1.16 shows examples of deterministic and random excitations.

1.6 Vibration Analysis Procedure

A vibratory system is a dynamic system for which the variables such as the excitations (inputs) and responses (outputs) are time-dependent. The response of a vibrating system generally depends on the initial conditions as well as the external excitations. Most practical vibrating systems are very complex, and it is impossible to consider all the details for a mathematical analysis. Only the most important features are considered in the analysis to predict the behavior of the system under specified input conditions. Often, the overall behavior of the system can be determined by considering even a simple model of the complex physical system. Thus the analysis of a vibrating system usually involves mathematical modeling, derivation of the governing equations, solution of the equations, and interpretation of the results.

Step 1: Mathematical Modeling. The purpose of mathematical modeling is to represent all the important features of the system for the purpose of deriving the mathematical (or analytical) equations governing the behavior of the system. The mathematical model should include enough details to be able to describe the system in terms of equations without making it too complex. The mathematical model may be linear or nonlinear, depending on the behavior of the components of the system. Linear models permit quick solutions and are simple to handle; however, nonlinear models sometimes reveal certain characteristics of the system that cannot be predicted using linear models. Thus a great deal of engineering judgment is needed to come up with a suitable mathematical model of a vibrating system.

Sometimes the mathematical model is gradually improved to obtain more accurate results. In this approach, first a very crude or elementary model is used to get a quick insight into the overall behavior of the system. Subsequently, the model is refined by including more components and/or details so that the behavior of the system can be observed more closely. To illustrate the procedure of refinement used in mathematical modeling, consider the forging hammer shown in Fig. 1.17(a). The forging hammer consists of a frame, a falling weight known as the tup, an anvil, and a foundation block. The anvil is a massive steel block on which material is forged into desired shape by the repeated blows of the tup. The anvil is usually mounted on an elastic pad to reduce the transmission of vibration to the foundation block and the frame [1.22]. For a first approximation, the frame, anvil, elastic pad, foundation block, and soil are modeled as a single degree of freedom system as shown in Fig. 1.17(b). For a refined approximation, the weights of the frame and anvil and the foundation block are represented separately with a two degree of freedom model as shown in Fig. 1.17(c). Further refinement of the model can be made by considering eccentric impacts of the tup, which cause each of the masses shown in Fig. 1.17(c) to have both vertical and rocking (rotation) motions in the plane of the paper.

Step 2: Derivation of Governing Equations. Once the mathematical model is available, we use the principles of dynamics and derive the equations that describe the vibration of the system. The equations of motion can be derived conveniently by drawing the free-body diagrams of all the masses involved. The free-body diagram of a mass can be obtained by isolating the mass and indicating all externally applied forces, the reactive forces, and the inertia forces. The equations of motion of a vibrating system are usually in the form of a set of ordinary differential equations for a discrete system and partial differential equations for a continuous system. The equations may be linear or nonlinear depending on the behavior of the components of the system. Several approaches are commonly used to derive the governing equations. Among them are Newton's second law of motion, d'Alembert's principle, and the principle of conservation of energy.

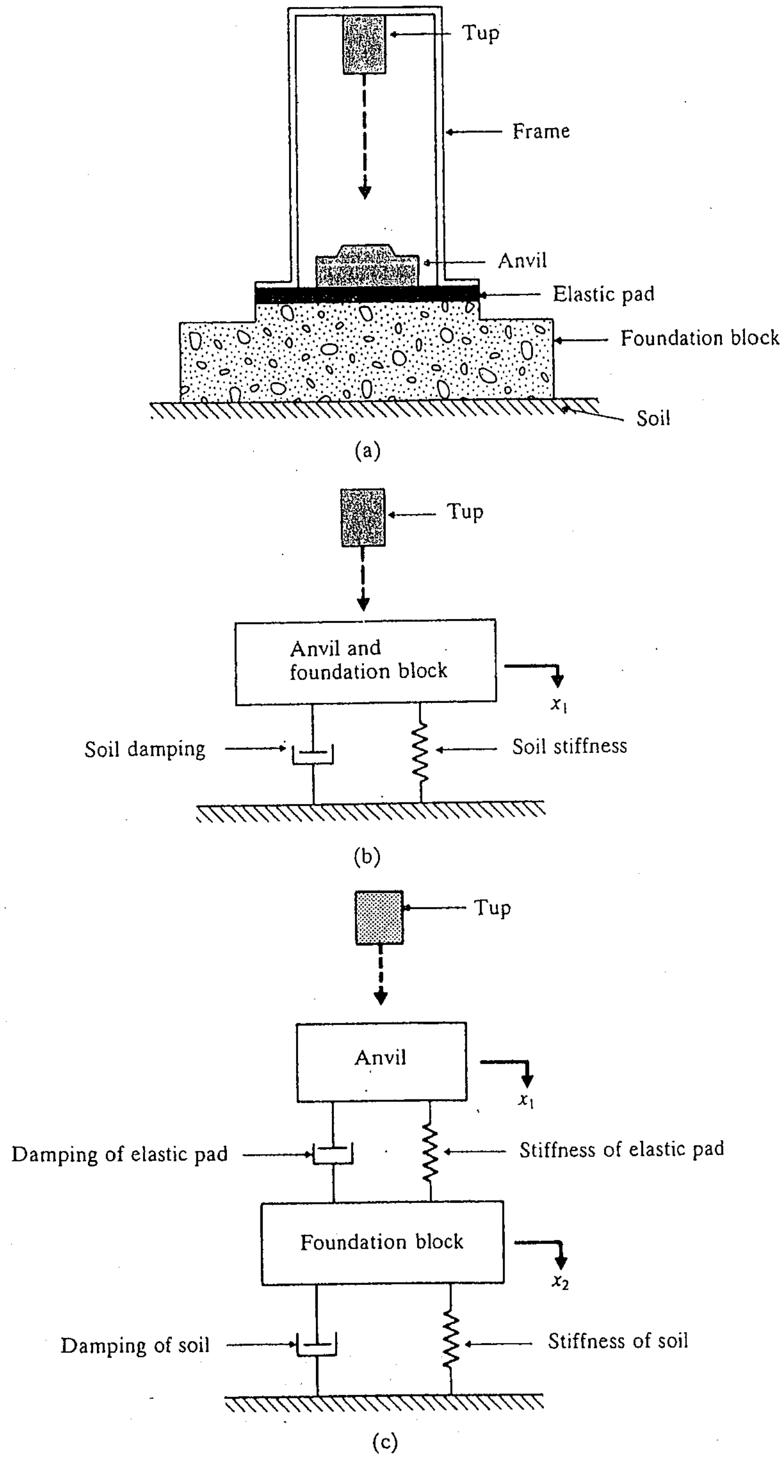


FIGURE 1.17 Modeling of a forging hammer.

Step 3: Solution of the Governing Equations. The equations of motion must be solved to find the response of the vibrating system. Depending on the nature of the problem, we can use one of the following techniques for finding the solution: standard methods of solving differential equations, Laplace transformation methods, matrix methods,[1] and numerical methods. If the governing equations are nonlinear, they can seldom be solved in closed form. Furthermore, the solution of partial differential equations is far more involved than that of ordinary differential equations. Numerical methods involving computers can be used to solve the equations. However, it will be difficult to draw general conclusions about the behavior of the system using computer results.

Step 4: Interpretation of the Results. The solution of the governing equations gives the displacements, velocities, and accelerations of the various masses of the system. These results must be interpreted with a clear view of the purpose of the analysis and the possible design implications of the results.

EXAMPLE 1.1 Mathematical Model of a Motorcycle

Figure 1.18(a) shows a motorcycle with a rider. Develop a sequence of three mathematical models of the system for investigating vibration in the vertical direction. Consider the elasticity of the tires, elasticity and damping of the struts (in the vertical direction), masses of the wheels, and elasticity, damping, and mass of the rider.

Given: Spring constants, damping constants, and masses of the various parts of the motorcycle and the rider.

Find: A sequence of three mathematical models.

Approach: Start with the simplest model and refine it gradually.

Solution: By using the equivalent values of the mass, stiffness, and damping of the system, a single degree of freedom model of the motorcycle with the rider can be obtained as indicated in Fig. 1.18(b). In this model, the equivalent stiffness (k_{eq}) includes the stiffnesses of the tires, struts, and rider. The equivalent damping constant (c_{eq}) includes the damping of the struts and the rider. The equivalent mass includes the masses of the wheels, vehicle body, and the rider. This model can be refined by representing the masses of wheels, elasticity of the tires, and elasticity and damping of the struts separately, as shown in Fig. 1.18(c). In this model, the mass of the vehicle body (m_v) and the mass of the rider (m_r) are shown as a single mass, $m_v + m_r$. When the elasticity (as spring constant, k_r) and damping (as damping constant, c_r) of the rider are considered, the refined model shown in Fig. 1.18(d) can be obtained.

Note that the models shown in Figs. 1.18(b) to (d) are not unique. For example, by combining the spring constants of both tires, the masses of both wheels, and the spring and damping constants of both struts as single quantities, the model shown in Fig. 1.18(e) can be obtained instead of Fig. 1.18(c).

[1] The basic definitions and operations of matrix theory are given in Appendix A.

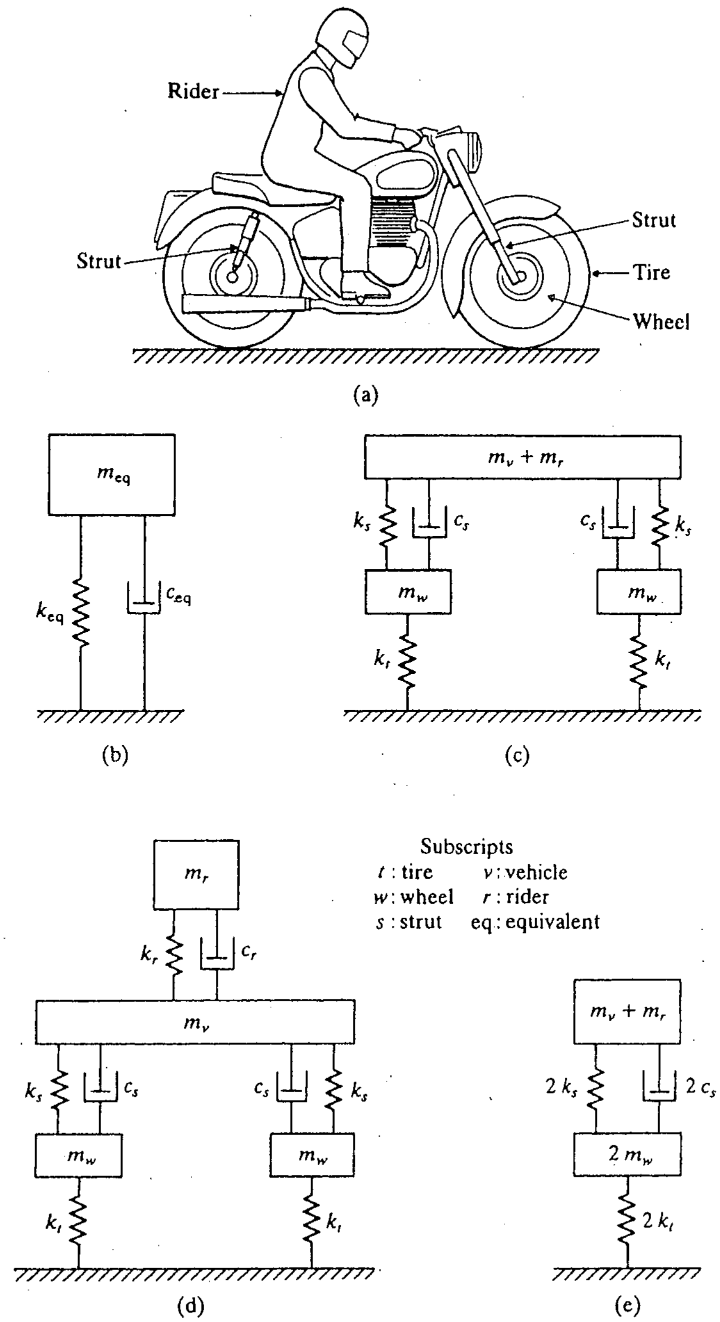


FIGURE 1.18 Motorcycle with a rider—a physical system and mathematical model.

1.7 Spring Elements

A linear spring is a type of mechanical link that is generally assumed to have negligible mass and damping. A force is developed in the spring whenever there is relative motion between the two ends of the spring. The spring force is proportional to the amount of deformation and is given by

$$F = kx \tag{1.1}$$

where F is the spring force, x is the deformation (displacement of one end with respect to the other), and k is the *spring stiffness* or *spring constant.* If we plot a graph between F and x, the result is a straight line according to Eq. (1.1). The work done (U) in deforming a spring is stored as strain or potential energy in the spring, and it is given by

$$U = \frac{1}{2}kx^2 \tag{1.2}$$

Actual springs are nonlinear and follow Eq. (1.1) only up to a certain deformation. Beyond a certain value of deformation (after point A in Fig. 1.19), the stress exceeds the yield point of the material and the force-deformation relation becomes nonlinear [1.23, 1.24]. In many practical applications we assume that the deflections are small and make use of the linear relation in Eq. (1.1). Even if the force-deflection relation of a spring is nonlinear, as shown in Fig. 1.20, we often approximate it as a linear one by using a linearization process [1.24, 1.25]. To illustrate the linearization process, let the static equilibrium load F acting on the spring cause a deflection of x^*. If an incremental force ΔF is added to F, the spring deflects by an additional quantity Δx. The new spring force $F + \Delta F$ can be expressed using Taylor's series expansion about the static equilibrium position x^* as

$$\begin{aligned} F + \Delta F &= F(x^* + \Delta x) \\ &= F(x^*) + \left.\frac{dF}{dx}\right|_{x^*} (\Delta x) + \frac{1}{2!}\left.\frac{d^2F}{dx^2}\right|_{x^*} (\Delta x)^2 + \ldots \end{aligned} \tag{1.3}$$

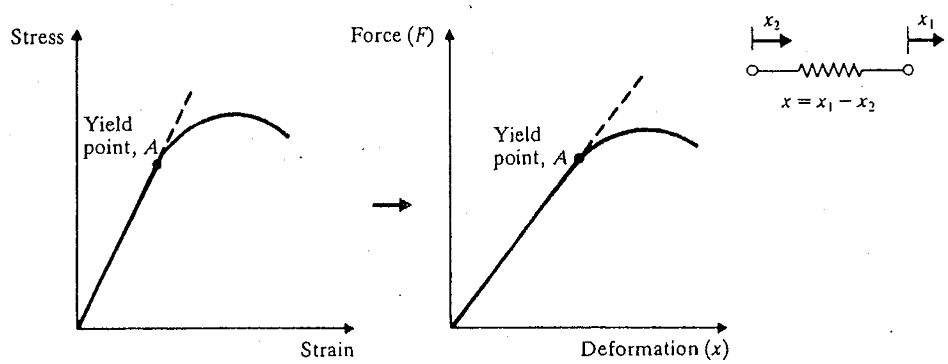


FIGURE 1.19 Nonlinearity beyond proportionality limit.

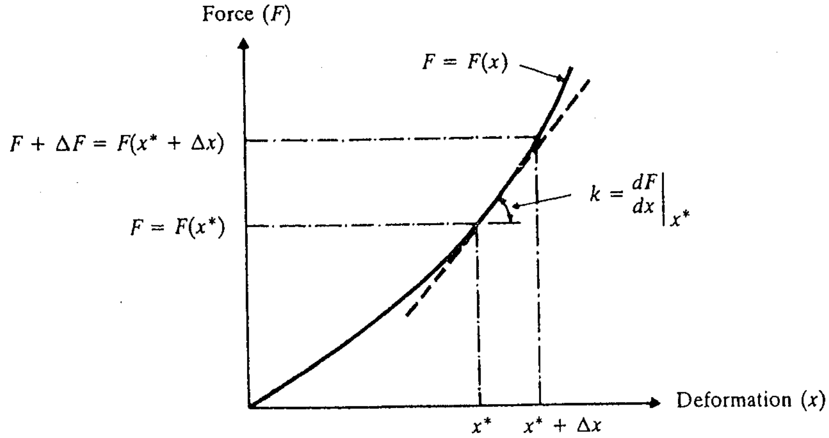


FIGURE 1.20 Linearization process.

For small values of Δx, the higher order derivative terms can be neglected to obtain

$$F + \Delta F = F(x^*) + \left.\frac{dF}{dx}\right|_{x^*} (\Delta x) \tag{1.4}$$

Since $F = F(x^*)$, we can express ΔF as

$$\Delta F = k\,\Delta x \tag{1.5}$$

where k is the linearized spring constant at x^* given by

$$k = \left.\frac{dF}{dx}\right|_{x^*}$$

We may use Eq. (1.5) for simplicity, but sometimes the error involved in the approximation may be very large.

Elastic elements like beams also behave as springs. For example, consider a cantilever beam with an end mass m, as shown in Fig. 1.21. We assume, for simplicity,

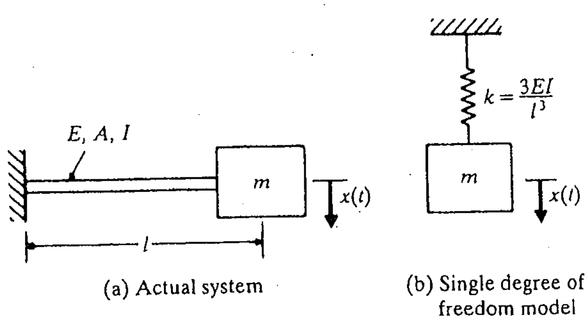


FIGURE 1.21 Cantilever with end mass.

that the mass of the beam is negligible in comparison with the mass m. From strength of materials [1.26], we know that the static deflection of the beam at the free end is given by

$$\delta_{st} = \frac{Wl^3}{3EI} \tag{1.6}$$

where $W = mg$ is the weight of the mass m, E is Young's modulus, and I is the moment of inertia of the cross section of the beam. Hence the spring constant is

$$k = \frac{W}{\delta_{st}} = \frac{3EI}{l^3} \tag{1.7}$$

Similar results can be obtained for beams with different end conditions.

The formulas given in Appendix B can be used to find the spring constants of beams and plates.

1.7.1 Combination of Springs

In many practical applications, several linear springs are used in combination. These springs can be combined into a single equivalent spring as indicated below.

Case 1: Springs in Parallel. To derive an expression for the equivalent spring constant of springs connected in parallel, consider the two springs shown in Fig. 1.22(a). When a load W is applied, the system undergoes a static deflection δ_{st} as shown in Fig. 1.22(b). Then the free body diagram, shown in Fig. 1.22(c), gives the equilibrium equation

$$W = k_1\,\delta_{st} + k_2\,\delta_{st} \tag{1.8}$$

If k_{eq} denotes the equivalent spring constant of the combination of the two springs, then for the same static deflection δ_{st}, we have

$$W = k_{eq}\delta_{st} \tag{1.9}$$

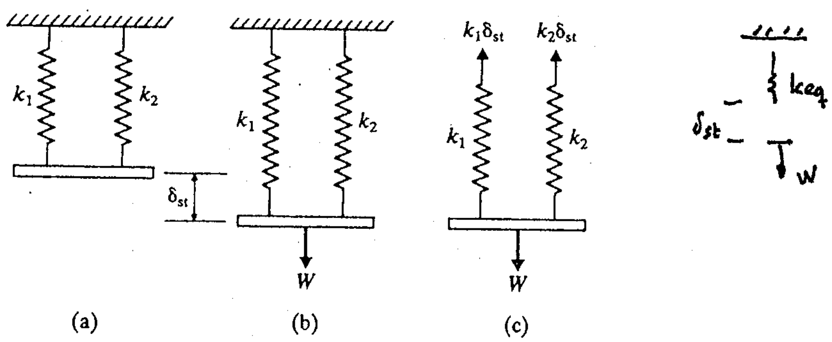


FIGURE 1.22 Springs in parallel.

Equations (1.8) and (1.9) give

$$k_{eq} = k_1 + k_2 \tag{1.10}$$

In general, if we have n springs with spring constants $k_1, k_2, \ldots, k_n$ in parallel, then the equivalent spring constant k_{eq} can be obtained:

$$k_{eq} = k_1 + k_2 + \cdots + k_n \tag{1.11}$$

Case 2: Springs in Series. Next we derive an expression for the equivalent spring constant of springs connected in series by considering the two springs shown in Fig. 1.23(a). Under the action of a load W, springs 1 and 2 undergo elongations δ_1 and δ_2, respectively, as shown in Fig. 1.23(b). The total elongation (or static deflection) of the system, δ_{st}, is given by

$$\delta_{st} = \delta_1 + \delta_2 \tag{1.12}$$

Since both springs are subjected to the same force W, we have the equilibrium shown in Fig. 1.23(c):

$$W = k_1\,\delta_1$$
$$W = k_2\,\delta_2 \tag{1.13}$$

If k_{eq} denotes the equivalent spring constant, then for the same static deflection,

$$W = k_{eq}\delta_{st} \tag{1.14}$$

Equations (1.13) and (1.14) give

$$k_1\delta_1 = k_2\delta_2 = k_{eq}\delta_{st}$$

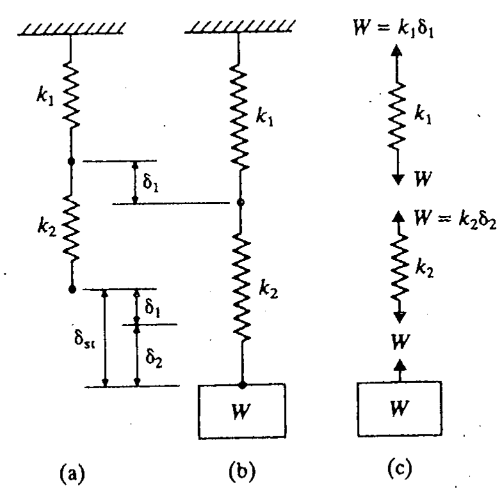


FIGURE 1.23 Springs in series.

or

$$\delta_1 = \frac{k_{eq}\delta_{st}}{k_1} \quad \text{and} \quad \delta_2 = \frac{k_{eq}\delta_{st}}{k_2} \tag{1.15}$$

Substituting these values of δ_1 and δ_2 into Eq. (1.12), we obtain

$$\frac{k_{eq}\delta_{st}}{k_1} + \frac{k_{eq}\delta_{st}}{k_2} = \delta_{st}$$

that is,

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} \tag{1.16}$$

Equation (1.16) can be generalized to the case of n springs in series:

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} \tag{1.17}$$

In certain applications, springs are connected to rigid components such as pulleys, levers, and gears. In such cases, an equivalent spring constant can be found using energy equivalence, as illustrated in Example 1.5.

EXAMPLE 1.2 Equivalent k of a Suspension System

Figure 1.24 shows the suspension system of a freight truck with a parallel-spring arrangement. Find the equivalent spring constant of the suspension if each of the three helical springs is

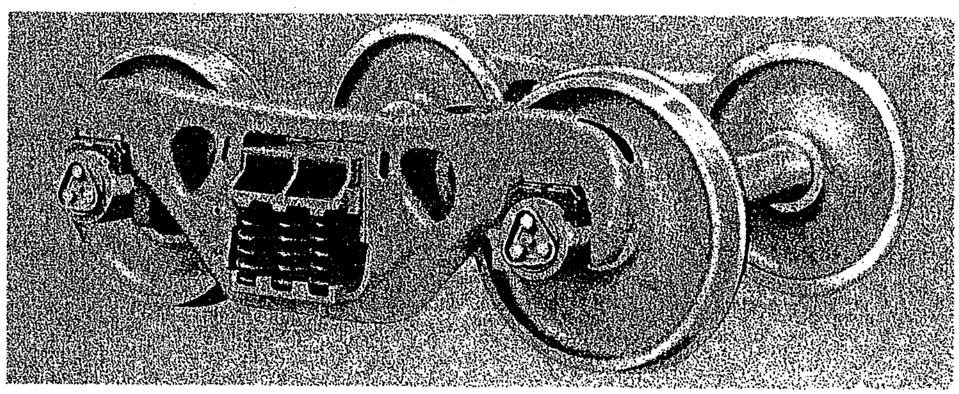


FIGURE 1.24 Parallel arrangement of springs in a freight truck. (Courtesy of Buckeye Steel Castings Company).

made of steel with a shear modulus $G = 80 \times 10^9$ N/m^2, and has five effective turns, mean coil diameter $D = 20$ cm, and wire diameter $d = 2$ cm.

Given: Suspension system with helical springs.

Find: Equivalent spring constant, k_{eq}.

Approach: Use the formula corresponding to springs in parallel.

Solution: The stiffness of each helical spring is given by

$$k = \frac{d^4 G}{8 D^3 n} = \frac{(0.02)^4 (80 \times 10^9)}{8 (0.2)^3 (5)} = 40{,}000.0 \text{ N/m}$$

(See inside front cover for the formula.) Since the springs are identical, the equivalent spring constant of the suspension system is given by

$$k_{eq} = 3k = 3 (40{,}000.0) = 120{,}000.0 \text{ N/m}$$

■

EXAMPLE 1.3 Torsional Spring Constant of a Propeller Shaft

Determine the torsional spring constant of the steel propeller shaft shown in Fig. 1.25.

Given: Geometry and material of a stepped shaft.

Find: Torsional spring constant, k_{eq}.

Approach: Consider the segments 12 and 23 of the shaft as springs in combination.

Solution: From Fig. 1.25, the torque induced at any cross section of the shaft (such as *AA* or *BB*) can be seen to be equal to the torque applied at the propeller, *T*. Hence the elasticities

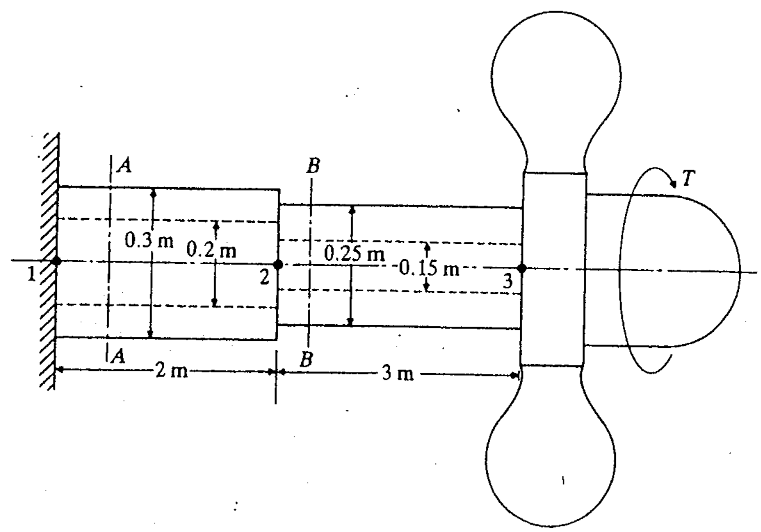


FIGURE 1.25

(springs) corresponding to the two segments 12 and 23 are to be considered as series springs. The spring constants of segments 12 and 23 of the shaft ($k_{t_{12}}$ and $k_{t_{23}}$) are given by

$$k_{t_{12}} = \frac{GJ_{12}}{l_{12}} = \frac{G\pi(D_{12}^4 - d_{12}^4)}{32\, l_{12}} = \frac{(80 \times 10^9)\pi(0.3^4 - 0.2^4)}{32\,(2)}$$
$$= 25.5255 \times 10^6 \text{ N-m/rad}$$

$$k_{t_{23}} = \frac{GJ_{23}}{l_{23}} = \frac{G\pi(D_{23}^4 - d_{23}^4)}{32\, l_{23}} = \frac{(80 \times 10^9)\pi(0.25^4 - 0.15^4)}{32\,(3)}$$
$$= 8.9012 \times 10^6 \text{ N-m/rad}$$

Since the springs are in series, Eq. (1.16) gives

$$k_{t_{eq}} = \frac{k_{t_{12}}\, k_{t_{23}}}{k_{t_{12}} + k_{t_{23}}} = \frac{(25.5255\ 10^6)\,(8.9012 \times 10^6)}{(25.5255 \times 10^6 + 8.9012 \times 10^6)} = 6.5997 \times 10^6 \text{ N-m/rad}$$

■

EXAMPLE 1.4 Equivalent k of Hoisting Drum

A hoisting drum, carrying a steel wire rope, is mounted at the end of a cantilever beam as shown in Fig. 1.26(a). Determine the equivalent spring constant of the system when the suspended length of the wire rope is l. Assume that the net cross-sectional diameter of the wire rope is d and the Young's modulus of the beam and the wire rope is E.

Given: Dimensions of the cantilever beam: length $= b$, width $= a$, and thickness $= t$. Young's modulus of the beam $= E$. Wire rope: length $= l$, diameter $= d$, and Young's modulus $= E$.

Find: Equivalent spring constant of the system.

Approach: Series springs.

Solution: The spring constant of the cantilever beam is given by

$$k_b = \frac{3EI}{b^3} = \frac{3E}{b^3}\left(\frac{1}{12}at^3\right) = \frac{Eat^3}{4b^3} \tag{E.1}$$

The stiffness of the wire rope subjected to axial loading is

$$k_r = \frac{AE}{l} = \frac{\pi d^2 E}{4l} \tag{E.2}$$

Since both the wire rope and the cantilever beam experience the same load W, as shown in Fig. 1.26(b), they can be modeled as springs in series, as shown in Fig. 1.26(c). The equivalent spring constant k_{eq} is given by

$$\frac{1}{k_{eq}} = \frac{1}{k_b} + \frac{1}{k_r} = \frac{4b^3}{Eat^3} + \frac{4l}{\pi d^2 E}$$

or

$$k_{eq} = \frac{E}{4}\left(\frac{\pi a t^3 d^2}{\pi d^2 b^3 + l a t^3}\right) \tag{E.3}$$

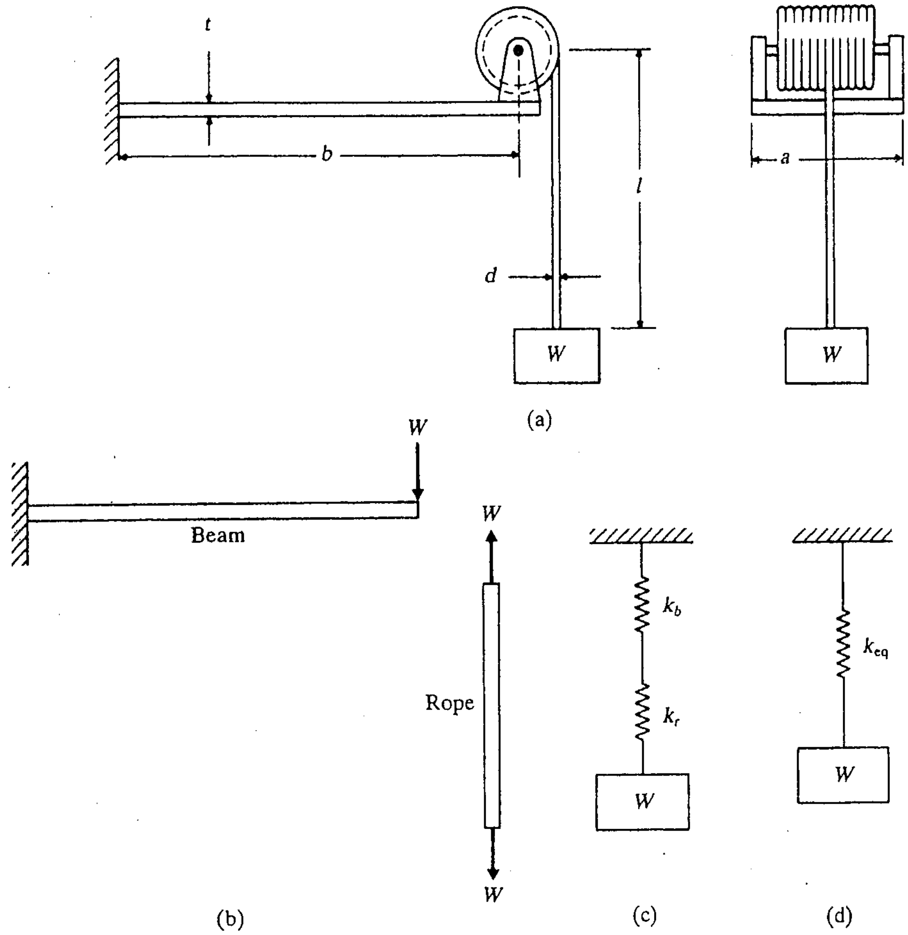


FIGURE 1.26 Hoisting drum.

EXAMPLE 1.5 Equivalent k of a Crane

The boom AB of the crane shown in Fig. 1.27(a) is a uniform steel bar of length 10 m and area of cross section 2500 mm^2. A weight W is suspended while the crane is stationary. The cable $CDEBF$ is made of steel and has a cross-sectional area of 100 mm^2. Neglecting the effect of the cable $CDEB$, find the equivalent spring constant of the system in the vertical direction.

Given: Steel boom: length = 10 m, cross-sectional area = 2500 mm^2, and material = steel. Cable FB: material = steel and cross-sectional area = 100 mm^2. Base: FA = 3 m.

A vertical displacement x of point B will cause the spring k_2 (boom) to deform by an amount $x_2 = x \cos 45°$ and the spring k_1 (cable) to deform by an amount $x_1 = x \cos (90° - \theta)$. The length of the cable FB, l_1, is given by Fig. 1.27(b):

$$l_1^2 = 3^2 + 10^2 - 2(3)(10)\cos 135° = 151.426, \qquad l_1 = 12.3055 \text{ m}$$

The angle θ satisfies the relation

$$l_1^2 + 3^2 - 2(l_1)(3)\cos\theta = 10^2, \quad \cos\theta = 0.8184, \quad \theta = 35.0736°$$

The total potential energy (U) stored in the springs k_1 and k_2 can be expressed, using Eq. (1.2), as

$$U = \tfrac{1}{2}k_1(x\cos 45°)^2 + \tfrac{1}{2}k_2[x\cos(90° - \theta)]^2 \tag{E.1}$$

where

$$k_1 = \frac{A_1E_1}{l_1} = \frac{(100 \times 10^{-6})(207 \times 10^9)}{12.3055} = 1.6822 \times 10^6 \text{ N/m}$$

and

$$k_2 = \frac{A_2E_2}{l_2} = \frac{(2500 \times 10^{-6})(207 \times 10^9)}{10} = 5.1750 \times 10^7 \text{ N/m}$$

Since the equivalent spring in the vertical direction undergoes a deformation x, the potential energy of the equivalent spring (U_{eq}) is given by

$$U_{eq} = \tfrac{1}{2}k_{eq}x^2 \tag{E.2}$$

By setting $U = U_{eq}$, we obtain the equivalent spring constant of the system as

$$k_{eq} = 26.4304 \times 10^6 \text{ N/m}$$

1.8 Mass or Inertia Elements

The mass or inertia element is assumed to be a rigid body; it can gain or lose kinetic energy whenever the velocity of the body changes. From Newton's second law of motion, the product of the mass and its acceleration is equal to the force applied to the mass. Work is equal to the force multiplied by the displacement in the direction of the force and the work done on a mass is stored in the form of kinetic energy of the mass.

In most cases, we must use a mathematical model to represent the actual vibrating system, and there are often several possible models. The purpose of the analysis often determines which mathematical model is appropriate. Once the model is chosen, the mass or inertia elements of the system can be easily identified. For example, consider again the cantilever beam with an end mass shown in Fig. 1.21(a). For a quick and reasonably accurate analysis, the mass and damping of the beam can be disregarded; the system can be modeled as a spring-mass system, as shown in Fig. 1.21(b). The tip mass m represents the mass element, and the elasticity of the beam denotes the stiffness of the spring. Next, consider a multistory building subjected to an earthquake. Assuming that the mass of the frame is negligible compared to the masses of the floors, the building can be modeled as a multidegree of freedom system, as shown in Fig. 1.28. The masses at the various floor levels represent the mass elements, and the elasticities of the vertical members denote the spring elements.

1.8.1 Combination of Masses

In many practical applications, several masses appear in combination. For a simple analysis, we can replace these masses by a single equivalent mass, as indicated below [1.27].

Case 1: Translational Masses Connected by a Rigid Bar. Let the masses be attached to a rigid bar that is pivoted at one end, as shown in Fig. 1.29(a). The equivalent mass can be assumed to be located at any point along the bar. To be specific, we assume the location of the equivalent mass to be that of mass m_1. The velocities of masses m_2 ($\dot{x}_2$) and m_3 ($\dot{x}_3$) can be expressed in terms of the velocity of mass m_1 ($\dot{x}_1$), by assuming small angular displacements for the bar, as

$$\dot{x}_2 = \frac{l_2}{l_1}\dot{x}_1, \quad \dot{x}_3 = \frac{l_3}{l_1}\dot{x}_1 \tag{1.18}$$

and

$$\dot{x}_{eq} = \dot{x}_1 \tag{1.19}$$

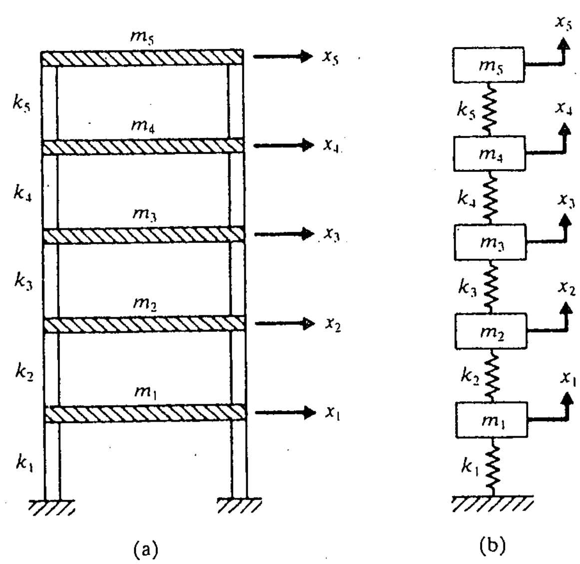


FIGURE 1.28 Idealization of a multistory building as a multidegree of freedom system.

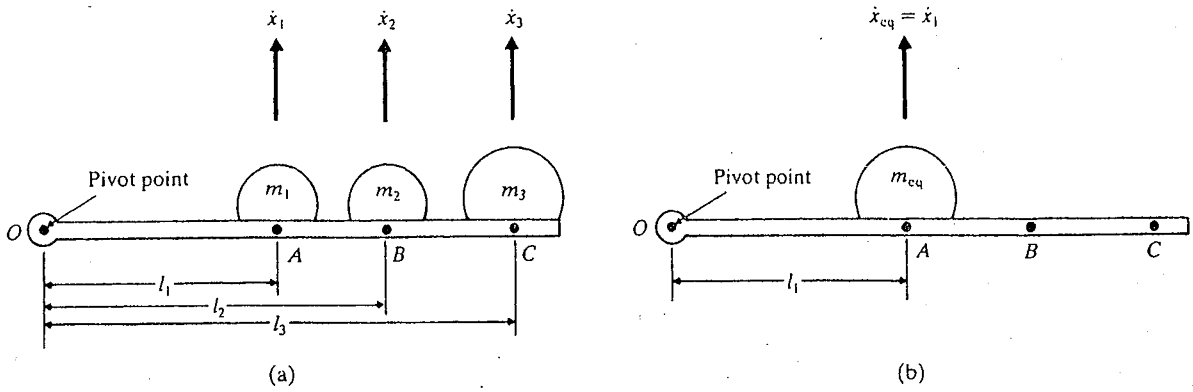


FIGURE 1.29 Translational masses connected by a rigid bar.

By equating the kinetic energy of the three mass system to that of the equivalent mass system, we obtain

$$\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2 = \frac{1}{2}m_{eq}\dot{x}_{eq}^2 \tag{1.20}$$

This equation gives, in view of Eqs. (1.18) and (1.19).

$$m_{eq} = m_1 + \left(\frac{l_2}{l_1}\right)^2 m_2 + \left(\frac{l_3}{l_1}\right)^2 m_3 \tag{1.21}$$

Case 2: Translational and Rotational Masses Coupled Together. Let a mass m having a translational velocity $\dot{x}$ be coupled to another mass (of mass moment of inertia J_0) having a rotational velocity $\dot{\theta}$, as in the rack and pinion arrangement shown in Fig. 1.30. These two masses can be combined to obtain either (1) a single equivalent translational mass m_{eq} or (2) a single equivalent rotational mass J_{eq}, as shown below.

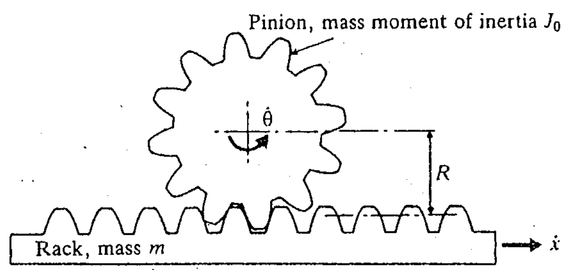


FIGURE 1.30 Translational and rotational masses in a rack and pinion arrangement.

1. *Equivalent translational mass.* The kinetic energy of the two masses is given by

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2 \tag{1.22}$$

and the kinetic energy of the equivalent mass can be expressed as

$$T_{eq} = \frac{1}{2}m_{eq}\dot{x}_{eq}^2 \tag{1.23}$$

Since $\dot{x}_{eq} = \dot{x}$ and $\dot{\theta} = \dot{x}/R$, the equivalence of T and T_{eq} gives

$$\frac{1}{2}m_{eq}\dot{x}^2 = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\left(\frac{\dot{x}}{R}\right)^2$$

that is,

$$m_{eq} = m + \frac{J_0}{R^2} \tag{1.24}$$

2. *Equivalent rotational mass.* Here $\dot{\theta}_{eq} = \dot{\theta}$ and $\dot{x} = \dot{\theta}R$, and the equivalence of T and T_{eq} leads to

$$\frac{1}{2}J_{eq}\dot{\theta}^2 = \frac{1}{2}m(\dot{\theta}R)^2 + \frac{1}{2}J_0\dot{\theta}^2$$

or

$$J_{eq} = J_0 + mR^2 \tag{1.25}$$

EXAMPLE 1.6 Equivalent Mass of a System

Find the equivalent mass of the system shown in Fig. 1.31, where the rigid link 1 is attached to the pulley and rotates with it.

Given: System composed of a mass, pulley, rigid links, and a cylinder, Fig. 1.31.

Find: Equivalent mass, m_{eq}.

Approach: Equivalence of kinetic energy (assuming small displacements).

Solution: When the mass m is displaced by a distance x, the pulley and the rigid link 1 rotate by an angle $\theta_p = \theta_1 = \frac{x}{r_p}$. This causes the rigid link 2 and the cylinder to be displaced by a distance $x_2 = \theta_p\, l_1 = \frac{x l_1}{r_p}$. Since the cylinder rolls without slippage, it rotates by an

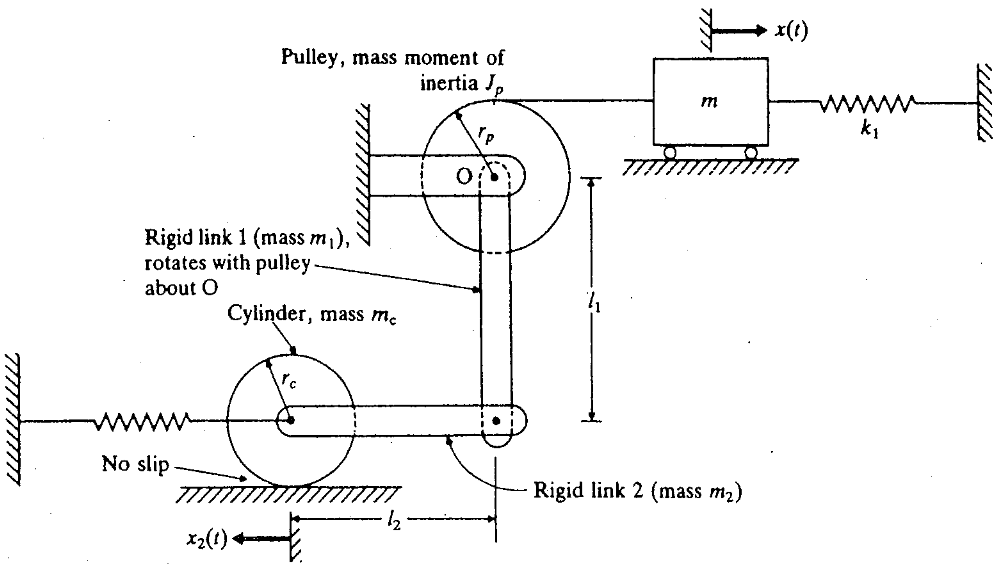


FIGURE 1.31

angle $\theta_c = \dfrac{x_2}{r_c} = \dfrac{x l_1}{r_p\, r_c}$. The kinetic energy of the system (T) can be expressed (for small displacements) as:

$$T = \frac{1}{2} m\, \dot{x}^2 + \frac{1}{2} J_p\, \dot{\theta}_p^2 + \frac{1}{2} J_1\, \dot{\theta}_1^2 + \frac{1}{2} m_2\, \dot{x}_2^2 + \frac{1}{2} J_c\, \dot{\theta}_c^2 \tag{E.1}$$

where J_p, J_1, and J_c denote the mass moments of inertia of the pulley, link 1 (about O), and cylinder, respectively, $\dot{\theta}_p$, $\dot{\theta}_1$, and $\dot{\theta}_c$ indicate the angular velocities of the pulley, link 1 (about O), and cylinder, respectively, and $\dot{x}$ and $\dot{x}_2$ represent the linear velocities of the mass m and link 2, respectively. Noting that $J_c = \dfrac{m_c\, r_c^2}{2}$ and $J_1 = \dfrac{m_1\, l_1^2}{3}$, Eq. (E.1) can be rewritten as

$$T = \frac{1}{2} m\, \dot{x}^2 + \frac{1}{2} J_p \left(\frac{\dot{x}}{r_p}\right)^2 + \frac{1}{2}\left(\frac{m_1\, l_1^2}{3}\right)\left(\frac{\dot{x}}{r_p}\right)^2 + \frac{1}{2} m_2 \left(\frac{\dot{x}\, l_1}{r_p}\right)^2 + \frac{1}{2}\left(\frac{m_c\, r_c^2}{2}\right)\left(\frac{\dot{x}\, l_1}{r_p\, r_c}\right)^2 \tag{E.2}$$

By equating Eq. (E.2) to the kinetic energy of the equivalent system,

$$T = \frac{1}{2} m_{eq}\, \dot{x}^2 \tag{E.3}$$

we obtain the equivalent mass of the system as

$$m_{eq} = m + \frac{J_p}{r_p^2} + \frac{1}{3}\frac{m_1\, l_1^2}{r_p^2} + \frac{m_2\, l_1^2}{r_p^2} + \frac{1}{2}\frac{m_c\, l_1^2}{r_p^2} \tag{E.4}$$

■

EXAMPLE 1.7 Cam-Follower Mechanism

A cam-follower mechanism (Fig. 1.32) is used to convert the rotary motion of a shaft into the oscillating or reciprocating motion of a valve. The follower system consists of a pushrod of mass m_p, a rocker arm of mass m_r, and mass moment of inertia J_r about its C.G., a valve of mass m_v, and a valve spring of negligible mass [1.28–1.30]. Find the equivalent mass (m_{eq}) of this cam-follower system by assuming the location of m_{eq} as (i) point A and (ii) point C.

Given: Mass of pushrod $= m_p$, mass of rocker arm $= m_r$, mass moment of inertia of rocker arm $= J_r$, and mass of valve $= m_v$. Linear displacement of pushrod $= x_p$.

Find: Equivalent mass of the cam-follower system (i) at point A, (ii) at point C.

Approach: Equivalence of kinetic energy.

Solution: Due to a vertical displacement x of the pushrod, the rocker arm rotates by an angle $\theta_r = x/l_1$ about the pivot point, the valve moves downward by $x_v = \theta_r l_2 = x l_2/l_1$,

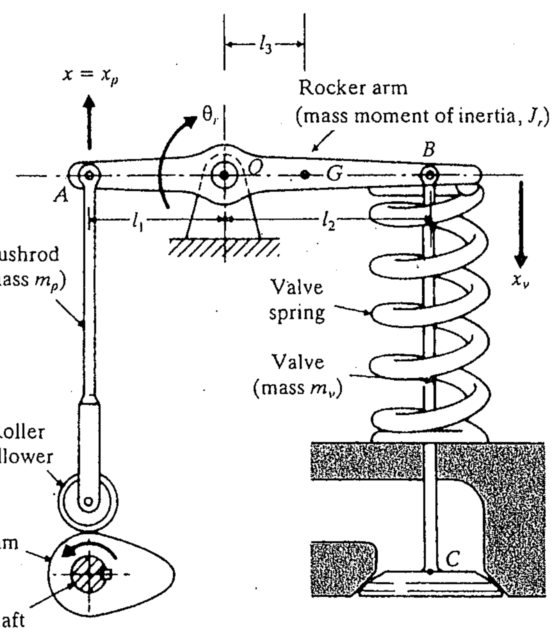


FIGURE 1.32 Cam-follower system.

and the C.G. of the rocker arm moves downward by $x_r = \theta_r l_3 = x l_3/l_1$. The kinetic energy of the system (T) can be expressed as[2]

$$T = \frac{1}{2}m_p\dot{x}_p^2 + \frac{1}{2}m_v\dot{x}_v^2 + \frac{1}{2}J_r\dot{\theta}_r^2 + \frac{1}{2}m_r\dot{x}_r^2 \tag{E.1}$$

where $\dot{x}_p$, $\dot{x}_r$, and $\dot{x}_v$ are the linear velocities of the pushrod, C.G. of the rocker arm and the valve, respectively, and $\dot{\theta}_r$ is the angular velocity of the rocker arm.

(i) If m_{eq} denotes the equivalent mass placed at point A, with $\dot{x}_{eq} = \dot{x}$, the kinetic energy of the equivalent mass system T_{eq} is given by

$$T_{eq} = \frac{1}{2}m_{eq}\dot{x}_{eq}^2 \tag{E.2}$$

By equating T and T_{eq}, and noting that

$$\dot{x}_p = \dot{x}, \qquad \dot{x}_v = \frac{\dot{x}l_2}{l_1}, \qquad \dot{x}_r = \frac{\dot{x}l_3}{l_1}, \qquad \text{and} \qquad \dot{\theta}_r = \frac{\dot{x}}{l_1}$$

we obtain

$$m_{eq} = m_p + \frac{J_r}{l_1^2} + m_v\frac{l_2^2}{l_1^2} + m_r\frac{l_3^2}{l_1^2} \tag{E.3}$$

(ii) Similarly, if the equivalent mass is located at point C, $\dot{x}_{eq} = \dot{x}_v$ and

$$T_{eq} = \frac{1}{2}m_{eq}\dot{x}_{eq}^2 = \frac{1}{2}m_{eq}\dot{x}_v^2 \tag{E.4}$$

Equating (E.4) and (E.1) gives

$$m_{eq} = m_v + \frac{J_r}{l_2^2} + m_p\left(\frac{l_1}{l_2}\right)^2 + m_r\left(\frac{l_3}{l_2}\right)^2 \tag{E.5}$$

■

1.9 Damping Elements

In many practical systems, the vibrational energy is gradually converted to heat or sound. Due to the reduction in the energy, the response, such as the displacement of the system, gradually decreases. The mechanism by which the vibrational energy is gradually converted into heat or sound is known as damping. Although the amount of energy converted into heat or sound is relatively small, the consideration of damping becomes important for an accurate prediction of the vibration response of a system. A damper is assumed to have neither mass nor elasticity, and damping force exists only if there is relative velocity between the two ends of the damper. It is difficult to determine the causes of damping in practical systems. Hence damping is modeled as one or more of the following types.

[2]If the valve spring has a mass m_s, then its equivalent mass will be $\frac{1}{3}m_s$ (see Example 2.7). Thus the kinetic energy of the valve spring will be $\frac{1}{2}(\frac{1}{3}m_s)\dot{x}_v^2$.

Viscous Damping. Viscous damping is the most commonly used damping mechanism in vibration analysis. When mechanical systems vibrate in a fluid medium such as air, gas, water, and oil, the resistance offered by the fluid to the moving body causes energy to be dissipated. In this case, the amount of dissipated energy depends on many factors, such as the size and shape of the vibrating body, the viscosity of the fluid, the frequency of vibration, and the velocity of the vibrating body. In viscous damping, the damping force is proportional to the velocity of the vibrating body. Typical examples of viscous damping include (1) fluid film between sliding surfaces, (2) fluid flow around a piston in a cylinder, (3) fluid flow through an orifice, and (4) fluid film around a journal in a bearing.

Coulomb or Dry Friction Damping. Here the damping force is constant in magnitude but opposite in direction to that of the motion of the vibrating body. It is caused by friction between rubbing surfaces that are either dry or have insufficient lubrication.

Material or Solid or Hysteretic Damping. When materials are deformed, energy is absorbed and dissipated by the material [1.31]. The effect is due to friction between the internal planes, which slip or slide as the deformations take place. When a body having material damping is subjected to vibration, the stress-strain diagram shows a hysteresis loop as indicated in Fig. 1.33(a). The area of this loop denotes the energy lost per unit volume of the body per cycle due to damping.[3]

1.9.1 Construction of Viscous Dampers

A viscous damper can be constructed using two parallel plates separated by a distance h, with a fluid of viscosity μ between the plates (see Fig. 1.34). Let one plate be fixed and let the other plate be moved with a velocity v in its own plane. The fluid layers in contact with the moving plate move with a velocity v, while those in contact with the fixed plate do not move. The velocities of intermediate fluid layers are assumed to vary linearly between 0 and v, as shown in Fig. 1.34. According to Newton's law of viscous flow, the shear stress (τ) developed in the fluid layer at a distance y from the fixed plate is given by

$$\tau = \mu \frac{du}{dy} \tag{1.26}$$

[3]When the load applied to an elastic body is increased, the stress (σ) and the strain (ε) in the body also increase. The area under the $\sigma - \varepsilon$ curve, given by

$$u = \int \sigma \, d\varepsilon$$

denotes the energy expended (work done) per unit volume of the body. When the load on the body is decreased, energy will be recovered. When the unloading path is different from the loading path, the area ABC in Fig. 1.33(b)—the area of the hysteresis loop in Fig. 1.33(a)—denotes the energy lost per unit volume of the body.

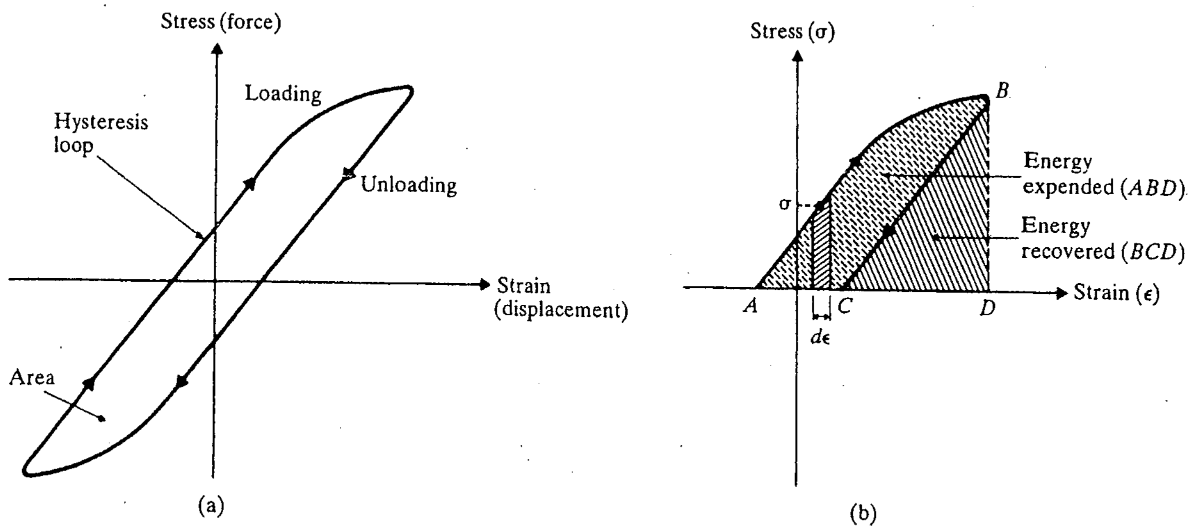


FIGURE 1.33 Hysteresis loop for elastic materials.

where $du/dy = v/h$ is the velocity gradient. The shear or resisting force (F) developed at the bottom surface of the moving plate is

$$F = \tau A = \frac{\mu A v}{h} = cv \tag{1.27}$$

where A is the surface area of the moving plate and

$$c = \frac{\mu A}{h} \tag{1.28}$$

is called the damping constant.

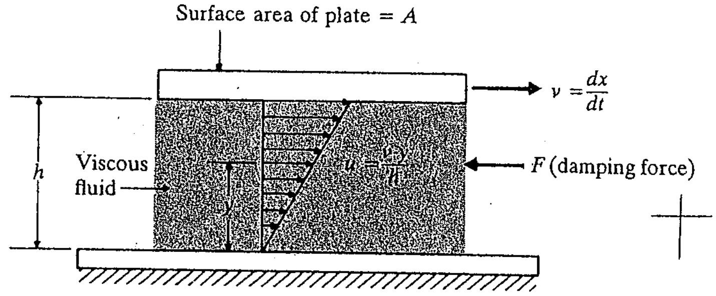


FIGURE 1.34 Parallel plates with a viscous fluid in between.

If a damper is nonlinear, a linearization procedure is generally used about the operating velocity (v^*), as in the case of a nonlinear spring. The linearization process gives the equivalent damping constant as

$$c = \left. \frac{dF}{dv} \right|_{v^*} \tag{1.29}$$

1.9.2 Combination of Dampers

When dampers appear in combination, they can be replaced by an equivalent damper by adopting a procedure similar to the one described in Sections 1.7 and 1.8 (see Problem 1.32).

EXAMPLE 1.8 Clearance in a Bearing

A bearing, which can be approximated as two flat plates separated by a thin film of lubricant (Fig. 1.35), is found to offer a resistance of 400 N when SAE30 oil is used as the lubricant and the relative velocity between the plates is 10 m/s. If the area of the plates *(A)* is 0.1 m^2, determine the clearance between the plates. Assume the absolute viscosity of SAE30 oil as 50 μ reyn or 0.3445 Pa-s.

Given: Characteristics of a bearing and the lubricant.

Find: Distance between the plates of the bearing.

Approach: Use the definition of damping constant.

Solution: Since the resisting force *(F)* can be expressed as $F = c\,v$, where c is the damping constant and v is the velocity, we have

$$c = \frac{F}{v} = \frac{400}{10} = 40 \text{ N-s/m} \tag{E.1}$$

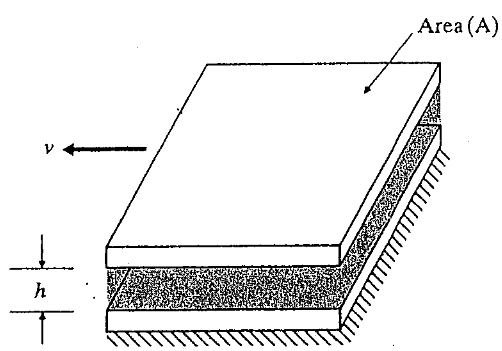


FIGURE 1.35

Let the total forces acting on all the springs and all the dampers be F_s and F_d, respectively (see Fig. 1.37(d)). The force equilibrium equations can thus be expressed as

$$F_s = F_{s1} + F_{s2} + F_{s3} + F_{s4}$$

$$F_d = F_{d1} + F_{d2} + F_{d3} + F_{d4} \qquad \text{(E.2)}$$

where $F_s + F_d = W$, with W denoting the total vertical force (including the inertia force) acting on the milling machine. From Fig. 1.37(d), we have

$$F_s = k_{eq}\, x$$

$$F_d = c_{eq}\, \dot{x} \qquad \text{(E.3)}$$

Equations (E.2) along with Eqs. (E.1) and (E.3), yield

$$k_{eq} = k_1 + k_2 + k_3 + k_4 = 4k$$

$$c_{eq} = c_1 + c_2 + c_3 + c_4 = 4c \qquad \text{(E.4)}$$

when $k_i = k$ and $c_i = c$ for $i = 1, 2, 3, 4$.

Note: If the center of mass, G, is not located symmetrically with respect to the four springs and dampers, the i^{th} spring experiences a displacement of x_i and the i^{th} damper experiences a velocity of $\dot{x}_i$ where x_i and $\dot{x}_i$ can be related to the displacement x and velocity $\dot{x}$ of the center of mass of the milling machine, G. In such a case, Eqs. (E.1) and (E.4) need to be modified suitably. ■

1.10 Harmonic Motion

Oscillatory motion may repeat itself regularly, as in the case of a simple pendulum, or it may display considerable irregularity, as in the case of ground motion during an earthquake. If the motion is repeated after equal intervals of time, it is called *periodic motion.* The simplest type of periodic motion is *harmonic motion.* The motion imparted to the mass m due to the Scotch yoke mechanism shown in Fig. 1.38 is an example of simple harmonic motion [1.24, 1.34, 1.35]. In this system, a crank of radius A rotates about the point O. The other end of the crank P slides in a slotted rod, which reciprocates in the vertical guide R. When the crank rotates at an angular velocity ω, the end point S of the slotted link and hence the mass m of the spring-mass system are displaced from their middle positions by an amount x (in time t) given by

$$x = A \sin \theta = A \sin \omega t \qquad (1.30)$$

This motion is shown by the sinusoidal curve in Fig. 1.38. The velocity of the mass m at time t is given by

$$\frac{dx}{dt} = \omega A \cos \omega t \qquad (1.31)$$

and the acceleration by

$$\frac{d^2x}{dt^2} = -\omega^2 A \sin \omega t = -\omega^2 x \qquad (1.32)$$

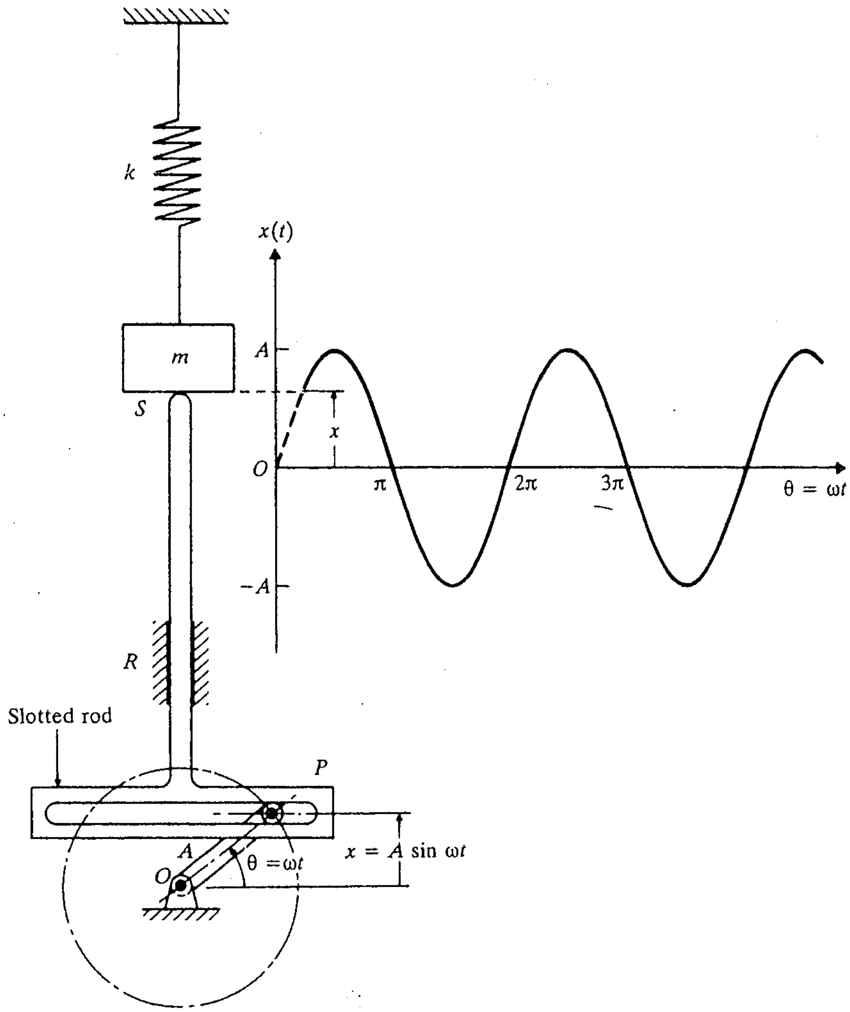


FIGURE 1.38 Scotch yoke mechanism.

It can be seen that the acceleration is directly proportional to the displacement. Such a vibration, with the acceleration proportional to the displacement and directed toward the mean position, is known as *simple harmonic motion.* The motion given by $x = A \cos \omega t$ is another example of a simple harmonic motion. Figure 1.38 clearly shows the similarity between cyclic (harmonic) motion and sinusoidal motion.

1.10.1 Vectorial Representation of Harmonic Motion

Harmonic motion can be represented conveniently by means of a vector $\vec{OP}$ of magnitude A rotating at a constant angular velocity ω. In Fig. 1.39, the projection of the tip of the vector $\vec{X} = \vec{OP}$ on the vertical axis is given by

$$y = A \sin \omega t \tag{1.33}$$

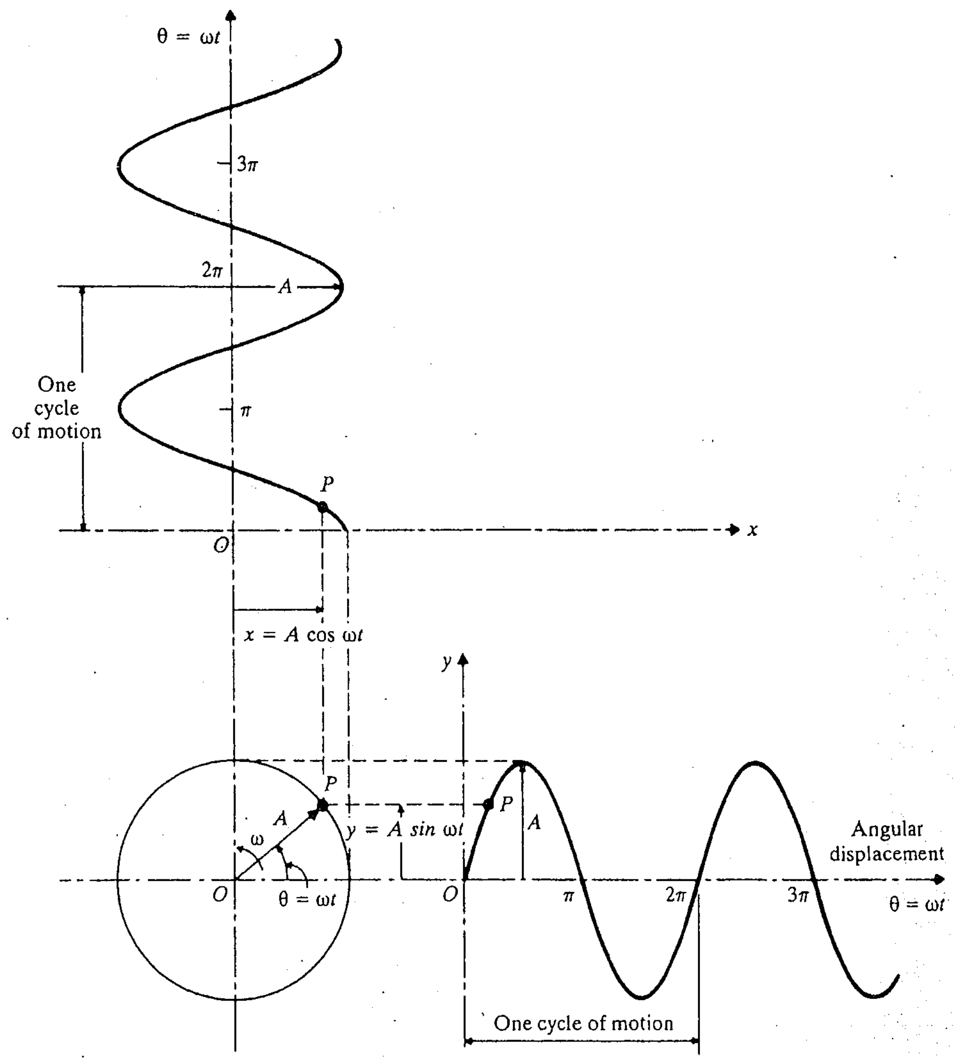


FIGURE 1.39 Harmonic motion as the projection of the end of a rotating vector.

and its projection on the horizontal axis by

$$x = A \cos \omega t \tag{1.34}$$

1.10.2 Complex Number Representation of Harmonic Motion

As seen above, the vectorial method of representing harmonic motion requires the description of both the horizontal and vertical components. It is more convenient to represent harmonic motion using a complex number representation. Any vector $\vec{X}$ in the xy plane can be represented as a complex number:

$$\vec{X} = a + ib \tag{1.35}$$

Problems

The problem assignments are organized as follows:

Problems	Section Covered	Topic Covered
1.1–1.6	1.6	Vibration analysis procedure
1.7–1.26	1.7	Spring elements
1.13, 1.26–1.31	1.8	Mass elements
1.32–1.38	1.9	Damping elements
1.39–1.59	1.10	Harmonic motion
1.60–1.70	1.11	Harmonic analysis
1.71–1.74	1.13	Computer program
1.75–1.80	—	Design projects

1.1* A study of the response of a human body subjected to vibration/shock is important in many applications. In a standing posture, the masses of head, upper torso, hips, and legs, and the elasticity/damping of neck, spinal column, abdomen, and legs influence the response characteristics. Develop a sequence of three improved approximations for modeling the human body.

1.2* Figure 1.54 shows a human body and a restraint system at the time of an automobile collision [1.47]. Suggest a simple mathematical model by considering the elasticity,

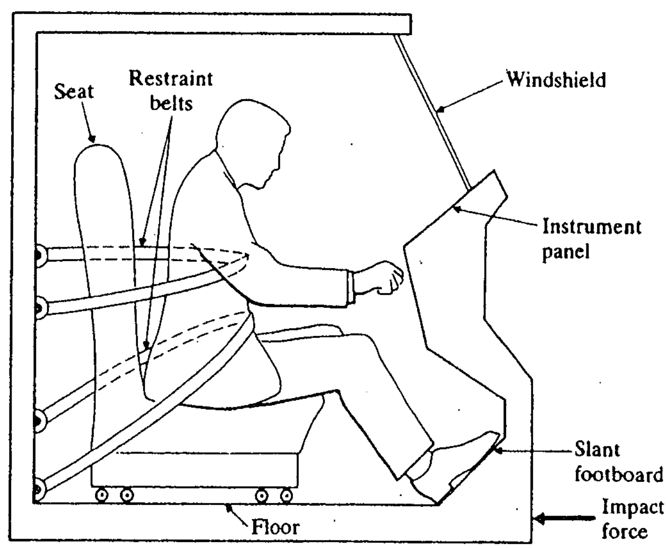


FIGURE 1.54 A human body and a restraint system.

*The asterisk denotes a design type problem or a problem with no unique answer.

mass, and damping of the seat, human body, and the restraints for a vibration analysis of the system.

1.3* A reciprocating engine is mounted on a foundation as shown in Fig. 1.55. The unbalanced forces and moments developed in the engine are transmitted to the frame and the foundation. An elastic pad is placed between the engine and the foundation block to reduce the transmission of vibration. Develop two mathematical models of the system using a gradual refinement of the modeling process.

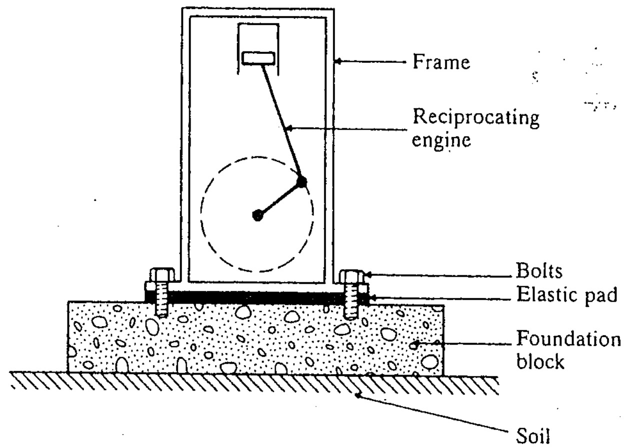


FIGURE 1.55 A reciprocating engine on a foundation.

1.4* An automobile moving over a rough road (Fig. 1.56) can be modeled considering (a) weight of the car body, passengers, seats, front wheels, and rear wheels; (b) elasticity of tires (suspension), main springs, and seats; and (c) damping of the seats, shock absorbers, and tires. Develop three mathematical models of the system using a gradual refinement in the modeling process.

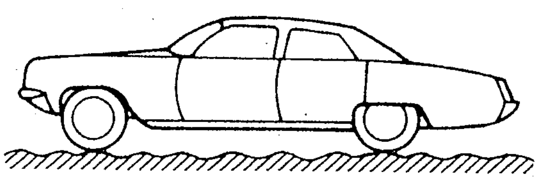


FIGURE 1.56 An automobile moving on a rough road.

1.5* The consequences of a head-on collision of two automobiles can be studied by considering the impact of the automobile on a barrier, as shown in Fig. 1.57. Construct a mathematical model by considering the masses of the automobile body, engine, transmission, and suspension, the elasticity of the bumpers, radiator, sheet metal body, driveline, and engine mounts.

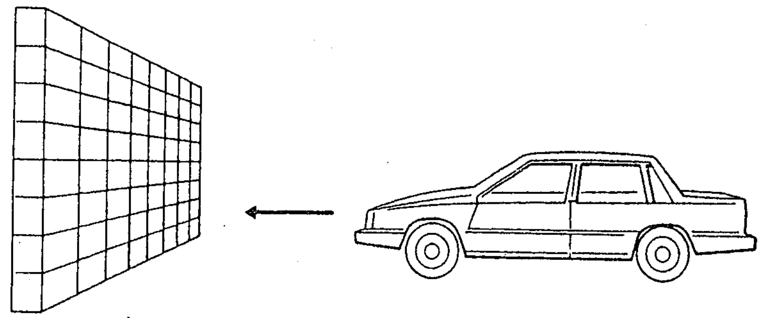


FIGURE 1.57 An automobile colliding on a barrier.

1.6* Develop a mathematical model for the tractor and plow shown in Fig. 1.58 by considering the mass, elasticity, and damping of the tires, shock absorbers, and the plows (blades).

1.7 Determine the equivalent spring constant of the system shown in Fig. 1.59.

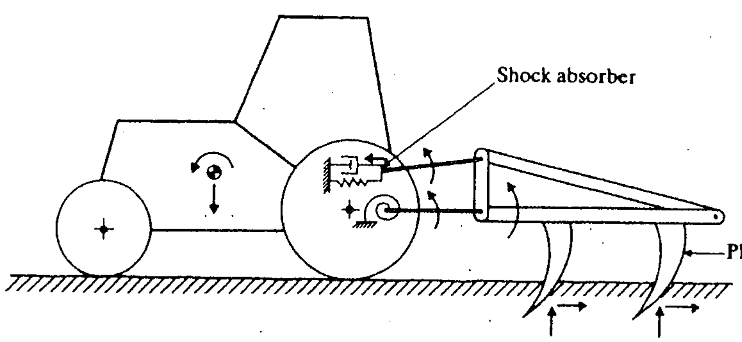


FIGURE 1.58 A tractor and plow.

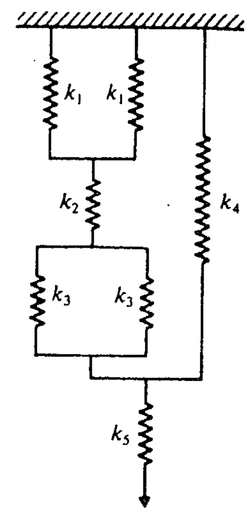


FIGURE 1.59

1.8 In Fig. 1.60, find the equivalent spring constant of the system in the direction of θ.

1.9 Find the equivalent torsional spring constant of the system shown in Fig. 1.61. Assume that k_1, k_2, k_3, and k_4 are torsional and k_5 and k_6 are linear spring constants.

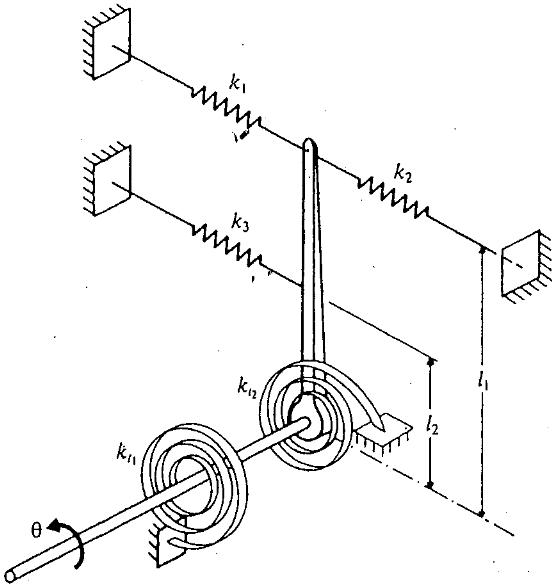


FIGURE 1.60

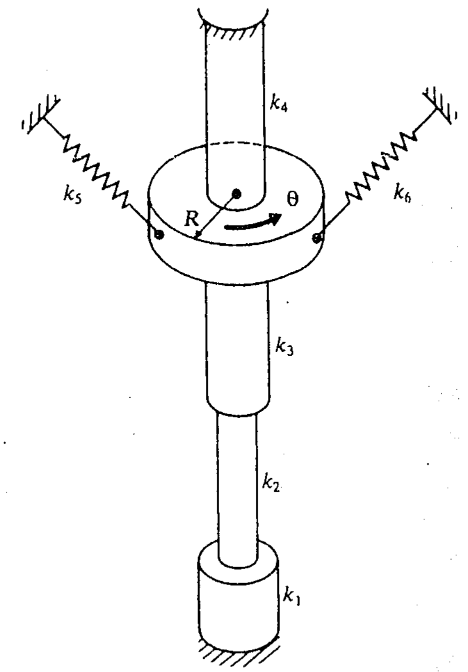


FIGURE 1.61

1.10 A machine of mass $m = 500$ kg is mounted on a simply supported steel beam of length $l = 2$ m having a rectangular cross section (depth = 0.1, m, width = 1.2 m) and Young's modulus $E = 2.06 \times 10^{11}$ N/m^2. To reduce the vertical deflection of the beam, a spring of stiffness k is attached at the mid-span, as shown in Fig. 1.62. Determine the value of k needed to reduce the deflection of the beam to one-third of its original value. Assume that the mass of the beam is negligible.

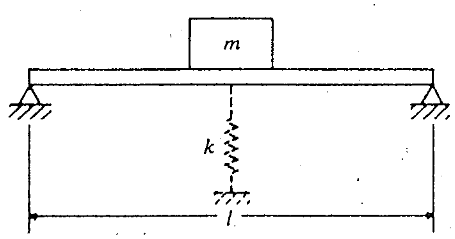


FIGURE 1.62

1.11 Four identical rigid bars—each of length a—are connected to a spring of stiffness k to form a structure for carrying a vertical load P, as shown in Figs. 1.63(a) and (b). Find the equivalent spring constant of the system k_{eq}, for each case, disregarding the masses of the bars and the friction in the joints.

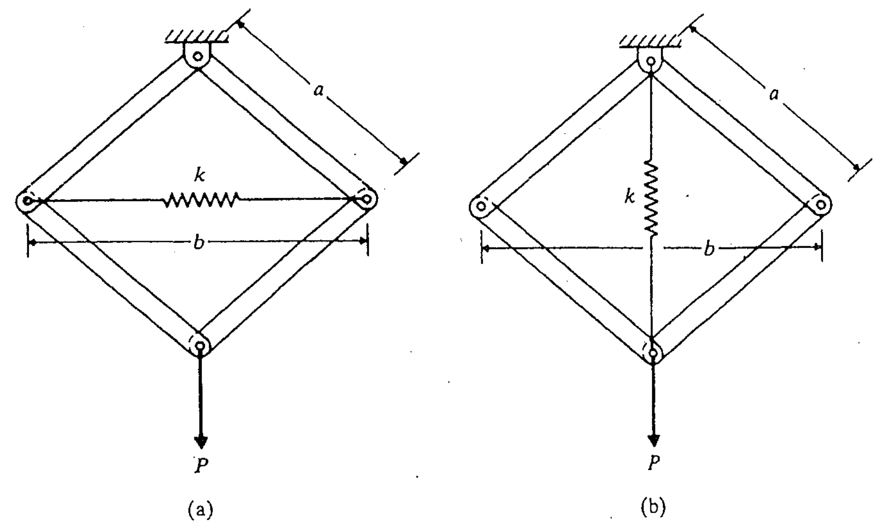


FIGURE 1.63

1.12 The tripod shown in Fig. 1.64 is used for mounting an electronic instrument that finds the distance between two points in space. The legs of the tripod are located symmetrically about the mid-vertical axis, each leg making an angle α with the vertical. If each leg has a length of l and axial stiffness of k, find the equivalent spring stiffness of the tripod in the vertical direction.

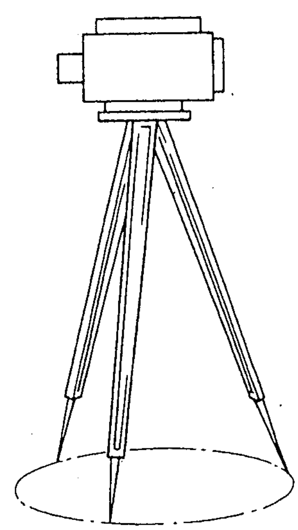


FIGURE 1.64

1.13 Find the equivalent spring constant and equivalent mass of the system shown in Fig. 1.65 with reference to θ. Assume that the bars AOB and CD are rigid with negligible mass.

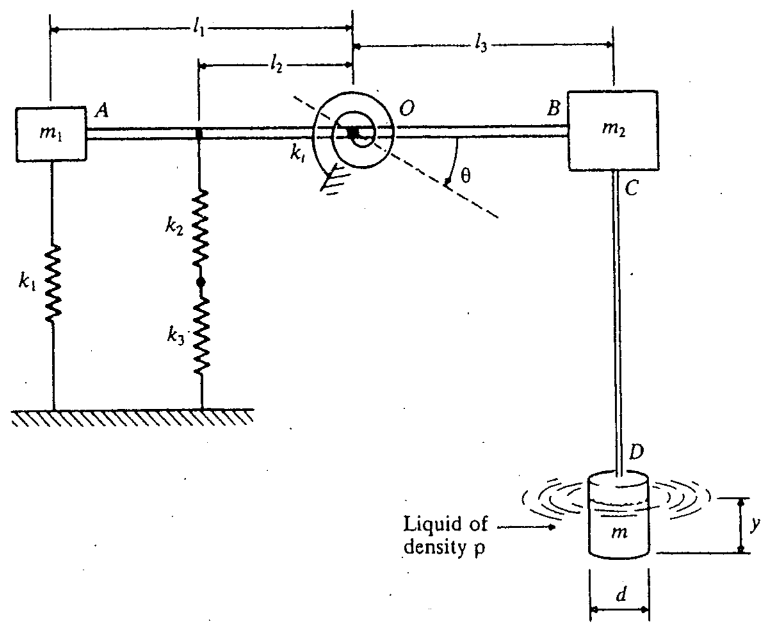


FIGURE 1.65

1.14 Find the length of the equivalent uniform hollow shaft of inner diameter d and thickness t that has the same axial spring constant as that of the solid conical shaft shown in Fig. 1.66.

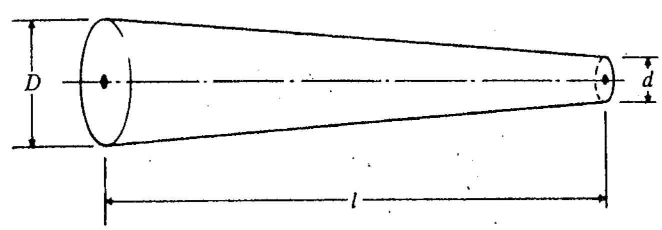


FIGURE 1.66

1.15 The force-deflection characteristic of a spring is described by $F = 500x + 2x^3$ where the force (F) is in Newtons and the deflection (x) is in millimeters. Find (a) the linearized spring constant at $x = 10$ mm, and (b) the spring forces at $x = 9$ mm and $x = 11$ mm using the linearized spring constant. Also find the error in the spring forces found in (b).

1.16 Figure 1.67 shows an air spring. This type of spring is generally used for obtaining very low natural frequencies while maintaining zero deflection under static loads. Find the spring constant of this air spring by assuming that the pressure p and volume v change adiabatically when the mass m moves.
Hint: $pv^{\gamma} =$ constant for an adiabatic process, where γ is the ratio of specific heats. For air, $\gamma = 1.4$.

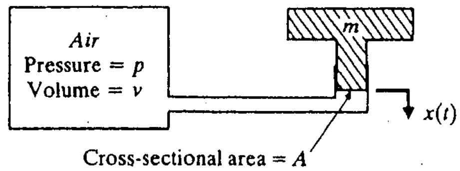


FIGURE 1.67

1.17 Find the equivalent spring constant of the system shown in Fig. 1.68 in the direction of the load P.

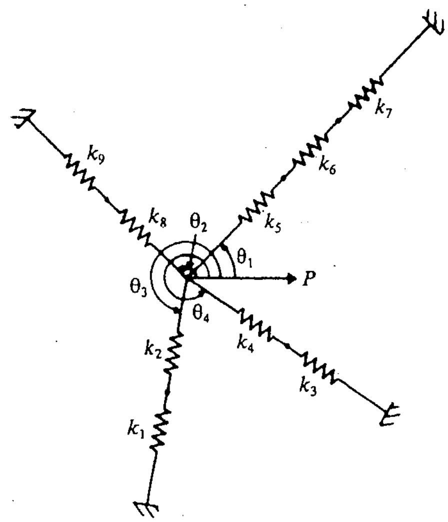


FIGURE 1.68

1.18* Design an air spring using a cylindrical container and a piston to achieve a spring constant of 75 lb/in. Assume that the maximum air pressure available is 200 psi.

1.19 The force (F)-deflection (x) relationship of a nonlinear spring is given by

$$F = ax + bx^3$$

where a and b are constants. Find the equivalent linear spring constant when the deflection is 0.01 m with $a = 20{,}000$ N/m and $b = 40 \times 10^6$ N/m^3.

1.20 Two nonlinear springs, S_1 and S_2, are connected in two different ways as indicated in Fig. 1.69. The force, F_i, in spring S_i is related to its deflection (x_i) as

$$F_i = a_i x_i + b_i x_i^3; \; i = 1, 2$$

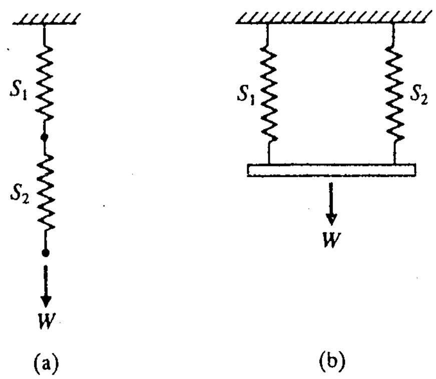


FIGURE 1.69

where a_i and b_i are constants. If an equivalent linear spring constant, k_{eq}, is defined by $W = k_{eq}\, x$ where x is the total deflection of the system, find an expression for k_{eq} in each case.

1.21* Design a steel helical compression spring to satisfy the following requirements:

Spring stiffness $(k) \geqslant 8000$ N/mm

Fundamental natural frequency of vibration $(f_1) \geqslant 0.4$ Hz

Spring index $(D/d) \geqslant 6$

Number of active turns $(N) \geqslant 10$.

The stiffness and fundamental natural frequency of the spring are given by [1.43]:

$$k = \frac{Gd^4}{8D^3N} \quad \text{and} \quad f_1 = \frac{1}{2}\sqrt{\frac{kg}{W}}$$

where G = shear modulus, d = wire diameter, D = coil diameter, W = weight of the spring, and g = acceleration due to gravity.

1.22 Find the spring constant of the bimetallic bar shown in Fig. 1.70 in axial motion.

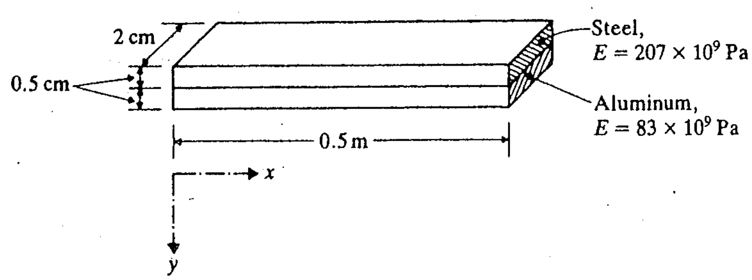


FIGURE 1.70

1.23 A tapered solid steel propeller shaft is shown in Fig. 1.71. Determine the torsional spring constant of the shaft.

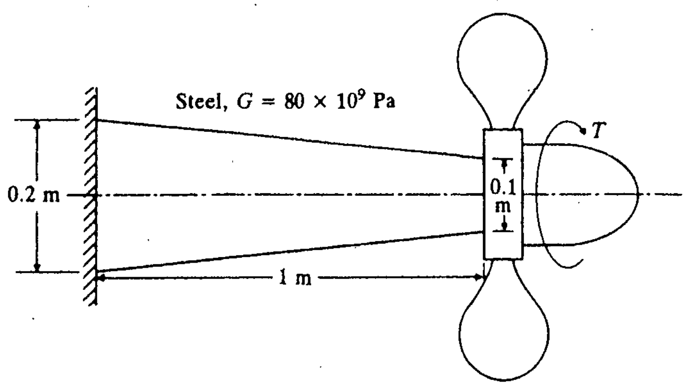


FIGURE 1.71

1.24 A composite propeller shaft, made of steel and aluminum, is shown in Fig. 1.72. Determine the torsional spring constant of the shaft.

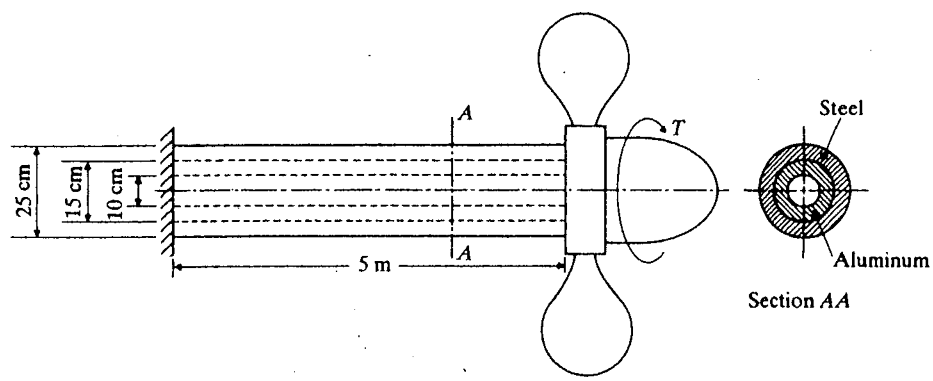


FIGURE 1.72

1.25 Consider two helical springs with the following characteristics:

Spring 1: material–steel; number of turns–10; mean coil diameter–12 in; wire diameter–2 in; free length–15 in; shear modulus–12×10^6 psi.

Spring 2: material–aluminum; number of turns–10; mean coil diameter–10 in; wire diameter–1 in; free length–15 in; shear modulus–4×10^6 psi.

Determine the equivalent spring constant when (a) spring 2 is placed inside spring 1, and (b) spring 2 is placed on top of spring 1.

1.26 Two sector gears, located at the ends of links 1 and 2, are engaged together and rotate about O_1 and O_2, as shown in Fig. 1.73. If links 1 and 2 are connected to springs k_1 to k_4 and k_{t1} and k_{t2} as shown, find the equivalent torsional spring stiffness and equivalent mass moment of inertia of the system with reference to θ_1. Assume (a) the mass moment of inertia of link 1 (including the sector gear) about O_1 as J_1 and that of link 2 (including the sector gear) about O_2 as J_2, and (b) the angles θ_1 and θ_2 to be small.

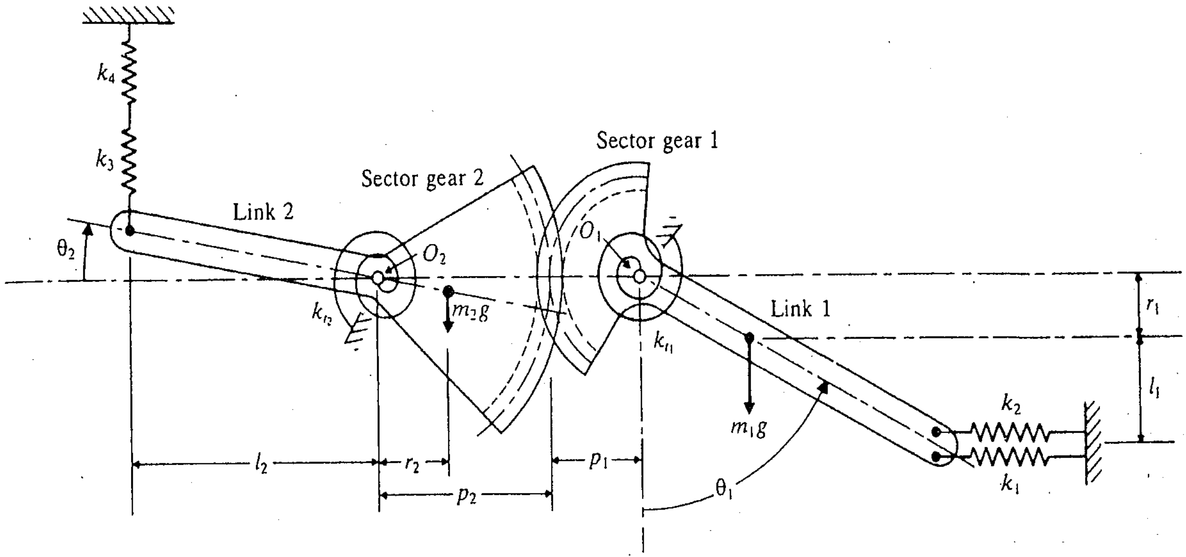


FIGURE 1.73

1.27 In Fig. 1.74 find the equivalent mass of the rocker arm assembly, referred to the x coordinate.

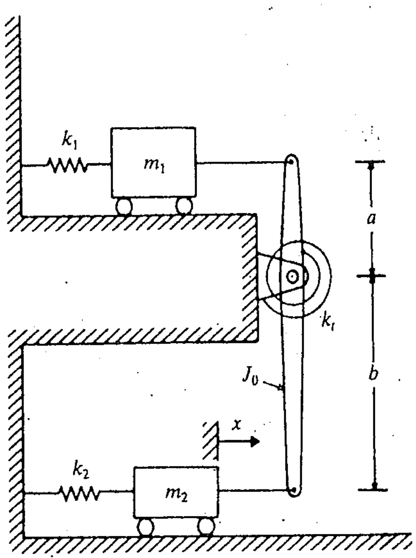


FIGURE 1.74

1.28 Find the equivalent mass moment of inertia of the gear train shown in Fig. 1.75 with reference to the driving shaft. In Fig. 1.75, J_i and n_i denote the mass moment of inertia and the number of teeth, respectively, of gear i, $i = 1,2, \ldots, 2N$.

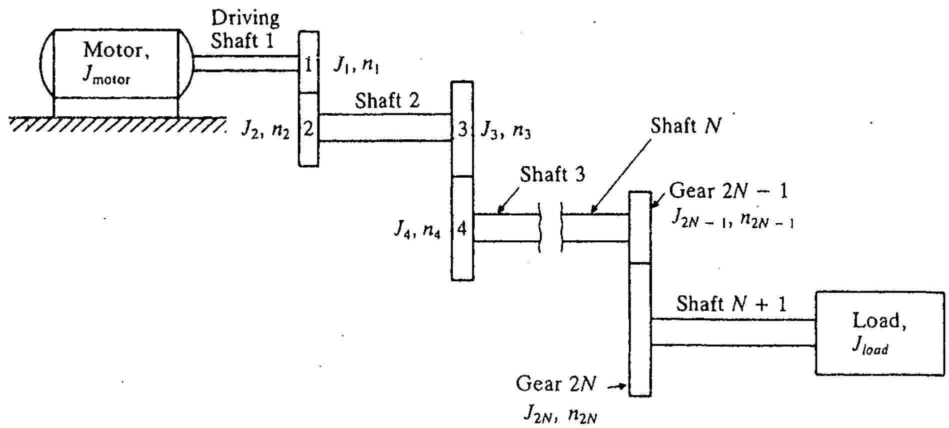


FIGURE 1.75

1.29 Two masses, having mass moments of inertia J_1 and J_2, are placed on rotating rigid shafts that are connected by gears, as shown in Fig. 1.76. If the number of teeth on gears 1 and 2 are n_1 and n_2, respectively, find the equivalent mass moment of inertia corresponding to θ_1.

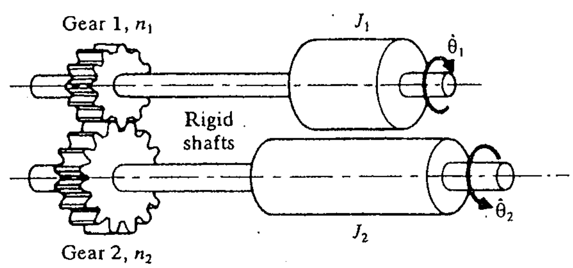


FIGURE 1.76 Rotational masses on geared shafts.

1.30 A simplified model of a petroleum pump is shown in Fig. 1.77, where the rotary motion of the crank is converted to the reciprocating motion of the piston. Find the equivalent mass, m_{eq}, of the system at location A.

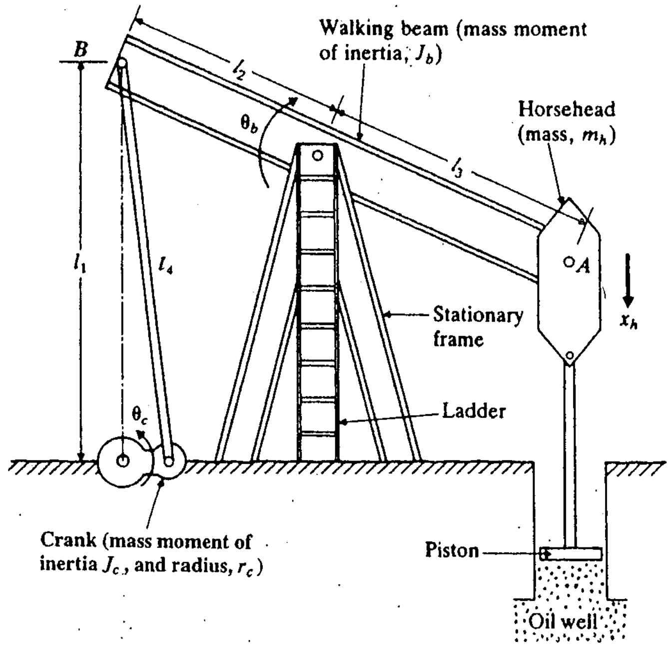


FIGURE 1.77

1.31 Find the equivalent mass of the system shown in Fig. 1.78.

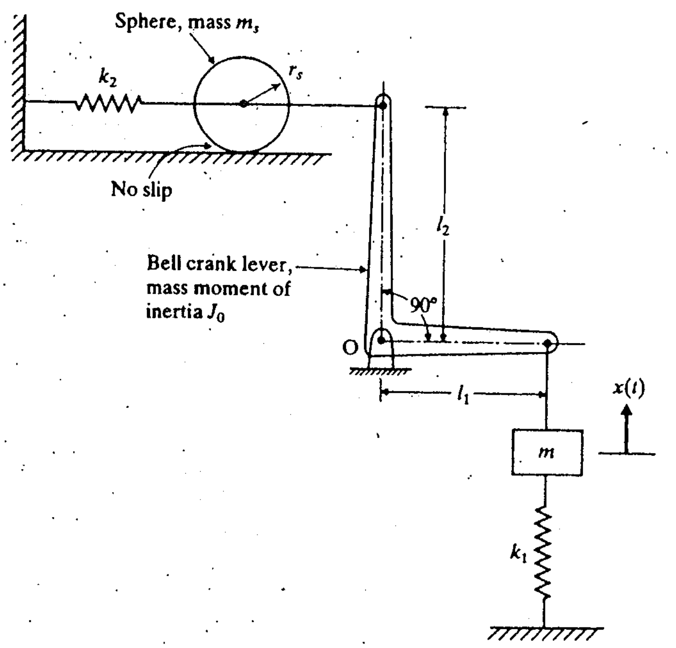


FIGURE 1.78

1.32 Find a single equivalent damping constant for the following cases:

a. When three dampers are parallel.
b. When three dampers are in series.
c. When three dampers are connected to a rigid bar (Fig. 1.79) and the equivalent damper is at site c_1.

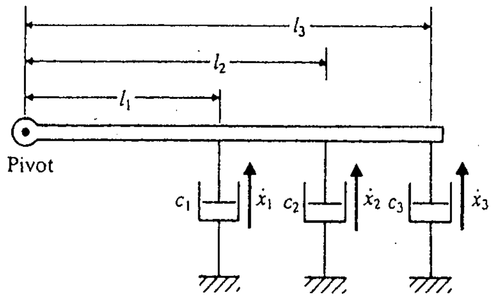


FIGURE 1.79 Dampers connected to a rigid bar.

d. When three torsional dampers are located on geared shafts (Fig. 1.80) and the equivalent damper is at location c_{t1}.

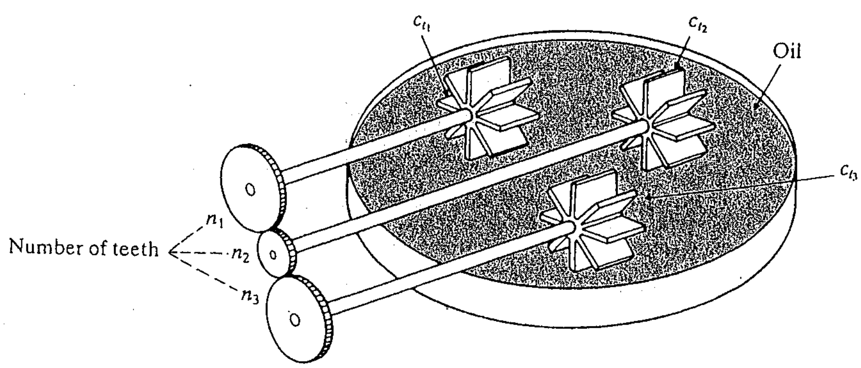


FIGURE 1.80 Dampers located on geared shafts.

Hint: The energy dissipated by a viscous damper in a cycle during harmonic motion is given by $\pi c \omega X^2$, where c is the damping constant, ω is the frequency, and X is the amplitude of oscillation.

1.33* Design a piston-cylinder type viscous damper to achieve a damping constant of 1 lbf-sec/in using a fluid of viscosity 4 μreyn (1 reyn = 1 lbf-sec/in^2).

1.34* Design a shock absorber (piston-cylinder type dashpot) to obtain a damping constant of 10^5 lb-sec/in using SAE 30 oil at 70° F. The diameter of the piston has to be less than 2.5 inches.

1.35 Develop an expression for the damping constant of the rotational damper shown in Fig. 1.81 in terms of D, d, l, h, ω, and μ, where ω denotes the constant angular velocity of the inner cylinder, and d and h represent the radial and axial clearances between the inner and outer cylinders.

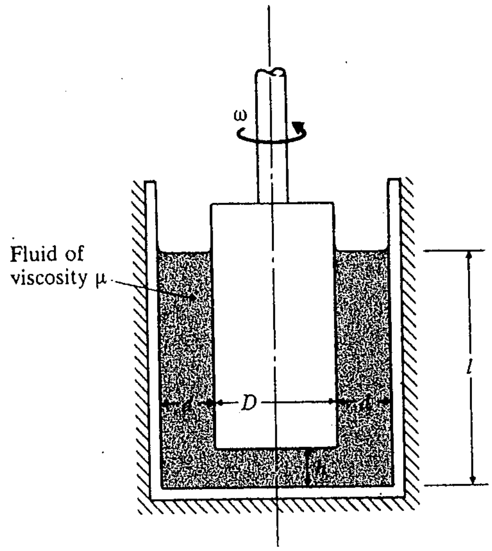


FIGURE 1.81

1.36 The force (F)-velocity ($\dot{x}$) relationship of a nonlinear damper is given by

$$F = a\,\dot{x} + b\,\dot{x}^2$$

where a and b are constants. Find the equivalent linear damping constant when the relative velocity is 5 m/s with a = 5N-s/m and b = 0.2N-s^2/m^2.

1.37 The damping constant (c) due to skin friction drag of a rectangular plate moving in a fluid of viscosity μ is given by (see Fig. 1.82):

$$c = 100\,\mu\,l^2\,d$$

Design a plate-type damper (shown in Fig. 1.35) that provides an identical damping constant for the same fluid.

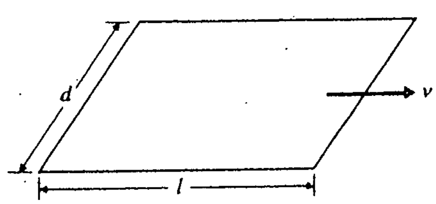


FIGURE 1.82

Sir Isaac Newton (1642–1727) was an English natural philosopher, a professor of mathematics at Cambridge University, and President of the Royal Society. His *Principia Mathematica* (1687), which deals with the laws and conditions of motion, is considered to be the greatest scientific work ever produced. The definitions of force, mass, and momentum, and his three laws of motion crop up continually in dynamics. Quite fittingly, the unit of force named "Newton" in SI units happens to be the approximate weight of an average apple, which inspired him to study the laws of gravity. (Photo courtesy of David Eugene Smith, *History of Mathematics*, Volume I—General Survey of the History of Elementary Mathematics, Dover Publications, Inc., New York, 1958.)

CHAPTER 2

Free Vibration of Single Degree of Freedom Systems

2.1 Introduction

A system is said to undergo free vibration when it oscillates only under an initial disturbance with no external forces acting after the initial disturbance. The oscillations of the pendulum of a grandfather clock, the vertical oscillatory motion felt by a bicyclist after hitting a road bump, and the motion of a child on a swing under an initial push represent a few examples of free vibration.

Figure 2.1(a) shows a spring-mass system that represents the simplest possible vibratory system. It is called a single degree of freedom system since one coordinate (x) is sufficient to specify the position of the mass at any time. There is no external force applied to the mass; hence the motion resulting from an initial disturbance will be a free vibration. Since there is no element that causes dissipation of energy during the motion of the mass, the amplitude of motion remains constant with time; it is an *undamped* system. In actual practice, except in a vacuum, the amplitude of free vibration diminishes gradually over time, due to the resistance offered by the surrounding medium (such as air). Such vibrations are said to be *damped*. The study

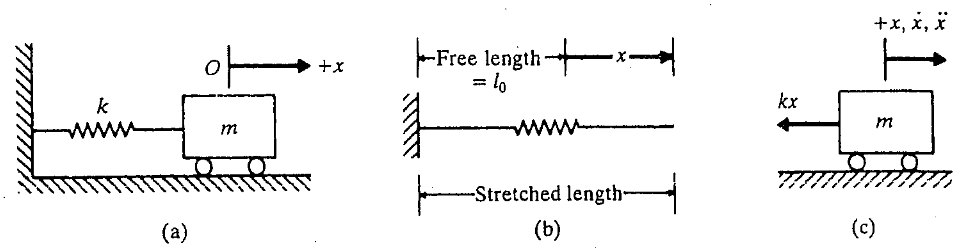


FIGURE 2.1 A spring-mass system in horizontal position.

of the free vibration of undamped and damped single degree of freedom systems is fundamental to the understanding of more advanced topics in vibrations.

Several mechanical and structural systems can be idealized as single degree of freedom systems. In many practical systems, the mass is distributed, but for a simple analysis, it can be approximated by a single point mass. Similarly, the elasticity of the system, which may be distributed throughout the system, can also be idealized by a single spring. For the cam-follower system shown in Fig. 1.32, for example, the various masses were replaced by an equivalent mass (m_{eq}) in Example 1.7. The elements of the follower system (pushrod, rocker arm, valve, and valve spring) are all elastic but can be reduced to a single equivalent spring of stiffness k_{eq}. For a simple analysis, the cam-follower system can thus be idealized as a single degree of freedom spring-mass system, as shown in Fig. 2.2.

Similarly, the structure shown in Fig. 2.3 can be considered a cantilever beam that is fixed at the ground. For the study of transverse vibration, the top mass can be considered a point mass and the supporting structure (beam) can be approximated as a spring to obtain the single degree of freedom model shown in Fig. 2.4. The building frame shown in Fig. 2.5(a) can also be idealized as a spring-mass system, as shown in Fig. 2.5(b). In this case, since the spring constant k is merely the ratio of force to deflection, it can be determined from the geometric and material properties of the columns. The mass of the idealized system is the same as that of the floor if we assume the mass of the columns to be negligible.

2.2 Free Vibration of an Undamped Translational System

2.2.1 Equation of Motion Using Newton's Second Law of Motion

Using Newton's second law of motion, we will consider the derivation of the equation of motion in this section. The procedure we will use can be summarized as follows:

1. Select a suitable coordinate to describe the position of the mass or rigid body in the system. Use a linear coordinate to describe the linear motion of a point mass or the centroid of a rigid body, and an angular coordinate to describe the angular motion of a rigid body.

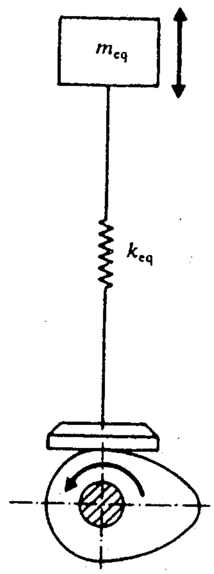


FIGURE 2.2 Equivalent spring-mass system for the cam-follower system of Fig. 1.32.

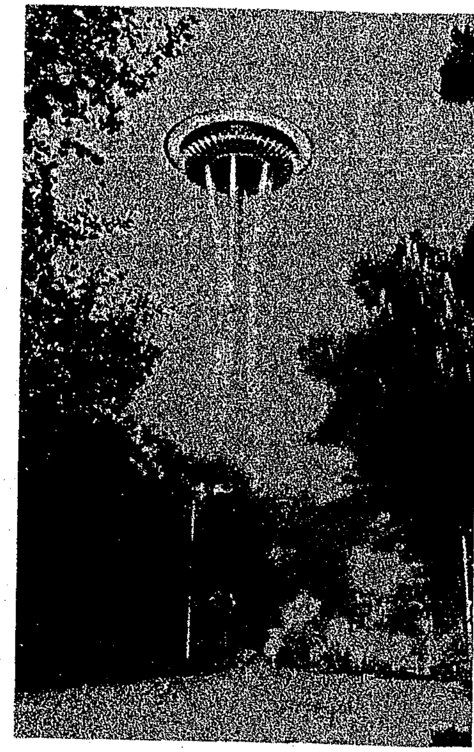
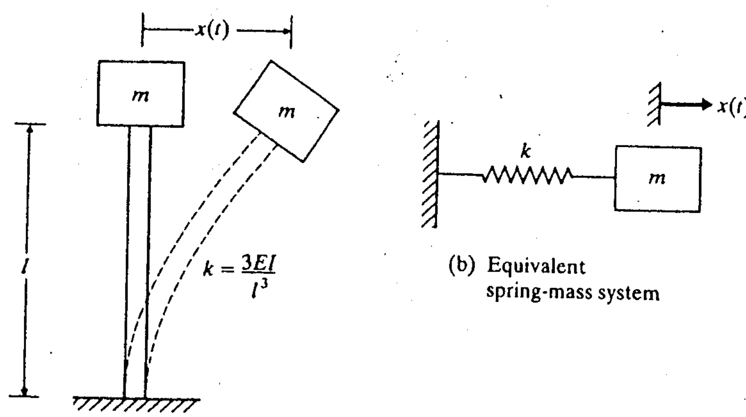


FIGURE 2.3



(a) Idealization of the tall structure

(b) Equivalent spring-mass system

FIGURE 2.4

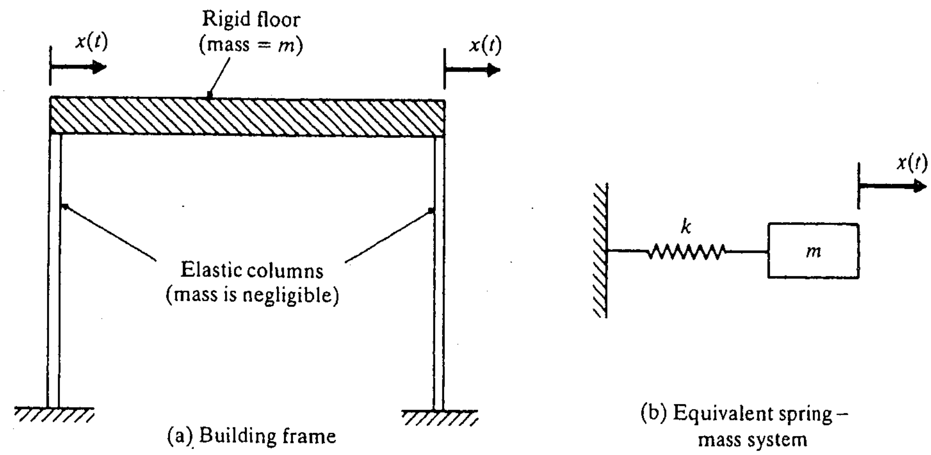


FIGURE 2.5 Idealization of a building frame.

2. Determine the static equilibrium configuration of the system and measure the displacement of the mass or rigid body from its static equilibrium position.
3. Draw the free-body diagram of the mass or rigid body when a positive displacement and velocity are given to it. Indicate all the active and reactive forces acting on the mass or rigid body.
4. Apply Newton's second law of motion to the mass or rigid body shown by the free-body diagram. Newton's second law of motion can be stated as follows:

The rate of change of momentum of a mass is equal to the force acting on it.

Thus, if mass m is displaced a distance $\vec{x}(t)$ when acted upon by a resultant force $\vec{F}(t)$ in the same direction, Newton's second law of motion gives

$$\vec{F}(t) = \frac{d}{dt}\left(m \frac{d\vec{x}(t)}{dt} \right)$$

If mass m is constant, this equation reduces to

$$\vec{F}(t) = m \frac{d^2\vec{x}(t)}{dt^2} = m \ddot{\vec{x}} \tag{2.1}$$

where

$$\ddot{\vec{x}} = \frac{d^2\vec{x}(t)}{dt^2}$$

is the acceleration of the mass. Equation (2.1) can be stated in words as

Resultant force on the mass = mass × acceleration

For a rigid body undergoing rotational motion, Newton's law gives

$$\vec{M}(t) = J\,\ddot{\vec{\theta}} \tag{2.2}$$

where $\vec{M}$ is the resultant moment acting on the body and $\vec{\theta}$ and $\ddot{\vec{\theta}} = \dfrac{d^2\theta(t)}{dt^2}$ are the resulting angular displacement and angular acceleration, respectively. Equation (2.1) or (2.2) represents the equation of motion of the vibrating system.

The procedure is now applied to the undamped single degree of freedom system shown in Fig. 2.1(a). Here the mass is supported on frictionless rollers and can have translatory motion in the horizontal direction. When the mass is displaced a distance $+x$ from its static equilibrium position, the force in the spring is kx and the free-body diagram of the mass can be represented as shown in Fig. 2.1(c). The application of Eq. (2.1) to mass m yields the equation of motion

$$F(t) = -k\,x = m\,\ddot{x}$$

or

$$m\,\ddot{x} + k\,x = 0 \tag{2.3}$$

2.2.2 Equation of Motion Using Other Methods

As stated in Section 1.6, the equations of motion of a vibrating system can be derived using several methods. The applications of D'Alembert's principle, the principle of virtual displacements, and the principle of conservation of energy are considered in this section.

D'Alembert's Principle The equations of motion, Eqs. (2.1) and (2.2), can be rewritten as

$$\vec{F}(t) - m\,\ddot{\vec{x}} = 0 \tag{2.4a}$$

$$\vec{M}(t) - J\,\ddot{\vec{\theta}} = 0 \tag{2.4b}$$

These equations can be considered equilibrium equations provided that $-m\ddot{\vec{x}}$ and $-J\ddot{\vec{\theta}}$ are treated as a force and a moment. This fictitious force (or moment) is known as the inertia force (or inertia moment) and the artificial state of equilibrium implied by Eq. (2.4a) or (2.4b) is known as dynamic equilibrium. This principle, implied in Eq. (2.4a) or (2.4b), is called the D'Alembert's principle. The application of D'Alembert's principle to the system shown in Fig. 2.1(c) yields the equation of motion:

$$-k\,x - m\,\ddot{x} = 0 \quad \text{or} \quad m\,\ddot{x} + k\,x = 0 \tag{2.3}$$

Principle of Virtual Displacements The principle of virtual displacements states that "if a system that is in equilibrium under the action of a set of forces is subjected to a virtual displacement, then the total virtual work done by the forces will be zero." Here the virtual displacement is defined as an imaginary infinitesimal displacement given instantaneously. It must be a physically possible displacement that

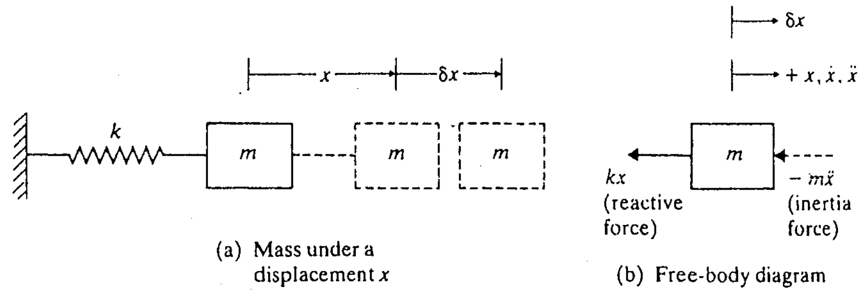


FIGURE 2.6

is compatible with the constraints of the system. The virtual work is defined as the work done by all the forces, including the inertia forces for a dynamic problem, due to a virtual displacement.

Consider a spring-mass system in a displaced position as shown in Fig. 2.6(a), where x denotes the displacement of the mass. Figure 2.6(b) shows the free-body diagram of the mass with the reactive and inertia forces indicated. When the mass is given a virtual displacement δx, as shown in Fig. 2.6(b), the virtual work done by each force can be computed as follows:

$$\text{Virtual work done by the spring force} = \delta W_s = -(k\,x)\,\delta x$$

$$\text{Virtual work done by the inertia force} = \delta W_i = -(m\,\ddot{x})\,\delta x$$

When the total virtual work done by all the forces is set equal to zero, we obtain

$$-m\,\ddot{x}\,\delta x - k\,x\,\delta x = 0 \tag{2.5}$$

Since the virtual displacement can have an arbitrary value, $\delta x \neq 0$, Eq. (2.5) gives the equation of motion of the spring-mass system as

$$m\,\ddot{x} + k\,x = 0 \tag{2.3}$$

Principle of Conservation of Energy A system is said to be conservative if no energy is lost due to friction or energy-dissipating nonelastic members. If no work is done on a conservative system by external forces (other than gravity or other potential forces), then the total energy of the system remains constant. Since the energy of a vibrating system is partly potential and partly kinetic, the sum of these two energies remains constant. The kinetic energy T is stored in the mass by virtue of its velocity, and the potential energy U is stored in the spring by virtue of its elastic deformation. Thus the principle of conservation of energy can be expressed as:

$$T + U = \text{constant}$$

or

$$\frac{d}{dt}(T + U) = 0 \tag{2.6}$$

The kinetic and potential energies are given by

$$T = \tfrac{1}{2}m\dot{x}^2 \tag{2.7}$$

and

$$U = \tfrac{1}{2}kx^2 \tag{2.8}$$

Substitution of Eqs. (2.7) and (2.8) into Eq. (2.6) yields the desired equation

$$m\ddot{x} + kx = 0 \tag{2.3}$$

2.2.3 Equation of Motion of a Spring-Mass System in Vertical Position

Consider the configuration of the spring-mass system shown in Fig. 2.7(a). The mass hangs at the lower end of a spring, which in turn is attached to a rigid support at its upper end. At rest, the mass will hang in a position called the *static equilibrium position*, in which the upward spring force exactly balances the downward gravita-

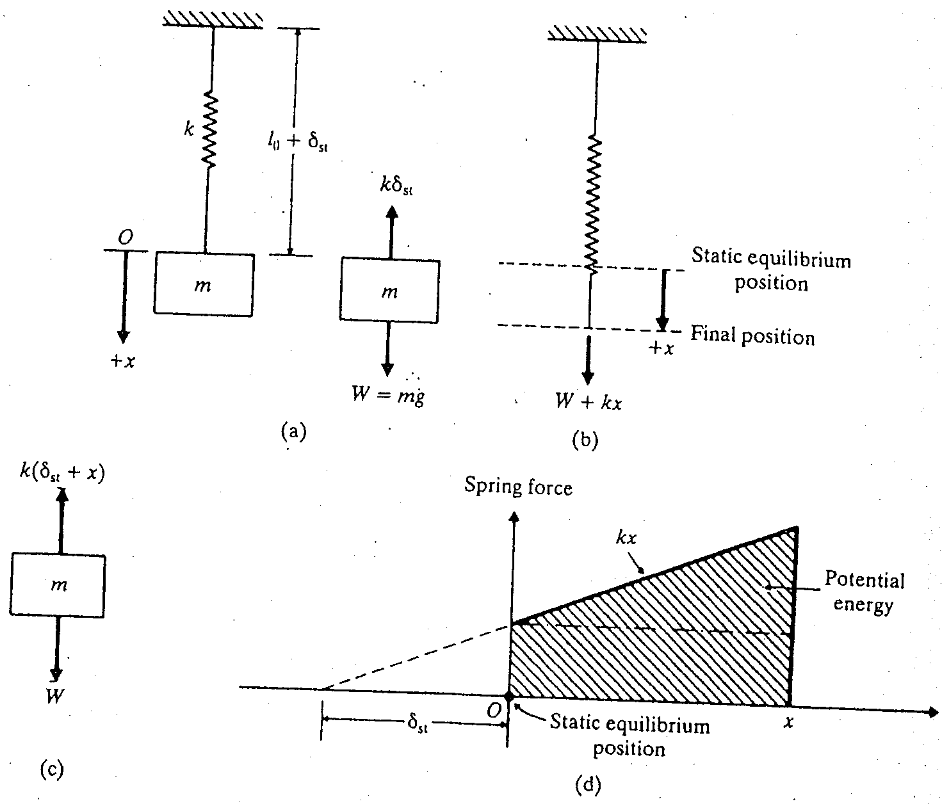


FIGURE 2.7 A spring-mass system in vertical position.

tional force on the mass. In this position the length of the spring is $l_0 + \delta_{st}$, where δ_{st} is the static deflection—the elongation due to the weight W of the mass m. From Fig. 2.7(a), we find that, for static equilibrium,

$$W = mg = k\delta_{st} \tag{2.9}$$

where g is the acceleration due to gravity. Let the mass be deflected a distance $+x$ from its static equilibrium position; then the spring force is $-k(x + \delta_{st})$, as shown in Fig. 2.7(c). The application of Newton's second law of motion to mass m gives

$$m\ddot{x} = -k(x + \delta_{st}) + W$$

and since $k\delta_{st} = W$, we obtain

$$m\ddot{x} + kx = 0 \tag{2.10}$$

Notice that Eqs. (2.3) and (2.10) are identical. This indicates that when a mass moves in a vertical direction, we can ignore its weight, provided we measure x from its static equilibrium position.

Note: Equation (2.10), the equation of motion of the system shown in Fig. 2.7, can also be derived using D'Alembert's principle, the principle of virtual displacements, or the principle of conservation of energy. For example, if the principle of conservation of energy is to be used, we note that the expression for the kinetic energy, T, remains the same as Eq. (2.7). However, the expression for the potential energy, U, is to be derived by considering the weight of the mass. For this we note that the spring force at static equilibrium position ($x = 0$) is mg. When the spring deflects by an amount x, its potential energy is given by (see Fig. 2.7d):

$$mg\,x + \frac{1}{2}kx^2$$

Furthermore, the potential energy of the system due to the change in elevation of the mass (note that $+x$ is downward) is $-mgx$. Thus the net potential energy of the system about the static equilibrium position is given by

$$\begin{aligned} U &= \text{potential energy of the spring} \\ &\quad + \text{change in potential energy due to change in elevation of the mass } m \\ &= mgx + \frac{1}{2}kx^2 - mgx = \frac{1}{2}kx^2 \end{aligned}$$

Since the expressions of T and U remain unchanged, the application of the principle of conservation of energy gives the same equation of motion, Eq. (2.3).

2.2.4 Solution

The solution of Eq. (2,3) can be found by assuming

$$x(t) = Ce^{st} \tag{2.11}$$

where C and s are constants to be determined. Substitution of Eq. (2.11) into Eq. (2.3) gives

$$C(ms^2 + k) = 0$$

Since C cannot be zero, we have

$$ms^2 + k = 0 \tag{2.12}$$

and hence

$$s = \pm\left(-\frac{k}{m}\right)^{1/2} = \pm i\omega_n \tag{2.13}$$

where $i = (-1)^{1/2}$ and

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} \tag{2.14}$$

Equation (2.12) is called the *auxiliary* or the *characteristic* equation corresponding to the differential Eq. (2.3). The two values of s given by Eq. (2.13) are the roots of the characteristic equation, also known as the *eigenvalues* or the *characteristic values* of the problem. Since both values of s satisfy Eq. (2.12), the general solution of Eq. (2.3) can be expressed as

$$x(t) = C_1 e^{i\omega_n t} + C_2 e^{-i\omega_n t} \tag{2.15}$$

where C_1 and C_2 are constants. By using the identities

$$e^{\pm i\alpha t} = \cos \alpha t \pm i \sin \alpha t$$

Eq. (2.15) can be rewritten as

$$x(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \tag{2.16}$$

where A_1 and A_2 are new constants. The constants C_1 and C_2 or A_1 and A_2 can be determined from the initial conditions of the system. Two conditions are to be specified to evaluate these constants uniquely. Note that the number of conditions to be specified is the same as the order of the governing differential equation. In the present case, if the values of displacement $x(t)$ and velocity $\dot{x}(t) = (dx/dt)(t)$ are specified as x_0 and $\dot{x}_0$ at $t = 0$, we have, from Eq. (2.16),

$$\begin{aligned} x(t = 0) &= A_1 = x_0 \\ \dot{x}(t = 0) &= \omega_n A_2 = \dot{x}_0 \end{aligned} \tag{2.17}$$

Hence $A_1 = x_0$ and $A_2 = \dot{x}_0/\omega_n$. Thus the solution of Eq. (2.3) subject to the initial conditions of Eq. (2.17) is given by

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t \tag{2.18}$$

2.2.5 Harmonic Motion

Equations (2.15), (2.16), and (2.18) are harmonic functions of time. The motion is symmetric about the equilibrium position of the mass m. The velocity is a maximum and the acceleration is zero each time the mass passes through this position. At the extreme displacements, the velocity is zero and the acceleration is a maximum. Since this represents simple harmonic motion (see Section 1.10), the spring-mass system itself is called a *harmonic oscillator*. The quantity ω_n, given by Eq. (2.14), represents the natural frequency of vibration of the system.

Equation (2.16) can be expressed in a different form by introducing the notation

$$\begin{aligned} A_1 &= A \cos \phi \\ A_2 &= A \sin \phi \end{aligned} \tag{2.19}$$

where A and ϕ are the new constants which can be expressed in terms of A_1 and A_2 as

$$\begin{aligned} A &= (A_1^2 + A_2^2)^{1/2} = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} = \text{amplitude} \\ \phi &= \tan^{-1}\left(\frac{A_2}{A_1} \right) = \tan^{-1}\left(\frac{\dot{x}_0}{x_0 \omega_n} \right) = \text{phase angle} \end{aligned} \tag{2.20}$$

Introducing Eq. (2.19) into Eq. (2.16), the solution can be written as

$$x(t) = A \cos(\omega_n t - \phi) \tag{2.21}$$

By using the relations

$$\begin{aligned} A_1 &= A_0 \sin \phi_0 \\ A_2 &= A_0 \cos \phi_0 \end{aligned} \tag{2.22}$$

Eq. (2.16) can also be expressed as

$$x(t) = A_0 \sin(\omega_n t + \phi_0) \tag{2.23}$$

where

$$A_0 = A = \left[x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right]^{1/2} \tag{2.24}$$

and

$$\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0} \right) \tag{2.25}$$

The nature of harmonic oscillation can be represented graphically as in Fig. 2.8(a). If $\vec{A}$ denotes a vector of magnitude A, which makes an angle $\omega_n t - \phi$ with respect to the vertical (x) axis, then the solution, Eq. (2.21), can be seen to be the projection of the vector $\vec{A}$ on the x-axis. The constants A_1 and A_2 of Eq. (2.16), given by Eq. (2.19), are merely the rectangular components of $\vec{A}$ along two orthogonal axes making angles ϕ and $-(\frac{\pi}{2} - \phi)$ with respect to the vector $\vec{A}$. Since the angle $\omega_n t - \phi$ is a linear function of time, it increases linearly with time; the entire diagram thus rotates anticlockwise at an angular velocity ω_n. As the diagram (Fig. 2.8a) rotates, the projection of $\vec{A}$ onto the x-axis varies harmonically so that the motion repeats itself every time the vector $\vec{A}$ sweeps an angle of 2π. The projection of $\vec{A}$, namely $x(t)$, is shown plotted in Fig. 2.8(b) as a function of time. The phase angle ϕ can also be interpreted as the angle between the origin and the first peak.

Thus, when the mass vibrates in a vertical direction, we can compute the natural frequency and the period of vibration by simply measuring the static deflection δ_{st}. It is not necessary that we know the spring stiffness k and the mass m.

2. From Eq. (2.21), the velocity $\dot{x}(t)$ and the acceleration $\ddot{x}(t)$ of the mass m at time t can be obtained as

$$\dot{x}(t) = \frac{dx}{dt}(t) = -\omega_n A \sin(\omega_n t - \phi) = \omega_n A \cos\left(\omega_n t - \phi + \frac{\pi}{2}\right)$$
$$\ddot{x}(t) = \frac{d^2x}{dt^2}(t) = -\omega_n^2 A \cos(\omega_n t - \phi) = \omega_n^2 A \cos(\omega_n t - \phi + \pi) \tag{2.31}$$

Equation (2.31) shows that the velocity leads the displacement by $\pi/2$ and the acceleration leads the displacement by π.

3. If the initial displacement (x_0) is zero, Eq. (2.21) becomes

$$x(t) = \frac{\dot{x}_0}{\omega_n}\cos\left(\omega_n t - \frac{\pi}{2}\right) = \frac{\dot{x}_0}{\omega_n}\sin \omega_n t \tag{2.32}$$

On the other hand, if the initial velocity ($\dot{x}_0$) is zero, the solution becomes

$$x(t) = x_0 \cos \omega_n t \tag{2.33}$$

4. The response of a single degree of freedom system can be represented in the displacement (x)-velocity $(\dot{x})$ plane, known as the state space or phase plane. For this we consider the displacement given by Eq. (2.21) and the corresponding velocity:

$$x(t) = A \cos(\omega_n t - \phi)$$

or

$$\cos(\omega_n t - \phi) = \frac{x}{A} \tag{2.34}$$

$$\dot{x}(t) = -A\,\omega_n \sin(\omega_n t - \phi)$$

or

$$\sin(\omega_n t - \phi) = -\frac{\dot{x}}{A\,\omega_n} = -\frac{y}{A} \tag{2.35}$$

where $y = \dot{x}/\omega_n$. By squaring and adding Eqs. (2.34) and (2.35), we obtain

$$\cos^2(\omega_n t - \phi) + \sin^2(\omega_n t - \phi) = 1$$

or

$$\frac{x^2}{A^2} + \frac{y^2}{A^2} = 1 \tag{2.36}$$

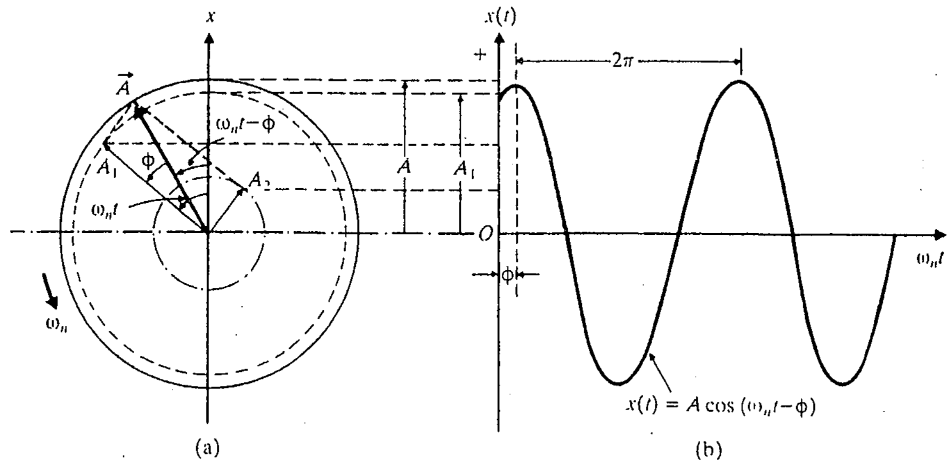


FIGURE 2.8 Graphical representation of the motion of a harmonic oscillator.

Note the following aspects of the spring-mass system:

1. If the spring-mass system is in a vertical position, as shown in Fig. 2.7(a), the circular natural frequency can be expressed as

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} \tag{2.26}$$

The spring constant k can be expressed in terms of the mass m from Eq. (2.9) as

$$k = \frac{W}{\delta_{st}} = \frac{mg}{\delta_{st}} \tag{2.27}$$

Substitution of Eq. (2.27) into Eq. (2.14) yields

$$\omega_n = \left(\frac{g}{\delta_{st}}\right)^{1/2} \tag{2.28}$$

Hence the natural frequency in cycles per second and the natural period are given by

$$f_n = \frac{1}{2\pi}\left(\frac{g}{\delta_{st}}\right)^{1/2} \tag{2.29}$$

$$\tau_n = \frac{1}{f_n} = 2\pi\left(\frac{\delta_{st}}{g}\right)^{1/2} \tag{2.30}$$

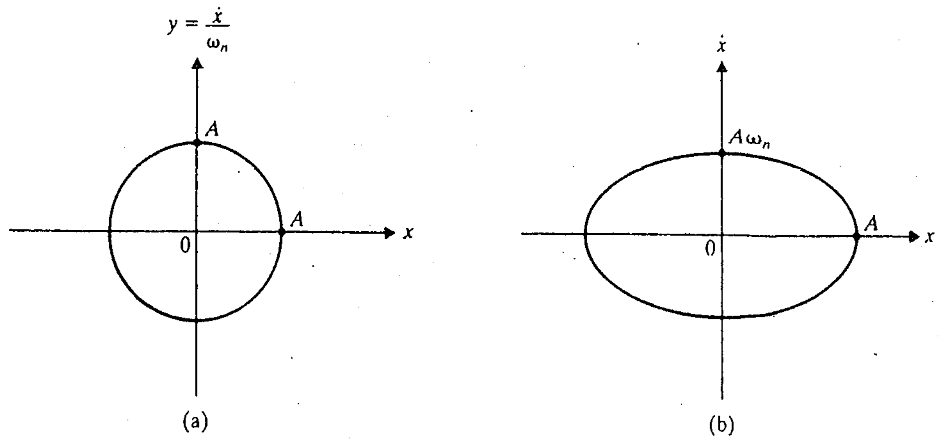


FIGURE 2.9

The graph of Eq. (2.36) in the (x, y)-plane is a circle, as shown in Fig. 2.9(a), and it constitutes the phase plane or state space representation of the undamped system. The radius of the circle, A, is determined by the initial conditions of motion. Note that the graph of Eq. (2.36) in the $(x, \dot{x})$ plane will be an ellipse, as shown in Fig. 2.9(b).

EXAMPLE 2.1 Natural Frequency of a Water Tank

The column of the water tank shown in Fig. 2.10 is 300 ft high and is made of reinforced concrete with a tubular cross section of inner diameter 8 ft and outer diameter 10 ft. The tank weighs 6×10^5 lb with water. Find the natural frequency of transverse vibration of the water tank by neglecting the mass of the column.

Given: Water tank of Fig. 2.10.

Find: Natural frequency of vibration of the tank in transverse direction.

Approach: Find the stiffness of the column and consider the tank as a single degree of freedom system.

Assumptions:

1. Water tank is a point mass.
2. Column has a uniform cross section.
3. Mass of the column is negligible.

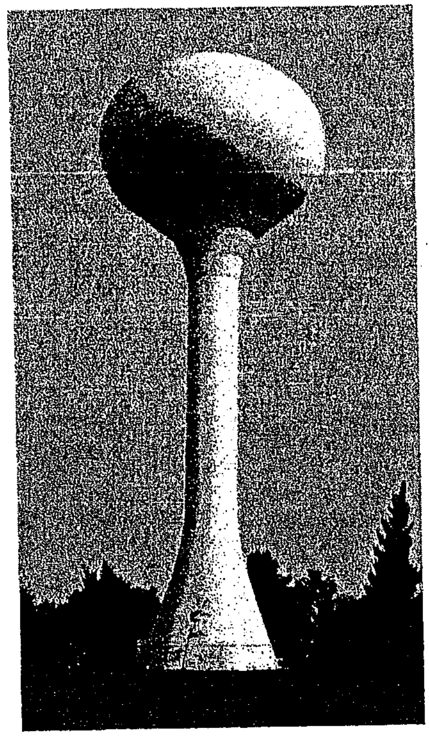


FIGURE 2.10 Elevated tank. (Photo courtesy of West Lafayette Water Company.)

Solution: The water tank can be considered as a cantilever beam with a concentrated load (weight) at the free end. The transverse deflection of the beam, δ, due to a load P is given by $\frac{Pl^3}{3EI}$, where l is the length, E is the Young's modulus and I is the area moment of inertia of the cross section of the beam. The stiffness of the beam (column of the tank) is given by

$$k = \frac{P}{\delta} = \frac{3\,EI}{l^3}$$

In the present case, $l = 3600$ in, $E = 4 \times 10^6$ psi,

$$I = \frac{\pi}{64}(d_0^4 - d_i^4) = \frac{\pi}{64}(120^4 - 96^4) = 600.9554 \times 10^4 \text{ in}^4$$

and hence

$$k = \frac{3\,(4 \times 10^6)\,(600.9554 \times 10^4)}{3600^3} = 1545.6672 \text{ lb/in}$$

The natural frequency of the water tank in transverse direction is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1545.6672 \times 386.4}{6 \times 10^5}} = 0.9977 \text{ rad/sec}$$

■

EXAMPLE 2.2 Natural Frequency of Cockpit of a Firetruck

The cockpit of a firetruck is located at the end of a telescoping boom, as shown in Fig. 2.11(a). The cockpit, along with the fireman, weighs 2000 N. Find the natural frequency of vibration of the cockpit in the vertical direction.

Data: Young's modulus of the material: $E = 2.1 \times 10^{11}$ N/m², Lengths: $l_1 = l_2 = l_3 =$ 3 m, cross-sectional areas: $A_1 = 20$ cm², $A_2 = 10$ cm², $A_3 = 5$ cm².

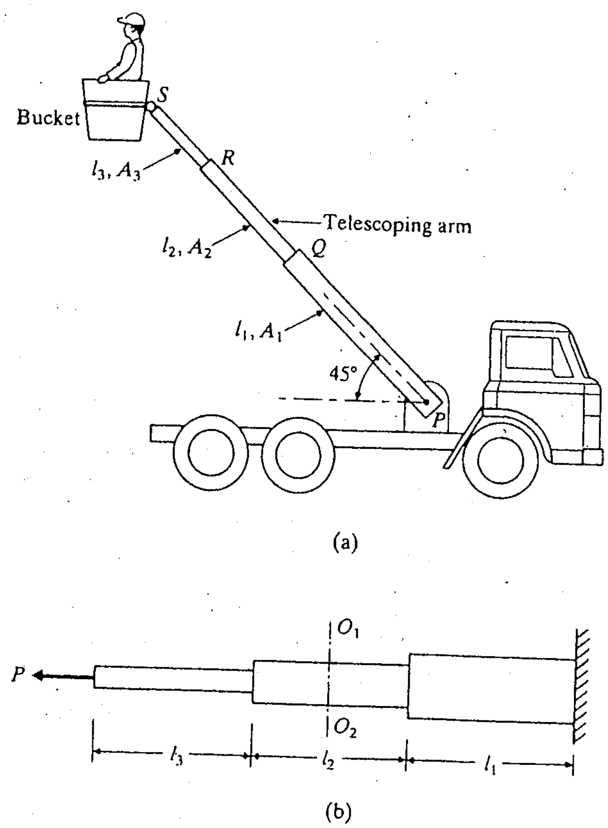


FIGURE 2.11

up by a distance $2\,W/k_1$, and the center of pulley 2 (point B) moves down by $2\,W/k_2$. Thus the total movement of the mass m (point O) is

$$2\left(\frac{2W}{k_1} + \frac{2W}{k_2}\right)$$

as the rope on either side of the pulley is free to move the mass downward. If k_{eq} denotes the equivalent spring constant of the system,

$$\frac{\text{Weight of the mass}}{\text{Equivalent spring constant}} = \text{Net displacement of the mass}$$

$$\frac{W}{k_{eq}} = 4W\left(\frac{1}{k_1} + \frac{1}{k_2}\right) = \frac{4W(k_1 + k_2)}{k_1 k_2}$$

$$k_{eq} = \frac{k_1 k_2}{4(k_1 + k_2)} \tag{E.1}$$

By displacing mass m from the static equilibrium position by x, the equation of motion of the mass can be written as

$$m\ddot{x} + k_{eq}x = 0 \tag{E.2}$$

and hence the natural frequency is given by

$$\omega_n = \left(\frac{k_{eq}}{m}\right)^{1/2} = \left[\frac{k_1 k_2}{4m(k_1 + k_2)}\right]^{1/2} \text{rad/sec} \tag{E.3}$$

or

$$f_n = \frac{\omega_n}{2\pi} = \frac{1}{4\pi}\left[\frac{k_1 k_2}{m(k_1 + k_2)}\right]^{1/2} \text{cycles/sec} \tag{E.4}$$

■

2.3 Free Vibration of an Undamped Torsional System

If a rigid body oscillates about a specific reference axis, the resulting motion is called *torsional vibration*. In this case, the displacement of the body is measured in terms of an angular coordinate. In a torsional vibration problem, the restoring moment may be due to the torsion of an elastic member or to the unbalanced moment of a force or couple.

Figure 2.13 shows a disc, which has a polar mass moment of inertia J_0, mounted at one end of a solid circular shaft, the other end of which is fixed. Let the angular rotation of the disc about the axis of the shaft be θ; θ also represents the angle of twist of the shaft. From the theory of torsion of circular shafts [2.1], we have the relation

$$M_t = \frac{GI_o}{l} \tag{2.37}$$

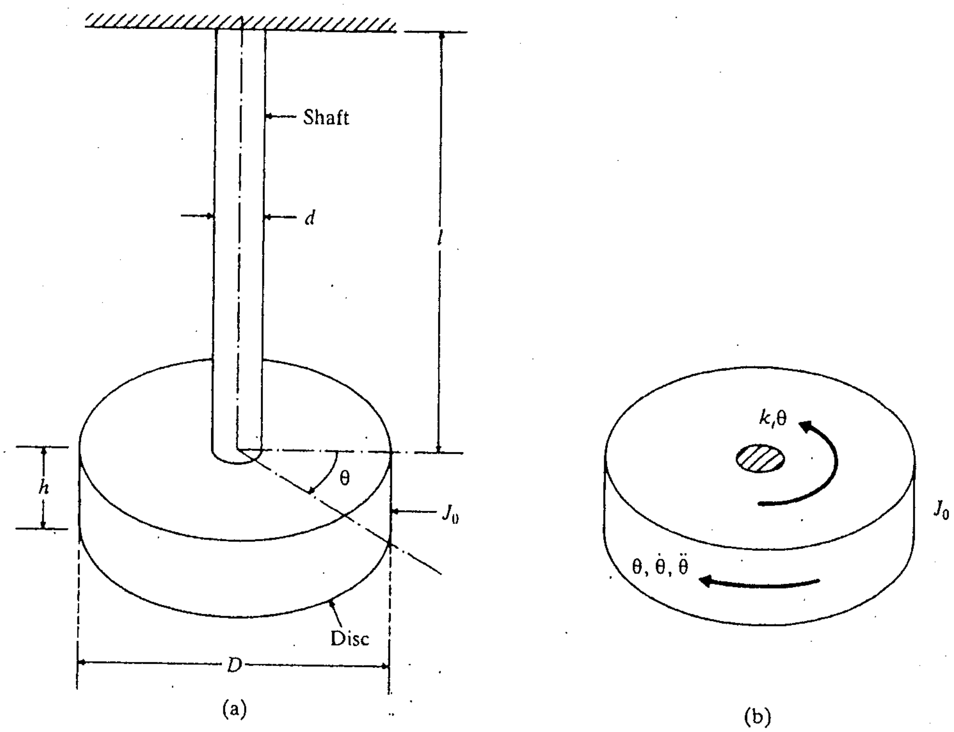


FIGURE 2.13 Torsional vibration of a disc.

where M_t is the torque that produces the twist θ, G is the shear modulus, l is the length of the shaft, I_o is the polar moment of inertia of the cross section of the shaft given by

$$I_o = \frac{\pi d^4}{32} \tag{2.38}$$

and d is the diameter of the shaft. If the disc is displaced by θ from its equilibrium position, the shaft provides a restoring torque of magnitude M_t. Thus the shaft acts as a torsional spring with a torsional spring constant

$$k_t = \frac{M_t}{\theta} = \frac{GI_o}{l} = \frac{\pi G d^4}{32 l} \tag{2.39}$$

2.3.1 Equation of Motion

The equation of the angular motion of the disc about its axis can be derived by using Newton's second law or any of the methods discussed in Section 2.2.2. By considering the free-body diagram of the disc (Fig. 2.13b), we can derive the equation of motion by applying Newton's second law of motion:

$$J_0 \ddot{\theta} + k_t \theta = 0 \tag{2.40}$$

which can be seen to be identical to Eq. (2.3) if the polar mass moment of inertia J_0, the angular displacement θ, and the torsional spring constant k_t are replaced by the mass m, the displacement x, and the linear spring constant k, respectively. Thus the natural circular frequency of the torsional system is

$$\omega_n = \left(\frac{k_t}{J_0}\right)^{1/2} \tag{2.41}$$

and the period and frequency of vibration in cycles per second are

$$\tau_n = 2\pi\left(\frac{J_0}{k_t}\right)^{1/2} \tag{2.42}$$

$$f_n = \frac{1}{2\pi}\left(\frac{k_t}{J_0}\right)^{1/2} \tag{2.43}$$

Note the following aspects of this system:

1. If the cross section of the shaft supporting the disc is not circular, an appropriate torsional spring constant is to be used [2.4, 2.5].
2. The polar mass moment of inertia of a disc is given by

$$J_0 = \frac{\rho h \pi D^4}{32} = \frac{WD^2}{8g}$$

   where ρ is the mass density, h is the thickness, D is the diameter, and W is the weight of the disc.
3. The torsional spring-inertia system shown in Fig. 2.13 is referred to as a *torsional pendulum.* One of the most important applications of a torsional pendulum is in a mechanical clock, where a ratchet and pawl convert the regular oscillation of a small torsional pendulum into the movements of the hands.

2.3.2 Solution

The general solution of Eq. (2.40) can be obtained, as in the case of Eq. (2.3):

$$\theta(t) = A_1 \cos \omega_n t + A_2 \sin \omega_n t \tag{2.44}$$

where ω_n is given by Eq. (2.41), and A_1 and A_2 can be determined from the initial conditions. If

$$\theta(t = 0) = \theta_0 \quad \text{and} \quad \dot{\theta}(t = 0) = \frac{d\theta}{dt}(t = 0) = \dot{\theta}_0 \tag{2.45}$$

the constants A_1 and A_2 can be found:

$$A_1 = \theta_0$$
$$A_2 = \dot{\theta}_0/\omega_n \tag{2.46}$$

Equation (2.44) can also be seen to represent a simple harmonic motion.

Equation (2.54) shows that $\theta(t)$ increases exponentially with time; hence the motion is unstable. The physical reason for this is that the restoring moment due to the spring ($2kl^2\theta$), which tries to bring the system to equilibrium position, is less than the nonrestoring moment due to gravity $[-W(l/2)\theta]$, which tries to move the mass away from the equilibrium position. Although the stability conditions are illustrated with reference to Fig. 2.17 in this section, similar conditions need to be examined in the vibration analysis of many engineering systems.

2.5 Rayleigh's Energy Method

For a single degree of freedom system, the equation of motion was derived using the energy method in Section 2.2.2. In this section, we shall use the energy method to find the natural frequencies of single degree of freedom systems. The principle of conservation of energy, in the context of an undamped vibrating system, can be restated as

$$T_1 + U_1 = T_2 + U_2 \tag{2.55}$$

where the subscripts 1 and 2 denote two different instants of time. Specifically, we use the subscript 1 to denote the time when the mass is passing through its static equilibrium position and choose $U_1 = 0$ as reference for the potential energy. If we let the subscript 2 indicate the time corresponding to the maximum displacement of the mass, we have $T_2 = 0$. Thus Eq. (2.55) becomes

$$T_1 + 0 = 0 + U_2 \tag{2.56}$$

If the system is undergoing harmonic motion, then T_1 and U_2 denote the maximum values of T and U, respectively, and Eq. (2.56) becomes

$$T_{max} = U_{max} \tag{2.57}$$

The application of Eq. (2.57), which is also known as *Rayleigh's energy method,* gives the natural frequency of the system directly, as illustrated in the following examples.

EXAMPLE 2.6 Manometer for Diesel Engine

The exhaust from a single-cylinder four-stroke diesel engine is to be connected to a silencer, and the pressure therein is to be measured with a simple U-tube manometer (see Fig. 2.18). Calculate the minimum length of the manometer tube so that the natural frequency of oscillation of the mercury column will be 3.5 times slower than the frequency of the pressure fluctuations in the silencer at an engine speed of 600 rpm. The frequency of pressure fluctuations in the silencer is equal to

$$\frac{\text{Number of cylinders} \times \text{Speed of the engine}}{2}$$

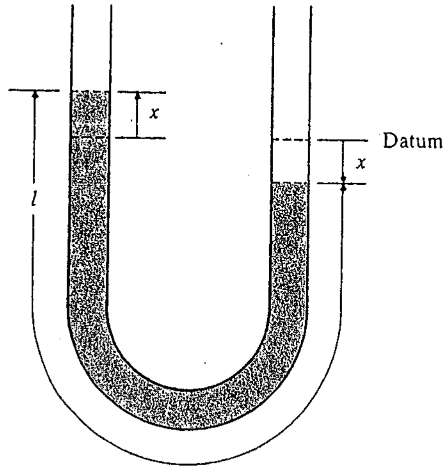


FIGURE 2.18

Given: U-tube manometer, engine speed = 600 rpm, and natural frequency of oscillation = 3.5 times slower than the frequency of pressure fluctuations.

Find: Minimum length of the manometer tube.

Approach: Use energy method to find the natural frequency.

Solution:

1. *Natural frequency of oscillation of the liquid column:* Let the datum in Fig. 2.18 be taken as the equilibrium position of the liquid. If the displacement of the liquid column from the equilibrium position is denoted by x, the change in potential energy is given by

   U = potential energy of raised liquid column + potential energy of depressed liquid column

   = (weight of mercury raised × displacement of the C.G. of the segment) + (weight of mercury depressed × displacement of the C.G. of the segment)

   $$= (Ax\gamma)\frac{x}{2} + (Ax\gamma)\frac{x}{2} = A\gamma x^2 \tag{E.1}$$

   where A is the cross-sectional area of the mercury column and γ is the specific weight of mercury. The change in kinetic energy is given by

   $$T = \frac{1}{2}(\text{mass of mercury})(\text{velocity})^2$$

   $$= \frac{1}{2}\frac{Al\gamma}{g}(\dot{x})^2 \tag{E.2}$$

where l is the length of the mercury column. By assuming harmonic motion, we can write

$$x(t) = X \cos \omega_n t \tag{E.3}$$

where X is the maximum displacement and ω_n is the natural frequency. By substituting Eq. (E.3) into Eqs. (E.1) and (E.2), we obtain

$$U = U_{\max} \cos^2 \omega_n t \tag{E.4}$$

$$T = T_{\max} \sin^2 \omega_n t \tag{E.5}$$

where

$$U_{\max} = A\gamma X^2 \tag{E.6}$$

and

$$T_{\max} = \frac{1}{2}\frac{A\gamma l \omega_n^2}{g} X^2 \tag{E.7}$$

By equating $U_{\max}$ to $T_{\max}$, we obtain the natural frequency:

$$\omega_n = \left(\frac{2g}{l}\right)^{1/2} \tag{E.8}$$

2. *Length of the mercury column:* The frequency of pressure fluctuations in the silencer

$$= \frac{1 \times 600}{2}$$

$$= 300 \text{ rev/min}$$

$$= \frac{300 \times 2\pi}{60} = 10\pi \text{ rad/sec} \tag{E.9}$$

Thus the frequency of oscillations of the liquid column in the manometer is $10\pi/3.5 = 9.0$ rad/sec. By using Eq. (E.8), we obtain

$$\left(\frac{2g}{l}\right)^{1/2} = 9.0 \tag{E.10}$$

or

$$l = \frac{2.0 \times 9.81}{(9.0)^2} = 0.243 \text{ m} \tag{E.11}$$

■

EXAMPLE 2.7 Effect of Mass on ω_n of a Spring

Determine the effect of the mass of the spring on the natural frequency of the spring-mass system shown in Fig. 2.19.

The maximum kinetic energy of the beam itself (T_{max}) is given by

$$T_{max} = \frac{1}{2}\int_0^l \frac{m}{l}\left\{\dot{y}(x)\right\}^2 dx \tag{E.2}$$

where m is the total mass and (m/l) is the mass per unit length of the beam. Equation (E.1) can be used to express the velocity variation, $\dot{y}(x)$, as

$$\dot{y}(x) = \frac{\dot{y}_{max}}{2l^3}(3\,x^2\,l - x^3) \tag{E.3}$$

and hence Eq. (E.2) becomes

$$T_{max} = \frac{m}{2\,l}\left(\frac{\dot{y}_{max}}{2\,l^3}\right)^2 \int_0^l (3\,x^2\,l - x^3)^2\,dx$$

$$= \frac{1}{2}\,\frac{m}{l}\,\frac{\dot{y}^2_{max}}{4\,l^6}\left(\frac{33}{35}\,l^7\right) = \frac{1}{2}\left(\frac{33}{35}m\right)\dot{y}^2_{max} \tag{E.4}$$

If m_{eq} denotes the equivalent mass of the cantilever (water tank) at the free end, its maximum kinetic energy can be expressed as

$$T_{max} = \frac{1}{2}\,m_{eq}\,\dot{y}^2_{max} \tag{E.5}$$

By equating Eqs. (E.4) and (E.5), we obtain

$$m_{eq} = \frac{33}{35}\,m \tag{E.6}$$

Thus the total effective mass acting at the end of the cantilever beam is given by

$$M_{eff} = M + m_{eq} \tag{E.7}$$

where M is the mass of the water tank. The natural frequency of transverse vibration of the water tank is given by

$$\omega_n = \sqrt{\frac{k}{M_{eff}}} = \sqrt{\frac{k}{M + \frac{33}{35}m}} \tag{E.8}$$

■

2.6 Free Vibration with Viscous Damping

2.6.1 Equation of Motion

As stated in Section 1.9, the viscous damping force F is proportional to the velocity $\dot{x}$ or v and can be expressed as

$$F = -c\dot{x} \tag{2.58}$$

where c is the damping constant or coefficient of viscous damping and the negative sign indicates that the damping force is opposite to the direction of velocity. A single degree of freedom system with a viscous damper is shown in Fig. 2.21. If x is measured from the equilibrium position of the mass m, the application of Newton's law yields the equation of motion:

$$m\ddot{x} = -c\dot{x} - kx$$

or

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{2.59}$$

2.6.2 Solution

To solve Eq. (2.59), we assume a solution in the form

$$x(t) = Ce^{st} \tag{2.60}$$

where C and s are undetermined constants. Inserting this function into Eq. (2.59) leads to the characteristic equation

$$ms^2 + cs + k = 0 \tag{2.61}$$

the roots of which are

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{2.62}$$

These roots give two solutions to Eq. (2.59):

$$x_1(t) = C_1 e^{s_1 t} \quad \text{and} \quad x_2(t) = C_2 e^{s_2 t} \tag{2.63}$$

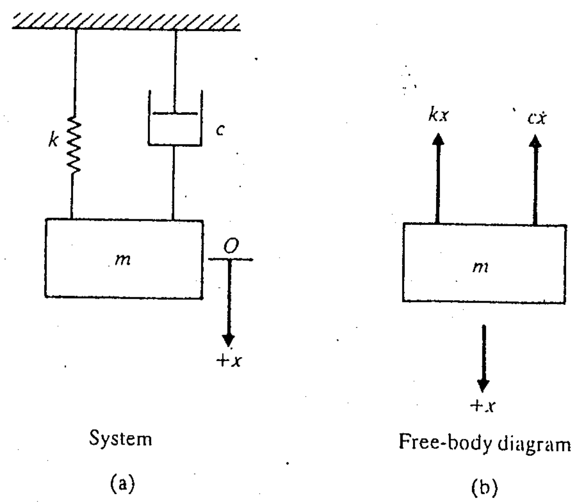


FIGURE 2.21 Single degree of freedom system with viscous damper.

Thus the general solution of Eq. (2.59) is given by a combination of the two solutions $x_1(t)$ and $x_2(t)$:

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$$
$$= C_1 e^{\left\{-\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\} t} + C_2 e^{\left\{-\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}\right\} t} \tag{2.64}$$

where C_1 and C_2 are arbitrary constants to be determined from the initial conditions of the system.

Critical Damping Constant and the Damping Ratio. The critical damping c_c is defined as the value of the damping constant c for which the radical in Eq. (2.62) becomes zero:

$$\left(\frac{c_c}{2m}\right)^2 - \frac{k}{m} = 0$$

or

$$c_c = 2m\sqrt{\frac{k}{m}} = 2\sqrt{km} = 2m\omega_n \tag{2.65}$$

For any damped system, the damping ratio ζ is defined as the ratio of the damping constant to the critical damping constant:

$$\zeta = c/c_c \tag{2.66}$$

Using Eqs. (2.66) and (2.65), we can write

$$\frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \zeta\omega_n \tag{2.67}$$

and hence

$$s_{1,2} = \left(-\zeta \pm \sqrt{\zeta^2 - 1}\right)\omega_n \tag{2.68}$$

Thus the solution, Eq. (2.64), can be written as

$$x(t) = C_1 e^{\left(-\zeta + \sqrt{\zeta^2 - 1}\right)\omega_n t} + C_2 e^{\left(-\zeta - \sqrt{\zeta^2 - 1}\right)\omega_n t} \tag{2.69}$$

The nature of the roots s_1 and s_2 and hence the behavior of the solution, Eq. (2.69), depends upon the magnitude of damping. It can be seen that the case $\zeta = 0$ leads to the undamped vibrations discussed in Section 2.2. Hence we assume that $\zeta \neq 0$ and consider the following three cases.

Case 1. *Underdamped system* ($\zeta < 1$ or $c < c_c$ or $c/2m < \sqrt{k/m}$). For this condition, $(\zeta^2 - 1)$ is negative and the roots s_1 and s_2 can be expressed as

$$s_1 = \left(-\zeta + i\sqrt{1 - \zeta^2}\right)\omega_n$$
$$s_2 = \left(-\zeta - i\sqrt{1 - \zeta^2}\right)\omega_n$$

and the solution, Eq. (2.69), can be written in different forms:

$$
\begin{aligned}
x(t) &= C_1 e^{(-\zeta + i\sqrt{1-\zeta^2})\omega_n t} + C_2 e^{(-\zeta - i\sqrt{1-\zeta^2})\omega_n t} \\
&= e^{-\zeta\omega_n t}\left\{ C_1 e^{i\sqrt{1-\zeta^2}\omega_n t} + C_2 e^{-i\sqrt{1-\zeta^2}\omega_n t} \right\} \\
&= e^{-\zeta\omega_n t}\left\{ (C_1 + C_2)\cos\sqrt{1-\zeta^2}\,\omega_n t + i(C_1 - C_2)\sin\sqrt{1-\zeta^2}\,\omega_n t \right\} \\
&= e^{-\zeta\omega_n t}\left\{ C_1'\cos\sqrt{1-\zeta^2}\,\omega_n t + C_2'\sin\sqrt{1-\zeta^2}\,\omega_n t \right\} \\
&= X e^{-\zeta\omega_n t}\sin\left(\sqrt{1-\zeta^2}\,\omega_n t + \phi \right) \\
&= X_0 e^{-\zeta\omega_n t}\cos\left(\sqrt{1-\zeta^2}\,\omega_n t - \phi_0 \right)
\end{aligned}
\tag{2.70}
$$

where (C_1', C_2'), (X, ϕ), and (X_0, ϕ_0) are arbitrary constants to be determined from the initial conditions.

For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, C_1' and C_2' can be found:

$$C_1' = x_0 \qquad \text{and} \qquad C_2' = \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n} \tag{2.71}$$

and hence the solution becomes

$$x(t) = e^{-\zeta\omega_n t}\left\{ x_0\cos\sqrt{1-\zeta^2}\,\omega_n t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\sqrt{1-\zeta^2}\,\omega_n}\sin\sqrt{1-\zeta^2}\,\omega_n t \right\} \tag{2.72}$$

The constants (X, ϕ) and (X_0, ϕ_0) can be expressed as

$$X = X_0 = \sqrt{(C_1')^2 + (C_2')^2} \tag{2.73}$$

$$\phi = \tan^{-1}(C_1'/C_2') \tag{2.74}$$

$$\phi_0 = \tan^{-1}(-C_2'/C_1') \tag{2.75}$$

The motion described by Eq. (2.72) is a damped harmonic motion of angular frequency $\sqrt{1-\zeta^2}\,\omega_n$, but because of the factor $e^{-\zeta\omega_n t}$, the amplitude decreases exponentially with time, as shown in Fig. 2.22. The quantity

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \tag{2.76}$$

is called the *frequency of damped vibration*. It can be seen that the frequency of damped vibration ω_d is always less than the undamped natural frequency ω_n. The

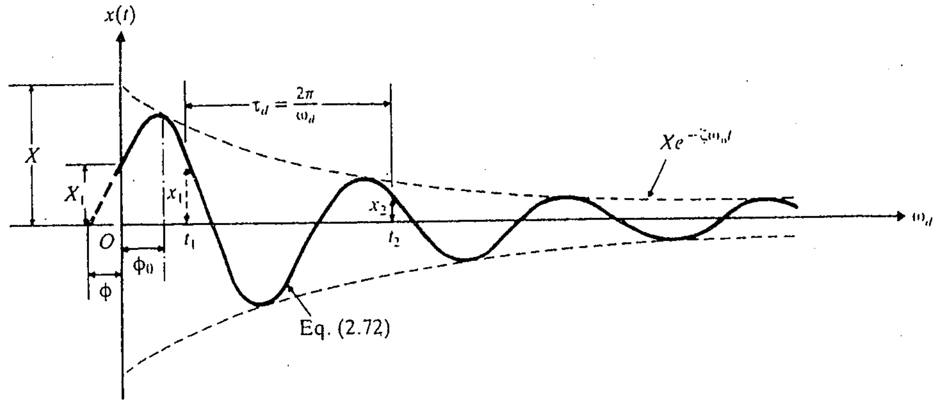


FIGURE 2.22 Underdamped solution.

decrease in the frequency of damped vibration with increasing amount of damping, given by Eq. (2.76), is shown graphically in Fig. 2.23. The underdamped case is very important in the study of mechanical vibrations, as it is the only case which leads to an oscillatory motion [2.10].

Case 2. *Critically damped system* ($\zeta = 1$ or $c = c_c$ or $c/2m = \sqrt{k/m}$). In this case the two roots s_1 and s_2 in Eq. (2.68) are equal:

$$s_1 = s_2 = -\frac{c_c}{2m} = -\omega_n \tag{2.77}$$

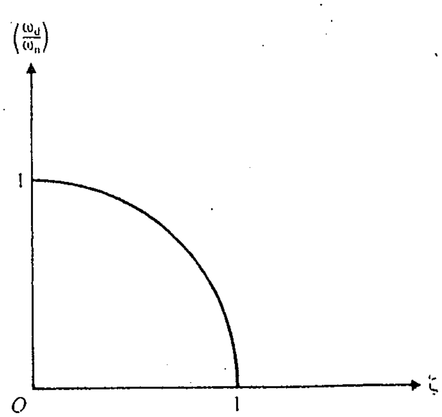


FIGURE 2.23 Variation of ω_d with damping.

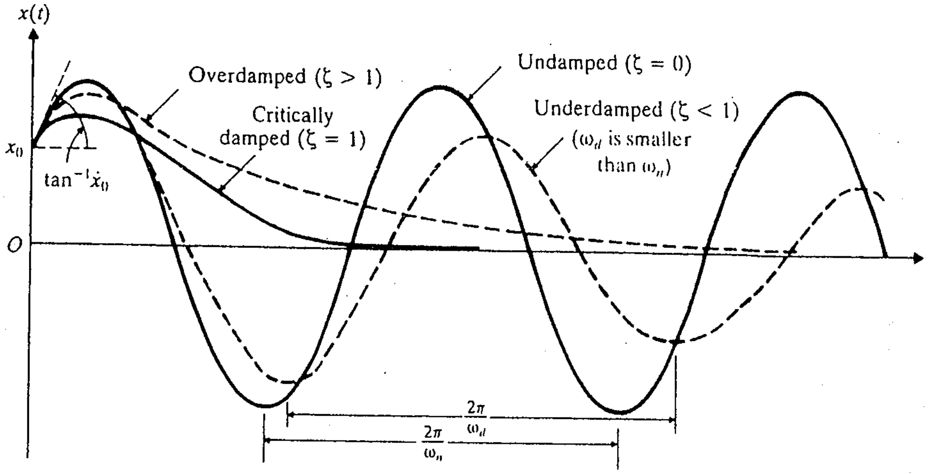


FIGURE 2.24 Comparison of motions with different types of damping.

Because of the repeated roots, the solution of Eq. (2.59) is given by [2.6][1]

$$x(t) = (C_1 + C_2 t)e^{-\omega_n t} \tag{2.78}$$

The application of the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$ for this case gives

$$\begin{aligned} C_1 &= x_0 \\ C_2 &= \dot{x}_0 + \omega_n x_0 \end{aligned} \tag{2.79}$$

and the solution becomes

$$x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t]e^{-\omega_n t} \tag{2.80}$$

It can be seen that the motion represented by Eq. (2.80) is *aperiodic* (i.e., nonperiodic). Since $e^{-\omega_n t} \to 0$ as $t \to \infty$, the motion will eventually diminish to zero, as indicated in Fig. 2.24.

Case 3. *Overdamped system* ($\zeta > 1$ or $c > c_c$ or $c/2m > \sqrt{k/m}$). As $\sqrt{\zeta^2 - 1} > 0$, Eq. (2.68) shows that the roots s_1 and s_2 are real and distinct and are given by

$$s_1 = (-\zeta + \sqrt{\zeta^2 - 1})\omega_n < 0$$
$$s_2 = (-\zeta - \sqrt{\zeta^2 - 1})\omega_n < 0$$

[1]Equation (2.78) can also be obtained by making ζ approach unity in the limit in Eq. (2.72). As $\zeta \to 1$, $\omega_d \to 0$; hence $\cos \omega_d t \to 1$ and $\sin \omega_d t \to \omega_d t$. Thus Eq. (2.72) yields

$$x(t) = e^{-\omega_n t}(C_1' + C_2'\omega_d t) = (C_1 + C_2 t)e^{-\omega_n t}$$

where $C_1 = C_1'$ and $C_2 = C_2'\omega_d$ are new constants.

with $s_2 << s_1$. In this case, the solution, Eq. (2.69), can be expressed as

$$x(t) = C_1 e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + C_2 e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t} \quad (2.81)$$

For the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, the constants C_1 and C_2 can be obtained:

$$C_1 = \frac{x_0\omega_n (\zeta + \sqrt{\zeta^2 - 1}) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}}$$

$$C_2 = \frac{-x_0\omega_n (\zeta - \sqrt{\zeta^2 - 1}) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2 - 1}} \quad (2.82)$$

Equation (2.81) shows that the motion is aperiodic regardless of the initial conditions imposed on the system. Since roots s_1 and s_2 are both negative, the motion diminishes exponentially with time, as shown in Fig. 2.24.

Note the following aspects of these systems:

1. The nature of the roots s_1 and s_2 with varying values of damping c or ζ can be shown in a complex plane. In Fig. 2.25, the horizontal and vertical axes are chosen as the real and imaginary axes. The semicircle represents the locus of the roots s_1 and s_2 for different values of ζ in the range $0 < \zeta < 1$. This figure permits us to see instantaneously the effect of the parameter ζ on the behavior of the system. We find that for $\zeta = 0$, we obtain the imaginary roots $s_1 = i\omega_n$ and $s_2 = -i\omega_n$, leading to the solution given in Eq. (2.15). For $0 < \zeta < 1$, the roots s_1 and s_2 are complex conjugate and are located symmetrically about the

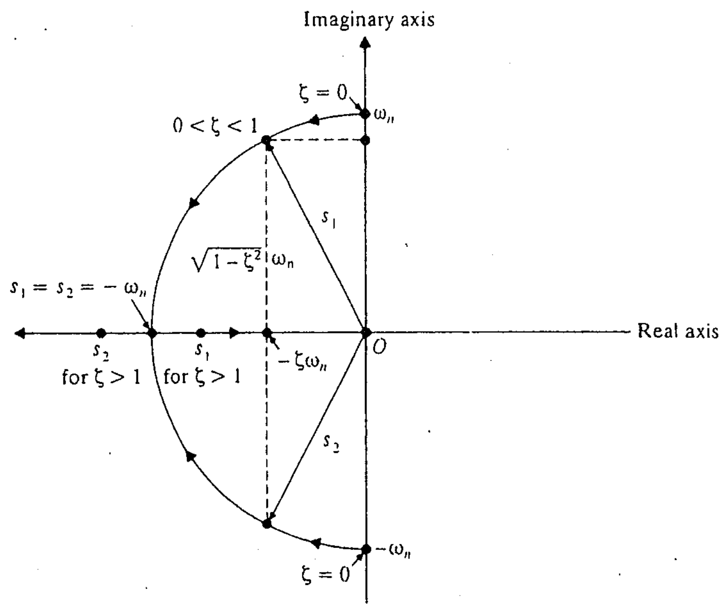


FIGURE 2.25 Locus of s_1 and s_2.

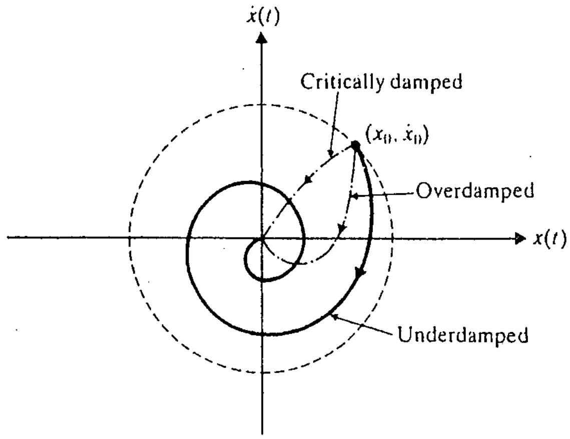


FIGURE 2.26

real axis. As the value of ζ approaches 1, both roots approach the point $-\omega_n$ on the real axis. If $\zeta > 1$, both roots lie on the real axis, one increasing and the other decreasing. In the limit when $\zeta \to \infty$, $s_1 \to 0$ and $s_2 \to -\infty$. The value $\zeta = 1$ can be seen to represent a transition stage, below which both roots are complex and above which both roots are real.

2. A critically damped system will have the smallest damping required for aperiodic motion; hence the mass returns to the position of rest in the shortest possible time without overshooting. The property of critical damping is used in many practical applications. For example, large guns have dashpots with critical damping value, so that they return to their original position after recoil in the minimum time without vibrating. If the damping provided were more than the critical value, some delay would be caused before the next firing.
3. The free damped response of a single degree of freedom system can be represented in phase plane or state space as indicated in Fig. 2.26.

2.6.3 Logarithmic Decrement

The logarithmic decrement represents the rate at which the amplitude of a free damped vibration decreases. It is defined as the natural logarithm of the ratio of any two successive amplitudes. Let t_1 and t_2 denote the times corresponding to two consecutive amplitudes (displacements), measured one cycle apart for an underdamped system, as in Fig. 2.22. Using Eq. (2.70), we can form the ratio

$$\frac{x_1}{x_2} = \frac{X_0 e^{-\zeta\omega_n t_1}\cos(\omega_d t_1 - \phi_0)}{X_0 e^{-\zeta\omega_n t_2}\cos(\omega_d t_2 - \phi_0)} \tag{2.83}$$

But $t_2 = t_1 + \tau_d$ where $\tau_d = 2\pi/\omega_d$ is the period of damped vibration. Hence $\cos(\omega_d t_2 - \phi_0) = \cos(2\pi + \omega_d t_1 - \phi_0) = \cos(\omega_d t_1 - \phi_0)$, and Eq. (2.83) can be written as

$$\frac{x_1}{x_2} = \frac{e^{-\zeta\omega_n t_1}}{e^{-\zeta\omega_n(t_1+\tau_d)}} = e^{\zeta\omega_n\tau_d} \tag{2.84}$$

The logarithmic decrement δ can be obtained from Eq. (2.84):

$$\delta = \ln\frac{x_1}{x_2} = \zeta\omega_n\tau_d = \zeta\omega_n\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi}{\omega_d}\cdot\frac{c}{2m} \tag{2.85}$$

For small damping, Eq. (2.85) can be approximated:

$$\delta \simeq 2\pi\zeta \qquad \text{if} \qquad \zeta << 1 \tag{2.86}$$

Figure 2.27 shows the variation of the logarithmic decrement δ with ζ as given by Eqs. (2.85) and (2.86). It can be noticed that for values up to $\zeta = 0.3$, the two curves are difficult to distinguish.

The logarithmic decrement is dimensionless and is actually another form of the dimensionless damping ratio ζ. Once δ is known, ζ can be found by solving Eq. (2.85):

$$\zeta = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} \tag{2.87}$$

If we use Eq. (2.86) instead of Eq. (2.85), we have

$$\zeta \simeq \frac{\delta}{2\pi} \tag{2.88}$$

If the damping in the given system is not known, we can determine it experimentally by measuring any two consecutive displacements x_1 and x_2. By taking the natural logarithm of the ratio of x_1 and x_2, we obtain δ. By using Eq. (2.87), we can compute

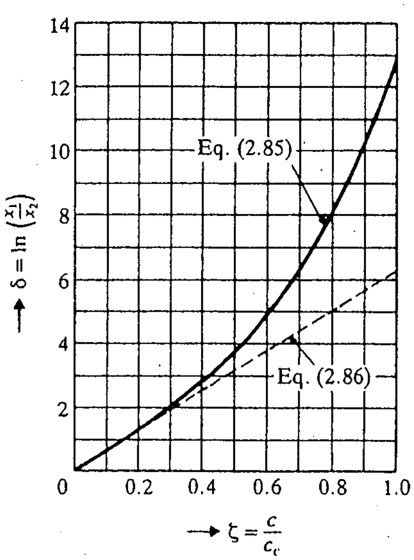


FIGURE 2.27 Variation of logarithmic decrement with damping.

the damping ratio ζ. In fact, the damping ratio ζ can also be found by measuring two displacements separated by any number of complete cycles. If x_1 and x_{m+1} denote the amplitudes corresponding to times t_1 and $t_{m+1} = t_1 + m\tau_d$ where m is an integer, we obtain

$$\frac{x_1}{x_{m+1}} = \frac{x_1}{x_2}\frac{x_2}{x_3}\frac{x_3}{x_4} \cdots \frac{x_m}{x_{m+1}} \tag{2.89}$$

Since any two successive displacements separated by one cycle satisfy the equation

$$\frac{x_j}{x_{j+1}} = e^{\zeta\omega_n\tau_d} \tag{2.90}$$

Eq. (2.89) becomes

$$\frac{x_1}{x_{m+1}} = (e^{\zeta\omega_n\tau_d})^m = e^{m\zeta\omega_n\tau_d} \tag{2.91}$$

Equations (2.91) and (2.85) yield

$$\delta = \frac{1}{m}\ln\left(\frac{x_1}{x_{m+1}}\right) \tag{2.92}$$

which can be substituted into Eq. (2.87) or Eq. (2.88) to obtain the viscous damping ratio ζ.

2.6.4 Energy Dissipated in Viscous Damping

In a viscously damped system, the rate of change of energy with time (dW/dt) is given by

$$\frac{dW}{dt} = \text{force} \times \text{velocity} = Fv = -cv^2 = -c\left(\frac{dx}{dt}\right)^2 \tag{2.93}$$

using Eq. (2.58). The negative sign in Eq. (2.93) denotes that energy dissipates with time. Assume a simple harmonic motion as $x(t) = X \sin \omega_d t$, where X is the amplitude of motion and the energy dissipated in a complete cycle is given by[2]

$$\Delta W = \int_{t=0}^{(2\pi/\omega_d)} c\left(\frac{dx}{dt}\right)^2 dt = \int_0^{2\pi} cX^2\omega_d\cos^2\omega_d t \cdot d(\omega_d t)$$
$$= \pi c\,\omega_d X^2 \tag{2.94}$$

[2] In the case of a damped system, simple harmonic motion $x(t) = X\cos\omega_d t$ is possible only when the steady-state response is considered under a harmonic force of frequency ω_d (see Section 3.4). The loss of energy due to the damper is supplied by the excitation under steady-state forced vibration [2.7].

as the maximum kinetic energy ($\frac{1}{2}mv_{\max}^2 = \frac{1}{2}mX^2\omega_d^2$), the two being approximately equal for small values of damping. Thus

$$\frac{\Delta W}{W} = \frac{\pi c \omega_d X^2}{\frac{1}{2}m\omega_d^2 X^2} = 2\left(\frac{2\pi}{\omega_d}\right)\left(\frac{c}{2m}\right) = 2\delta \simeq 4\pi\zeta = \text{constant} \quad (2.99)$$

using Eqs. (2.85) and (2.88). The quantity $\Delta W/W$ is called the *specific damping capacity* and is useful in comparing the damping capacity of engineering materials. Another quantity known as the *loss coefficient* is also used for comparing the damping capacity of engineering materials. The loss coefficient is defined as the ratio of the energy dissipated per radian and the total strain energy:

$$\text{loss coefficient} = \frac{(\Delta W/2\pi)}{W} = \frac{\Delta W}{2\pi W} \quad (2.100)$$

2.6.5 Torsional Systems with Viscous Damping

The methods presented in Sections 2.6.1 through 2.6.4 for linear vibrations with viscous damping can be extended directly to viscously damped torsional (angular) vibrations. For this, consider a single degree of freedom torsional system with a viscous damper, as shown in Fig. 2.29(a). The viscous damping torque is given by (Fig. 2.29b):

$$T = -c_t\dot{\theta} \quad (2.101)$$

where c_t is the torsional viscous damping constant, $\dot{\theta} = d\theta/dt$ is the angular velocity of the disc, and the negative sign denotes that the damping torque is opposite the direction of angular velocity. The equation of motion can be derived as

$$J_0\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0 \quad (2.102)$$

where J_0 = mass moment of inertia of the disc, k_t = spring constant of the system (restoring torque per unit angular displacement), and θ = angular displacement of the disc. The solution of Eq. (2.102) can be found exactly as in the case of linear vibrations. For example, in the underdamped case, the frequency of damped vibration is given by

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n \quad (2.103)$$

where

$$\omega_n = \sqrt{\frac{k_t}{J_0}} \quad (2.104)$$

and

$$\zeta = \frac{c_t}{c_{tc}} = \frac{c_t}{2J_0\omega_n} = \frac{c_t}{2\sqrt{k_tJ_0}} \quad (2.105)$$

where c_{tc} is the critical torsional damping constant.

i.e.,

$$0.4 = -\left(\frac{v_{a2} - v_{t2}}{0 - 6.26099}\right)$$

i.e.,

$$v_{a2} = v_{t2} + 2.504396 \qquad \text{(E.5)}$$

The solution of Eqs. (E.3) and (E.5) gives

$$v_{a2} = 1.460898 \text{ m/s} \; ; \; v_{t2} = -1.043498 \text{ m/s}$$

Thus the initial conditions of the anvil are given by

$$x_0 = 0 \; ; \; \dot{x}_0 = 1.460898 \text{ m/s}$$

The damping coefficient is equal to

$$\zeta = \frac{c}{2\sqrt{k M}} = \frac{1000}{2\sqrt{(5 \times 10^6)\left(\frac{5000}{9.81}\right)}} = 0.0989949$$

The undamped and damped natural frequencies of the anvil are given by

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{5 \times 10^6}{\left(\frac{5000}{9.81}\right)}} = 98.994949 \text{ rad/s}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 98.994949 \sqrt{1 - 0.0989949^2} = 98.024799 \text{ rad/s}$$

The displacement response of the anvil is given by Eq. (2.72):

$$x(t) = e^{-\zeta \omega_n t} \left\{ \cos \omega_d t + \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d} \sin \omega_d t \right\}$$

$$= e^{-9.799995t} \{ \cos 98.024799\, t + 0.01490335 \sin 98.024799\, t \} \text{ m}$$ ■

EXAMPLE 2.10 Shock Absorber for a Motorcycle

An underdamped shock absorber is to be designed for a motorcycle of mass 200 kg (Fig. 2.31a). When the shock absorber is subjected to an initial vertical velocity due to a road bump, the resulting displacement-time curve is to be as indicated in Fig. 2.31(b). Find the necessary stiffness and damping constants of the shock absorber if the damped period of vibration is to be 2 sec and the amplitude x_1 is to be reduced to one-fourth in one half cycle (i.e., $x_{1.5} = x_1/4$). Also find the minimum initial velocity that leads to a maximum displacement of 250 mm.

Given: Mass = 200 kg; displacement-time curve of the system (Fig. 2.31b); damped period of vibration = 2 sec, $x_{1.5} = x_1/4$; and maximum displacement = 250 mm.

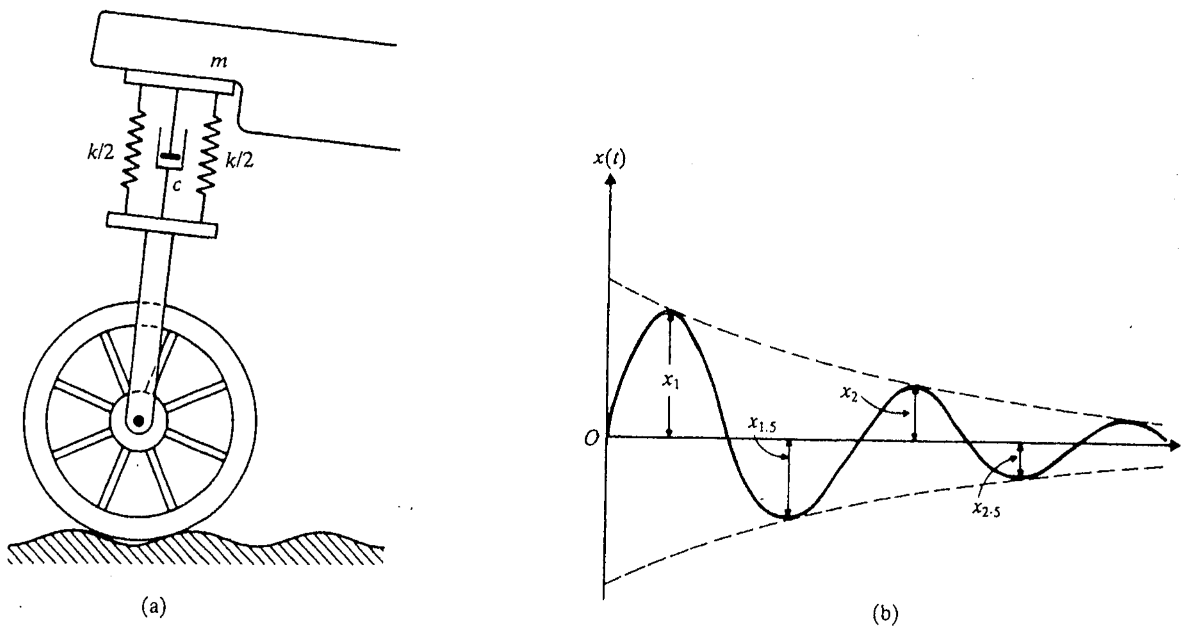


FIGURE 2.31

Find: Stiffness (k), damping constant (c), and initial velocity $(\dot{x}_0)$, which results in a maximum displacement of 250 mm.

Approach: Equation for the logarithmic decrement in terms of the damping ratio, equation for the damped period of vibration, time corresponding to maximum displacement for an underdamped system, and envelope passing through the maximum points of an underdamped system.

Solution: Since $x_{1.5} = x_1/4, x_2 = x_{1.5}/4 = x_1/16$. Hence the logarithmic decrement becomes

$$\delta = \ln\left(\frac{x_1}{x_2}\right) = \ln(16) = 2.7726 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \tag{E.1}$$

from which the value of ζ can be found as $\zeta = 0.4037$. The damped period of vibration is given to be 2 sec. Hence

$$2 = \tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\omega_n = \frac{2\pi}{2\sqrt{1-(0.4037)^2}} = 3.4338 \text{ rad/sec}$$

The critical damping constant can be obtained:

$$c_c = 2m\omega_n = 2(200)(3.4338) = 1373.54 \text{ N-s/m}$$

Thus the damping constant is given by

$$c = \zeta c_c = (0.4037)(1373.54) = 554.4981 \text{ N-s/m}$$

and the stiffness by

$$k = m\omega_n^2 = (200)(3.4338)^2 = 2358.2652 \text{ N/m}$$

The displacement of the mass will attain its maximum value at time t_1, given by

$$\sin \omega_d t_1 = \sqrt{1 - \zeta^2}$$

(See Problem 2.77.) This gives

$$\sin \omega_d t_1 = \sin \pi t_1 = \sqrt{1 - (0.4037)^2} = 0.9149$$

or

$$t_1 = \frac{\sin^{-1}(0.9149)}{\pi} = 0.3678 \text{ sec}$$

The envelope passing through the maximum points (see Problem 2.77) is given by

$$x = \sqrt{1 - \zeta^2}\, X e^{-\zeta\omega_n t} \tag{E.2}$$

Since $x = 250$ mm, Eq. (E.2) gives at t_1

$$0.25 = \sqrt{1 - (0.4037)^2}\, X e^{-(0.4037)(3.4338)(0.3678)}$$

or

$$X = 0.4550 \text{ m}.$$

The velocity of the mass can be obtained by differentiating the displacement

$$x(t) = X e^{-\zeta\omega_n t} \sin \omega_d t$$

as

$$\dot{x}(t) = X e^{-\zeta\omega_n t}(-\zeta\omega_n \sin \omega_d t + \omega_d \cos \omega_d t) \tag{E.3}$$

When $t = 0$, Eq. (E.3) gives

$$\dot{x}(t = 0) = \dot{x}_0 = X\omega_d = X\omega_n\sqrt{1 - \zeta^2} = (0.4550)(3.4338)\left(\sqrt{1 - (0.4037)^2}\right)$$
$$= 1.4294 \text{ m/s}$$

■

EXAMPLE 2.11 Analysis of Cannon

The schematic diagram of a large cannon is shown in Fig. 2.32 [2.8]. When the gun is fired, high-pressure gases accelerate the projectile inside the barrel to a very high velocity. The reaction force pushes the gun barrel in the opposite direction of the projectile. Since it is desirable to bring the gun barrel to rest in the shortest time without oscillation, it is made to translate backward against a critically damped spring-damper system called the *recoil mechanism*. In a particular case, the gun barrel and the recoil mechanism have a mass of 500 kg with a recoil spring of stiffness 10,000 N/m. The gun recoils 0.4 m upon firing. Find

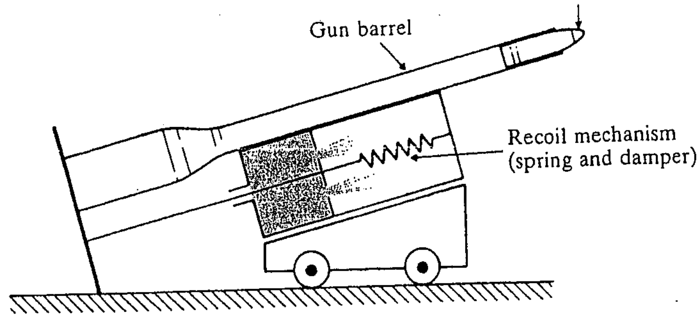


FIGURE 2.32

(1) the critical damping coefficient of the damper, (2) the initial recoil velocity of the gun, and (3) the time taken by the gun to return to a position 0.1 m from its initial position.

Given: Critically damped recoil mechanism with $m = 500$ kg, $k = 10{,}000$ N/m, and recoil distance = 0.4 m.

Find: Critical damping coefficient, recoil velocity, and time taken by the gun to return to a position 0.1 m from its initial position.

Approach: Use the response equation of a critically damped system.

Solution

1. The undamped natural frequency of the system is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10{,}000}{500}} = 4.4721 \text{ rad/sec}$$

and the critical damping coefficient (Eq. 2.65) of the damper is

$$c_c = 2m\omega_n = 2(500)(4.4721) = 4472.1 \text{ N-s/m}$$

2. The response of a critically damped system is given by Eq. (2.78):

$$x(t) = (C_1 + C_2 t)\, e^{-\omega_n t} \tag{E.1}$$

where $C_1 = x_0$ and $C_2 = \dot{x}_0 + \omega_n x_0$. The time t_1 at which $x(t)$ reaches a maximum value can be obtained by setting $\dot{x}(t) = 0$. The differentiation of Eq. (E.1) gives

$$\dot{x}(t) = C_2 e^{-\omega_n t} - \omega_n (C_1 + C_2 t) e^{-\omega_n t}$$

Hence $\dot{x}(t) = 0$ yields

$$t_1 = \left(\frac{1}{\omega_n} - \frac{C_1}{C_2}\right) \tag{E.2}$$

In this case, $x_0 = C_1 = 0$; hence Eq. (E.2) leads to $t_1 = 1/\omega_n$. Since the maximum value of $x(t)$ or the recoil distance is given to be $x_{max} = 0.4$ m, we have

$$x_{max} = x(t = t_1) = C_2 t_1 e^{-\omega_n t_1} = \frac{\dot{x}_0}{\omega_n} e^{-1} = \frac{\dot{x}_0}{e\,\omega_n}$$

or

$$\dot{x}_0 = x_{max}\,\omega_n e = (0.4)(4.4721)(2.7183) = 4.8626 \text{ m/s}$$

3. If t_2 denotes the time taken by the gun to return to a position 0.1 m from its initial position, we have

$$0.1 = C_2 t_2 e^{-\omega_n t_2} = 4.8626 t_2 e^{-4.4721 t_2} \tag{E.3}$$

The solution of Eq. (E.3) gives $t_2 = 0.8258$ sec. ■

2.7 Free Vibration with Coulomb Damping

In many mechanical systems, *Coulomb* or *dry-friction* dampers are used because of their mechanical simplicity and convenience [2.9]. Also in vibrating structures, whenever the components slide relative to each other, dry-friction damping appears internally. As stated in Section 1.9, Coulomb damping arises when bodies slide on dry surfaces. Coulomb's law of dry friction states that when two bodies are in contact, the force required to produce sliding is proportional to the normal force acting in the plane of contact. Thus the friction force F is given by

$$F = \mu N = \mu W = \mu mg \tag{2.106}$$

where N is the normal force and μ is the coefficient of friction. The friction force acts in a direction opposite to the direction of velocity. Coulomb damping is sometimes called *constant damping,* since the damping force is independent of the displacement and velocity; it depends only on the normal force N between the sliding surfaces.

2.7.1 Equation of Motion

Consider a single degree of freedom system with dry friction as shown in Fig. 2.33(a). Since the friction force varies with the direction of velocity, we need to consider two cases, as indicated in Figs. 2.33(b) and (c).

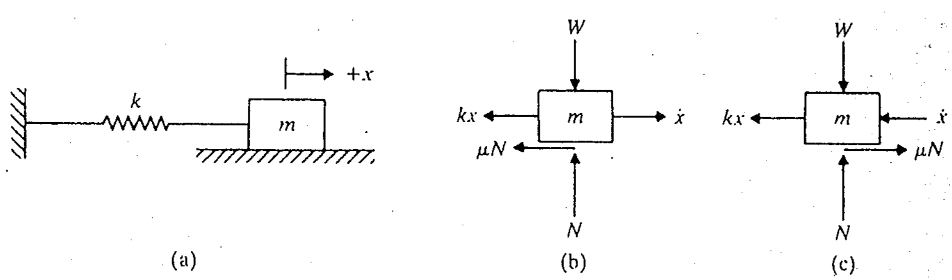


FIGURE 2.33 Spring-mass system with Coulomb damping.

2.11 Is the frequency of a damped free vibration smaller or greater than the natural frequency of the system?

2.12 What is the use of logarithmic decrement?

2.13 Is hysteresis damping a function of the maximum stress?

2.14 What is critical damping and what is its importance?

2.15 What happens to the energy dissipated by damping?

2.16 What is equivalent viscous damping? Is the equivalent viscous damping factor a constant?

2.17 What is the reason for studying the vibration of a single degree of freedom system?

2.18 How can you find the natural frequency of a system by measuring its static deflection?

2.19 Give two practical applications of a torsional pendulum.

2.20 Define these terms: damping ratio, logarithmic decrement, loss coefficient, and specific damping capacity.

2.21 In what ways is the response of a system with Coulomb damping different from that of systems with other types of damping?

2.22 What is complex stiffness?

2.23 Define the hysteresis damping constant.

2.24 Give three practical applications of the concept of center of percussion.

Problems

The problem assignments are organized as follows:

Problems	Section Covered	Topic Covered
2.1–2.50	2.2	Undamped translational systems
2.51–2.64	2.3	Undamped torsional systems
2.65–2.74	2.5	Energy method
2.75–2.97, 2.111	2.6	Systems with viscous damping
2.98–2.107	2.7	Systems with Coulomb damping
2.108–2.110	2.8	Systems with hysteretic damping
2.112–2.115	2.9	Computer program
2.116–2.120	—	Projects

2.1 An industrial press is mounted on a rubber pad to isolate it from its foundation. If the rubber pad is compressed 5 mm by the self-weight of the press, find the natural frequency of the system.

2.2 A spring-mass system has a natural period of 0.21 sec. What will be the new period if the spring constant is (a) increased by 50% and (b) decreased by 50%?

2.3 A spring-mass system has a natural frequency of 10 Hz. When the spring constant is reduced by 800 N/m, the frequency is altered by 45 percent. Find the mass and spring constant of the original system.

2.4 A helical spring, when fixed at one end and loaded at the other, requires a force of 100 N to produce an elongation of 10 mm. The ends of the spring are now rigidly fixed, one end vertically above the other, and a mass of 10 kg is attached at the middle point of its length. Determine the time taken to complete one vibration cycle when the mass is set vibrating in the vertical direction.

2.5 An air-conditioning chiller unit weighing 2000 lb is to be supported by four air springs (Fig. 2.39). Design the air springs such that the natural frequency of vibration of the unit lies between 5 rad/s and 10 rad/s.

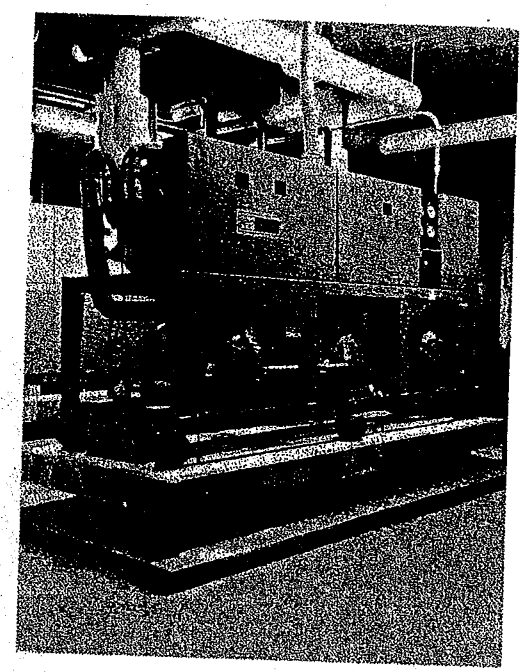


FIGURE 2.39 (Courtesy of *Sound and Vibration*)

2.6 The maximum velocity attained by the mass of a simple harmonic oscillator is 10 cm/sec, and the period of oscillation is 2 sec. If the mass is released with an initial

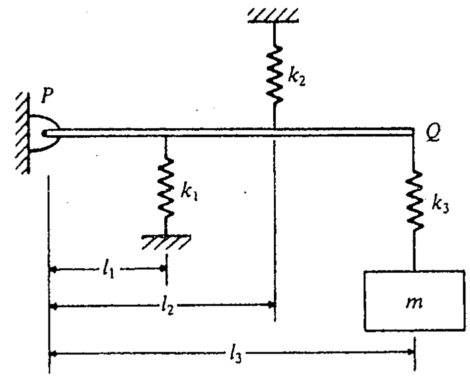


FIGURE 2.40

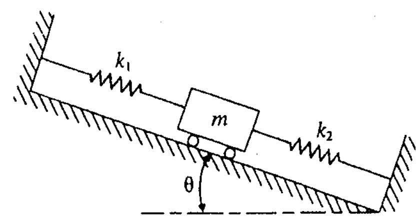


FIGURE 2.41

displacement of 2 cm, find (a) the amplitude, (b) the initial velocity, (c) the maximum acceleration, and (d) the phase angle.

2.7 Three springs and a mass are attached to a rigid, weightless, bar PQ as shown in Fig. 2.40. Find the natural frequency of vibration of the system.

2.8 An automobile having a mass of 2000 kg deflects its suspension springs 0.02 m under static conditions. Determine the natural frequency of the automobile in the vertical direction by assuming damping to be negligible.

2.9 Find the natural frequency of vibration of a spring-mass system arranged on an inclined plane, as shown in Fig. 2.41.

2.10 A loaded mine cart, weighing 5,000 lb, is being lifted by a frictionless pulley and a wire rope, as shown in Fig. 2.42. Find the natural frequency of vibration of the cart in the given position.

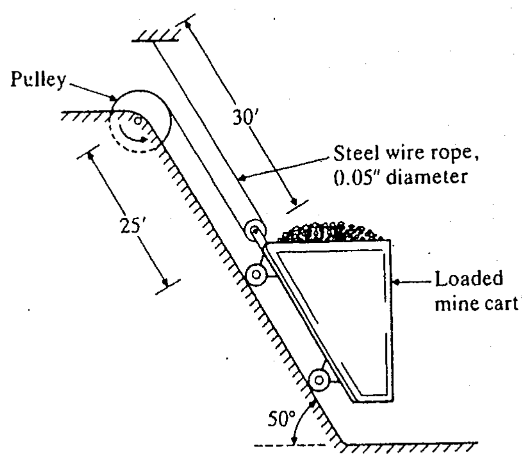


FIGURE 2.42

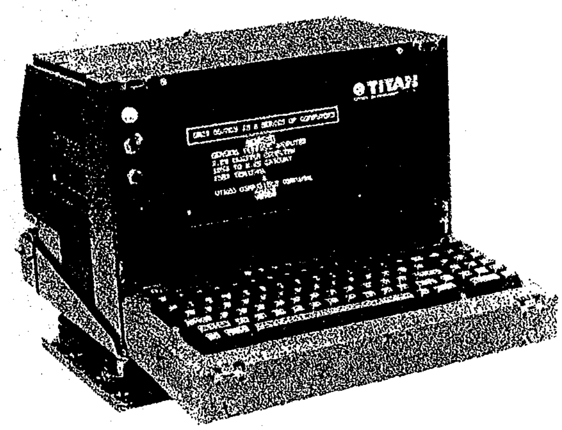


FIGURE 2.43 An electronic chassis mounted on vibration isolators. (Courtesy of Titan SESCO.)

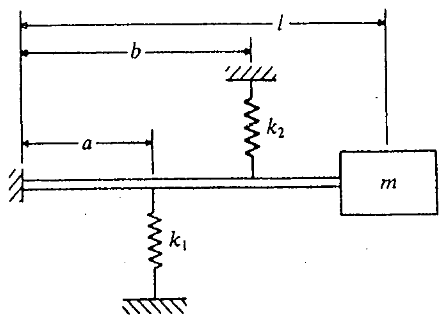


FIGURE 2.44

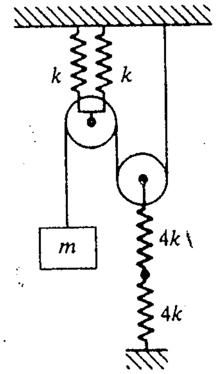


FIGURE 2.45

2.11 An electronic chassis, weighing 500 N, is isolated by supporting it on four helical springs, as shown in Fig. 2.43. Design the springs so that the unit can be used in an environment in which the vibratory frequency ranges from 0 to 5 Hz.

2.12 Find the natural frequency of the system shown in Fig. 2.44 with and without the springs k_1 and k_2 in the middle of the elastic beam.

2.13 Find the natural frequency of the pulley system shown in Fig. 2.45 by neglecting the friction and the masses of the pulleys.

2.14 A weight W is supported by three frictionless and massless pulleys and a spring of stiffness k, as shown in Fig. 2.46. Find the natural frequency of vibration of weight W for small oscillations.

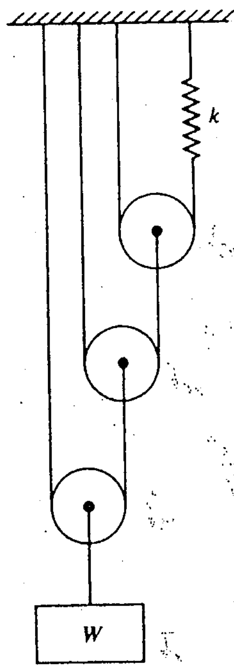


FIGURE 2.46

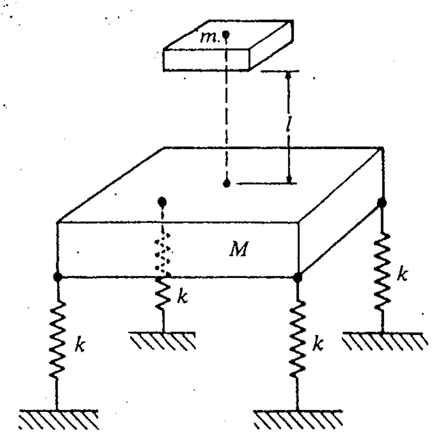


FIGURE 2.47

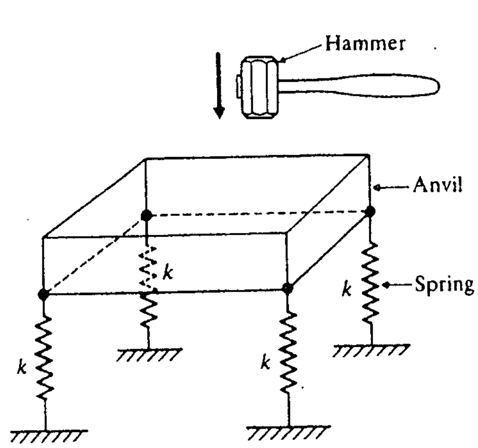


FIGURE 2.48

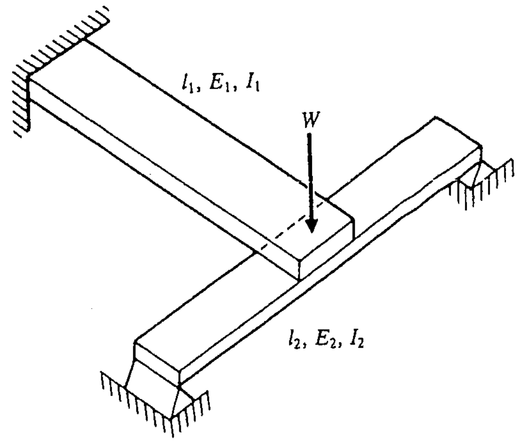


FIGURE 2.49

2.15 A rigid block of mass M is mounted on four elastic supports, as shown in Fig. 2.47. A mass m drops from a height l and adheres to the rigid block without rebounding. If the spring constant of each elastic support is k, find the natural frequency of vibration of the system (a) without the mass m, and (b) with the mass m. Also find the resulting motion of the system in case (b).

2.16 A sledgehammer strikes an anvil with a velocity of 50 ft/sec (Fig. 2.48). The hammer and the anvil weigh 12 lb and 100 lb, respectively. The anvil is supported on four springs, each of stiffness k = 100 lb/in. Find the resulting motion of the anvil (a) if the hammer remains in contact with the anvil, and (b) if the hammer does not remain in contact with the anvil after the initial impact.

2.17 Derive the expression for the natural frequency of the system shown in Fig. 2.49. Note that the load W is applied at the tip of beam 1 and midpoint of beam 2.

2.18 A heavy machine weighing 9810 N is being lowered vertically down by a winch at a uniform velocity of 2 m/sec. The steel cable supporting the machine has a diameter of 0.01 m. The winch is suddenly stopped when the steel cable's length is 20 m. Find the period and amplitude of the ensuing vibration of the machine.

2.19 The natural frequency of a spring-mass system is found to be 2 Hz. When an additional mass of 1 kg is added to the original mass m, the natural frequency is reduced to 1 Hz. Find the spring constant k and the mass m.

2.20 An electrical switchgear is supported by a crane through a steel cable of length 4 m and diameter 0.01 m (Fig. 2.50). If the natural time period of axial vibration of the switchgear is found to be 0.1 s, find the mass of the switchgear.

2.21 Four weightless rigid links and a spring are arranged to support a weight W in two different ways, as shown in Fig. 2.51. Determine the natural frequencies of vibration of the two arrangements.

2.22 A scissors jack is used to lift a load W. The links of the jack are rigid and the collars can slide freely on the shaft against the springs of stiffnesses k_1 and k_2

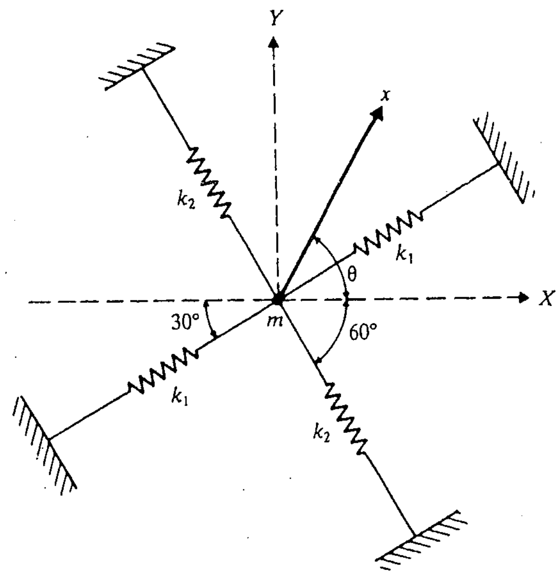


FIGURE 2.55

2.25 A mass m is supported by two sets of springs oriented at 30° and 120° with respect to the X axis, as shown in Fig. 2.55. A third pair of springs, with a stiffness of k_3 each, is to be designed so as to make the system have a constant natural frequency while vibrating in any direction x. Determine the necessary spring stiffness k_3 and the orientation of the springs with respect to the X axis.

2.26 A mass m is attached to a cord that is under a tension T, as shown in Fig. 2.56. Assuming that T remains unchanged when the mass is displaced normal to the cord, (a) write the differential equation of motion for small transverse vibrations, and (b) find the natural frequency of vibration.

2.27 A bungee jumper weighing 160 lb ties one end of an elastic rope of length 200 ft and stiffness 10 lb/in to a bridge and the other end to himself and jumps from the bridge (Fig. 2.57). Assuming the bridge to be rigid, determine the vibratory motion of the jumper about his static equilibrium position.

2.28 An acrobat weighing 120 lb walks on a tightrope, as shown in Fig. 2.58. If the natural frequency of vibration in the given position, in vertical direction, is 10 rad/s, find the tension in the rope.

2.29 The schematic diagram of a centrifugal governor is shown in Fig. 2.59. The length of each rod is l, the mass of each ball is m and the free length of the spring is h. If the shaft speed is ω, determine the equilibrium position and the frequency for small oscillations about this position.

2.30 In the Hartnell governor shown in Fig. 2.60, the stiffness of the spring is 10^4 N/m and the weight of each ball is 25 N. The length of the ball arm is 20 cm and that

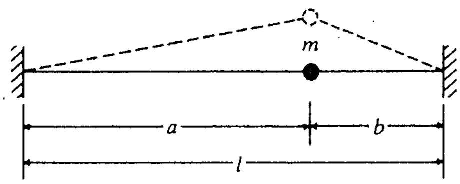


FIGURE 2.56

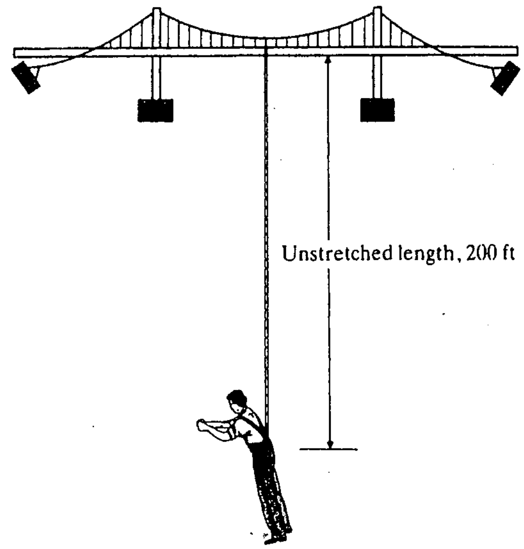


FIGURE 2.57

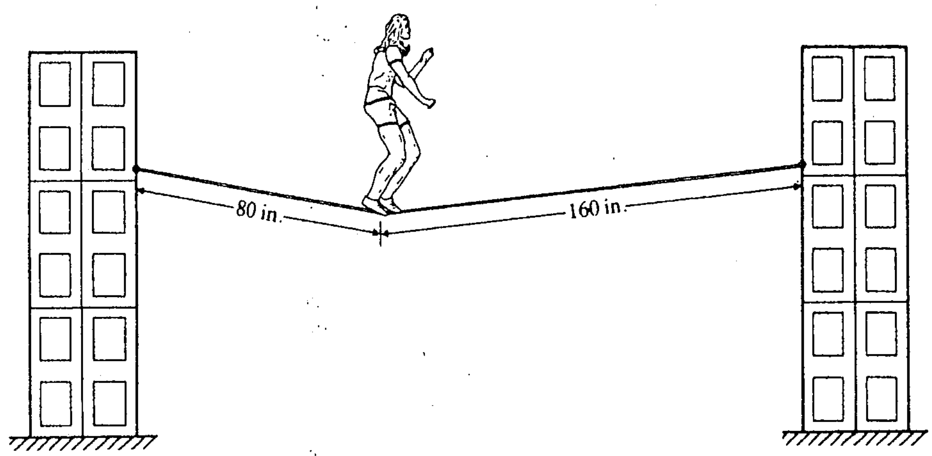


FIGURE 2.58

of the sleeve arm is 12 cm. The distance between the axis of rotation and the pivot of the bell crank lever is 16 cm. The spring is compressed by 1 cm when the ball arm is vertical. Find (a) the speed of the governor at which the ball arm remains vertical, and (b) the natural frequency of vibration for small displacements about the vertical position of the ball arms.

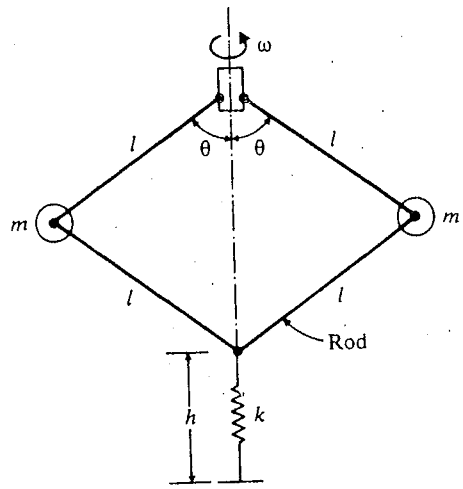


FIGURE 2.59

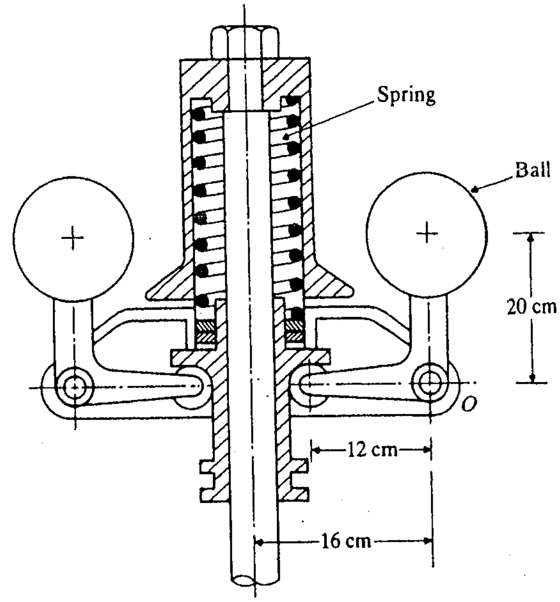


FIGURE 2.60 Hartnell governor.

2.31 A square platform *PQRS* and a car that it is supporting have a combined mass of *M*. The platform is suspended by four elastic wires from a fixed point *O*, as indicated in Fig. 2.61. The vertical distance between the point of suspension *O* and the horizontal equilibrium position of the platform is *h*. If the side of the platform is *a* and the stiffness of each wire is *k*, determine the period of vertical vibration of the platform.

2.32 The inclined manometer, shown in Fig. 2.62, is used to measure pressure. If the total length of mercury in the tube is *L*, find an expression for the natural frequency of oscillation of the mercury.

2.33 The crate, of mass 250 kg, hanging from a helicopter (shown in Fig. 2.63a) can be modeled as shown in Fig. 2.63b. The rotor blades of the helicopter rotate at 300 rpm. Find the diameter of the steel cables so that the natural frequency of vibration of the crate is at least twice the frequency of the rotor blades.

2.34 A pressure vessel head is supported by a set of steel cables of length 2 m as shown in Fig. 2.64. The time period of axial vibration (in vertical direction) is found to vary from 5 s to 4.0825 s when an additional mass of 5,000 kg is added to the pressure vessel head. Determine the equivalent cross-sectional area of the cables and the mass of the pressure vessel head.

2.35 A flywheel is mounted on a vertical shaft, as shown in Fig. 2.65. The shaft has a diameter *d* and length *l* and is fixed at both ends. The flywheel has a weight of *W* and a radius of gyration of *r*. Find the natural frequency of the longitudinal, the transverse, and the torsional vibration of the system.

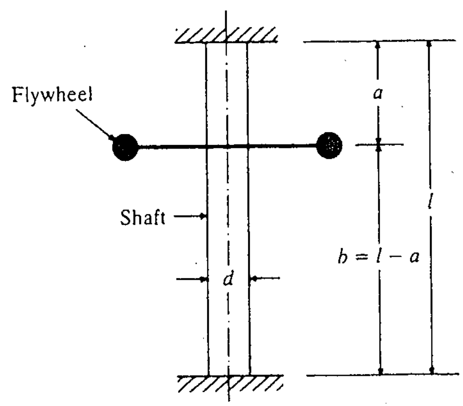


FIGURE 2.65

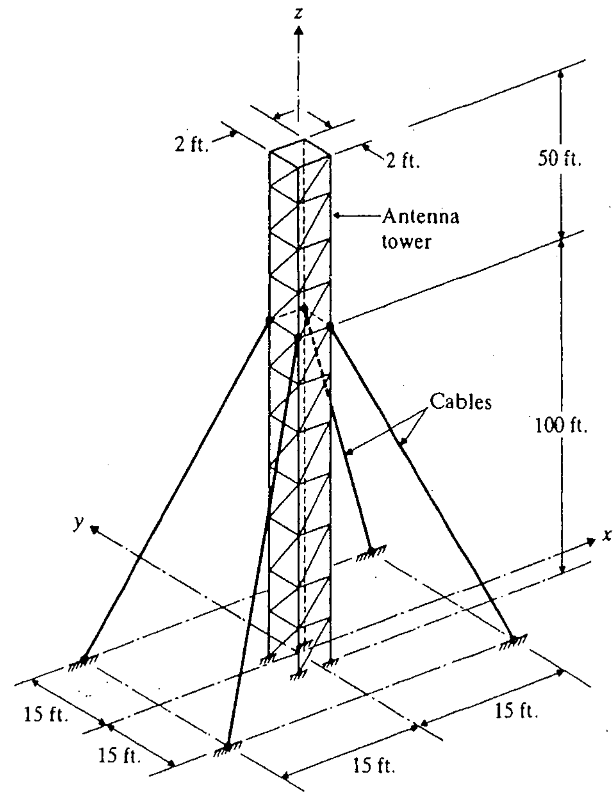


FIGURE 2.66

2.38 A building frame is modeled by four identical steel columns, of weight w each, and a rigid floor of weight W, as shown in Fig. 2.68. The columns are fixed at the ground and have a bending rigidity of EI each. Determine the natural frequency of horizontal vibration of the building frame by assuming the connection between the floor and the columns to be (a) pivoted as shown in Fig. 2.68(a), and (b) fixed against rotation as shown in Fig. 2.68(b). Include the effect of self weights of the columns.

2.39 A pick and place robot arm, shown in Fig. 2.69, carries an object weighing 10 lb. Find the natural frequency of the robot arm in the axial direction for the following data: $l_1 = 12$ in., $l_2 = 10$ in., $l_3 = 8$ in., $E_1 = E_2 = E_3 = 10^7$ psi, $D_1 = 2$ in., $D_2 = 1.5$ in., $D_3 = 1$ in., $d_1 = 1.75$ in., $d_2 = 1.25$ in., $d_3 = 0.75$ in.

2.40 A helical spring of stiffness k is cut into two halves and a mass m is connected to the two halves as shown in Fig. 2.70(a). The natural time period of this system is found to be 0.5 sec. If an identical spring is cut so that one part is 1/4 and the other part 3/4 of the original length, and the mass m is connected to the two parts as shown in Fig. 2.70(b), what would be the natural period of the system?

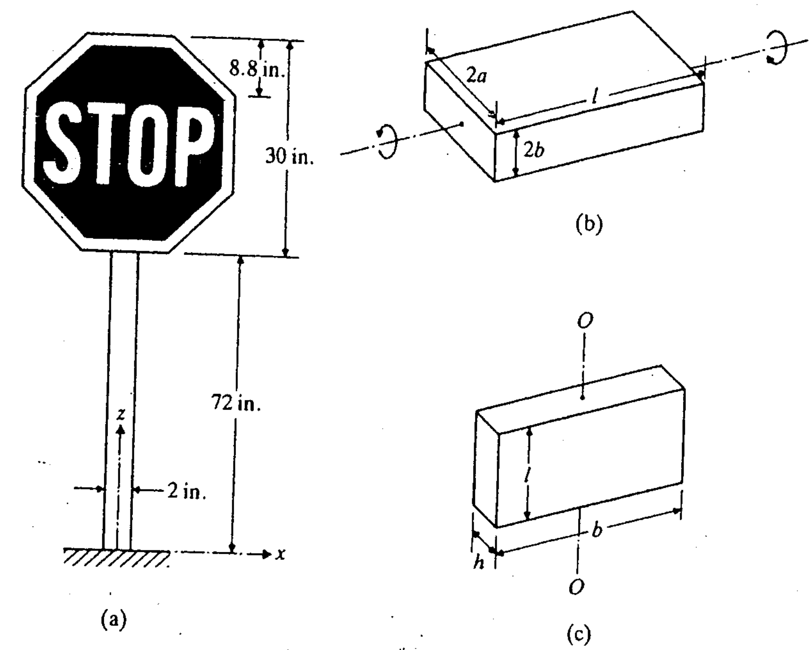


FIGURE 2.67

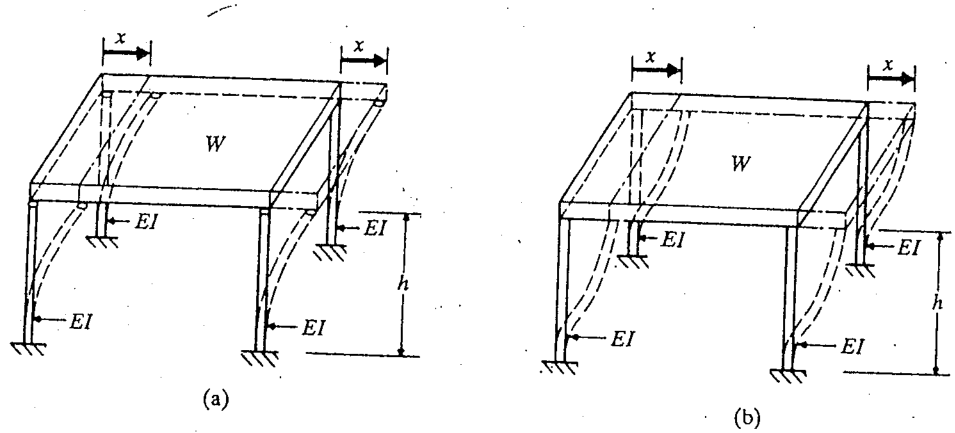


FIGURE 2.68

2.41* Figure 2.71 shows a metal block supported on two identical cylindrical rollers rotating in opposite directions at the same angular speed. When the center of gravity of the block is initially displaced by a distance x, the block will be set into simple harmonic

*The asterisk denotes a design problem or a problem with no unique answer.

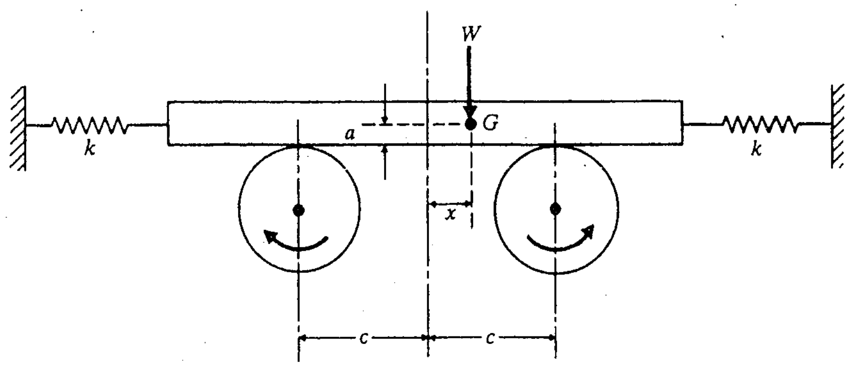


FIGURE 2.72

2.43 An electromagnet weighing 3000 lb is at rest while holding an automobile of weight 2000 lb in a junkyard. The electric current is turned off and the automobile is dropped. Assuming that the crane and the supporting cable have an equivalent spring constant of 10,000 lb/in, find the following: (a) the natural frequency of vibration of the electromagnet; (b) the resulting motion of the electromagnet; and (c) the maximum tension developed in the cable during the motion.

2.44 Derive the equation of motion of the system shown in Fig. 2.73 using the following methods: (a) Newton's second law of motion, (b) D'Alembert's principle, (c) principle of virtual work, and (d) principle of conservation of energy.

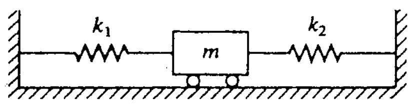


FIGURE 2.73

2.45–
2.46 Draw the free-body diagram and derive the equation of motion using Newton's second law of motion for each of the systems shown in Figs. 2.74 and 2.75.

2.47–
2.48 Derive the equation of motion using the principle of conservation of energy for each of the systems shown in Figs. 2.74 and 2.75.

2.49 A steel beam of length 1 m carries a mass of 50 kg at its free end, as shown in Fig. 2.76. Find the natural frequency of transverse vibration of the mass by modeling it as a single degree of freedom system.

2.50 A steel beam of length 1 m carries a mass of 50 kg at its free end, as shown in Fig. 2.77. Find the natural frequency of transverse vibration of the system by modeling it as a single degree of freedom system.

2.51 A pulley 250 mm in diameter drives a second pulley 1000 mm in diameter by means of a belt (see Fig. 2.78). The moment of inertia of the driven pulley is 0.2 kg-m^2. The belt connecting these pulleys is represented by two springs, each of stiffness k. For what value of k will the natural frequency be 6 Hz?

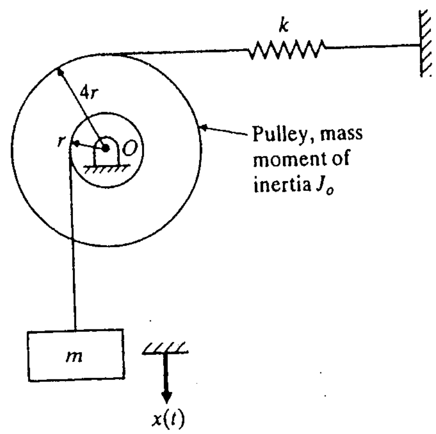


FIGURE 2.74

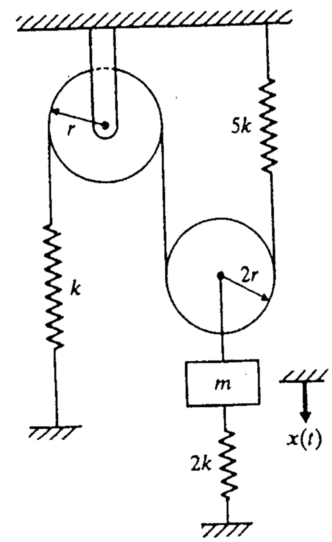


FIGURE 2.75

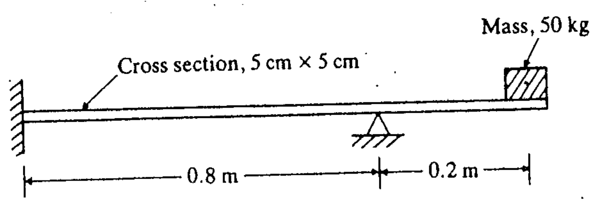


FIGURE 2.76

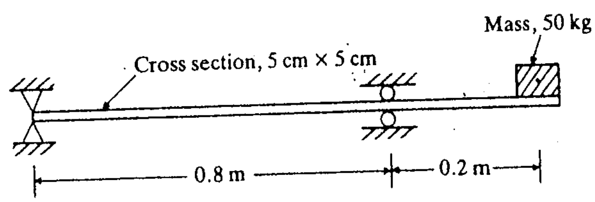


FIGURE 2.77

2.52 Derive an expression for the natural frequency of the simple pendulum shown in Fig. 1.11. Determine the period of oscillation of a simple pendulum having a mass $m = 5$ kg and a length $l = 0.5$ m.

2.53 A mass m is attached at the end of a bar of negligible mass and is made to vibrate in three different configurations, as indicated in Fig. 2.79. Find the configuration corresponding to the highest natural frequency.

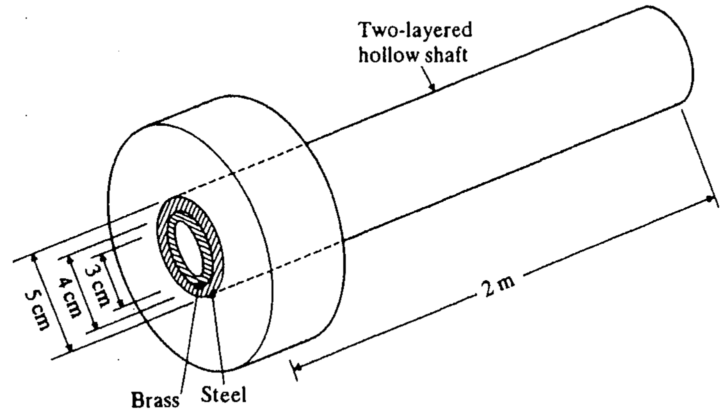


FIGURE 2.82

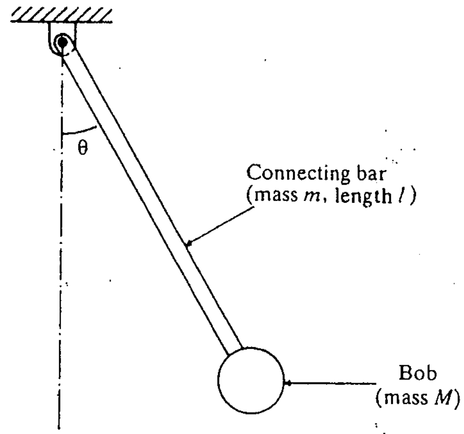


FIGURE 2.83

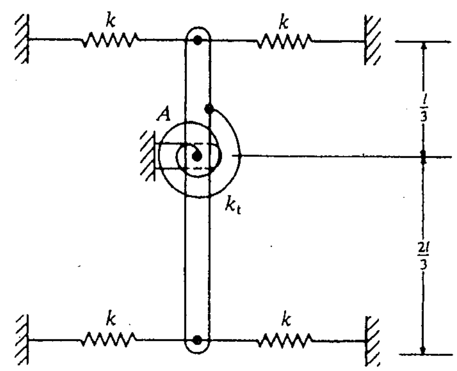


FIGURE 2.84

frequency of the system if $k = 2000$ N/m, $k_t = 1000$ N-m/rad, $m = 10$ kg, and $l = 5$ m.

2.60 A cylinder of mass m and mass moment of inertia J_0 is free to roll without slipping but is restrained by two springs of stiffnesses k_1 and k_2, as shown in Fig. 2.85. Find its natural frequency of vibration. Also find the value of a that maximizes the natural frequency of vibration.

2.61 If the pendulum of Problem 2.52 is placed in a rocket moving vertically with an acceleration of 5 m/s^2, what will be its period of oscillation?

2.62 Find the equation of motion of the uniform rigid bar OA of length l and mass m shown in Fig. 2.86. Also find its natural frequency.

2.63 A uniform circular disc is pivoted at point O, as shown in Fig. 2.87. Find the natural

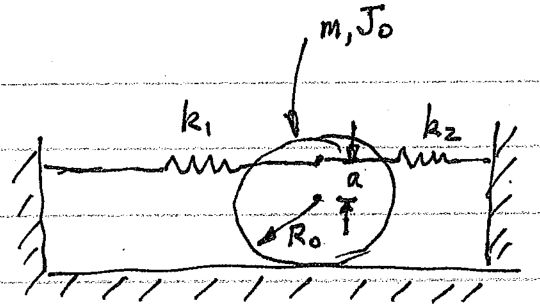


Fig 2.85

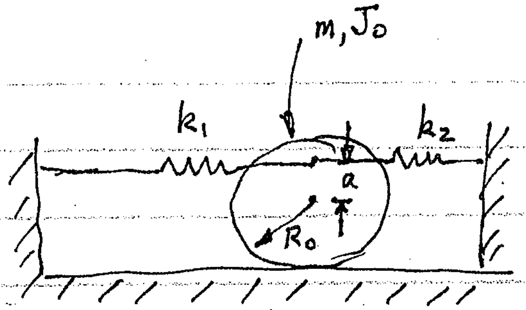


Fig 2.85

$\frac{x_0}{x_{\frac{1}{2}}} = 14.2888$, $x_{\frac{1}{2}} = 0.0700\ x_0$

$\therefore$ overshoot = 7%

(2.80) (i) (a) Viscous damping, (b) Coulomb damping.

(iii) (a) $\tau_d = 0.2$ sec, $f_d = 5$ Hz, $\omega_d = 31.416$ rad/sec.
(b) $\tau_n = 0.2$ sec, $f_n = 5$ Hz, $\omega_n = 31.416$ rad/sec.

(ii) (a) $\frac{x_i}{x_{i+1}} = e^{\zeta \omega_n \tau_d}$

$$\ln\left(\frac{x_i}{x_{i+1}}\right) = \ln 2 = 0.6931 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}.sp$$

$\delta / \sqrt{4\pi^2+\delta^2} = \zeta = 0.1096$

or $39.9590\ \zeta^2 = 0.4804$ or $\zeta = 0.1096$

Since $\omega_d = \omega_n \sqrt{1-\zeta^2}$, we find

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{31.416}{\sqrt{0.98798}} = 31.6065 \text{ rad/sec}$$

$$k = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(31.6065)^2 = 5.0916\ (10^4) \text{ N/m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\,m\,\omega_n}$$

Hence $c = 2\,m\,\omega_n\,\zeta = 2\left(\frac{500}{9.81}\right)(31.6065)(0.1096) = 353.1164$ N–s/m

(b) From Eq. (2.116):

$$k = m\,\omega_n^2 = \frac{500}{9.81}(31.416)^2 = 5.0304\ (10^4) \text{ N/m}$$

now $X_n - X_{n+1} = \frac{4f}{k} = \frac{4\mu N}{k} = \frac{4\mu W}{k}$

Using N = W = 500 N,

$$\mu = \frac{0.002\ k}{4\ W} = \frac{(0.002)\,(5.0304\,(10^4))}{4\,(500)} = 0.0503$$

(2.81) (a) $c_c = 2\sqrt{km} = 2\sqrt{5000 \times 50} = 1000$ N-s/m

(b) $c = c_c/2 = 500$ N-s/m

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m}}\sqrt{1-\left(\frac{c}{c_c}\right)^2} = \sqrt{\frac{5000}{50}}\sqrt{1-\left(\tfrac{1}{2}\right)^2}$$
$$= 8.6603 \text{ rad/sec}$$

(c) From Eq. (2.85), $\delta = \frac{2\pi}{\omega_d}\left(\frac{c}{2m}\right) = \frac{2\pi}{8.6603}\left(\frac{500}{2\times 50}\right)$
$= 3.6276$

(2.82) $m = 2000$ kg, $v = \dot{x}_0 = 10$ m/sec, $k = 40{,}000$ N/m
$c = 20{,}000$ N-sec/m

2.79 A shock absorber is to be designed to limit its overshoot to 15 percent of its initial displacement when released. Find the damping ratio ζ_0 required. What will be the overshoot if ζ is made equal to (a) $\frac{3}{4}\zeta_0$, and (b) $\frac{5}{4}\zeta_0$?

2.80 The free vibration response of an electric motor of weight 500 N mounted on different types of foundations are shown in Figs. 2.91(a) and (b). Identify the following in each case: (i) the nature of damping provided by the foundation; (ii) the spring constant and damping coefficient of the foundation; and (iii) the undamped and damped natural frequencies of the electric motor.

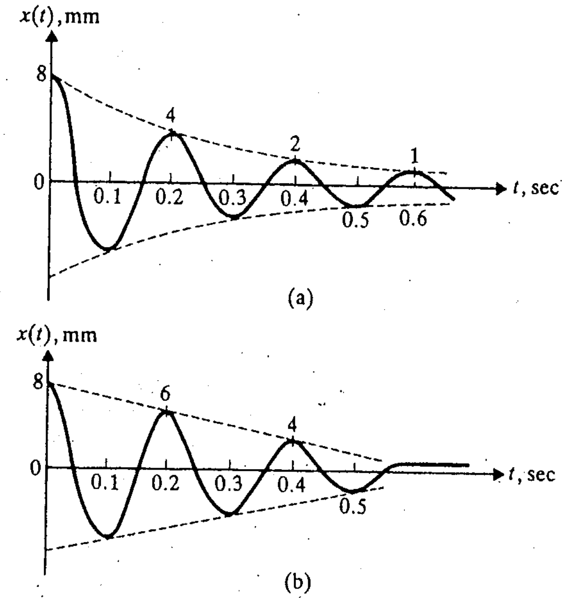


FIGURE 2.91

2.81 For a spring-mass-damper system, $m = 50$ kg and $k = 5000$ N/m. Find the following: (a) critical damping constant c_c; (b) damped natural frequency when $c = c_c/2$; and (c) logarithmic decrement.

2.82 A locomotive car of mass 2000 kg traveling at a velocity $v = 10$ m/sec is stopped at the end of tracks by a spring-damper system, as shown in Fig. 2.92. If the stiffness of the spring is $k = 40$ N/mm and the damping constant is $c = 20$ N-s/mm, determine (a) the maximum displacement of the car after engaging the springs and damper and (b) the time taken to reach the maximum displacement.

2.83 A torsional pendulum has a natural frequency of 200 cycles/min when vibrating in vacuum. The mass moment of inertia of the disc is 0.2 kg-m^2. It is then immersed in oil and its natural frequency is found to be 180 cycles/min. Determine the damping constant. If the disc, when placed in oil, is given an initial displacement of 2°, find its displacement at the end of the first cycle.

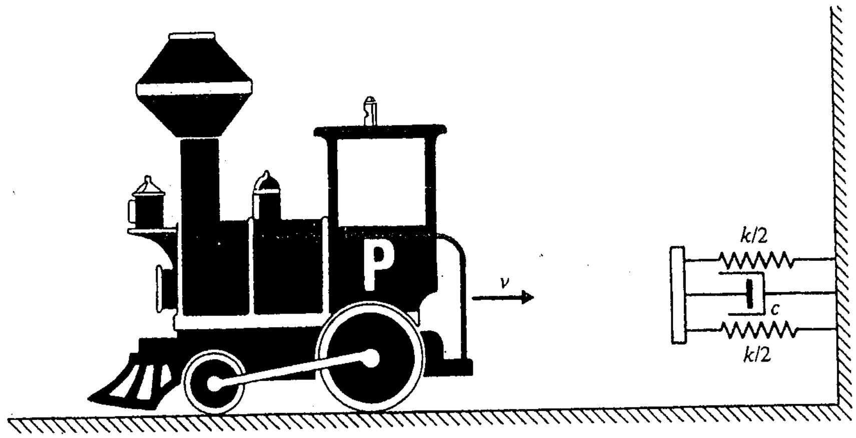


FIGURE 2.92

2.84 A boy riding a bicycle can be modeled as a spring-mass-damper system with an equivalent weight, stiffness and damping constant of 800 N, 50000 N/m, and 1000 N-s/m, respectively. The differential setting of the concrete blocks on the road caused the level surface to decrease suddenly as indicated in Fig. 2.93. If the speed of the bicycle is 5 m/s (18 km/hr), determine the displacement of the boy in the vertical direction. Assume that the bicycle is free of vertical vibration before encountering the step change in the vertical displacement.

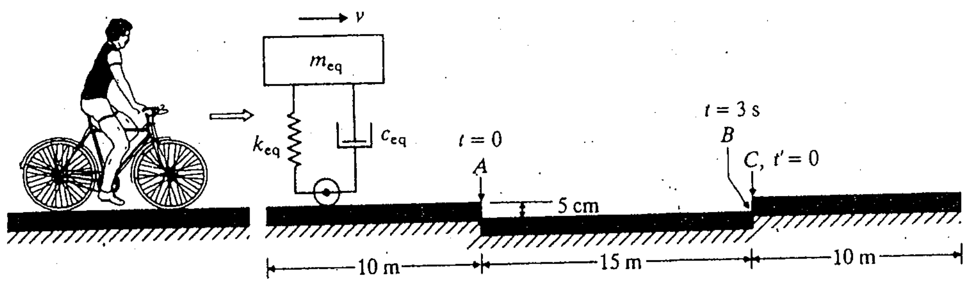


FIGURE 2.93

2.85 A wooden rectangular prism of weight 20 lb, height 3 ft. and cross section 1 ft. × 2 ft. floats and remains vertical in a tub of oil. The frictional resistance of the oil can be assumed to be equivalent to a viscous damping coefficient ζ. When the prism is depressed by a distance of 6 in. from its equilibrium and released, it is found to reach a depth of 5.5 in. at the end of its first cycle of oscillation. Determine the value of the damping coefficient of the oil.

2.86 A body vibrating with viscous damping makes five complete oscillations per second, and in 50 cycles its amplitude diminishes to 10 percent. Determine the logarithmic decrement and the damping ratio. In what proportion will the period of vibration be decreased if damping is removed?

2.87* The maximum permissible recoil distance of a gun is specified as 0.5 m. If the initial recoil velocity is to be between 8 m/sec and 10 m/sec, find the mass of the gun and the spring stiffness of the recoil mechanism. Assume that a critically damped dashpot is used in the recoil mechanism and the mass of the gun has to be at least 500 kg.

2.88 A viscously damped system has a stiffness of 5000 N/m, critical damping constant of 0.2 N-s/mm, and a logarithmic decrement of 2.0. If the system is given an initial velocity of 1 m/sec, determine the maximum displacement of the system.

2.89 Explain why an overdamped system never passes through the static equilibrium position when it is given (a) an initial displacement only and (b) an initial velocity only.

2.90–2.92 Derive the equation of motion and find the natural frequency of vibration of each of the systems shown in Figs. 2.94 to 2.96.

2.93–2.95 Using the principle of virtual work, derive the equation of motion for each of the systems shown in Figs. 2.94 to 2.96.

2.96 A wooden rectangular prism of cross section 40 cm × 60 cm, height 120 cm, and mass 40 kg floats in a fluid, as shown in Fig. 2.90. When disturbed, it is observed to vibrate freely with a natural period of 0.5 s. Determine the density of the fluid.

2.97 The system shown in Fig. 2.97 has a natural frequency of 5 Hz for the following data: $m = 10$ kg, $J_0 = 5$ kg − m^2, $r_1 = 10$ cm, $r_2 = 25$ cm. When the system is disturbed by giving it an initial displacement, the amplitude of free vibration is reduced by 80 percent in 10 cycles. Determine the values of k and c.

2.98 A single degree of freedom system consists of a mass of 20 kg and a spring of stiffness 4000 N/m. The amplitudes of successive cycles are found to be 50, 45, 40, 35, . . . mm. Determine the nature and magnitude of the damping force and the frequency of the damped vibration.

2.99 A mass of 20 kg slides back and forth on a dry surface due to the action of a spring having a stiffness of 10 N/mm. After four complete cycles, the amplitude has been

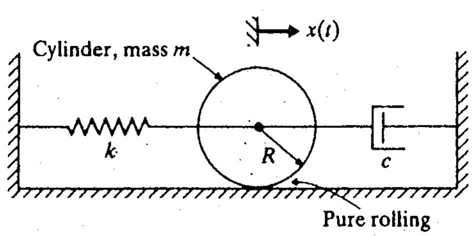


FIGURE 2.94

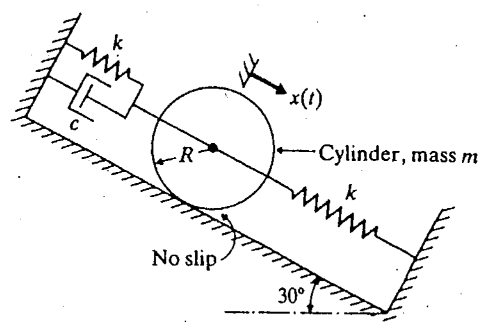


FIGURE 2.95

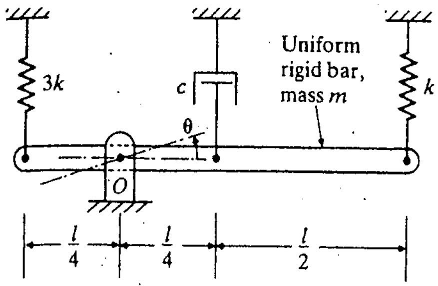


FIGURE 2.96

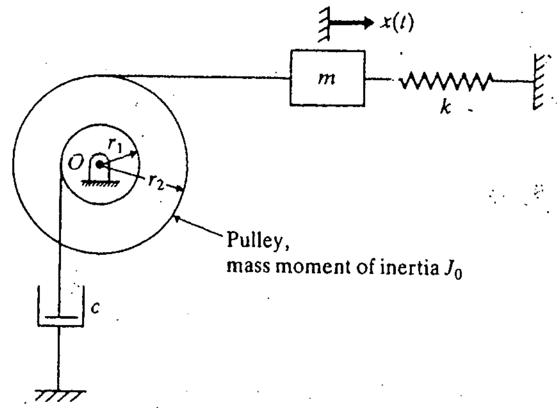


FIGURE 2.97

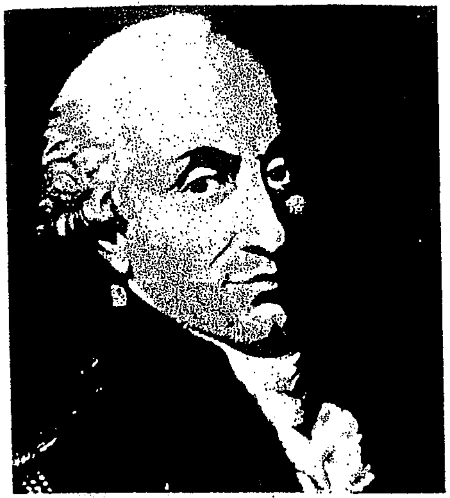
found to be 100 mm. What is the average coefficient of friction between the two surfaces if the original amplitude was 150 mm? How much time has elapsed during the four cycles?

2.100 A 10-kg mass is connected to a spring of stiffness 3000 N/m and is released after giving an initial displacement of 100 mm. Assuming that the mass moves on a horizontal surface, as shown in Fig. 2.33(a), determine the position at which the mass comes to rest. Assume the coefficient of friction between the mass and the surface to be 0.12.

2.101 A weight of 25 N is suspended from a spring that has a stiffness of 1000 N/m. The weight vibrates in the vertical direction under a constant damping force. When the weight is initially pulled downward a distance of 10 cm from its static equilibrium position and released, it comes to rest after exactly two complete cycles. Find the magnitude of the damping force.

2.102 A mass of 20 kg is suspended from a spring of stiffness 10,000 N/m. The vertical motion of the mass is subject to Coulomb friction of magnitude 50 N. If the spring is initially displaced downward by 5 cm from its static equilibrium position, determine (a) the number of half cycles elapsed before the mass comes to rest, (b) the time elapsed before the mass comes to rest, and (c) the final extension of the spring.

2.103 The Charpy impact test is a dynamic test in which a specimen is struck and broken by a pendulum (or hammer) and the energy absorbed in breaking the specimen is measured. The energy values serve as a useful guide for comparing the impact strengths of different materials. As shown in Fig. 2.98, the pendulum is suspended from a shaft, is released from a particular position, and is allowed to fall and break the specimen. If the pendulum is made to oscillate freely (with no specimen), find (a) an expression for the decrease in the angle of swing for each cycle caused by friction, (b) the solution for $\theta(t)$ if the pendulum is released from an angle θ_0, and (c) the number of cycles after which the motion ceases. Assume the mass of the pendulum as m and the coefficient of friction between the shaft and the bearing of the pendulum as μ.



Charles Augustin de Coulomb (1736–1806) was a French military engineer and physicist. His early work on statics and mechanics was presented in 1779 in his great memoir *The Theory of Simple Machines*, which describes the effect of resistance and the so-called "Coulomb's law of proportionality" between friction and normal pressure. In 1784, he obtained the correct solution to the problem of the small oscillations of a body subjected to torsion. He is well known for his laws of force for electrostatic and magnetic charges. His name is remembered through the unit of electric charge. (Courtesy of *Applied Mechanics Reviews*).

CHAPTER 3

Harmonically Excited Vibration

3.1 Introduction

A mechanical or structural system is said to undergo forced vibration whenever external energy is supplied to the system during vibration. External energy can be supplied to the system through either an applied force or an imposed displacement excitation. The applied force or displacement excitation may be harmonic, nonharmonic but periodic, nonperiodic, or random in nature. The response of a system to a harmonic excitation is called *harmonic response*. The nonperiodic excitation may have a long or short duration. The response of a dynamic system to suddenly applied nonperiodic excitations is called *transient response*.

In this chapter, we shall consider the dynamic response of a single degree of freedom system under harmonic excitations of the form $F(t) = F_0 e^{i(\omega t + \phi)}$ or $F(t) = F_0 \cos(\omega t + \phi)$ or $F(t) = F_0 \sin(\omega t + \phi)$, where F_0 is the amplitude, ω is the frequency, and ϕ is the phase angle of the harmonic excitation. The value of ϕ depends on the value of $F(t)$ at $t = 0$ and is usually taken to be zero. Under a harmonic excitation, the response of the system will also be harmonic. If the frequency of excitation coincides with the natural frequency of the system, the response of the system will be very large. This condition, known as resonance, is to be avoided to prevent failure of the system. The vibration produced by an unbalanced rotating machine, the oscillations of a tall chimney due to vortex shedding in a steady wind, and the vertical motion of an automobile on a sinusoidal road surface are examples of harmonically excited vibration.

3.2 Equation of Motion

If a force $F(t)$ acts on a viscously damped spring-mass system as shown in Fig. 3.1, the equation of motion can be obtained using Newton's second law:

$$m\ddot{x} + c\dot{x} + kx = F(t) \tag{3.1}$$

Since this equation is nonhomogeneous, its general solution $x(t)$ is given by the sum of the homogeneous solution, $x_h(t)$, and the particular solution, $x_p(t)$. The homogeneous solution, which is the solution of the homogeneous equation

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{3.2}$$

represents the free vibration of the system and was discussed in Chapter 2. As seen in Section 2.6.2, this free vibration dies out with time under each of the three possible conditions of damping (underdamping, critical damping, and overdamping) and under all possible initial conditions. Thus the general solution of Eq. (3.1) eventually reduces to the particular solution $x_p(t)$, which represents the steady-state vibration. The steady-state motion is present as long as the forcing function is present. The variations of homogeneous, particular, and general solutions with time for a typical case are shown in Fig. 3.2. It can be seen that $x_h(t)$ dies out and $x(t)$ becomes $x_p(t)$ after some time (τ in Fig. 3.2). The part of the motion that dies out due to damping (the free vibration part) is called *transient*. The rate at which the transient motion decays depends on the values of the system parameters k, c, and m. In this chapter, except in Section 3.3, we ignore the transient motion and derive only the particular solution of Eq. (3.1), which represents the steady-state response, under harmonic forcing functions.

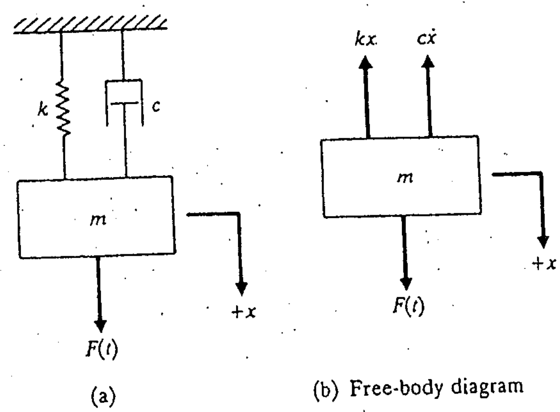


FIGURE 3.1 A spring-mass-damper system.

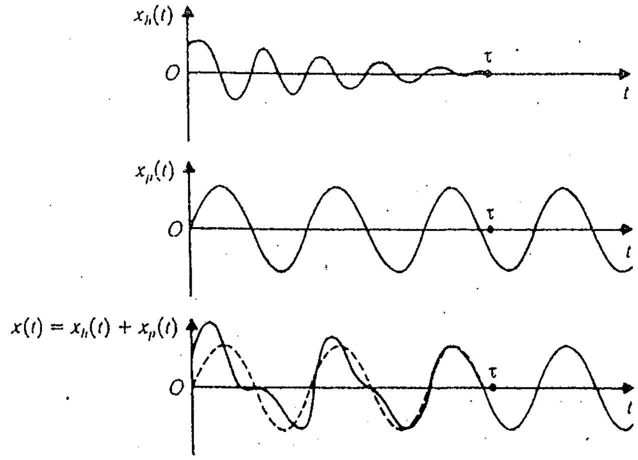


FIGURE 3.2 Homogenous, particular, and general solutions of Eq. (3.1) for an underdamped case.

3.3 Response of an Undamped System under Harmonic Force

Before studying the response of a damped system, we consider an undamped system subjected to a harmonic force, for the sake of simplicity. If a force $F(t) = F_0 \cos \omega t$ acts on the mass m of an undamped system, the equation of motion, Eq. (3.1), reduces to

$$m\ddot{x} + kx = F_0 \cos \omega t \tag{3.3}$$

The homogeneous solution of this equation is given by

$$x_h(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t \tag{3.4}$$

where $\omega_n = (k/m)^{1/2}$ is the natural frequency of the system. Because the exciting force $F(t)$ is harmonic, the particular solution $x_p(t)$ is also harmonic and has the same frequency ω. Thus we assume a solution in the form

$$x_p(t) = X \cos \omega t \tag{3.5}$$

where X is a constant that denotes the maximum amplitude of $x_p(t)$. By substituting Eq. (3.5) into Eq. (3.3) and solving for X, we obtain

$$X = \frac{F_0}{k - m\omega^2} \tag{3.6}$$

Thus the total solution of Eq. (3.3) is

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{F_0}{k - m\omega^2} \cos \omega t \tag{3.7}$$

Using the initial conditions $x(t = 0) = x_0$ and $\dot{x}(t = 0) = \dot{x}_0$, we find that

$$C_1 = x_0 - \frac{F_0}{k - m\omega^2}, \qquad C_2 = \frac{\dot{x}_0}{\omega_n} \tag{3.8}$$

and hence

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right)\sin\omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\cos\omega t \tag{3.9}$$

The maximum amplitude X in Eq. (3.6) can also be expressed as

$$\frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \tag{3.10}$$

where $\delta_{st} = F_0/k$ denotes the deflection of the mass under a force F_0 and is sometimes called "static deflection" since F_0 is a constant (static) force. The quantity X/δ_{st} represents the ratio of the dynamic to the static amplitude of motion and is called the *magnification factor, amplification factor,* or *amplitude ratio.* The variation of the amplitude ratio, X/δ_{st}, with the frequency ratio $r = \omega/\omega_n$ (Eq. 3.10) is shown in Fig. 3.3. From this figure, the response of the system can be identified to be of three types.

Case 1. When $0 < \omega/\omega_n < 1$, the denominator in Eq. (3.10) is positive and the response is given by Eq. (3.5) without change. The harmonic response of the system $x_p(t)$ is said to be in phase with the external force as shown in Fig. 3.4.

Case 2. When $\omega/\omega_n > 1$, the denominator in Eq. (3.10) is negative, and the steady-state solution can be expressed as

$$x_p(t) = -X\cos\omega t \tag{3.11}$$

where the amplitude of motion X is redefined to be a positive quantity as

$$X = \frac{\delta_{st}}{\left(\frac{\omega}{\omega_n}\right)^2 - 1} \tag{3.12}$$

The variations of $F(t)$ and $x_p(t)$ with time are shown in Fig. 3.5. Since $x_p(t)$ and $F(t)$ have opposite signs, the response is said to be 180° out of phase with the

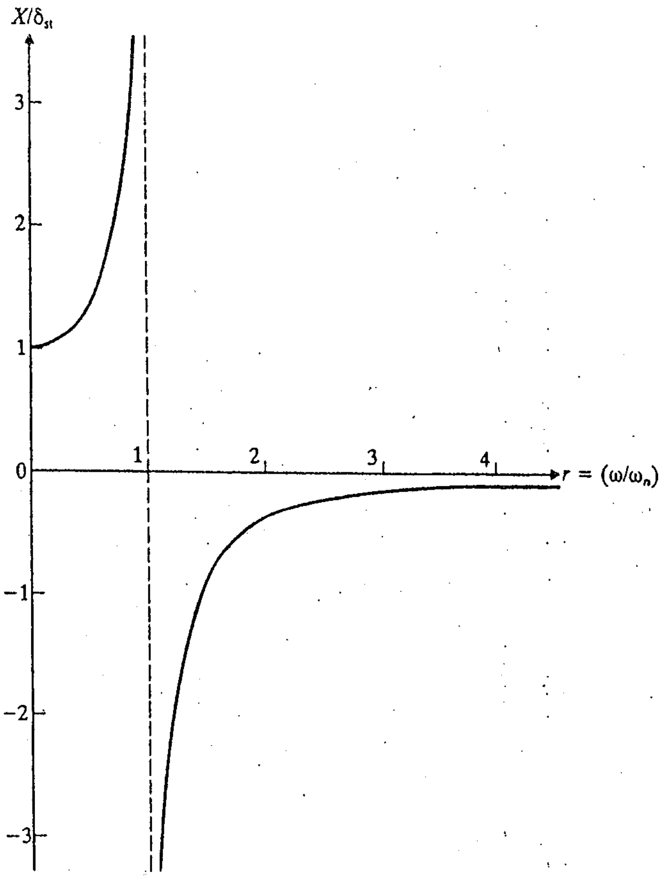


FIGURE 3.3

external force. Further, as $\omega/\omega_n \rightarrow \infty$, $X \rightarrow 0$. Thus the response of the system to a harmonic force of very high frequency is close to zero.

Case 3. When $\omega/\omega_n = 1$, the amplitude X given by Eq. (3.10) or (3.12) becomes infinite. This condition, for which the forcing frequency ω is equal to the natural frequency of the system ω_n, is called *resonance*. To find the response for this condition, we rewrite Eq. (3.9) as

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \delta_{st} \left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\dfrac{\omega}{\omega_n} \right)^2} \right] \qquad (3.13)$$

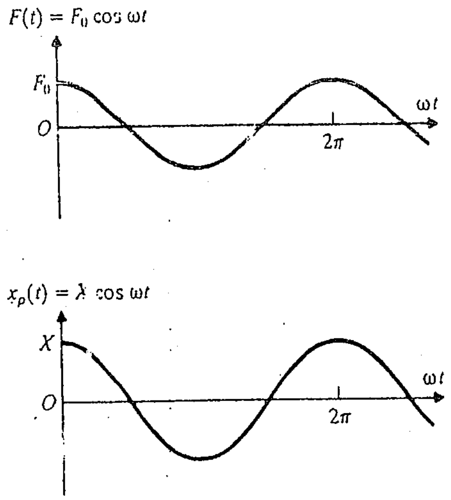


FIGURE 3.4

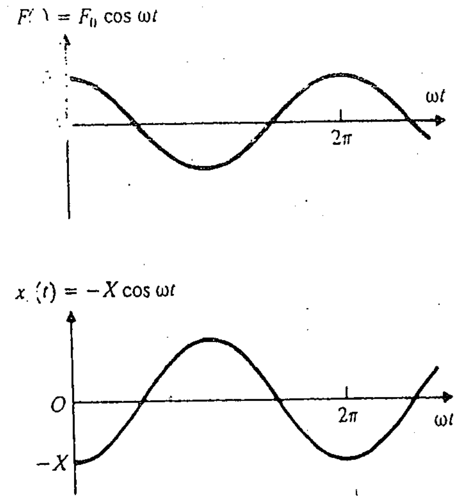


FIGURE 3.5

Since the last term of this equation takes an indefinite form for $\omega = \omega_n$, we apply L'Hospital's rule [3.1] to evaluate the limit of this term:

$$\lim_{\omega \to \omega_n} \left[\frac{\cos \omega t - \cos \omega_n t}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \right] = \lim_{\omega \to \omega_n} \left[\frac{\frac{d}{d\omega}(\cos \omega t - \cos \omega_n t)}{\frac{d}{d\omega}\left(1 - \frac{\omega^2}{\omega_n^2}\right)} \right]$$

$$= \lim_{\omega \to \omega_n} \left[\frac{t \sin \omega t}{2 \frac{\omega}{\omega_n^2}} \right] = \frac{\omega_n t}{2} \sin \omega_n t. \tag{3.14}$$

Thus the response of the system at resonance becomes

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st} \omega_n t}{2} \sin \omega_n t \tag{3.15}$$

It can be seen from Eq. (3.15) that at resonance, $x(t)$ increases indefinitely. The last term of Eq. (3.15) is shown in Fig. 3.6, from which the amplitude of the response can be seen to increase linearly with time.

3.3.1 Total Response

The total response of the system, Eq. (3.7) or Eq. (3.9), can also be expressed as

$$x(t) = A \cos(\omega_n t - \phi) + \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \cos \omega t; \qquad \text{for } \frac{\omega}{\omega_n} < 1 \tag{3.16}$$

$$x(t) = A \cos(\omega_n t - \phi) - \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \cos \omega t; \qquad \text{for } \frac{\omega}{\omega_n} > 1 \tag{3.17}$$

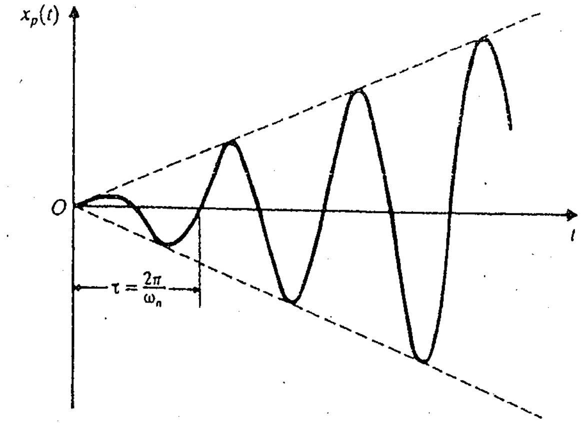


FIGURE 3.6

where A and ϕ can be determined as in the case of Eq. (2.21). Thus the complete motion can be expressed as the sum of two cosine curves of different frequencies. In Eq. (3.16), the forcing frequency ω is smaller than the natural frequency, and the total response is shown in Fig. 3.7(a). In Eq. (3.17), the forcing frequency is greater than the natural frequency, and the total response appears as shown in Fig. 3.7(b).

3.3.2 Beating Phenomenon

If the forcing frequency is close to, but not exactly equal to, the natural frequency of the system, a phenomenon known as *beating* may occur. In this kind of vibration, the amplitude builds up and then diminishes in a regular pattern. The phenomenon of beating can be explained by considering the solution given by Eq. (3.9). If the initial conditions are taken as $x_0 = \dot{x}_0 = 0$, Eq. (3.9) reduces to

$$x(t) = \frac{(F_0/m)}{\omega_n^2 - \omega^2}(\cos \omega t - \cos \omega_n t)$$

$$= \frac{(F_0/m)}{\omega_n^2 - \omega^2}\left[2 \sin \frac{\omega + \omega_n}{2} t \cdot \sin \frac{\omega_n - \omega}{2} t\right] \tag{3.18}$$

Let the forcing frequency ω be slightly less than the natural frequency:

$$\omega_n - \omega = 2\varepsilon \tag{3.19}$$

where ε is a small positive quantity. Then $\omega_n \approx \omega$ and

$$\omega + \omega_n \simeq 2\omega \tag{3.20}$$

Multiplication of Eqs. (3.19) and (3.20) gives

$$\omega_n^2 - \omega^2 = 4\varepsilon\omega \tag{3.21}$$

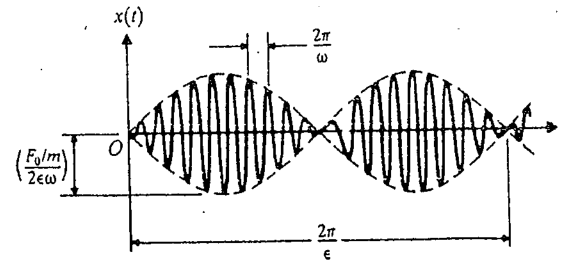


FIGURE 3.8

or the points of maximum amplitude is called the *period of beating* (τ_b) and is given by

$$\tau_b = \frac{2\pi}{2\varepsilon} = \frac{2\pi}{\omega_n - \omega} \tag{3.23}$$

with the frequency of beating defined as

$$\omega_b = 2\varepsilon = \omega_n - \omega$$

EXAMPLE 3.1 Plate Supporting a Pump

A reciprocating pump, weighing 150 lb, is mounted at the middle of a steel plate of thickness 0.5 in., width 20 in., and length 100 in., clamped along two edges as shown in Fig. 3.9. During operation of the pump, the plate is subjected to a harmonic force, $F(t) = 50 \cos 62.832\, t$ lb. Find the amplitude of vibration of the plate.

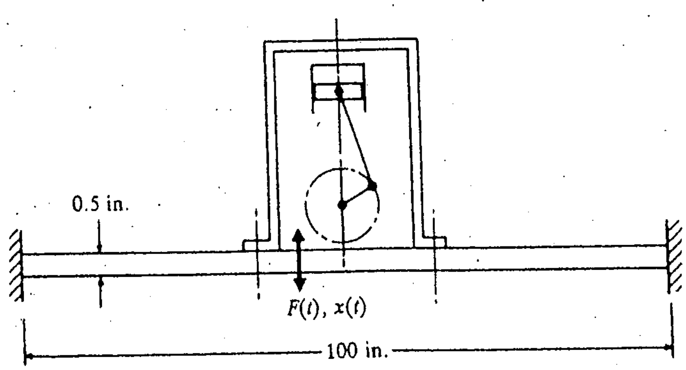


FIGURE 3.9

Given: Pump weight = 150 lb; plate dimensions: thickness (t) = 0.5 in., width (w) = 20 in., and length (l) = 100 in.; and harmonic force: $F(t) = 50 \cos 62.832\, t$ lb.

Find: Amplitude of vibration of the plate, X.

Approach: Find the stiffness of the plate by modeling it as a clamped beam. Use the equation for the response under harmonic excitation.

Solution: The plate can be modeled as a fixed-fixed beam having Young's modulus (E) = 30×10^6 psi, length (l) = 100 in., and area moment of inertia (I) = $\frac{1}{12}(20)(0.5)^3$ = 0.2083 in^4. The bending stiffness of the beam is given by

$$k = \frac{192EI}{l^3} = \frac{192(30 \times 10^6)(0.2083)}{(100)^3} = 1200.0 \text{ lb/in.} \tag{E.1}$$

The amplitude of harmonic response is given by Eq. (3.6) with $F_0 = 50$ lb, $m = 150/386.4$ lb-sec^2/in. (neglecting the weight of the steel plate), $k = 1200.0$ lb/in., and $\omega = 62.832$ rad/sec. Thus Eq. (3.6) gives

$$X = \frac{F_0}{k - m\omega^2} = \frac{50}{1200.0 - (150/386.4)(62.832)^2} = -0.1504 \text{ in.} \tag{E.2}$$

The negative sign indicates that the response $x(t)$ of the plate is out of phase with the excitation $F(t)$. ■

3.4 Response of a Damped System under Harmonic Force

If the forcing function is given by $F(t) = F_0 \cos \omega t$, the equation of motion becomes

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t \tag{3.24}$$

The particular solution of Eq. (3.24) is also expected to be harmonic; we assume it in the form[1]

$$x_p(t) = X \cos(\omega t - \phi) \tag{3.25}$$

where X and ϕ are constants to be determined. X and ϕ denote the amplitude and phase angle of the response, respectively. By substituting Eq. (3.25) into Eq. (3.24), we arrive at

$$X[(k - m\omega^2)\cos(\omega t - \phi) - c\omega \sin(\omega t - \phi)] = F_0 \cos \omega t \tag{3.26}$$

Using the trigonometric relations

$$\cos(\omega t - \phi) = \cos \omega t \cos \phi + \sin \omega t \sin \phi$$
$$\sin(\omega t - \phi) = \sin \omega t \cos \phi - \cos \omega t \sin \phi$$

[1]Alternatively, we can assume $x_p(t)$ to be of the form $x_p(t) = C_1 \cos \omega t + C_2 \sin \omega t$, which also involves two constants C_1 and C_2. But the final result will be the same in both the cases.

in Eq. (3.26) and equating the coefficients of cos ωt and sin ωt on both sides of the resulting equation, we obtain

$$X\left[(k - m\omega^2)\cos\phi + c\omega\sin\phi\right] = F_0$$
$$X\left[(k - m\omega^2)\sin\phi - c\omega\cos\phi\right] = 0 \tag{3.27}$$

Solution of Eqs. (3.27) gives

$$X = \frac{F_0}{\left[(k - m\omega^2)^2 + c^2\omega^2\right]^{1/2}} \tag{3.28}$$

and

$$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \tag{3.29}$$

By inserting the expressions of X and ϕ from Eqs. (3.28) and (3.29) into Eq. (3.25) we obtain the particular solution of Eq. (3.24). Figure 3.10 shows typical plots of the forcing function and (steady-state) response. Dividing both the numerator and denominator of Eq. (3.28) by k and making the following substitutions

$$\omega_n = \sqrt{\frac{k}{m}} = \text{undamped natural frequency,}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{mk}};\quad \frac{c}{m} = 2\zeta\omega_n,$$

$$\delta_{st} = \frac{F_0}{k} = \text{deflection under the static force } F_0, \text{ and}$$

$$r = \frac{\omega}{\omega_n} = \text{frequency ratio}$$

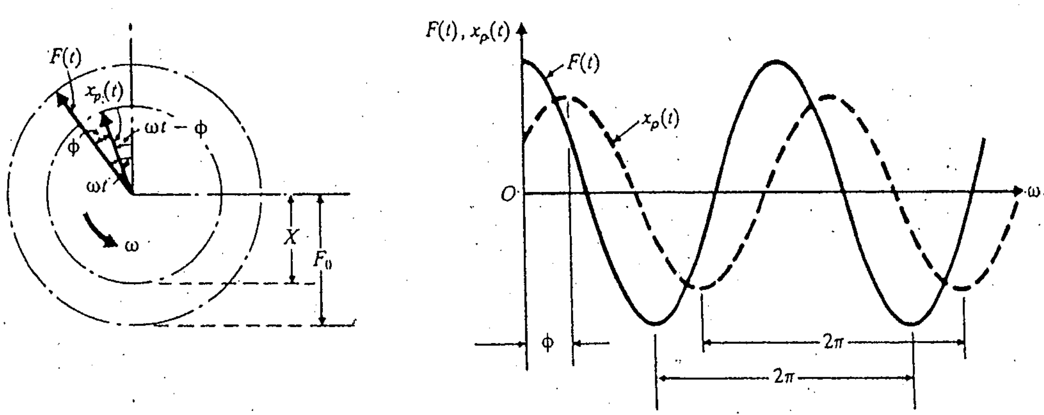


FIGURE 3.10 Graphical representation of forcing function and response.

we obtain

$$\frac{X}{\delta_{st}} = \frac{1}{\left\{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left[2\zeta\frac{\omega}{\omega_n}\right]^2\right\}^{1/2}} = \frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{3.30}$$

and

$$\phi = \tan^{-1}\left\{\frac{2\zeta\frac{\omega}{\omega_n}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) \tag{3.31}$$

As stated in Section 3.3, the quantity $M = X/\delta_{st}$ is called the *magnification factor, amplification factor,* or *amplitude ratio.* The variations of X/δ_{st} and ϕ with the frequency ratio r and the damping ratio ζ are shown in Fig. 3.11.

The following characteristics of the magnification factor (M) can be noted from Eq. (3.30) and Fig. 3.11(a):

1. For an undamped system ($\zeta = 0$), Eq. (3.30) reduces to Eq. (3.10), and $M \to \infty$ as $r \to 1$.

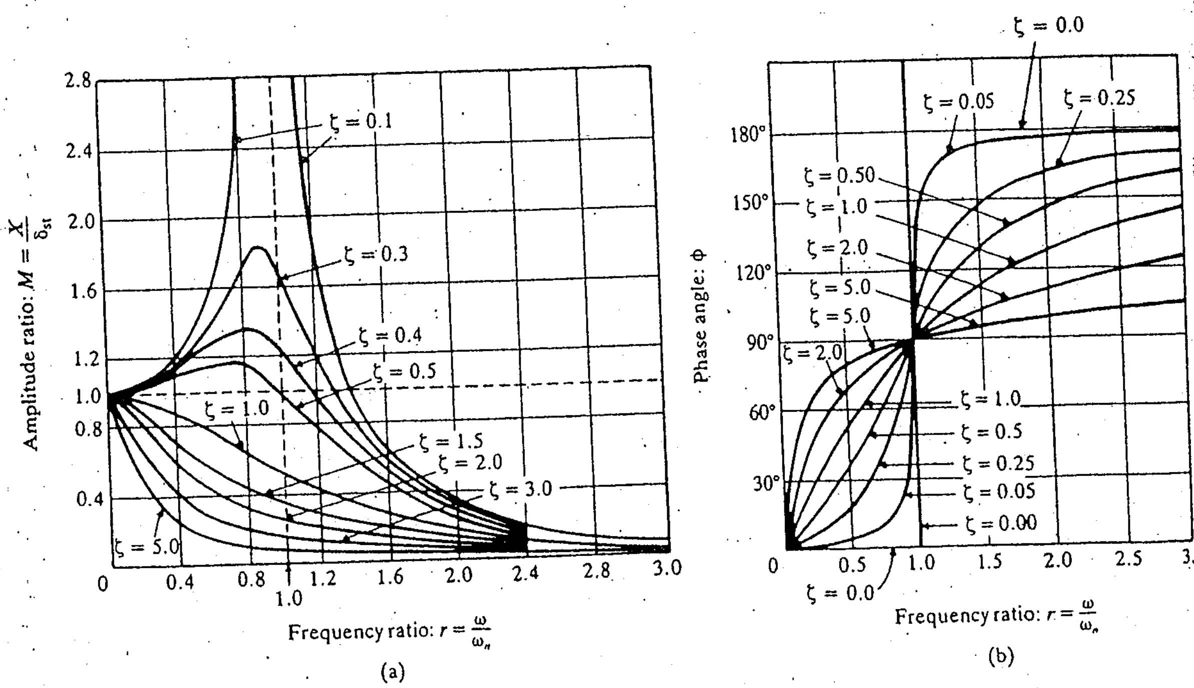


FIGURE 3.11 Variation of X and ϕ with frequency ratio r.

2. Any amount of damping ($\zeta > 0$) reduces the magnification factor (M) for all values of the forcing frequency.
3. For any specified value of r, a higher value of damping reduces the value of M.
4. In the degenerate case of a constant force (when $r = 0$), the value of $M = 1$.
5. The reduction in M in the presence of damping is very significant at or near resonance.
6. The amplitude of forced vibration becomes smaller with increasing values of the forcing frequency (that is, $M \to 0$ as $r \to \infty$).
7. For $0 < \zeta < \frac{1}{\sqrt{2}}$, the maximum value of M occurs when (see Problem 3.19)

$$r = \sqrt{1 - 2\zeta^2} \quad \text{or} \quad \omega = \omega_n \sqrt{1 - 2\zeta^2} \tag{3.32}$$

which can be seen to be lower than the undamped natural frequency ω_n and the damped natural frequency $\omega_d = \omega_n \sqrt{1 - \zeta^2}$.

8. The maximum value of X (when $r = \sqrt{1 - 2\zeta^2}$) is given by

$$\left(\frac{X}{\delta_{st}}\right)_{max} = \frac{1}{2\zeta\sqrt{1 - \zeta^2}} \tag{3.33}$$

and the value of X at $\omega = \omega_n$ by

$$\left(\frac{X}{\delta_{st}}\right)_{\omega = \omega_n} = \frac{1}{2\zeta} \tag{3.34}$$

Equation (3.33) can be used for the experimental determination of the measure of damping present in the system. In a vibration test, if the maximum amplitude of the response $(X)_{max}$ is measured, the damping ratio of the system can be found using Eq. (3.33). Conversely, if the amount of damping is known, one can make an estimate of the maximum amplitude of vibration.

9. For $\zeta = \frac{1}{\sqrt{2}}, \frac{dM}{dr} = 0$ when $r = 0$. For $\zeta > \frac{1}{\sqrt{2}}$, the graph of M monotonically decreases with increasing values of r.

The following characteristics of the phase angle can be observed from Eq. (3.31) and Fig. 3.11(b):

1. For an undamped system ($\zeta = 0$), Eq. (3.31) shows that the phase angle is 0 for $0 < r < 1$ and 180° for $r > 1$. This implies that the excitation and response are in phase for $0 < r < 1$ and out of phase for $r > 1$ when $\zeta = 0$.
2. For $\zeta > 0$ and $0 < r < 1$, the phase angle is given by $0 < \phi < 90°$, implying that the response lags the excitation.
3. For $\zeta > 0$ and $r > 1$, the phase angle is given by $90° < \phi < 180°$, implying that the response leads the excitation.
4. For $\zeta > 0$ and $r = 1$, the phase angle is given by $\phi = 90°$, implying that the phase difference between the excitation and the response is 90°.

5. For $\zeta > 0$ and large values of r, the phase angle approaches 180°, implying that the response and the excitation are out of phase.

3.4.1 Total Response

The complete solution is given by $x(t) = x_h(t) + x_p(t)$ where $x_h(t)$ is given by Eq. (2.64). Thus

$$x(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0) + X \cos(\omega t - \phi) \tag{3.35}$$

where

$$\omega_d = \sqrt{1 - \zeta^2} \cdot \omega_n \tag{3.36}$$

$$r = \frac{\omega}{\omega_n} \tag{3.37}$$

X and ϕ are given by Eqs. (3.30) and (3.31), respectively, and X_0 and ϕ_0 can be determined from the initial conditions.

3.4.2 Quality Factor and Bandwidth

For small values of damping ($\zeta < 0.05$), we can take

$$\left(\frac{X}{\delta_{st}}\right)_{max} \simeq \left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta} = Q \tag{3.38}$$

The value of the amplitude ratio at resonance is also called *Q factor* or *quality factor* of the system, in analogy with some electrical-engineering applications, such as the tuning circuit of a radio, where the interest lies in an amplitude at resonance that is as large as possible [3.2]. The points R_1 and R_2, where the amplification factor falls to $Q/\sqrt{2}$, are called *half power points* because the power absorbed (ΔW) by the damper (or by the resistor in an electrical circuit), responding harmonically at a given frequency, is proportional to the square of the amplitude (see Eq. 2.94):

$$\Delta W = \pi c \omega X^2 \tag{3.39}$$

The difference between the frequencies associated with the half power points R_1 and R_2 is called the *bandwidth* of the system (see Fig. 3.12). To find the values of R_1 and R_2, we set $X/\delta_{st} = Q/\sqrt{2}$ in Eq. (3.30) so that

$$\frac{1}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = \frac{Q}{\sqrt{2}} = \frac{1}{2\sqrt{2}\zeta}$$

or

$$r^4 - r^2(2 - 4\zeta^2) + (1 - 8\zeta^2) = 0 \tag{3.40}$$

The solution of Eq. (3.40) gives

$$r_1^2 = 1 - 2\zeta^2 - 2\zeta\sqrt{1 + \zeta^2}, \qquad r_2^2 = 1 - 2\zeta^2 + 2\zeta\sqrt{1 + \zeta^2} \tag{3.41}$$

3.13 How does the force transmitted to the base change as the speed of the machine increases?

3.14 If a vehicle vibrates badly while moving on a uniformly bumpy road, will a change in the speed improve the condition?

3.15 Is it possible to find the maximum amplitude of a damped forced vibration for any value of r by equating the energy dissipated by damping to the work done by the external force?

3.16 What assumptions are made about the motion of a forced vibration with nonviscous damping in finding the amplitude?

3.17 Is it possible to find the approximate value of the amplitude of a damped forced vibration without considering damping at all? If so, under what circumstances?

3.18 Is dry friction effective in limiting the reasonant amplitude?

3.19 How do you find the response of a viscously damped system under rotating unbalance?

3.20 What is the frequency of the response of a viscously damped system when the external force is $F_0 \sin \omega t$? Is this response harmonic?

3.21 What is the difference between the peak amplitude and the resonant amplitude?

3.22 Why is viscous damping used in most cases rather than other types of damping?

3.23 What is self-excited vibration?

Problems

The problem assignments are organized as follows:

Problems	Section Covered	Topic Covered
3.1–3.16	3.3	Undamped systems
3.17–3.32	3.4	Damped systems
3.33–3.41	3.6	Base excitation
3.42–3.52	3.7	Rotating unbalance
3.53–3.55	3.8	Response under Coulomb damping
3.56–3.57	3.9	Response under hysteresis damping
3.58–3.61	3.10	Response under other types of damping
3.62–3.65	3.11	Self excitation and stability
3.66–3.69	3.12	Computer program
3.70–3.71	—	Projects

3.1 A weight of 50 N is suspended from a spring of stiffness 4000 N/m and is subjected to a harmonic force of amplitude 60 N and frequency 6 Hz. Find (a) the extension of the spring due to the suspended weight, (b) the static displacement of the spring due to the maximum applied force, and (c) the amplitude of forced motion of the weight.

3.2 A spring-mass system is subjected to a harmonic force whose frequency is close to the natural frequency of the system. If the forcing frequency is 39.8 Hz and the natural frequency is 40.0 Hz, determine the period of beating.

3.3 A spring-mass system consists of a mass weighing 100 N and a spring with a stiffness of 2000 N/m. The mass is subjected to resonance by a harmonic force $F(t) = 25 \cos \omega t$ N. Find the amplitude of the forced motion at the end of (a) $\frac{1}{4}$ cycle, (b) $2\frac{1}{2}$ cycles, and (c) $5\frac{3}{4}$ cycles.

3.4 A mass m is suspended from a spring of stiffness 4000 N/m and is subjected to a harmonic force having an amplitude of 100 N and a frequency of 5 Hz. The amplitude of the forced motion of the mass is observed to be 20 mm. Find the value of m.

3.5 A spring-mass system with $m = 10$ kg and $k = 5000$ N/m is subjected to a harmonic force of amplitude 250 N and frequency ω. If the maximum amplitude of the mass is observed to be 100 mm, find the value of ω.

3.6 In Fig. 3.1(a), a periodic force $F(t) = F_0 \cos \omega t$ is applied at a point on the spring that is located at a distance of 25 percent of its length from the fixed support. Assuming that $c = 0$, find the steady-state response of the mass m.

3.7 An aircraft engine has a rotating unbalanced mass m at radius r. If the wing can be modeled as a cantilever beam of uniform cross section $a \times b$, as shown in Fig. 3.34(b), determine the maximum deflection of the wing at an engine speed of N rpm. Assume damping to be negligible.

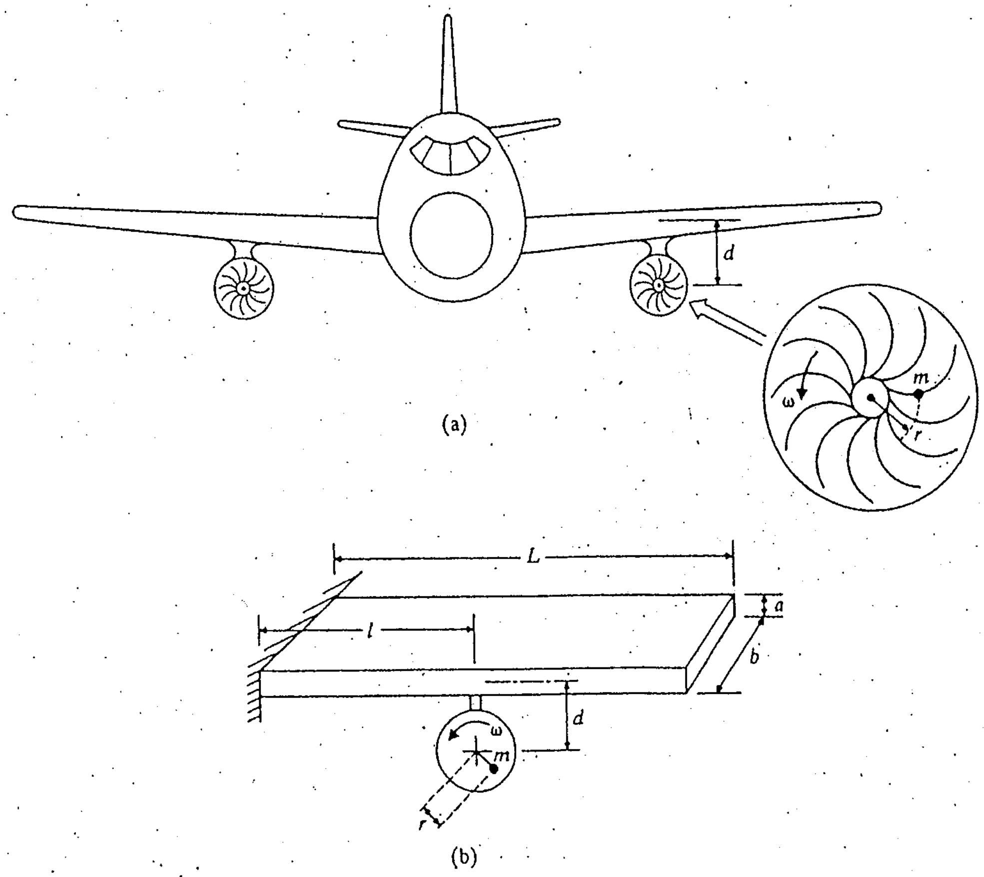


FIGURE 3.34

3.8 A three-bladed wind turbine (Fig. 3.35a) has a small unbalanced mass m located at a radius r in the plane of the blades. The blades are located from the central vertical (y) axis at a distance R and rotate at an angular velocity of ω. If the supporting truss can be modeled as a hollow steel shaft of outer diameter 0.1 m and inner diameter 0.08 m, determine the maximum stresses developed at the base of the support (point A). The mass moment of inertia of the turbine system about the vertical (y) axis is J_0. Assume $R = 0.5$ m, $m = 0.1$ kg, $r = 0.1$ m, $J_0 = 100$ kg-m^2, $h = 8$ m, and $\omega = 31.416$ rad/sec.

3.9 An electromagnetic fatigue testing machine is shown in Fig. 3.36 in which an alternating force is applied to the specimen by passing an alternating current of frequency f through the armature. If the weight of the armature is 40 lb, the stiffness of the spring (k_1) is 10,217.0296 lb/in and the stiffness of the steel specimen is 75×10^4 lb/in, determine the frequency of the a.c. current that induces a stress in the specimen that is twice the amount generated by the magnets.

3.10 The spring actuator shown in Fig. 3.37 operates by using the air pressure from a pneumatic controller (p) as input and providing an output displacement to a valve (x) proportional to the input air pressure. The diaphragm, made of a fabric-base rubber, has an area A and deflects under the input air pressure against a spring of stiffness k. Find the response of the valve under a harmonically fluctuating input air pressure $p(t) = p_0 \sin \omega t$ for the following data: $p_0 = 10$ psi, $\omega = 8$ rad/s, $A = 100$ in^2, $k = 400$ lb/in, weight of spring $= 15$ lb, and weight of valve and valve rod $= 20$ lb.

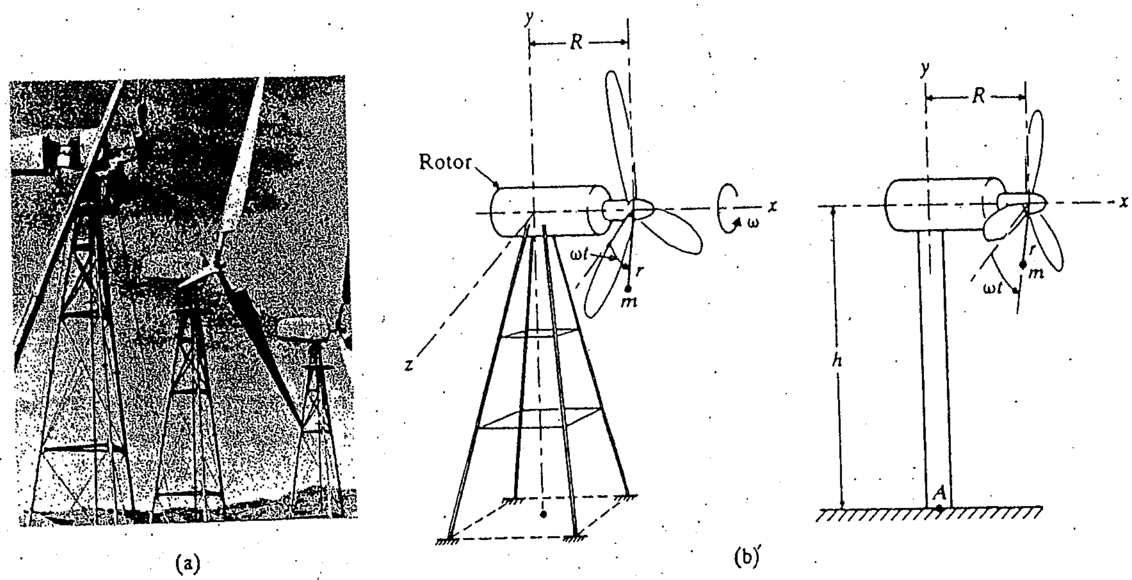


FIGURE 3.35 (Photo courtesy of *Power Transmission Design*)

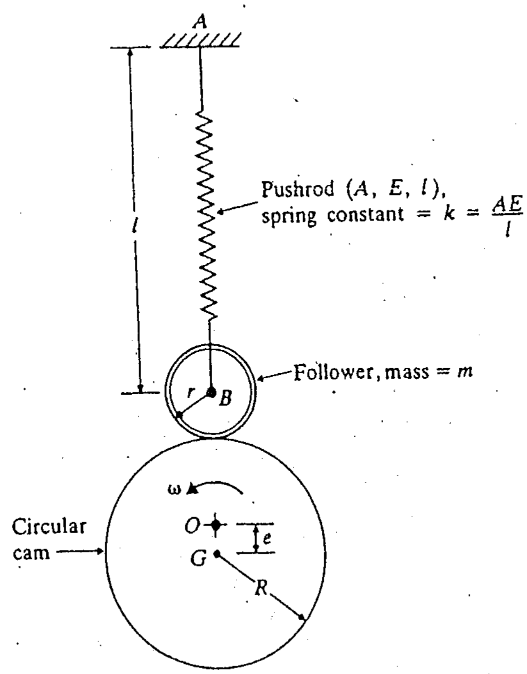


FIGURE 3.38

3.11 In the cam-follower system shown in Fig. 3.38, the rotation of the cam imparts a vertical motion to the follower. The pushrod, which acts as a spring, has been compressed by an amount x_0 before assembly. Determine the following: (a) equation of motion of the follower, including the gravitational force; (b) force exerted on the follower by the cam; and (c) conditions under which the follower loses contact with the cam.

3.12* Design a solid steel shaft supported in bearings which carries the rotor of a turbine at the middle. The rotor weighs 500 lb and delivers a power of 200 hp at 3000 rpm. In order to keep the stress due to the unbalance in the rotor small, the critical speed of the shaft is to be made one-fifth of the operating speed of the rotor. The length of the shaft is to be made equal to at least 30 times its diameter.

3.13 A hollow steel shaft, of length 100 in., outer diameter 4 in. and inner diameter 3.5 in., carries the rotor of a turbine, weighing 500 lb, at the middle and is supported at the ends in bearings. The clearance between the rotor and the stator is 0.5 in. The rotor has an eccentricity equivalent to a weight of 0.5 lb at a radius of 2 in. A limit switch is installed to stop the rotor whenever the rotor touches the stator. If the rotor operates at resonance, how long will it take to activate the limit switch? Assume the initial displacement and velocity of the rotor perpendicular to the shaft to be zero.

3.14 A steel cantilever beam, carrying a weight of 0.1 lb at the free end, is used as a frequency meter.[6] The beam has a length of 10 in., width of 0.2 in., and thickness of

*The asterisk denotes a design type problem or a problem with no unique answer.

[6]The use of cantilever beams as frequency meters is discussed in detail in Section 10.4.

0.05 in. The internal friction is equivalent to a damping ratio of 0.01. When the fixed end of the beam is subjected to a harmonic displacement $y(t) = 0.05 \cos \omega t$, the maximum tip displacement has been observed to be 2.5 in. Find the forcing frequency.

3.15 Derive the equation of motion and find the steady-state response of the system shown in Fig. 3.39 for rotational motion about the hinge O for the following data: $k_1 = k_2 = 5000$ N/m, $a = 0.25$ m, $b = 0.5$ m, $l = 1$ m, $M = 50$ kg, $m = 10$ kg, $F_0 = 500$ N, $\omega = 1000$ rpm.

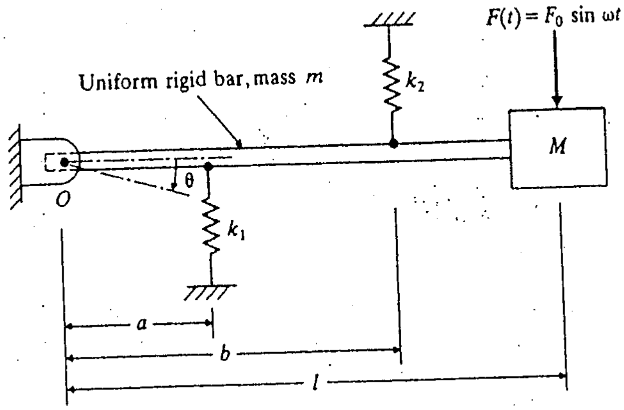


FIGURE 3.39

3.16 Derive the equation of motion and find the steady-state solution of the system shown in Fig. 3.40 for rotational motion about the hinge O for the following data: $k = 5000$ N/m, $l = 1$ m, $m = 10$ kg, $M_0 = 100$ N-m, $\omega = 1000$ rpm.

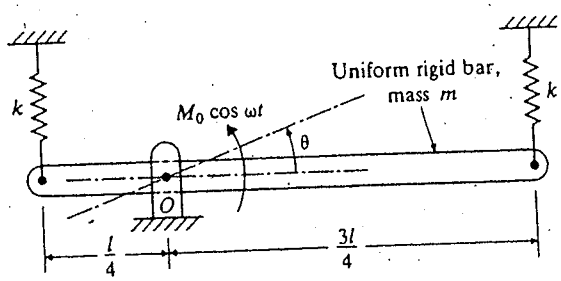


FIGURE 3.40

3.17 A four-cylinder automobile engine is to be supported on three shock mounts, as indicated in Fig. 3.41. The engine block assembly weighs 500 lb. If the unbalanced force generated by the engine is given by $200 \sin 100 \pi t$ lb, design the three shock mounts (each of stiffness k and viscous damping constant c) such that the amplitude of vibration is less than 0.1 in.

3.18 The propeller of a ship, of weight 10^5 N and polar mass moment of inertia 10,000 kg-m^2, is connected to the engine through a hollow stepped steel propeller shaft, as shown in Fig. 3.42. Assuming that water provides a viscous damping ratio of 0.1,

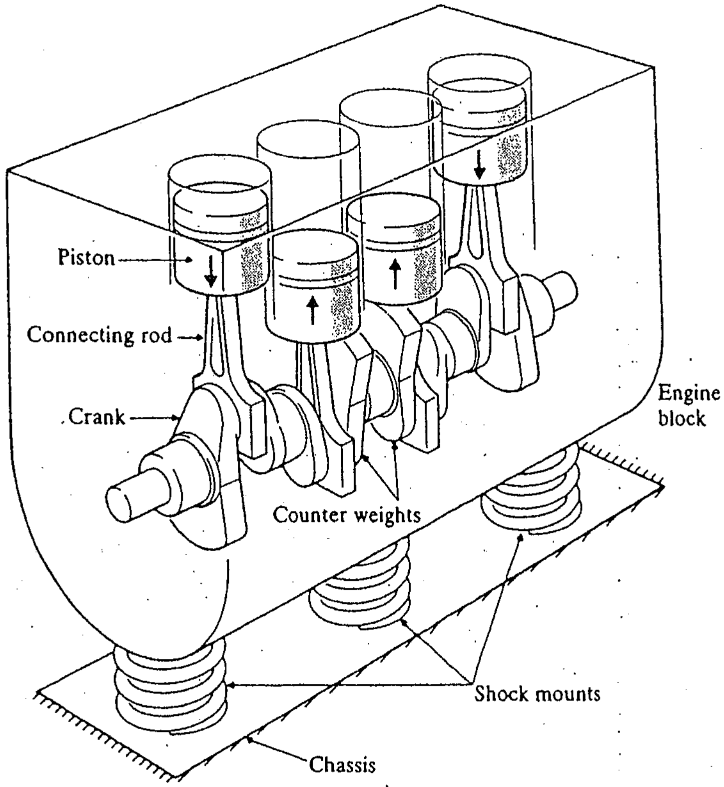


FIGURE 3.41

determine the torsional vibratory response of the propeller when the engine induces a harmonic angular displacement of 0.05 sin 314.16 t rad at the base (point A) of the propeller shaft.

3.19 Find the frequency ratio $r = \omega/\omega_n$ at which the amplitude of a single degree of freedom damped system attains the maximum value. Also find the value of the maximum amplitude.

3.20 Figure 3.43 shows a permanent-magnet moving coil ammeter. When current (I) flows through the coil wound on the core, the core rotates by an angle proportional to the magnitude of the current that is indicated by the pointer on a scale. The core, with the coil, has a mass moment of inertia J_0, the torsional spring constant of k_t, and the torsional damper has a damping constant of c_t. The scale of the ammeter is calibrated such that when a d.c. current of magnitude 1 ampere is passed through the coil, the pointer indicates a current of 1 ampere. The meter has to be recalibrated for measuring the magnitude of a.c. current. Determine the steady-state value of the current indicated by the pointer when an a.c. current of magnitude 5 amperes and frequency 50 Hz is passed through the coil. Assume $J_0 = 0.001$ N-m^2, $k_t = 62.5$ N-m/rad and $c_t = 0.5$ N-m-s/rad.

3.21 A spring-mass-damper system is subjected to a harmonic force. The amplitude is found to be 20 mm at resonance and 10 mm at a frequency 0.75 times the resonant frequency. Find the damping ratio of the system.

3.22 For the system shown in Fig. 3.44, x and y denote, respectively, the absolute displacements of the mass m and the end Q of the dashpot c_1. (a) Derive the equation of motion of the mass m, (b) find the steady state displacement of the mass m, and (c) find the force transmitted to the support at P, when the end Q is subjected to the harmonic motion $y(t) = Y \cos \omega t$.

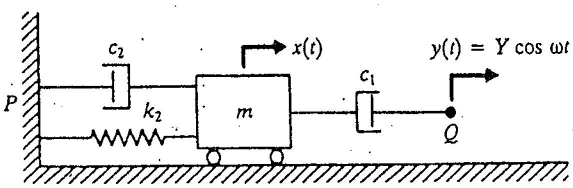


FIGURE 3.44

3.23 Show that, for small values of damping, the damping ratio ζ can be expressed as

$$\zeta = \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1}$$

where ω_1 and ω_2 are the frequencies corresponding to the half power points.

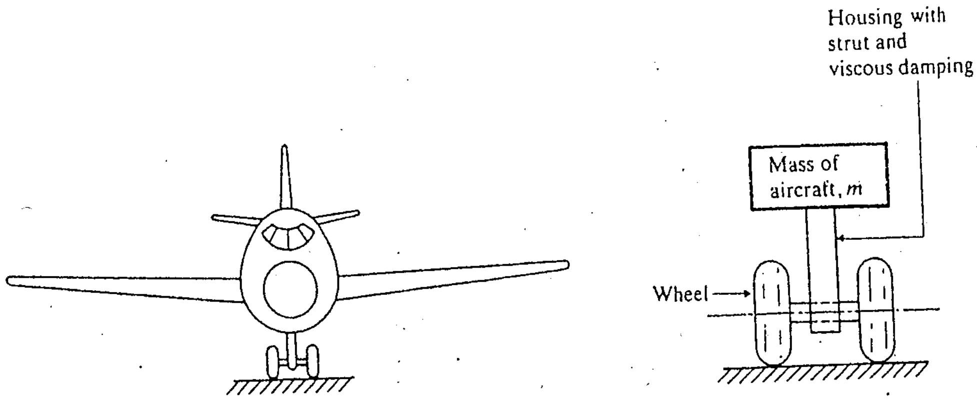
3.24 A torsional system consists of a disc of mass moment of inertia $J_0 = 10$ kg-m^2, a torsional damper of damping constant $c_t = 300$ N-m-s/rad, and a steel shaft of diameter 4 cm and length 1 m (fixed at one end and attached to the disc at the other end). A steady angular oscillation of amplitude 2° is observed when a harmonic torque of magnitude 1000 N-m is applied to the disc. (a) Find the frequency of the applied torque, and (b) find the maximum torque transmitted to the support.

3.25 For a vibrating system, $m = 10$ kg, $k = 2500$ N/m, and $c = 45$ N-s/m. A harmonic force of amplitude 180 N and frequency 3.5 Hz acts on the mass. If the initial displacement and velocity of the mass are 15 mm and 5 m/s, find the complete solution representing the motion of the mass.

3.26 The peak amplitude of a single degree of freedom system, under a harmonic excitation, is observed to be 0.2 in. If the undamped natural frequency of the system is 5 Hz, and the static deflection of the mass under the maximum force is 0.1 in., (a) estimate the damping ratio of the system, and (b) find the frequencies corresponding to the amplitudes at half power.

3.27 The landing gear of an airplane can be idealized as the spring-mass-damper system shown in Fig. 3.45. If the runway surface is described $y(t) = y_0 \cos \omega t$, determine the values of k and c that limit the amplitude of vibration of the airplane (x) to 0.1 m. Assume $m = 2000$ kg, $y_0 = 0.2$ m and $\omega = 157.08$ rad/s.

3.28 A precision grinding machine (Fig. 3.46) is supported on an isolator that has a stiffness of 1 MN/m and a viscous damping constant of 1 kN-s/m. The floor on which the machine is mounted is subjected to a harmonic disturbance due to the operation of an unbalanced engine in the vicinity of the grinding machine. Find the maximum acceptable displacement amplitude of the floor if the resulting amplitude of vibration



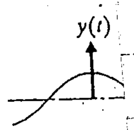


FIGURE 3.45

$T_0 = 1000$ N-m

3.24 $I_0 = 10$ kg-m^2 $\quad c_t = 300 \frac{\text{N-m-s}}{\text{rad}}$ $\quad k_t = \frac{JG}{l}$

$J = \frac{\pi d^4}{32} = \frac{\pi (.04)^4}{32} = 2.51\times10^{-7}$ $\quad G = \frac{E}{2(1+\nu)} = 79.3 \times 10^9$ N/m^2

$\frac{JG}{l} = k_t = 19.903 \times 10^3 \frac{\text{N-m}}{\text{rad}}$

$\omega_n = \sqrt{\frac{k_t}{I_0}} = 44.64$ rad/s $\quad \Theta_0 = \frac{T_0}{k_t} = .0502$ rad

$\zeta = \frac{c_t}{2 I_0 \omega_n} = .336$

$\Theta_{ss} = \frac{2^\circ}{(180/\pi)} = \frac{\Theta_0}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = .03491$

$(1-r^2)^2 + (2\zeta r)^2 = \left(\frac{.0502}{.03491}\right)^2 = 2.0678$

$1 - 2r^2 + r^4 + .4516 r^2 = 2.0679$

$-1.0679 - 1.5484 r^2 + r^4 = 0$

$r^2 = 2.0655, \; -.5171$ X $\qquad r = \sqrt{2.0655} = 1.4372$

$\omega_f = r\omega_n = 64.16$ rad/s

$T_{max,trans} = k_t \Theta_{ss} \sqrt{1 + (2\zeta r)^2} = (19.903\times10^3)(.03491)\sqrt{1+4\zeta^2 r^2}$

$= 967.2$ N-m

machine … a

3.29 D… i… N… … shown … = 5000 … rpm.

3.30 A… o… th… D… … has been … compressor, … 0 rpm.

3.31 Fi… da… 1 … following … J_0 =

3.32 A … Fig… sta… the … ted in … steady- … dle of

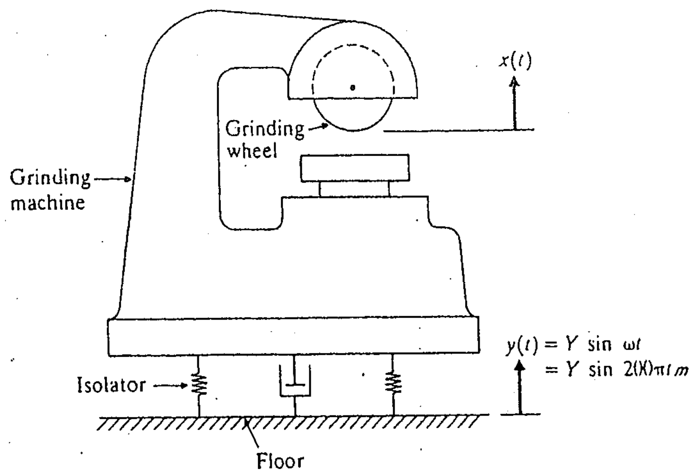


FIGURE 3.46

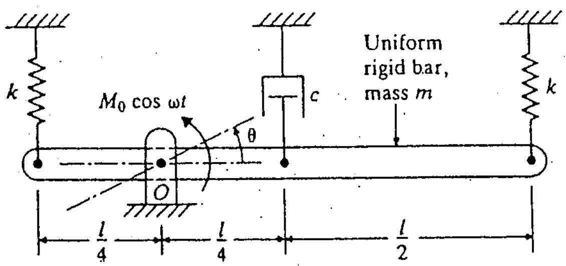


FIGURE 3.47

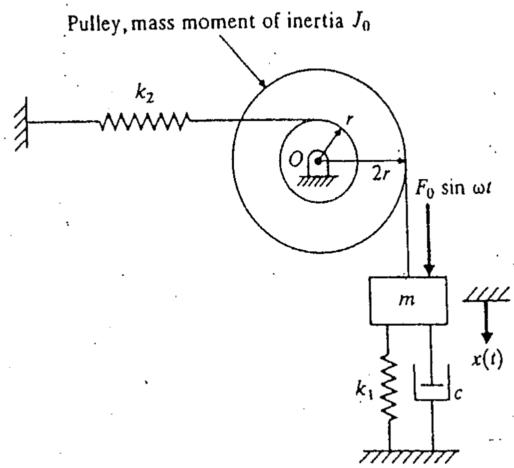


FIGURE 3.48

: machine

:m shown
: = 5000
rpm.
has been
npressor,
300 rpm.

following
:m, J_0 =

licated in
d steady-
middle of

3.33 A single story building frame is subjected to a harmonic ground acceleration as shown in Fig. 3.50. Find the steady-state motion of the floor (mass m).

3.34 Find the horizontal displacement of the floor (mass m) of the building frame shown in Fig. 3.50 when the ground acceleration is given by $\ddot{x}_g = 100 \sin \omega t$ mm/sec^2. Assume $m = 2000$ kg, $k = 0.1$ MN/m, $\omega = 25$ rad/sec, and $x_g(t = 0) = \dot{x}_g(t = 0) = x(t = 0) = \dot{x}(t = 0) = 0$.

Jean Baptiste Joseph Fourier (1768–1830) was a French mathematician and a professor at the Ecole Polytechnique in Paris. His works on heat flow, published in 1822, and on trigonometric series are well known. The expansion of a periodic function in terms of harmonic functions has been named after him as the "Fourier series." (Reprinted with permission from *Applied Mechanics Reviews*).

CHAPTER 4

Vibration Under General Forcing Conditions

4.1 Introduction

The response of a single degree of freedom system under general, nonharmonic, forcing functions is considered in this chapter. A general forcing function may be periodic (nonharmonic) or nonperiodic. A nonperiodic forcing function may be acting for a short, long, or infinite duration. If the duration of the forcing function or excitation is small compared to the natural time period of the system, the forcing function or excitation is called a shock. The motion imparted by a cam to the follower, the vibration felt by an instrument when its package is dropped from a height, the force applied to the foundation of a forging press, the motion of an automobile when it hits a pothole, and the ground vibration of a building frame during an earthquake are examples of general forcing functions.

If the forcing function is periodic but not harmonic, it can be replaced by a sum of harmonic functions using the harmonic analysis procedure discussed in Section 1.11. Using the principle of superposition, the response of the system can then be determined by superposing the responses due to the individual harmonic forcing functions. On the other hand, if the system is subjected to a suddenly applied nonperiodic force, the response will involve transient vibration. The transient response of a system can be found by using what is known as the *convolution integral*.

4.2 Response Under a General Periodic Force

When the external force $F(t)$ is periodic with period $\tau = 2\pi/\omega$, it can be expanded in a Fourier series (see Section 1.11):

$$F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t \tag{4.1}$$

where

$$a_j = \frac{2}{\tau}\int_0^{\tau} F(t) \cos j\omega t \, dt, \qquad j = 0, 1, 2, \ldots \tag{4.2}$$

and

$$b_j = \frac{2}{\tau}\int_0^{\tau} F(t) \sin j\omega t \, dt, \qquad j = 1, 2, \ldots \tag{4.3}$$

The equation of motion of the system can be expressed as

$$m\ddot{x} + c\dot{x} + kx = F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t \tag{4.4}$$

The right-hand side of this equation is a constant plus a sum of harmonic functions. Using the principle of superposition, the steady-state solution of Eq. (4.4) is the sum of the steady-state solutions of the following equations:

$$m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} \tag{4.5}$$

$$m\ddot{x} + c\dot{x} + kx = a_j \cos j\omega t \tag{4.6}$$

$$m\ddot{x} + c\dot{x} + kx = b_j \sin j\omega t \tag{4.7}$$

Noting that the solution of Eq. (4.5) is given by

$$x_p(t) = \frac{a_0}{2k} \tag{4.8}$$

and using the results of Section 3.4, we can express the solutions of Eqs. (4.6) and (4.7), respectively, as

$$x_p(t) = \frac{(a_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta j r)^2}} \cos(j\omega t - \phi_j) \tag{4.9}$$

$$x_p(t) = \frac{(b_j/k)}{\sqrt{(1 - j^2r^2)^2 + (2\zeta j r)^2}} \sin(j\omega t - \phi_j) \tag{4.10}$$

where

$$\phi_j = \tan^{-1}\left(\frac{2\zeta j r}{1 - j^2 r^2}\right) \tag{4.11}$$

and

$$r = \frac{\omega}{\omega_n} \tag{4.12}$$

Thus the complete steady-state solution of Eq. (4.4) is given by

$$x_p(t) = \frac{a_0}{2k} + \sum_{j=1}^{\infty} \frac{(a_j/k)}{\sqrt{(1 - j^2 r^2)^2 + (2\zeta j r)^2}} \cos(j\omega t - \phi_j) + \sum_{j=1}^{\infty} \frac{(b_j/k)}{\sqrt{(1 - j^2 r^2)^2 + (2\zeta j r)^2}} \sin(j\omega t - \phi_j) \tag{4.13}$$

It can be seen from the solution, Eq. (4.13), that the amplitude and phase shift corresponding to the jth term depend on j. If $j\omega = \omega_n$, for any j, the amplitude of the corresponding harmonic will be comparatively large. This will be particularly true for small values of j and ζ. Further, as j becomes larger, the amplitude becomes smaller and the corresponding terms tend to zero. Thus the first few terms are usually sufficient to obtain the response with reasonable accuracy.

The solution given by Eq. (4.13) denotes the steady-state response of the system. The transient part of the solution arising from the initial conditions can also be included to find the complete solution. To find the complete solution, we need to evaluate the arbitrary constants by setting the value of the complete solution and its derivative to the specified values of initial displacement $x(0)$ and the initial velocity $\dot{x}(0)$. This results in a complicated expression for the transient part of the total solution.

EXAMPLE 4.1 Periodic Vibration of a Hydraulic Valve

In the study of vibrations of valves used in hydraulic control systems, the valve and its elastic stem are modeled as a damped spring-mass system as shown in Fig. 4.1(a). In addition to the spring force and damping force, there is a fluid pressure force on the valve that changes with the amount of opening or closing of the valve. Find the steady-state response of the valve when the pressure in the chamber varies as indicated in Fig. 4.1(b). Assume $k = 2500$ N/m, $c = 10$ N-s/m, and $m = 0.25$ kg.

Given: Hydraulic control valve with $m = 0.25$ kg, $k = 2500$ N/m, and $c = 10$ N-s/m and pressure on the valve as given in Fig. 4.1(b).

The natural frequency of the valve is given by

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{0.25}} = 100 \text{ rad/sec} \tag{E.14}$$

and the forcing frequency ω by

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{2} = \pi \text{ rad/sec} \tag{E.15}$$

Thus the frequency ratio can be obtained:

$$r = \frac{\omega}{\omega_n} = \frac{\pi}{100} = 0.031416 \tag{E.16}$$

and the damping ratio:

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n} = \frac{10.0}{2(0.25)(100)} = 0.2 \tag{E.17}$$

The phase angles ϕ_1 and ϕ_3 can be computed as follows:

$$\begin{aligned}\phi_1 &= \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)\\ &= \tan^{-1}\left(\frac{2 \times 0.2 \times 0.031416}{1 - 0.031416^2}\right) = 0.0125664 \text{ rad}\end{aligned} \tag{E.18}$$

and

$$\begin{aligned}\phi_3 &= \tan^{-1}\left(\frac{6\zeta r}{1 - 9r^2}\right)\\ &= \tan^{-1}\left(\frac{6 \times 0.2 \times 0.031416}{1 - 9(0.031416)^2}\right) = 0.0380483 \text{ rad}\end{aligned} \tag{E.19}$$

In view of Eqs. (E.2) and (E.14) to (E.19), the solution can be written as

$$\begin{aligned}x_p(t) = 0.019635 &- 0.015930 \cos(\pi t - 0.0125664)\\ &- 0.0017828 \cos(3\pi t - 0.0380483) \text{ m}\end{aligned} \tag{E.20}$$

■

4.3 Response Under a Periodic Force of Irregular Form

In some cases, the force acting on a system may be quite irregular and may be determined only experimentally. Examples of such forces include wind-and earthquake-induced forces. In such cases, the forces will be available in graphical form and no analytical expression can be found to describe $F(t)$. Sometimes, the value of $F(t)$ may be available only at a number of discrete points $t_1, t_2, \ldots, t_N$. In all these cases, it is possible to find the Fourier coefficients by using a numerical

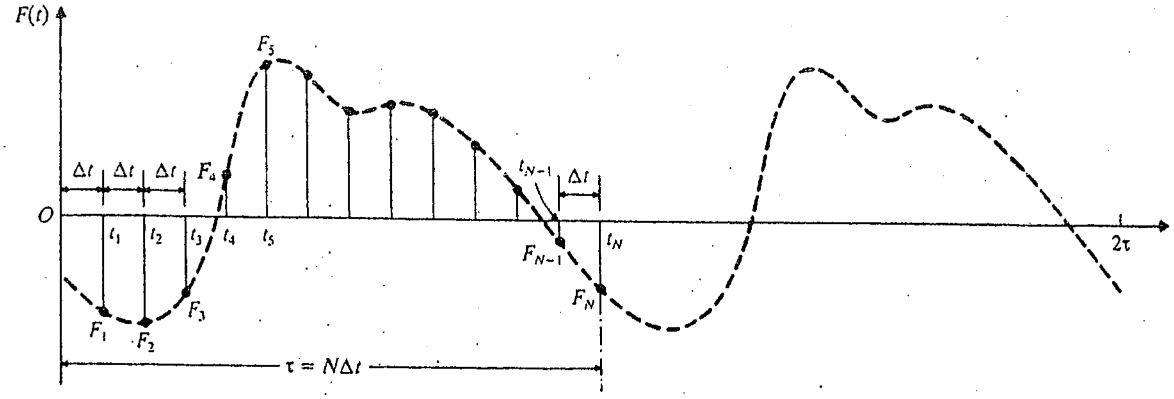


FIGURE 4.2

integration procedure, as described in Section 1.11. If $F_1, F_2, \ldots, F_N$ denote the values of $F(t)$ at $t_1, t_2, \ldots, t_N$, respectively, where N denotes an even number of equidistant points in one time period $\tau(\tau = N\Delta t)$, as shown in Fig. 4.2, the application of trapezoidal rule [4.1] gives

$$a_0 = \frac{2}{N}\sum_{i=1}^{N} F_i \tag{4.14}$$

$$a_j = \frac{2}{N}\sum_{i=1}^{N} F_i \cos\frac{2j\pi t_i}{\tau}, \qquad j = 1, 2, \ldots \tag{4.15}$$

$$b_j = \frac{2}{N}\sum_{i=1}^{N} F_i \sin\frac{2j\pi t_i}{\tau}, \qquad j = 1, 2, \ldots \tag{4.16}$$

Once the Fourier coefficients a_0, a_j, and b_j are known, the steady-state response of the system can be found using Eq. (4.13) with

$$r = \left(\frac{2\pi}{\tau\omega_n}\right)$$

EXAMPLE 4.2 Steady-State Vibration of a Hydraulic Valve

Find the steady-state response of the valve in Example 4.1 if the pressure fluctuations in the chamber are found to be periodic. The values of pressure measured at 0.01 second intervals in one cycle are given below.

Time, t_i (seconds)	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10	0.11	0.12
$p_i = p(t_i)$ (kN/m^2)	0	20	34	42	49	53	70	60	36	22	16	7	0

Given: Arbitrary pressure fluctuations on the valve, shown in Fig. 4.1(a).

Find: Steady-state response of the valve.

Approach: Find Fourier series expansion of the pressure acting on the valve using numerical procedure. Add the responses due to individual harmonic force components.

Solution: The Fourier analysis of the pressure fluctuations (see Example 1.13) gives the result

$$p(t) = 34083.3 - 26996.0 \cos 52.36t + 8307.7 \sin 52.36t + 1416.7 \cos 104.72t + 3608.3 \sin 104.72t - 5833.3 \cos 157.08t + 2333.3 \sin 157.08t + \ldots \text{ N/m}^2 \quad \text{(E.1)}$$

Other quantities needed for the computation are

$$\omega = \frac{2\pi}{\tau} = \frac{2\pi}{0.12} = 52.36 \text{ rad/sec}$$

$$\omega_n = 100 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = 0.5236$$

$$\zeta = 0.2$$

$$A = 0.000625\pi \text{ m}^2$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right) = \tan^{-1}\left(\frac{2 \times 0.2 \times 0.5236}{1 - 0.5236^2}\right) = 16.1°$$

$$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1 - 4r^2}\right) = \tan^{-1}\left(\frac{4 \times 0.2 \times 0.5236}{1 - 4 \times 0.5236^2}\right) = -77.01°$$

$$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1 - 9r^2}\right) = \tan^{-1}\left(\frac{6 \times 0.2 \times 0.5236}{1 - 9 \times 0.5236^2}\right) = -23.18°$$

The steady-state response of the valve can be expressed, using Eq. (4.13), as

$$x_p(t) = \frac{34083.3A}{k} - \frac{(26996.0A/k)}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}\cos(52.36t - \phi_1) + \frac{(8309.7A/k)}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}\sin(52.36t - \phi_1)$$

$$+ \frac{(1416.7A/k)}{\sqrt{(1 - 4r^2)^2 + (4\zeta r)^2}} \cos(104.72t - \phi_2)$$

$$+ \frac{(3608.3A/k)}{\sqrt{(1 - 4r^2)^2 + (4\zeta r)^2}} \sin(104.72t - \phi_2)$$

$$- \frac{(5833.3A/k)}{\sqrt{(1 - 9r^2)^2 + (6\zeta r)^2}} \cos(157.08t - \phi_3)$$

$$+ \frac{(2333.3A/k)}{\sqrt{(1 - 9r^2)^2 + (6\zeta r)^2}} \sin(157.08t - \phi_3)$$

■

4.4 Response Under a Nonperiodic Force

We have seen that periodic forces of any general wave form can be represented by Fourier series as a superposition of harmonic components of various frequencies. The response of a linear system is then found by superposing the harmonic response to each of the exciting forces. When the exciting force $F(t)$ is nonperiodic, such as that due to the blast from an explosion, a different method of calculating the response is required. Various methods can be used to find the response of the system to an arbitrary excitation. Some of these methods are as follows:

1. Representing the excitation by a Fourier integral
2. Using the method of convolution integral
3. Using the method of Laplace transformation
4. First approximating $F(t)$ by a suitable interpolation model and then using a numerical procedure
5. Numerically integrating the equations of motion

We shall discuss Methods 2, 3, and 4 in the following sections and Method 5 in Chapter 11.

4.5 Convolution Integral

A nonperiodic exciting force usually has a magnitude that varies with time; it acts for a specified period of time and then stops. The simplest form of such a force is the impulsive force. An impulsive force is one that has a large magnitude F and acts for a very short period of time Δt. From dynamics we know that impulse can be measured by finding the change in momentum of the system caused by it [4.2].

If $\dot{x}_1$ and $\dot{x}_2$ denote the velocities of the mass m before and after the application of the impulse, we have

$$\text{Impulse} = F\Delta t = m\dot{x}_2 - m\dot{x}_1 \tag{4.17}$$

By designating the magnitude of the impulse $F\Delta t$ by $\underset{\sim}{F}$, we can write, in general,

$$\underset{\sim}{F} = \int_t^{t+\Delta t} F\, dt \tag{4.18}$$

A unit impulse ($\underset{\sim}{f}$) is defined as

$$\underset{\sim}{f} = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} F\, dt = F\, dt = 1 \tag{4.19}$$

It can be seen that in order for $F\, dt$ to have a finite value, F tends to infinity (since dt tends to zero). Although the unit impulse function has no physical meaning, it is a convenient tool in our present analysis.

4.5.1 Response to an Impulse

We first consider the response of a single degree of freedom system to an impulse excitation; this case is important in studying the response under more general excitations. Consider a viscously damped spring-mass system subjected to a unit impulse at $t = 0$, as shown in Figs. 4.3(a) and (b). For an underdamped system, the solution of the equation of motion

$$m\ddot{x} + c\dot{x} + kx = 0 \tag{4.20}$$

is given by Eq. (2.72) as follows:

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\} \tag{4.21}$$

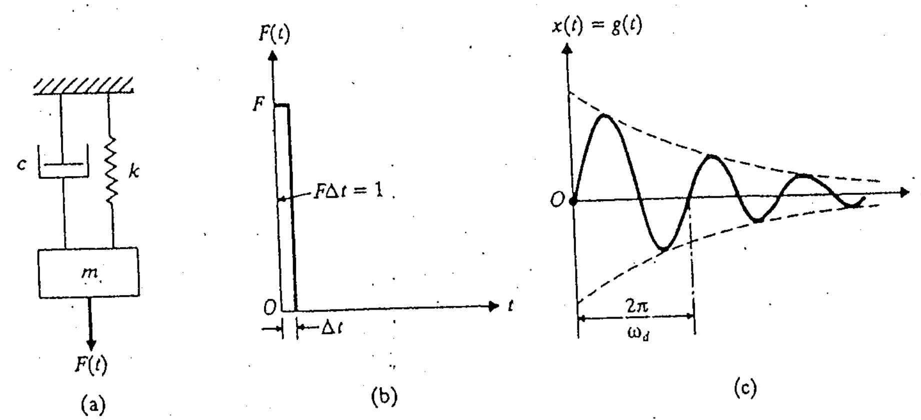


FIGURE 4.3

where

$$\zeta = \frac{c}{2m\omega_n} \tag{4.22}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} \tag{4.23}$$

$$\omega_n = \sqrt{\frac{k}{m}} \tag{4.24}$$

If the mass is at rest before the unit impulse is applied ($x = \dot{x} = 0$ for $t < 0$ or at $t = 0^-$), we obtain, from the impulse-momentum relation,

$$\text{Impulse} = \underset{\sim}{f} = 1 = m\dot{x}\,(t=0) - m\dot{x}\,(t=0^-) = m\dot{x}_0 \tag{4.25}$$

Thus the initial conditions are given by

$$x(t=0) = x_0 = 0$$
$$\dot{x}(t=0) = \dot{x}_0 = \frac{1}{m} \tag{4.26}$$

In view of Eq. (4.26), Eq. (4.21) reduces to

$$x(t) = g(t) = \frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t \tag{4.27}$$

Equation (4.27) gives the response of a single degree of freedom system to a unit impulse, which is also known as the *impulse response function*, denoted by $g(t)$. The function $g(t)$, Eq. (4.27), is shown in Fig. 4.3(c).

If the magnitude of the impulse is $\underset{\sim}{F}$ instead of unity, the initial velocity $\dot{x}_0$ is $\underset{\sim}{F}/m$ and the response of the system becomes

$$x(t) = \frac{\underset{\sim}{F}e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t = \underset{\sim}{F}g(t) \tag{4.28}$$

If the impulse $\underset{\sim}{F}$ is applied at an arbitrary time $t = \tau$, as shown in Fig. 4.4(a), it will change the velocity at $t = \tau$ by an amount $\underset{\sim}{F}/m$. Assuming that $x = 0$ until the impulse is applied, the displacement x at any subsequent time t, caused by a change in the velocity at time τ, is given by Eq. (4.28) with t replaced by the time elapsed after the application of the impulse, that is, $t - \tau$. Thus we obtain

$$x(t) = \underset{\sim}{F}g(t-\tau) \tag{4.29}$$

This is shown in Fig. 4.4(b).

4.5.2 Response to General Forcing Condition

Now we consider the response of the system under an arbitrary external force $F(t)$, shown in Fig. 4.5. This force may be assumed to be made up of a series of impulses of varying magnitude. Assuming that at time τ, the force $F(\tau)$ acts on the system for a short period of time $\Delta\tau$, the impulse acting at $t = \tau$ is given by $F(\tau)\,\Delta\tau$. At

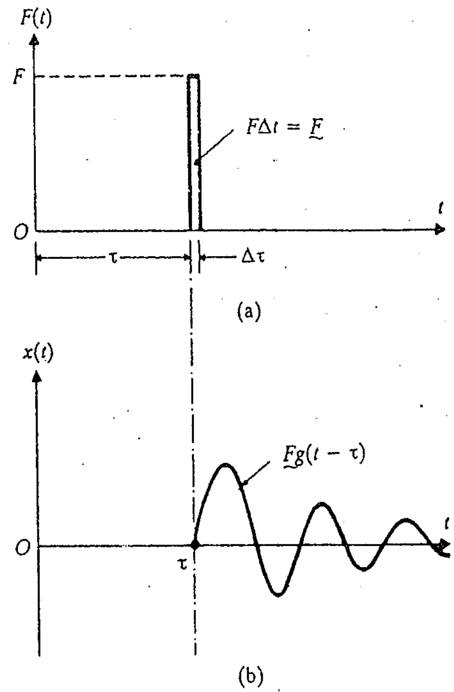


FIGURE 4.4

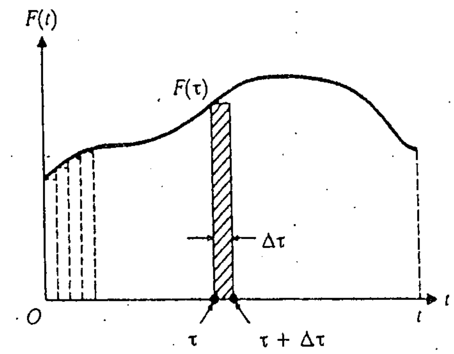


FIGURE 4.5 An arbitrary (nonperiodic) forcing function.

any time t, the elapsed time since the impulse is $t - \tau$, so the response of the system at t due to this impulse alone is given by Eq. (4.29) with $\underset{\sim}{F} = F(\tau)\,\Delta\tau$:

$$\Delta x(t) = F(\tau)\,\Delta\tau g\,(t - \tau) \tag{4.30}$$

The total response at time t can be found by summing all the responses due to the elementary impulses acting at all times τ:

$$x(t) \simeq \Sigma F(\tau) g(t - \tau)\,\Delta\tau \tag{4.31}$$

Letting $\Delta\tau \to 0$ and replacing the summation by integration, we obtain

$$x(t) = \int_0^t F(\tau) g(t - \tau)\,d\tau \tag{4.32}$$

By substituting Eq. (4.27) into Eq. (4.32), we obtain

$$x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau) e^{-\zeta\omega_n(t-\tau)} \sin \omega_d\,(t - \tau)\,d\tau \tag{4.33}$$

which represents the response of an underdamped single degree of freedom system to the arbitrary excitation $F(t)$. Note that Eq. (4.33) does not consider the effect of initial conditions of the system. The integral in Eq. (4.32) or Eq. (4.33) is called the *convolution* or *Duhamel integral.* In many cases the function $F(t)$ has a form that permits an explicit integration of Eq. (4.33). In case such integration is not

4.3 What is the Duhamel integral? What is its use?

4.4 How are the initial conditions determined for a single degree of freedom system subjected to an impulse at $t = 0$?

4.5 Derive the equation of motion of a system subjected to base excitation.

4.6 What is a response spectrum?

4.7 What are the advantages of the Laplace transformation method?

4.8 What is the use of the pseudo spectrum?

4.9 How is the Laplace transform of a function $x(t)$ defined?

4.10 Define these terms: generalized impedance and admittance of a system.

4.11 State the interpolation models that can be used for approximating an arbitrary forcing function.

4.12 How many resonant conditions are there when the external force is not harmonic?

4.13 How do you compute the frequency of the first harmonic of a periodic force?

4.14 What is the relation between the frequencies of higher harmonics and the frequency of the first harmonic for a periodic excitation?

Problems

The problem assignments are organized as follows:

Problems	Section Covered	Topic Covered
4.1–4.10	4.2	Response under general periodic force
4.11–4.13	4.3	Periodic force of irregular form
4.14–4.34	4.5	Convolution integral
4.35–4.44	4.6	Response spectrum
4.45–4.47	4.7	Laplace transformation
4.48–4.51	4.8	Irregular forcing conditions using numerical methods
4.52–4.57	4.9	Computer program
4.58–4.60	—	Projects

4.1–4.4 Find the steady-state response of the hydraulic control valve shown in Fig. 4.1(a) to the forcing functions obtained by replacing $x(t)$ with $F(t)$ and A with F_0 in Figs. 1.87–1.90.

4.5 Find the steady-state response of a viscously damped system to the forcing function obtained by replacing $x(t)$ and A with $F(t)$ and F_0, respectively, in Fig. 1.46(a).

4.6 The torsional vibrations of a driven gear mounted on a shaft (see Fig. 4.29) under steady conditions are governed by the equation:

$$J_0 \ddot{\theta} + k_t \theta = M_t$$

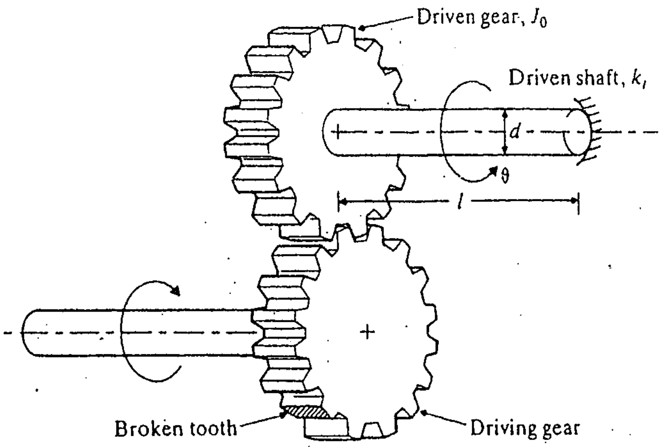


FIGURE 4.29

where k_t is the torsional stiffness of the driven shaft, M_t is the torque transmitted, J_0 is the mass moment of inertia, and θ is the angular deflection of the driven gear. If one of the 16 teeth on the driving gear breaks, determine the resulting torsional vibration of the driven gear for the following data.

Driven gear: $J_0 = 0.1$ N-m-s^2, speed $= 1000$ rpm, driven shaft: material - steel, solid circular section with diameter 5 cm and length 1 m, $M_{t0} = 1000$ N-m.

4.7 A slider crank mechanism is used to impart motion to the base of a spring-mass-damper system, as shown in Fig. 4.30. Approximating the base motion $y(t)$ as a series of harmonic functions, find the response of the mass for $m = 1$ kg, $c = 10$ N-s/m, $k = 100$ N/m, $r = 10$ cm, $l = 1$ m, and $\omega = 100$ rad/s.

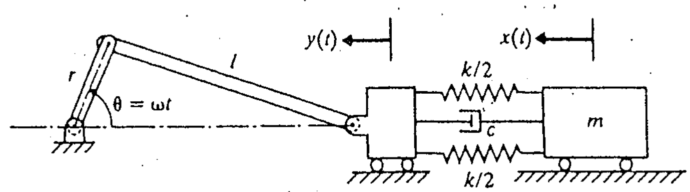


FIGURE 4.30

4.8 The base of a spring-mass-damper system is subjected to the periodic displacement shown in Fig. 4.31. Determine the response of the mass using the principle of superposition.

4.9 The base of a spring-mass system, with Coulomb damping, is connected to the slider crank mechanism shown in Fig. 4.32. Determine the response of the system for a coefficient of friction μ between the mass and the surface by approximating the motion $y(t)$ as a series of harmonic functions for $m = 1$ kg, $k = 100$ N/m, $r = 10$ cm, $l = 1$ m, $\mu = 0.1$, and $\omega = 100$ rad/s. Discuss the limitations of your solution.

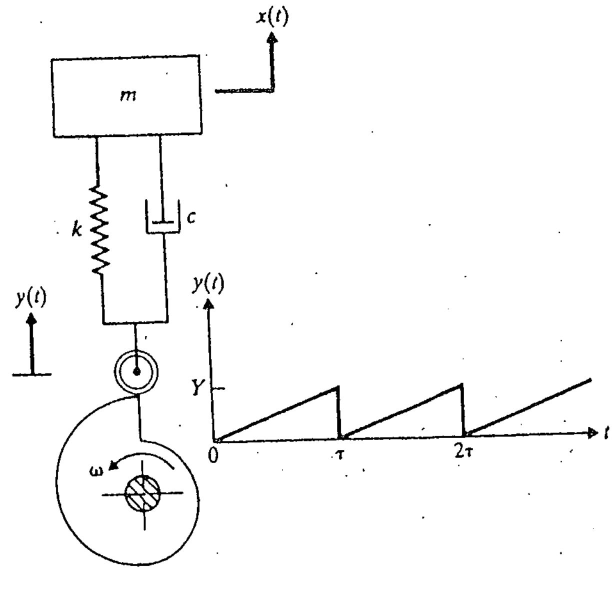


FIGURE 4.31

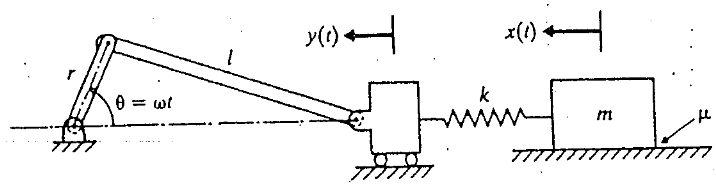


FIGURE 4.32

4.10 A roller cam is used to impart a periodic motion to the base of the spring-mass system shown in Fig. 4.33. If the coefficient of friction between the mass and the surface is μ, find the response of the system using the principle of superposition. Discuss the validity of the result.

4.11 Find the response of a damped system with $m = 1$ kg, $k = 15$ kN/m. and $\zeta = 0.1$ under the action of a periodic forcing function, as shown in Fig. 1.92.

4.12 Find the response of a viscously damped system under the periodic force whose values are given in Problem 1.69. Assume that M_t denotes the value of the force in Newtons at time t_i seconds. Use $m = 0.5$ kg, $k = 8000$ N/m, and $\zeta = 0.06$.

4.13 Find the displacement of the water tank shown in Fig. 4.34(a) under the periodic force shown in Fig. 4.34(b) by treating it as an undamped single degree of freedom system. Use the numerical procedure described in Section 4.3.

4.14 Sandblasting is a process in which an abrasive material, entrained in a jet, is directed onto the surface of a casting to clean its surface. In a particular setup for sandblasting, the casting of mass m is placed on a flexible support of stiffness k as shown in Fig.

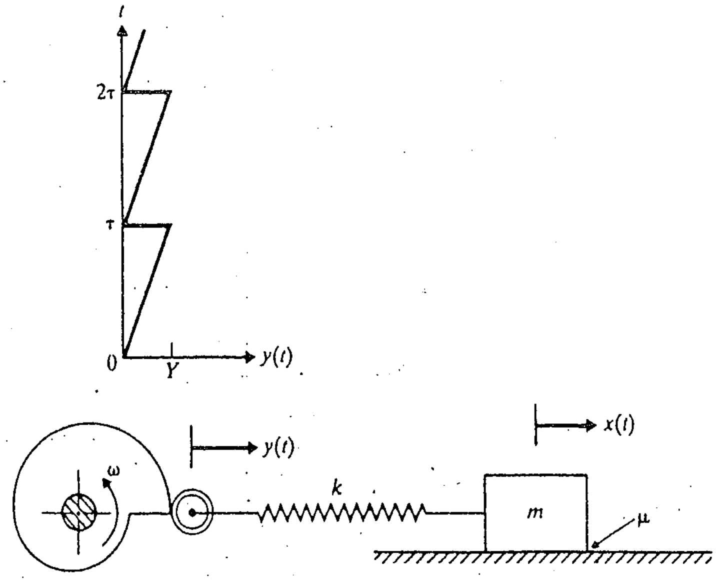


FIGURE 4.33

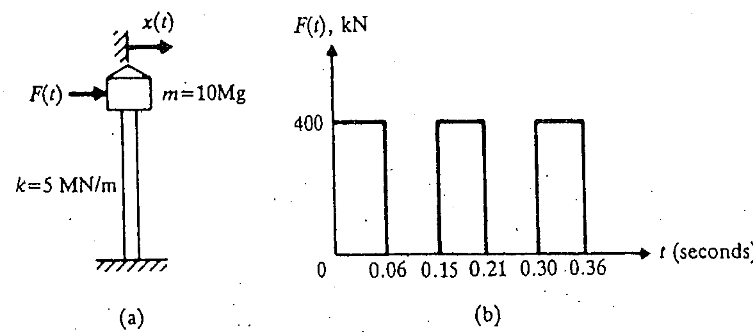


FIGURE 4.34

4.35(a). If the force exerted on the casting due to the sandblasting operation varies as shown in Fig. 4.35(b), find the response of the casting.

4.15 The frame, anvil, and the base of the forging hammer, shown in Fig. 4.36(a), have a total mass of m. The support elastic pad has a stiffness of k. If the force applied by the hammer is given by Fig. 4.36(b), find the response of the anvil.

4.16 Find the displacement of a damped single degree of freedom system under the forcing function $F(t) = F_0 e^{-\alpha t}$ where α is a constant.

4.17 A compressed air cylinder is connected to the spring-mass system shown in Fig. 4.37(a). Due to a small leak in the valve, the pressure on the piston, $p(t)$, builds up

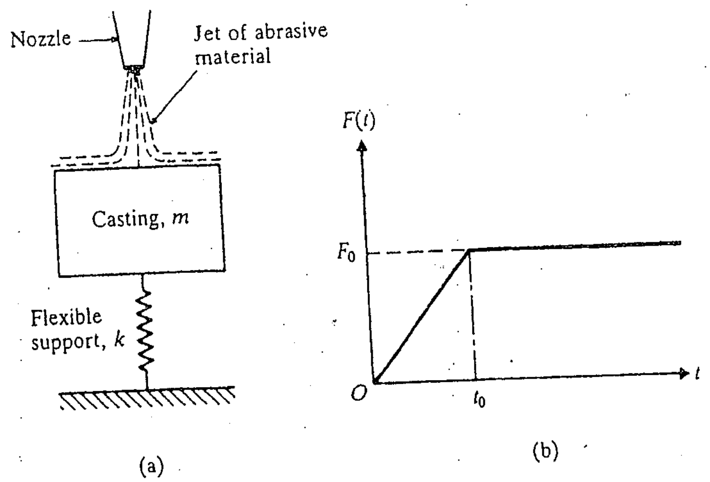


FIGURE 4.35

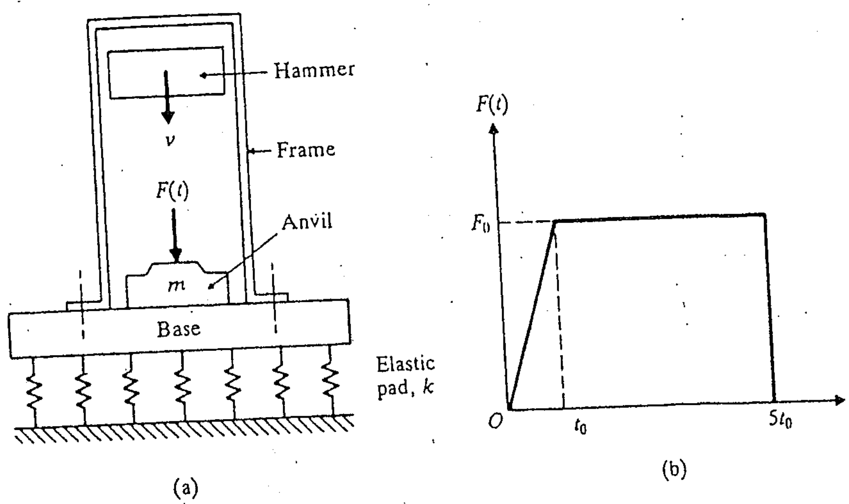


FIGURE 4.36

as indicated in Fig. 4.37(b). Find the response of the piston for the following data: $m = 10$ kg, $k = 1000$ N/m, and $d = 0.1$ m.

4.18 Find the transient response of an undamped spring-mass system for $t > \pi/\omega$ when the mass is subjected to a force

$$F(t) = \begin{cases} \dfrac{F_0}{2}(1 - \cos \omega t) & \text{for } 0 \leq t \leq \dfrac{\pi}{\omega} \\ F_0 & \text{for } t > \dfrac{\pi}{\omega} \end{cases}$$

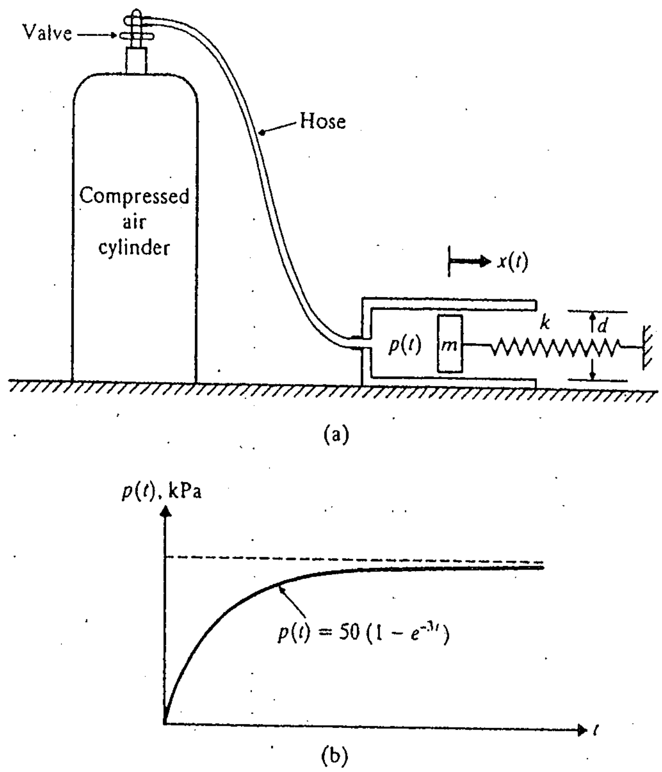


FIGURE 4.37

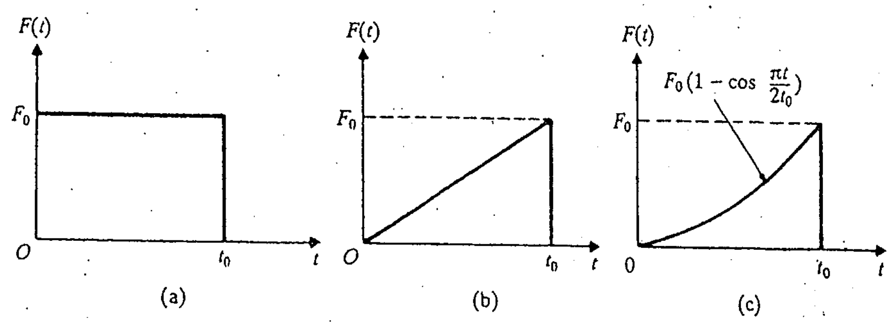


FIGURE 4.38

Assume that the displacement and velocity of the mass are zero at $t = 0$.

4.19–4.21 Use the Dahamel integral method to derive expressions for the response of an undamped system subjected to the forcing functions shown in Figs. 4.38(a) to (c).

4.22 Figure 4.39 shows a one degree of freedom model of a motor vehicle traveling in the horizontal direction. Find the relative displacement of the vehicle as it travels over a road bump of the form $y(s) = Y \sin \pi s/\delta$.

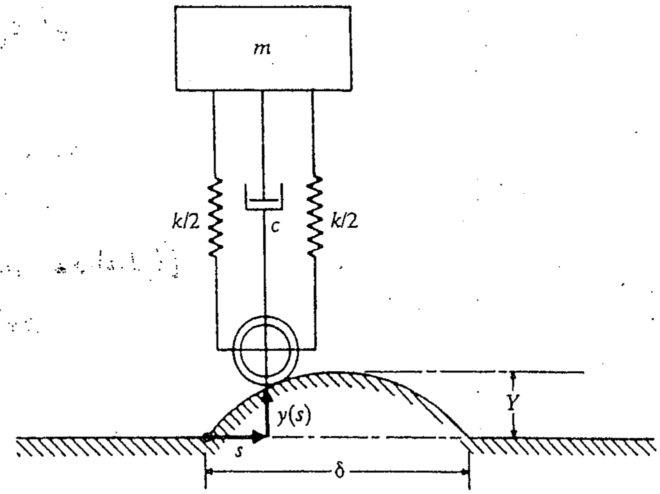


FIGURE 4.39

4.23 A vehicle traveling at a constant speed v in the horizontal direction encounters a triangular road bump, as shown in Fig. 4.40. Treating the vehicle as an undamped spring-mass system, determine the response of the vehicle in the vertical direction.

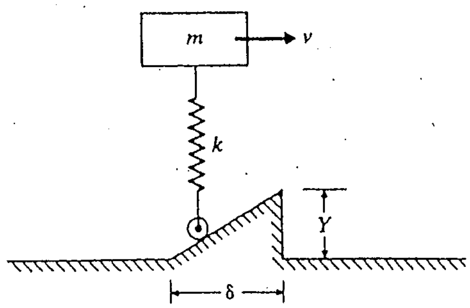


FIGURE 4.40

4.24 An automobile, having a mass of 1000 kg, runs over a road bump of the shape shown in Fig. 4.41. The speed of the automobile is 50 km/hr. If the undamped natural period of vibration in the vertical direction is 1.0 second, find the response of the car by assuming it as a single degree of freedom undamped system vibrating in the vertical direction.

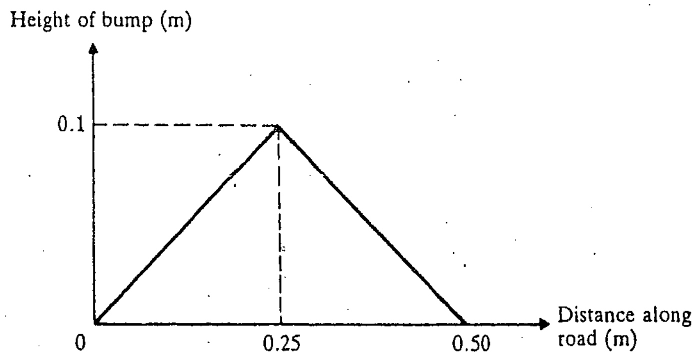


FIGURE 4.41

4.25 A camcorder of mass m is packed in a container using a flexible packing material. The stiffness and damping constant of the packing material are given by k and c, respectively, and the mass of the container is negligible. If the container is dropped accidentally from a height of h onto a rigid floor (see Fig. 4.42), find the motion of the camcorder.

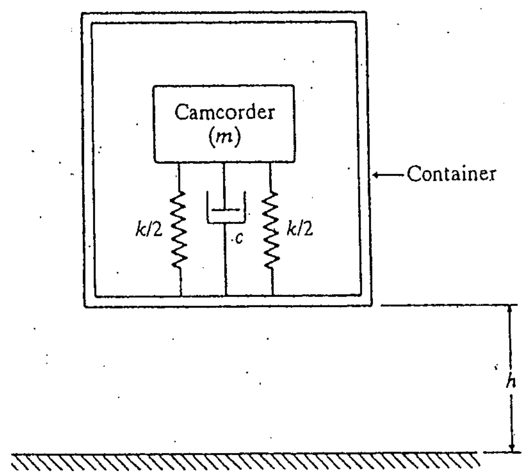


FIGURE 4.42

4.27 in new pg 369

4.26 An airplane, taxiing on a runway, encounters a bump. As a result, the root of the wing is subjected to a displacement that can be expressed as

$$y(t) = \begin{cases} Y(t^2/t_0^2), & 0 \leq t \leq t_0 \\ 0, & t > t_0 \end{cases}$$

Find the response of the mass located at the tip of the wing if the stiffness of the wing is k (see Fig. 4.43).

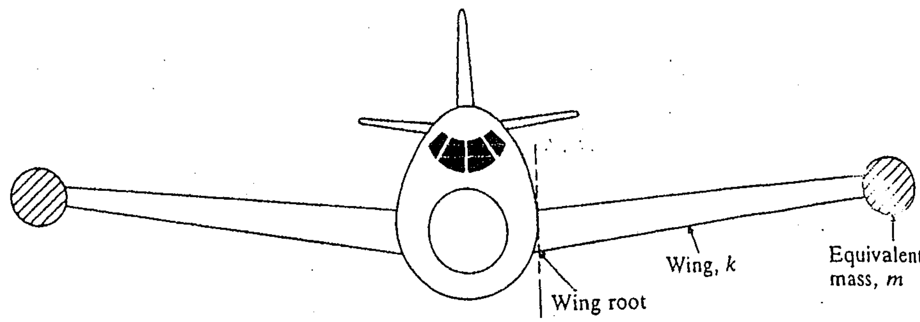


FIGURE 4.43

4.27 Derive Eq. (E.1) of Example 4.6.

4.28 In a static firing test of a rocket, the rocket is anchored to a rigid wall by a spring-damper system, as shown in Fig. 4.44(a). The thrust acting on the rocket reaches its maximum value F in a negligibly short time and remains constant until the burnout time t_0, as indicated in Fig. 4.44(b). The thrust acting on the rocket is given by $F = m_0 v$ where m_0 is the constant rate at which fuel is burnt and v is the velocity of the jet stream. The initial mass of the rocket is M, so that its mass at any time t is given by $m = M - m_0 t$, $0 \leq t \leq t_0$. If the data are $k = 7.5 \times 10^6$ N/m, $c = 0.1 \times 10^6$ N-s/m, $m_0 = 10$ kg/s, $v = 2000$ m/s, $M = 2000$ kg, and $t_0 = 100$ s, (1) derive the equation of motion of the rocket, and (2) find the maximum steady-state displacement of the rocket by assuming an average (constant) mass of $(M - \frac{1}{2} m_0 t_0)$.

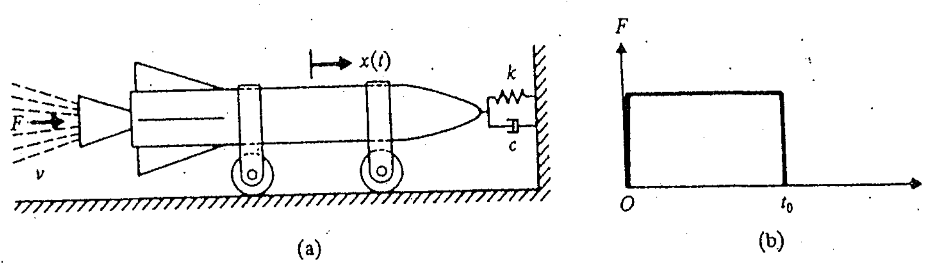


FIGURE 4.44

4.29 Show that the response to a unit step function $h(t)$ ($F_0 = 1$ in Fig. 4.6b) is related to the impulse response function $g(t)$, Eq. (4.27), as follows:

$$g(t) = \frac{dh(t)}{dt}$$

4.30 Show that the convolution integral, Eq. (4.33), can also be expressed in terms of the response to a unit step function $h(t)$ as

$$x(t) = F(0)\, h(t) + \int_0^t \frac{dF(\tau)}{d\tau} h(t - \tau)\, d\tau$$

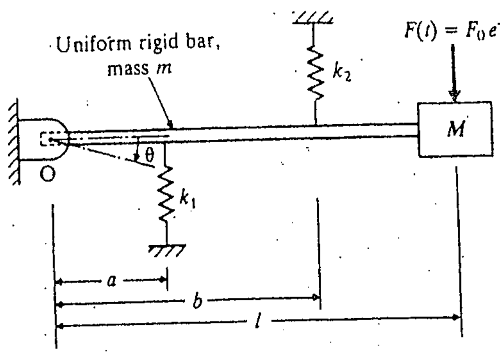


FIGURE 4.45

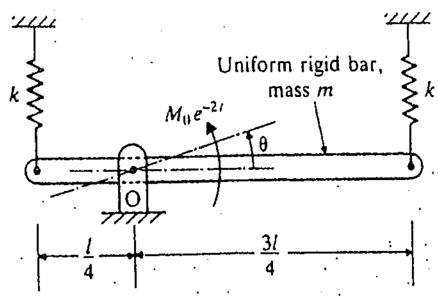


FIGURE 4.46

4.31 Find the response of the rigid bar shown in Fig. 4.45 using convolution integral for the following data: $k_1 = k_2 = 5000$ N/m, $a = 0.25$ m, $b = 0.5$ m, $l = 1.0$ m, $M = 50$ kg, $m = 10$ kg, $F_0 = 500$ N.

4.32 Find the response of the rigid bar shown in Fig. 4.46 using convolution integral for the following data: $k = 5000$ N/m, $l = 1$ m, $m = 10$ kg, $M_0 = 100$ N-m.

4.33 Find the response of the rigid bar shown in Fig. 4.47 using convolution integral when the end P of the spring PQ is subjected to the displacement, $x(t) = x_0\, e^{-t}$. Data: $k = 5000$ N/m, $l = 1$ m, $m = 10$ kg, $x_0 = 1$ cm.

4.34 Find the response of the mass shown in Fig. 4.48 under the force $F(T) = F_0\, e^{-t}$ using convolution integral. Data: $k_1 = 1000$ N/m, $k_2 = 500$ N/m, $r = 5$ cm, $m = 10$ kg, $J_0 = 1$ kg-m^2, $F_0 = 50$ N.

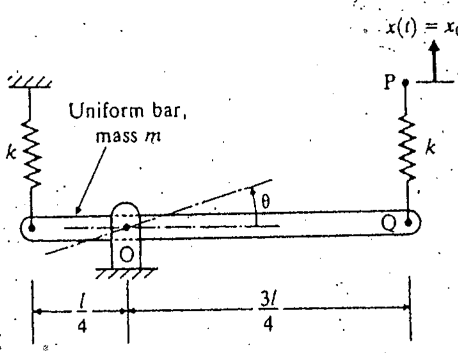


FIGURE 4.47

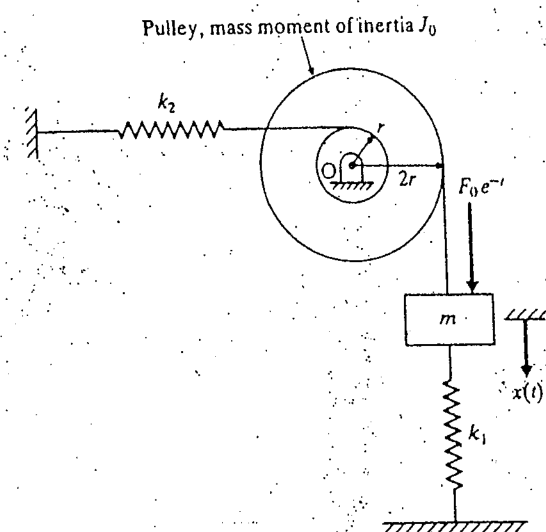


FIGURE 4.48

The damping ratios obtainable with different types of construction/arrangement are indicated below:

Type of Construction/Arrangement	Equivalent Viscous Damping Ratio (%)
Welded construction	1–4
Bolted construction	3–10
Steel frame	5–6
Unconstrained viscoelastic layer on steel-concrete girder	4–5
Constrained viscoelastic layer on steel-concrete girder	5–8

9.9 Vibration Isolation

Vibration isolation is a procedure by which the undesirable effects of vibration are reduced [9.21–9.24]. Basically, it involves the insertion of a resilient member (or isolator) between the vibrating mass (or equipment or payload) and the source of vibration so that a reduction in the dynamic response of the system is achieved under specified conditions of vibration excitation. An isolation system is said to be active or passive depending on whether or not external power is required for the isolator to perform its function. A passive isolator consists of a resilient member (stiffness) and an energy dissipator (damping). Examples of passive isolators include metal springs, cork, felt, pneumatic springs, and elastomer (rubber) springs. Figure 9.16 shows typical spring and pneumatic mounts that can be used as passive isolators, and Fig. 9.17 illustrates the use of passive isolators in the mounting of a high-speed punch press [9.25]. The optimal synthesis of vibration isolators is presented in Refs. [9.26–9.30].

An active isolator is comprised of a servomechanism with a sensor, signal processor, and an actuator. The effectiveness of an isolator is stated in terms of its transmissibility. The transmissibility (T_r) is defined as the ratio of the amplitude of the force transmitted to that of the exciting force.

Vibration isolation can be used in two types of situations. In the first type, the foundation or base of a vibrating machine is protected against large unbalanced forces (as in the case of reciprocating and rotating machines) or impulsive forces (as in the case of forging and stamping presses). In these cases, if the system is modeled as a single degree of freedom system as shown in Fig. 9.18(a), the force is transmitted to the foundation through the spring and the damper. The force transmitted to the base (F_t) is given by

$$F_t(t) = kx(t) + c\dot{x}(t) \tag{9.79}$$

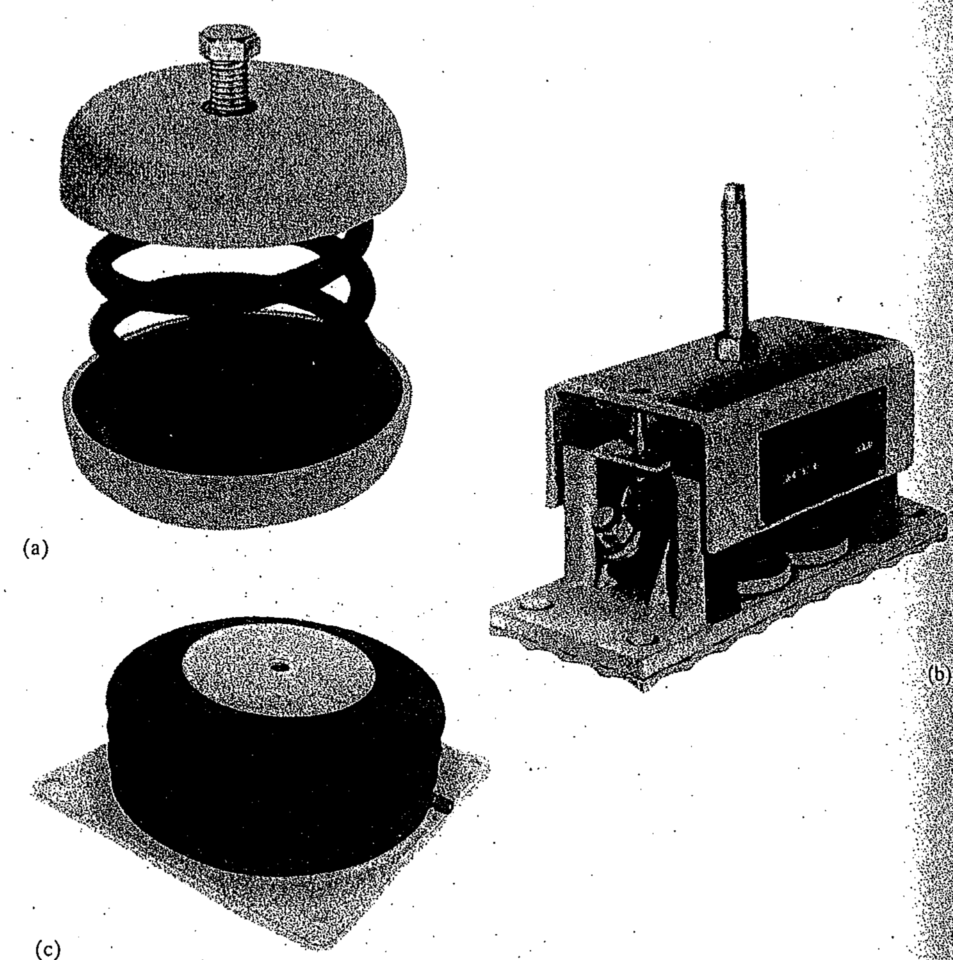


FIGURE 9.16 (a) Undamped spring mount; (b) damped spring mount; (c) pneumatic rubber mount. *(Courtesy of* Sound and Vibration.*)*

If the force transmitted to the base $F_t(t)$ varies harmonically, as in the case of unbalanced reciprocating and rotating machines, the resulting stresses in the foundation bolts also vary harmonically, which might lead to fatigue failure. Even if the force transmitted is not harmonic, its magnitude is to be limited to safe permissible values.

In the second type, the system is protected against the motion of its foundation or base (as in the case of protection of a delicate instrument or equipment from the motion of its container). If the delicate instrument is modeled as a single degree of freedom system, as shown in Fig. 9.18(b), the force transmitted to the instrument (mass m in Fig. 9.18b) is given by

$$F_t(t) = m\ddot{x}(t) \equiv k[x(t) - y(t)] + c[\dot{x}(t) - \dot{y}(t)] \tag{9.80}$$

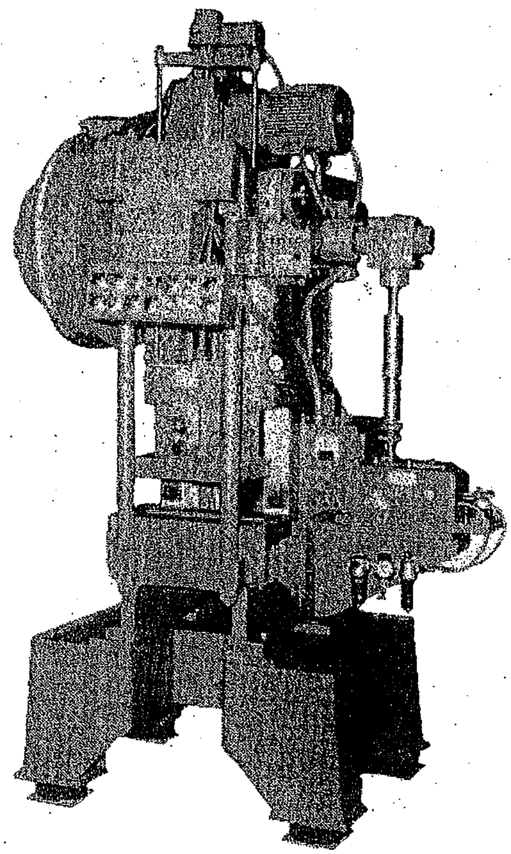


FIGURE 9.17 High-speed punch press mounted on pneumatic rubber mounts. (*Courtesy of* Sound and Vibration.)

where $(x - y)$ and $(\dot{x} - \dot{y})$ denote the relative displacement and relative velocity of the spring and the damper, respectively. In many practical problems, the package is to be designed properly to avoid transmission of large forces to the delicate instrument to avoid damage.

9.9.1 Vibration Isolation System with Rigid Foundation

Reduction of the Force Transmitted to Foundation. When a machine is bolted directly to a rigid foundation or floor, the foundation will be subjected to a harmonic load due to the unbalance in the machine in addition to the static load due to the weight of the machine. Hence an elastic or resilient member is placed between the machine and the rigid foundation to reduce the force transmitted to the foundation. The system can then be idealized as a single degree of freedom system, as shown

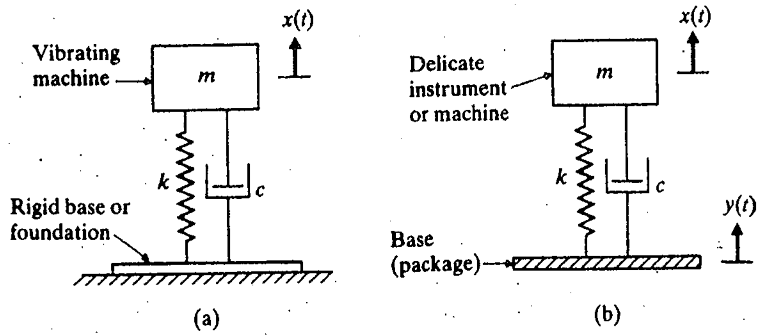


FIGURE 9.18

in Fig. 9.19(a). The resilient member is assumed to have both elasticity and damping and is modeled as a spring k and a dashpot c, as shown in Fig. 9.19(b). It is assumed that the operation of the machine gives rise to a harmonically varying force $F(t) = F_0\cos \omega t$. The equation of motion of the machine (of mass m) is given by

$$m\ddot{x} + c\dot{x} + kx = F_0\cos \omega t \tag{9.81}$$

Since the transient solution dies out after some time, only the steady-state solution will be left. The steady-state solution of Eq. (9.81) is given by (see Eq. 3.25)

$$x(t) = X \cos (\omega t - \phi) \tag{9.82}$$

where

$$X = \frac{F_0}{[(k - m\omega^2)^2 + \omega^2 c^2]^{1/2}} \tag{9.83}$$

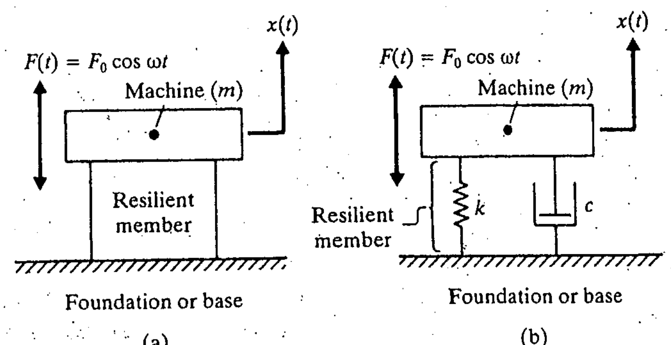


FIGURE 9.19 Machine and resilient member on rigid foundation.

and

$$\phi = \tan^{-1}\left(\frac{\omega c}{k - m\omega^2}\right) \tag{9.84}$$

The force transmitted to the foundation through the spring and the dashpot, $F_t(t)$, is given by

$$F_t(t) = kx(t) + c\dot{x}(t) = kX\cos(\omega t - \phi) - c\omega X\sin(\omega t - \phi) \tag{9.85}$$

The magnitude of the total transmitted force (F_T) is given by

$$F_T = [(kx)^2 + (c\dot{x})^2]^{1/2} = X\sqrt{k^2 + \omega^2 c^2}$$
$$= \frac{F_0(k^2 + \omega^2 c^2)^{1/2}}{[(k - m\omega^2)^2 + \omega^2 c^2]^{1/2}} \tag{9.86}$$

The transmissibility or transmission ratio of the isolator (T_r) is defined as the ratio of the magnitude of the force transmitted to that of the exciting force:

$$T_r = \frac{F_T}{F_0} = \left\{\frac{k^2 + \omega^2 c^2}{(k - m\omega^2)^2 + \omega^2 c^2}\right\}^{1/2}$$
$$= \left\{\frac{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}{\left[1 - \left(\frac{\omega}{\omega_n}\right)^2\right]^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}\right\}^{1/2} \tag{9.87}$$

where $r = \frac{\omega}{\omega_n}$ is the frequency ratio. The variation of T_r with the frequency ratio $r = \frac{\omega}{\omega_n}$ is shown in Fig. 9.20. In order to achieve isolation, the force transmitted to the foundation needs to be less than the excitation force. It can be seen, from Fig. 9.20, that the forcing frequency has to be greater than $\sqrt{2}$ times the natural frequency of the system in order to achieve isolation of vibration.

Notes

1. The magnitude of the force transmitted to the foundation can be reduced by decreasing the natural frequency of the system (ω_n).
2. The force transmitted to the foundation can also be reduced by decreasing the damping ratio. However, since vibration isolation requires $r > \sqrt{2}$, the machine should pass through resonance during start-up and stopping. Hence, some damping is essential to avoid infinitely large amplitudes at resonance.
3. Although damping reduces the amplitude of the mass (X) for all frequencies, it reduces the maximum force transmitted to the foundation (F_t) only if $r < \sqrt{2}$. Above that value, the addition of damping increases the force transmitted.
4. If the speed of the machine (forcing frequency) varies, we must compromise in choosing the amount of damping to minimize the force transmitted. The

$T_r = \frac{F_{trans,max}}{P_0}$

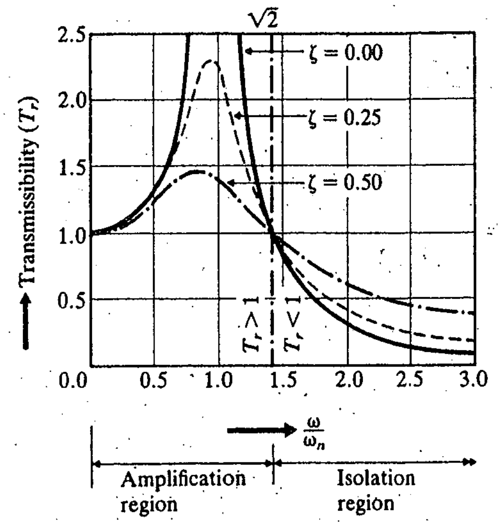


FIGURE 9.20 Variation of transmission ratio (T_r) with ω.

amount of damping should be sufficient to limit the amplitude X and the force transmitted F_t while passing through the resonance, but not so much to increase unnecessarily the force transmitted at the operating speed.

Reduction of the Force Transmitted to the Mass. If a sensitive instrument or machine of mass m is to be isolated from the unwanted harmonic motion of its base, the governing equation is given by Eq. (3.75):

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} \tag{9.88}$$

where $z = x - y$ denotes the displacement of the mass relative to the base. If the base motion is harmonic, then the motion of the mass will also be harmonic. Hence the displacement transmissibility, $T_d = \frac{X}{Y}$, is given by Eq. (3.68)

$$T_d = \frac{X}{Y} = \left\{ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right\}^{\frac{1}{2}} \tag{9.89}$$

where the right-hand side expression in Eq. (9.89) can be identified to be same as that in Eq. (9.87). Note that Eq. (9.89) is also equal to the ratio of the maximum steady-state accelerations of the mass and the base.

Reduction of the Force Transmitted to the Foundation Due to Rotating Unbalance. The excitation force caused by a rotating unbalance is given by

$$F(t) = F_0 \sin \omega t \equiv me\omega^2 \sin \omega t \tag{9.90}$$

The natural frequencies of the system are given by the roots of the equation

$$\begin{vmatrix} (k - m_1\omega^2) & -k \\ -k & (k - m_2\omega^2) \end{vmatrix} = 0 \tag{9.97}$$

The roots of Eq. (9.97) are given by

$$\omega_1^2 = 0, \qquad \omega_2^2 = \frac{(m_1 + m_2)\,k}{m_1 m_2} \tag{9.98}$$

The value $\omega_1 = 0$ corresponds to rigid-body motion since the system is unconstrained. In the steady state, the amplitudes of m_1 and m_2 are governed by Eq. (9.96), whose solution yields

$$X_1 = \frac{(k - m_2\omega^2)F_0}{[(k - m_1\omega^2)(k - m_2\omega^2) - k^2]} \tag{9.99}$$

$$X_2 = \frac{kF_0}{[(k - m_1\omega^2)(k - m_2\omega^2) - k^2]} \tag{9.100}$$

The force transmitted to the supporting structure (F_t) is given by the amplitude of $m_2\ddot{x}_2$:

$$F_t = -m_2\omega^2 X_2 = \frac{-m_2 k\omega^2 F_0}{[(k - m_1\omega^2)(k - m_2\omega^2) - k^2]} \tag{9.101}$$

The transmissibility of the isolator (T_r) is given by

$$T_r = \frac{F_t}{F_0} = \frac{-m_2 k\omega^2}{[(k - m_1\omega^2)(k - m_2\omega^2) - k^2]}$$

$$= \frac{1}{\left(\dfrac{m_1 + m_2}{m_2} - \dfrac{m_1\omega^2}{k}\right)} = \frac{m_2}{(m_1 + m_2)}\left(\frac{1}{1 - \dfrac{\omega^2}{\omega_2^2}}\right) \tag{9.102}$$

where ω_2 is the natural frequency of the system given by Eq. (9.98). Equation (9.102) shows, as in the case of an isolator on a rigid base, that the force transmitted to the foundation becomes less as the natural frequency of the system ω_2 is reduced.

EXAMPLE 9.3 Spring Support for Exhaust Fan

An exhaust fan, rotating at 1000 rpm, is to be supported by four springs, each having a stiffness of K. If only 10 percent of the unbalanced force of the fan is to be transmitted to the base, what should be the value of K? Assume the mass of the exhaust fan to be 40 kg.

Given: Exhaust fan with mass = 40 kg, rotational speed = 1000 rpm, and permissible shaking force to be transmitted to base = 10 percent.

Find: Stiffness (K) of each of the four supporting springs.

Approach: Use transmissibility equation.

Solution: Since the transmissibility has to be 0.1, we have, from Eq. (9.87),

$$0.1 = \left[\frac{1 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2}{\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left(2\zeta\frac{\omega}{\omega_n}\right)^2} \right]^{1/2} \quad \text{(E.1)}$$

where the forcing frequency is given by

$$\omega = \frac{1000 \times 2\pi}{60} = 104.72 \text{ rad/sec} \quad \text{(E.2)}$$

and the natural frequency of the system by

$$\omega_n = \left(\frac{k}{m}\right)^{1/2} = \left(\frac{4K}{40}\right)^{1/2} = \frac{\sqrt{K}}{3.1623} \quad \text{(E.3)}$$

By assuming the damping ratio to be $\zeta = 0$, we obtain from Eq. (E.1),

$$0.1 = \frac{\pm 1}{\left\{1 - \left(\frac{104.72 \times 3.1623}{\sqrt{K}}\right)^2\right\}} \quad \text{(E.4)}$$

To avoid imaginary values, we need to consider the negative sign on the right-hand side of Eq. (E.4). This leads to

$$\frac{331.1561}{\sqrt{K}} = 3.3166$$

or

$$K = 9969.6365 \text{ N/m}$$

■

EXAMPLE 9.4 Isolation of Vibrating System

A vibrating system is to be isolated from its supporting base. Find the required damping ratio that must be achieved by the isolator to limit the transmissibility at resonance to $T_r = 4$. Assume the system to have a single degree of freedom.

Given: Transmissibility at resonance = 4.

Find: Damping ratio of the isolator.

Approach: Find the equation for the transmissibility at resonance.

Solution: By setting $\omega = \omega_n$, Eq. (9.87) gives

$$T_r = \frac{\sqrt{1 + (2\zeta)^2}}{2\zeta}$$

or

$$\zeta = \frac{1}{2\sqrt{T_r^2 - 1}} = \frac{1}{2\sqrt{15}} = 0.1291$$

9.9.3 Vibration Isolation System with Partially Flexible Foundation

Figure 9.22 shows a more realistic situation in which the base of the isolator, instead of being completely rigid or completely flexible, is partially flexible [9.34]. We can define the mechanical impedance of the base structure, $Z(\omega)$, as the force at frequency ω required to produce a unit displacement of the base (as in Section 3.5):

$$Z(\omega) = \frac{\text{Applied force of frequency } \omega}{\text{Displacement}}$$

The equations of motion are given by[6]

$$m_1\ddot{x}_1 + k\,(x_1 - x_2) = F_0 \cos \omega t \tag{9.103}$$

$$k\,(x_2 - x_1) = -x_2 Z(\omega) \tag{9.104}$$

By substituting the harmonic solution

$$x_j(t) = X_j \cos \omega t, \qquad j = 1, 2 \tag{9.105}$$

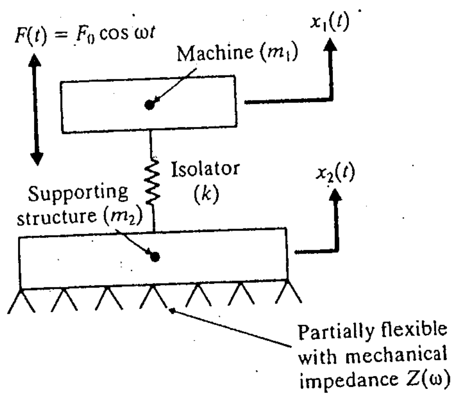


FIGURE 9.22 Machine with isolator on a partially flexible foundation.

[6]If the base is completely flexible with an unconstrained mass of m_2, $Z(\omega) = -\omega^2 m_2$, and Eqs. (9.104) and (9.105) lead to Eq. (9.94).

points can be determined by setting

$$\frac{d}{dr}\left(\frac{X}{m_0 e/m}\right) = 0$$

This results in defining

$$\frac{X}{m_0 e/m} = 0$$

as the initial minimum point of all curves. Also, all curves approach unity as r becomes large. Finally, the maximum point occurs at

$$r = \frac{1}{\sqrt{1 - 2\zeta^2}} > 1 \qquad (4\text{-}83)$$

Accordingly, the peaks occur to the right of the resonance value of $r = 1$. For $\zeta = 0.707$ the curve rises through its entirety, with the maximum equal to 1 as r approaches infinity.

Figure 4-20 is adequate, provided that the variation in r is limited to changing ω_f. Note that small amplitude occurs only at low operating frequencies, as would be expected.

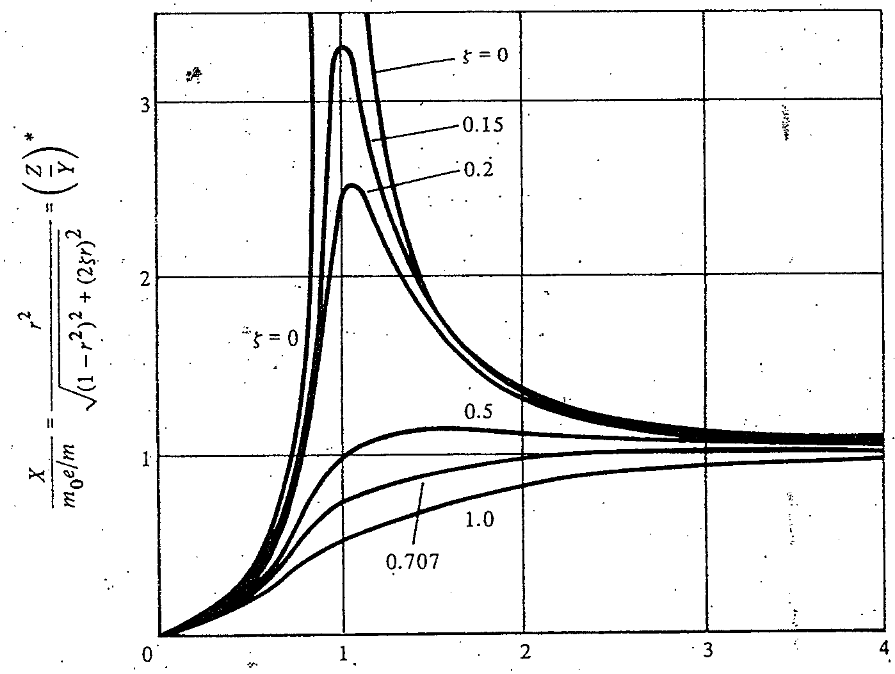
Since ζ (as well as r) is dependent on k and m, Fig. 4-20 does not properly show the effect of varying k or m. Also, the reference $m_0 e/m$ is affected by altering m. The effect of varying k or m can be observed by writing the amplitude relation here in the form given by Eq. 4-79. The amplitude X can then be plotted against either k or m for various values of the damping constant c. The resulting families of curves will be identical to those of Figs. 4-17 and 4-18, provided P_0 is replaced by $m_0 e\omega_f^2$.

In all the preceding discussion, it should be noted that amplitude is dependent on the quantity $m_0 e$ and that if either m_0 or e is small, the amplitude will become small. This merely emphasizes the importance of reducing the eccentric condition insofar as may be possible.

EXAMPLE 4-6 A machine with a rotating shaft has a total weight of 200 lb and is supported by springs. The damping constant for the system is found to be 3 lb sec/in. The resonant speed is determined experimentally to be 1200 rpm, and the corresponding amplitude of the main mass of the machine is 0.50 in. Determine the amplitude for a speed of 2400 rpm. Also determine the fixed value the amplitude will eventually approach at high speed.

SOLUTION Since the machine oscillates when in operation, the rotating part must contain an eccentric mass. At resonance, Eq. 4-79 becomes

$$X = \frac{m_0 e\omega_f}{c}$$



*For oscillating support case.

Figure 4-20

so that

$$m_0 e = \frac{Xc}{\omega_f} = \frac{0.5 \times 3}{40\pi} = 0.011\,93 \text{ lb sec}^2$$

$$k = m\omega_f^2 = \frac{200}{386} \times (40\pi)^2 = 8180 \text{ lb/in.}$$

Then, at 2400 rpm,

$$X = \frac{m_0 e\omega_f^2}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}}$$

$$= \frac{0.011\,93 \times (80\pi)^2}{\sqrt{[8180 - \frac{200}{386} \times (80\pi)^2]^2 + (3 \times 80\pi)^2}}$$

$$= 0.0306 \text{ in.}$$

For further increase in speed, the amplitude approaches the value defined by

$$X = \frac{m_0 e}{m} = \frac{0.011\,93 \times 386}{200} = 0.0230 \text{ in.}$$

OLUTION

$$X = \frac{P_0}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} = \frac{0.54}{\sqrt{[k - 0.1 \times (12)^2]^2 + (0.24 \times 12)^2}}$$

$$= \frac{0.54}{\sqrt{(k - 14.4)^2 + (2.88)^2}}$$

a. For $k = 2$, $X = 0.042\,42$ m $= 4.242$ cm
For $k = 25$, $X = 4.916$ cm
For $k = 90$, $X = 0.7138$ cm

b. $X_{max} = \dfrac{P_0}{c\omega_f} = \dfrac{0.54}{0.24 \times 12} = 0.1875$ m $= 18.75$ cm

for $k = m\omega_f^2 = 0.1 \times (12)^2 = 14.4$ N/m

-13. ROTATING UNBALANCE

. common source of forced vibration is caused by the rotation of a small ccentric mass such as that represented by m_0 in Fig. 4-19(a). This condition esults from a setscrew or a key on a rotating shaft, crankshaft rotation, and nany other simple but unavoidable situations. Rotating unbalance is inherent in rotating parts, because it is virtually impossible to place the axis of the nass center on the axis of rotation.

For the system shown, the total mass is m and the eccentric mass is m_0, o the mass of the machine body is $(m - m_0)$. The length of the eccentric rm, or the eccentricity of m_0, is represented by e. If the arm rotates with an ngular velocity ω_f rad/sec, then the angular position of the arm is defined y $\omega_f t$ with respect to the indicated horizontal reference, where t is time, in econds. The free-body diagram for this system is shown in Fig. 4-19(b), ositive x having been taken as upward. The horizontal motion of $(m - m_0)$ considered to be prevented by guides. The vertical displacement of m_0 is $x + e \sin \omega_f t$). From Eq. 1-8, the differential equation of motion can then e written as

$$(m - m_0)\frac{d^2x}{dt^2} + m_0\frac{d^2}{dt^2}(x + e \sin \omega_f t) = -kx - c\frac{dx}{dt} \tag{4-76}$$

hich can be rearranged in the form

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = m_0 e\omega_f^2 \sin \omega_f t \tag{4-77}$$

xamination of this and comparison to the differential equation (Eq. 4-38) or motion forced by $P = P_0 \sin \omega_f t$ enable the steady-state solution to be et down, from Eq. 4-48, as

$$x = X \sin(\omega_f t - \psi) \tag{4-78}$$

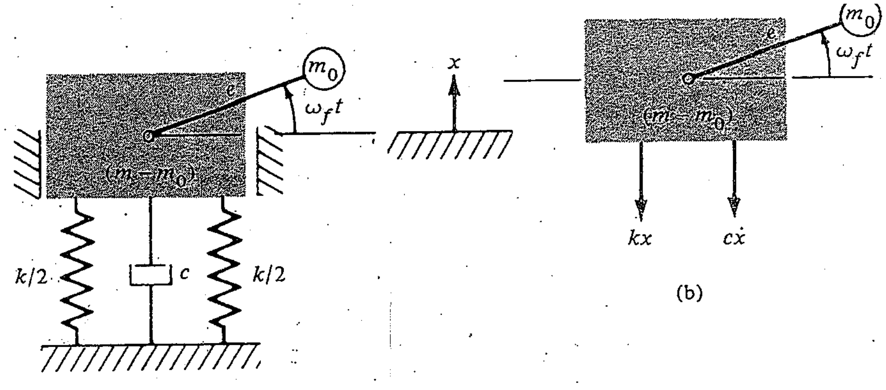


Figure 4-19

where

$$X = \frac{m_0 e\omega_f^2}{\sqrt{(k - m\omega_f^2)^2 + (c\omega_f)^2}} \tag{4-79}$$

$$= \frac{(m_0 e/m)\omega_f^2(m/k)}{\sqrt{[(k - m\omega_f^2)/k]^2 + (c\omega_f/k)^2}} = \frac{m_0 e}{m}\frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4-80}$$

and

$$\frac{X}{m_0 e/m} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \tag{4-81}$$

Also

$$\tan \psi = \frac{2\zeta r}{1 - r^2} \tag{4-82}$$

Here $\omega = \sqrt{k/m}$ represents the natural circular frequency of the undamped system (including the mass m_0), but x defines the forced motion of the main mass $(m - m_0)$. It should be noted that for this case ψ will be represented physically by the angle of the eccentric arm relative to the horizontal reference of $\omega_f t$. Thus for a value of ψ determined by Eq. 4-82, the arm would be at this angle when the main body is at its neutral position, moving upward. (Since the motion lags the forcing condition, the arm then leads the motion by the angle ψ determined.) The steady-state amplitude is generally significant, and this can be studied by plotting

$$\frac{X}{m_0 e/m}$$

against the frequency ratio r for various values of the damping factor ζ, resulting in the family of curves shown in Fig. 4-20. Maximum and minimum

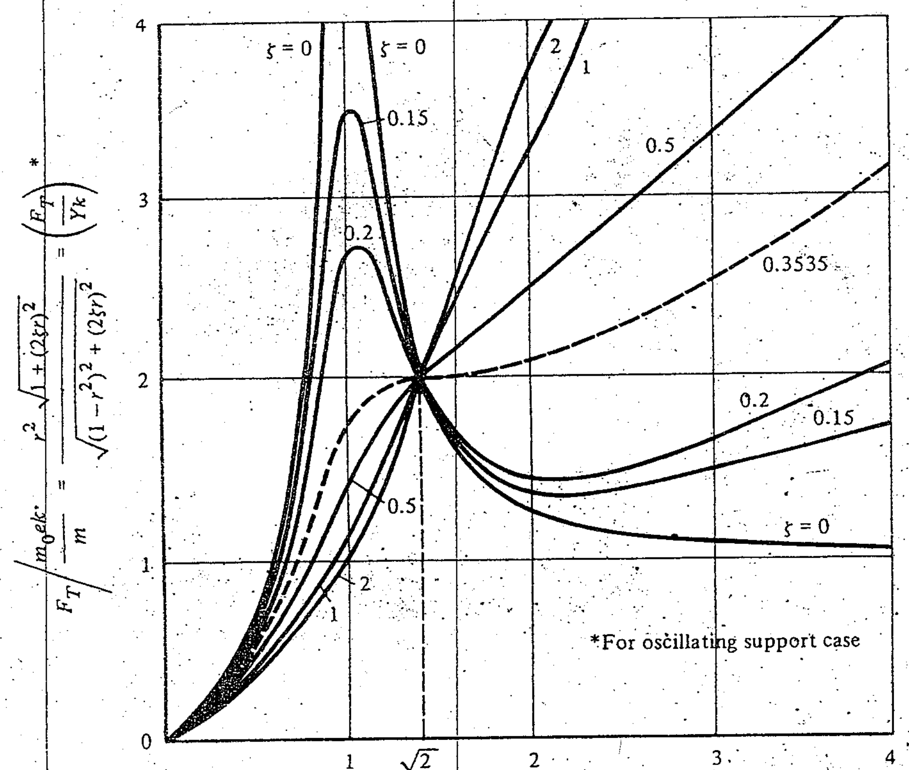


Figure 4-25

$m_0 e\omega_f^2$ for P_0 in Eqs. 4-88 and 4-89. Then

$$F_T = m_0 e\omega_f^2 \frac{\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}} = m_0 e\omega_f^2 \frac{\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \tag{4-93}$$

Multiplying the numerator and denominator by k/m and rearranging gives

$$F_T = \frac{m_0 ek}{m} \cdot \frac{r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \tag{4-94}$$

whence

$$\frac{F_T}{m_0 ek/m} = \frac{r^2\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \tag{4-95}$$

The effect of varying ω_f on the transmitted force can be shown by plotting $F_T/(m_0 ek/m)$ against r for various values of the damping factor ζ. In so doing, k and m are taken as constant. The reference $m_0 ek/m$ is then fixed. The resulting family of curves is shown in Fig. 4-25. Damping serves to limit the transmitted force in the region of resonance. A crossover point occurs at $r = \sqrt{2}$, for which $F_T/(m_0 ek/m)$ has a value of 2. For no damping, the curve approaches a value of 1 as r approaches infinity. When damping is present, the force becomes very large as r increases, and the greater the damping, the more rapidly this occurs. Even for small damping, the increase in the transmitted force is significant. Since frequency ratios of 10 or more are common in practice, the seriousness of damping is evident.

The maximum and minimum points for the family of curves here can be determined by setting

$$\frac{d}{dr}\left(\frac{F_T}{m_0 ek/m}\right) = 0$$

The resulting expression is satisfied by the following conditions:

1. For $r = 0$. This defines the initial point of $F_T/(m_0 ek/m) = 0$ for all curves.
2. By the roots of the relation

$$2\zeta^2 r^6 + (16\zeta^4 - 8\zeta^2)r^4 + (8\zeta^2 - 1)r^2 + 1 = 0$$

If $0 < \zeta < \sqrt{2}/4$, there are two positive real roots of this relation. One of these will be between $r = 0$ and $r = \sqrt{2}$, and will define a maximum point on the curve. The other will be $r > \sqrt{2}$ and will define a minimum point on the curve. If $\zeta > \sqrt{2}/4$, there is no maximum point on the curve.

If it is desired to determine the effect of varying k and m on the transmitted force, this can be done by using Eq. 4-93 and arranging it as

$$\frac{F_T}{m_0 e\omega_f^2} = \frac{\sqrt{k^2+(c\omega_f)^2}}{\sqrt{(k-m\omega_f^2)^2+(c\omega_f)^2}} \tag{4-96}$$

Since the forcing frequency ω_f, the mass m_0, and the eccentricity e are to be held constant, this case becomes identical to those shown in Figs. 4-23 and 4-24, and no further analysis is needed here.

EXAMPLE 4-7 For the rotating eccentric of Example 4-6, calculate the maximum dynamic force transmitted to the foundation at the resonant speed. Also obtain F_T for the crossover point of Fig. 4-25.

SOLUTION At resonance, Eq. 4-93 reduces to

$$F_T = \frac{m_0 e\omega_f}{c}\sqrt{k^2+(c\omega_f)^2} = X_{res}\sqrt{k^2+(c\omega_f)^2}$$

$$= 0.5\sqrt{(8180)^2 + (3 \times 40\pi)^2} = 4094 \text{ lb}$$

For the crossover,

$$F_T = \frac{2m_0 e}{m}k = 2 \times 0.0230 \times 8180 = 376 \text{ lb}$$