FLORIDA INTERNATIONAL UNIVERSITY
Mechanical Engineering Department

Spring 2015 | System Dynamics | EML3222

Instructor: Professor Cesar Levy (levyez@fiu.edu)
Telephone: 305-348-3643
Office hours: **to be determined.**
TA: Hari Adluru, hadlu001@fiu.edu, office hours to be determined

COURSE OBJECTIVES
Understand Undamped SDOF systems $d^2x/dt^2+\omega^2x=0$ and its relation to a vibrating system
Understand Damped SDOF systems-viscous (underdamped, critically damped and overdamped) and coulomb friction and their differences
Understand Forced Motion due to external input
Understand the essentials of modeling Understand the lumped parameter concept Understand the similarities and differences between: a) linear mechanical systems b) rotational mechanical systems c) fluid systems d) electrical systems e) thermal systems Understand transformers and transducers Understand system graphs Understand how to get the equations of motion Understand how to solve the equations of motion using: closed form, numerical methods, transform methods, state variable-matrix methods

MME Program Educational Objectives
Broad and in-depth knowledge of engineering science and principles in the major fields of MME for effective engineering practice, professional growth, and as a base for life-long learning.
The ability to utilize analytical and experimental methods and modern computer technology for decision-making and engineering design and to solve realistic engineering problems.
The ability to work effectively with others in a team while simultaneously maintaining independent and creative thought.
The ability to communicate effectively and to articulate technical matters using verbal, written, and graphic techniques.
An adequate background to pursue graduate studies in engineering and other fields.
A sense of professional and social responsibility, including a commitment to protect both occupational and public health and safety, developed through consideration of moral, social, and ethical paradigms related to the engineering profession and practice.

MME Student Outcomes
A. Ability to apply knowledge of mathematics, science, and engineering.
E. Ability to identify, formulate, and solve engineering problems.
F. Understanding of professional and ethical responsibility.
I. Recognition of the need for, and an ability to engage in, life long learning.
J. Knowledge of contemporary issues.
K. Ability to use the techniques, skills, and modern engineering tools necessary for engineering practice.

PREREQUISITES: EGN3321, EMA3702, EML2032 with a grade of C or better. Students not having the required passing grade in the prerequisite courses should drop EML3222 during the drop/add period. Violators will be dropped out automatically later on during the semester. This may result in their loss of course tuition. **You must also be registered for 3 credits not 2.**

COURSE CONTENT

Topics

1. What is vibrations and its importance
2. Equivalent Systems and Equations of Motion
3. Free Vibrations of M-K system, Energy Methods
4. SDOF with damping
5. Forced SDOF without damping
6. Forced SDOF with damping Q1
7. Other Systems-Fluidic, thermal
8. Understand the essentials of modeling, lumped parameter concept
9. Understand the similarities and differences between:
   a) linear mechanical systems
   b) rotational mechanical systems
   c) fluid systems
   d) electrical systems
   e) thermal systems
10. Understand transformers and transducers
11. Understand system graphs Q2
12. Understand how to get the equations of motion
13. Understand how to solve the equations of motion using:
    closed form, numerical methods, transform methods, state variable-matrix methods Q3

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1. Importance of Vibrations; Basic Concepts--period, amplitude, circular frequency, units; Classification of Vibrations-- random, periodic, harmonic, aperiodic; Vibrational Analysis Procedures; Quick Review of Dynamics--Kinetics and Principle of Linear and Angular Motion of a Particle, Mass Center and a System of Particles.

2. Undamped Free Vibrations for a Single Degree of Freedom (SDOF) System: Spring-Mass System; Equivalent Springs and Masses; Energy Method--KE<->PE transfer.

3. Damped Free Vibrations for an SDOF System; Spring-Mass-Dashpot System; Overdamped, Underdamped, Critically Damped System, damped frequency, damping factor, general solutions, Quick Review of second order linear constant coeff. Diff. Eqs.; Coulomb Damping, frequency displacement decrease, differences between coulomb and viscous damping.

4. Forced Vibrations (FV) of an SDOF System; Undamped and Damped Vibrations--magnification factor, resonance conditions; beats; force transmission; Duhamel's Integral; Response of a Damped System.

5. Essentials of Modelling--Capturing the gist of the real-life system
   - o Modelling of mechanical systems by
     springs, masses, dampers, v - F relationships
   - o Modelling of electrical systems by
     inductance, capacitance, resistances, v - i relationships
   - o Modelling of fluidic systems by
     inertance, capacitance, resistances, p - Q relationships
   - o Modelling of thermal systems by

capacitance, resistances, T - q relationships

6. Thru and Across Variables; systems equations; system graphs
7. Determination of governing equation for the system
8. Solution of the governing differential equation --
   - o State Variables and Matrix Methods of Solution
   - o Closed form solutions (reduction of order and variation of parameter methods)
   - o Laplace Transform Method
   - o Numerical Methods (Euler and Runge-Kutta Methods)

Textbook: Introduction to System Dynamics by Derek Rowell and David Wormley, Prentice-Hall, 1st Edition ISBN 978-0132108089, 1996

*** ALSO NOTES WILL BE DISTRIBUTED IN CLASS ***

Other helpful information

HW's will be assigned but none will be collected for grading. Website will provide solutions to many problems two lessons after they are assigned. However, it is to your advantage to do it since similar problems will appear on the exams and final exam.

Important information:

1. **You are required to send me an email using an email address from which you can receive class information.**
2. **If you come in late, find a seat quickly and quietly.**
3. **Please turn off your cellphones in class.** They are an unnecessary interruption to both your fellow students and to the instructor. **Turn off your cellphones especially during quizzes/exams. Anyone caught using their cellphone/text messaging during quiz/exam will fail the quiz/exam.**
4. **Cheating of any kind especially during quizzes/examinations will result in automatic failure of the exam/quiz. Cheating during final exam will result in failure of the course and possible expulsion from the university.**

Please note: *Florida International University is a community of faculty, staff and students dedicated to generating and imparting knowledge through 1) excellent teaching and research, 2) the rigorous and respectful exchange of ideas, and 3) community service. All students should respect the right of others to have an equitable opportunity to learn and honestly demonstrate the quality of their learning. Therefore, all students are expected to adhere to a standard of academic conduct, which demonstrates respect for themselves, their fellow students, and the educational mission of the University. All students are deemed by the University to understand that if they are found responsible for academic misconduct, they will be subject to the Academic Misconduct procedures and sanctions, as outlined in the Student Handbook.*

5. **No make up quizzes/exams will be given.** Exceptions are if you are sick (provide a note from your doctor), or if you are being called up for military duty (provide a copy of your orders).
6. **No attendance will be taken. HOWEVER, if you do not come to class, it is your responsibility to get the material you've missed and to learn the material.**
7. **If you don't understand something in class-ASK.** You may also come to my office during office hours. You may call me at my office and, if I am not in the middle of something, we can discuss your question.

Grades will be determined on the basis of

3 Exams	25, 20, 20% each
Final exam	35%

Grading Scheme: 93 and above	A	77 - 79	B-	63 - 66	D+
90 - 92	A-	74 - 76	C+	60 - 62	D
84 - 89	B+	70 - 73	C	Below 60	F
80 - 84	B	67 - 69	C-		

We will be meeting 2 times a week TR 920-1045 in EC1105

My office will be in EAS 3443 and
Office hours: **to be determined.**
TA: Hari Adluru, hadlu001@fiu.edu, office hours to be determined

The Drug-Free Schools and Communities Act Amendments of 1989 and the Drug-Free Workplace Act of 1988 (collectively the "Act"), require that Florida International University (FIU) maintain a program to prevent the use of illicit drugs and abuse of alcohol by students, faculty, and staff. Under the Act, FIU is also required to annually distribute the following information about its drug and alcohol program to all students, faculty, and staff. Please read this notification carefully. Click here for a more detailed version of this notification.

POLICY: The university is committed to maintaining a safe, productive, and drug-free work and educational environment. As such, FIU strictly prohibits the unlawful manufacture, distribution, dispensation, possession, trade, sell, or offer for sale of a controlled substance or alcohol, or otherwise engaging in the unlawful use of controlled substances or alcohol on campus. No person may report to classes, work, or related assignments "under the influence" of controlled substances, alcohol, or prescription drugs taken illegally. Furthermore, all employees are required to notify the university of any criminal drug statute conviction for a violation occurring in the workplace no later than five (5) days after such conviction. Click here to read the complete Drug-Free Campus/Workplace Drug and Alcohol Abuse Prevention Policy, which can be found in the University Policies and Procedures Library.

SANCTIONS: Any student, faculty or staff found to have abused drugs and/or alcohol in the workplace or campus shall be subject to disciplinary action in accordance with university regulations, policies and any applicable collective bargaining agreement.

DRUGS: The unlawful possession, use, distribution, dispensation, manufacture, or sale of Controlled Substances is governed by federal law, Florida law, university regulation (FIU-2501 Student Code of Conduct), and university policy.

ALCOHOL: The use, possession, or distribution of beverages containing alcohol on university property, including residence halls, is governed by Florida law and university regulations (FIU-2505 Alcoholic Beverages, FIU-2501 Student Code of Conduct).

HEALTH RISKS: Drugs have different effects on each individual, depending on their age, size and sex. The manner in which a particular drug interacts with a person can cause temporary or permanent damage to a person's body and brain, and can lead to addiction or death. Description of drugs and effects can be found in the DEA Drug Fact Sheets.

Alcohol is a depressant which leads to the loss of control over judgment, resulting in a loss of inhibitions. It affects physical coordination, causing blurred vision, slurred speech and loss of

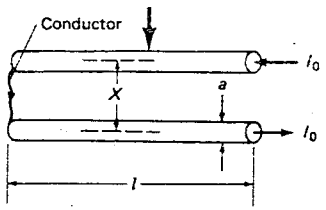
balance. Click here for a summary of health problems and conditions associated with excessive drinking over time.

RESOURCES: Click here for a list of on-campus and community resources for drug and alcohol prevention and assistance.

For additional information, please contact Dr. Kate Kominars, Director of the Office of Employee Assistance and Chair of the Alcohol and Drug Abuse Prevention Task Force, at 305-348-2469 or kominars@fiu.edu.

THIS IS A PRELIMINARY SCHEDULE--ALL CHANGES WILL BE ANNOUNCED IN CLASS.

Table 2.4.1-1 (cont.)



6. Parallel conductors

$$L = \frac{l}{\pi}\ln(X/a)$$

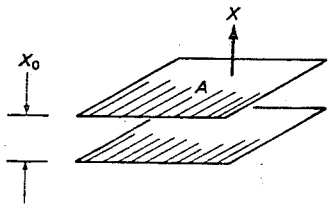
$$F = \frac{\frac{1}{2}I_0^2 la}{\pi X}$$

$$K_{eq} = \frac{\frac{1}{2}I_0^2 la}{\pi X^2}$$

$$D_{eq} = \frac{1}{R}\left[\frac{I_0 la}{\pi X}\right]^2$$

units in mks system
L_0 = nominal current (A)
V = voltage (V)
R_c = coil resistance (Ω)
B = mag. field (Wb/m^2)
l = length(m)

Table 2.4.2-1 ELECTROSTATIC DAMPERS AND SPRINGS



1. Parallel planes

$$C = \frac{\epsilon A}{X}$$

$$F_0 = \frac{V_0^2 \epsilon A}{2X_0^2}$$

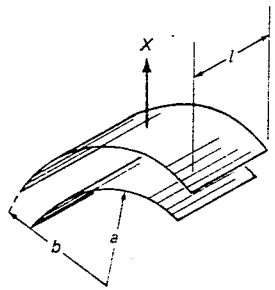
$$K_{eq} = \frac{V_0^2 \epsilon A}{X_0^3}$$

$$D_{eq} = \frac{(V_0 \epsilon A)^2 R}{X_0^4}$$

C = capacitance (farad)

ϵ = permittivity

See Appendix 6



2. Concentric Cylinders

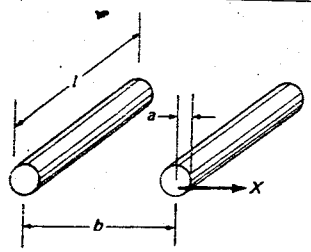
$$C = \frac{2\pi\epsilon l}{\ln(b/a)}$$

$$F_0 = \frac{V_0^2 \pi\epsilon l}{b(\ln X_0/a)^2}$$

$$K_{eq} = \frac{V_0^2 2\pi\epsilon l}{b^2(\ln X_0/a)^3}$$

$$D_{eq} = \frac{4(V_0 \pi\epsilon l)^2 R}{b^2(\ln X_0/a)^4}$$

$X_0 = b - a$



3. Parallel rods

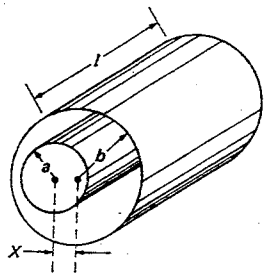
$$C = \frac{\pi\epsilon l}{\ln\left(\frac{b + \sqrt{b^2 - 4a^2}}{2a}\right)}$$

$$F_0 = \frac{V_0^2 \pi\epsilon l}{2(\ln X_0/a)^2}$$

$$K_{eq} = \frac{V_0^2 \pi\epsilon l}{(\ln X_0/a)^3}$$

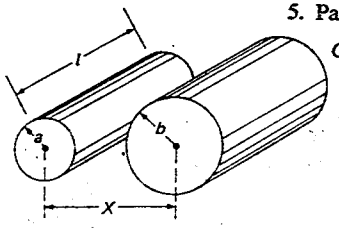
$$D_{eq} = \frac{2(V_0 \pi\epsilon l)^2 R}{(\ln X_0/a)^4}$$

$X_0 = b$



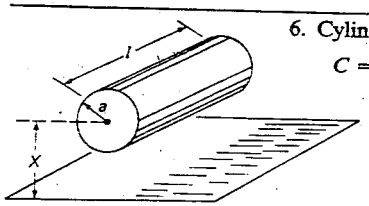
4. Eccentric cylinders

$$C = \frac{2\pi\epsilon l}{\cosh^{-1}\left(\frac{a^2 + b^2 - X^2}{2ab}\right)}$$



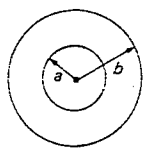
5. Parallel cylinders

$$C = \frac{2\pi\epsilon l}{\cosh^{-1}\left(\frac{a^2 + b^2 - X^2}{2ab}\right)}$$



6. Cylinder and plane

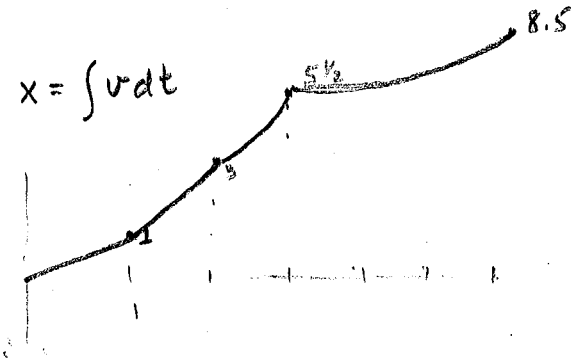
$$C = \frac{2\pi\epsilon l}{\ln(2X/a)}$$



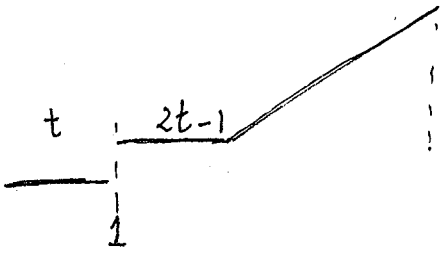
7. Concentric spheres

$$C = \frac{4\pi\epsilon ab}{b - a}$$

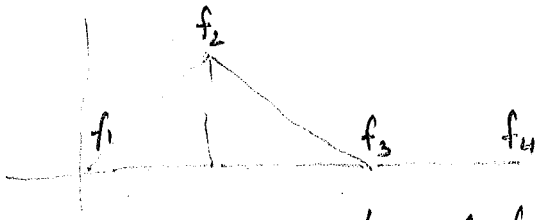
HW
do 1-8 b, c
1-12, 1-9



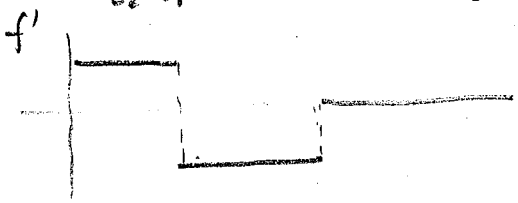
$x = \int v\,dt$

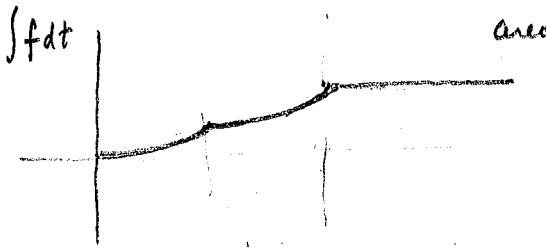


$\frac{.5}{45} = \frac{1}{90}$



$\frac{f_2 - f_1}{t_2 - t_1} = 1 = f'$ $\quad \frac{f_3 - f_2}{t_3 - t_2} = -1$ $\quad \frac{f_4 - f_3}{t_4 - t_3} = 0$





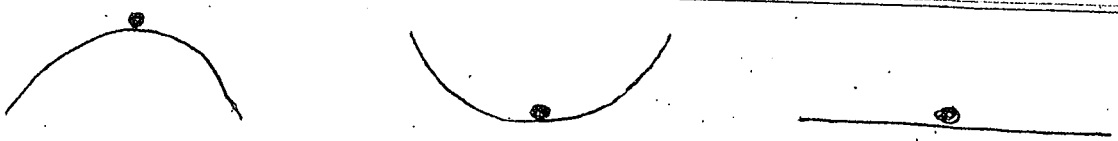
area $= \frac{1 \cdot 1}{2}, \frac{1 \cdot 1}{2}, 0$

$f = t$ $\qquad f = 2 - t$

$\frac{t^2}{2}\Big|_0^1 = \frac{1}{2}$ $\qquad -\frac{(2-t)^2}{2}\Big|_1^2 = \frac{1}{2}$

LECTURE #1 - INTRODUCTION

- DYNAMIC SYSTEMS CAN BE CLASSIFIED AS STABLE OR UNSTABLE
- IN GROSS SENSE STABILITY CAN BE DEFINED AS



UNSTABLE STABLE NEUTRAL

- FOR UNSTABLE SYSTEMS OUR JOB IS TO CONTROL THE INSTABILITY & BRING SYSTEM UNDER CONTROL

  EXAMPLES:
  - HEATING UNIT WITHOUT FEED BACK
  - F-16

- BUT BEFORE ONE CAN CONTROL THE SYSTEM
  - MODEL THE SYSTEM
  - ANALYZE THE MODEL

- ANALYSIS, MODELING & CONTROL HAVE INTERESTED ENGINEERS FOR A LONG TIME
- WITH ADVENT OF COMPUTERS THIS SUBJECT HAS BECOME IMPORTANT SINCE IT ALLOWS US TO LOOK AT MORE COMPLEX SYSTEMS
  - BEFORE COMPUTER - ONLY THE MOST SIMPLIFIED MODELS

- WE CAN BETTER USE LIMITED MATERIALS & ENERGY TO DESIGN (REQUIRES MORE DETAILED MODEL)
- CAN USE COMPUTERS FOR MEASUMENT & CONTROL, DATA ANALYSIS & DECISION MAKING (i.e., MICROPROCESSORS TO CALIBRATE THEMSELVES

MODELLING OF DYNAMIC SYSTEMS

- OBJECTIVE OF ENGINEERING ANALYSIS OF PHYSICAL SYSTEMS

  $\Longrightarrow$ PREDICTION OF BEHAVIOR

- REAL SYSTEMS ARE TOO COMPLEX FOR EXACT ANALYSIS

  SO WE MAKE SIMPLIFYING ASSUMPTIONS BASED ON JUDGEMENT EXPERIENCE

- MODEL IS IDEALIZED VERSION OF THE SYSTEM

  WITH BEHAVIOR THAT APPROX. THE IMPORTANT BEHAVIOR OF REAL SYS

- NEGLECT EFFECTS THAT HAVE SMALL IMPACT ON PROPERTIES TO BE STUD

  - EXAMPLES
    - MECHANICAL VIBRATIONS OF AN ELECTRICAL CIRCUIT
    - GRAVITATIONAL PULL OF NEARBY MASSES COMPARED TO EARTH

DFN THE PROCESS OF SIMPLIFYING A PHYSICAL SYSTEM TO OBTAIN A MATHEMATICALLY SOLVABLE SITUATION IS CALLED MODELLING

DFN THE RESULTING SIMPLIFIED VERSION OF THE SYSTEM IS CALLED THE MATHEMATICAL MODEL.

- THE ACCURACY OF THE MODEL DEPENDS ON THE VALIDITY OF THE ASSUMPTIONS

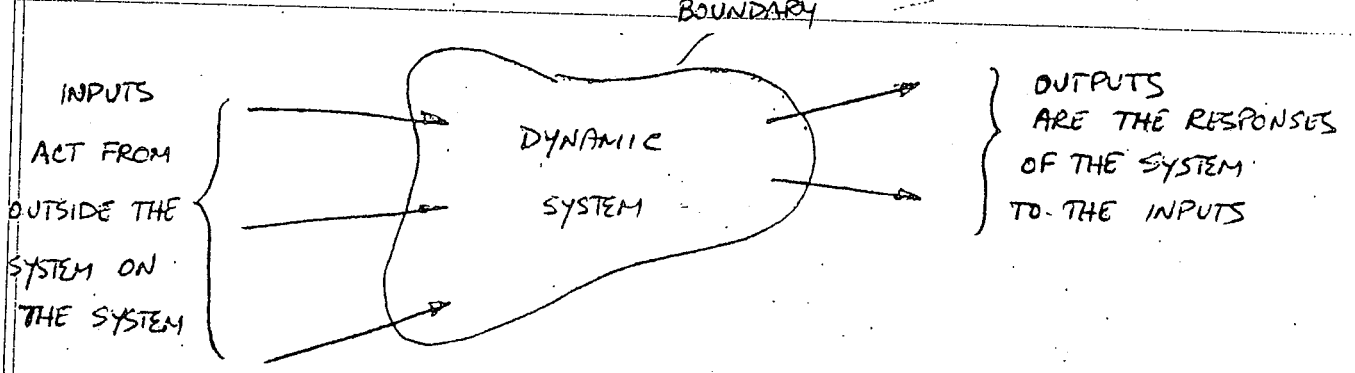
A MATHEMATICAL ANALYSIS IS USELESS, NO MATTER HOW CLEVER OR COMPLETE, IF BASED ON A MODEL THAT DOESN'T CORRECTLY REPRESENT THE BEHAVIOR OF THE PHYSICAL SYSTEM

OFTEN - EXPERIMENTATION WITH THE REAL SYSTEM IS REQUIRED

DFN A SYSTEM IS A COLLECTION OF MATTER, PARTS OR COMPONENTS WHICH ARE INCLUDED INSIDE A SPECIFIED OFTEN ARBITRARY BOUNDARY

DEF A DYNAMIC SYSTEM - A SYSTEM WHERE ONE OR MORE QUANTITIES OF THE SYSTEM CHANGE W/ TIME

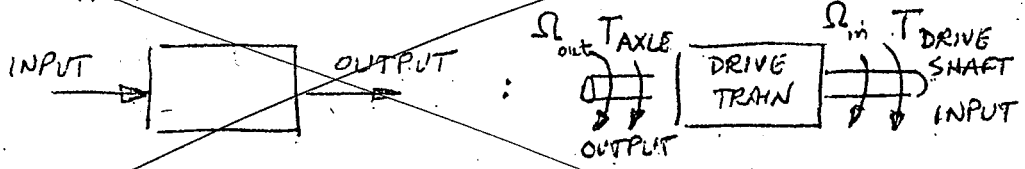
- THE SYSTEMS BOUNDARY IS CHOSEN FOR CONVENIENT CONCEPTUAL SEPARATIC OF THE SYSTEM FROM ITS SURROUNDING ENVIRONMENT



- DEFINE A SYSTEM, ALONG W/ INPUTS & OUTPUTS, TO UNDERSTAND THE CAUSE & EFFECT RELATIONSHIPS AT WORK IN THE REAL PHYSICAL SYSTEMS - WE WANT TO KNOW
  - (1) WHAT CHANGES IN WHAT INPUTS CAUSE WHAT CHANGES IN WHAT OUTPUTS
  - (2) CAN SYSTEM OUTPUT BE REGULATED OR CONTROLLED BY CONTROLLING SYSTEM INPUTS.
  - (3) HOW CAN THE CAUSE & EFFECT RELATIONSHIP BETWEEN INPUT & OUTPUT BE CONTROLLED

BLOCK DIAGRAMS - SYSTEMS APPROACH

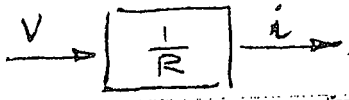
- MODEL THE SYSTEM AS A SERIES OF BLOCKS EACH AFFECTED BY INPUT & OUTPUT



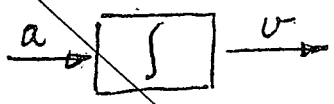
ELEMENTS REPRESENTED BY

- ANALYSIS FOCUSES ON THE CONNECTION BETWEEN THESE ELEMENTS THESE CONNECTIONS DETERMINE SYSTEM BEHAVIOR

EXAMPLES OF BLOCK DIAGRAMS



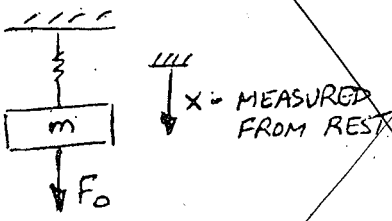
RESISTOR RELATING VOLTAGE TO CURRENT
PHYSICAL ELEMENT $i = V/R$



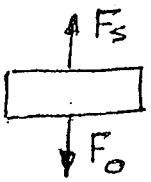
PROCESS - INTEGRATOR RELATES ACCEL
TO VELOCITY $v = \int a\,dt$

- WHENEVER OUTPUT IS TIME INTEGRAL OF INPUT THIS IS INTEGRAL CA
- INTEGRAL CAUSALITY CONSTITUTES A BASIC FORM OF CAUSALITY FOR A
  PHYSICAL SYSTEMS

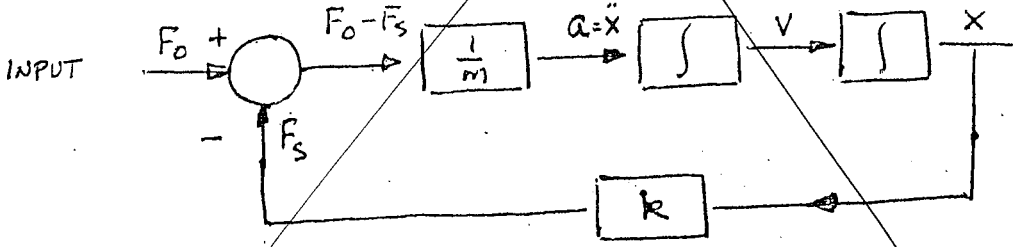
EXAMPLE



DYNAMIC FBD ON MASS IS

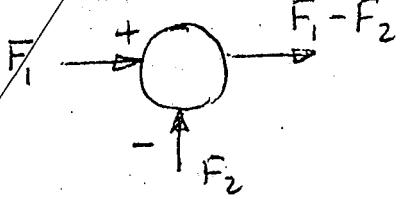


$$\Sigma F = F_o - F_s = ma = m\ddot{x}$$



$F_o = m\ddot{x} + F_s = m\ddot{x} + kx$ relates input F_o to output x

- THIS IS A BLOCK DIAGRAM OF THE SYSTEM NOT A CONTRO
  DIAGRAM



COMPARATOR REPRESENTS THE ELEMENT THAT INTRODUCES ADDITION & SUBTRACTION
BLOCKS ONLY ALLOW FOR MULT, DIV &

STATIC & DYNAMIC SYSTEMS

DFN STATIC ELEMENT ONE WHOSE OUTPUT AT ANY TIME DEPENDS ONLY
DEPENDS ON INPUT AT THAT TIME

- DFN DYNAMIC ELEMENT - ONE WHOSE OUTPUT DEPENDS ON PRESENT AN PAST INPUTS

EXAMPLE: STATIC - RESISTOR - CURRENT DEPENDS ONLY ON PRESENT APPLIED VOLTAGE $i = \frac{V}{R}$

DYNAMIC - CAPACITOR - CURRENT DEPENDS ON RATE OF CHA OF APPLIED VOLTAGE $i = C\frac{dV}{dt}$

$$V = \frac{1}{C}\int i\,dt + V_0$$

- DFN A DYNAMIC SYSTEM IS ONE THAT HAS AT LEAST ONE DYNAMI ELEMENT

- AS WE HAVE SAID OUR JOB IS TO MODEL, ANALYZE & CONTROL A SY
  - WE'VE BRIEFLY DISCUSSED MODELING
- DFN ANALYSIS - PREDICTING THE BEHAVIOR OF THE SYSTEM FROM TH DEVELOPED MATH MODEL
- DFN CONTROL - PROCESS OF DELIBERATELY INFLUENCING THE BEHAV OF A SYSTEM TO ACHIEVE DESIRED RESULT

  THERMOSTAT

  AUTO PILOT, CRUISE CONTROL

- CONTROLLER SHOULD
  (1) BRING OPERATING CONDITIONS TO DESIRED VALUE
  (2) MAINTAIN DESIRED CONDITIONS IN THE PRESENCE OF VARIATIONS OF THE EXTERNAL CONDITIONS (DISTURBANC

TYPES OF MODELS

- STATIC ELEMENTS - LEAD TO ALGEBRAIC EQNS
- DYNAMIC ELEMENTS - " " DIFFERENTIAL EQNS
- LUMPED PARAMETER - NON-SPACIALLY DEPENDENT BUT TIME DEPENDE
- DISTRIBUTED PARAMETER - SPACE & TIME DEPENDENCE MAKES COMPLE

EXAMPLE POINT MASS VS DISTRIBUTED MASS (1) VS (2) EQS OF

- NONLINEAR - WHEN DEPENDENT VARIABLES OR DERIVATIVES INVOLVE POWERS OR TRANSCENDAL FNS: $\frac{d^2y}{dt^2} + y^2 = 0 \qquad y = y(t)$

- LINEAR - WHEN DEPENDENT VAR. & DERIV'S ARE TO THE FIRST POWER

$$\ddot{y} + 5\dot{y} + 6y = 0$$

- CAN APPLY SUPERPOSITION PRINCIPLE

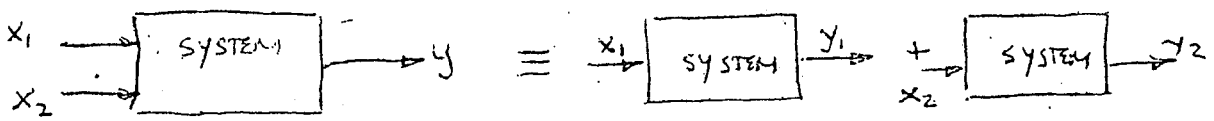
- TIME INVARIANT - MODELS W/ CONSTANT COEFFS (STATIONARY)
- TIME VARIANT (NONSTATIONARY) - MODELS W/ NON CONST COEFFS, $\ddot{y} + c(t)$
- DISCRETE - GOVERNING EQNS ARE REPLACED BY DIFFERENCE EQNS
- CONTINUOUS TIME - " " ARE DIFFERENTIAL OR ALGEBRAIC EQNS
- DETERMINISTIC - MODELS COEFFS/INPUTS ARE KNOWN
- STOCHASTIC - MODELS COEFFICIENTS/INPUTS HAVE AN UNCERTAINTY

- FOR EASE OF ANALYSIS USE LUMPED & STATIC ELEMENTS
- FOR MORE REALISM USE DISTRIBUTED & DYNAMIC ELEMENTS
- ENGINEERS TRY TO SIMPLIFY MODELS TO MAKE ANALYSIS EASIER
  - ONE WAY TO DO THIS IS TO DEVELOP A LINEAR MODEL
  - THIS ALLOWS FOR USE OF PRINCIPLE OF SUPERPOSITION

DFN - THE RESPONSE $y(t)$ OF A LINEAR SYSTEM DUE TO SEVERAL INPUTS $x_1(t)$, $x_2(t)$, $x_3(t)$ .... ACTING SIMULTANEOUSLY IS EQUAL TO THE SUM OF THE RESPONSES OF EACH INPUT, IF EACH WERE TO ACT ALONE.

IF $y_i(t)$ is response due to $x_i(t)$ THEN

$$y(t) = \sum y_i(t)$$



- YOU CAN TELL A SYSTEM IS LINEAR IF WHEN YOU MULTIPLY THE INPUT BY A CONSTANT THEN THE OUTPUT WILL BE MULTIPLIED BY SAME CONSTANT.

EXAMPLE OF LINEARITY: $y = f(x)$ $\quad x \rightarrow \boxed{f(x)} \rightarrow y$

and now if $x = ax_1 + bx_2$ & $y_1 = f(x_1)$ $\quad y_2 = f(x_2)$

THEN f is LINEAR IF $y = f(ax_1 + bx_2) = af(x_1) + bf(x_2) = ay_1 + by_2$

$x \rightarrow \boxed{m} \rightarrow y = mx$

$x \rightarrow \boxed{D} \rightarrow y = Dx = \frac{d}{dt}x$

$x \rightarrow \boxed{\int} \rightarrow y = \int x\,dt$

$x \rightarrow \boxed{(\,)^2} \rightarrow y = x^2$ is not linear ; $y = \sin x$ not linear

- CAN EXTEND THIS DEFN TO FNS OF MORE THAN ONE VARIABLE
- SUPPOSE WE HAVE A SYSTEM THAT IS DESCRIBED BY

$$\frac{d^2y}{dt^2} + 4x\frac{dy}{dt} + 2x^2 = 0$$

WHERE $x(t)$ is THE INPUT

$y(t)$ is THE OUTPUT

IF IT IS LINEAR $\Rightarrow$ IF INPUT IS REPLACED BY cx THEN OUTPUT IS REPLACED BY cy

$$\frac{d^2cy}{dt^2} + 4(cx)\frac{dcy}{dt} + 2(cx)^2 = 0$$

or $\quad c\ddot{y} + 4c^2x\dot{y} + 2c^2x^2 \neq c\left[\ddot{y} + 4x\dot{y} + 2x^2\right] = 0$

- IF YOU ARE GIVEN AN ODE, LOOK AT THE HOMOGENEOUS EQN TO DETERMINE LINEARITY

- IS $\frac{d^2}{dt^2}y + 4t\frac{dy}{dt} + 4y = \sin t$ LINEAR?

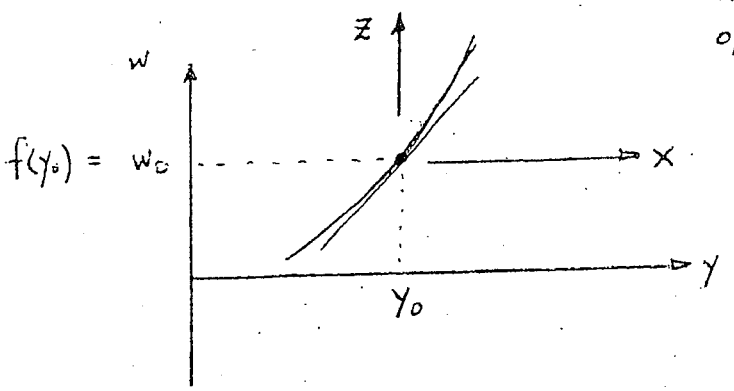
- FOR DYNAMIC SYSTEMS - NO "real" physical system, can be described by a linear constant coefficient differential equation. HOWEVER IN A LIMITED RANGE OF OPERATION THEY CAN BE APPROXIMATED BY SUCH MODELS

- BECAUSE OF THE USEFULNESS OF SUPERPOSITION, WE TRY TO OBTAIN THE LINEAR MODELS. HOW DO WE DO THIS? BY LINEARIZING ABOUT AN OPERATING PT.
  - TO DO THIS WE USE TAYLOR SERIES (APPENDIX A PALM)

- CONSIDER THE STATIC ELEMENT W/ INPUT/OUTPUT MODEL

$w = f(y)$

y - input
w - output
operating pt (y_0, w_0)



$$w = f(y) = f(y_0) + \left(\frac{df}{dy}\right)\Big|_{y=y_0} (y-y_0) + \left(\frac{d^2f}{dy^2}\right)\Big|_{y=y_0} \frac{(y-y_0)^2}{2} + \cdots$$

IF y is close to y_0 THEN

$$w \cong f(y_0) + \left(\frac{df}{dy}\right)\Big|_{y=y_0} (y-y_0)$$

$\left(\frac{df}{dy}\right)\Big|_{y=y_0}$ is slope at oper point = const

$$w \cong w_0 + C(y-y_0) \; ; \quad \text{NOW LET} \quad w - w_0 = z, \quad y - y_0 = x$$

$$\therefore \quad z = Cx$$

THUS WE HAVE REPLACED THE CURVE $w = f(y)$

BY A LINEAR EQ MEASURED FROM THE OPERATING PT

EXAMPLE - ANGULAR OSCILLATION OF A PENDULUM

$$\frac{d^2y}{dt^2} + k\sin y = 0$$ note $\frac{dy}{dt}$ is linear but $\sin y$ isn't

---

WE CAN LINEARIZE $\sin y$ ABOUT THE OPERATING PT $y=0$

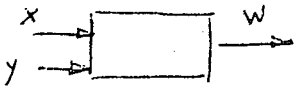
$$\sin y = y - y^3/3! + y^5/5! - \ldots$$

FOR y close to $y=0$

$$\frac{d^2y}{dt^2} + k\sin y \Rightarrow \frac{dy}{dt} + ky = 0$$ THIS IS A LINEAR ODE

- LINEARIZING FOR A FN OF 2 VARIABLES x, y are inputs W is outp

$$W = f(x,y)$$



LET THE OPERATING PT BE $W=W_0$ WHEN $X=X_0, Y=Y_0$

$$W = f(x,y) \cong f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}} (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\y=y_0}} (y-y_0)$$

if $x - x_0 = X$ $\quad a = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}}$

$y - y_0 = Y$

$w - w_0 = Z$ $\quad b = \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\y=y_0}}$

$$Z = aX + bY$$

- WE CAN USE THIS PROCESS FOR FNS OF n-VARIABLES

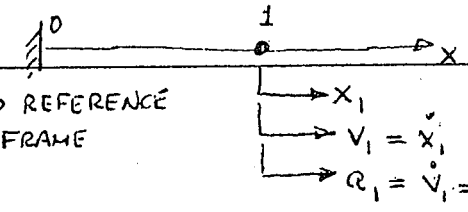
→ WILL DISCUSS LATER ON IN COURSE ←

DYNAMIC SYSTEM ELEMENTS - MODELLING

- REAL PHYSICAL SYSTEMS ARE CONTROLLED BY FLOW, STORAGE AND EXCHANGE OF VARIOUS FORMS OF ENERGY
- MODEL OR DYNAMIC SYSTEMS ARE MADE UP OF BASIC BUILDING BL THAT IDEALIZE THE ESSENTIAL PHYSICAL PHENOMENA OF REAL SYS
- BUILDING BLOCKS WE WILL STUDY - SIMPLE MECHANICAL & ELECTRICAL SYSTEM ELEMENTS THAT STORE, DISSIPATE & TRANSFO ENERGY

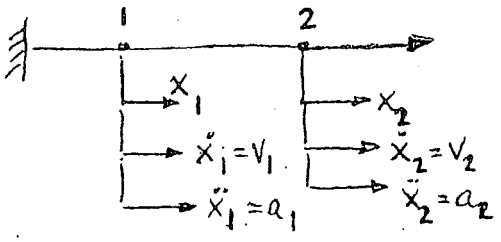
MECHANICAL ELEMENTS IN TRANSLATION

- LOOK AT SIMPLE TRANSLATIONAL MOTION OF A PARTICLE FROM DYNAMICS



FIXED REFERENCE FRAME

DISPLACEMENT IN DIRECTION SHOWN MEASURED FROM REFERENCE PT 0



SPRINGS, DAMPERS, MASSES IN GENERAL WILL HAVE MOTION: THE REQUIRE KNOWLEDGE OF RELATIVE MOTION

THUS $x_{21} = x_2 - x_1$

$v_{21} = v_2 - v_1 = \dot{x}_{21}$

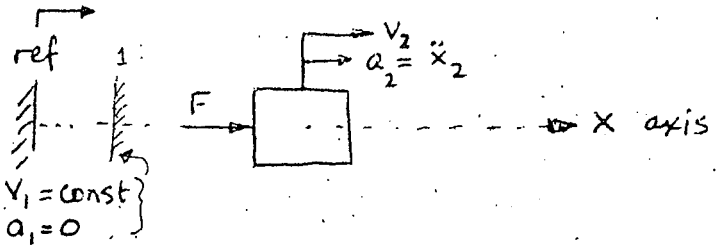
$a_{21} = a_2 - a_1 = \ddot{x}_{21}$

NOTE THAT THIS IS TRUE FOR CASE WHERE EITHER FRAME OF REF. IS STATIONARY OR TRANSLATING ONLY.

CAN DEFINE $\mathcal{P} = \vec{F} \cdot \vec{v}$ and work $= \int \vec{F} \cdot \vec{v}\, dt$

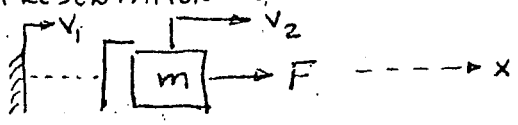
PURE TRANSLATIONAL MASS

- NEWTON'S SECOND LAW $F = \frac{dp}{dt}$ p = momentum = mv
- FOR THE IDEAL MASS CASE $\frac{dp}{dt} = m\frac{dv}{dt} = m\frac{d^2x}{dt^2} = ma$



- a is absolute acceleral
- $\frac{dv_2}{dt} \equiv \frac{dv_{21}}{dt}$ IF $\equiv$

- V_1 is THE VELOCITY OF THE FRAME OF REF & IT IS EITHER 0 or a co
- SYMBOLIC REPRESENTATION IS



- PURE TRANSLATIONAL MASS - ALLOWS FOR NO ROTATION, FRICTION OR ELAST
  - PARTICLES OF BODY ARE RIGIDLY CONNECTE
  - $V_2 <<$ SPEED OF LIGHT, V_1 = Const or (
  - F

THUS $\vec{p} = \int_0^t \vec{F}\,dt + \vec{p}(t=0)$ & $\vec{v}_{21} = \frac{1}{m}\int_0^t \vec{F}\,dt + \vec{v}_{21}(t=0)$

ELEMENTAL EQUATION

- Power delivered to or removed from mass

$$\mathbb{P} = \vec{F} \cdot \vec{v}$$

- Work done on the mass to move it from pt 1 to pt 2

$$W = \int_{x_1}^{x_2} \vec{F} \cdot d\vec{x}_{21} = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}_{21}\,dt = \int_{p_1}^{p_2} \vec{v}_{21} \cdot d\vec{p}$$

for spring like elements; for damper like element; for mass like elements

- MASS STORES WORK AS A FORM OF ENERGY ASSOCIATED WITH VELOCITIES OF MASS PARTICLES (KINETIC ENERGY $= E_k$)
- FOR A NEWTONIAN MASS $\vec{p} = m\vec{v}_2 = m\vec{v}_{21}$ or $d\vec{p} = m\,d\vec{v}_{21}$

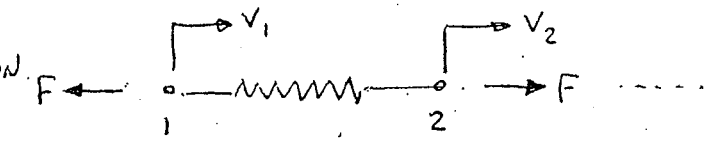
$$E_k = W = \frac{1}{2} m v_2^2 \geq 0 \quad \text{ALSO} \quad E_k = \frac{1}{2m} p^2$$

- SIGN OF $\vec{v}$ IS IMPORTANT AS IT TELL US HOW TO RETRIEVE STORED ENERGY

PURE TRANSLATIONAL SPRING - A ZERO MASS MECHANICAL ELEMENT WHICH DEFORMS BY STEADY AMOUNTS WHEN LOADED BY STEADY FORCES

- SINCE NO MASS $\Rightarrow$ $\Sigma \vec{F} = \vec{0}$ AND

SYMBOLIC REPRESENTATION



$x_{21} = f_n$ of F

$v_2 > v_1$

$$\bar{v}_{21} = \frac{d\bar{x}_{21}}{dt} \quad ; \quad \bar{x}_{21} = \int_0^t \bar{v}_{21}\,dt + \bar{x}_{21}(t=0)$$

- SINCE WE HAVE EQUILIBRIUM OF FORCES NOTE F IS THOUGHT OF AS FLOWING THROUGH THE SPRING

- FOR AN IDEAL SPRING : $\bar{x}_{21} = \frac{\bar{F}}{k}$ $\therefore$ $\bar{v}_{21} = \frac{1}{k}\frac{d\bar{F}}{dt}$

or $\bar{F} = k\int^t \bar{v}_{21}\,dt + \bar{F}(t=0)$

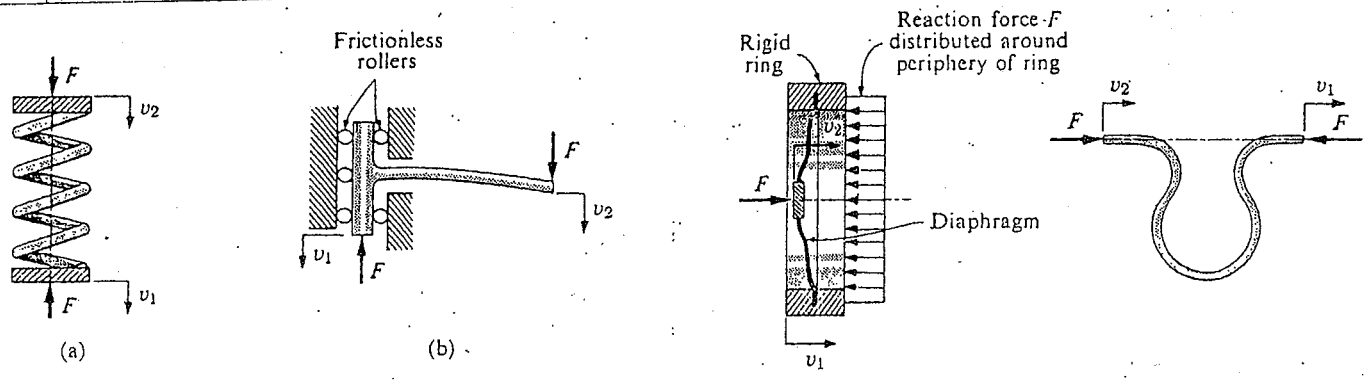
Work done on the spring by the forces.

$$W = \int \vec{F} \cdot \vec{v}_{21}\, dt = \int \vec{F} \cdot d\vec{x}_{21}$$

FOR AN IDEAL SPRING. $d\vec{x}_{21} = \frac{1}{k} d\vec{F}$

$$E_p = W = \frac{1}{2} k x_{21}^2 = \frac{1}{2k} F^2 \geq 0$$

- THIS WORK IS STORED AS POTENTIAL ENERGY IN THE SPRING



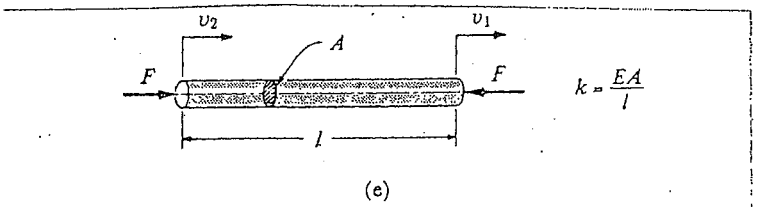


FIG. 2-8. Typical translational springs. (a) Coil spring. (b) Cantilever beam. (c) Corrugated circular diaphragm. (d) Curved bar. (e) Uniform circular bar.

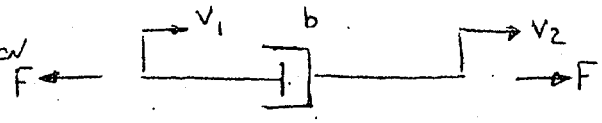
$v_2 > v_1 \Rightarrow$

$\frac{dx}{dt} \sim v_1 - v_2 < 0$

$\Rightarrow$ F is compressive

- PURE TRANSLATIONAL DAMPER – HAS NO MASS OR SPRING EFFECTS, REPRESENTS ONLY THE EFFECTS OF RESISTANCE TO <u>RATE</u> OF DEFORMATION

SYMBOLIC REPRESENTATION



$\Sigma \vec{F} = \vec{0}$

F = fn of v_{21}

sign of F & v_{21} are same

FOR AN IDEAL DAMPER

$$F = b v_{21} = b \dot{x}_{21}$$

FOR A GIVEN CHANGE IN POSITION $x_{21} = \frac{1}{b} \int F\, dt + x_{21}(t=0)$

WORK DONE $W = \int \vec{F} \cdot d\vec{v}_{21} = \int \vec{F} \cdot \vec{v}_{21}\, dt$

Power ... $\mathcal{P} = \vec{F} \cdot \vec{v}_{21}$ supplies to the damper

LECTURE #1 - INTRODUCTION

- DYNAMIC SYSTEMS CAN BE CLASSIFIED AS STABLE OR UNSTABLE
- IN GROSS SENSE STABILITY CAN BE DEFINED AS



UNSTABLE STABLE NEUTRAL

- FOR UNSTABLE SYSTEMS OUR JOB IS TO CONTROL THE INSTABILITY & BRING SYSTEM UNDER CONTROL

EXAMPLES:

- HEATING UNIT WITHOUT FEED BACK
- F-16

- BUT BEFORE ONE CAN CONTROL THE SYSTEM
  - MODEL THE SYSTEM
  - ANALYZE THE MODEL

- ANALYSIS, MODELING & CONTROL HAVE INTERESTED ENGINEERS FOR A LONG TIME
- WITH ADVENT OF COMPUTERS THIS SUBJECT HAS BECOME IMPORTANT SINCE IT ALLOWS US TO LOOK AT MORE COMPLEX SYSTEMS
  - BEFORE COMPUTER - ONLY THE MOST SIMPLIFIED MODELS

- WE CAN BETTER USE LIMITED MATERIALS & ENERGY TO DESIGN } REQUIRE MORE DETAILED MODEL
- CAN USE COMPUTERS FOR MEASUMENT & CONTROL, DATA ANALYSIS & DECISION MAKING (i.e. MICROPROCESSORS TO CALIBRATE THEMSELVES,

MODELLING OF DYNAMIC SYSTEMS

- OBJECTIVE OF ENGINEERING ANALYSIS OF PHYSICAL SYSTEMS

  $\Longrightarrow$ PREDICTION OF BEHAVIOR

- REAL SYSTEMS ARE TOO COMPLEX FOR EXACT ANALYSIS

  SO WE MAKE SIMPLIFYING ASSUMPTIONS BASED ON JUDGEMENT EXPERIENCE

- MODEL IS IDEALIZED VERSION OF THE SYSTEM

  WITH BEHAVIOR THAT APPROX. THE IMPORTANT BEHAVIOR OF REAL SYSTEM

- NEGLECT EFFECTS THAT HAVE SMALL IMPACT ON PROPERTIES TO BE STUDIED

  - EXAMPLES
    - MECHANICAL VIBRATIONS OF AN ELECTRICAL CIRCUIT
    - GRAVITATIONAL PULL OF NEARBY MASSES COMPARED TO EARTH

DFN THE PROCESS OF SIMPLIFYING A PHYSICAL SYSTEM TO OBTAIN A MATHEMATICALLY SOLVABLE SITUATION IS CALLED MODELLING

DFN THE RESULTING SIMPLIFIED VERSION OF THE SYSTEM IS CALLED THE MATHEMATICAL MODEL.

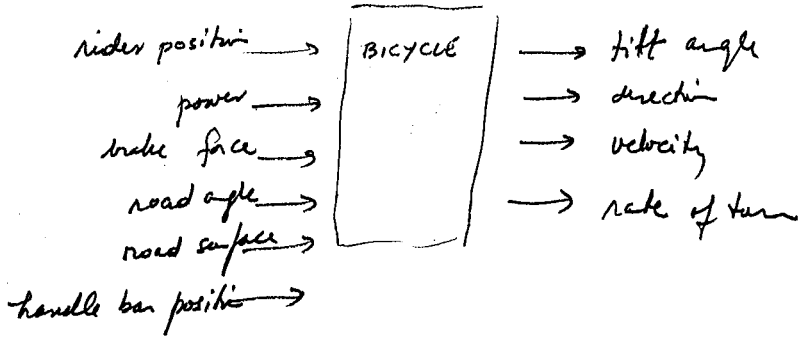
- THE ACCURACY OF THE MODEL DEPENDS ON THE VALIDITY OF THE ASSUMPTIONS

A MATHEMATICAL ANALYSIS IS USELESS, NO MATTER HOW CLEVER OR COMPLETE, IF BASED ON A MODEL THAT DOESN'T CORRECTLY REPRESENT THE BEHAVIOR OF THE PHYSICAL SYSTEM

OFTEN - EXPERIMENTATION WITH THE REAL SYSTEM IS REQUIRED

DFN A SYSTEM IS A COLLECTION OF MATTER, PARTS OR COMPONENTS WHICH ARE INCLUDED INSIDE A SPECIFIED, OFTEN ARBITRARY BOUNDARY

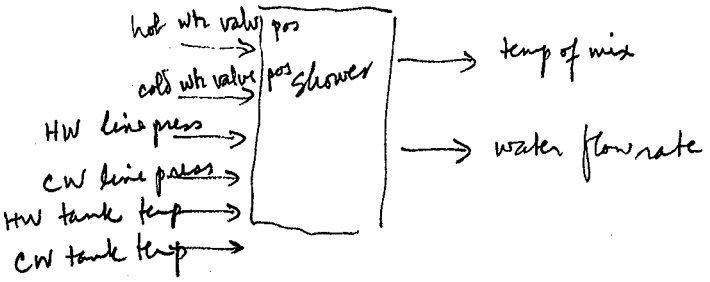
1-1



System param

mass & inertia (bicycle & rider)
sprocket & wheel size
CG
gyroscopic action (speed)

1-3



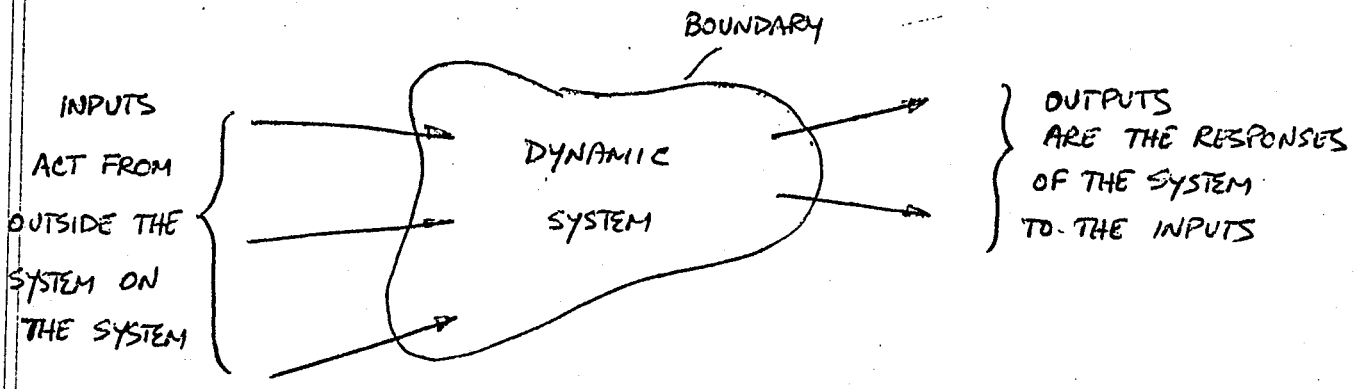
Syste param

valve sensitivity
time delays - pipe length
capacity of HW tank
heating efficiency HW tank.

DO 1-1 1-3 1-4 1-7

DFN A DYNAMIC SYSTEM - A SYSTEM WHERE ONE OR MORE QUANTITIES OF THE SYSTEM CHANGE W/ TIME

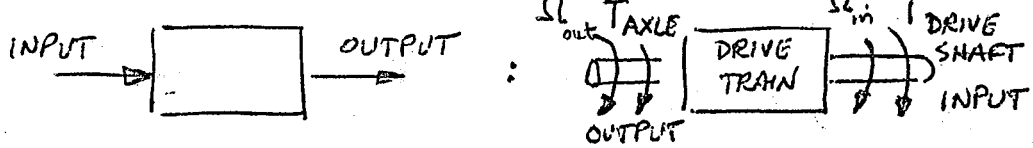
- THE SYSTEM'S BOUNDARY IS CHOSEN FOR CONVENIENT CONCEPTUAL SEPARATION OF THE SYSTEM FROM ITS SURROUNDING ENVIRONMENT



- DEFINE A SYSTEM, ALONG W/ INPUTS & OUTPUTS, TO UNDERSTAND THE CAUSE & EFFECT RELATIONSHIPS AT WORK IN THE REAL PHYSICAL SYSTEMS - WE WANT TO KNOW
  - (1) WHAT CHANGES IN WHAT INPUTS CAUSE WHAT CHANGES IN WHAT OUTPUTS
  - (2) CAN SYSTEM OUTPUT BE REGULATED OR CONTROLLED BY CONTROLLING SYSTEM INPUTS.
  - (3) HOW CAN THE CAUSE & EFFECT RELATIONSHIP BETWEEN INPUT & OUTPUT BE CONTROLLED

BLOCK DIAGRAMS - SYSTEMS APPROACH

- MODEL THE SYSTEM AS A SERIES OF BLOCKS (ELEMENTS REPRESENTED BY) EACH AFFECTED BY INPUT & OUTPUT



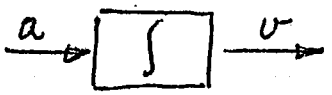
- ANALYSIS FOCUSES ON THE CONNECTION BETWEEN THESE ELEMENTS THESE CONNECTIONS DETERMINE SYSTEM BEHAVIOR

EXAMPLES OF BLOCK DIAGRAMS



RESISTOR RELATING VOLTAGE TO CURRENT

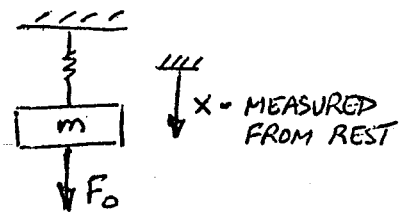
PHYSICAL ELEMENT $i = V/R$



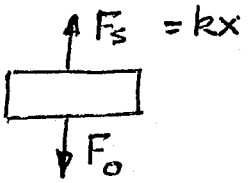
PROCESS - INTEGRATOR RELATES ACCEL. TO VELOCITY $v = \int a\,dt$

- WHENEVER OUTPUT IS TIME INTEGRAL OF INPUT THIS IS INTEGRAL CAUSAL
- INTEGRAL CAUSALITY CONSTITUTES A BASIC FORM OF CAUSALITY FOR ALL PHYSICAL SYSTEMS

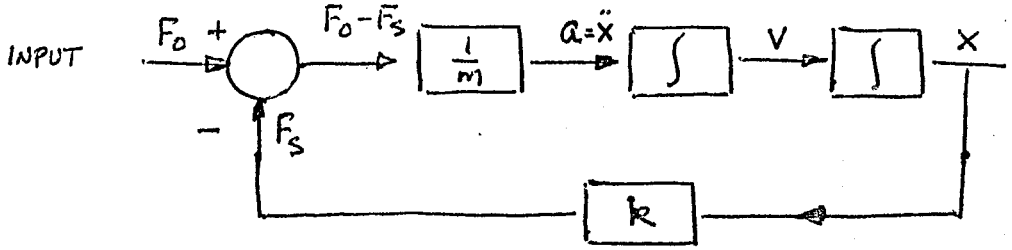
EXAMPLE



DYNAMIC FBD ON MASS IS

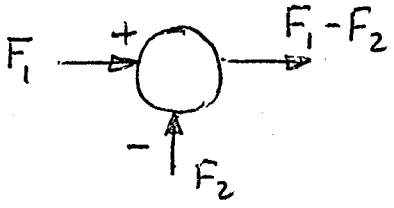


$$\Sigma \bar{F} = F_o - F_s = ma = m\ddot{x}$$



$F_o = m\ddot{x} + F_s = m\ddot{x} + kx$ relates input F_o to output x

- THIS IS A BLOCK DIAGRAM OF THE SYSTEM NOT A CONTROL DIAGRAM



COMPARATOR REPRESENTS THE ELEMENT THAT INTRODUCES ADDITION & SUBTRACTION

BLOCKS ONLY ALLOW FOR MULT, DIV & $\int$

STATIC & DYNAMIC SYSTEMS

DEFN STATIC ELEMENT ONE WHOSE OUTPUT AT ANY TIME DEPENDS ONLY ON INPUT AT THAT TIME

- DFN DYNAMIC ELEMENT - ONE WHOSE OUTPUT DEPENDS ON PRESENT AND PAST INPUTS

EXAMPLE: STATIC - RESISTOR - CURRENT DEPENDS ONLY ON PRESENT APPLIED VOLTAGE $i = \frac{V}{R}$

DYNAMIC - CAPACITOR - CURRENT DEPENDS ON RATE OF CHANG OF APPLIED VOLTAGE $i = C\frac{dV}{dt}$

$$V = \frac{1}{C}\int i\,dt + V_0$$

- DFN A DYNAMIC SYSTEM IS ONE THAT HAS AT LEAST ONE DYNAMIC ELEMENT

- AS WE HAVE SAID OUR JOB IS TO MODEL, ANALYZE & CONTROL A SYST
  - WE'VE BRIEFLY DISCUSSED MODELLING
- DFN ANALYSIS - PREDICTING THE BEHAVIOR OF THE SYSTEM FROM THE DEVELOPED MATH MODEL
- DFN CONTROL - PROCESS OF DELIBERATELY INFLUENCING THE BEHAVIOR OF A SYSTEM TO ACHIEVE DESIRED RESULT

  THERMOSTAT

  AUTO PILOT, CRUISE CONTROL

- CONTROLLER SHOULD
  - (1) BRING OPERATING CONDITIONS TO DESIRED VALUE
  - (2) MAINTAIN DESIRED CONDITIONS IN THE PRESENCE OF VARIATIONS OF THE EXTERNAL CONDITIONS (DISTURBANCES

TYPES OF MODELS

- STATIC ELEMENTS - LEAD TO ALGEBRAIC EQNS
- DYNAMIC ELEMENTS - " " DIFFERENTIAL EQNS
- LUMPED PARAMETER - NON-SPACIALLY DEPENDENT BUT TIME DEPENDENT
- DISTRIBUTED PARAMETER - SPACE & TIME DEPENDENCE MADE SIMPLER

EXAMPLE POINT MASS VS. DISTRIBUTED MASS (1) VS (2) EQS OF MO

- NONLINEAR - WHEN DEPENDENT VARIABLES OR DERIVATIVES INVOLVE POWERS OR TRANSCENDAL FNS: $\frac{d^2y}{dt^2} + y^2 = 0$ $y = y(t)$

- LINEAR - WHEN DEPENDENT VAR. & DERIVS ARE TO THE FIRST POWER

$$\ddot{y} + 5\dot{y} + 6y = 0 \quad \cdots$$

- CAN APPLY SUPERPOSITION PRINCIPLE

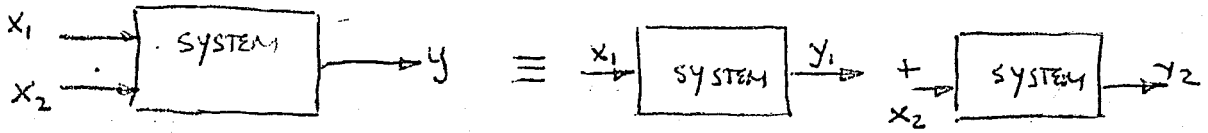
- TIME INVARIANT - MODELS W/ CONSTANT COEFFS (STATIONARY)
- TIME VARIANT (NONSTATIONARY) - MODELS W/ NON CONST COEFFS. $\ddot{y} + c(t)y =$
- DISCRETE - GOVERNING EQNS ARE REPLACED BY DIFFERENCE EQNS
- CONTINUOUS TIME - " " ARE DIFFERENTIAL OR ALGEBRAIC EQNS
- DETERMINISTIC - MODELS COEFFS/INPUTS ARE KNOWN
- STOCHASTIC - MODELS COEFFICIENTS/INPUTS HAVE AN UNCERTAINTY

- FOR EASE OF ANALYSIS USE LUMPED & STATIC ELEMENTS
- FOR MORE REALISM USE DISTRIBUTED & DYNAMIC ELEMENTS
- ENGINEERS TRY TO SIMPLIFY MODELS TO MAKE ANALYSIS EASIER
  - ONE WAY TO DO THIS IS TO DEVELOP A LINEAR MODEL
    - THIS ALLOWS FOR USE OF PRINCIPLE OF SUPERPOSITION

DFN - THE RESPONSE $y(t)$ OF A LINEAR SYSTEM DUE TO SEVERAL INPUTS $x_1(t)$, $x_2(t)$, $x_3(t)$ .... ACTING SIMULTANEOUSLY IS EQUAL TO THE SUM OF THE RESPONSES OF EACH INPUT, IF EACH WERE TO ACT ALONE.

IF $y_i(t)$ is response due to $x_i(t)$ THEN

$$y(t) = \sum y_i(t)$$



- YOU CAN TELL A SYSTEM IS LINEAR IF WHEN YOU MULTIPLY THE INPUT BY A CONSTANT THEN THE OUTPUT WILL BE MULTIPLIED BY SAME CONSTANT.

EXAMPLE OF LINEARITY: $y = f(x)$ $\quad x \rightarrow \boxed{f(x)} \rightarrow y$

and now if $x = ax_1 + bx_2$ & $y_1 = f(x_1)$ $\quad y_2 = f(x_2)$

THEN f is LINEAR IF $y = f(ax_1 + bx_2) = af(x_1) + bf(x_2) = ay_1 + by_2$

$x \rightarrow \boxed{m} \rightarrow y = mx$

$\dot{x} \rightarrow \boxed{D} \rightarrow y = Dx = \frac{d}{dt}x$

$x \rightarrow \boxed{\int} \rightarrow y = \int x\,dt$

$x \rightarrow \boxed{(\,)^2} \rightarrow y = x^2$ is not linear ; $y = \sin x$ not linear

- CAN EXTEND THIS DEFN TO FNS OF MORE THAN ONE VARIABLE
- SUPPOSE WE HAVE A SYSTEM THAT IS DESCRIBED BY

$$\frac{d^2y}{dt^2} + 4x\frac{dy}{dt} + 2x^2 = 0$$

WHERE $x(t)$ is THE INPUT
$y(t)$ is THE OUTPUT

IF IT IS LINEAR $\Rightarrow$ IF INPUT IS REPLACED BY cx THEN OUTPUT IS REPLACED BY cy

$$\frac{d^2 cy}{dt^2} + 4(cx)\frac{dcy}{dt} + 2(cx)^2 = 0$$

or $\quad c\ddot{y} + 4c^2x\dot{y} + 2c^2x^2 \neq c\left[\ddot{y} + 4x\dot{y} + 2x^2\right] = 0$

- IF YOU ARE GIVEN AN ODE, LOOK AT THE HOMOGENEOUS EQN TO DETERMINE LINEARITY

- IS $\frac{d^2}{dt^2}y + 4t\frac{dy}{dt} + 4y = \sin t$ LINEAR?

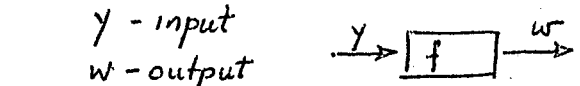
- FOR DYNAMIC SYSTEMS - no "real" physical system, can be described by a linear constant coefficient differential equation. HOWEVER IN A LIMITED RANGE OF OPERATION THEY CAN BE APPROXIMATED BY SUCH MODELS

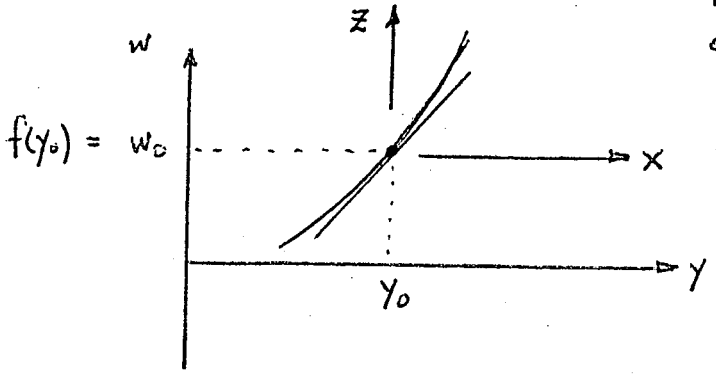
- BECAUSE OF THE USEFULNESS OF SUPERPOSITION, WE TRY TO OBTAIN THE LINEAR MODELS. HOW DO WE DO THIS? BY LINEARIZING ABOUT AN OPERATING PT.
  - TO DO THIS WE USE TAYLOR SERIES (APPENDIX A PALM)

- CONSIDER THE STATIC ELEMENT W/ INPUT/OUTPUT MODEL

$w = f(y)$

y - input
w - output
operating pt (y_0, w_0)





$$w = f(y) = f(y_0) + \left(\frac{df}{dy}\right)\Big|_{y=y_0} (y-y_0) + \left(\frac{d^2f}{dy^2}\right)\Big|_{y=y_0} \frac{(y-y_0)^2}{2} + \cdots$$

IF y is close to y_0 THEN

$$w \cong f(y_0) + \left(\frac{df}{dy}\right)\Big|_{y=y_0} (y-y_0)$$

$\left(\frac{df}{dy}\right)\Big|_{y=y_0}$ is slope at operati point = constan

$$w \cong w_0 + C(y-y_0) \quad ; \quad \text{NOW LET} \quad w - w_0 = z, \quad y - y_0 = x$$

$$\therefore \quad z = Cx$$

THUS WE HAVE REPLACED THE CURVE $w = f(y)$

BY A LINEAR EN MEASURED FROM THE OPERATING PT

EXAMPLE - ANGULAR OSCILLATION OF A PENDULUM

$$\frac{d^2y}{dt^2} + k\sin y = 0$$ note $\frac{dy}{dt}$ is linear but $\sin y$ isn't

WE CAN LINEARIZE $\sin y$ ABOUT THE OPERATING PT $y=0$

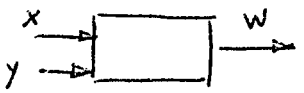
$$\sin y = y - y^3/3! + y^5/5! - \ldots$$

FOR y close to $y=0$

$$\frac{d^2y}{dt^2} + k\sin y \Rightarrow \frac{dy}{dt} + ky = 0$$ THIS IS A LINEAR ODE

- LINEARIZING FOR A FN OF 2 VARIABLES x, y are inputs W is output

$$W = f(x,y)$$



LET THE OPERATING PT BE $W=W_0$ WHEN $X=X_0, Y=Y_0$

$$W = f(x,y) \cong f(x_0,y_0) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}} (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\y=y_0}} (y-y_0)$$

if $x - x_0 = X$ $\quad a = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_0\\y=y_0}}$

$y - y_0 = Y$

$w - w_0 = Z$ $\quad b = \left.\frac{\partial f}{\partial y}\right|_{\substack{x=x_0\\y=y_0}}$

$$Z = aX + bY$$

- WE CAN USE THIS PROCESS FOR FNS OF n-VARIABLES

→ WILL DISCUSS LATER ON IN COURSE ←

DYNAMIC SYSTEM ELEMENTS - MODELLING

- REAL PHYSICAL SYSTEMS ARE CONTROLLED BY FLOW, STORAGE AND EXCHANGE OF VARIOUS FORMS OF ENERGY
- MODEL OR DYNAMIC SYSTEMS ARE MADE UP OF BASIC BUILDING BLOCKS THAT IDEALIZE THE ESSENTIAL PHYSICAL PHENOMENA OF REAL SYSTEMS
- BUILDING BLOCKS WE WILL STUDY - SIMPLE MECHANICAL & ELECTRICAL SYSTEM ELEMENTS THAT STORE, DISSIPATE & TRANSFORM ENERGY

EXAMPLE

$y'' + y = \tan x$ let $y_p = u_1(x)\cos x + u_2(x)\sin x$

$y_p' = u_1'\cos + u_2'\sin = 0$

$-u_1'\sin + u_2'\cos = \tan x.$

$$\begin{pmatrix} \cos & \sin \\ -\sin & \cos \end{pmatrix}\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ \tan x \end{pmatrix}$$

$$-\frac{\tan x \sin x}{1} = u_1' \qquad \boxed{u_2' = \frac{\cos \tan}{1} = \sin x}$$

$$-\frac{\sin^2 x}{\cos x} = -\left[\frac{1-\cos^2 x}{\cos x}\right] = -\frac{1}{\cos x} + \cos x = u_1'$$

$u_2 = -\cos x$ $\qquad u_1 = -\ln|\sec x + \tan x| + \sin x$

$\therefore \quad y_p = -\ln|\quad|\cos x + \cos x \sin x - \sin x \cos x$

$y_p = -(\cos x)\ln|\sec x + \tan x|$

BASIC CONCEPTS

- <u>VIBRATION OR OSCILLATION</u> Any motion which repeats itself after an interval of time

  MOTION MAY BE REGULAR OR DETERMINISTIC

- IRREGULAR OR NONDETERMINISTIC (STOCHASTIC)
- WE STUDY THE MOTION OF BODIES OR SYSTEMS AS WELL AS THE FORCES THAT ACCOMPANY THIS MOTION OR IS CAUSED BY THIS OSCILLATORY MOTION

- VIBRATIONS EXIST IN ELECTRIC CIRCUITS
  ACOUSTICS
  CAN BE ELECTROMAGNETIC WAVES
  MECHANICAL SYSTEMS

pluck a string — Ella Fitzgerald & glass

- MECHANICAL VIB CAN CAUSE ACOUSTICAL VIBRATION (VICE VERSA)
- " CAN CAUSE AN ELECTRIC OSCILLATION (")

- BASIC PRINCIPLES ARE THE SAME

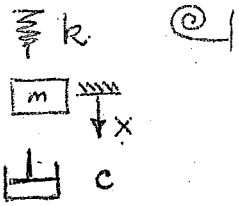
- FOR VIBRATIONS TO OCCUR NEED AT LEAST TWO ENERGY STORAGE ELEMENTS

POSSIBILITIES

INERTIA OR MASS – STORES KINETIC ENERGY

ELASTIC OR SPRING MEMBER – " POTENTIAL ENERGY

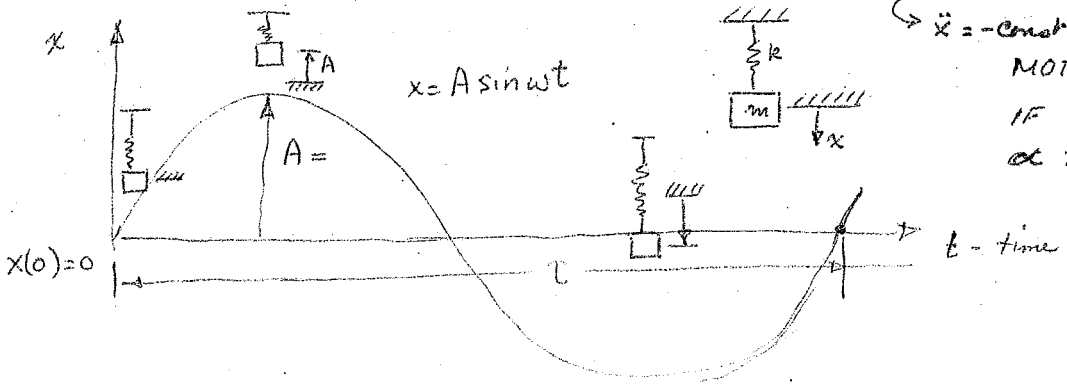
DAMPER (DASHPOT) – DISSIPATES ENERGY

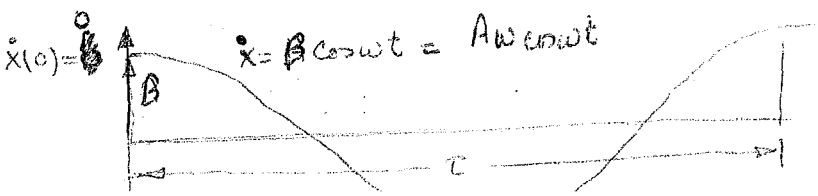


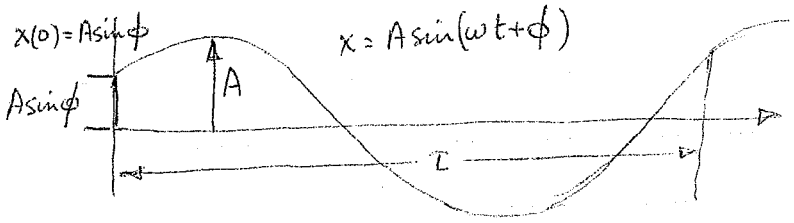
VERTICAL VIBRATIONS OF MASS

- ASSUMPTION – NO HORIZONTAL MOTION OF MASS
- VIBRATION OR OSCILLATION INVOLVES TRANSFER OF ENERGY FROM KE ⇄ PE
- MOTION OF MASS WILL BE CYCLIC OR PERIODIC

PERIODIC HARMONIC MOTION
$\ddot{x} = -\text{const}\cdot x$
MOTION OF MASS IF ELASTIC MEMBER ∝ TO DEFORMATION





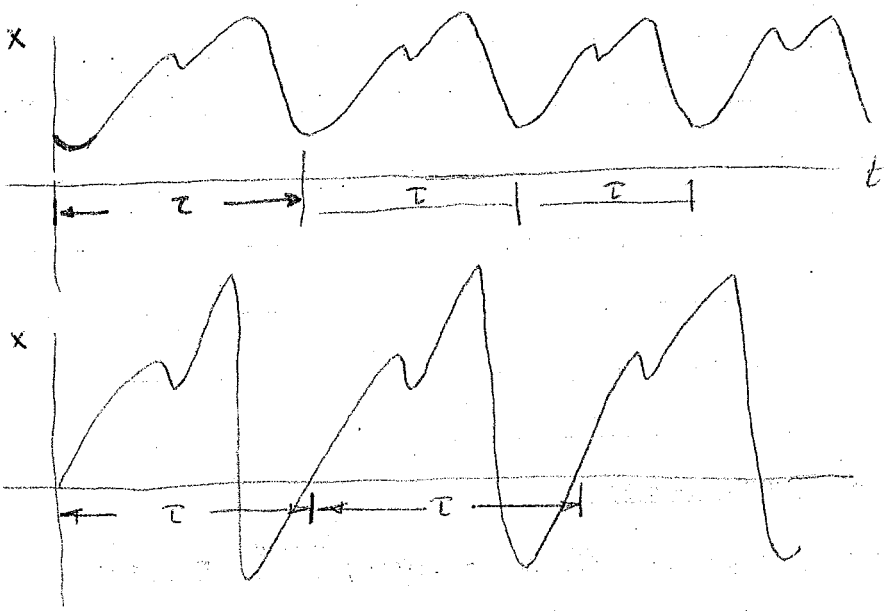


- MOTION IS DEPENDENT ON THE INITIAL CONDITIONS
- A - AMPLITUDE = MAX DISP.

  ϕ - PHASE ANGLE rad or degrees

  ω - CIRCULAR FREQUENCY rad/sec or degrees/sec

NON HARMONIC PERIODIC MOTION



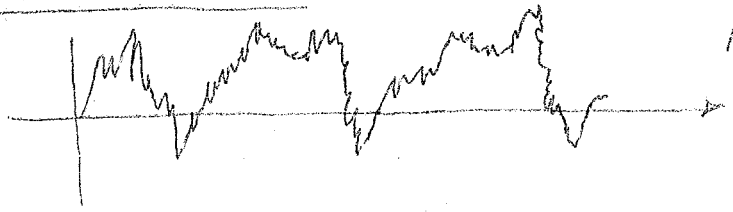
ONE COMPLETE MOVEMENT IS A CYCLE

TIME OF 1 CYCLE IS A PERIOD = τ (seconds)

FREQUENCY (# OF TIMES) OF CYCLE IN A UNIT TIME = f (cycles/second, Hz)

$\tau f = 1$.

RANDOM VIBRATIONS



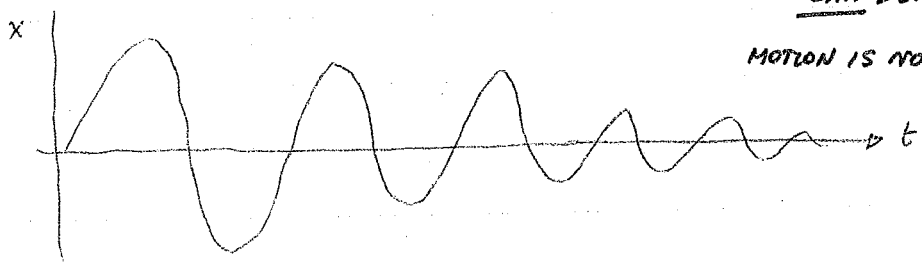
PRODUCED BY FORCES THAT ARE IRREGULAR

- CAN OCCUR DURING TRANSPORTATION OF GOODS
- IMPACT CONDITIONS

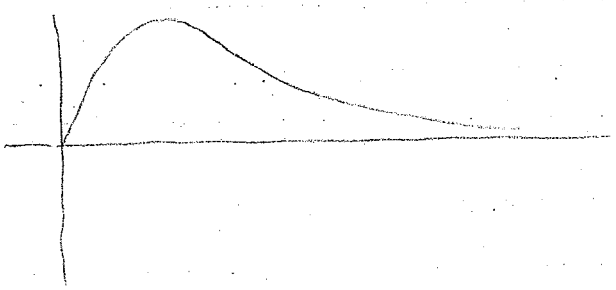
WHEN SYSTEM UNDERGOES RESISTANCE (DAMPING) OSCILLATORY BEHAVIOR DIES OUT

CAN DEFINE A PERIOD

MOTION IS NOT HARMONIC



OR



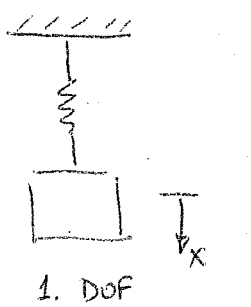
APERIODIC - NONPERIODIC

IF $\exists$ EXTERNAL FORCES CAUSING OSCILLATION - FORCED VIB

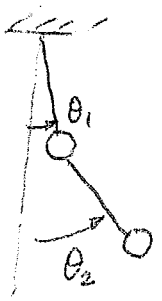
$\nexists$ " " - FREE VIBS

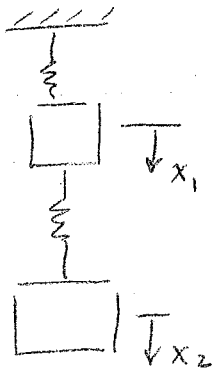
NO. OF DEGREES OF FREEDOM

NO. OF INDEP COORDINATES NEEDED TO FULLY DESCRIBE MOTION

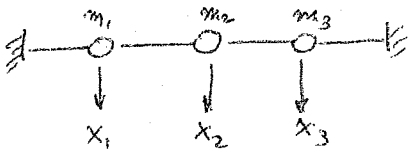


1. DOF





2. DOF



FOR EACH DISCRETE POINT MASS 3 DOF

- IN GENERAL SYSTEMS THAT UNDERGO VIBS ARE COMPLICATED
- CAN BE REPLACED BY SIMPLER SYSTEMS INVOLVING masses, springs, dampers
- THE MODEL IS AN EQUIVALENT SYSTEM
- EACH ELEMENT IS CONSIDERED AS LUMPED
  - SPRING HAS NO MASS, DAMPING
  - MASS IS RIGID - DOES NOT DEFORM, NO DAMPING
  - DAMPER HAS NO MASS, ELASTICITY

SEE PG 12 for car
20 for motorcycle 4DE.

STATEMENT OF VIBRATIONS PROBLEM

1. RECOGNIZE THAT VIBS CAN OCCUR — USE CAR AS EXAMPLE
2. DETERMINE WHAT VIBS ARE SIGNIFICANT
3. FORMULATE A SIMPLE MODEL THAT CAPTURES GIST OF PROBLEM
4. DERIVE GOVERNING EQS.
5. SOLVE EQS
6. INTERPRET RESULTS
7. MAKE APPROPRIATE RECOMENDATIONS

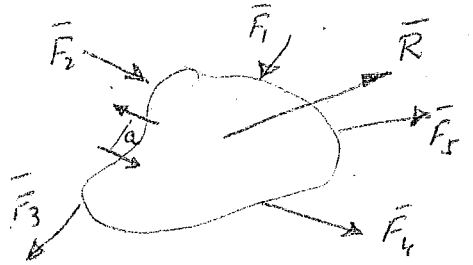
GOVERNING EQS DEPENDENT ON NEWTON'S LAWS

FIRST LAW - BODY REMAINS AT REST OR CONTINUES MOVING A CONST. VEL. UNLESS ACTED UPON BY AN UNBALANCED EXTERNAL FORCE

SECOND LAW - $\bar{F} = m\bar{a}$ $\bar{a}$ is in direction of $\bar{F}$

THIRD LAW - FOR EVERY ACTION IS AN EQUAL & OPPOSITE REACTION

BODY ACTED UPON BY A SET OF FORCES & COUPLES



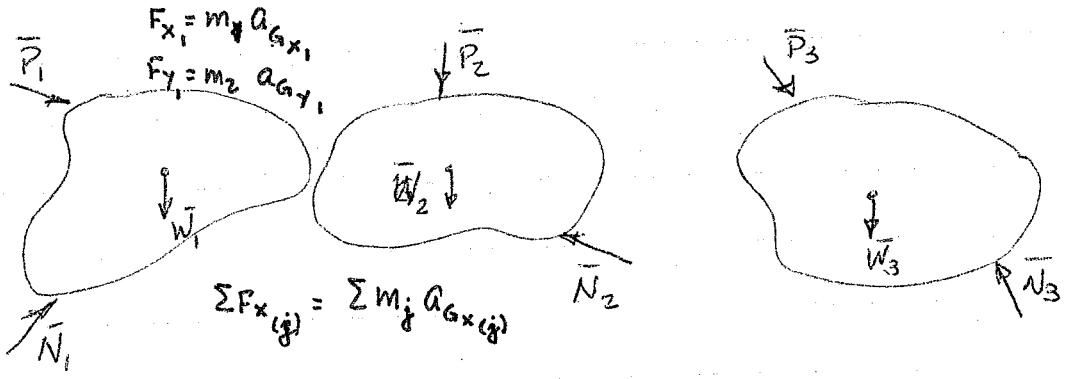
$\Sigma \bar{F}_i = \bar{R} = m\bar{a}_G$

where $\bar{a}_G$ is the accel of the mass center

SESSION #4

$\bar{a}_G$ & $\bar{R}$ have same direction but different lines of action

FOR A SYSTEM OF RIGID BODIES UNDER EXTERNAL FORCES



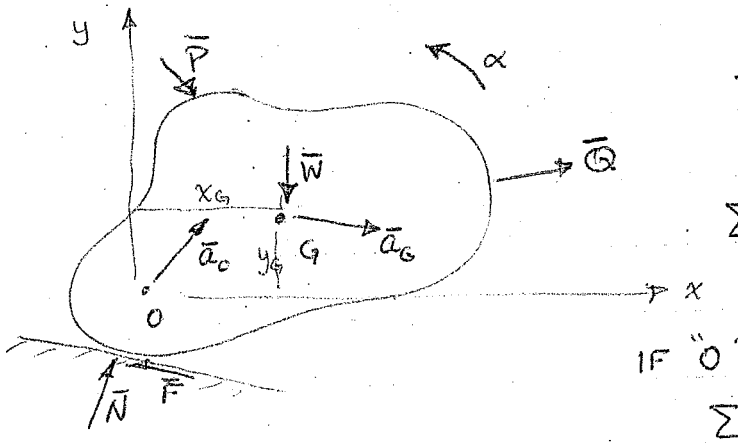
FOR EACH BODY

$$\bar{R} = \sum m_i \bar{a}_G^{(i)}$$

$\bar{R}$ resultant of EXTERNAL FORCES

BODIES MAY OR MAY NOT BE IN CONTACT – CONTACT FORCES CANCEL

PRINCIPLE OF ANGULAR MOTION – PICK AXES THROUGH A PT



$$\Sigma T_O = I_O \alpha + m a_{O_y} x_G - m a_{O_x} y_G$$

$\bar{r}_{G/O} \times m \bar{a}_O$

$$\Sigma T_G = I_G \alpha \quad \text{ABOUT G}$$

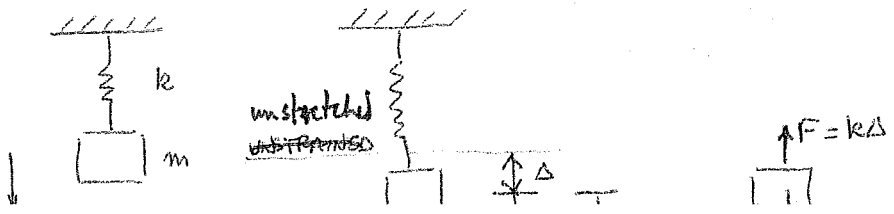
IF "O" IS FIXED $\bar{a}_O = 0$

$$\Sigma T_O = I_O \alpha$$

DO EQUIV SYSTEMS HERE FIRST (LESSON #5).

UNDAMPED FREE VIBS FOR SDOF SYS

SIMPLEST IS AN ELASTIC MEMBER + MASS

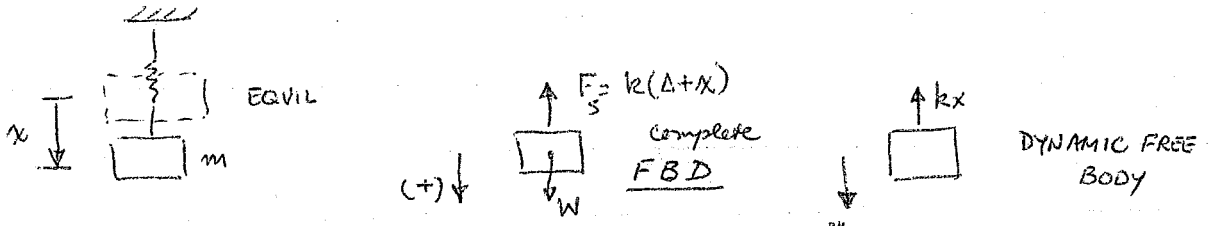


NO EXTERNAL FORCES - FREE VIBS

SDOF - SINCE x IS ONLY VAR. TO DEFINE POSITION OF MASS

BY STATIC EQUILIB $F = k\Delta_{st} = W$ $\quad \Delta_{st} = \frac{W}{k}$ STATIC DISP. δ_{ST} in RAO

- ASSUME NO HORIZONTAL MOTION OF MASS
-
- SUPPOSE WE DISPLACE MASS & LET GO
- MASS MOVES VERTICALLY ABOUT EQUILIB POSITION



$$\Sigma F = -F_s + W = ma = m\ddot{x}$$

$$= -(\Delta+x)k + W = -kx = m\ddot{x}$$

THIS SHOWS VIB TAKES PLACE ABOUT THE EQUIL POSITION y_0

LET $\omega^2 = \frac{k}{m}$ $\qquad \ddot{x} + \omega_n^2 x = 0$ $\qquad \omega_n^2$ in RAO

SOLUTION: $\quad x = A\sin\omega_n t + B\cos\omega_n t$

LET $\quad A = X\cos\phi \qquad B = X\sin\phi$

$$x = X[\cos\phi\sin\omega_n t + \sin\phi\cos\omega_n t] = X\sin(\omega_n t + \phi)$$

$$X = \sqrt{A^2+B^2} \qquad \phi = \tan^{-1} B/A$$

ϕ is the phase angle

ω_n - CIRCULAR FREQ. [rad/sec] $\quad \omega_n t = 2\pi$ 1 cycle $\quad \tau = \frac{2\pi}{\omega_n}$ period

TO SOLVE COMPLETELY NEED IC

$$x(0) = x_0 \implies B = x_0$$

$$\dot{x}(0) = \dot{x}_0 \implies A = \dot{x}_0/\omega$$

$$x = \frac{\dot{x}_0}{\omega_n}\sin\omega_n t + x_0\cos\omega_n t = X\sin(\omega_n t + \phi)$$

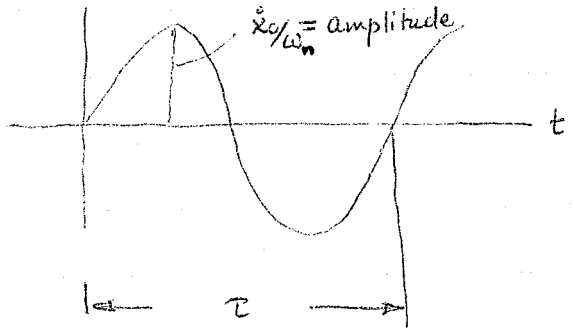
$$X = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} \qquad \phi = \tan^{-1}\frac{x_0}{\dot{x}_0/\omega_n}$$

MOTION IS SINUSOIDAL OR HARMONIC

TIME FOR 1 CYCLE $\omega_n t = 2\pi$ $\Rightarrow \frac{2\pi}{\omega_n} = \tau$ PERIOD

$\tau f = 1 \quad \therefore \quad f = \frac{1}{\tau} = \frac{\omega_n}{2\pi}$ FREQUENCY [cycles/sec = Hz]

ω_n = CIRCULAR FREQ $= 2\pi f = \sqrt{\frac{k}{m}} = \sqrt{\frac{g}{mg/k}} = \sqrt{\frac{g}{W/k}} = \sqrt{\frac{g}{\Delta_{st}}}$



FOR $x_0 = 0 \quad \dot{x}_0 \neq 0$

- amplitude - max. travel about equil position
- ω_n^2 is prop. to g & inversely prop. to static disp

units of ω_n is radians/sec

τ is seconds

f is cycles/sec or Hertz (Hz)

FOR $x = \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + x_0 \cos \omega_n t = X \sin(\omega_n t + \phi) \quad \therefore \quad X = \sqrt{\left(\frac{\dot{x}_0}{\omega_n}\right)^2 + x_0^2} \quad \phi = \tan^{-1} \frac{x_0}{\dot{x}_0/\omega_n}$

$\dot{x} = \dot{x}_0 \cos \omega_n t - x_0 \omega_n \sin \omega_n t = X \omega_n \cos(\omega_n t + \phi) = X \omega_n \sin(\omega_n t + \phi + \pi/2)$

$= \dot{X} \cos(\omega_n t + \phi) \qquad \underline{\dot{X} = X \omega_n}$ amplitude of veloc.

- max velocity is ω_n times the displacement (amplitude)
- this shows that the velocity is out of phase with the displ by $\pi/2$

$\ddot{x} = -\dot{x}_0 \omega_n \sin \omega_n t - x_0 \omega_n^2 \cos \omega_n t = -X \omega_n^2 \sin(\omega_n t + \phi)$

this is governing DE $= -\omega_n^2 x$

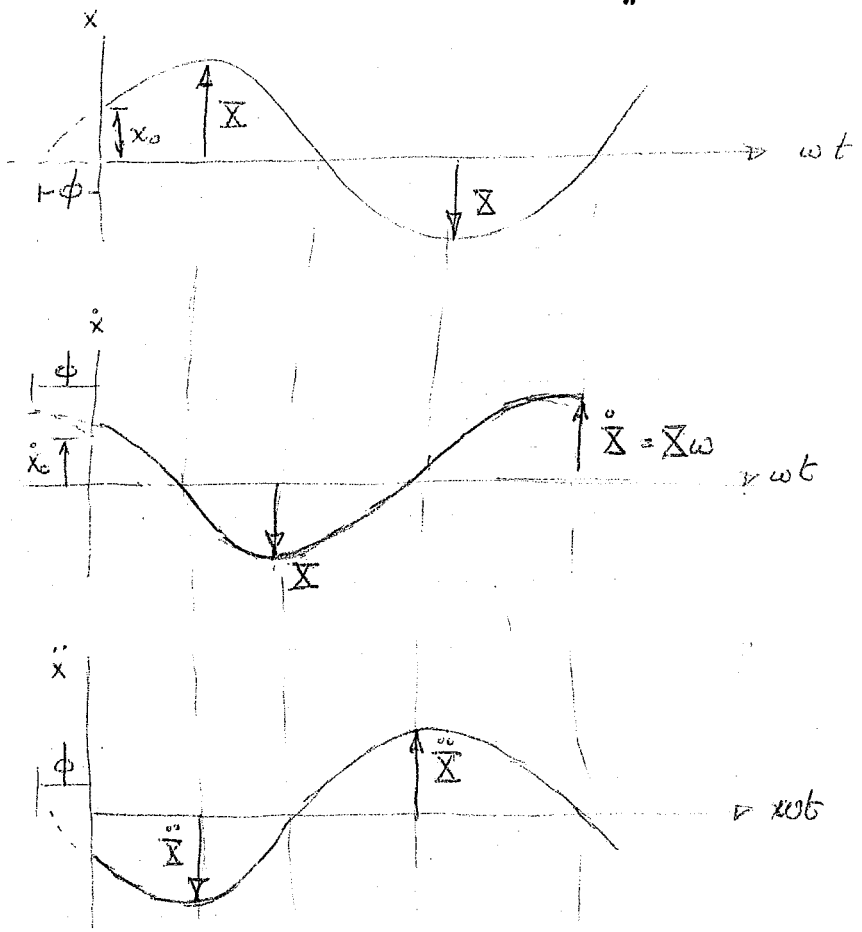
$= -\ddot{X} \sin(\omega_n t + \phi) \qquad \underline{\ddot{X} = -X \omega_n^2 = \dot{X} \omega_n}$ Amplitude of accel

- accel. is negative of displacement curve mult. by ω_n^2

- since ω_n is const better to plot x vs. $\omega_n t$
- ϕ is the shift of the sine curve along horiz axis

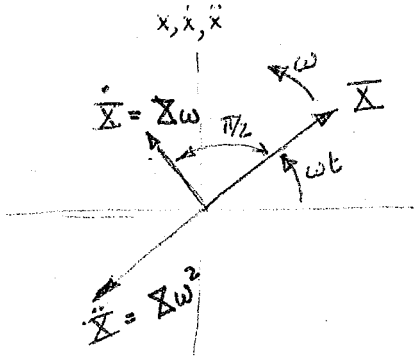
$$\omega_n t_o + \phi = 0$$

$$t_o = -\phi/\omega_n$$ defines how much of shift.



if $\dot{x}=0$

GRAPHICAL REPRESENTATION PG. #23 (PG 30 RAO)

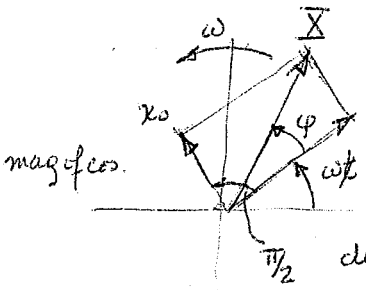


AS VECTOR $\bar{X}$ moves with angular velocity ω

$\dot{\bar{X}}$ ~~which~~ leads $\bar{X}$ by $\pi/2$

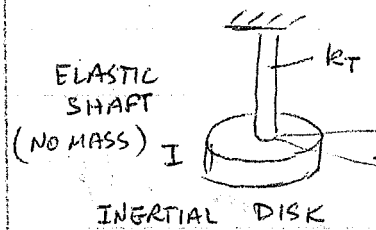
$\ddot{\bar{X}}$ which is $-\bar{X}$

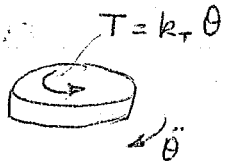
LENGTH OF VECTOR = AMPLITUDE OF $X, \dot{X}, \ddot{X}$



GRAPH. REP. OF EACH PART OF SOLUTION x

TORSIONAL VIBRATIONS





J polar moment of inertia

$$T = \frac{JG\theta}{L} = k_T\theta \qquad k_T = \frac{JG}{L}$$

I $\left[lb\text{-}in\text{-}sec^2\right]$ $\left[kg\text{-}m^2\right]$ mass moment of inertia

$$I\ddot{\theta} = -k_T\theta$$

I ⇒ J_o in Rao

$$\theta = A\sin\omega_n t + B\cos\omega_n t$$

$$= \textcircled{X}\sin(\omega_n t + \phi)$$

$$\omega_n = \sqrt{\frac{k_T}{I}}$$

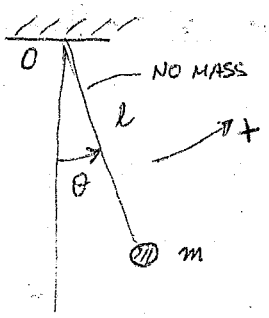
$$\textcircled{X} = \sqrt{\theta_0^2 + \frac{\Omega_0^2}{\omega_n^2}}$$

here $\theta(t=0) = \theta_0$

$\dot{\theta}(t=0) = \Omega_0$

REPLACE θ BY x I BY m k_T by k SAME AS BEFORE

ANGULAR OSCILLATIONS



$$I_o = I_{CG}(\text{of mass}) + m\,d_\perp^2 = md_\perp^2$$

$= 0$ since mass is assumed at a pt.



$$I_o\ddot{\theta} = -mg\,l\sin\theta = \Sigma T_o$$

$I_o = ml^2$ if dimensional character is small wrt l

- if we look at this

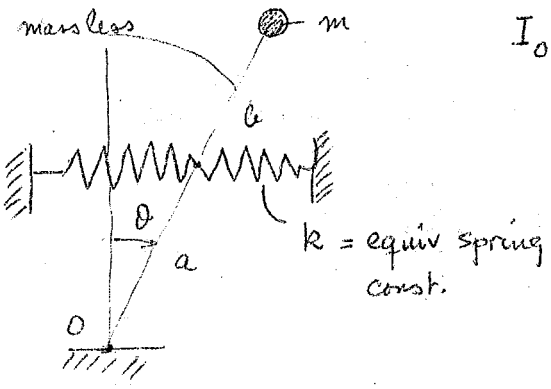
$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \qquad 2^{nd} \text{ order } \underline{\text{nonlin}} \text{ dif eq.}$$

assume θ is small. $\theta \gg \theta^3/3!$ in sine exp.

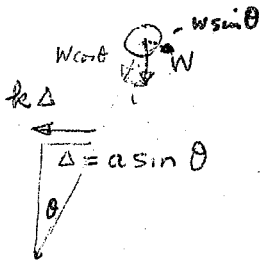
$$ml^2\ddot{\theta} + mgl\left[\theta - \theta^3/3! + \cdots\right] = 0$$

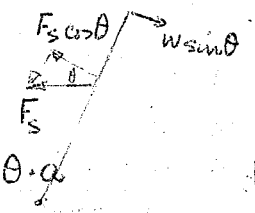
$$ml^2\ddot{\theta} + mgl\theta = 0 \Rightarrow \ddot{\theta} + g/l\,\theta = 0 \qquad \text{let } \omega = \sqrt{g/l}$$

CONSIDER INVERTED PENDULUM - ASSUME SPRING UNSTRAINED WHEN PEND. IS VERTICAL



$$I_O = ml^2$$





$$I_O\ddot{\theta} = \sum T_O$$

$$= +W\sin\theta \cdot l - F_S\cos\theta \cdot a \qquad F_S = k\Delta = ka\sin\theta$$

$$= -ka^2\cos\theta\sin\theta + Wl\sin\theta$$

for <u>small angles</u> $\cos\theta \approx 1$

$$I_O\ddot{\theta} = -ka^2\theta + Wl\theta = \theta\left[Wl - ka^2\right]$$

or

$$\ddot{\theta} + \left[\frac{ka^2 - Wl}{I_O}\right]\theta = 0$$

if $ka^2 - Wl > 0$

$$\ddot{\theta} + \left[\frac{ka^2 - Wl}{ml^2}\right]\theta = 0 \qquad \omega = \left[\frac{ka^2 - Wl}{ml^2}\right]^{1/2}$$

$\theta = C\sin(\omega t + \varphi)$ STABLE OSCIL.

if $ka^2 = Wl$ $\theta = C_1 t + C_2$ CONSTANT-VELOCITY EQUIL.

if $ka^2 - Wl < 0$ $\theta = C_1\sinh\gamma t + C_2\cosh\gamma t$ $\gamma = \left[\frac{Wl - ka^2}{ml^2}\right]^{1/2}$

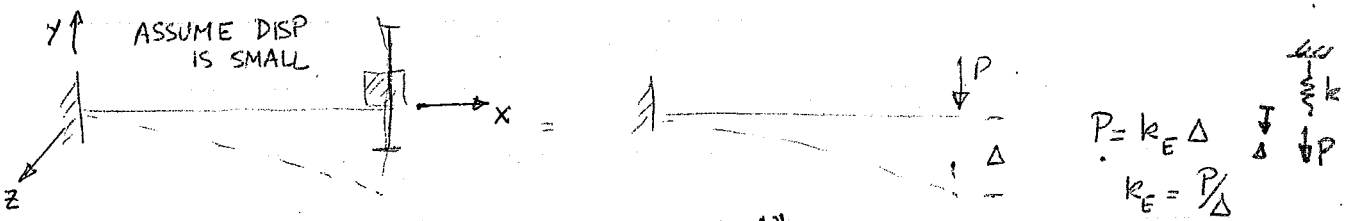
torque (weight) > torque due to spring UNSTABLE - NONOSCIL

SESSION #5

HAND IN HW # 2

- EQUIVALENT SPRING

- REPLACE ELASTIC MEMBER BY AN EQUIV. SPRING
- REDUCE TO SPRING-MASS SYSTEM.
- EXPRESS FORCE OR MOMENT FOR UNIT DISP AT LOCATION OF MASS OR INERTIAL ELEMENT

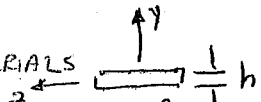


$P = k_E \Delta$

$k_E = P/\Delta$

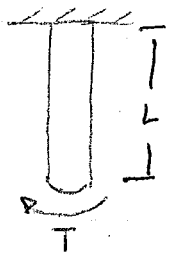
$\Delta = \frac{Pl^3}{3EI_z}$

STRENGTH OF MATERIALS



$I_z = \frac{bh^3}{12}$

$k_E = \frac{3EI}{l^3} =$



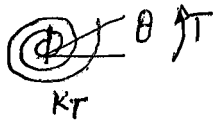
$\theta = Tl/GJ$

$T = k_t \theta$

$k_T = GJ/l$

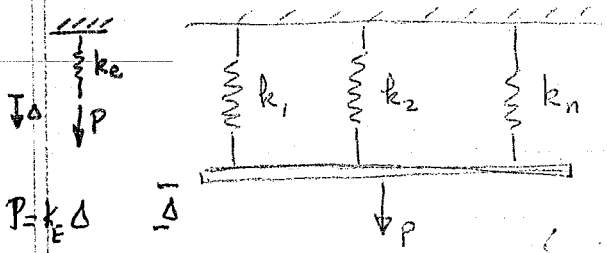
$k_T = T/\theta$

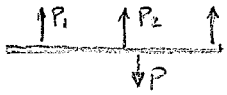
$\frac{lb}{in^2} \cdot \frac{in^4}{in} = lb\cdot in$



- IN A SYSTEM W/ MANY SPRINGS WANT TO FIND EQUIV. SPRING

SPRINGS IN PARALLEL





$\Sigma F = 0 \Rightarrow \Sigma P_i = P$

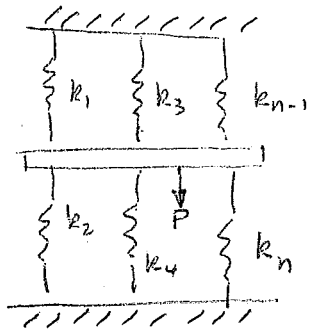
Assume bar remains horiz such that Δ for each is SAME

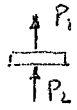
$k_1\Delta = P_1$, $k_2\Delta = P_2$ ... $k_n\Delta = P_n$

$P_1 + P_2 + \ldots + P_n = P = \Delta(k_1 + k_2 + \ldots + k_n) = \Delta k_E$

$$k_E = \sum_{i=1}^{n} k_i$$

IN PARALLEL NOTE ALL SPRINGS UNDERGOE SAME DEFLECTION



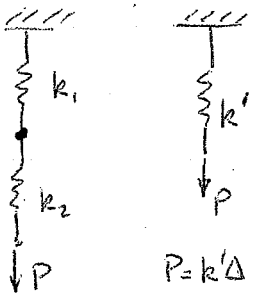


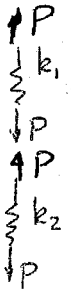
$$P_1 + P_2 = k_1\Delta + k_2\Delta = (k_1 + k_2)\Delta$$

now $$\sum P_i = P = \sum k_i \Delta = \Delta \sum k_i = k_E \Delta$$

$$k_E = \sum k_i$$

SPRINGS IN SERIES





$P = k_1\Delta_1$

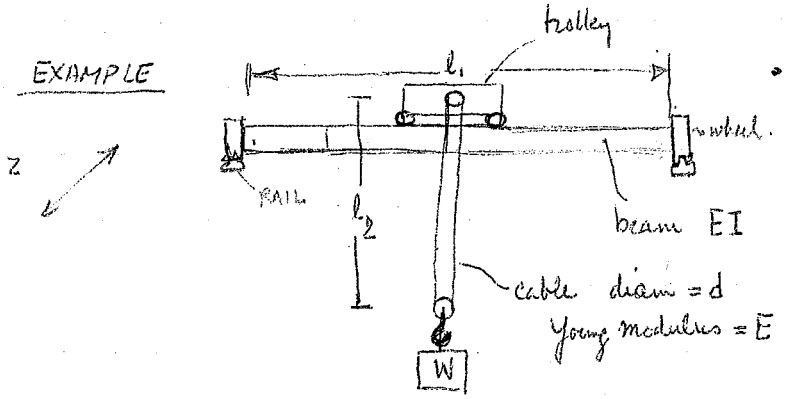
$P = k_2\Delta_2$

$$\Delta = \Delta_1 + \Delta_2 = \frac{P}{k_1} + \frac{P}{k_2} = \frac{P}{k'}$$

$$\sum \frac{1}{k_i} = \frac{1}{k'}$$

NOTE ALL SPRINGS UNDERGO SAME LOAD

EXAMPLE



- weight W is being lifted overhead by a traveling crane
- mass of trolley, cables, hook small

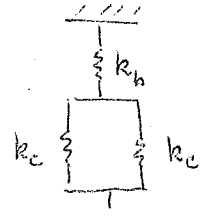
(1) Assume trolley acts as a point load at center

$$\Delta = \frac{Pl^3}{48EI} = \frac{P}{k_b} \qquad k_b = \frac{48EI}{l_b^3}$$

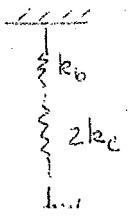
(2) Each cable holds $\frac{W}{2} = P'$ $\qquad \Delta = \frac{P' l_2}{AE} = \frac{P'}{k_c} \Rightarrow k_c = \frac{AE}{l_2} = \frac{\pi d^2 E}{4 l_2}$

since same disp. cables $\Rightarrow$ eqiv spring of $2k_c$

$$\omega_n = \sqrt{\frac{k_E}{m}}$$



=



$$\frac{1}{k_E} = \frac{1}{k_b} + \frac{1}{2k_c}$$

$$\frac{1}{k_E} = \frac{l_b^3}{48EI} + \frac{l_2}{2AE}$$

ENERGY METHOD FOR CONSERVATIVE SYSTEMS

- DE. CAN BE OBTAINED BY USE OF CONSERVATION OF ENERGY

- FOR FREE VIB. SYS. NO ENERGY IS REMOVED $\Rightarrow$ TOTAL ENERGY = CONS

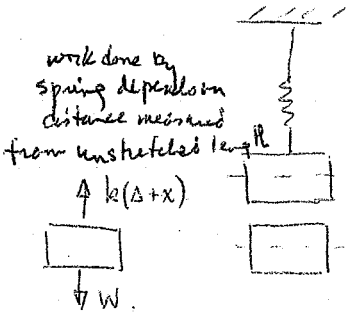
T = KE V = PE

- $$\frac{d}{dt}(T+V) = 0$$

- FOR SMALL VIBS $T = T(\text{veloc})$ $V = V(\text{displ.})$

FOR MASS – SPRING OSCILLATOR work done $= \int_0^x [W - k(\Delta + x')]dx'$

$V = (\text{potential}) = -\text{work} = -\int_0^x [\quad]dx'$



$T = \frac{1}{2}m\dot{x}^2$ $\dot{x}$ = veloc. of mass.

$$V = \int_x^0 [W - k(\Delta + x')]\,dx' = -\int_x^0 kx'dx' = \frac{kx^2}{2}$$

V: work done to move system from displaced config back to equil. posit

$$T + V = \frac{1}{2}m\dot{x}^2 + \frac{kx^2}{2} = \text{const.} \Rightarrow \frac{d}{dt}(T+V) = \dot{x}(m\ddot{x} + kx) = 0$$

$\dot{x} = 0 \Rightarrow$ no motion (trivial solution)

$m\ddot{x} + kx = 0 \Rightarrow$ non trivial sol.

Rayleigh states that $T_{max} = V_{max}$: if solution is harmonic $x = A\sin(\omega t + \phi)$, $\dot{x} = A\omega\cos(\omega t + \phi)$

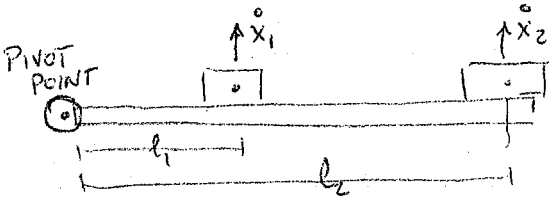
$T_{max} = \frac{1}{2}mA^2\omega^2\cos^2(\quad) \Rightarrow T_{max} = \frac{1}{2}mA^2\omega^2$

$V = \frac{1}{2}kA^2\sin^2(\quad) \Rightarrow V_{max} = \frac{1}{2}kA^2$

$\Rightarrow \omega^2 = k/m$ or $\omega = \sqrt{k/m}$

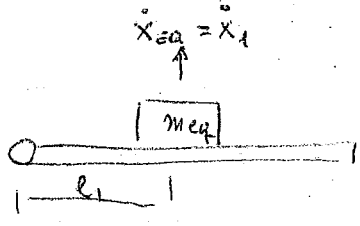
using equivalent K. energies

EQUIVALENT MASSES

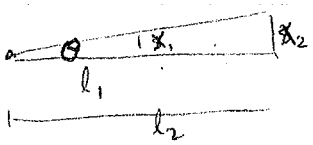


RIGID LEVER

=



FOR SMALL ANGULAR DISPL.



$x_2 = l_2\theta$

$x_1 = l_1\theta$

$\theta = \frac{x_2}{l_2} = \frac{x_1}{l_1}$

$$x_2 = \frac{x_1}{l_1}l_2 \Rightarrow \dot{x}_2 = \dot{x}_1\frac{l_2}{l_1}$$

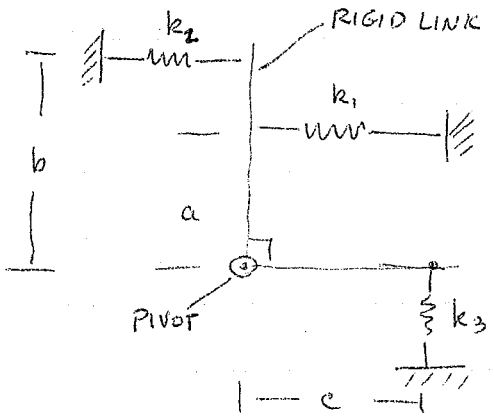
EQUATE KE'S

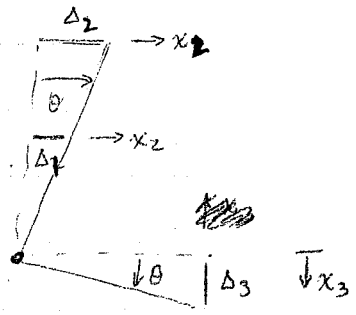
$$\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 = \frac{1}{2} m_{eq} \dot{x}_{EQ}^2 = \frac{1}{2} m_{eq} \dot{x}_1^2$$

$$\frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \left[\dot{x}_1^2 \frac{l_2^2}{l_1^2} \right] = \frac{1}{2} m_{eq} \dot{x}_1^2$$

$$\boxed{m_1 + m_2 \frac{l_2^2}{l_1^2} = m_{EQ}}$$

EXAMPLE # 2 — equating PE's w/ k_{eq} at x_1





$\Delta_1 = a\theta$

$\Delta_2 = b\theta \qquad \theta = \frac{\Delta_2}{b} = \frac{\Delta_1}{a} \Rightarrow \Delta_2 = \frac{b}{a}\Delta_1$

$\Delta_3 = c\theta \qquad \theta = \frac{\Delta_3}{c} = \frac{\Delta_1}{a} \Rightarrow \Delta_3 = \frac{c}{a}\Delta_1$

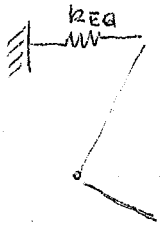
$$U = \frac{1}{2} k_1 \Delta_1^2 + \frac{1}{2} k_2 \Delta_2^2 + \frac{1}{2} k_3 (\Delta_3)^3 = \frac{1}{2} k_{EQ} \Delta_{EQ}^2$$

let $\Delta_{EQ} = \Delta_1$

$$U = \frac{1}{2} k_{EQ} \Delta_1^2 = \frac{1}{2} k_1 \Delta_1^2 + \frac{1}{2} k_2 \left(\frac{b^2}{a^2} \right) \Delta_1^2 + \frac{1}{2} k_3 \left(\frac{c^2}{a^2} \right) \Delta_1^2$$

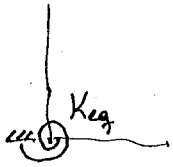
$$= \frac{1}{2} \left[k_1 + k_2 \frac{b^2}{a^2} + k_3 \frac{c^2}{a^2} \right] \Delta_1^2$$

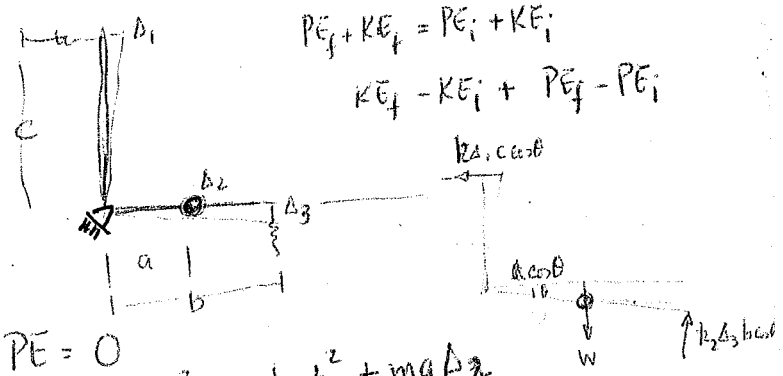
$$k_{EQ} = k_1 + k_2 \frac{b^2}{a^2} + k_3 \frac{c^2}{a^2}$$



EQUATE PE's.

$$U = \frac{1}{2} \left(k_1 a^2 + k_2 b^2 + k_3 c^2 \right) \theta^2 = \frac{1}{2} K_{eq} \theta^2$$





$PE_f + KE_f = PE_i + KE_i$

$KE_f - KE_i + PE_f - PE_i$

$PE = 0$

$PE_f = \frac{1}{2}k_1\Delta_1^2 + \frac{1}{2}k_2\Delta_3^2 + mg\Delta_2$

$KE_i = 0$

$KE_f = \frac{1}{2}m\dot{\Delta}_2^2$

$I\ddot{\theta} = -k_1c^2\sin 2\theta + Wa\cos\theta - k_2b^2\frac{\sin 2\theta}{2}$

$ma^2\ddot{\theta} + (k_1c^2 + k_2b^2)\theta - Wa = 0$

$\left(\frac{1}{2}m\dot{\Delta}_2^2\right) + \left(\frac{1}{2}k_1\Delta_1^2 + \frac{1}{2}k_2\Delta_2^2 + mg\Delta_2\right)$

$\Delta_2 = -a\sin\theta$ $\quad \Delta_1 = c\sin\theta$ $\quad \Delta_2 = -a\sin\theta$ $\quad \dot{\Delta}_2 = -a\cos\theta\,\dot{\theta}$

$\dot{\Delta}_2 = -a\cos\theta\,\dot{\theta}$ $\quad \Delta_2 = -a\sin\theta$

$\frac{1}{2}ma^2\cos^2\theta\,\dot{\theta}^2 + \left(\frac{1}{2}k_1c^2\sin^2\theta + \frac{1}{2}k_2b^2\sin^2\theta - mg\Delta_2\sin\theta\right)$

$\frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}\left[(k_1c^2 + k_2b^2)\sin^2\theta\right] - mgb\sin\theta\Big]$

$-2\cos\theta\sin\theta\,\dot{\theta}$ $\quad \frac{1}{2}($ $\quad)2\sin\theta\cos\theta\,\dot{\theta} - mga\cos\theta\,\dot{\theta}$

$2\cos^2\theta\,\dot{\theta}\ddot{\theta}$

$\frac{1}{2}\cdot 2\cdot\frac{1}{2}ma^2\dot{\theta}\ddot{\theta} + 2\cdot\frac{1}{2}(k_1c^2 + k_2b^2)\sin\theta\cos\theta\,\dot{\theta} - mga\cos\theta\,\dot{\theta}$

$\dot{\theta}\left[ma^2\ddot{\theta} + (k_1c^2 + k_2b^2)\theta - mga\right] = 0$

must replace lecture 5

see if it can be taped Friday - call Mercy

Topics

start showing Cochin document keq for beam w/ diff. BCs

equiv mass of beams $J_{mass} = J_{polar} \cdot \rho \cdot$ thickness

$\bar{J} = \int r^2 dA \quad J_m = \int r^2 dm = \int r^2 \rho dV = \int r^2 dA \cdot \rho \cdot L$

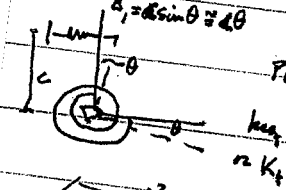


keq for 

meq for that using 

work out in detail → $KE_{DS} = KE_{eq}$

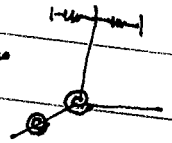
clarify that $\omega_n \neq \sqrt{\frac{k_{eq}}{m}}$ or $\sqrt{\frac{m_{eq}}{k}}$ and must use eq of motion



$PE_{DS} = PE_{eq}$

$k_{eq} = k\left(\frac{a}{b}\right)^2 + K_t$ also

or $K_{t_{eq}} = ka^2 + K_t$



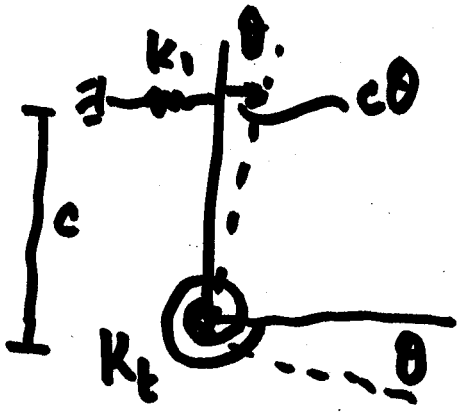


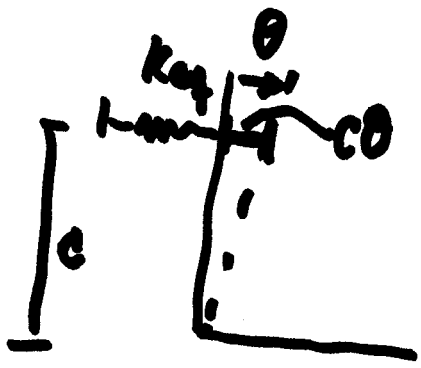
m? $k_{eq} = \frac{1}{4}k$ $\tau = 0.4$ $\tau = \frac{1}{f}$ $\tau\omega_n = 2\pi$ add $m = 30$ kg $\tau = .5$ sec

Proble

$\omega_n' = \sqrt{\frac{k}{m+30}}$ $k_{eq} = 13160$

$m = 58.3$ kg





$$PE_{\text{o.s.}} = \frac{1}{2} k_1 c^2 \theta^2 + \frac{1}{2} K_t \theta^2$$

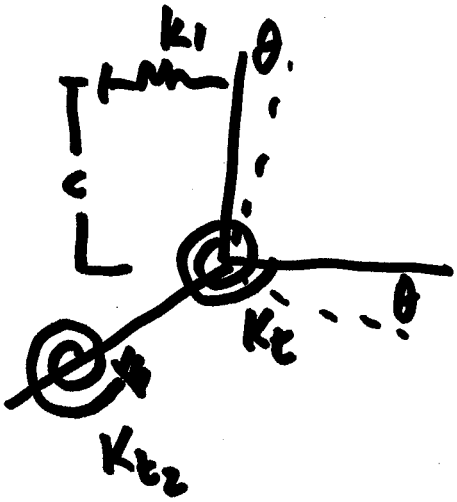
$$PE_{equiv} = \frac{1}{2} k_{eq} (c\theta)^2$$

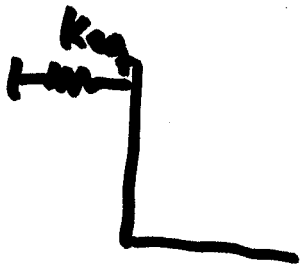
$$PE_{\text{o.s.}} = PE_{equiv}$$

$$k_1 c^2 + K_t = k_{eq} c^2$$

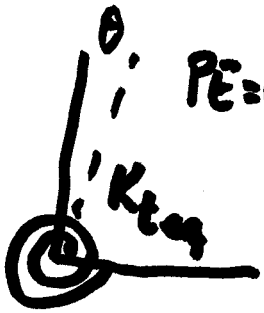
$$k_{eq} = k_1 + K_t/c^2$$

$$T = K_t \theta \quad \frac{N\text{-}m}{rad}$$



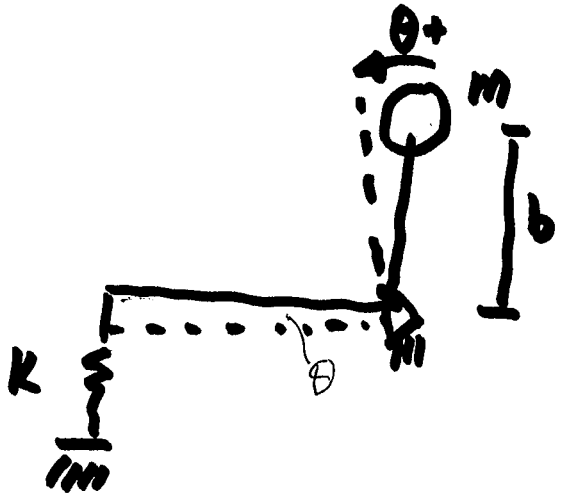


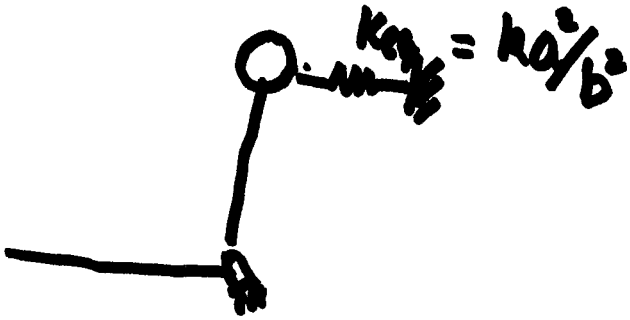
$$k_{eq} = k_1 + \frac{K_t}{c^2} + \frac{K_{t_2}}{c^2}$$

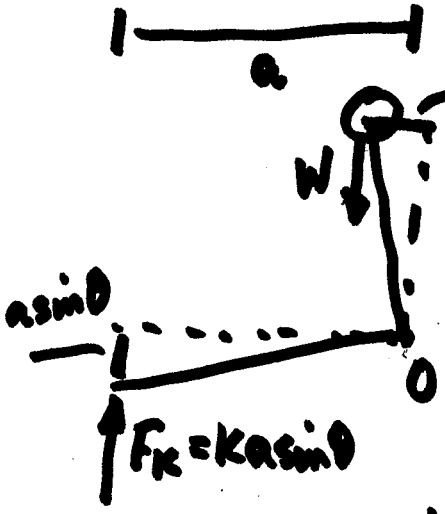


$$K_{t_{eq}} = K_t + K_{t_2} + k_1 c^2$$









$\Sigma T_o = I_o\ddot{\theta}$ $\quad I_o = I_{cm}^{0} + mb^2$

$Wb\sin\theta - F_K \cdot a\cos\theta = mb^2\ddot{\theta}$

$Wb\theta - ka^2\theta = mb^2\ddot{\theta}$

$mb^2\ddot{\theta} + (ka^2 - Wb)\theta = 0$ $\quad \omega_n = \sqrt{\dfrac{ka^2 - Wb}{mb^2}}$

$\ddot{\theta} + \omega_n^2\theta = 0$

$\omega_n \overset{?}{=} \sqrt{\dfrac{K_{eq}}{m}} = \sqrt{\dfrac{ka^2}{mb^2}}$ ✗



$A = 2\text{cm}$ $\quad \tau_n = 3\text{sec}$

FIND: V_{max}, a_{max}

$x(t) = A\sin(\omega_n t + \phi)$

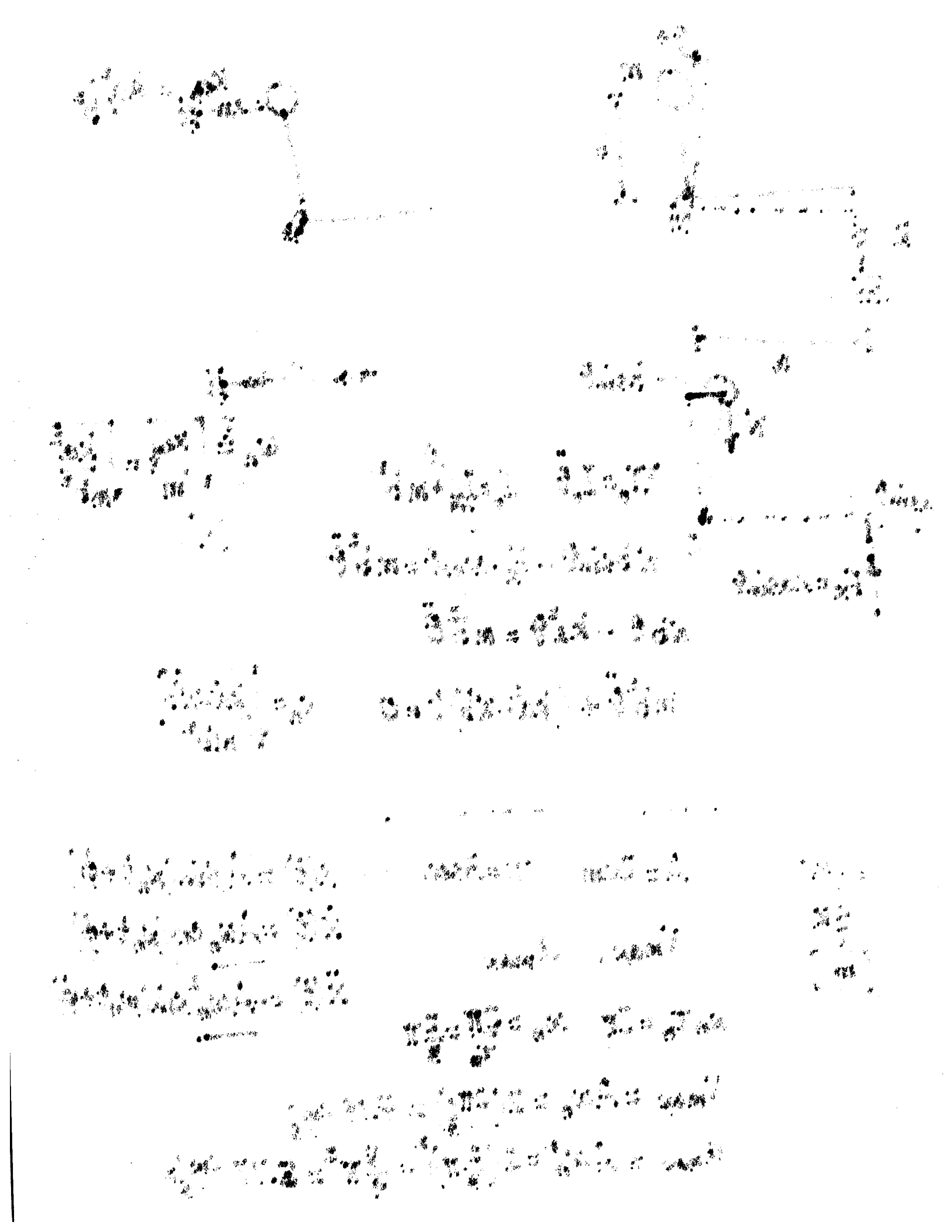
$\dot{x}(t) = A\omega_n\cos(\omega_n t + \phi)$

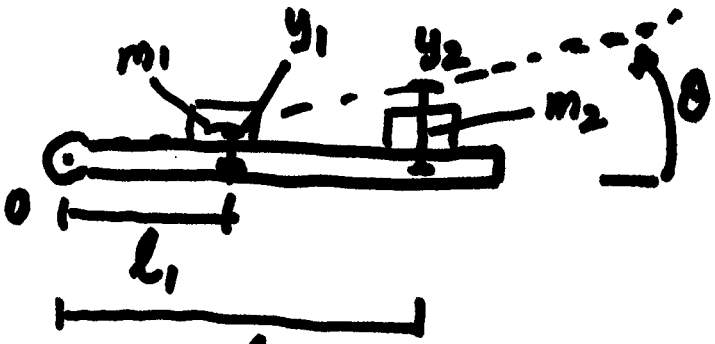
$\ddot{x}(t) = -A\omega_n^2\sin(\omega_n t + \phi)$

$\omega_n \tau_n = 2\pi$ $\quad \omega_n = \dfrac{2\pi}{\tau_n} = \dfrac{2}{3}\pi \ \dfrac{rad}{s}$

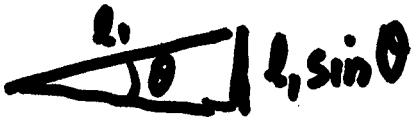
$V_{max} = A\omega_n = 2(2\pi/3) = 4.19 \ cm/s$

$a_{max} = A\omega_n^2 = 2\left(\dfrac{2}{3}\pi\right)^2 = \dfrac{8}{9}\pi^2 = 8.77 \ cm/s^2$

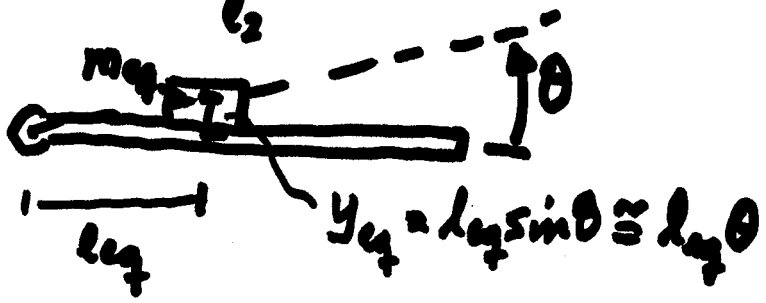




$y_1 = l_1 \sin\theta \cong l_1\theta$



$y_2 = l_2 \sin\theta \cong l_2\theta$



$KE_{o.s.} = KE_{equiv}$

$V_1 = \dot{y}_1 = l_1\dot{\theta}$

$V_2 = \dot{y}_2 = l_2\dot{\theta}$

$$KE_{o.s.} = \frac{1}{2}m_1V_1^2 + \frac{1}{2}m_2V_2^2$$

$$= \frac{1}{2}m_1(l_1\dot{\theta})^2 + \frac{1}{2}m_2(l_2\dot{\theta})^2$$

$$KE_{equiv} = \frac{1}{2}m_{eq}V_{eq}^2 = \frac{1}{2}m_{eq}(l_{eq}\dot{\theta})^2 \qquad V_{eq} = \dot{y}_{eq} = l_{eq}\dot{\theta}$$

$$\frac{1}{2}m_{eq}(l_{eq}\dot{\theta})^2 = \frac{1}{2}m_1(l_1\dot{\theta})^2 + \frac{1}{2}m_2(l_2\dot{\theta})^2$$

$$m_{eq}l_{eq}^2 = m_1l_1^2 + m_2l_2^2$$

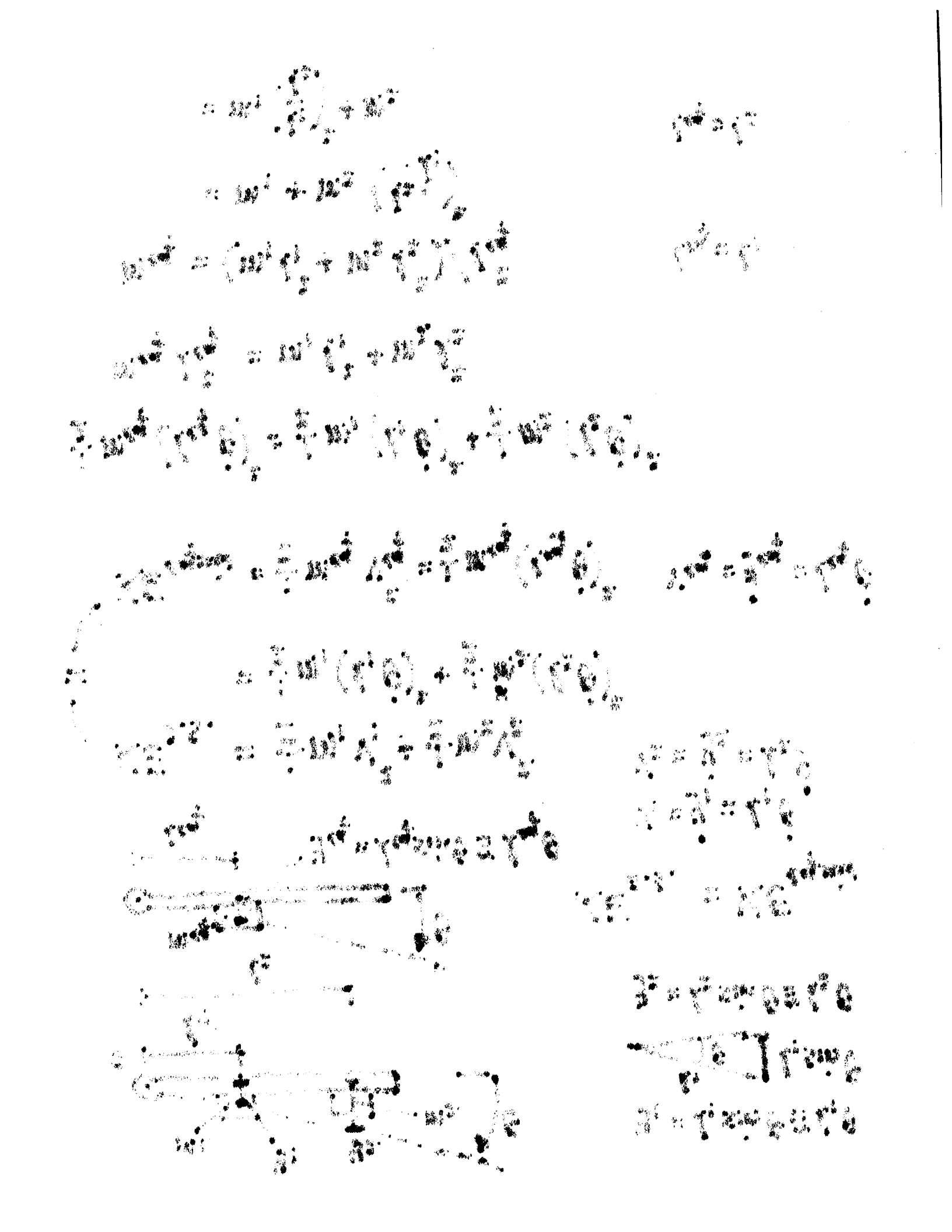
$$m_{eq} = (m_1l_1^2 + m_2l_2^2)/l_{eq}^2$$

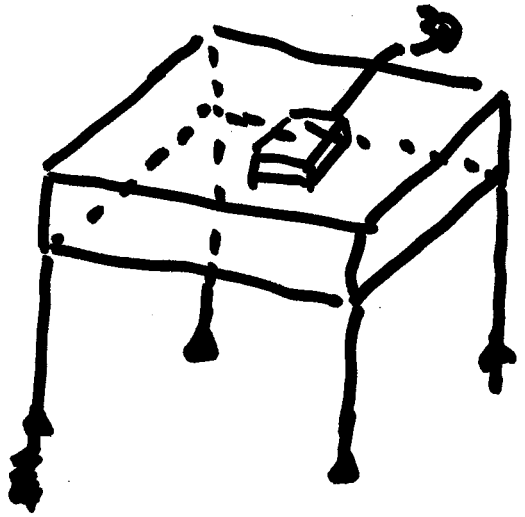
$l_{eq} = l_1$

$$= m_1 + m_2(l_2/l_1)^2$$

$l_{eq} = l_2$

$$= m_1\left(\frac{l_1}{l_2}\right)^2 + m_2$$





$? = k_{eq} = 4k_{leg}$

$m = ?$

$\tau_n = 0.4\,sec \Rightarrow \tau_n \omega_n = 2\pi$

$\omega_n = \frac{2\pi}{0.4\,sec} = 5\pi = \sqrt{\frac{k}{m}}$

$\tau_n' = 0.5\,sec \quad \omega_n' = \frac{2\pi}{\tau_n'} = 4\pi = \sqrt{\frac{k}{m+30}}$

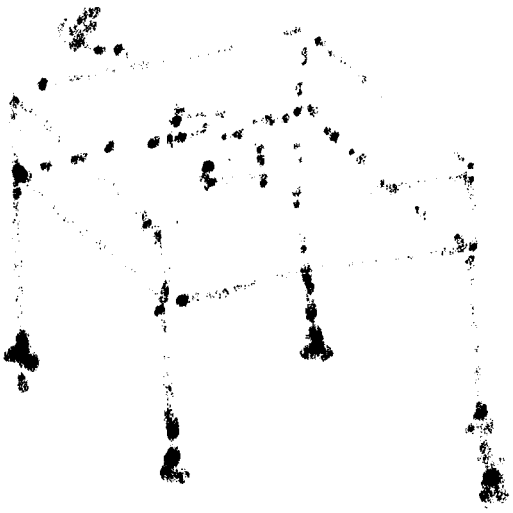
$\frac{m}{k} = \frac{1}{25\pi^2} \qquad \frac{m+30}{k} = \frac{1}{16\pi^2} = \frac{m}{k} + \frac{30}{k}$

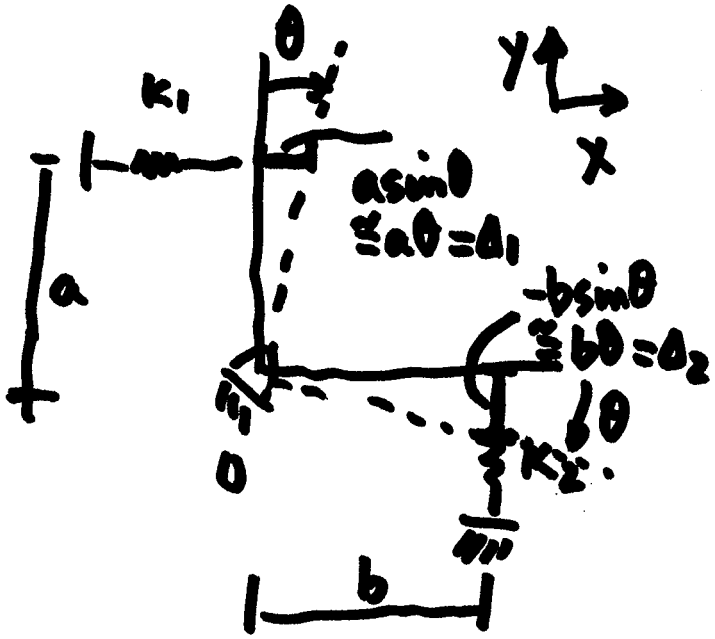
$\frac{30}{k} = \left[\frac{1}{16\pi^2} - \frac{1}{25\pi^2}\right]$

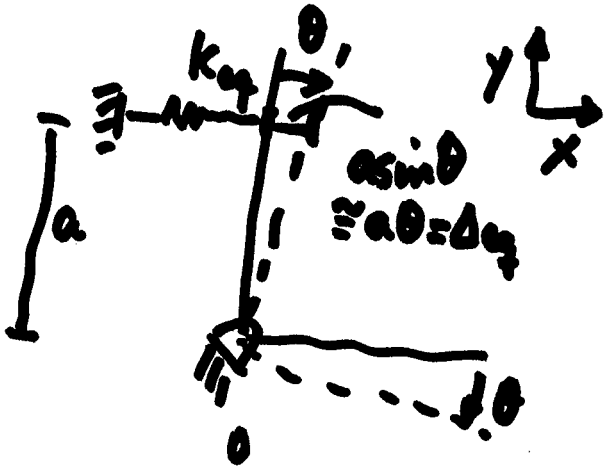
$k = 30\left[\frac{1}{16\pi^2} - \frac{1}{25\pi^2}\right]^{-1} = 13160\,\frac{N}{m}$

$k_{leg} = \frac{k}{4} = 3290\,\frac{N}{m}$

TO FIND m: $\omega_1^2 = 25\pi^2 = \frac{k}{m} \qquad m = \frac{k}{25\pi^2} = 53.3\,kg$







$$PE_{o.s.} = \frac{1}{2}k_1\Delta_1^2 + \frac{1}{2}k_2\Delta_2^2$$

$$= \frac{1}{2}k_1(a\theta)^2 + \frac{1}{2}k_2(-b\theta)^2$$

$$PE_{equiv} = \frac{1}{2}k_{eq}\Delta_{eq}^2$$

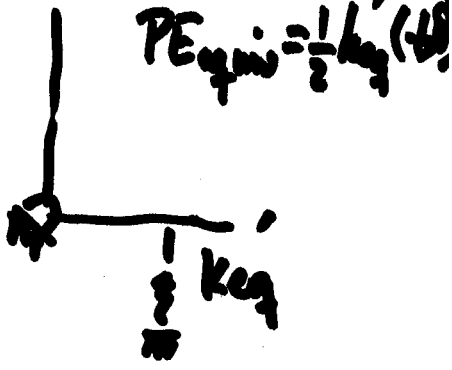
$$= \frac{1}{2}k_{eq}(a\theta)^2$$

$$PE_{o.s.} = PE_{equiv}$$

$$\frac{1}{2}k_1a^2\theta^2 + \frac{1}{2}k_2b^2\theta^2 = \frac{1}{2}k_{eq}a^2\theta^2$$

$$\frac{k_1a^2 + k_2b^2}{a^2} = k_{eq}$$

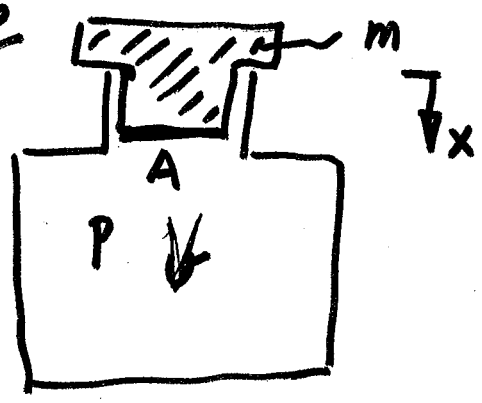
$$\frac{k_1a^2 + k_2b^2}{b^2} = k'_{eq}$$



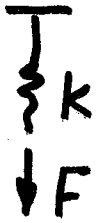


9/15/04

1.16



$$pV^{\gamma} = \text{constant}$$



$$F = kx$$

$$\Delta V = -A\,\Delta x$$

$$\Delta F = k_{eq}\,\Delta x$$

$$\Delta F = k_{eq} \cdot \frac{\Delta V}{-A}$$

$$\Delta F = A\,\Delta p$$

$$A\,\Delta p = k_{eq}\,\frac{\Delta V}{-A}$$

$$\Delta p \cdot V^{\gamma} + p\gamma V^{\gamma-1}\Delta V = 0$$

$$\Delta p = \left[-\frac{V}{\gamma p}\right]\Delta V$$

$$k_{eq} = A^2\frac{\gamma p}{V}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

$Q = -Av$

$\frac{dF}{dt} = k_{eq}\,v$

$\frac{dF}{dt} = k_{eq} \cdot \frac{Q}{-A}$

$A\frac{\Delta p}{\Delta t} = k_{eq} \cdot \frac{Q}{-A}$

$\frac{\Delta p}{\Delta t} = \frac{k_{eq}}{-A^2}\,Q$

$Q = \frac{-A^2}{k_{eq}}\frac{dp}{dt}$



$Q = p$

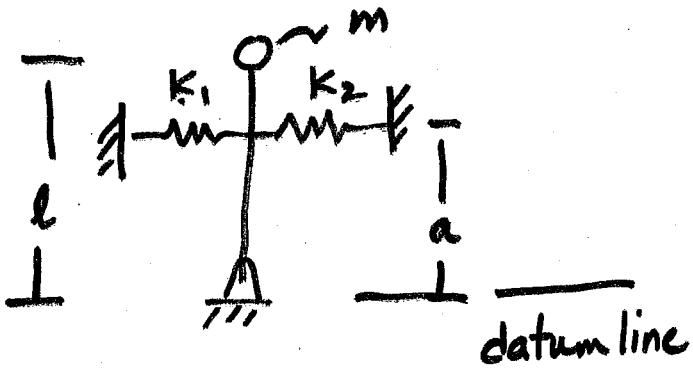
Energy Method

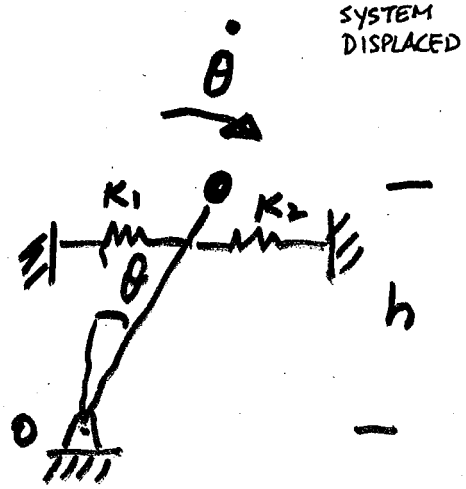
Conservative systems (no damping)

$$(KE + PE) = \text{constant}$$

$$\frac{d}{dt}(KE + PE) = 0 \Rightarrow \text{Eqn. of motion}$$

Inverted Pendulum





$$KE_i = 0$$

$$PE_i = PE_{\text{springs}} + PE_{\text{mass}}$$

O unstretched

$$= mgl$$

$$KE_f = \frac{1}{2}m(l\dot{\theta})^2 = \frac{1}{2}I_0\dot{\theta}^2$$

$$PE_f = PE_{\text{springs}} + PE_{\text{mass}}$$

$$= \frac{1}{2}k_1(a\sin\theta)^2 + \frac{1}{2}k_2(-a\sin\theta)^2 + mg \cdot l\cos\theta$$

$$PE_{\text{of mass}} = mgh$$

$$PE_{\text{of springs}} = \frac{1}{2}kx^2$$

$$(KE_f - KE_i) + (PE_f - PE_i) = 0$$

$$KE_f + PE_f = KE_i + PE_i = \text{const}$$

$$\left(\frac{1}{2}ml^2\dot{\theta}^2 + 0\right) + \frac{1}{2}(k_1+k_2)a^2\sin^2\theta + mgl\cos\theta - mgl = 0$$

$$0 = \frac{1}{2} m l^2 \left[2\dot{\theta}\ddot{\theta}\right] + \frac{1}{2}(k_1+k_2)a^2\left[2\sin\theta\cos\theta\cdot\dot{\theta}\right] + mgl\left[\sin\theta\cdot\dot{\theta}\right]$$

$$0 = \dot{\theta}\left[ml^2\ddot{\theta} + (k_1+k_2)a^2\sin\theta\cos\theta - mgl\sin\theta\right]$$

$\dot{\theta} \neq 0$ or $[\quad\downarrow\quad] = 0$

after $\frac{d}{dt}$, make small angle assumption, e.g.,

$\cos\theta \sim 1 \quad \sin\theta \sim \theta$

$\cos\theta - 1 \sim \theta^2/2$

RAYLEIGH METHOD – from energy you can get the natural frequency. For Previous Prob.

$\frac{1}{2} m l^2 \dot{\theta}^2$ = KE of the system

$\frac{1}{2}(k_1+k_2)a^2\sin^2\theta + mgl(1-\cos\theta)$ = PE of ~~stem~~ system

$\cos\theta \sim 1 - \theta^2/2$ ONLY FOR COS θ - KEEP 2 TERMS

$KE_{max} = PE_{max} \Longrightarrow \omega_n^2$

$\frac{1}{2}(k_1+k_2)a^2\theta^2 - mgl\cdot\frac{\theta^2}{2}$ = PE of the sys. for small θ

if $\theta = A\sin(\omega_n t+\phi)$ for an undamped harmonic system

$\dot{\theta} = A\omega_n\cos(\omega_n t+\phi)$

$$KE = \frac{1}{2} m \ell^2 \dot{\theta}^2 = \frac{1}{2} m \ell^2 \left[A \omega_n \cos(\omega_n t + \phi) \right]^2$$

$$= \frac{1}{2} m \ell^2 A^2 \omega_n^2 \cos^2(\ \ )$$

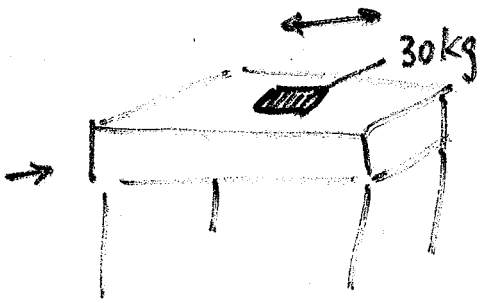
$$\underline{KE_{max} = \frac{1}{2} m \ell^2 A^2 \omega_n^2}$$

$$PE = \frac{1}{2} \left[(k_1 + k_2) a^2 - mg\ell \right] \theta^2 = \frac{1}{2} \left[(k_1 + k_2) a^2 - mg\ell \right] \left[A \sin(\omega_n t + \phi) \right]$$

$$\underline{PE_{max} = \frac{1}{2} \left[(k_1 + k_2) a^2 - mg\ell \right] A^2}$$

$$KE_{max} = \frac{1}{2} m \ell^2 A^2 \omega_n^2 = \frac{1}{2} \left[(k_1 + k_2) a^2 - mg\ell \right] A^2 = PE_{max}$$

Solve for $\omega_n^2 = \dfrac{\left[(k_1 + k_2) a^2 - mg\ell \right]}{m \ell^2}$



$\tau_n = 0.4$ sec

$f\tau = 1$

$\omega_n = 2\pi f$

$2\pi f \tau = 2\pi$

$\omega_n \tau_n = 2\pi$

$$\sqrt{\frac{k}{m}} = \omega_n = \frac{2\pi}{\tau_n} = \frac{2\pi}{.4}$$

$\tau_n' = 0.5$ sec

Q: m, k ?

$$\sqrt{\frac{k}{m+30}} = \omega_n' = \frac{2\pi}{\tau_n'} = \frac{2\pi}{0.5}$$

$$\sqrt{\frac{K}{m}} = \frac{2\pi}{.4} \quad \Rightarrow \quad \frac{K}{m} = \frac{4\pi^2}{.16}$$

$$\sqrt{\frac{K}{m+30}} = \frac{2\pi}{.5} \quad \Rightarrow \quad \frac{K}{m+30} = \frac{4\pi^2}{.25}$$

$$\frac{m+30}{K} - \frac{m}{K} = \frac{0.25 - 0.16}{4\pi^2}$$

$$\frac{30}{K} = \frac{0.09}{4\pi^2} \qquad K_{sys} = 13160 \frac{N}{m}$$

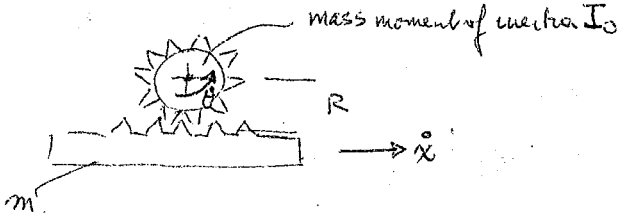
$$K_{leg} = \frac{1}{4} K_{sys} = 3290 \frac{N}{m}$$

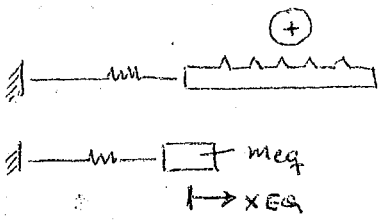
$$\frac{m}{K} = \frac{.16}{4\pi^2} \qquad m = \frac{.16}{4\pi^2} K_{sys} = 53.3 \text{ kg}$$

2-4, 6, 7, 13, 15

TRANSLATIONAL & ROTATIONAL MASSES COUPLED TOGETHER

FOR BODIES IN CONTACT





1. EQUIV. TRANSLATING mass.

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_0 \dot{\theta}^2 = T_{EQ} = \frac{1}{2} m_{EQ} \dot{x}_{EQ}^2$$

IF WE CHOOSE $\dot{x}_{EQ} = \dot{x}$ $\quad R\theta = x \quad R\dot{\theta} = \dot{x} \quad \dot{\theta} = \dot{x}/R$.

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_0 \left(\frac{\dot{x}^2}{R^2}\right) = \frac{1}{2}\left[m + \frac{I_0}{R^2}\right]\dot{x}^2 = \frac{1}{2} m_{EQ} \dot{x}^2$$

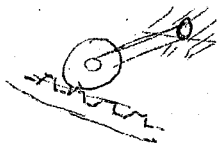
WHAT IS [redacted]? 0 SINCE WORK BY INTERNAL FORCES = 0

WORK DO TO CONTACT FORCES

$$m_{EQ} = m + \frac{I_0}{R^2}$$

2. EQUIVALENT ROTATING MASS

$$T_{EQ} = \frac{1}{2} I_{EQ} \dot{\theta}_{EQ}^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} I_0 \dot{\theta}^2 = T$$



LET $\dot{\theta}_{EQ} = \dot{\theta} \Rightarrow \dot{x} = R\dot{\theta}$

$$\frac{1}{2} I_{EQ} \dot{\theta}^2 = \frac{1}{2} m R^2 \dot{\theta}^2 + \frac{1}{2} I_0 \dot{\theta}^2 = \frac{1}{2}\left[m R^2 + I_0\right]\dot{\theta}^2$$

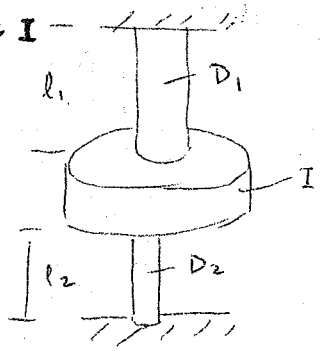
OR $\quad I_{EQ} = I_0 + m R^2$

HW 2-3, 2-10, 2-13, 2-23 // RAO 2nd ed: 1.7, 1.9, 1.17, 1.16

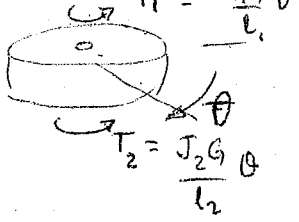
SESSION #6

PROBLEM 2-30

disk - mass moment I about its axis
shafts are fixed
find ODE & freq



UNDERGO SAME ANGULAR DISP



$$J_i = \frac{\pi D_i^4}{32}$$

ΣT_i = total torque

$$I\ddot{\theta} = -\left(\frac{J_1}{l_1} + \frac{J_2}{l_2}\right) G\theta$$

$$\omega = \sqrt{\left(\frac{J_1}{l_1} + \frac{J_2}{l_2}\right)\frac{G}{I}} = \sqrt{\frac{G}{I}\cdot\frac{(J_1 l_2 + J_2 l_1)}{l_1 l_2}} = \sqrt{\frac{G\pi}{32 I}\cdot\frac{(D_1^4 l_2 + D_2^4 l_1)}{l_1 l_2}}$$

$$\tau = \frac{1}{f} = \frac{2\pi}{\omega_n} = 2\pi \Big/ \sqrt{\frac{G\pi}{32 I l_1 l_2}\left(l_1 D_2^4 + l_2 D_1^4\right)}$$

$$= 2\pi \sqrt{\frac{32 I l_1 l_2}{G\pi} \frac{1}{\left(l_1 D_2^4 + l_2 D_1^4\right)}} = 8\sqrt{\frac{2\pi I l_1 l_2}{G\left(l_1 D_2^4 + l_2 D_1^4\right)}}$$

PROBLEM 2-11

suppose you are told a system represented by a spring-mass had a freq of 2 Hz by subtracting 0.4 kg freq increases by 25%. what is k, m

Given: mass decreased by 0.4 kg ⇒ freq ↑ by 25%

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}} = 2\,\text{Hz} \qquad f' = \frac{\omega'}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m-.4}} = 1.25 f$$

$$(2\pi f)^2 = [2\pi(2)]^2 = \frac{k}{m} \Rightarrow \frac{m}{k} = \frac{1}{16\pi^2}$$

$$(2\pi f')^2 = [2\pi(2.5)]^2 = \frac{k}{m-.4} \Rightarrow \frac{m-.4}{k} = \frac{1}{25\pi^2}$$

$$-\frac{.4}{k} = \frac{1}{25\pi^2} - \frac{m}{k} = \frac{1}{25\pi^2} - \frac{1}{16\pi^2} = \frac{1}{\pi^2}\left[\frac{-9}{16\cdot 25}\right]$$

$$k = \frac{\pi^2 (16\cdot 25)}{9}(.4) = 175.46\ \text{N/m}$$

$$m = \frac{k}{16\pi^2} = 1.11\ \text{kg}$$

PROBLEM 2-16

A vibrating SDOF system shows an amplitude of 2cm and a period of 3 sec. Determine the max. velocity & accel

given $X = 2\,\text{cm}$ $\tau = 3\,\text{sec} = \frac{2\pi}{\omega}$ $\omega = \frac{2}{3}\pi = 2.094\ \text{rad/sec}$

max velocity $\dot{X} = X\omega = 4.19\ \text{cm/sec}$

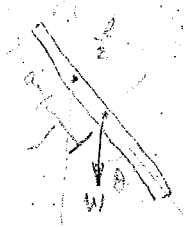
$\ddot{X} = -X\omega^2 = 8.77\ \text{cm/sec}^2$

PROBLEM 2-21

A slender uniform bar pivots in a vertical plane about A for small vibs obtain the DE & freq

center of gravity is at $l/2$





$$I_0 = I_G + m\left(\tfrac{l}{2} - a\right)^2$$

$$I_G = \int_{-l/2}^{l/2} x^2\,dm = \int_{-l/2}^{l/2} x^2 \rho\,dx = \rho\frac{l^3}{12} = \frac{m l^2}{12}$$

$dm = \rho\,dx$

$$I_o = \frac{ml^2}{12} + m\left(\frac{l^2}{4} - \frac{2l}{2}a + a^2\right) = m\left[\frac{l^2}{3} - la + a^2\right]$$

Torque of weight about O $= -W\sin\theta\left(\frac{l}{2}-a\right)$

$$I_o\ddot{\theta} = -mg\sin\theta\left(\frac{l}{2}-a\right) \quad \text{or}$$

$$m\left[\frac{l^2}{3} - la + a^2\right]\ddot{\theta} + mg\left(\frac{l}{2}-a\right)\theta = 0 \qquad \text{for small } \theta$$

$$\omega = \sqrt{\frac{mg\left(\frac{l}{2}-a\right)}{\left(\frac{l^2}{3} - la + a^2\right)m}} = \sqrt{\frac{g\left(\frac{l}{2}-a\right)}{l^2/3 - la + a^2}}$$

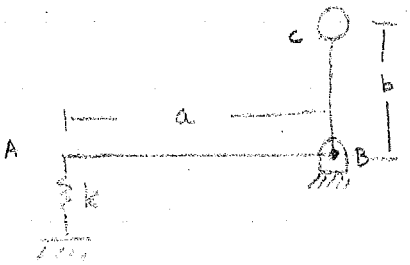
$$f = \frac{1}{2\pi}\omega$$

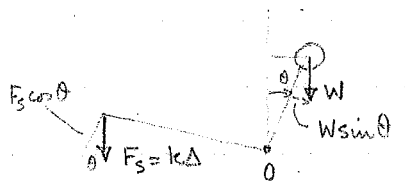
now $\omega_n = \sqrt{g/l'}$ for simple pendulum $l' = \dfrac{l^2/3 - la + a^2}{l/2 - a}$

center of percussion

PROBLEM 2-24

WRITE EQ wrt B





Torque due to weight $W\sin\theta \cdot b$

$\Delta = a\sin\theta$

$F_s = ka\sin\theta$ Torque due to $F_s = (-ka\sin\theta)\cos\theta\, a$

$I_o = mb^2$ if I_G of ball is small

$$I_o\ddot{\theta} = -ka^2\sin\theta\cos\theta + Wb\sin\theta$$

or $\quad mb^2\ddot{\theta} + (ka^2 - Wb)\theta = 0 \qquad$ for small θ

$$\omega = \sqrt{\frac{ka^2 - Wb}{mb^2}} \qquad f = \frac{1}{2\pi}\omega.$$

energy $\frac{1}{2}k\Delta_{eq}^2 = \frac{1}{2}k\Delta^2 + mgb(\cos\theta$

$\frac{1}{2}k_{eq}\Delta_{eq}^2 = \frac{1}{2}ka^2\theta^2 - \frac{mgb}{2}\theta^2$

$(a\theta)^2$

$\therefore k_{eq} = k - \frac{mgb}{a^2}$

DO PROBLEMS 2-31, 2-32, 2-33

energy $\frac{1}{2}m\dot{\Delta}_{eq}^2 = \frac{1}{2}m(b\dot{\theta})^2$

$m_{eq} = m\left(\frac{b}{a}\right)^2$

$$\therefore \omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{k - mgb/a^2}{mb^2/a^2}}$$

A heavy table is supported by flat steel legs. Its $\tau_n = .4$ sec in horiz motion by adding a 30 kg plate clamped to the table then $\tau_n = .5$ sec
what are the effective k & mass of table

$$\tau_n = \frac{1}{f_n} \qquad f_n = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

$$\Rightarrow \quad \frac{m}{k} = \left(\frac{\tau_n}{2\pi}\right)^2_1 \qquad \frac{m+m_0}{k} = \left(\frac{\tau_n}{2\pi}\right)^2_2$$

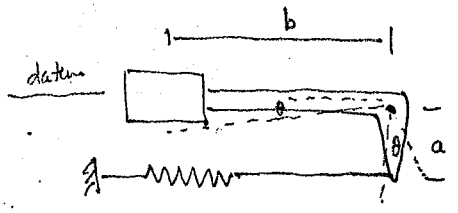
$$\therefore \quad \frac{m_0}{k} = \frac{1}{4\pi^2}\left[\tau_{n_2}^2 - \tau_{n_1}^2\right]$$

$$\text{or} \quad k = \frac{m_0 \cdot 4\pi^2}{\tau_{n_2}^2 - \tau_{n_1}^2} \quad \& \quad m = \frac{k \cdot \tau_{n_1}^2}{4\pi^2}$$

$$= \frac{30 \cdot 4 \cdot 10}{.25 - .16} = \frac{1200}{.09} \cong 13160 \text{ N/m}$$

$$m = \frac{13160 \cdot .16}{4 \cdot \pi^2} = 53.3 \text{ kg}$$

find the natural freq



$$k\Delta_{st}\, a \overset{\cos\theta_{st}}{\wedge} F_s \cdot a = mg \cdot b \cdot \cos\theta_{st} \qquad \frac{\Delta_{st}}{a} = \frac{y}{b}$$

$$g/\Delta_{st} = \frac{ka}{mb}$$

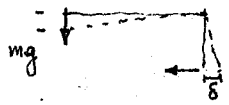
$$\omega_n = \sqrt{g/\Delta_{st}}$$

$$ka^2 \sin\theta_{st} = mgb$$

$$\sin\theta_{st} = \frac{mgb}{ka^2}$$

$$\theta_{st} = \frac{mgb}{ka^2}$$

$$\Sigma M = I_o \ddot{\theta}$$



$$\frac{\delta}{a} = \frac{y}{b}$$

θ measured from horizontal

$$mgb\cos\theta - k\delta a\cos\theta = I_o\ddot{\theta} = mb^2\ddot{\theta}$$

$$\frac{a^2}{b} \cdot y\cos\theta$$

$$mgb\cos\theta - ka^2 \sin\theta\cos\theta = mb^2\ddot{\theta}$$

$$(mgb) = \ddot{\theta}mb^2 + ka^2\theta$$

forcing funct

$$\omega_n = \frac{a}{b}\sqrt{\frac{k}{m}}$$

$$\Rightarrow \ddot{\theta}\frac{mb^2}{ka^2} + (\theta - \theta_{st}) = 0$$

$$\text{or} \quad \ddot{\theta} - \ddot{\theta}_{st} + \frac{ka^2}{mb^2}(\theta - \theta_{st}) = 0$$

also $\Delta V = \frac{1}{2}kx^2 - mgy = \frac{1}{2}k\left[a\sin\theta\right]^2 - mgb\sin\theta$

$\Delta T = \frac{1}{2}mv^2 = \frac{1}{2}m\dot{y}^2 = \frac{1}{2}mb^2\left[\cos\theta\,\dot{\theta}\right]^2$

$\frac{1}{2}mb^2\dot{\theta}^2 + \frac{1}{2}ka^2\theta^2 - mgb\theta$

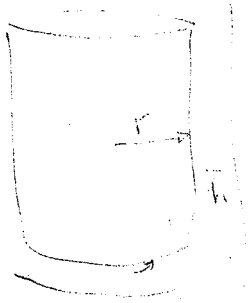
$\left[mb^2\ddot{\theta} + ka^2\theta - mgb\right]\dot{\theta} = 0 \quad \Rightarrow mb^2\ddot{\theta} + ka^2\theta$

if let $\theta - \theta_{st} = \psi$

$$\ddot{\psi} + \frac{ka^2}{mb^2}\psi = 0$$

ψ measured from equil

1.35



Torque $= F \cdot r$

$F = \tau \, 2\pi r l = \tau A$

$\tau = \mu \frac{du}{dy} = \mu \frac{v}{h} = \mu \frac{\Omega r}{h}$

$\therefore \quad F = 2\pi r l \cdot \frac{\mu \Omega r}{h} = \frac{2\pi r^2 l \mu}{h} \Omega$

$T = Fr = \frac{2\pi r^3 l \mu}{h} \Omega$

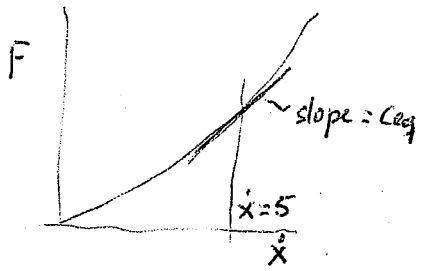
$\therefore \quad T = k_t \Omega$

$T = \int r \, dF = \int \frac{\mu \Omega r}{h} \cdot 2\pi r \, dr$

$T = \int r \, dF = \frac{\mu \Omega r}{h} 2\pi r^2 dr$

$\frac{\mu \Omega r^3}{3h} 2\pi$

1.36 $F = a\dot{x} + b\dot{x}^2$

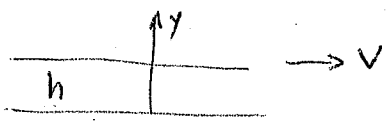


$F = c_{eq} \dot{x} \qquad c_{eq} = \frac{\partial F}{\partial \dot{x}}$

$c_{eq} = a + 2b\dot{x}$

near $\dot{x} = 5 \quad c_{eq} = 5 + 0.2(5) \cdot 2 = 7 \ \frac{N\text{-}s}{m}$

$\therefore \quad F = 7\dot{x}$ near $\dot{x} = 5$



$F = \tau \cdot A = c\,v$

$\tau = \mu \frac{du}{dy} = \mu \frac{v-0}{h-0} = \mu \frac{v}{h}$

$A = l b$

$\therefore \quad F = \left(\frac{\mu}{h} \cdot lb\right) v \qquad c_{eq} = \frac{\mu A}{h}$

DO 1.7, 1.9, 1.17, 1.16

in series

$$\frac{dP}{dA} = \sigma = E\epsilon$$

$$dP =$$

$$\sigma A = (\sigma + d\sigma)(A + dA)$$

$$A\,d\sigma = -\sigma\,dA$$

$$\frac{d\sigma}{\sigma} = -\frac{dA}{A}$$

$$\ln\sigma = -\ln A + C$$

$$\frac{P}{A} = \epsilon E$$

$$\frac{P}{AE}\,dx = du$$

1.11

$$\frac{P}{E}\left[\frac{4\,dx}{\pi\left\{\left(\frac{D-d}{l}\right)x + D\right\}^2}\right] = \Delta$$

$$\ln\sigma_D - \ln\sigma_d = \ln A_d - \ln A_D$$

$$\frac{P}{E\pi}\int_0^l \frac{dx}{\left(-\frac{D+d}{l}x + D\right)^2} = \Delta$$

$$\frac{P}{E\pi}\left[\frac{4P}{E\pi}\left(-\frac{D+d}{l}x + D\right)^{-1}\right]_0^l = \Delta$$

$$\frac{4P}{E\pi}\left[\frac{-1}{d\left(-\frac{D+d}{l}\right)} + \frac{1}{D\left(-\frac{D+d}{l}\right)}\right]$$

$$\left(\frac{d-D}{l}\right)^{-1}\frac{4P}{E\pi}\left[\frac{-D+d}{dD}\right]$$

$$\frac{4Pl}{E\pi\,dD} = \Delta$$

$$k = \frac{dD\,E\pi}{4l}$$

$$\frac{dD}{4l} = \frac{t(d+t)}{l_2}$$

$$\therefore \boxed{l_2 = \frac{4t(d+t)}{dD}\,l}$$

$$k = \frac{\pi\left(d+2t\right)^2}{4} - \frac{\pi d^2}{4}$$

$$k = \frac{\pi}{4}\left(2t \cdot 2(d+t)\right)$$

$$= \pi t\,(d+t)$$

$$\frac{P}{A} = \sigma = E \cdot \frac{\Delta}{l_2}$$

$$\therefore P = \frac{AE}{l_2}\Delta$$

$$P = \frac{E\pi\,t(d+t)}{l_2}\Delta$$

X **1.31** When mass m is displaced by x, the bell crank lever rotates by the angle $\theta_b = \frac{x}{\ell_1}$. This makes the center of the sphere displace by $x_s = \theta_b \ell_2$. Since the sphere rotates with out slip, it rotates by an angle

$$\theta_s = \frac{x_s}{r_s} = \frac{\theta_b \ell_2}{r_s} = \frac{x \ell_2}{\ell_1 r_s}$$

The kinetic energy of the system can be expressed as

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} J_s \dot{\theta}_s^2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \left(\frac{\dot{x}}{\ell_1}\right)^2 + \frac{1}{2} \left(\frac{2}{5} m_s r_s^2\right) \dot{x}^2 \left(\frac{\ell_2}{\ell_1 r_s}\right)^2$$

since for a sphere, $J_s = \frac{2}{5} m_s r_s^2$. Equating this to $T = \frac{1}{2} m_{eq} \dot{x}^2$, we obtain

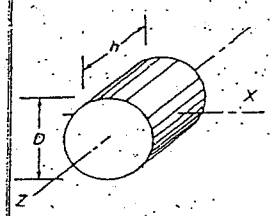
$$m_{eq} = m + J_0 \frac{1}{\ell_1^2} + \frac{2}{5} m_s \frac{\ell_2^2}{\ell_1^2}$$

Appendix 2
Modeling of System Elements

COCHIN "ANALYSIS & DESIGN OF DYNAMIC SYSTEMS"

Table 2.1.3-1 MASSES AND MOMENTS OF INERTIA OF RIGID BODIES

1. Solid disk

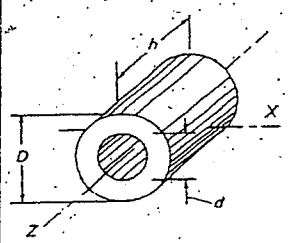


$$M = \frac{\rho\pi D^2 h}{4}$$

$$J_z = \frac{\rho\pi D^4 h}{32} = \frac{MD^2}{8}$$

$$J_x = \frac{\rho\pi D^2 h}{16}\left[\frac{h^2}{3} + \frac{D^2}{4}\right] = M\left[\frac{h^2}{12} + \frac{D^2}{16}\right]$$

2. Tube



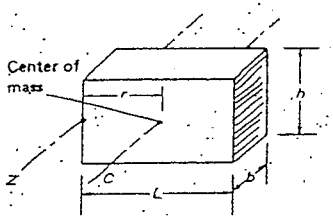
$$M = \frac{\rho\pi h(D^2 - d^2)}{4}$$

$$J_z = \frac{\rho\pi h(D^4 - d^4)}{32} = \frac{M(D^2 + d^2)}{8}$$

$$J_x = \frac{\rho\pi h}{16}(D^2 - d^2)\left[\frac{h^2}{3} + \frac{D^2}{4} + \frac{d^2}{4}\right]$$

$$= M\left[\frac{h^2}{12} + \frac{D^2}{16} + \frac{d^2}{16}\right]$$

3. Rectangular solid



$$M = \rho Lbh$$

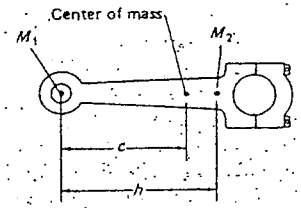
$$J_z = \frac{\rho Lbh}{12}(L^2 + h^2 + 12r^2)$$

$$= M\left[\frac{L^2}{12} + \frac{h^2}{12} + r^2\right]$$

$$J_c = \frac{\rho Lbh(L^2 + h^2)}{12}$$

$$= \frac{M[L^2 + h^2]}{12}$$

4. Connecting rod



$$M_1 = M\left(1 - \frac{c}{h}\right)$$

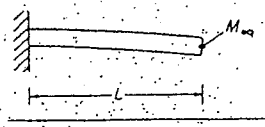
$$M_2 = \frac{Mc}{h}$$

$$h = \frac{J_1}{Mc}$$

J_1 = moment of inertia about M_1

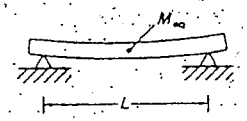
Table 2.1.4-1 MASSES AND MOMENTS OF INERTIA OF ELASTIC BODIES

1. Cantilever beam



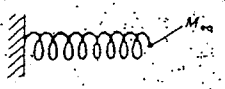
$$M_{eq} \approx \frac{M}{4}$$

2. Simply supported beam



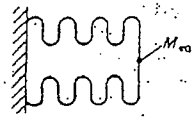
$$M_{eq} = \tfrac{1}{2}M$$

3. Coiled spring



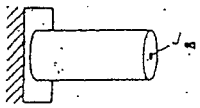
$$M_{eq} = \frac{M}{3}$$

4. Bellows



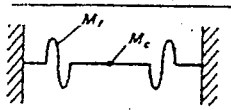
$$M_{eq} = \frac{M}{3}$$

5. Shaft in torsion



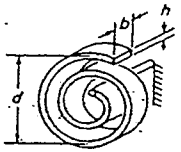
$$J_{eq} = \frac{J}{3}$$

Table 2.1.4-1 (cont.)



6. Diaphragm

$$M_{eq} = M_d + \frac{M_f}{4}$$



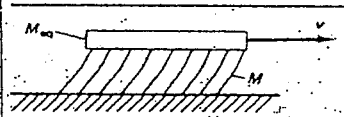
7. Clock spring

$M = \rho bhL$

$$J_{eq} \approx \frac{Md^2}{16}$$

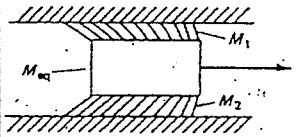
L = total length of spiral

Table 2.1.5-1 FLUID MASSES



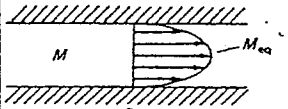
1. Translating surfaces

$$M_{eq} = \frac{M}{3}$$



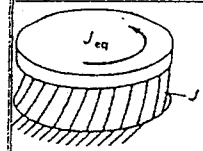
2. Translating body

$$M_{eq} = \frac{M_1 + M_2}{3}$$



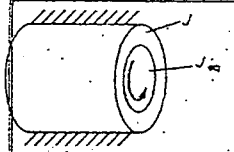
3. Laminar flow in pipe

$$M_{eq} = \frac{4M}{3}$$



4. Rotating parallel surfaces

$$J_{eq} = \frac{J}{3}$$



5. Rotating concentric cylinders

$$J_{eq} = \frac{J}{3}$$

Table 2.2.3-1 MECHANICAL SPRINGS (TRANSLATORY)

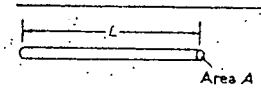


1. Coiled spring (round wire)

$$K_{eq} = \frac{d^4G}{64R^3N}$$

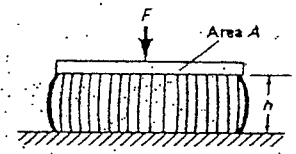
where

G = modulus in shear
N = number of coils



2. Rod or wire in tension

$$K_{eq} = \frac{EA}{L}$$

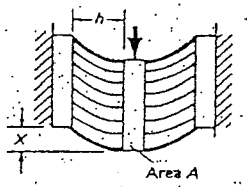


3. Rubber in compression

$$K_{eq} = \frac{1 + F/AE}{h/AE}$$

where

A = area of plate

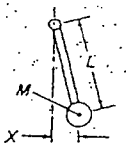


4. Rubber in double shear

$$K_{eq} = \frac{2AG}{h}$$

where

A = area of plate providing shear surface



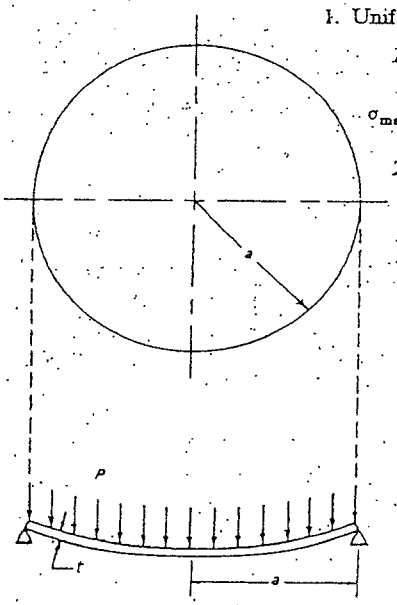
5. Pendulum

$$K_{eq} = \frac{gM}{L}$$

where

g = acceleration due to gravity

Table 2.2.3-2 DISK SPRINGS



1. Uniform load, edges simply supported

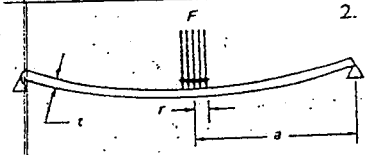
$$K = \frac{64\pi Z(1 + \nu)}{a^2(5 + \nu)}$$

$$\sigma_{max} = \frac{3Pa^2(3 + \nu)}{8t^2}$$

$$Z = \frac{Et^3}{12(1 - \nu^2)}$$

Table 2.2.3-2 (cont.)

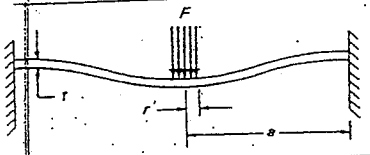
2. Concentrated load, edges simply supported



$$K = \frac{16\pi Z(1+\nu)}{a^2(3+\nu)}$$

$$\sigma_{max} = \frac{3F}{2\pi t^2}\left[(1+\nu)\ln\frac{a}{r} + 1\right]$$

3. Concentrated load, edges clamped

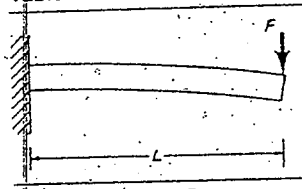


$$K = \frac{16\pi Z}{a^2}$$

$$\sigma_{max} = \frac{3F}{2\pi t^2}(1+\nu)\ln\frac{a}{r}$$

Table 2.2.3-3 EQUIVALENT SPRINGS FOR BEAMS

1. Cantilever

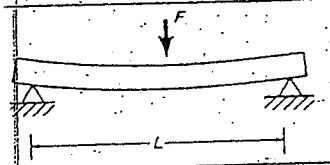


$$K_{eq} = \frac{3EI}{L^3}$$

where

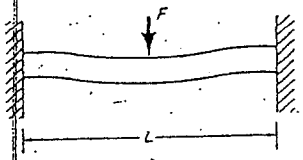
$$I = \frac{bh^3}{12}$$

2. Simply supported



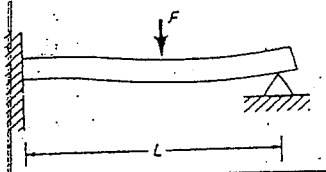
$$K_{eq} = \frac{48EI}{L^3}$$

3. Both ends fixed or clamped.



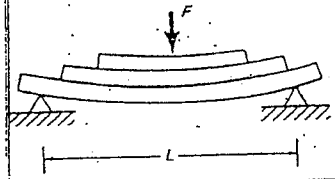
$$K_{eq} = \frac{192EI}{L^3}$$

4. Cantilever with pinned end



$$K_{eq} = \frac{110EI}{L^3}$$

5. Leaf spring



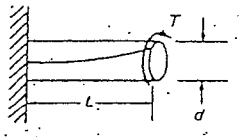
$$K_{eq} = \sum K_i$$

where

K_i = spring constant for each leaf

Table 2.2.3-4 TORSION SPRINGS

1. Round rod

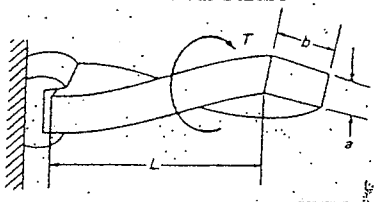


$$K_{eq} = \frac{\pi d^4 G}{32L}$$

where

G = modulus in shear

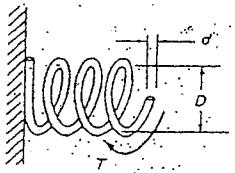
2. Rectangular bar



$$K_{eq} = \frac{C_3 a^3 b G}{L}$$

C_3 = Saint Venant's coefficient

3. Coiled spring in torsion (round wire)



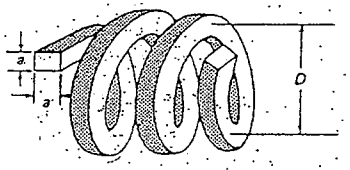
$$K_{eq} = \frac{Ed^4}{64DN}$$

where

E = Young's modulus

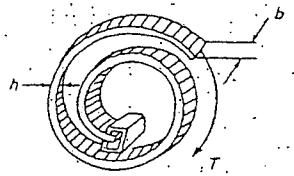
N = number of coils

4. Square wire coiled spring



$$K_{eq} = \frac{Ea^4}{12\pi DN}$$

5. Clock spring

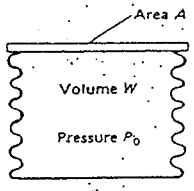


$$K_{eq} = \frac{EI}{L}$$

or

$$K_{eq} = \frac{Ebh^3}{12L}$$

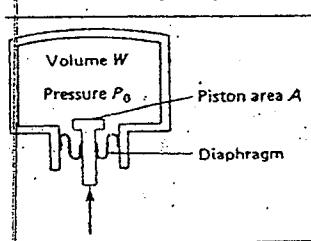
Table 2.2.4-1 FLUID SPRINGS



1. Air spring, bellows

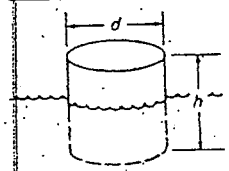
$$K = \frac{A^2 P_0}{W}$$

Table 2.2.4-1 (cont.)



2. Air spring, piston

$$K = \frac{A^2 P_0}{W}$$

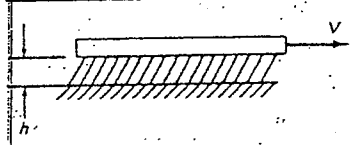


3. Floating cylinder

$$K = \frac{\pi d^2 \rho_f g}{4}$$

ρ_f = mass density of fluid
g = acceleration due to gravity

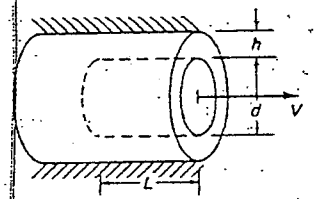
Table 2.3.3-1 FLUID DAMPERS USING VISCOUS SHEAR



1. Translating surfaces

$$D_{eq} = \frac{\mu A}{h}$$

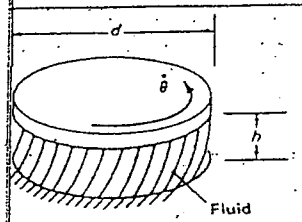
where
A = wetted surface area
μ = coefficient of viscosity (absolute)



2. Translating cylinder

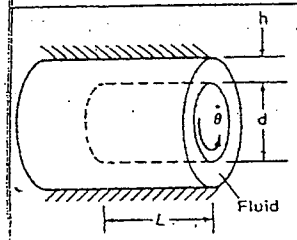
$$D_{eq} = \frac{\mu A}{h}$$

where
A = wetted surface area
$= \pi L d$



3. Rotating disks

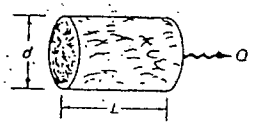
$$D_{eq} = \frac{\pi \mu d^4}{16h}$$



4. Rotating cylinders

$$D_{eq} = \frac{\pi \mu d^3 L}{4h}$$

Table 2.3.4-1 DAMPING DUE TO FLOW THROUGH RESTRICTION

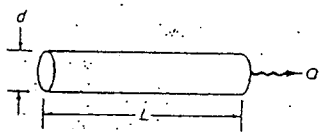


1. Porous material[a]

$$R = \frac{4 C_1 L}{\pi d^2 \rho g}$$

C_1 = coefficient of fluid resistance

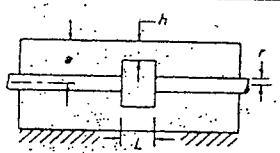
$$D_{eq} = \frac{A^2 4 C_1 L}{\pi d^2}$$



2. Pipe[a]

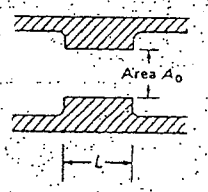
$$R = \frac{128 \mu L}{\pi d^4 \rho g}$$

$$D_{eq} = \frac{A^2 128 \mu L}{\pi d^4}$$



3. Dashpot

$$D_{eq} = \frac{6\pi\mu L}{h^3}\left[\left(a - \frac{h}{2}\right)^2 - r^2\right]\left[\frac{a^2 - r^2}{a - h/2} - h\right]$$



4. Orifice

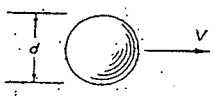
$$Q = C_d A_0 \sqrt{\frac{2\Delta p}{\rho}}$$

A_0 = area of orifice
C_d = discharge coefficient
ρ = fluid density
Δp = pressure drop through orifice

$$N_r = \frac{\rho Q d}{A_0 \mu}$$ Reynolds #

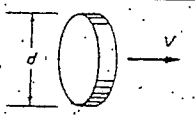
[a] A = area of piston in Fig. 2.3.4-1. μ = absolute (dynamic) viscosity. See Appendix 6.

Table 2.3.5-1 DAMPING DUE TO DRAG IN A VISCOUS MEDIUM



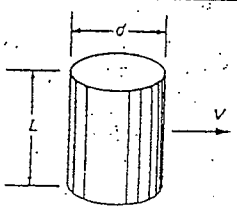
1. Sphere[a]

$$D_{eq} = 3\pi\mu d$$



2. Disk[a]

$$D_{eq} = 30\mu d$$

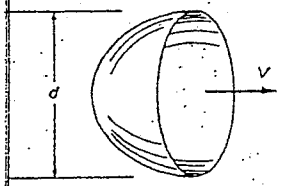


3. Cylinder[a]

$$D_{eq} = 3\mu L$$

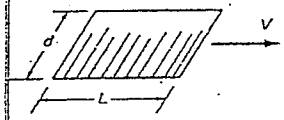
Table 2.3.5-1 (cont.)

4. Cup or parachute[a]



$$D_{eq} = 100\mu d$$

5. Skin friction[a]

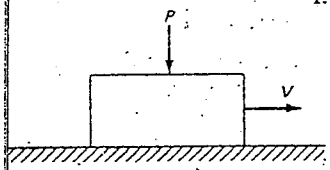


$$D_{eq} = 100\mu L^2 d$$

[a]Fair approximation valid for low Reynold's number.

Table 2.3.6-1 MECHANICAL DAMPERS

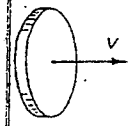
1. Coulomb friction



$$D_{eq} = \frac{4\mu P}{\pi\omega X_m}$$

μ = coefficient of dry friction
P = normal force
ω = input frequency
X_m = amplitude of oscillation of mass relative to base

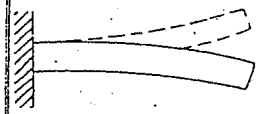
2. Velocity squared



$$D_{eq} = \frac{4\omega C_d X_m}{3\pi}$$

C_d = drag coefficient

3. Structural damping

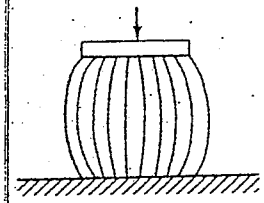


$$D_{eq} = \frac{C}{\pi\omega}$$

$$C = \frac{E}{X_m^2}$$

E = energy lost per cycle

4. Rubber pad



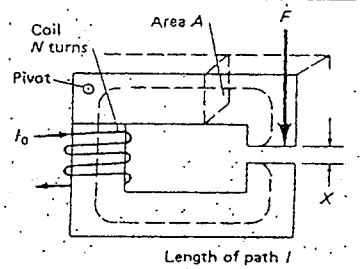
$$D_{eq} = \frac{E}{\pi\omega X_m^2}$$

Rubber acts like a spring damper network Z where

$$Z(s) = DS + K$$

Table 2.4.1-1 MAGNETIC DAMPERS AND SPRINGS

1. Small air gap



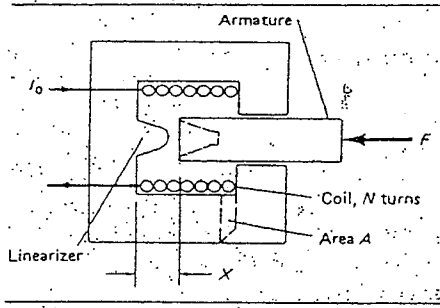
$$L = \frac{\mu N^2 A}{l(1 + \mu X/l)}$$

$$F = \frac{A}{2}\left[\frac{\mu N I_0}{l(1 + \mu X/l)}\right]^2$$

$$K_{eq} = AN^2 I_0^2\left[\frac{\mu}{l(1 + \mu X/l)}\right]^3$$

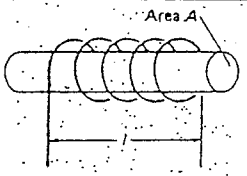
$$D_{eq} = \frac{I_0^2 A^2}{R}\left[\frac{\mu N}{l(1 + \mu X/l)}\right]^4$$

2. Solenoid



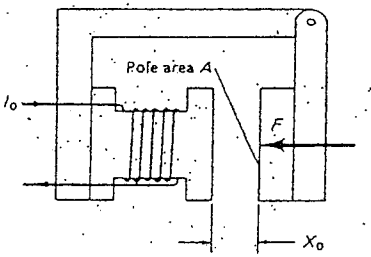
All relations are identical to those for either small air gap (Item 1) or large air gap (Item 4) depending upon the stroke. The protrusion and corresponding well in the armature tends to linearize the system.

3. Open Core



$$L = \mu N^2 A/l$$

4. Large air gap



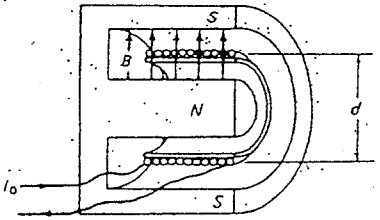
$$F = \frac{AN^2 I_0^2}{2X_0^2}$$

$$K_m = \frac{AN^2 I_0^2}{X_0^3}$$

$$D_m = \frac{A^2 I_0^2 N^4}{R_c X_0^4}$$

$$L = \frac{N^2 A}{X_0}$$

5. Voice coil



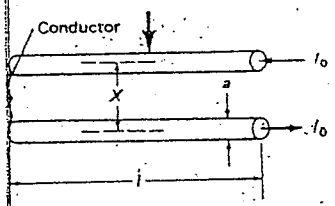
$$F = \pi dNBI_0$$

$$V = \pi dNBX$$

$$K = 0$$

$$D_m = \frac{(\pi dNB)^2}{10^8 R_c}$$

Table 2.4.1-1 (cont.)



6. Parallel conductors

$$L = \frac{l}{\pi}\ln(X/a)$$

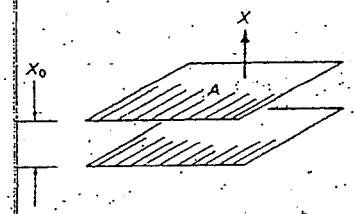
$$F = \frac{\frac{1}{2}I_0^2 la}{\pi X}$$

$$K_{eq} = \frac{\frac{1}{2}I_0^2 la}{\pi X^2}$$

$$D_{eq} = \frac{1}{R}\left[\frac{I_0 la}{\pi X}\right]^2$$

units in mks system
L_0 = nominal current (A)
V = voltage (V)
R_c = coil resistance (Ω)
B = mag. field (Wb/m²)
l = length(m)

Table 2.4.2-1 ELECTROSTATIC DAMPERS AND SPRINGS



1. Parallel planes

$$C = \frac{\epsilon A}{X}$$

$$F_0 = \frac{V_0^2 \epsilon A}{2X_0^2}$$

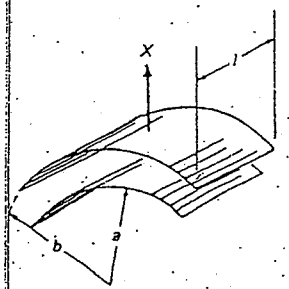
$$K_{eq} = \frac{V_0^2 \epsilon A}{X_0^3}$$

$$D_{eq} = \frac{(V_0 \epsilon A)^2 R}{X_0^4}$$

C = capacitance (farad)

ϵ = permittivity

See Appendix 6.



$X_0 = b - a$

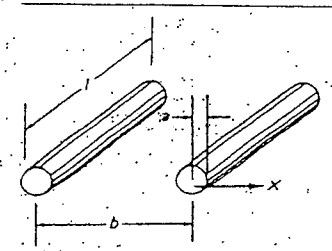
2. Concentric Cylinders

$$C = \frac{2\pi\epsilon l}{\ln(b/a)}$$

$$F_0 = \frac{V_0^2 \pi\epsilon l}{b(\ln X_0/a)^2}$$

$$K_{eq} = \frac{V_0^2 2\pi\epsilon l}{b^2(\ln X_0/a)^3}$$

$$D_{eq} = \frac{4(V_0 \pi\epsilon l)^2 R}{b^2(\ln X_0/a)^4}$$



$X_0 = b$

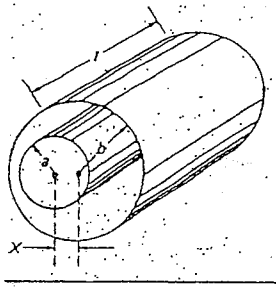
3. Parallel rods

$$C = \frac{\pi\epsilon l}{\ln\left(\frac{b + \sqrt{b^2 - 4a^2}}{2a}\right)}$$

$$F_0 = \frac{V_0^2 \pi\epsilon l}{2(\ln X_0/a)^2}$$

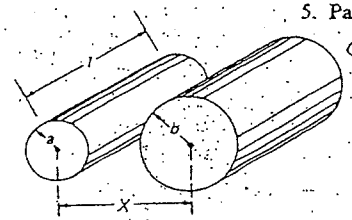
$$K_{eq} = \frac{V_0^2 \pi\epsilon l}{(\ln X_0/a)^3}$$

$$D_{eq} = \frac{2(V_0 \pi\epsilon l)^2 R}{(\ln X_0/a)^4}$$



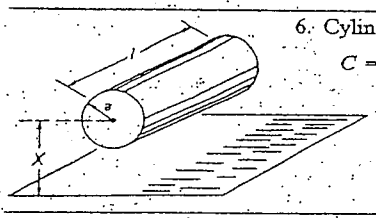
4. Eccentric cylinders

$$C = \frac{2\pi\epsilon l}{\cosh^{-1}\left(\frac{a^2 + b^2 - X^2}{2ab}\right)}$$



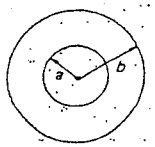
5. Parallel cylinders

$$C = \frac{2\pi\epsilon l}{\cosh^{-1}\left(\frac{a^2 + b^2 - X^2}{2ab}\right)}$$



6. Cylinder and plane

$$C = \frac{2\pi\epsilon l}{\ln(2X/a)}$$



7. Concentric spheres

$$C = \frac{4\pi\epsilon ab}{b - a}$$

$$\tau = \frac{1}{f} = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{G\pi}{32 I l_1 l_2}\left(l_1 D_2^4 + l_2 D_1^4\right)}$$

$$= 2\pi\sqrt{\frac{32 I l_1 l_2}{G\pi}\cdot\frac{1}{\left(l_1 D_2^4 + l_2 D_1^4\right)}} = 8\sqrt{\frac{2\pi I l_1 l_2}{G\left(l_1 D_2^4 + l_2 D_1^4\right)}}$$

DAMPED FREE VIBS FOR SDOF

- SO FAR – VIBS HAVE BEEN SELF-SUSTAINING
- REAL LIFE VIBS DIE AWAY (VIBS ARE DAMPED)
- THREE TYPES OF DAMPING

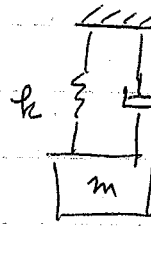
1. VISCOUS DAMPING $F_d = -c\dot{x}$ c = CONST. F_d - DAMPING FORCE
   F_d - lbs, N kg/sec, $\frac{N\,sec}{m}$, $\frac{lb\text{-}sec}{in}$
   - EXAMPLE – BODY MOVING THRU FLUIDS AT LOW VELOC.
   - FORCE OPPOSES MOTION
     - BODIES SLIDE OVER LUBRICATED SURFACE
     - BODIES THROUGH AIR, OIL

   SHOCK ABSORBERS, DASHPOTS

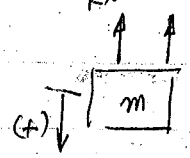
2. COULOMB FRICTION $F = \mu N$ SLIDING OVER DRY SURFACE

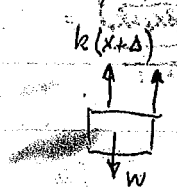
3. HYSTERESIS DAMPING DUE TO INTERNAL FRICTION IN MATERIAL

CONSIDER MASS-SPRING-DASHPOT SYSTEM



dynamic



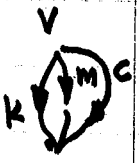


measure every thing from unstretched length of spring

$$\frac{d^2}{dt^2}(x+\Delta) = \ddot{x}$$

$$m\ddot{x} = -kx - c\dot{x}$$

$$\ddot{x} + \frac{k}{m}x + \frac{c}{m}\dot{x} = 0$$



$$m\frac{dv}{dt} + \int k\frac{dv}{dt} + cv = 0$$

$$m\ddot{v} + k\dot{v} + c\dot{v} = 0$$

$$x = Ce^{st} \Rightarrow Ce^{st}\left(s^2 + \frac{k}{m}s + \frac{c}{m}s\right) = 0$$

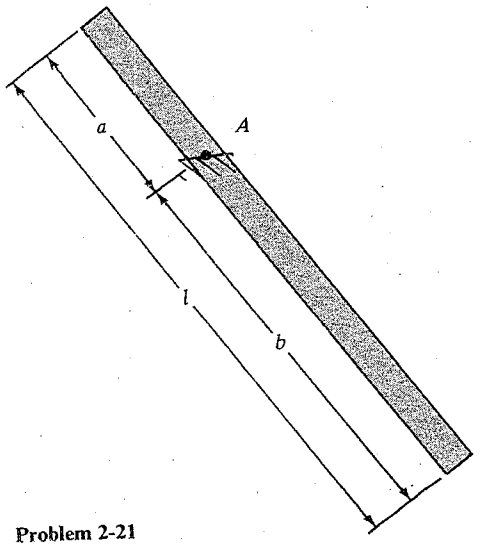
$$s = \frac{-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - 4\left(\frac{k}{m}\right)}}{2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k}{m}\right)}$$

Determine (a) the displacement amplitude and (b) the maximum velocity during motion of the structure.

2-19. An accelerometer is used to measure a vibratory motion. If the frequency is found to be 20 Hz and the maximum acceleration is 12 m/s^2, determine (a) the period, (b) the amplitude, and (c) the maximum velocity of the motion.

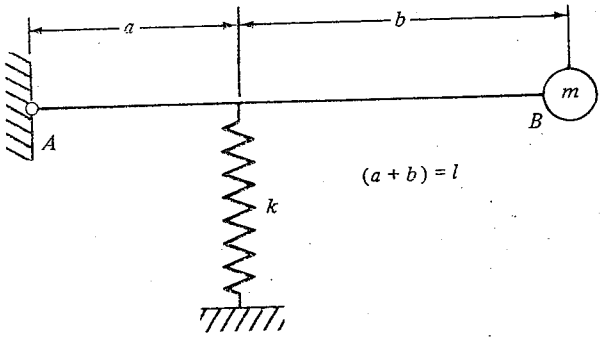
2-20. If the frequency of a vibrating body is 5 Hz and its maximum velocity is 90 in./sec, determine (a) the amplitude, (b) the maximum acceleration, and (c) the period of the motion.

2-21. A slender uniform bar pivots in a vertical plane about axis A located distance a from the upper end. For small angular vibration, obtain the differential equation of motion and express the frequency.



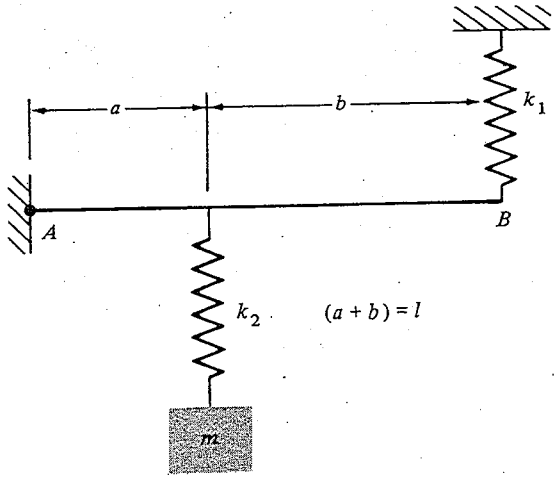
Problem 2-21

2-22. A rigid bar AB is weightless and pivots at A as shown in the accompanying figure. Obtain (a) the differential equation of motion and (b) the expression for the natural frequency of angular vibration of the system.



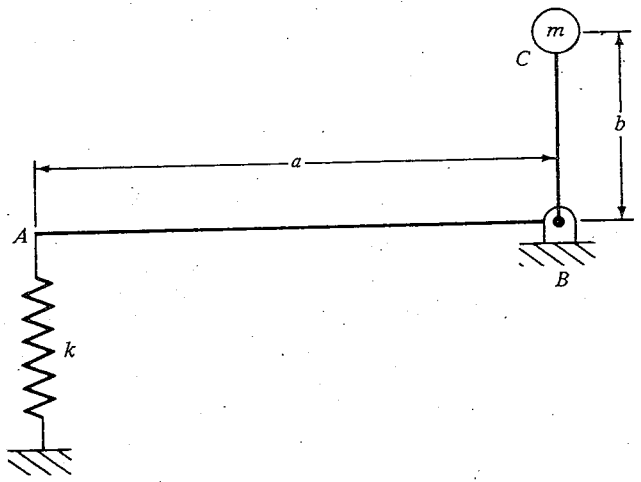
Problem 2-22

2-23. A rigid bar AB is weightless and pivots at A as shown in the accompanying figure. Determine (a) the differential equation for the vertical motion of m and (b) the expression for the natural frequency of oscillation of m.



Problem 2-23

2-24. A rigid member ABC is weightless and pivots at B as shown in the accompanying figure. Obtain (a) the differential equation of motion and (b) the expression for the natural frequency of angular vibration of the system. (c) Indicate the condition for stability of the vibration.



Problem 2-24

2-25. Determine (a) the differential equation of motion and (b) the expression for th natural frequency of vibration of the system shown herewith.

SOLUTION

$$k = \frac{AE}{l} = \frac{(0.5)^2 \times 10^7}{100} = 2.5 \times 10^4 \text{ lb/in.}$$

a. $$\omega = \sqrt{\frac{k}{M}} = \sqrt{\frac{2.5 \times 10^4 \times 386}{5}} = 1389 \text{ rad/sec} \qquad M = \frac{W}{g}$$

$$f = \frac{\omega}{2\pi} = \frac{1389}{2\pi} = 221 \text{ Hz}$$

b. $$\omega' = \sqrt{\frac{k}{M + m/3}} = \sqrt{\frac{2.5 \times 10^4 \times 386}{5 + 2.5/3}} = 1286 \text{ rad/sec}$$

$$f' = \frac{\omega'}{2\pi} = \frac{1286}{2\pi} = 205 \text{ Hz}$$

The exact results† are

$$\omega'' = 1283 \text{ rad/sec}$$

$$f'' = 204 \text{ Hz}$$

PROBLEMS

2-1. A body weighing 12 lb is suspended by a spring having a modulus of 4 lb/in. Determine the natural frequency for the system.

2-2. A 5-lb weight, attached to a spring, causes a static equilibrium elongation of 0.143 in. Determine the natural frequency of oscillation that the system will exhibit if it is displaced from its equilibrium position and then released.

2-3. A body having a mass of 9 kg is suspended by a spring having a modulus of 800 N/m. Determine the natural period of vibration for the system.

2-4. A 1.4-kg mass causes a 4-mm static equilibrium elongation of a certain spring. Determine the natural frequency of vibration for this system.

2-5. A weight of 20 lb, attached to a spring, is found to oscillate at 330 cycles/min. Determine the spring constant k.

2-6. A motor is placed on rubber mountings to isolate it from its foundation. If the dead weight of the motor compresses the mountings 3.1 mm, determine the natural frequency of the system.

2-7. Determine the mass which must be attached to a spring having a modulus of 1750 N/m in order that the resulting system will have a frequency of 21 Hz.

2-8. Determine the length of a simple pendulum so that the half period will be 0.5 sec.

2-9. A mass-spring system has a natural period of 0.152 sec. Determine the period if the spring constant is increased by 60%.

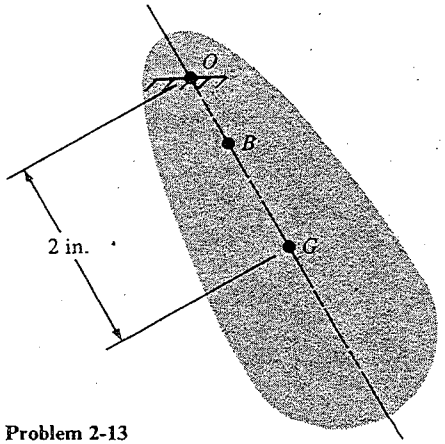
2-10. A weight W is suspended from the end of spring k, and the system has a natural frequency of 90 cycles/min. The weight W is removed and replaced by a weight $W/4$ at the middle of the spring, and the upper and lower ends of the spring are then fixed. Determine the frequency of the new system.

† See Example 10-6.

2-11. A mass-spring system has a frequency of 2 Hz. When the mass is decreased by 0.4 kg, the frequency is changed by 25%. Determine (a) the mass m, and (b) the spring constant k for the original system.

2-12. A mass-spring system has a frequency of 5 Hz. When the spring constant is reduced by 4 lb/in., the frequency is altered by 30%. Determine (a) the spring constant k and (b) the weight W for the original system.

2-13. The cam-shaped member shown in the accompanying figure is free to rotate about the axis at O perpendicular to the view and exhibits a natural frequency of angular oscillation of 54 cycles/min with respect to this axis. The rotation axis is moved to point B, located somewhere between O and the mass center G. The frequency about the axis at B is found to be 36 cycles/min. Determine the distance OB.



Problem 2-13

2-14. A harmonic motion is expressed by the relation $x = 0.5 \sin (10t + 4)$, where x is measured in inches, t in seconds, and the phase angle in radians. Determine (a) the circular and natural frequencies of vibration and (b) the displacement, velocity, and acceleration at $t = 0.3$ sec.

2-15. A vibratory motion has an amplitude of 0.30 in. and a period of 0.4 sec. Determine (a) the frequency of vibration, (b) the maximum velocity, and (c) the maximum acceleration.

2-16. From experimental measurements it has been found that a structure is vibrating harmonically. If the amplitude of motion was measured to be 2 cm and the period was 3 sec, determine (a) the maximum velocity and (b) the maximum acceleration to which the structure is subjected.

2-17. A simple harmonic motion has a maximum amplitude of 50 mm and a period of 10 sec. Determine (a) the displacement, (b) the velocity, and (c) the acceleration, at the time $t = 8$ sec, if the time is zero when the motion passes through the equilibrium position.

2-18. Vibration-measuring instruments indicate that a structure is vibrating harmonically at a frequency of 4 Hz, with a maximum acceleration of 324 in./sec^2.

SOLUTION

$$k = \frac{AE}{l} = \frac{(0.5)^2 \times 10^7}{100} = 2.5 \times 10^4 \text{ lb/in.}$$

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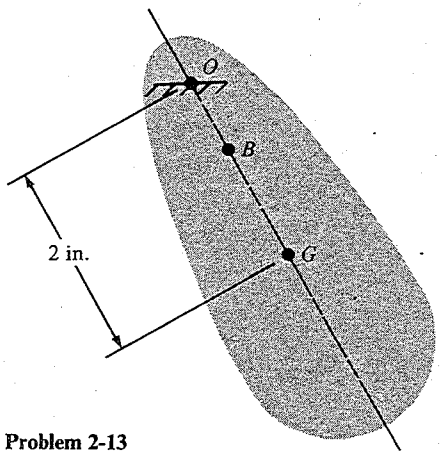
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SESSION #7

if $\frac{c}{2m} = \sqrt{\frac{k}{m}} = \omega_n$ then $c = c_{cr}$ critical damping cond.

$$x(t) = (c_1 + c_2 t) e^{-\frac{c_c}{2m} t} = (c_1 + c_2 t) e^{-\omega_n t} \qquad c_c = 2\sqrt{km} = 2m\omega_n$$

CAN DEFINE DAMPING FACTOR

$$\frac{c}{c_c} = \zeta \quad \therefore \quad \frac{c}{2m} = \frac{c}{c_c} \cdot \frac{c_c}{2m} = \zeta(\omega_n) \qquad c = 2m\omega_n\zeta$$

now $\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} = \sqrt{\zeta^2\omega_n^2 - \omega_n^2} = \sqrt{\zeta^2 - 1}\,\omega_n$

General Solution $= c_1 e^{\left(-\zeta + \sqrt{\zeta^2-1}\right)\omega_n t} + c_2 e^{\left(-\zeta - \sqrt{\zeta^2-1}\right)\omega_n t}$ for $\zeta \neq 1$

$c_1 e^{s_1 t} + c_2 e^{s_2 t}$

- if $\zeta > 1$ DAMPING CONST > CRIT. DAMP. CONST.

$\zeta > 1$; $\sqrt{\zeta^2 - 1} < \zeta$ $\quad \therefore \quad -\zeta + \sqrt{\zeta^2 - 1} < 0$

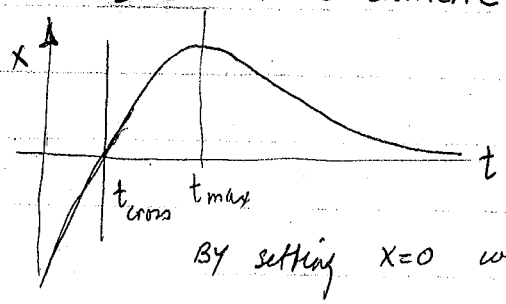
$-\zeta - \sqrt{\quad} < 0$

$x(t)$ = sum of 2 decreasing exponential $= c_1 e^{-\alpha t} + c_2 e^{-\beta t}$

$\alpha = (\zeta - \sqrt{\zeta^2 - 1})\omega_n \qquad \beta = (\zeta + \sqrt{\zeta^2 - 1})\omega_n$

$x(t)$ is not periodic or aperiodic

- $\zeta > 1$ ABOVE CRITICAL DAMPING overdamped



$x_0 = c_1 + c_2$

$\dot{x}_0 = -\alpha c_1 - \beta c_2$ @ $t = 0$

$\frac{\alpha x_0 + \dot{x}_0}{\alpha - \beta} = c_2 \qquad \frac{\beta x_0 + \dot{x}_0}{\beta - \alpha} = c_1$

BY setting $x = 0$ we get first crossing pt. $t_{crossing} = \frac{\ln(-c_2/c_1)}{\beta - \alpha}$

$$t_{max} = \frac{\ln(-\beta c_2 / \alpha c_1)}{\beta - \alpha} \qquad \beta - \alpha = 2\omega_n\sqrt{\zeta^2 - 1}$$

$\frac{dx}{dt} = 0 \Rightarrow -c_1\alpha e^{-\alpha t} - c_2\beta e^{-\beta t}$

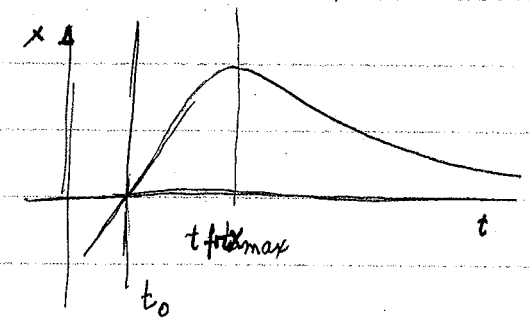
- IF $\zeta = 1$ $\quad x(t) = (A+Bt)e^{-\omega t}$ APERIODIC FN

$x_0 = A$

$\dot{x}_0 = B - A\omega$

$B = \dot{x}_0 + A\omega = \dot{x}_0 + x$

FOR THE CRITICALLY DAMPED CASE $\zeta = 1$



$x = 0 \Rightarrow A + Bt_0 = 0 \quad t_0 = -\frac{A}{B}$

t_0 is first crossing

$x(t) = (A+Bt)e^{-\omega t} = (x_0[1+\omega t] + \dot{x}_0 t)e^{-\omega t}$

$\dot{x} = \left[(B - A\omega) - B\omega t\right]e^{-\omega t} \qquad \dot{x} = 0 \Rightarrow$

$t_{max} = \frac{1}{\omega} - \frac{A}{B}$

- FOR THE SUBCRITICAL DAMPING CASE $\zeta < 1$ underdamped.

$$x(t) = C_1 e^{(-\zeta + i\sqrt{1-\zeta^2})\omega_n t} + C_2 e^{(-\zeta - i\sqrt{1-\zeta^2})\omega_n t} = e^{-\zeta\omega_n t}\left[C_1 \sin(\omega_d t) + C_2 \cos(\omega_d t)\right]$$

$$= C e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n$ damped circular freq

- FOR $x = 0$

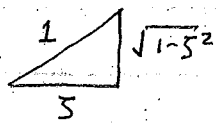
Crossing points $\omega_d t + \phi = n\pi \qquad t = (n\pi - \phi)/\omega_d$

- TO FIND THE MAX OR MIN

$$\dot{x}(t^*) = 0 = Ce^{-\zeta\omega t^*}\left[-\zeta\omega \sin(\omega_d t^* + \phi) + \omega_d \cos(\omega_d t^* + \phi)\right]$$

IF $t^* \neq \infty$ $e^{-\zeta\omega t^*}$ and C are not zero

$$\Rightarrow \quad \tan(\omega_d t^* + \phi) = \frac{\omega_d}{\zeta\omega_n} = \frac{\sqrt{1-\zeta^2}}{\zeta}$$



since $\zeta < 1$ tan is (+) in 1ST & 3rd QUAD.

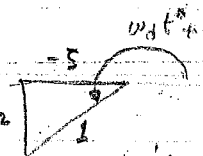
S	A
T	C

in 1ST QUAD $\quad \sin(\omega_d t_1^* + \phi) = \sqrt{1-\zeta^2} < 1$ MAXS

$$\sin(\omega_d t_1^* + \phi) = \sin(\omega_d t_1^* + \phi + 2\pi) = \ldots$$

$$t_1^* = \left[\sin^{-1}\sqrt{1-\zeta^2} - \phi - 2n\pi\right]/\omega_d \qquad \text{max.}$$

First max is at $t_1^* = \left[(\sin^{-1}\sqrt{1-\zeta^2}) - \phi\right]/\omega_d$

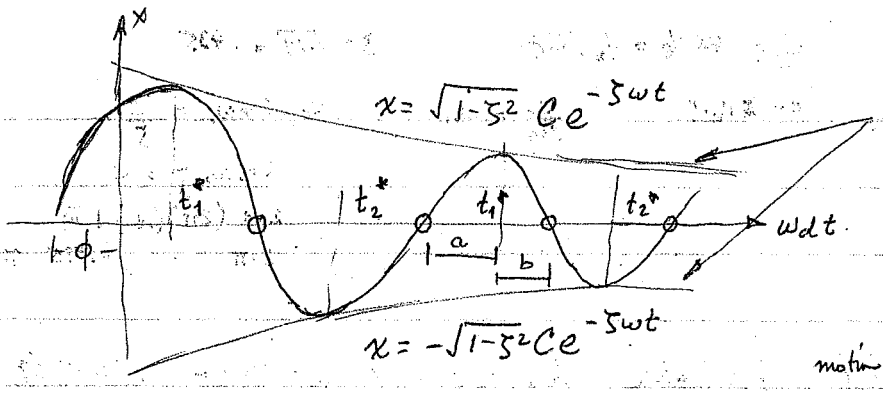


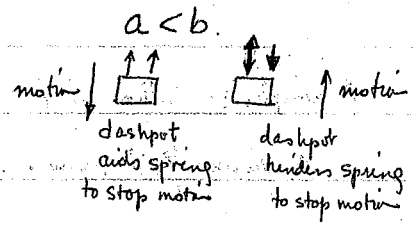
in 3rd QUAD $\quad \sin(\omega_d t_2^* + \phi) = -\sqrt{1-\zeta^2} > -1$ MINS

$$\sin(\omega_d t_2^* + \phi) = \sin(\omega_d t_2^* + \phi + 2\pi) = \ldots$$

$$t_2^* = \left[\sin^{-1}(-\sqrt{1-\zeta^2}) - \phi - 2n\pi\right]/\omega_d = \left[\sin^{-1}(\sqrt{1-\zeta^2}) - \phi - (2n+1)\pi\right]/\omega_d$$

$$\omega_d t_2^* = \omega_d t_1^* + \pi$$





- period between maximums. $\omega_d \tau = 2\pi$

$$\tau = \frac{2\pi}{\omega_d} = \frac{2\pi}{\sqrt{1-\zeta^2}\,\omega}.$$

ENVELOPES

- since at max. $\sin(\omega_d t_1^* + \phi) = \sqrt{1-\zeta^2}$

$$x \underset{ENV.}{=} C e^{-\zeta\omega t} \sin(\omega_d t_1^* + \phi) = C e^{-\zeta\omega t}\sqrt{1-\zeta^2}$$

- at min $\sin(\omega_d t_2^* + \phi) = -\sqrt{1-\zeta^2}$

$$x \underset{ENV.}{=} C e^{-\zeta\omega t} \sin(\omega_d t_2^* + \phi) = -\sqrt{1-\zeta^2}\, C e^{-\zeta\omega t}$$

EFFECTS OF DAMPING. $\zeta > 0$

- REDUCES THE AMPLITUDE OF THE MOTION
- REDUCES TIME TO FIRST PEAK $\omega t_1^* = \frac{\sin^{-1}\sqrt{1-\zeta^2} - \phi}{\sqrt{1-\zeta^2}} \cong 1 + \frac{(1-\zeta^2)}{6} + \frac{3}{40}(1-\zeta^2)^2 + \ldots$

$\omega t_1^* = \pi/2 - \phi$ vs. $\pi/2 - \zeta - \phi$

- INCREASES TIME OF MOTION TO RETURN TO NEUTRAL POSITION

$\omega t_{cross} = \pi - \phi$ vs. $\omega t_{cross} = \frac{\pi - \phi}{\sqrt{1-\zeta^2}}$ FIRST crossing

$\sin(\omega_n t + \phi)$ $\sin(\omega_d t + \phi)$

- PERIOD OF DAMPED MOTION > PERIOD OF UNDAMPED MOTION

$\omega_n \tau_n = 2\pi$ vs $\omega_n \tau_d = \frac{2\pi}{\sqrt{1-\zeta^2}}$ $\tau_n = \frac{2\pi}{\omega_n}$ & $\tau_d = \frac{2\pi}{\omega_d}$ but $\omega_d < \omega_n \Rightarrow \tau_d > \tau_n$

- PEAKS STARTED TO THE LEFT OF THE UNDAMPED PEAKS
  - SINCE PERIOD INCREASES $\Rightarrow$ PEAKS MOVE TO RIGHT OF UNDAMPED
- DAMPING DUE TO CRITICAL DAMPING CAUSES AMPLITUDE TO DROP TO ZERO MOST QUICKLY FOR SAME I.C.
- FREQ DECREASES $\omega_d = \sqrt{1-\zeta^2}\,\omega$



SESSION #8

LOGARITHMIC DECREMENT

- WANT TO MEASURE THE RATE OF DECAY OF SYSTEM
- TAKE 2 POINTS IN PHASE (i.e. amplitudes)

$$x_j = C\sqrt{1-\zeta^2}\, e^{-\zeta\omega t_j} \quad \text{or} \quad Ce^{-\zeta\omega t_j}\sin(\omega_d t_j + \phi)$$

$$x_{j+1} = C\sqrt{1-\zeta^2}\, e^{-\zeta\omega(t_j+\tau)} \quad \text{or} \quad Ce^{-\zeta\omega(t_j+\tau)}\sin[\omega_d(t_j+\tau)+\phi]$$

$$\sin[\omega_d(t_j+\tau)+\phi] = \sin(\omega_d t_j + \phi + 2\pi)$$

$$\frac{x_j}{x_{j+1}} = e^{\zeta\omega_n\tau_d} = \text{constant}; \quad \ln\frac{x_j}{x_{j+1}} = \zeta\omega_n\tau_d = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} \qquad \tau_d = 2\pi/\omega_d$$

LOGARITHMIC DECREMENT $\quad \delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \ln\left(\dfrac{x_j}{x_{j+1}}\right)$ SAME FOR ANY SUCCESSIVE AMPLITUDES

- FOR THE DECAY AFTER n cycles

$$\frac{x_0}{x_n} = \frac{x_0}{x_1}\frac{x_1}{x_2}\frac{x_2}{x_3}\cdots\frac{x_{n-1}}{x_n} = \left(\frac{x_j}{x_{j+1}}\right)^n$$

$$\ln\left(\frac{x_0}{x_n}\right) = n\ln\left(\frac{x_j}{x_{j+1}}\right) = n\delta$$

$$\delta = \frac{1}{n}\ln\left(\frac{x_0}{x_n}\right)$$

or

- DETERMINE NO. OF CYCLES TO CAUSE DECAY FROM x_0 TO x_n

$$n = \frac{1}{\delta}\ln\left(\frac{x_0}{x_n}\right) = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta}\ln\left(\frac{x_0}{x_n}\right)$$ NOT INTEGER

- TIME FOR THIS TO OCCUR IS (τ = period between cycles)

$$n\tau = \frac{\sqrt{1-\zeta^2}}{2\pi\zeta}\ln\left(\frac{x_0}{x_n}\right)\frac{2\pi}{\sqrt{1-\zeta^2}\,\omega_n}$$

$$\Delta\tau = n\tau = \frac{1}{\zeta\omega_n}\ln\left(\frac{x_0}{x_n}\right)$$

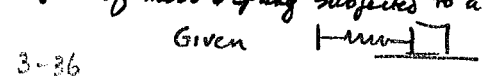
if from exper we can get $\delta = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \zeta = \dfrac{\delta}{\sqrt{(2\pi)^2+\delta^2}}$

if k & m are known $\omega_n = \sqrt{\frac{k}{m}}$

$$\frac{c}{2m} = \zeta\omega_n \Rightarrow c = 2m\zeta\omega_n$$

HW 3-26, 3-27, 3-30, 3-35

A system is composed of mass & spring subjected to a coulomb force. If initial amplitude is find amplitude after eight cycles & frequency

Problem 3-36 Given  FIND X AFTER 8 cycles

$m = 57.61\ kg \quad k = 9100\ N/m \quad F_c = 6.825N \quad X_0 = 4cm$ find X_8

$$\Delta X = \frac{4F_c}{k} = \frac{4(6.825N)}{9100\ N/m} = .003m = .3cm$$

at end of 8 cycles decrease by $8(.3cm) = 2.4cm \quad \therefore \quad X_8 = X_0 - 8(\Delta X) = 1.6cm$

- $f = \frac{\omega}{2\pi} \qquad \omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{9100\ N/m}{57.61\ kg}} = 12.5682\ rad/sec$

$$f = 2\ Hz = \frac{\omega}{2\pi}$$

CESSATION OF MOTION

- $X_{max} = \frac{F_c}{k} = .075cm \qquad \frac{X_0 - .075}{n} = .3 \quad \text{or} \quad \frac{4 - .075}{.3} = \frac{3.925}{.3} \cong 13.1$ cycles TO CESSATION

TOPIC IV Harmonically Forced Vibs – UNDAMPED CASE

- LOOK AT CASE OF MASS-SPRING SYSTEM INFLUENCED BY A TIME DEPENDENT FORCING FUNCTION

- FORCE MAY BE HARMONIC, OR NONHARMONIC BUT PERIODIC, OR NONPERIODIC OR RANDOM

- WANT DYNAMIC RESPONSE OF SYSTEM CAUSED BY FORCE

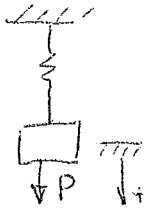
- LOOK AT SDOF W/ HARMONIC FORCING CONDITION OF FORM $P(t) = P_0 \sin(\omega_f t + \psi)$ RAO use $F_0 \cos\omega t$

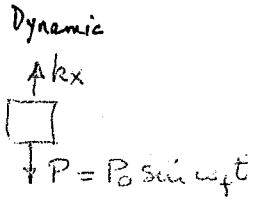
P_0 is amplitude of $P(t)$

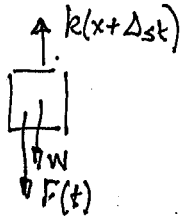
ω_f is the circular freq of FORCING FUNC ω in RAO

ψ is the phase angle ψ is DEPENDENT ON I.C. of $P(t)$

LOOK AT CAUSE WHERE $\psi = 0$

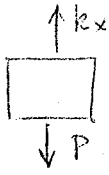






SESSION #10

FOR DYNAMIC FREE-BODY DIAG.



$m\ddot{x} = -kx + P$

$m\ddot{x} + kx = P_o \sin \omega_f t$

solution is $x_c + x_p = x$

FREE VIBS SOLN : x_c satisfies $m\ddot{x} + kx = 0$ $\quad x_c = A \sin \omega_n t + B \cos \omega_n t$ $\quad \omega_n = \sqrt{\frac{k}{m}}$

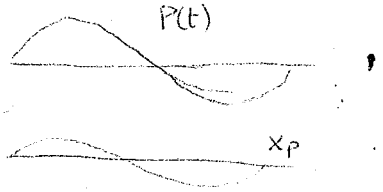
FORCED VIBS : x_p " $m\ddot{x} + kx = P_o \sin \omega_f t$ or $P_o \cos \omega_f t$

- IF we choose $x_p = \begin{cases} C \sin \omega_f t \\ C \cos \omega_f t \end{cases}$ $\quad C = \frac{P_o}{k - m\omega_f^2} = \frac{P_o}{m} \frac{1}{\omega_n^2 - \omega_f^2}$

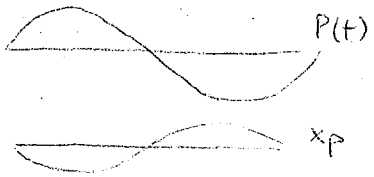
$$\frac{P_o}{m} \frac{1}{\omega_n^2 \left[1 - \left(\frac{\omega_f}{\omega_n}\right)^2\right]} = \frac{P_o}{m \cdot k/m} \frac{1}{[1 - r^2]} = \frac{P_o/k}{1 - r^2} = X = \frac{X_o}{1 - r^2}$$

- $X_o = P_o/k$ STATIC DISPL OF SPRING DUE TO P_o $\quad \delta_{ST}$ in RAO
- $r = \omega_f/\omega_n$ IS THE FREQ ratio $\quad \frac{\omega}{\omega_n}$ in RAO

- if $r < 1$ $\quad 0 < 1 - r^2 < 1$ $\quad X_p > 0$
  and x_p is in phase w/ $P(t)$



- if $r > 1$ $\quad 1 - r^2 < 0$ $\quad X_p < 0$
  and x_p is opposite to $P(t)$



- if $r = 1$ $\quad 1 - r^2 = 0$ and we have <u>resonance</u>

if $x(0) = \dot{x}(0) = 0$ $\quad \Rightarrow \dot{x}(0) = -\frac{X_o \omega_n}{2}$

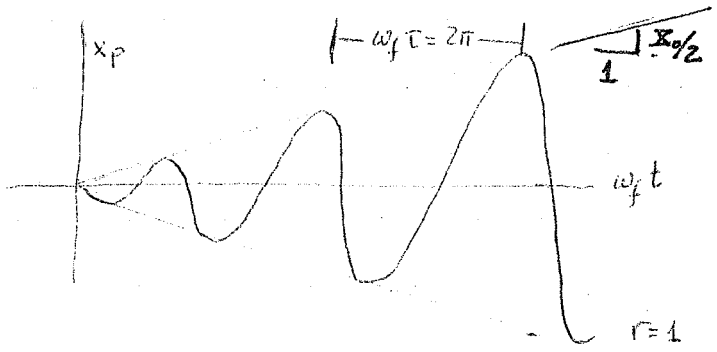
$$x_p = \frac{P_o}{2m\omega_f} t [-\cos \omega_f t] = \frac{P_o/k}{2\omega_f \left(\frac{1}{\omega_f}\right)^2} t [-\cos \omega_f t] = \frac{X_o \omega_f}{2} t [-\cos \omega_f t] \; [+\sin \omega_f t]$$

Note 1: amplitude of $x_p = \frac{X_o \omega_f t}{2}$ increases linearly w/ time

2: since $\sin(\omega_f t + \pi/2) = -\cos \omega_f t$ $\quad$ <u>displ</u> lags <u>force</u> by $\pi/2$
$\cos(\omega_f t - \pi/2) = \sin \omega_f t$

3. solution to X_h has already been discussed under FREE UNDAMPED motion

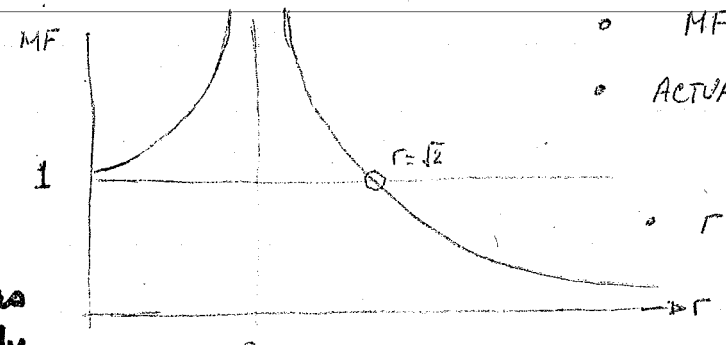
4. $x_p = \frac{X_0 \omega_f t}{2} \begin{matrix} \sin(\omega_f t - \pi/2) \\ \cos(\omega_f t - \pi/2) \end{matrix}$ if $P = P_0 \sin \omega_f t$ / $P_0 \cos \omega_f t$



MAGNIFICATION FACTOR

- NORMALLY AMPLITUDE OF FORCED MOTION IS IMPORTANT
- DEFINE MAGNIFICATION RATIO $MF = \frac{X}{X_0} = \frac{X_0/1-r^2}{X_0} = \frac{1}{1-r^2}$ $r<1$

  or $\frac{X}{X_0} = \left|\frac{1}{r^2-1}\right|$ $r>1$ RATIO $\frac{\text{DYNAMIC}}{\text{STATIC}}$ AMPLITUDE = **MF** OF FORCING FN



- $MF \to \infty$ as $r \to 1$
- ACTUALLY TIME FOR $MF \to \infty$ IS VERY LARGE
- r CAN BE ALTERED BY CHANGING k, m, ω_f
- CHANGING $k \to X_0$ CHANGES

MF varies as r varies only through changes in ω_f

$r<1$

1. IF $\tau_f > \tau$ / $\omega_f < \omega_n$ FREE MOTION OSCILLATES ABOUT FORCED VIB
2. $\omega_f > \omega_n$ / $\tau_f < \tau$ FORCED MOTION " ABOUT FREE VIB

SEE PG 135

$r>1$

3. IF $\omega = \omega_f$ $r = 1$ for small TIME FREE VIBS DOMINATES

for LARGE TIME FORCED VIBS DOMINATES

ASSUME $x_c = 0$ & $x = x_p$

4. IF FORCED MOTION CEASES – FREE VIBS CONTINUES

- CONTINUES w/ x, $\dot{x}$ BEING VALUES WHEN FORCED MOTION CEASES
- CHECK OUT SECT 4-5 ON THIS $x = x_0 \cos \omega t + \frac{\dot{x}_0}{\omega} \sin \omega t$
- IF $\dot{x} = 0$ AMPLITUDE REMAINS SAME ω CHANGES
- IF $x = 0$ AMPLITUDE CHANGES

  IF $\omega_f < \omega$ AMPLITUDE ↓ $\tau_f > \tau$ CYCLES occur FASTER

  IF $\omega_f > \omega$ " ↑ " $\tau_f < \tau$ " SLOWER
- IF $\dot{x} \neq 0$, $x \neq 0$ RESULT IS IN BETWEEN

DON'T DO

FOR THE RESONANT CONDITION TO $m\ddot{x} + kx = P_0 \sin \omega_f t$

- $x_{TOTAL} = x_c + x_p = A \sin \omega_n t + B \cos \omega_n t + \frac{X_0}{1 - r^2} \sin \omega_f t$
- IF $x = x_0$ @ $t = 0$ & $\dot{x} = \dot{x}_0$ @ $t = 0$

  $B = x_0$ and $A = \left[\dot{x}_0 - \frac{X_0 \omega_f}{1 - r^2} \right] \frac{1}{\omega_n} = \frac{\dot{x}_0}{\omega_n} - \frac{X_0 r}{1 - r^2}$

Thus $x = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{X_0}{1 - r^2} \left[\sin \omega_f t - r \sin \omega_n t \right]$

SUPPOSE $x_0 = \dot{x}_0 = 0$; @ resonance $r = 1$ $\frac{P_0}{m[\omega_n^2 - \omega^2]}$ $x \neq x_p$ under these conditions

$$x = \frac{X_0}{1 - r^2} \left[\sin \omega_f t - r \sin \omega_n t \right] = \frac{2 X_0}{1 - r^2} \sin \left(\frac{\omega_f - \omega_n}{2} \right) t \cdot \cos \left(\frac{\omega_f + \omega_n}{2} \right) t$$

if $\omega_n - \omega_f = 2\Delta \ll 1$ $\omega_n + \omega_f \cong 2\omega_f$ $\omega_n^2 - \omega_f^2 = 4\omega_f \Delta$

$1 - r^2 \cong 4\Delta / \omega_n \cdot r$

$$x = -\frac{2 X_0}{1 - r^2} \sin \Delta t \cdot \cos \omega_f t = \frac{-2 X_0}{(4\Delta/\omega) r} \sin \Delta t \cdot \cos \omega_f t = -\frac{X_0}{2} \omega_f t \cos \omega_f t$$

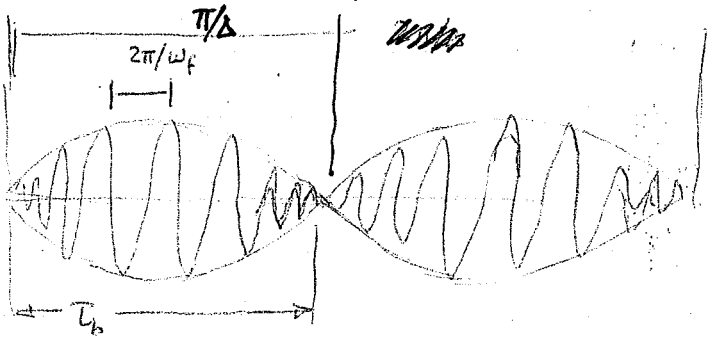
as $r \to 1$

- IF $\omega_f \approx \omega$ but $\neq$ BEATING OCCURS
  - AMPLITUDE ↑ THEN ↓

$r \cong 1$

$$X = -\frac{X_0 \omega}{2\Delta r} \sin \Delta t \cos \omega_f t = \left(-\frac{X_0 \omega}{2\Delta} \cos \omega_f t\right) \sin \Delta t = \left(-\frac{X_0 \omega}{2\Delta} \sin \Delta t\right) \cos \omega_f t$$

- since $\omega_f >> \Delta$ cosine goes through many cycles before sine goes through one
  $\tau_f << \tau$
- $-\frac{X_0 \omega}{2\Delta r} \cos \omega_f t$ is a cyclic varying amplitude
- $\sin \Delta t$ represents envelope of the amplitude.

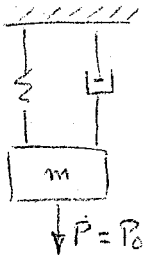


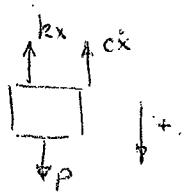
- BEAT PERIOD $\tau_b = \frac{\pi}{\Delta}$ TIME BETWEEN POINTS OF ZERO MOTION
  $= \frac{2\pi}{\omega_n - \omega_f}$ OR BETWEEN MAXIMUM MOTIONS

→ HW 4-2, 6, 13

SESSION #11 + PROBLEMS

FORCED VIBRATIONS w/ VISCOUS DAMPING





DYNAMIC F.B.D.

$$m\ddot{x} = -kx - c\dot{x} + P$$

$$m\ddot{x} + c\dot{x} + kx = P_0 \sin \omega_f t$$

HAS SOLUTION $x = x_p + x_h$

x_h solves $m\ddot{x} + c\dot{x} + kx = 0$

x_p " $= P_0 \sin \omega_f t$

HAVE STUDIED HOMOGENEOUS – GIVES RISE TO

- UNDERDAMPED, CRITICAL DAMPED, OVERDAMPED
- AFTER SOME TIME – THE SOLUTION OF HOMOGENEOUS DE DIES OUT
- PARTICULAR REMAINS

- SOLUTION TO HOMOG → TRANSIENT
- SOLUTION TO PARTIC → STEADY STATE

- WILL CONCENTRATE ON STEADY STATE – **PARTICULAR SOLN**

$\zeta \geq 1$ $\quad C_1 e^{s_1 t} + C_2 e^{s_2 t}$

- IF $\zeta < 1$ $\quad x_h = C e^{-\zeta \omega_n t} \sin(\omega_d t + \phi)$ $\quad \zeta = \frac{c}{c_c}$; $\omega_n = \sqrt{\frac{k}{m}} = \frac{c_{cr}}{2m}$

$\zeta = 1$ $\quad x_h = (C_1 + C_2 t) e^{-\omega_n t}$ $\quad \omega_d = \sqrt{1-\zeta^2}\,\omega_n$

- FOR STEADY STATE TAKE $x_p = X_{ss} \sin(\omega_f t - \psi) \Rightarrow X_{ss} \cos\psi = A \quad B = -X_{ss} \sin\psi$

choose $x_p = A \sin\omega_f t + B \cos\omega_f t$ since DE INVOLVES $\ddot{x}, \dot{x}, x$

$$\dot{x}_p = A\omega_f \cos\omega_f t - B\omega_f \sin\omega_f t$$

$$\ddot{x}_p = -A\omega_f^2 \sin\omega_f t - B\omega_f^2 \cos\omega_f t$$

$$m\ddot{x}_p + c\dot{x}_p + kx_p = P_0 \sin\omega_f t + 0 \cdot \cos\omega_f t$$

$$\left[-mA\omega_f^2 - cB\omega_f + kA\right]\sin\omega_f t + \left[-mB\omega_f^2 + cA\omega_f + kB\right]\cos\omega_f t = P_0 \sin\omega_f t$$

skip

$$\Rightarrow \quad \left[-m\omega_f^2 + k\right]A - c\omega_f B = P_0$$

$$\left[+c\omega_f\right]A + \left[-m\omega_f^2 + k\right]B = 0$$

$$A = \frac{\begin{vmatrix} P_0 & -c\omega_f \\ 0 & k - m\omega_f^2 \end{vmatrix}}{\text{denom}}$$

$$\text{denominator} = \left(k - m\omega_f^2\right)^2 + c^2\omega_f^2 = m^2\left(\left[\omega_n^2 - \omega_f^2\right]^2 + 4\omega_n^2\omega_f^2\zeta^2\right) \quad B = \frac{\begin{vmatrix} k - m\omega_f^2 & P_0 \\ c\omega_f & 0 \end{vmatrix}}{\text{denom}}$$

thus $\quad A = \dfrac{P_0\left(k - m\omega_f^2\right)}{\text{denom}} \qquad B = \dfrac{-c\omega_f P_0}{\text{denom}} \qquad \text{denom} = \left(k - m\omega_f^2\right)^2 + \left(c\omega_f\right)^2$

skip this derivation → $\text{denom} = m^2\omega_n^4\left(\left[1 - r^2\right]^2 + \left(2r\zeta\right)^2\right) = k^2\left[\left(1 - r^2\right)^2 + \left(2r\zeta\right)^2\right]$

just write this

$$\mathbb{X}_{ss} = \sqrt{A^2+B^2} = \frac{P_0}{\sqrt{\text{denom}}} = \frac{P_0/k}{\sqrt{(1-r^2)^2+(2r\zeta)^2}} = \frac{\mathbb{X}_0}{\sqrt{(1-r^2)^2+(2r\zeta)^2}}$$

$$\tan\psi = -\frac{B}{A} = \frac{c\omega_f}{k-m\omega_f^2} = \frac{2\zeta r}{1-r^2}$$

BOTH $\mathbb{X}$ & ψ are fns of ζ & r

$x_p = \mathbb{X}\sin(\omega_f t - \psi)$ <—> $P_0 \sin\omega_f t = P$ TIME LAG IN RESPONSE $r<1$ LEAD $r>1$

SESSION #12

- SAME MOTION BUT x_p lags $P(t)$ by ψ
- AMPLITUDE IS $\mathbb{X}$
- $\forall \zeta > 0$ REDUCES $\mathbb{X}$ and for

NOTE IF $\zeta = 0$ (NO DAMPING) $\psi = 0$ & $\mathbb{X} = \frac{\mathbb{X}_0}{|1-r^2|}$ AS BEFORE

- THE LAG TIME t' IS TIME x_p LAGS $P(t)$
  - IE when $t=0$ $\sin\omega_f t = 0$ but $\sin(\omega_f t - \psi) \neq 0$
  - $\sin(\omega_f t' - \psi) = 0$ when $t' = \frac{\psi}{\omega_f}$

IF UNDERDAMPED

TOTAL SOL'N is $x = Ce^{-\zeta\omega t}\sin(\omega_d t + \phi) + \mathbb{X}\sin(\omega_f t - \psi)$

GIVE HANDOUT

2nd ed. RAO 129

$x = (A+Bt)e^{-\omega_n t} + \mathbb{X}\sin(\omega_f t - \psi)$ for critical

- LOOK AT PAGE 107 FOR TRANSIENT —> STEADY STATE CONVERSION

ASSUMED: $\omega_f < \omega_d$ = $\omega_f < \sqrt{1-\zeta^2}\,\omega_n \Rightarrow r < \sqrt{1-\zeta^2} < 1$

$\tau_f > \tau_d$

- NOTE THAT $\mathbb{X}_{\text{DAMPED}} < \mathbb{X}_{\text{UNDAMPED}}$ FOR ALL ω_f
- NOTE WHEN $r=1$ $\mathbb{X}_{RES} = \frac{\mathbb{X}_0}{2\zeta} \neq \infty$ EXCEPT FOR UNDAMPED

EXTREMUMS OF MF

- $MF = \frac{\mathbb{X}}{\mathbb{X}_0} = \frac{1}{\sqrt{(1-r^2)^2+(2r\zeta)^2}}$ $MF = MF(r,\zeta)$

take $\frac{d(MF)}{dr} = -\frac{1}{2}\frac{[2(1-r^2)(-2r)+4\zeta(2r\zeta)]}{[\quad]^{3/2}} = \frac{4r[-1+r^2+2\zeta^2]}{2[\quad]^{3/2}} = 0$

$dMF/dr = 0$

when $r = 0$

when $r \to \infty$ $\quad d(MF)/dr \to -\frac{(-2r^3)}{r^6} = \frac{2}{r^3} \to 0 \quad r \to \infty \quad \forall \zeta$

when $1 - r^2 + 2\zeta^2 = 0 \quad r = \sqrt{1-2\zeta^2} < 1$ for $0 \le \zeta \le \frac{1}{\sqrt{2}} = .707$

for $0 \le \zeta \le .707$ there is a relative max. $\Rightarrow \omega = \omega_n\sqrt{1-2\zeta^2} < \omega_d$.

- THIS DEFINES MAX POINT IN THE RESONANT REGION & SINCE $r<1$

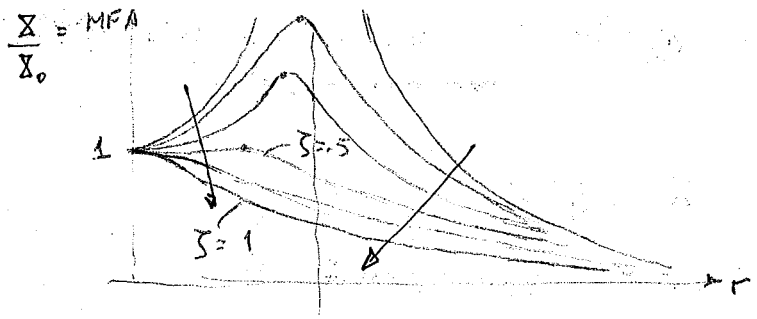
Pg 125 OCCURS TO THE LEFT OF THE RESONANT VALUE $r=1$

PUT $r = \sqrt{1-2\zeta^2}$ in $MF = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

$$(MF)_{max} = \frac{X_{max}}{X_0} = \frac{1}{\sqrt{(2\zeta^2)^2 + (2\zeta)^2[1-2\zeta^2]}} = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \quad \text{for } \zeta < 0.707$$

SOLN TO 3.9 $\Rightarrow \dfrac{X_{max}}{X_0} = \dfrac{X_{RES}}{X_0}\dfrac{1}{\sqrt{1-\zeta^2}} \Rightarrow \dfrac{X_{max}}{X_{res}} = \dfrac{1}{\sqrt{1-\zeta^2}}$

- FOR $\zeta \ge 0.707$ $(MF)_{max}$ is at $r = 0$



3 extremums for $\zeta < .707$
2 extremum for $\zeta > .707$

- $(MF)_{max}$ occurs when $r = \sqrt{1-2\zeta^2}$ or when $\omega_f = \omega_n\sqrt{1-2\zeta^2}$

$$\omega_f = \omega_n\sqrt{1-2\zeta^2} < \omega_n\sqrt{1-\zeta^2} < \omega_n$$
$$\omega_d$$

- MF will decrease if r is large
- look at this as ω_f change only when r changes

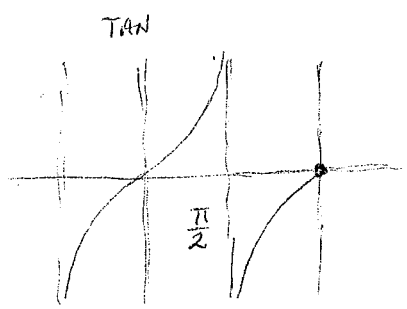
* GO TO MASS & VISCOSITY VARIATIONS NEXT - THEN RETURN

- PHASE ANGLE VARIATION with r

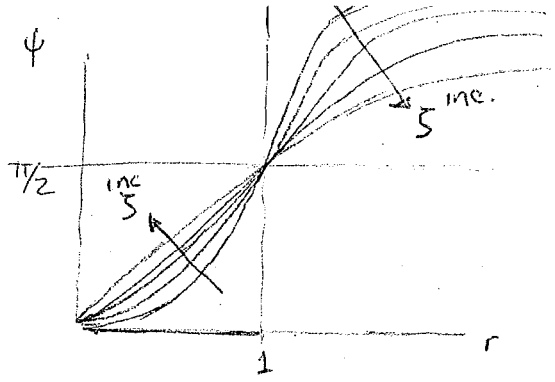
$$\tan\psi = \frac{2\zeta r}{1-r^2} = \frac{c\omega_f}{k - m\omega_f^2}$$; for ANY const ζ as $r \uparrow$ $\tan\psi \uparrow$ $r<1$

when $r = 1$ $\tan\psi = \infty \Rightarrow \psi = \pi/2$

when $r > 1$ when $r \uparrow$ $\tan\psi \to -0$



$\Rightarrow$



$$\tan\psi = \frac{2\zeta r}{1-r^2}$$

keeping r fixed: as $\zeta\uparrow$ $\tan\psi\uparrow$ for $r<1$

as $\zeta\uparrow$ for $r>1$ $\tan\psi\downarrow$ TOWARDS $-\infty$ $(\psi\to\pi/2)$

FOR $\zeta=0$ and $r<1$ $\tan\psi=0$

$\zeta=0$ and $r>1$ $\tan\psi=0$ but from negative side is $(\psi\to\pi)$
DISCONTINUITY AT $r=1$

- NOTE $\psi = \psi(\zeta,r) = \psi(c, c_c, \omega_f, \omega) = \psi(c,m,k,\omega_f)$
  BUT NOT P_0
- as $r\to\infty$ $\tan\psi\to -0$ and $\psi\to\pi$ $x_p \to \bar{X}\sin(\omega_f t-\pi)$
- as $r\to 0$ $\tan\psi\to +0$ and $x_p = \bar{X}\sin\omega_f t$ $P = P_0\sin\omega_f t$
- x_p will be in phase for $r<<1$ but will be out of phase for $r>>1$
- $\forall\zeta$ at $r=1$ $x_p = \bar{X}\sin(\omega_f t - \underline{\underline{\pi/2}}) = \frac{\bar{X}_0}{2\zeta}\sin(\omega t-\pi/2)$

- for $r<1$ $\omega_f<\omega_n$ $\psi\uparrow$ when $\zeta\uparrow$
  $r>1$ $\omega_f>\omega$ $\psi\uparrow$ when $\zeta\downarrow$

FIRST PART OF SESSION # 13 TIL m & k Variations

- QUALITY FACTOR & BANDWIDTH

  REMEMBER $\frac{\bar{X}_{RES}}{\bar{X}_0} = \frac{1}{2\zeta}$ $\frac{\bar{X}_{max}}{\bar{X}_0} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$

  FOR $\zeta<<1$ $\frac{\bar{X}_{max}}{\bar{X}_0} \cong \boxed{\frac{\bar{X}_{res}}{\bar{X}_0} = \frac{1}{2\zeta} = Q}$ (Q factor or quality factor)

  $\bar{X}_{max}/\bar{X}_{0_{RES}}$

  in EE - when a filter has an max gain at resonance

- DEFINE BANDWIDTH AS DIFFERENCE in FORCED FREQ's when $MF = \frac{Q}{\sqrt{2}}$

$$MF = \frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} = \frac{1}{2\sqrt{2}\zeta} = \frac{Q}{\sqrt{2}}$$

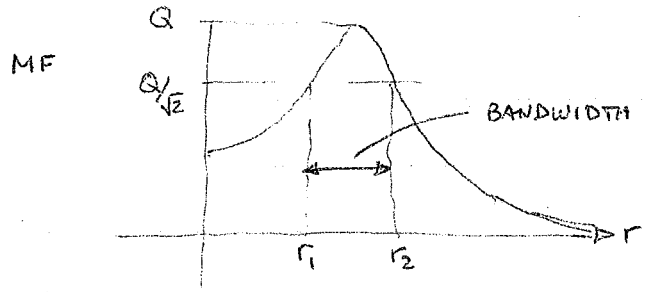
SOLVING FOR $r^2 = (1-2\zeta^2) \pm 2\zeta\sqrt{1+\zeta^2}$

FOR small ζ $\quad r^2 \sim 1 \pm 2\zeta \Rightarrow r \sim 1+\zeta,\ 1-\zeta$

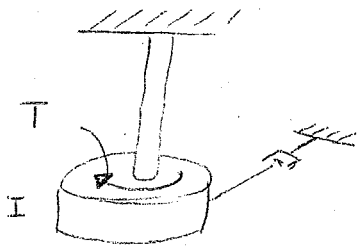
$$r_2 - r_1 = \frac{\omega_{f_2}}{\omega} - \frac{\omega_{f_1}}{\omega} \cong 1+\zeta - (1-\zeta) = 2\zeta$$

$$\text{or} \quad \omega_{f_2} - \omega_{f_1} = 2\zeta\omega_n \qquad \text{BUT} \quad Q = \frac{1}{2\zeta} = \frac{\omega_n}{\omega_{f_2} - \omega_{f_1}}$$

- Q CAN BE USED TO ESTIMATE ζ



r_1, r_2 are the half power pts since power absorbed by the damper is related to the value of Σ^2 & hence MF



$$I\ddot{\theta} = -k_T\theta - c_T\dot{\theta} + T_0 \sin\omega_f t$$
$$I\ddot{\theta} + c_T\dot{\theta} + k_T\theta = T_0 \sin\omega_f t$$

HW 4-21, 4-28, 4-27, 4-29a FIND BAND-WIDTH, MF_{max}, MF_{RES}

PART OF LESSON #13 TO HERE

- MASS & SPRING CONST. VARIATIONS
  - when r varies m, k vary
  - IF k varies Σ_0 Varies
  - when k varies ζ varies $\quad \frac{c}{c_c} = \frac{c}{2m\omega} = \frac{c}{2\sqrt{mk}}$
  - IF m varies ζ varies

CANNOT USE ORIGINAL GRAPHS → JUMP FROM ONE ζ CURVE TO OTHER

- SPRING k VARIATIONS
  - LOOK AT $\quad \Sigma = \dfrac{P_0}{\sqrt{(k-m\omega_f^2)^2 + (c\omega_f)^2}}$

$$\frac{d\Sigma}{dk} = -\frac{(P_0)\,2(k-m\omega_f^2)}{2\left[\quad\right]^{3/2}}$$

- $\dfrac{d\Sigma}{dk} = 0 \Rightarrow \Sigma_{max}$ occurs when $k = m\omega_f^2$ or $\Sigma_{max} = \dfrac{P_0}{c\omega_f}$

the spring due to the maximum impressed force. (c) Obtain the amplitude of the forced motion for the following forced frequencies f_f: 1.5, 3, 3.25, and 7.5 Hz.

4-2. The mass of an undamped mass-spring system is subjected to a harmonic force having a maximum value of 45 lb and a period of 0.25 sec. The body weighs 2.5 lb and the spring constant is 15 lb/in. Determine the amplitude of the forced motion.

4-3. An undamped system is composed of mass $m = 1.1$ kg and spring $k = 4400$ N/m. It is acted on by a harmonic force having maximum value of 440 N and a frequency of 180 cycles/min. Determine the amplitude of the forced motion.

4-4. A weight of 8 lb is suspended from a spring having a constant of 25 lb/in. The system is undamped but the mass is subjected to a harmonic force with a frequency of 7 Hz, resulting in a forced-motion amplitude of 1.59 in. Determine the maximum value of the impressed force.

4-5. The body of an undamped system is driven by a harmonic force having an amplitude of 36 N and a frequency of 450 cycles/min. The body has a mass of 8 kg and exhibits a forced-displacement amplitude of 4.06 mm. Obtain the value of the spring modulus.

4-6. An undamped system consists of an 8.75-kg mass suspended from a spring having a constant of 3500 N/m. A harmonic force acting on the mass and having a maximum value of 187 N causes the system to vibrate with a forced amplitude of 7.6 cm. Determine the frequency of the impressed force.

4-7. A weight W is suspended from a spring having a constant of 25 lb/in. The system is undamped, but the mass is driven by a harmonic force having a frequency of 2 Hz and a maximum value of 10 lb, causing a forced-motion amplitude of 0.5 in. Determine the value of weight W.

4-8. The mass of an undamped system is acted on by force $P = P_0 \cos \omega_f t$, having a maximum value of 10 lb. The spring constant is 3 lb/in. The forced frequency can be any integral multiple of the natural frequency of the system other than 1, thereby avoiding resonance. Only a single forced frequency can exist at a time. Determine the maximum amplitude of the forced motion that could occur.

4-9. Express the relation for the complete motion of an undamped system, excited by a force $P = P_0 \sin \omega_f t$, for the initial conditions of $x = 0$, $\dot{x} = 0$, at $t = 0$. $(r \neq 1.)$

4-10. Write the relation that defines the complete motion of an undamped system, subjected to the harmonic force $P = P_0 \sin \omega_f t$, for the initial conditions of $x = 0$, $\dot{x} = \dot{x}_0$, at $t = 0$. $(r \neq 1.)$

4-11. Express the equation for the complete motion of an undamped system, subjected to the forcing condition $P = P_0 \sin \omega_f t$, for the initial conditions of $x = x_0$, $\dot{x} = 0$, at $t = 0$. $(r \neq 1.)$

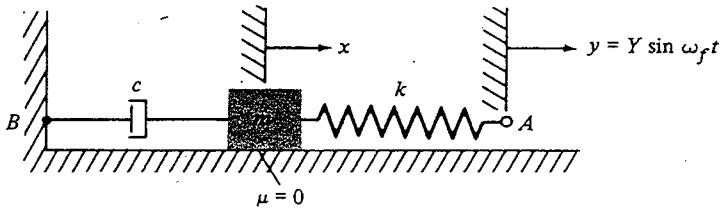
4-12. Write the complete solution for the motion of an undamped system, driven by the force $P = P_0 \sin \omega_f t$, for the initial conditions of $x = x_0$, $\dot{x} = \dot{x}_0$, at $t = 0$. $(r \neq 1.)$

4-13. An undamped system consists of a weight of 19.3 lb and a spring having a modulus of 10 lb/in. The system mass is driven at resonance by harmonic force $P = P_0 \sin \omega_f t$ having a maximum value of 4 lb. Determine the amplitude of the forced motion at the end of (a) $\frac{1}{2}$ cycle, (b) $2\frac{1}{2}$ cycles, (c) $4\frac{1}{2}$ cycles, and (d) $6\frac{1}{2}$ cycles.

4-14. An undamped system is composed of a 4.375-kg mass and a spring having a modulus of 3500 N/m. The mass is driven at resonance by a harmonic force having a maximum value of 14 N. Determine the amplitude of the forced motion at the end of (a) 1 cycle, (b) 5 cycles, and (c) 10 cycles.

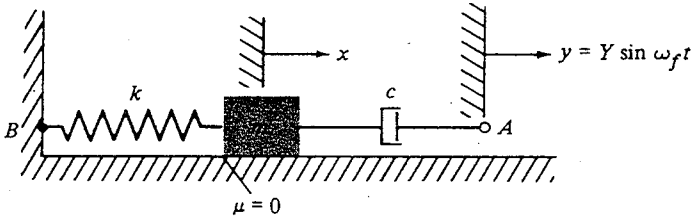
4-15. An undamped system is harmonically forced near resonance, resulting in a beating condition. The natural frequency of the system is 1765 cycles/min, and the forced frequency is 1752 cycles/min. Determine the beat period of the motion.

4-16. For the arrangement shown, x represents the absolute displacement of the mass m, and y is the absolute displacement of the end A of the spring k. Point A is moved according to the relation $y = Y \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation of motion for m. (c) Obtain the solution for the steady-state motion of m. (d) Determine the relation for the impressed force at A. (e) Obtain the expression for the force transmitted to the support at B.



Problem 4-16

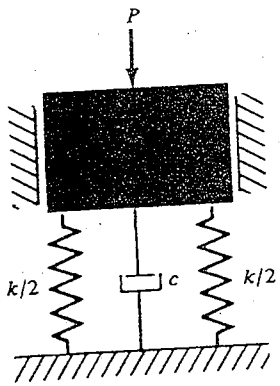
4-17. For the arrangement shown, x represents the absolute displacement of the mass m, and y is the absolute displacement of the end A of the dashpot c. The motion of point A is defined by $y = Y \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation of motion for m. (c) Obtain the solution for the steady-state motion of m. (d) Determine the relation for the impressed force at A. (e) Obtain the expression for the force transmitted to the support at B.



Problem 4-17

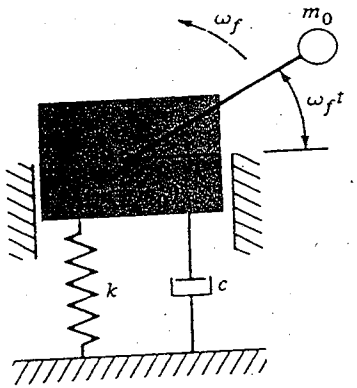
4-18. The spring k and the dashpot c of the accompanying diagram are fastened together at A; x represents the absolute displacement of m, and y is the absolute displacement of the point A. The motion of A is defined by $y = Y \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation of motion for m. (c) Obtain the solution for the steady-state motion of m. (d) Determine the relation for the impressed force at A.

4-31. A machine having a mass of 70 kg is mounted as shown on springs having a total stiffness of 33 880 N/m. The damping factor is $\zeta = 0.20$. A harmonic force $P = 450 \sin 13.2t$ (where P is in newtons and t is in seconds) acts on the mass. For the sustained or steady-state vibration, determine (a) the amplitude of the motion of the machine, (b) its phase with respect to the exciting force, (c) the transmissibility, (d) the maximum dynamic force transmitted to the foundation, and (e) the maximum velocity of the motion.



Problem 4-31

4-32. A machine having a total weight of 96.5 lb is mounted as shown on a spring having a modulus of 900 lb/in. and is connected to a dashpot with a damping *factor* of 0.25. The machine contains a rotating unbalance ($w_0 e$) of 5 lb in. If the speed of rotation is 401.1 rpm, obtain (a) the amplitude of the steady-state motion, (b) the maximum dynamic force transmitted to the foundation, and (c) the angular position of the arm when the structure goes upward through its neutral position.



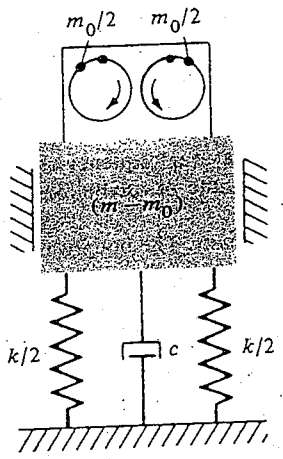
Problem 4-32

• **4-33.** A machine having a total weight of 128.67 lb contains a part that rotates at 859.44 rpm. The machine is supported equally on four identical springs and is

connected to a dashpot for which the damping constant is 10 lb sec/in. operation the machine has a steady amplitude of 0.116 in. and transmits maximum dynamic force of 174 lb to the supporting base. (a) Determine t modulus for the springs. (b) Calculate the unbalanced moment ($w_0 e$) for rotating part.

4-34. A machine with a total mass of 50 kg contains a shaft mechanism that rota at 1800 rpm. The machine is supported by springs for which the equival modulus is 20 000 N/m but the damping constant is unknown. Due to rotating unbalance, the steady-state amplitude is 1.32 cm and the maxim dynamic force carried by the base is 278 N. Determine (a) the damping c stant and (b) the unbalanced moment $m_0 e$.

• **4-35.** A typical vibration exciter is composed of two eccentric masses that rot oppositely, as represented in the top of the accompanying figure. Thus oscillation is produced in the vertical direction only, since the horizontal eff cancels. This device is used to determine the vibrational characteristics of structure to which it is attached. The unbalance ($ew_0/2$) of each exciter whee 3 lb in. The total arrangement has a weight $mg = 200$ lb. The exciter speed eccentric rotation) was adjusted until a stroboscope showed the structure to moving upward through its equilibrium position at the instant the eccen weights were at their top position. The exciter speed then was 840 rpm, and steady amplitude was 0.75 in. Determine (a) the natural frequency of the str ture and the damping factor for the structure. If the speed were changed 1260 rpm, (b) obtain the steady-state amplitude of the structure and (c) angular position of the eccentrics as the structure moves upward through equilibrium position.



Problem 4-35

• **4-36.** A small compressor weighs 69 lb and runs at 875 rpm. It is to be supporte four springs (equally) and no damping is provided. Design the springs, determining k, so that only 15% of the shaking force is transmitted to supporting foundation or structure.

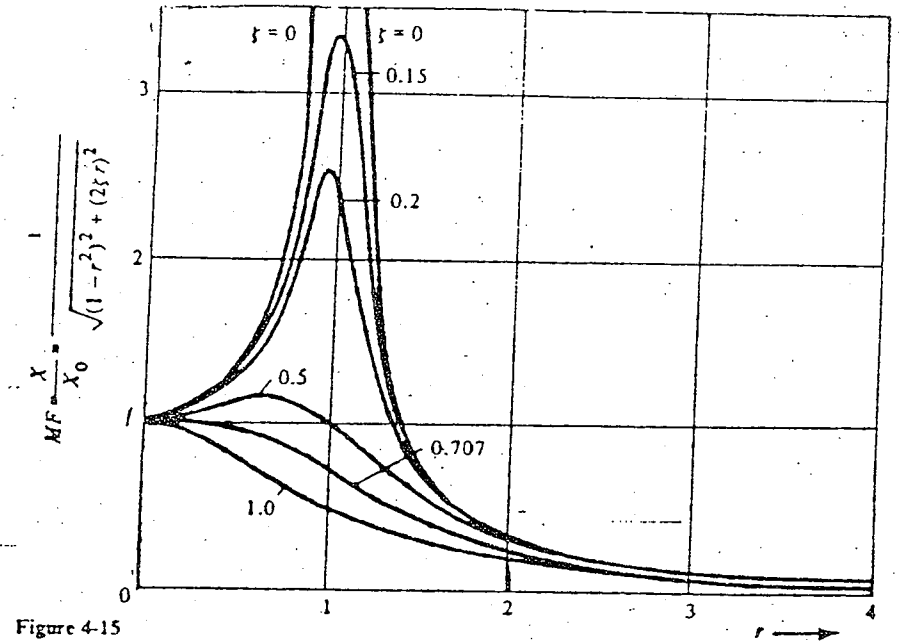


Figure 4-15

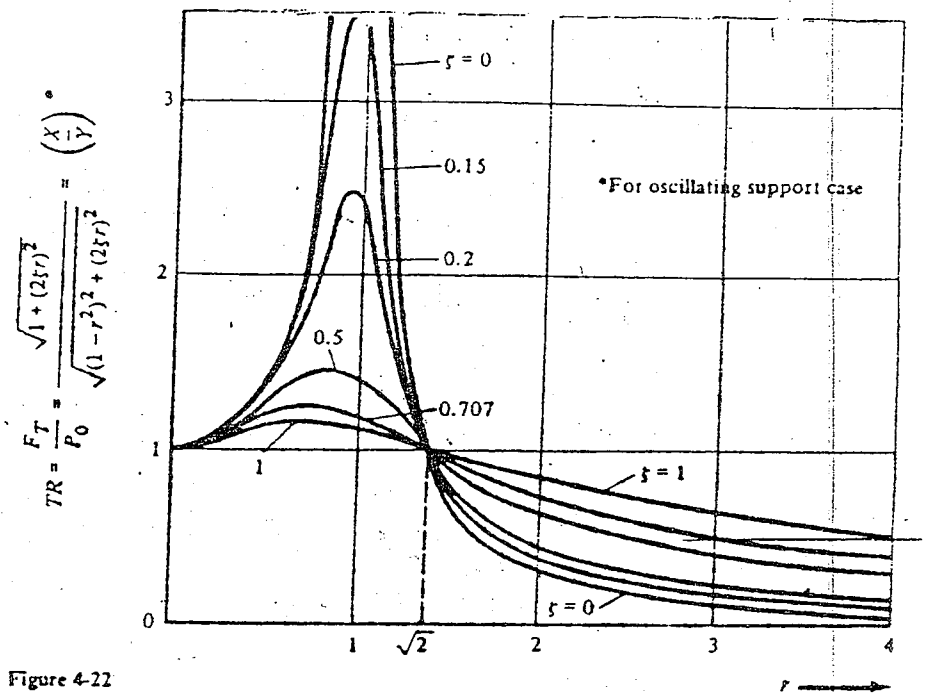


Figure 4-22

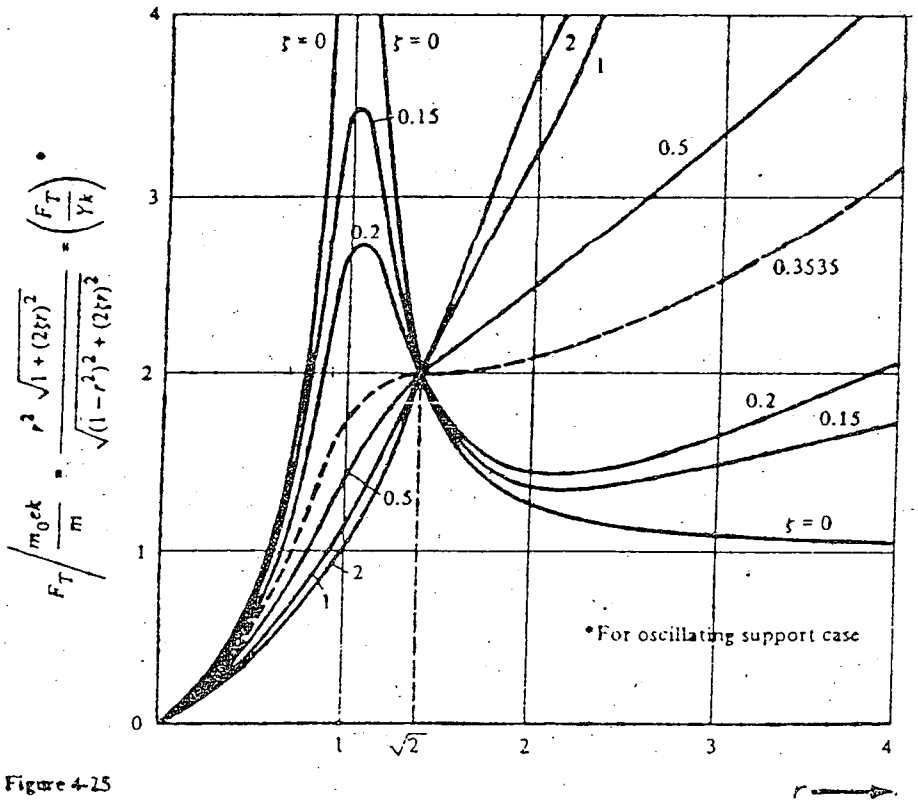


Figure 4-25

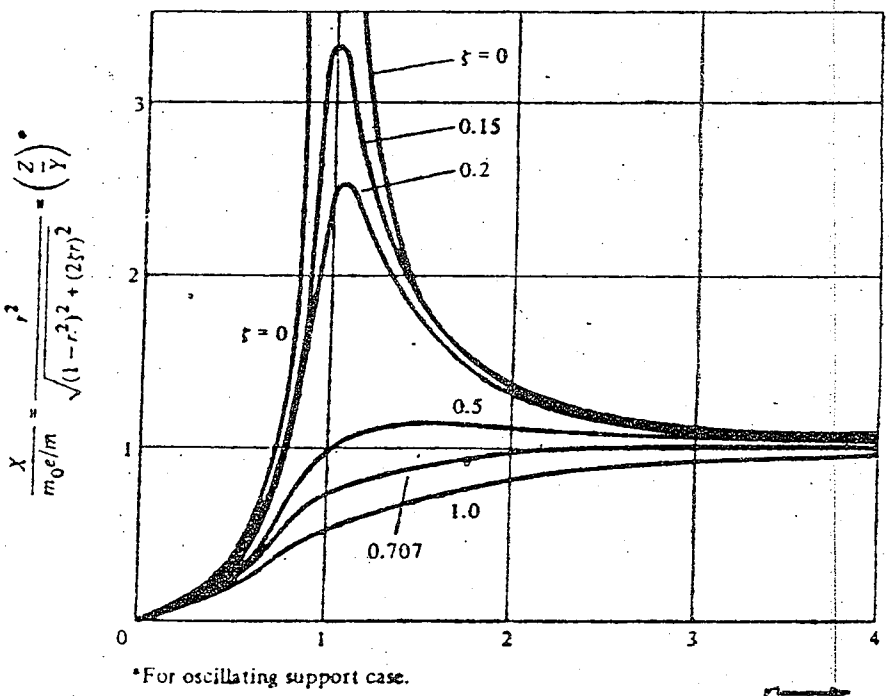


Figure 4-20

the spring due to the maximum impressed force. (c) Obtain the amplitude of the forced motion for the following forced frequencies f_f: 1.5, 3, 3.25, and 7.5 Hz.

4-2. The mass of an undamped mass-spring system is subjected to a harmonic force having a maximum value of 45 lb and a period of 0.25 sec. The body weighs 2.5 lb and the spring constant is 15 lb/in. Determine the amplitude of the forced motion.

4-3. An undamped system is composed of mass $m = 1.1$ kg and spring $k = 4400$ N/m. It is acted on by a harmonic force having maximum value of 440 N and a frequency of 180 cycles/min. Determine the amplitude of the forced motion.

4-4. A weight of 8 lb is suspended from a spring having a constant of 25 lb/in. The system is undamped but the mass is subjected to a harmonic force with a frequency of 7 Hz, resulting in a forced-motion amplitude of 1.59 in. Determine the maximum value of the impressed force.

4-5. The body of an undamped system is driven by a harmonic force having an amplitude of 36 N and a frequency of 450 cycles/min. The body has a mass of 8 kg and exhibits a forced-displacement amplitude of 4.06 mm. Obtain the value of the spring modulus.

4-6. An undamped system consists of an 8.75-kg mass suspended from a spring having a constant of 3500 N/m. A harmonic force acting on the mass and having a maximum value of 187 N causes the system to vibrate with a forced amplitude of 7.6 cm. Determine the frequency of the impressed force.

4-7. A weight W is suspended from a spring having a constant of 25 lb/in. The system is undamped, but the mass is driven by a harmonic force having a frequency of 2 Hz and a maximum value of 10 lb, causing a forced-motion amplitude of 0.5 in. Determine the value of weight W.

4-8. The mass of an undamped system is acted on by force $P = P_0 \cos \omega_f t$, having a maximum value of 10 lb. The spring constant is 3 lb/in. The forced frequency can be any integral multiple of the natural frequency of the system other than 1, thereby avoiding resonance. Only a single forced frequency can exist at a time. Determine the maximum amplitude of the forced motion that could occur.

4-9. Express the relation for the complete motion of an undamped system, excited by a force $P = P_0 \sin \omega_f t$, for the initial conditions of $x = 0$, $\dot{x} = 0$, at $t = 0$. ($r \neq 1$.)

4-10. Write the relation that defines the complete motion of an undamped system, subjected to the harmonic force $P = P_0 \sin \omega_f t$, for the initial conditions of $x = 0$, $\dot{x} = \dot{x}_0$, at $t = 0$. ($r \neq 1$.)

4-11. Express the equation for the complete motion of an undamped system, subjected to the forcing condition $P = P_0 \sin \omega_f t$, for the initial conditions of $x = x_0$, $\dot{x} = 0$, at $t = 0$. ($r \neq 1$.)

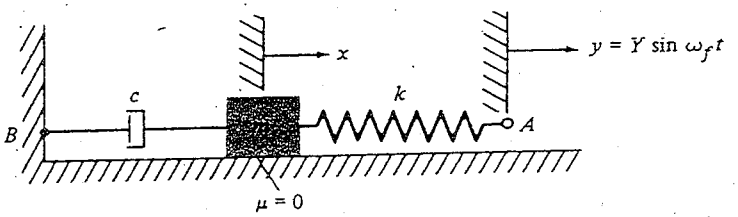
4-12. Write the complete solution for the motion of an undamped system, driven by the force $P = P_0 \sin \omega_f t$, for the initial conditions of $x = x_0$, $\dot{x} = \dot{x}_0$, at $t = 0$. ($r \neq 1$.)

4-13. An undamped system consists of a weight of 19.3 lb and a spring having a modulus of 10 lb/in. The system mass is driven at resonance by harmonic force $P = P_0 \sin \omega_f t$ having a maximum value of 4 lb. Determine the amplitude of the forced motion at the end of (a) $\frac{1}{2}$ cycle, (b) $2\frac{1}{2}$ cycles, (c) $4\frac{1}{2}$ cycles, and (d) $6\frac{1}{2}$ cycles.

4-14. An undamped system is composed of a 4.375-kg mass and a spring having a modulus of 3500 N/m. The mass is driven at resonance by a harmonic force having a maximum value of 14 N. Determine the amplitude of the forced motion at the end of (a) 1 cycle, (b) 5 cycles, and (c) 10 cycles.

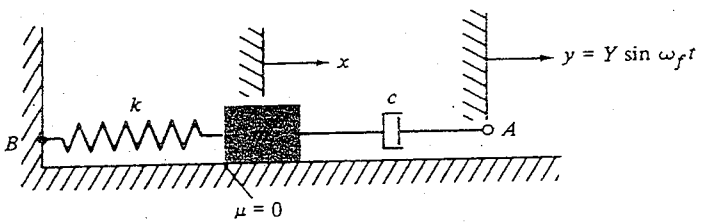
4-15. An undamped system is harmonically forced near resonance, resulting in a beating condition. The natural frequency of the system is 1765 cycles/min, and the forced frequency is 1752 cycles/min. Determine the beat period of the motion.

4-16. For the arrangement shown, x represents the absolute displacement of the mass m, and y is the absolute displacement of the end A of the spring k. Point A is moved according to the relation $y = Y \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation of motion for m. (c) Obtain the solution for the steady-state motion of m. (d) Determine the relation for the impressed force at A. (e) Obtain the expression for the force transmitted to the support at B.



Problem 4-16

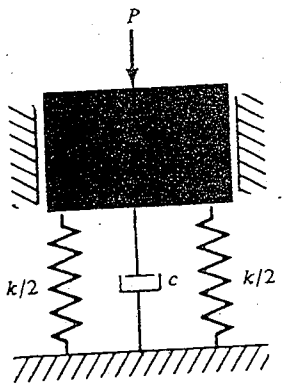
4-17. For the arrangement shown, x represents the absolute displacement of the mass m, and y is the absolute displacement of the end A of the dashpot c. The motion of point A is defined by $y = Y \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation of motion for m. (c) Obtain the solution for the steady-state motion of m. (d) Determine the relation for the impressed force at A. (e) Obtain the expression for the force transmitted to the support at B.



Problem 4-17

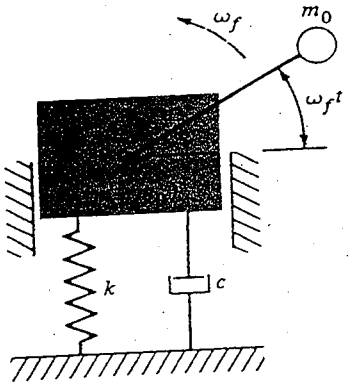
4-18. The spring k and the dashpot c of the accompanying diagram are fastened together at A; x represents the absolute displacement of m, and y is the absolute displacement of the point A. The motion of A is defined by $y = Y \sin \omega_f t$. (a) Construct the free-body diagram for m. (b) Write the differential equation of motion for m. (c) Obtain the solution for the steady-state motion of m. (d) Determine the relation for the impressed force at A.

4-31. A machine having a mass of 70 kg is mounted as shown on springs having a total stiffness of 33 880 N/m. The damping factor is $\zeta = 0.20$. A harmonic force $P = 450 \sin 13.2t$ (where P is in newtons and t is in seconds) acts on the mass. For the sustained or steady-state vibration, determine (a) the amplitude of the motion of the machine, (b) its phase with respect to the exciting force, (c) the transmissibility, (d) the maximum dynamic force transmitted to the foundation, and (e) the maximum velocity of the motion.



Problem 4-31

4-32. A machine having a total weight of 96.5 lb is mounted as shown on a spring having a modulus of 900 lb/in. and is connected to a dashpot with a damping *factor* of 0.25. The machine contains a rotating unbalance ($w_0 e$) of 5 lb in. If the speed of rotation is 401.1 rpm, obtain (a) the amplitude of the steady-state motion, (b) the maximum dynamic force transmitted to the foundation, and (c) the angular position of the arm when the structure goes upward through its neutral position.

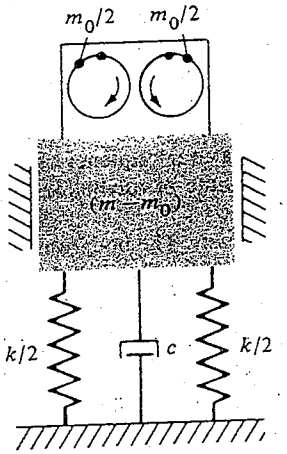


Problem 4-32

• **4-33.** A machine having a total weight of 128.67 lb contains a part that rotates at 859.44 rpm. The machine is supported equally on four identical springs and is connected to a dashpot for which the damping constant is 10 lb sec/in. operation the machine has a steady amplitude of 0.116 in. and transmits maximum dynamic force of 174 lb to the supporting base. (a) Determine t modulus for the springs. (b) Calculate the unbalanced moment ($w_0 e$) for t rotating part.

4-34. A machine with a total mass of 50 kg contains a shaft mechanism that rota at 1800 rpm. The machine is supported by springs for which the equival modulus is 20 000 N/m but the damping constant is unknown. Due to rotating unbalance, the steady-state amplitude is 1.32 cm and the maxim dynamic force carried by the base is 278 N. Determine (a) the damping c stant and (b) the unbalanced moment $m_0 e$.

• **4-35.** A typical vibration exciter is composed of two eccentric masses that rot oppositely, as represented in the top of the accompanying figure. Thus oscillation is produced in the vertical direction only, since the horizontal eff cancels. This device is used to determine the vibrational characteristics of structure to which it is attached. The unbalance ($ew_0/2$) of each exciter whee 3 lb in. The total arrangement has a weight $mg = 200$ lb. The exciter speed eccentric rotation) was adjusted until a stroboscope showed the structure to moving upward through its equilibrium position at the instant the eccen weights were at their top position. The exciter speed then was 840 rpm, and steady amplitude was 0.75 in. Determine (a) the natural frequency of the str ture and the damping factor for the structure. If the speed were changed 1260 rpm, (b) obtain the steady-state amplitude of the structure and (c) angular position of the eccentrics as the structure moves upward through equilibrium position.



Problem 4-35

• **4-36.** A small compressor weighs 69 lb and runs at 875 rpm. It is to be supporte four springs (equally) and no damping is provided. Design the springs, determining k, so that only 15% of the shaking force is transmitted to supporting foundation or structure.

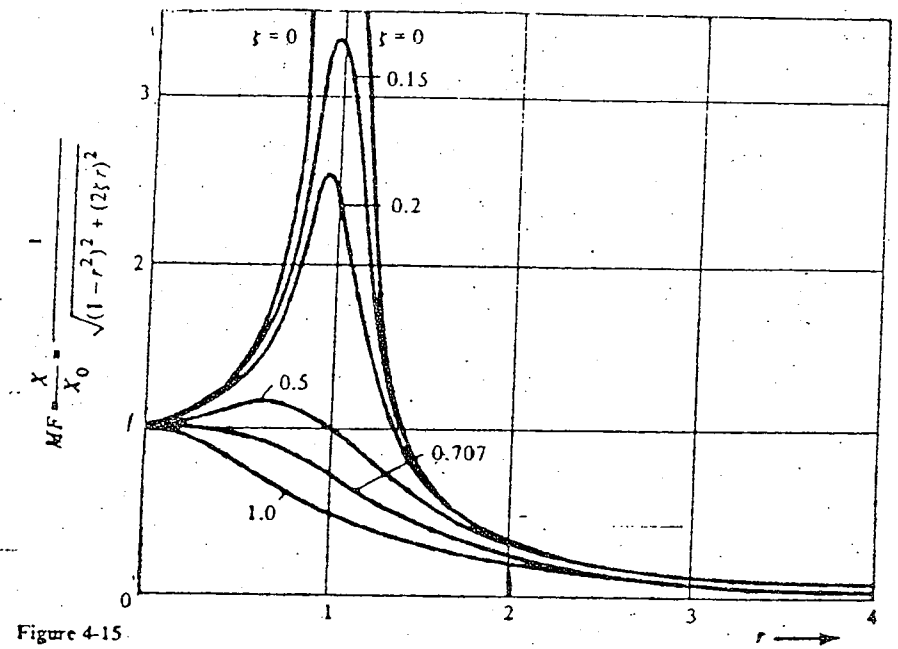


Figure 4-15

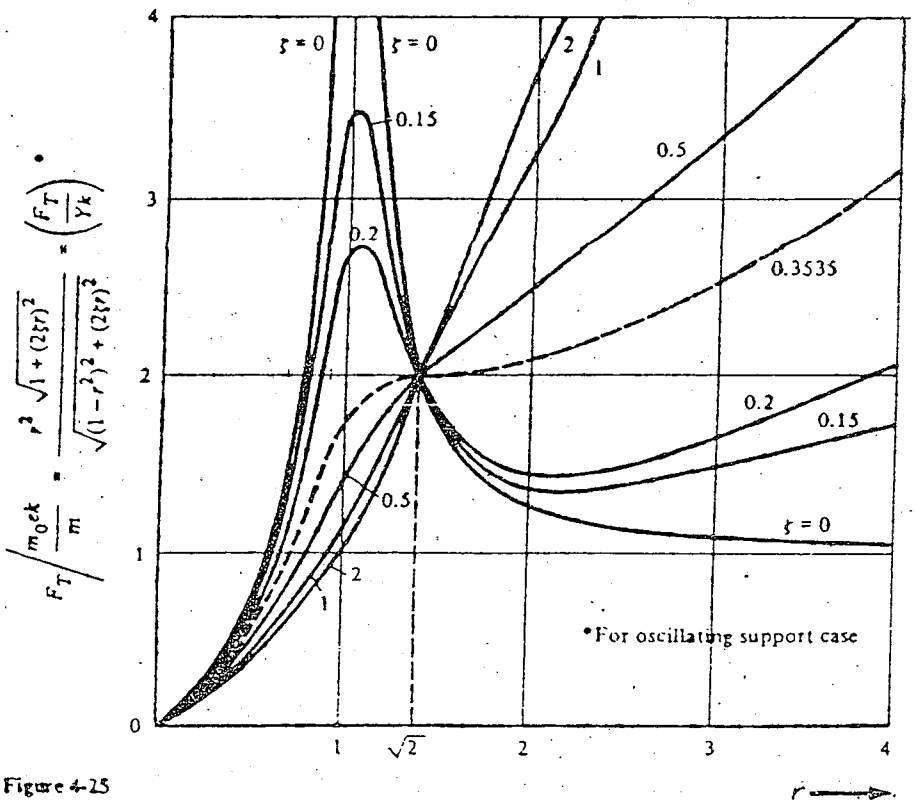


Figure 4-25

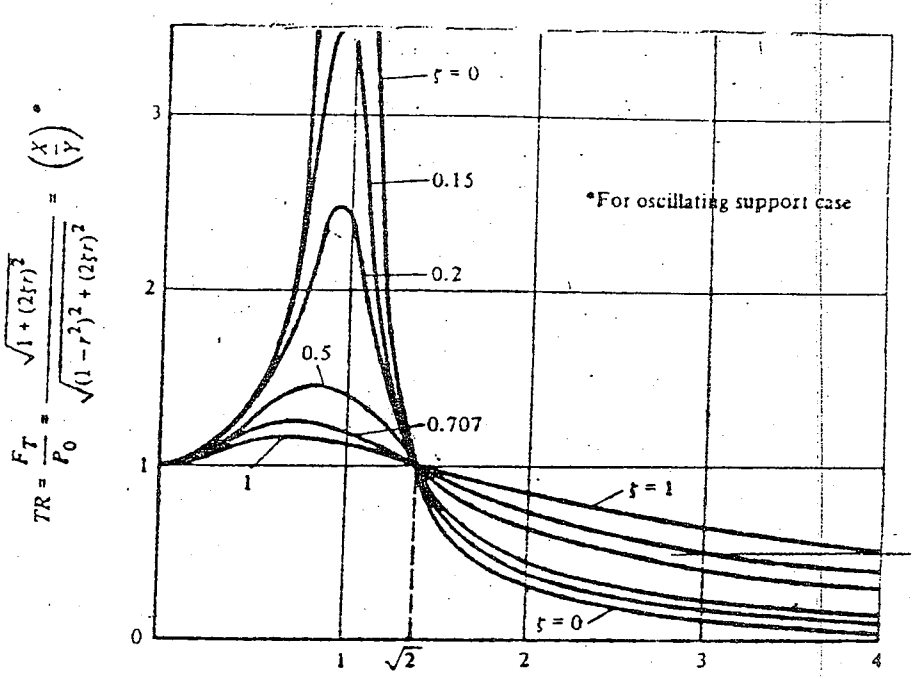
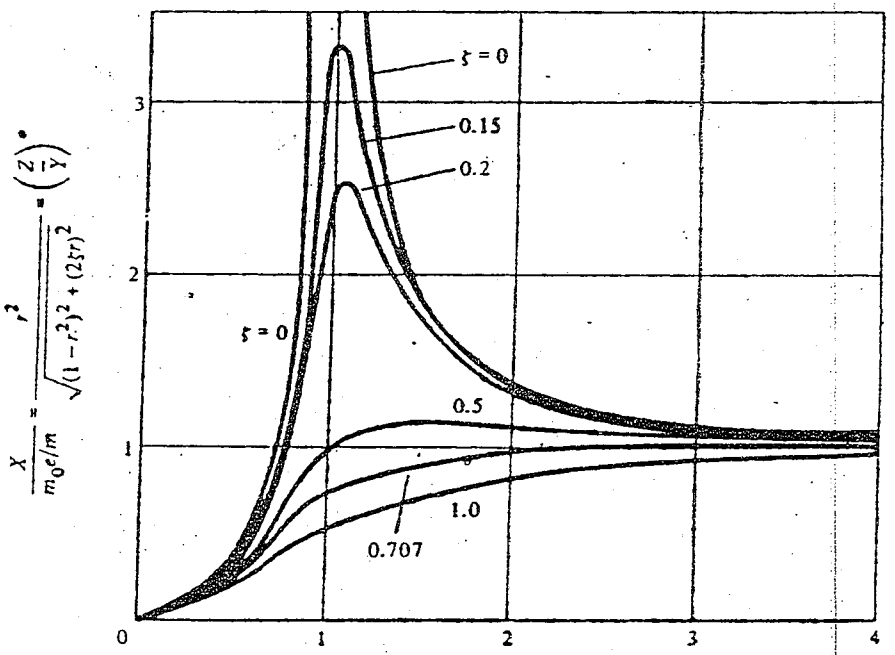


Figure 4-22

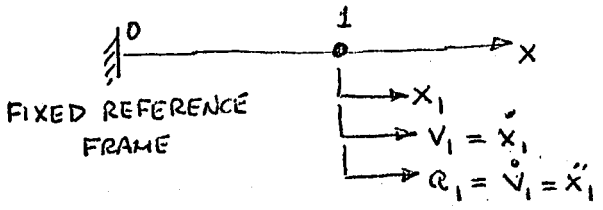


*For oscillating support case.

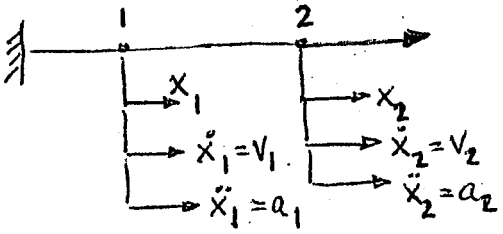
Figure 4-20

MECHANICAL ELEMENTS IN TRANSLATION

- LOOK AT SIMPLE TRANSLATIONAL MOTION OF A PARTICLE FROM DYNAMICS



DISPLACEMENT IN DIRECTION SHOWN MEASURED FROM REFERENCE PT 0



SPRINGS, DAMPERS, MASSES IN GENERAL WILL HAVE MOTIONS THAT REQUIRE KNOWLEDGE OF RELATIVE MOTION

THUS $X_{21} = X_2 - X_1$

$V_{21} = V_2 - V_1 = \dot{X}_{21}$

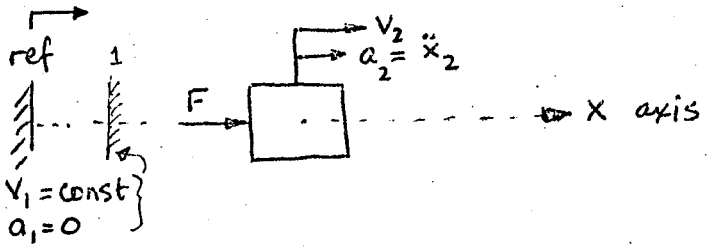
$a_{21} = a_2 - a_1 = \ddot{X}_{21}$

NOTE THAT THIS IS TRUE FOR CASE WHERE EITHER FRAME OF REF. IS STATIONARY OR TRANSLATING (at const velocity) ONLY.

CAN DEFINE $\mathcal{P} = \vec{F} \cdot \vec{v}$ and work $= \int \vec{F} \cdot \vec{v}\, dt$

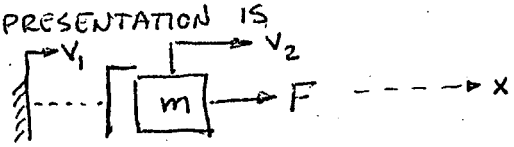
PURE TRANSLATIONAL MASS

- NEWTON'S SECOND LAW $F = \frac{dp}{dt}$ $\quad p = \text{momentum} = mv$
- FOR THE IDEAL MASS CASE $\frac{dp}{dt} = m\frac{dv}{dt} = m\frac{d^2x}{dt^2} = ma$



- a is absolute acceleration
- $\frac{dV_2}{dt} \equiv \frac{dV_{21}}{dt}$ IF

- V_1 IS THE VELOCITY OF THE FRAME OF REF & IT IS EITHER 0 or a const
- SYMBOLIC REPRESENTATION IS



- PURE TRANSLATIONAL MASS - ALLOWS FOR NO ROTATION, FRICTION OR ELASTIC
  - PARTICLES OF BODY ARE RIGIDLY CONNECTED
  - $V_2 \ll$ SPEED OF LIGHT, $V_1 = \text{const}$ or 0
  - F is resultant force acting on mass

THUS $\vec{p} = \int_0^t \vec{F}\,dt + \vec{p}(t=0)$ & $\vec{v}_{21} = \frac{1}{m}\int_0^t \vec{F}\,dt + \vec{v}_{21}(t=0)$

ELEMENTAL EQUATION

- Power delivered to or removed from mass

$$\mathcal{P} = \vec{F}\cdot\vec{v}$$

- Work done on the mass to move it from pt 1 to pt 2

$$W = \int_{x_1}^{x_2} \vec{F}\cdot d\vec{x}_{21} = \int_{t_1}^{t_2} \vec{F}\cdot\vec{v}_{21}\,dt = \int_{p_1}^{p_2} \vec{v}_{21}\cdot d\vec{p}$$

for spring like elements; for damper like element; for mass like elements

- MASS STORES WORK AS A FORM OF ENERGY ASSOCIATED WITH VELOCITIES OF MASS PARTICLES (KINETIC ENERGY $= E_k$)
- FOR A NEWTONIAN MASS $\vec{p} = m\vec{v}_2 = m\vec{v}_{21}$ or $d\vec{p} = m\,d\vec{v}_{21}$

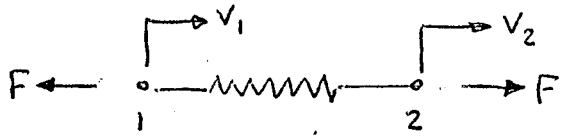
$$E_k = W = \frac{1}{2} m v_2^2 \geq 0 \quad \text{ALSO} \quad E_k = \frac{1}{2m} p^2$$

- SIGN OF $\vec{v}$ IS IMPORTANT AS IT TELL US HOW TO RETRIEVE STORED ENERGY

PURE TRANSLATIONAL SPRING - A ZERO MASS MECHANICAL ELEMENT WHICH DEFORMS BY STEADY AMOUNTS WHEN LOADED BY STEADY FORCES

- SINCE NO MASS $\Rightarrow$ $\Sigma\vec{F} = \vec{0}$ AND

SYMBOLIC REPRESENTATION



$x_{21} = $ fn of F

$V_2 > V_1$

$$\bar{v}_{21} = \frac{d\bar{x}_{21}}{dt} \quad ; \quad \bar{x}_{21} = \int_0^t \bar{v}_{21}\,dt + \bar{x}_{21}(t=0)$$

- SINCE WE HAVE EQUILIBRIUM OF FORCES NOTE F IS THOUGHT OF AS FLOWING THROUGH THE SPRING

- FOR AN IDEAL SPRING : $\bar{x}_{21} = \frac{\bar{F}}{k}$ $\therefore$ $\bar{v}_{21} = \frac{1}{k}\frac{d\bar{F}}{dt}$

or $\bar{F} = k\int^t \bar{v}_{21}\,dt + \bar{F}(t=0)$

Work done on the spring by the forces.

$$W = \int \vec{F} \cdot \vec{v}_{21}\, dt = \int \vec{F} \cdot d\vec{x}_{21}$$

FOR AN IDEAL SPRING. $d\vec{x}_{21} = \frac{1}{k} d\vec{F}$

$$E_p = W = \frac{1}{2} k x_{21}^2 = \frac{1}{2k} F^2 \geq 0$$

- THIS WORK IS STORED AS POTENTIAL ENERGY IN THE SPRING

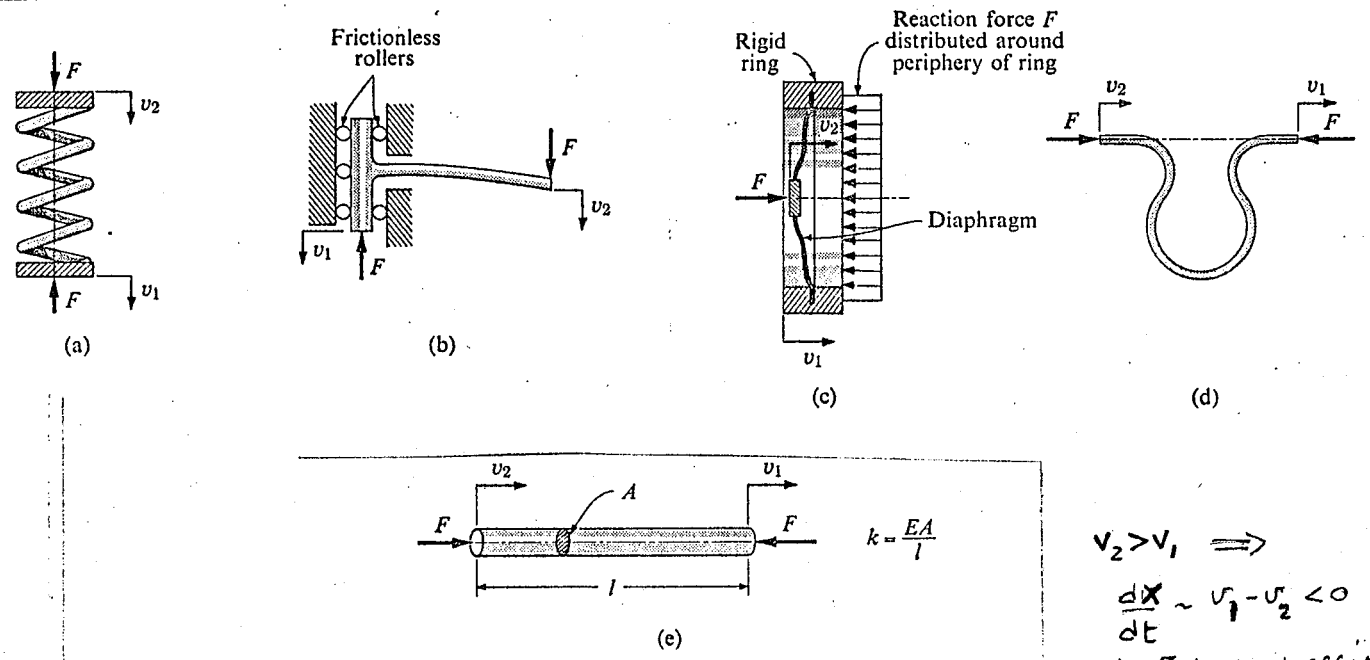


FIG. 2-8. Typical translational springs. (a) Coil spring. (b) Cantilever beam. (c) Corrugated circular diaphragm. (d) Curved bar. (e) Uniform circular bar.

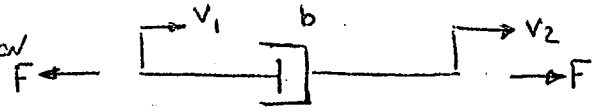
$v_2 > v_1 \Rightarrow$

$\frac{dx}{dt} \sim v_1 - v_2 < 0$

$\Rightarrow$ F is compressive

- PURE TRANSLATIONAL DAMPER – HAS NO MASS OR SPRING EFFECTS, REPRESENTS ONLY THE EFFECTS OF RESISTANCE TO <u>RATE</u> OF DEFORMATION

SYMBOLIC REPRESENTATION



$\Sigma \vec{F} = \vec{0}$

F = fn of v_{21}

sign of F & v_{21} are same

FOR AN IDEAL DAMPER

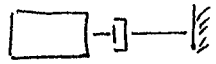
$$F = b v_{21} = b \dot{x}_{21}$$

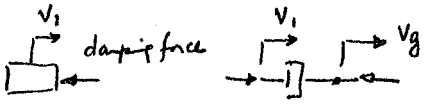
FOR A GIVEN CHANGE IN POSITION $x_{21} = \frac{1}{b}\int F\, dt + x_{21}(t=0)$

WORK DONE $W = \int \vec{F} \cdot d\vec{v}_{21} = \int \vec{F} \cdot \vec{v}_{21}\, dt$

Power $\mathcal{P} = \vec{F} \cdot \vec{v}_{21}$ supplies to the damper

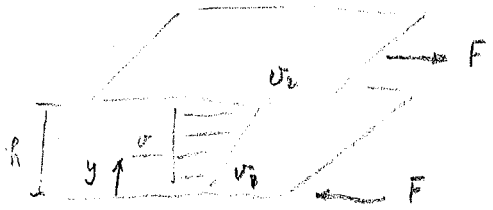
Problem 2.2





$$m\frac{dv_1}{dt} = -F_b \qquad F_b = b v_{1g}$$

$$m\dot{v}_1 + b v_1 = 0 \qquad \dot{v}_1 = e^{-\frac{b}{m}t} + v(0)$$



$$\tau = \mu \frac{dv}{dy}$$

$$v = v_1 + \frac{v_2 - v_1}{h} y$$

$$\frac{dy}{dy} = \frac{v_2 - v_1}{h}$$

$$\tau = \mu \frac{v_2 - v_1}{h}$$

$$F = \tau A = \frac{\mu A}{h}(v_2 - v_1) \qquad \frac{\mu A}{h} = b.$$

if $v >>$ $\quad \tau = \mu (v_2 - v_1)|v_2 - v_1| \quad$ or $\quad \tau = \pm c(v_2 - v_1)^2 = \pm c(\Delta v)^2 = f(\Delta v)$

$$\pm\left[c \cdot f(5) + 2c\,\Delta v\Big|_{\Delta v = 5} \Delta v\right]$$

$$\hat{\tau} = \tau \pm c f(5) = \pm 10c\,\Delta\hat{v}$$

FOR AN IDEAL DAMPER

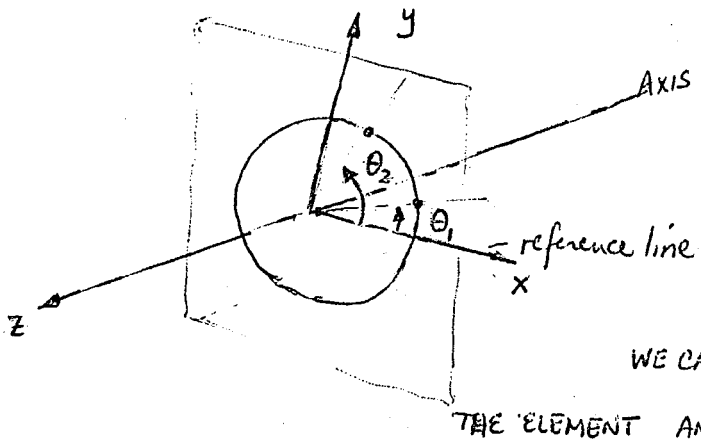
$$\mathcal{P} = b v_{21}^2 \geq 0$$

$$W = b \int v_{21}^2 \, dt$$

- NOTE WORK DONE INCREASE W/ TIME & CANNOT BE REMOVED
- DAMPER DISSIPATES THE WORK DONE OR ENERGY DELIVERED
- POWER FLOW IS NOT REVERSIBLE & ALWAYS FLOWS INTO DAMPER
- DAMPER UNLIKE MASS & SPRING, BUT IMPORTANT

ROTATIONAL MECHANICAL SYSTEMS

- FOR SYSTEMS THAT UNDERGO ROTATIONAL MOTION (TURBINES, GALVANOME PUMPS ETC) WE NEED TO DEFINE IDEAL ROTARY MECHANICAL ELEMEN WHICH ARE THE COUNTERPARTS OF THE TRANSLATIONAL ELEMENTS



$$\theta_{21} = \theta_2 - \theta_1$$

$$\omega_{21} = \dot{\theta}_2 - \dot{\theta}_1 = \omega_2 - \omega_1$$

$$\alpha_{21} = \alpha_2 - \alpha_1 = \dot{\omega}_2 - \dot{\omega}_1 = \ddot{\theta}_2 - \ddot{\theta}_1$$

WE CAN DEFINE $W = \int \vec{T} \cdot \vec{\omega} \, dt$, WORK DONE ON THE ELEMENT AND $\mathcal{P} = \vec{T} \cdot \vec{\omega}$

PURE ROTATIONAL MASS

$J_m = mr^2$ POINT MASS rotating at distance r from axis of rotation

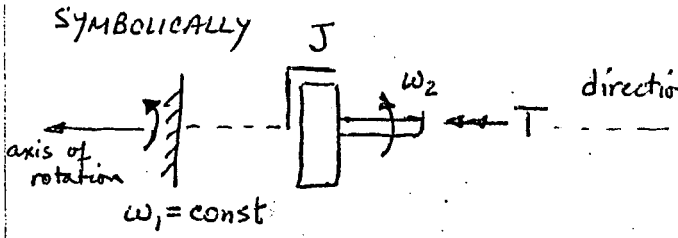
$J_m = \sum m_k r_k^2$ for n masses

$J_m = \int r^2 dm$ for distributed mass

units $kg \cdot m^2$ or $lb \cdot in \cdot sec$

CAN DEFINE THE ANGULAR MOMENTUM h SO THAT $T = \frac{dh}{dt} = \dot{h}$

SYMBOLICALLY



direction of T determined by rh screw rule

- IDEAL ROTATIONAL MASS IS A COLLECTION OF IDEAL TRANSLATIONAL MASSES RIGIDLY CONNECTED TOGETHER OR CONSTRAINED TO ROTATE AT ω_2 ABOUT A FIXED AXIS

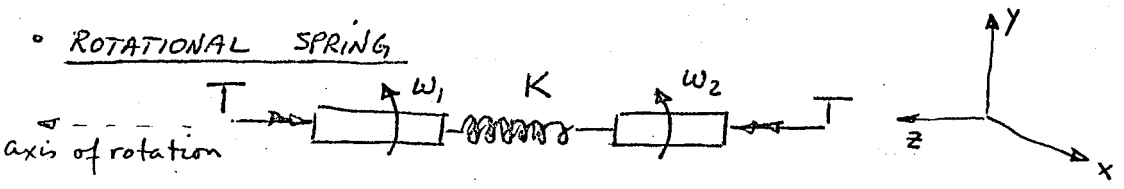
THUS $\vec{h} = J\vec{\omega}_2$. Since $\vec{T} = \frac{d\vec{h}}{dt} = J\dot{\vec{\omega}}_{21} = J\vec{\alpha}_{21} = J\vec{\alpha}_{21}$

since $\vec{\alpha}_1 = \vec{0}$

- $E_k = \frac{1}{2} J\omega_2^2 = \frac{1}{2}\frac{h^2}{J}$ n·m or lb·in

- THIS ENERGY CAN BE RECOVERED. INDEPENDENT OF T

- ROTATIONAL SPRING



- HAS NO INERTIA & TRANSMITS TORQUE

$$\theta_{21} = \frac{1}{K} T$$

$$\omega_{21} = \omega_2 - \omega_1 = \frac{d\theta_{21}}{dt} = \frac{1}{K}\frac{dT}{dt} \Rightarrow T = K\int_0^t \omega_{21}\, dt + T(t=0)$$

- $E_p = \int \vec{T}\cdot\vec{\omega}_{21}\, dt = \int \vec{T}\cdot d\vec{\theta}_{21} = \frac{1}{2K}T^2 = \frac{1}{2}K\theta_{21}^2 \geq 0$

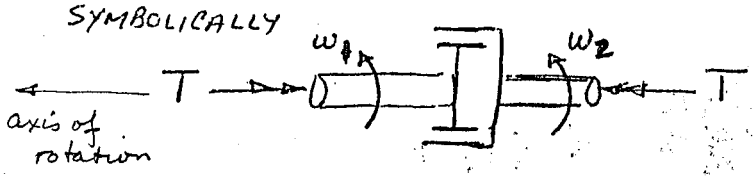
- ENERGY ALWAYS POSITIVE & RECOVERABLE BY AFFECTING THE TORQUE

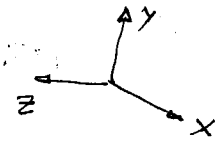
- ROTATIONAL DAMPER

  - ENCOUNTERED IN BEARINGS OR SURFACES OF ROTATING ELEMENTS IMMERSED IN FLUIDS (AIR IS A FLUID ALSO!!)

- PROVIDES RESISTANCE TO THE RATE OF RELATIVE ANGULAR MOTION

SYMBOLICALLY





- HAS NO MASS OR COMPLIANCE
- FOR THE IDEAL DAMPER $T = B\omega_{21}$

$$P = \vec{T} \cdot \vec{\omega}_{21} = B\omega_{21}^2 \geq 0$$
$$= T^2/B$$

B { N·m·sec/rad, lb·in·sec/rad

- DISSIPATES ENERGY LIKE ITS LINEAR COUNTERPART

- IN ALL THESE CASES WE HAVE LINEARIZED

$$p = \frac{mv_2}{\sqrt{1-(v/c)^2}} \qquad v_2 \ll c$$

k
b } LINEARIZED ABOUT OPERATING RANGE

$h = rp$ $\qquad v_2 \ll c$

K
B } LINEARIZED ABOUT OPERATING RANGE

MECHANICAL TRANSFORMERS

- THIRD CLASS OF ELEMENTS THAT DO NOT STORE OR DISSIPATE ENERGY BUT TRANSFORMES ENERGY - PURE TRANSFORMERS
- MAGNITUDE OF FORCE (OR TORQUE) OR VELOCITY (OR ANGULAR VELOCITY) ARE CHANGED SO THAT NET POWER FLOW INTO THE ELEMEN IS ZERO.

EXAMPLE : SIMPLE GEAR TRAIN (ROTATIONAL TRANSFORMER)

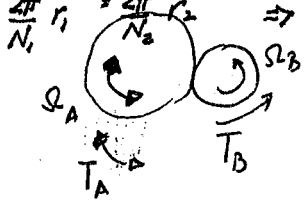
- MODEL AS 2 CIRCLES ROLLING W/O SLIPPAGE.

$\frac{2\pi r_i}{\text{no of teeth}_i} = \theta_i r_i \Rightarrow \quad \frac{2\pi r_1}{N_1} = \frac{2\pi r_2}{N_2} \quad \therefore \quad \frac{r_2}{r_1} = \frac{N_2}{N_1}$

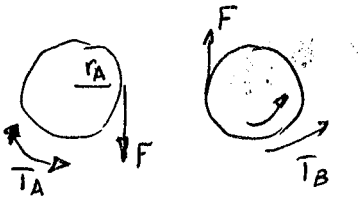
$s_i = \Delta\theta_i r_i \qquad s_1 = s_2$

$\Delta\theta_1 r_1 = \Delta\theta_2 \cdot r_2$

$\frac{2\pi}{N_1} r_1 = \frac{2\pi}{N_2} r_2 \Rightarrow \frac{r_2}{r_1} = \frac{N_2}{N_1}$ w/ $\Omega_A r_A = V_A = -V_B = -\Omega_B r_B$



$\frac{\Omega_B}{\Omega_A} = -\frac{r_A}{r_B} = -\frac{N_A}{N_B}$ = Gear ratio



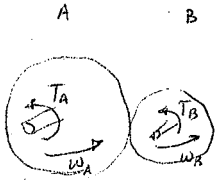
$N_B \Delta\theta_B = N_A \Delta\theta_A = 2\pi$

$r_A \Delta\theta_A = r_B \Delta\theta_B = \Delta s$

$\frac{r_A}{r_B} = \frac{\Delta\theta_B}{\Delta\theta_A} = \frac{N_A}{N_B}$

$F \cdot r_B = T_B \qquad \frac{T_B}{T_A} = +\frac{r_B}{r_A} = +\frac{N_B}{N_A}$

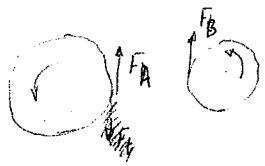
$+ F r_A = T_A$



$\Delta\theta_A r_A = -\Delta\theta_B r_B$

$\omega_A = -\omega_B \cdot \frac{r_B}{r_A} = -\frac{\omega_B}{N} \qquad N = \frac{r_A}{r_B}$

$v_t = \omega_A r_A = -\omega_B r_B$



$r_B F_B = T_B$

$-r_A \cdot F_A = T_A$

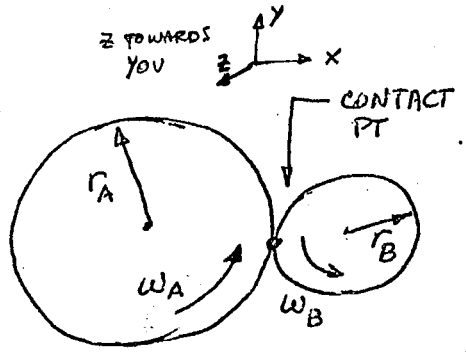
$\therefore F_A = -F_B$ for equil.

$\therefore \quad +\frac{T_B}{r_B} = F_B = -F_A = \frac{T_A}{r_A}$

$\therefore T_A = \frac{r_A}{r_B} \cdot T_B = N T_B$

$T_A \omega_A + T_B \omega_B$

$N T_B \cdot \left(-\frac{1}{N}\omega_B\right) + T_B \omega_B = 0$



AXIS OF ROTATION IS OUT OF PAPER

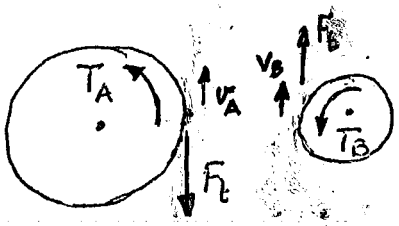
PITCH RADII RATIO r_A/r_B ARE EQUAL TO GEAR TEETH RATIO $\frac{N_A}{N_B}$

AT CONTACT $+V_A = -V_B$ or $+\omega_A r_A = -\omega_B r_B$

$$\therefore \frac{-\omega_B}{\omega_A} = \frac{+r_A}{r_B} = \frac{+N_A}{N_B} = +\text{GEAR RATIO}$$

ASSUME $\omega_{\text{FRAME OF REF}} = 0$

N_A, N_B : # TEETH



AT CONTACT PT $\Sigma F = 0$

$$+F_t r_A = T_A \quad \Big\} \quad \frac{T_B}{T_A} = +\frac{r_B}{r_A} = \frac{+1}{\text{GEAR RATIO}}$$
$$F_t r_B = +T_B$$

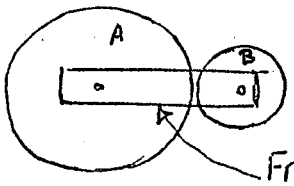
- IF FRICTION BETWEEN GEAR TOOTH SURFACES IS ZERO
- IF GEARS HAVE NO CHANGE IN ANGULAR MOMENTUM (ie $J = 0$ or $\omega = \text{const}$)
- IF BEARING TORQUE IS NEGLIGIBLE & ITS FRICTION IS ZERO

$$\mathcal{P} = +\bar{T}_A \cdot \bar{\omega}_A + \bar{T}_B \cdot \bar{\omega}_B = T_A \omega_A + (+T_A/GR)(-\omega_A \cdot GR) = 0$$

- FOR THE CASE WHERE THE FRAME OF REFERENCE IS MOVING

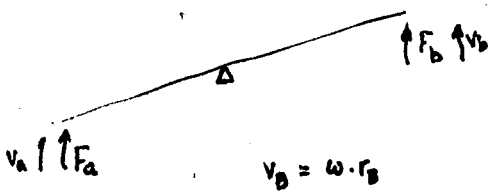
$$\frac{-\omega_{Bi}}{\omega_{Ai}} = +\frac{r_A}{r_B} = +\frac{N_A}{N_B} = +\frac{T_A}{T_B} = +\text{GEAR RATIO} \quad \text{BUT } \mathcal{P} = 0$$

where $\omega_{Bi} = \omega_B - \omega_1$ $\quad \omega_{Ai} = \omega_A - \omega_1$



Frame of reference is moving w/ speed ω_1

- THUS AN IDEAL OR PURE ROTATIONAL TRANSFORMER STORES OR DISSIPATES NO ENERGY AND DETERMINES ω_B GIVEN ω_A. The IDEAL TRANSFORMER



$$v_B = \omega \cdot r_B \qquad \frac{-v_A}{v_B} = \frac{r_A}{r_B} =$$

$$-v_A = \omega \cdot r_A$$

$$v_A = -v_B \cdot \frac{r_A}{r_B} = -v_B \cdot \frac{1}{N}$$

$$F_B \cdot r_B = F_A \cdot r_A$$

$$F_A = F_B \cdot \frac{r_B}{r_A} = F_B \cdot N$$

Speed ratio[edit]

Gear teeth are distributed along the circumference of the pitch circle so the thickness t of each tooth and the space between neighboring teeth are the same. The pitch p of the gear, which is the distance between equivalent points on neighboring teeth along the pitch circle, is equal to twice the thickness of a tooth,

p = 2 t . {\displaystyle p=2t.\!} p=2t.\!

The pitch of a gear GA can be computed from the number of teeth NA and the radius rA of its pitch circle

p = 2 π r A /N A . {\displaystyle p={\frac {2\pi r_{A}}{N_{A}}}.} p={\frac {2\pi r_{A}}{N_{A}}}.

In order to mesh smoothly two gears GA and GB must have the same sized teeth and therefore they must have the same pitch p, which means

p = 2 π r A /N A = 2 π r B /N B . {\displaystyle p={\frac {2\pi r_{A}}{N_{A}}}={\frac {2\pi r_{B}}{N_{B}}}.} p={\frac {2\pi r_{A}}{N_{A}}}={\frac {2\pi r_{B}}{N_{B}}}.

This equation shows that the ratio of the circumference, the diameters and the radii of two meshing gears is equal to the ratio of their number of teeth,

r B / r A = N B / N A . {\displaystyle {\frac {r_{B}}{r_{A}}}={\frac {N_{B}}{N_{A}}}.} {\frac {r_{B}}{r_{A}}}={\frac {N_{B}}{N_{A}}}.

The speed ratio of two gears rolling without slipping on their pitch circles is given by,

R = ω A / ω B = r B / r A , {\displaystyle R={\frac {\omega _{A}}{\omega _{B}}}={\frac {r_{B}}{r_{A}}},} R={\frac {\omega _{A}}{\omega _{B}}}={\frac {r_{B}}{r_{A}}},

therefore

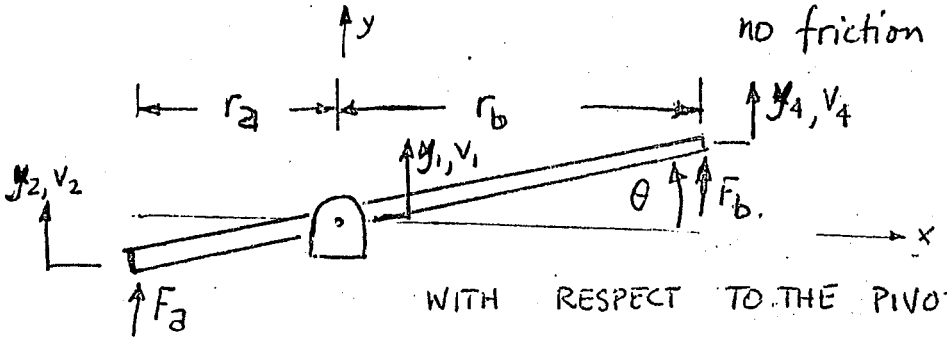
R = ω A / ω B = N B / N A . {\displaystyle R={\frac {\omega _{A}}{\omega _{B}}}={\frac {N_{B}}{N_{A}}}.} R={\frac {\omega _{A}}{\omega _{B}}}={\frac {N_{B}}{N_{A}}}.

In other words, the gear ratio, or speed ratio, is inversely proportional to the radius of the pitch circle and the number of teeth of the input gear

HAS A LINEAR RELATION BETWEEN OUTPUT (ω_{B1}) & INPUT (ω_{A1}): GEAR RATIO

- USES
  - (1) CHANGE SPEED OF MOTOR OR OTHER HIGH POWER SOURCE
  - (2) OBTAIN NON-UNIFORM MOTION FROM UNIFORM ONE
  - (3) ACHIEVE MECHANICAL ADVANTAGE THROUGH TORQUE CHANGE

A <u>PURE TRANSLATIONAL TRANSFORMER</u> - no mass, infinitely stiff
no friction



WITH RESPECT TO THE PIVOT PT VELOCITY

$$\frac{-(V_4 - V_1)}{V_2 - V_1} = \frac{-V_{41}}{V_{21}} = \frac{r_b \omega}{+r_a \omega} = n \quad \text{(LEVER RATIO)}$$

$\sum \text{moments} = I\alpha^{\nearrow 0}$

ALSO SINCE IT IS <u>MASSLESS</u> $\sum$ moments must be 0 about pt. 1

$$\therefore \quad F_a r_a = F_b r_b \quad \text{or} \quad F_b/F_a = \frac{r_a}{r_b} = +\frac{1}{n}$$

AN IDEAL LINEAR TRANSFORMER HAS RATIO n BEING CONSTANT AND ALSO HAS ZERO POWER FLOW.

<u>MECHANICAL TRANSDUCERS</u>

DFN - A DEVICE THAT TRANSFORMS ENERGY OR POWER IN ONE FORM INTO ENERGY OR POWER IN ANOTHER

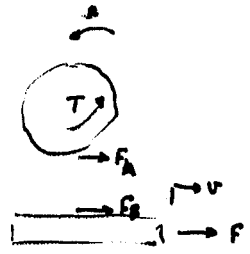
DFN - A PURE TRANSDUCER DOES THIS WITHOUT ENERGY LOSS OR STORAGE, & NET POWER FLOW INTO THE TRANSDUCER IS ZERO

<u>EXAMPLES</u>

ELECTRIC MOTORS - (ELECTRICAL TO MECH.)

GAS TURBINES (FLUID TO MECH)

THERMO COUPLES (THERMAL TO ELECTRICAL)



$v = \omega R$

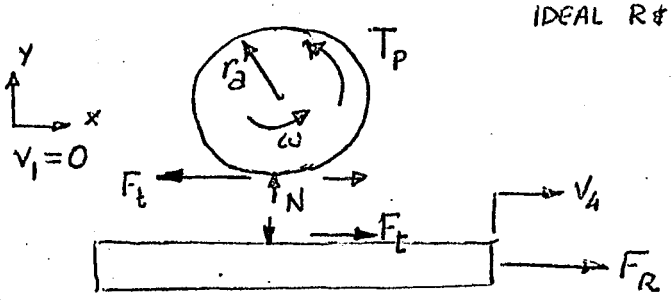
$R F_A = -T$

$F = -F_B$ $\qquad F_A = -F_B$

$RF = T$ $\qquad F = \frac{-1}{R} T$ here R

EXAMPLE OF RACK & PINION

- RACK IS A GEAR OF INFINITE RADIUS



IDEAL R&P : ASSUME NO MOMENTUM, $J, m = 0$

- FRICTION TORQUES = 0
- GEARS ARE PERFECT & RIGID
- PINION ROTATES ONLY
- RACK TRANSLATES ONLY

$$r_a \omega = V_4 \quad \Rightarrow \quad \frac{V_4}{\omega} = r_a = n \quad \text{TRANSDUCTION RATIO}$$

$$F_t = -F_R \qquad T_P = F_t r_a \qquad \therefore \quad F_R / T_P = \frac{-1}{r_a} = \frac{-1}{n}$$

A PURE ROTARY TO TRANSLATORY ENERGY TRANDUCER PRODUCES A DISPLACEMENT (VELOCITY) WHICH IS A SINGLE VALUED FN OF THE ROTARY DISPLACEMENT (OR ANGULAR VELOCITY). NET POWER = 0

AN IDEAL R - TO - T TRANDUCER HAS ROTARY MOTION PROPORTIONAL TO TRANSLATORY MOTION

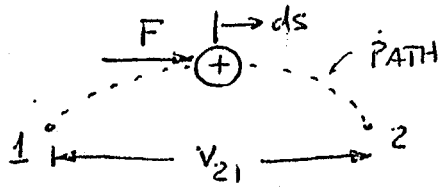
ELECTRICAL SYSTEM ELEMENTS

WE WILL FIND THE SAME CONCEPTS OF WORK, ENERGY & POWER USEFUL IN STUDYING MECHANICAL SYSTEMS ALSO HAVE THEIR COUNTERPARTS IN ELECTRICAL

RECALL VOLTAGE – ELECTRIC FIELD POTENTIAL

CURRENT – FLOW OF ELECTRIC CHARGE

DFN VOLTAGE (OR POTENTIAL DIFFERENCE) V_{21} BETWEEN TWO PTS – THE WORK DONE IN CARRYING A UNIT POSITIVE CHARGE FROM PT 1 TO PT 2



$$dq \cdot V_{21} = \int \bar{F} \cdot d\bar{s}$$

IF $V_{21} > 0 \Rightarrow V_2 > V_1$ PT 2 IS AT A HIGHER POTENTIAL.

FROM THE VOLTAGE DFN

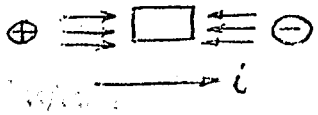
THE AMOUNT OF WORK DONE WHEN AN INFINITESIMAL CHARGE dq IS MOVED FROM PT 1 TO 2.

$$dW_{21} = V_{21}\, dq \qquad \text{or} \qquad V_{21} = \frac{dW_{21}}{dq}$$

$$[V_{21}] = \text{VOLTS} = \frac{\text{JOULES}}{\text{COUL.}} = \frac{\text{WATTS} \cdot \text{sec}}{\text{COULOMB}}$$

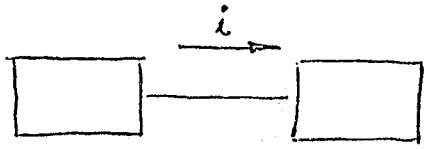
DFN CURRENT IS THE RATE OF FLOW OF CHARGE ACROSS A GIVEN AREA

$$i = \frac{dq}{dt} \qquad [i] = \text{AMP} = \frac{\text{COULOMB}}{\text{sec.}}$$



CONVENTION FOR ⊕ CURRENT

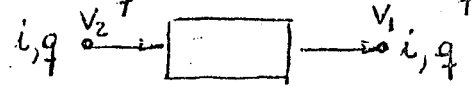
- NEWTON'S 3rd LAW HAS AN EQUIVALENT — PRINCIPLE OF CONSERVATION OF ELECTRIC CHARGE : ① NET CHARGE LEAVING ONE ELEMENT MUST ENTER ADJACENT ELEMENT



: ② CHARGE ENTERING ELEMENT — CHARGE LEAVING ELEMENT = CHANGE OF NET CHARGE STORED IN THE ELEMENT

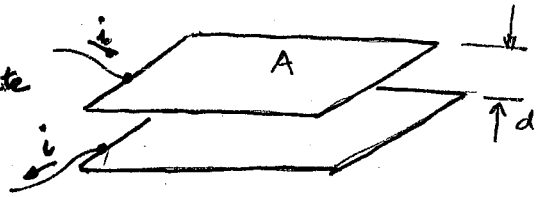
FIRST WE WILL RESTRICT OURSELVES TO CASES WHERE NO ACCUMULATION OCCURS IN AN ELEMENT

- THUS i & q in $=$ i & q OUT LIKE DAMPERS, SPRINGS



- ELECTRICAL ELEMENTS TRANSMIT CURRENT AS MECHANICAL ELEMENTS TRANSMIT FORCE

flate plate

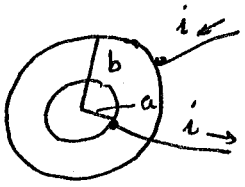


$$C = \frac{\epsilon A}{d}$$

parallel plate capacitor

ϵ - permittivity

Concentric spheres



$$C = \frac{4\pi\epsilon}{(1/a - 1/b)}$$

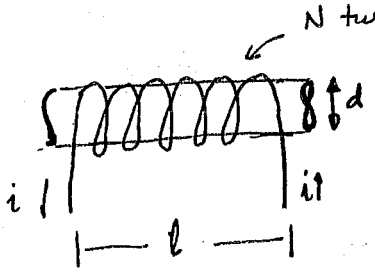
ϵ - permittivity of material

Concentric cylinders



$$C = \frac{2\pi\epsilon l}{\cosh^{-1}\left(\frac{a^2+b^2}{2ab}\right)}$$

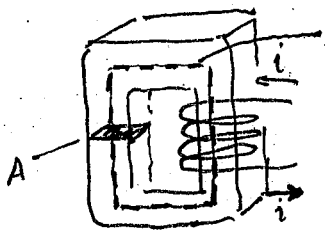
solenoid



$$L = \frac{\pi\mu d^2 N^2}{4l} = \frac{\pi A N^2}{l}$$

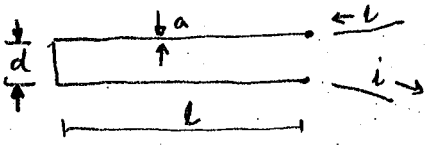
μ - permeability

permeable core



$$L = \frac{\mu A N^2}{l}$$

thin parallel wires



$$L = \frac{\mu l}{\pi} \ln\left(\frac{d}{a}\right);$$

a = wire diameter

POWER, WORK & ENERGY

$\mathcal{P}$ = rate of change of work $= \frac{dW_{21}}{dt} = V_{21}\frac{dq}{dt} = i\,V_{21}$ (WATTS)

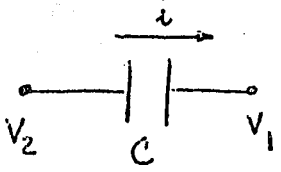
electrical energy $= \int \mathcal{P}\,dt = \int i V_{21}\,dt$ (joules)

$= \int V_{21}\,dq$

ELECTRICAL ELEMENTS in general $q = f(V_{21})$

→ CAPACITANCE (STORES ENERGY IN AN ELECTRIC FIELD BUT NOT CHARGE)

LIKE MASS (STORES ENERGY IN GRAVITATIONAL FIELD)



VOLTAGE DROP IN DIRECTION OF CURRENT FLOW

IDEAL CAPACITANCE

similar to { $p = mV_{21}$; $F = m\frac{dV_{21}}{dt}$ }

integ thru var → $q = CV_{21}$; $i = C\frac{dV_{21}}{dt}$

[C] - farads = $\frac{\text{amp.sec}}{\text{volt}}$

- ZERO RESISTANCE TO FLOW
- DIELECTRIC HAS INFINITE RESISTANCE TO FLOW
- MAGNETIC FIELD EFFECTS ARE ZERO

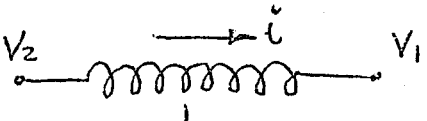
$$V_{21} = \frac{1}{C}\int_0^t i\,dt + V_{21}(t=0)$$

$$E_e = \int V_{21}\,dq = \frac{1}{2}C V_{21}^2 \geq 0$$

similar to [$E_k = \frac{1}{2}mV_{21}^2$]

→ INDUCTANCE (STORES ENERGY IN A MAGNETIC FIELD) LIKE SPRIN

- CURRENT FLOW INDUCES MAGNETIC FIELD
- CHANGE IN CURRENT, INTENSITY OF MAGNETIC FIELD CHANGES
- LENZ'S LAW - CHANGE IN MAGNETIC FIELD INDUCES VOLTAGE DIFFERENC

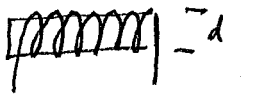


$V_{21} = \frac{d\lambda_{21}}{dt}$ λ_{21} - FLUX LINKAGE $[\lambda_{21}]$ - WEBERS = VOLTS·sec

integrated across var. $\lambda_{21} = \int_0^t V_{21}\,dt + \lambda_{21}(t=0)$

like [$x_{21} = \int_0^t V_{21}\,dt + x_{21}(t=0)$]

$$R = \rho \frac{Nd}{r^2}$$

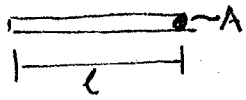


$$\frac{\overbrace{\rho N \cdot \pi d}^{l}}{\underbrace{\pi r^2}_{A}}$$

ρ - resistivity N = no. of loops

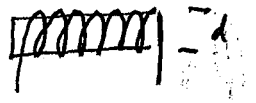
r = ~~diameter~~ radius d = diameter of wire

$$R = \frac{\rho l}{A}$$



σ = ~~resistivity~~ conductivity $= \frac{1}{\text{resistivity}}$

$$R = \rho \frac{Nd}{r^2}$$

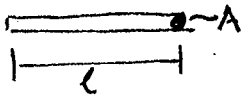


$$\frac{\overbrace{\rho N \cdot \pi d}^{l}}{\underbrace{\pi r^2}_{A}}$$

ρ - resistivity N = no. of turns

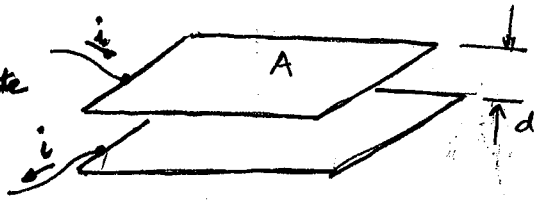
r = ~~diameter~~ radius d = diameter of wire

$$R = \frac{\rho l}{A}$$



σ = ~~resistivity~~ conductivity $= \frac{1}{\text{resistivity}}$

flate plate

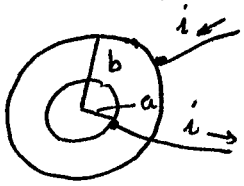


$$C = \frac{\epsilon A}{d}$$

parallel plate capacitor

ϵ - permittivity

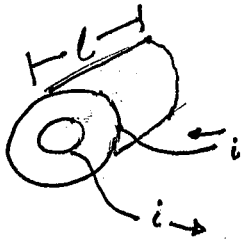
concentric spheres



$$C = \frac{4\pi\epsilon}{(1/a - 1/b)}$$

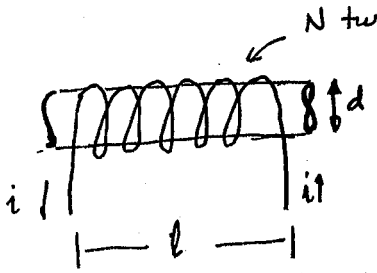
ϵ - permittivity of material

Concentric cylinders



$$C = \frac{2\pi\epsilon l}{\cosh^{-1}\left(\frac{a^2+b^2}{2ab}\right)}$$

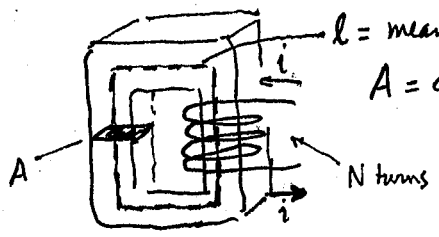
solenoid



$$L = \frac{\pi\mu d^2 N^2}{4l}$$

μ - permeability

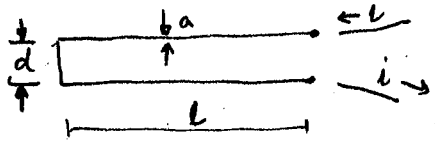
permeable core



A = cross-sectional area

$$L = \frac{\mu A N^2}{l}$$

thin parallel wires



$$L = \frac{\mu l}{\pi} \ln\left(\frac{d}{a}\right);$$

a = wire diameter

in general $\lambda_{21} = f(i)$

AN IDEAL INDUCTANCE $\lambda_{21} = Li$ [L] - HENRIES = $\frac{\text{VOLT·SEC}}{\text{AMP}}$

or $i = \frac{1}{L}\lambda_{21} \Rightarrow i = \frac{1}{L}\int_0^t V_{21}\,dt + i(t=0)$

$V_{21} = L\frac{di}{dt}$

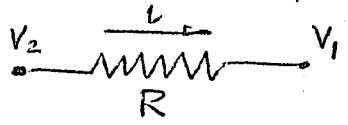
like spring: $F = kx_{21}$, $V_{21} = \frac{1}{k}dF/dt$

$E_m = \int V_{21}\, i\, dt = \int i\, d\lambda_{21} = \frac{1}{2L}\lambda_{21}^2 = \frac{1}{2}i^2 L \geq 0$ like $E_p = \frac{1}{2}\frac{F^2}{k}$

ENERGY STORED BY VIRTUE OF CURRENT. ENERGY IS RECOVERABLE. NOT DIRECTLY DEPENDENT ON V_{21}. L DEPENDENT ON GEOMETRY & MATERIAL

RESISTANCE (DISSIPATES ENERGY) LIKE DAMPER

in general $i = f(V_{21})$



$i = \frac{1}{R}V_{21}$ like damper $F = bV_{21}$

RESISTORS EXHIBIT NO CAPACITANCE OR INDUCTANCE

$P = iV_{21} = \frac{1}{R}V_{21}^2 = i^2R > 0$ WATTS

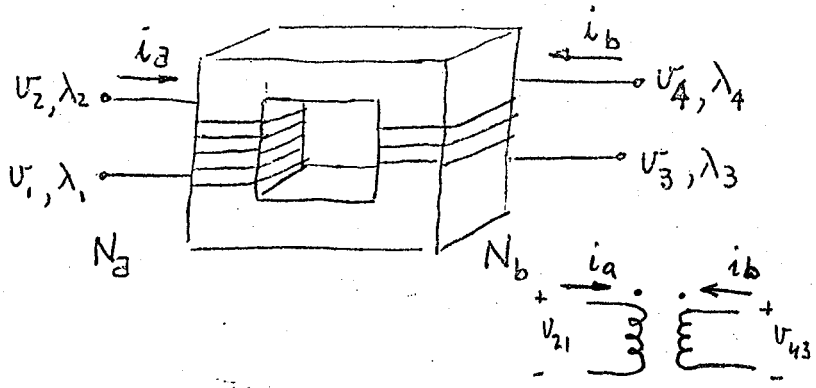
like damper $P = F^2/b$

POWER FLOWS INTO A RESISTOR BUT NEVER OUT - ENERGY IS DISSIPATED ... POWER IS TRANSFORMED TO HEAT

TRANSFORMERS (TRANSFORM ENERGY W/O DISSIPATION)

- PURE ELECTRICAL TRANSFORMER – DOESN'T STORE OR DISSIPATE ENERGY
- DETERMINES OUTPUT VOLTAGE DIFF. AS A FN OF INPUT VOLTAGE DIFF.

- ASSUME TWO COILS OF N_a, N_b TURNS WOUND AROUND A MAGNET WHOSE PERMEABILITY IS LARGE IN COMPARISON TO AIR



- ASSUME MAGNETIC FLUX Φ IS NOT LOST
- FARADAY'S LAW

$\lambda_{21} = \Phi N_a$

$\lambda_{43} = \Phi N_b$

$\lambda_{43}/\lambda_{21} = N_b/N_a$

since v is related to λ $\Rightarrow$ $\frac{V_{43}}{V_{21}} = \frac{N_b}{N_a} = n$ turns ratio

SINCE NO net POWER IN AN IDEAL TRANSFORMER THEN

$$i_a V_{21} = -i_b V_{43} \Rightarrow \frac{i_b}{i_a} = \frac{-V_{21}}{V_{43}} = \frac{-1}{n} = -\frac{N_a}{N_b}$$

ELECTROMECHANICAL TRANSDUCERS

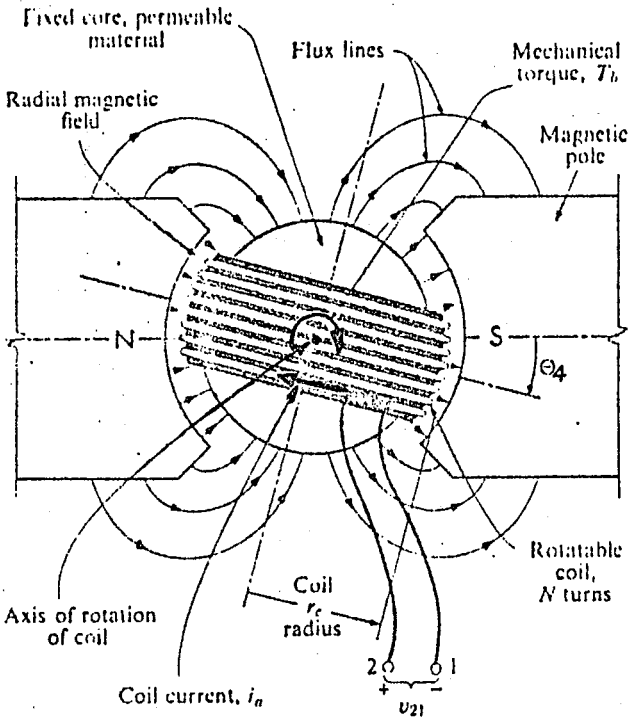


FIG. 2-38. Schematic diagram of rotary electromagnetic electrical-to-mechanical transducer.

- CONVERTS ELECTRICAL CURRENT TO MECHANICAL TORQUE
- VOLTAGE INTO ANGULAR VELOCITY

- A RADIAL MAGNETIC FIELD OF STRENGTH B (WEBERS/M^2) IS ESTABLISHED BETWEEN MAGNETS.
- MOVABLE N-TURN COIL ROTATES ABOUT AXIS
- AS CURRENT PASSES THROUGH COIL, MAGNETIC FIELD PRODUCES TORQUE
- BIOT-SAVART LAW GIVES

$$T_b = -2N\ell\, B\, r_c\, i_a = -\frac{1}{n} i_a$$

$2N\ell$ - LENGTH OF CONDUCTOR

r_c - COIL RADIUS

n - TRANSDUCTION RATIO OR ELECTROMECHANICAL COUPLING CONST

- IF COIL MOVES W/ $\omega_4 = \dot{\theta}_4$, VOLTAGE DROP IS INDUCED ACROSS TERMINALS
- BY FARADAY'S LAW

$$v_{21} = N\frac{d\Phi}{dt} = 2N\ell\, B\, r_c\, \omega_4 = \frac{\omega_4}{n}$$

Φ MAGNETIC FLUX

FLUID SYSTEMS

- ANALOGOUS TO ELECTRICAL SYSTEMS

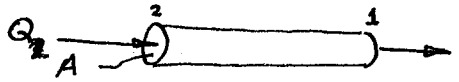
  Q VOLUMETRIC FLOWRATE EQUIV. TO CURRENT FLOW i

  p FLUID PRESSURE EQUIV. TO ELECTRIC POTENTIAL (voltage) V

- FLUID MAY BE LIQUID OR GAS
- PNEUMATIC ELEMENTS ARE MORE HIGHLY NONLINEAR THAN HYDRAULIC ELEMENTS

- ANY FORCE ACTING ON FLUID CAN BE RESOLVED INTO NORMAL / TANGENTIAL
- NORMAL FORCE (TO SURFACE) PRODUCES PRESSURE ; TANGENTIAL FORCE (TO SURFACE) PRODUCES SHEAR
- CONSIDER PRESSURE CONSTANT ACROSS PIPES SO THAT $F_n = PA \cong \int p\,dA = \bar{p}A$
- MOVING A FLUID ELEMENT A DISTANCE dx produces $dW = F_n\,dx$ WORK

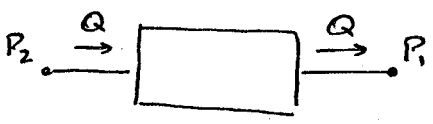
  THUS $dW = \bar{p}\,A\,dx = \bar{p}\,d(\text{Volume})$ $\quad [dW = V_{21}\,d(\text{charge})]$



- pressure is considered positive WHEN FLUID IS IN STATE OF COMPRESSION
- now $\frac{dW}{dt} = \mathcal{P} = \bar{p}\,\frac{d}{dt}(\text{Volume}) = \bar{p}\,Q$ $\quad Q$ volumetric FLOW RATE $\quad [i]$ [Voltage]
  p pressure
- IF FLOW IS INCOMPRESSIBLE & PIPE IS RIGID FLOW IN = FLOW OUT = Q

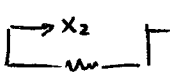
- WHEN WE CONNECT TWO PIPES TOGETHER CONTINUITY SAYS mass is conserved

  CONSERVATION OF MASS $\Rightarrow$ net rate of Q INTO SYSTEM = rate of increase of Q IN SYSTEM. IF FLUID IS INCOMPRESSIBLE (or COMPRESSIBLE) $\text{mass}_{in} - \text{mass}_{out} = \frac{d}{dt}(\rho V)$
- Q IS A THROUGH VARIABLE LIKE F, T, i
- P IS AN ACROSS variable like Veloc, ω, Voltage



POWER DELIVERED IS $\mathcal{P} = Q(P_2 - P_1)$
$= QP_{21}$

IF Q IN = Q OUT

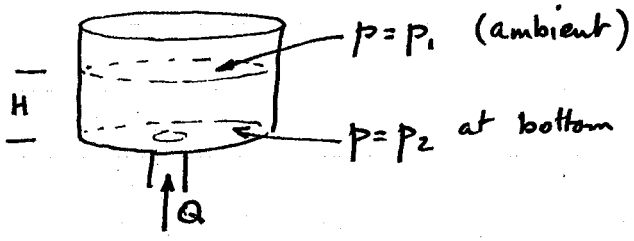
HW's  $x_2 > x_1 \Rightarrow$ compression

if v is continuous $x = \int v\,dt$ is continuous

- FLUID SYSTEMS CAN BE IDEALIZED BY ELEMENTS THAT MODEL FLUID ENERGY STORAGE, DISSIPATION & FLUID TRANSFER
  - FLUID CAPACITANCE STORES ENERGY VIA PRESSURE (ACROSS VAR)
  - FLUID INERTANCE " " VIA FLOW RATE (THROUGH VAR)
  - FLUID RESISTANCE DISSIPATES ENERGY

CAPACITANCE

IN A FLUID RESERVOIR - OPEN TANK IN A GRAVITY FIELD



$$\rho Q = \frac{d}{dt}(\rho A H)$$ CONSERV. OF MASS

- FOR $\rho = \text{const}$ $Q = \frac{d}{dt}(AH)$

- for a reservoir (quasistatic)

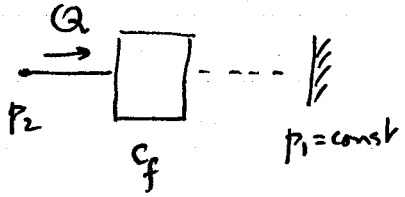
$$p_{21} = p_2 - p_1 = \rho g H = g\rho\frac{AH}{A}$$

$$\therefore \frac{A}{g\rho}\frac{d}{dt}(p_{21}) = \frac{d}{dt}(AH) = Q \qquad \text{if we define } \frac{A}{g\rho} = C_f$$ FLUID CAPACITANCE

$$\frac{dV}{dt} = \boxed{Q = C_f \frac{d}{dt}(p_{21})} \Rightarrow \boxed{\text{volume} = C_f \cdot p_{21}}$$

p = "mom." = mass·velocity
q = charge = C · voltage
volume = $C_f \cdot \Delta p$.

- Note that fluid capacitance only has one port through which flow takes place. Pressure is measured with RESPECT TO AMBIENT PRESSURE HERE



SYMBOLIC REPRESENTATION

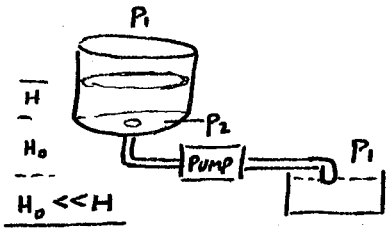
since $V = C_f\, p_{21}$ & $\frac{dV}{dt} = Q \Rightarrow p_{21} = \frac{1}{C_f}\int Q\,dt + p_{21}^{(t}$

$V_{21} = \frac{1}{m}\int F\,dt + V_{21}$

- also we CAN DEFINE FLUID POTENTIAL ENERGY E_p

- $E_p = \int \mathcal{P}\,dt = \int P_{21}\,\overbrace{Q\,dt}^{dV} = \int P_{21}\,dV$ if ρ = constant

- THIS IS THE ENERGY SUPPLIED BY A PUMP (CONSIDERED A SOURCE) IN RAISING THE PRESSURE OF THE FLUID FROM P_1 TO P_2



$$E_p = \int P_{21}\,dV = \int C_f\,p_{21}\,dp_{21} = C_f\,\frac{P_{21}^2}{2}$$

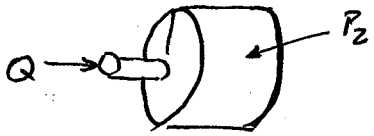
ENERGY: RECOVERABLE

P_{21} is the state variable here

IN A PRESSURIZED TANK FLUID ENERGY IS STORED DUE TO COMPRESSIBILITY OF FLUID

ASSUMPTIONS

- GRAVITY VARIATIONS ARE NEGLECTED
- RIGID WALL TANK OF VOLUME $\mathcal{V}$
- FLUID INERTIA & FRICTION NEGLECT
- LOOK AT VARIATIONS OF ρ IN TANK



CONSERV. OF MASS $\rho Q = \frac{d}{dt}(\rho \mathcal{V}) = \mathcal{V}\frac{d\rho}{dt}$

FOR MOST LIQUIDS $d\rho = \frac{\rho}{\beta}\,dP_2$ β is bulk modulus, P_2 is absolute press.

FOR MOST GASES $d\rho = \frac{\rho}{nP_2}\,dP_2$ n = ratio of C_p/C_v

if ΔP is small $\Rightarrow \beta \sim nP_2$

$$\therefore \quad \rho Q = \mathcal{V}\frac{d\rho}{dt} = \frac{\mathcal{V}\rho}{\beta}\frac{dP_2}{dt} \quad \text{or} \quad Q = \frac{\mathcal{V}}{\beta}\frac{dP_2}{dt} = \frac{\mathcal{V}}{\beta}\frac{dP_{21}}{dt}$$

P_1 is @ AMBIENT PRESSURE, not FN OF t

- now Volume of fluid $V = \int Q\,dt = \frac{\mathcal{V}}{\beta}P_{21}$ for a liquid where $\frac{\mathcal{V}}{\beta} = C_f$

IF β depends on P_2 THEN $V = \mathcal{V}\int \frac{dP_{21}}{\beta}$

3.1

$T = p_s A \cdot r$

Ωr V

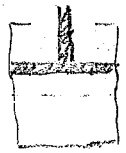
$Q = VA = \Omega A \cdot r \qquad \frac{Q}{\Omega} = \frac{T}{p_s}$

A = gear tooth height · gt width

$P = \quad Q \cdot p_s =$

~~3.2~~ $p \cdot 2\pi r = \sigma t \qquad \sigma = p$

Do. 3-4, 3-3,



$\Delta F = A \Delta p$

$\Delta V = -A \Delta x$

$pV^\gamma = \text{const}$

$\Delta p V^\gamma + p\gamma V^{\gamma-1} \Delta V = 0$

$\Delta p = -\frac{\gamma p \Delta V}{V}$

$\therefore \quad \Delta F = A \cdot -\frac{\gamma p \Delta V}{V} = A\left(-\frac{\gamma p}{V} \cdot -A\Delta x\right)$

$\Delta F = \frac{A^2 \gamma p}{V} \Delta x$

$\frac{\Delta F}{\Delta t} = \frac{A^2 \gamma p}{V} \frac{\Delta x}{\Delta t}$ or $\frac{dF}{dt} = kv \qquad k = \frac{\gamma p}{V} \cdot A^2$

let $Q = Av$ $\qquad A\frac{dp}{dt} = \frac{A\gamma p}{V} Q$

$\frac{dp}{dt} = \frac{\gamma p}{V} Q$

$\frac{V}{\gamma p} \frac{dp}{dt} = Q$

field, a fluid may have a shear-rate-shear-stress relation which includes magnetohydrodynamic effects. For low density gases the fluid may have to be considered as made up of discrete particles, and not be treated as a continuum. These are typical of fluids in this fourth category.

Most of the work on non-Newtonian fluids during the past decade has been done by chemical engineers, but interest is being shown by people in other engineering areas. An informative discussion of the civil engineering aspects of non-Newtonian fluids is given by Bugliarello *et al* [4]. Engineers in many areas are becoming increasingly confronted with the flow of non-Newtonian fluids in pipes as well as in the design and selection of pumps for these fluids [5]. Experiments have shown that frictional effects for some non-Newtonian fluids are less than for a Newtonian fluid under equivalent turbulent flow conditions, resulting in a lower pressure drop in pipes and reduced drag on bodies submerged in liquids. This latter situation is of interest in naval hydrodynamics.

The study of non-Newtonian fluids is a part of the general science of rheology [6].

2–3. SURFACE TENSION AND CAPILLARITY

If a spoon is held under a dripping faucet, the water may rise nearly ⅛ in. above the upper edges of the spoon before spilling. Similarly, water may be poured into a glass to a level nearly ⅛ in. above the lip of the glass. If a small-bore tube is placed vertically in a free water surface, the water will rise in the tube; if the liquid is mercury, the mercury will be depressed within the tube. These are examples of the effects of surface tension and capillarity of liquids.

Surface tension is a characteristic of a liquid surface or skin, and its effects occur at liquid-gas or liquid-liquid interfaces. It is expressed in terms of energy per unit area, or its equivalent, force per unit length. The surface tension of water, for example, varies slightly with temperature, but it is about 0.005 lb_f/ft at ordinary temperatures. It may be reduced to about one-half that value by the addition of wetting agents.

Typical values of surface tension for some liquids are given in Table 2–8. It is seen that the surface tension is essentially the same for a liquid in contact with air or its own vapor. The higher the surface tension, the better is the interface in a manometer (see Sec. 4–3).

Attempts are usually made to avoid the effects of surface tension. Manometers (see Sec. 4–3) contain fluids which are chosen so as to have a readable meniscus (interface), that of mercury under water being exceptionally good, and that of carbon tetrachloride under water being poor. Manometer tubing, especially when an air-liquid interface is used in plastic tubing, should be selected carefully in order to avoid erroneous results.

TABLE 2–8

SURFACE TENSION OF LIQUIDS AT 77 F IN CONTACT WITH AIR, WATER, OR THEIR OWN VAPOR*

Substance	Surface Tension (lb/ft)
Carbon tetrachloride-air	0.0018
Water-air	0.0050
Mercury-air	0.032
Carbon tetrachloride-water	0.0031
Mercury-water	0.026
Carbon tetrachloride-vapor	0.0018
Water-vapor	0.0050

**Handbook of Chemistry and Physics* (38th ed.; Cleveland: Chemical Rubber Publishing Company, 1956–1957).

Some plastic tubes also exhibit hysteresis effects when used in this way—a falling liquid column comes to rest at a different position than a rising liquid column for the same external pressure condition. Hydraulic models are made large enough so that shallow depths, which otherwise might be affected by surface tension, are avoided. Wetting agents may be used to reduce these effects if the size of the model is made as large as possible, yet smaller than would be desirable. Surface tension plays a role in deterring the growth of small gas nuclei in liquids when they pass through low-pressure regions, yet the precise role is not understood. This rapid growth and collapse of bubbles in liquids is one form of cavitation.

Forces due to surface tension are, with the preceding exceptions, generally small compared with the forces due to gravity, viscosity, and pressure in engineering practice.

2–4. COMPRESSIBILITY OR ELASTICITY

Fluids may be deformed by viscous shear or compressed by an external pressure applied to a volume of fluid. All fluids are compressible by this method, liquids to a much smaller degree, however, than gases.

The compressibility is defined in terms of an average bulk modulus of elasticity

$$\overline{K} = -\frac{p_2 - p_1}{(V_2 - V_1)/V_1} \tag{2–11}$$

What we call β →

where V_2 and V_1 are the volumes of the substance at pressure p_2 and p_1, respectively. The bulk modulus varies with the pressure for gases, and with both pressure and temperature (though but slightly) for liquids. Thus, the true bulk modulus of elasticity is the limiting value of Eq. 2–11 when the pressure and volume changes become infinitesimal.

$$K = -\frac{dp}{dV/V} \tag{2–12a}$$

If a unit mass of substance is considered,

$$K = -\frac{dp}{dv/v} \tag{2–12b}$$

and

$$K = +\frac{dp}{d\rho/\rho} \tag{2–12c}$$

The denominators of Eqs. 2–11 and 2–12 are dimensionless, so that K has the dimensions of a pressure, or force per unit area.

The value of K for water at 68 F is about 320,000 $lb_f/in.^2$ at atmospheric pressure, increasing essentially linearly to about 410,000 $lb_f/in.^2$ at a pressure of 15,000 $lb_f/in.^2$ Thus, in this range,

$$K = 320{,}000 + 6\,p \quad lb_f/in.^2$$

where p is the gage pressure in $lb_f/in.^2$

The change in volume of a gas for a change in pressure depends on the compression process. If *isothermal* (constant temperature), the gas equation may be expressed in logarithmic form as $\ln p = \ln \rho + \ln (RT)$ and differentiated to obtain $dp/p = d\rho/\rho$. Thus

$$K_{\text{isothermal}} = \frac{dp}{d\rho/\rho} = \frac{dp}{dp/p} = p \tag{2–13}$$

and the elastic modulus equals the absolute pressure during an isothermal compression.

If the compression is adiabatic and is carried out slowly so that equilibrium conditions exist, the compression may be considered reversible and adiabatic, or *isentropic* (see Chapter 3). For this process, $p/\rho^k = \text{constant}$ (k is the ratio of specific heat capacities). The logarithmic form, $\ln p - k \ln \rho = \ln C$, may be differentiated to obtain $dp/p = k\,d\rho/\rho$. Thus

$$K_{\text{isentropic}} = \frac{dp}{d\rho/\rho} = \frac{dp}{dp/kp} = kp \tag{2–14}$$

and the elastic modulus equals the absolute pressure times the ratio of specific heat capacities ($k = c_p/c_v = 1.4$ for air) during an isentropic compression.

The bulk modulus of elasticity K is of interest in acoustics as well as in fluid mechanics. The velocity of sound in any medium is

$$c = \sqrt{\frac{K}{\rho}} \tag{2–15}$$

and for a gas, sound waves are transmitted essentially isentropically (see Sec. 9–1), so that the velocity of sound in a perfect gas is

$$c = \sqrt{\frac{kp}{\rho}} = \sqrt{kRT} \tag{2–16}$$

EXAMPLE 2–5. What is the speed of sound in water at 68 F and atmospheric pressure?

Solution:

$$c = \sqrt{\frac{K}{\rho}} = \sqrt{\frac{(320{,}000\ lb_f/in.^2)\ (144\ in.^2/ft^2)}{1.94\ slugs/ft^3}} = 4870\ ft/sec$$

EXAMPLE 2–6. What is the speed of sound in air at 68 F at sea level and at an altitude where the pressure is 10 $lb_f/in.^2$ absolute?

Solution: At either pressure,

$$c = \sqrt{kRT} = \sqrt{(1.4)\left(1715\,\frac{ft\ lb_f}{slug\ R}\right)(528\ R)} = 1126\ ft/sec$$

2–5. VAPOR PRESSURE

If a liquid and its vapor coexist in equilibrium, the vapor is called a saturated vapor, and the pressure exerted by this saturated vapor is called the *vapor pressure*. The vapor pressure is a function of temperature for a given substance.

The vapor pressure of liquids is of practical importance in barometers (see Sec. 4–3), pump-piping systems, and from an elementary point of view, in the formation of cavities in low-pressure regions within a liquid.

Values of vapor pressure for some liquids at various temperatures are shown in Table 2–9.

TABLE 2–9
VAPOR PRESSURE OF SOME LIQUIDS

Temperature (deg F)	Liquid Vapor Pressure (psia)				
	Water	Mercury	Kerosene	Propane	Methyl Alcohol
40	0.122		0.32	78	0.71
60	0.26	0.000025	0.44	107	1.42
100	0.95		0.86	187	4.42
160	4.74		1.23		

REFERENCES

1. W. L. Wilkinson, *Non-Newtonian Fluids* (New York: Pergamon Press, 1960).

2. A. B. Metzner, "Flow of Non-Newtonian Fluids," Section 7 of *Handbook*

field, a fluid may have a shear-rate-shear-stress relation which includes magnetohydrodynamic effects. For low density gases the fluid may have to be considered as made up of discrete particles, and not be treated as a continuum. These are typical of fluids in this fourth category.

Most of the work on non-Newtonian fluids during the past decade has been done by chemical engineers, but interest is being shown by people in other engineering areas. An informative discussion of the civil engineering aspects of non-Newtonian fluids is given by Bugliarello *et al* [4]. Engineers in many areas are becoming increasingly confronted with the flow of non-Newtonian fluids in pipes as well as in the design and selection of pumps for these fluids [5]. Experiments have shown that frictional effects for some non-Newtonian fluids are less than for a Newtonian fluid under equivalent turbulent flow conditions, resulting in a lower pressure drop in pipes and reduced drag on bodies submerged in liquids. This latter situation is of interest in naval hydrodynamics.

The study of non-Newtonian fluids is a part of the general science of rheology [6].

2–3. SURFACE TENSION AND CAPILLARITY

If a spoon is held under a dripping faucet, the water may rise nearly ⅛ in. above the upper edges of the spoon before spilling. Similarly, water may be poured into a glass to a level nearly ⅛ in. above the lip of the glass. If a small-bore tube is placed vertically in a free water surface, the water will rise in the tube; if the liquid is mercury, the mercury will be depressed within the tube. These are examples of the effects of surface tension and capillarity of liquids.

Surface tension is a characteristic of a liquid surface or skin, and its effects occur at liquid-gas or liquid-liquid interfaces. It is expressed in terms of energy per unit area, or its equivalent, force per unit length. The surface tension of water, for example, varies slightly with temperature, but it is about 0.005 lb_f/ft at ordinary temperatures. It may be reduced to about one-half that value by the addition of wetting agents.

Typical values of surface tension for some liquids are given in Table 2–8. It is seen that the surface tension is essentially the same for a liquid in contact with air or its own vapor. The higher the surface tension, the better is the interface in a manometer (see Sec. 4–3).

Attempts are usually made to avoid the effects of surface tension. Manometers (see Sec. 4–3) contain fluids which are chosen so as to have a readable meniscus (interface), that of mercury under water being exceptionally good, and that of carbon tetrachloride under water being poor. Manometer tubing, especially when an air-liquid interface is used in plastic tubing, should be selected carefully in order to avoid erroneous results.

TABLE 2–8

SURFACE TENSION OF LIQUIDS AT 77 F IN CONTACT WITH AIR, WATER, OR THEIR OWN VAPOR*

Substance	Surface Tension (lb/ft)
Carbon tetrachloride-air	0.0018
Water-air	0.0050
Mercury-air	0.032
Carbon tetrachloride-water	0.0031
Mercury-water	0.026
Carbon tetrachloride-vapor	0.0018
Water-vapor	0.0050

**Handbook of Chemistry and Physics* (38th ed.; Cleveland: Chemical Rubber Publishing Company, 1956–1957).

Some plastic tubes also exhibit hysteresis effects when used in this way—a falling liquid column comes to rest at a different position than a rising liquid column for the same external pressure condition. Hydraulic models are made large enough so that shallow depths, which otherwise might be affected by surface tension, are avoided. Wetting agents may be used to reduce these effects if the size of the model is made as large as possible, yet smaller than would be desirable. Surface tension plays a role in deterring the growth of small gas nuclei in liquids when they pass through low-pressure regions, yet the precise role is not understood. This rapid growth and collapse of bubbles in liquids is one form of cavitation.

Forces due to surface tension are, with the preceding exceptions, generally small compared with the forces due to gravity, viscosity, and pressure in engineering practice.

2–4. COMPRESSIBILITY OR ELASTICITY

Fluids may be deformed by viscous shear or compressed by an external pressure applied to a volume of fluid. All fluids are compressible by this method, liquids to a much smaller degree, however, than gases.

The compressibility is defined in terms of an average bulk modulus of elasticity

$$\overline{K} = -\frac{p_2 - p_1}{(V_2 - V_1)/V_1} \tag{2–11}$$

where V_2 and V_1 are the volumes of the substance at pressure p_2 and p_1, respectively. The bulk modulus varies with the pressure for gases, and with both pressure and temperature (though but slightly) for liquids. Thus, the true bulk modulus of elasticity is the limiting value of Eq. 2–11 when the pressure and volume changes become infinitesimal.

$$K = -\frac{dp}{dV/V} \tag{2–12a}$$

If a unit mass of substance is considered,

$$K = -\frac{dp}{dv/v} \tag{2–12b}$$

and

$$K = +\frac{dp}{d\rho/\rho} \tag{2–12c}$$

The denominators of Eqs. 2–11 and 2–12 are dimensionless, so that K has the dimensions of a pressure, or force per unit area.

The value of K for water at 68 F is about 320,000 $lb_f/in.^2$ at atmospheric pressure, increasing essentially linearly to about 410,000 $lb_f/in.^2$ at a pressure of 15,000 $lb_f/in.^2$ Thus, in this range,

$$K = 320{,}000 + 6\,p \quad lb_f/in.^2$$

where p is the gage pressure in $lb_f/in.^2$

The change in volume of a gas for a change in pressure depends on the compression process. If *isothermal* (constant temperature), the gas equation may be expressed in logarithmic form as $\ln p = \ln \rho + \ln (RT)$ and differentiated to obtain $dp/p = d\rho/\rho$. Thus

$$K_{\text{isothermal}} = \frac{dp}{d\rho/\rho} = \frac{dp}{dp/p} = p \tag{2–13}$$

and the elastic modulus equals the absolute pressure during an isothermal compression.

If the compression is adiabatic and is carried out slowly so that equilibrium conditions exist, the compression may be considered reversible and adiabatic, or *isentropic* (see Chapter 3). For this process, $p/\rho^k = $ constant (k is the ratio of specific heat capacities). The logarithmic form, $\ln p - k \ln \rho = \ln C$, may be differentiated to obtain $dp/p = k\,d\rho/\rho$. Thus

$$K_{\text{isentropic}} = \frac{dp}{d\rho/\rho} = \frac{dp}{dp/kp} = kp \tag{2–14}$$

and the elastic modulus equals the absolute pressure times the ratio of specific heat capacities ($k = c_p/c_v = 1.4$ for air) during an isentropic compression.

The bulk modulus of elasticity K is of interest in acoustics as well as in fluid mechanics. The velocity of sound in any medium is

$$c = \sqrt{\frac{K}{\rho}} \tag{2–15}$$

and for a gas, sound waves are transmitted essentially isentropically (see Sec. 9–1), so that the velocity of sound in a perfect gas is

$p v^\gamma = \text{const}$

$dp \cdot v^\gamma + p \cdot \gamma v^{\gamma-1} dv = 0$

$\therefore \quad dv = \frac{-dp \cdot v}{p\gamma}$

now $v \cdot \rho = \text{const} = \text{mass}$

$dv \cdot \rho + v \cdot d\rho = 0$

$\therefore dv = -\frac{v}{\rho} \cdot d\rho$

$\therefore \quad -\frac{v}{\rho} d\rho = \frac{-dp \cdot v}{p\gamma} \quad \& \quad d\rho = \frac{\rho}{p\gamma} dp$

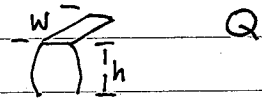
$P \cdot WH = F$

$F \cdot R = T$ 3-1

$T = p\,RWH$

$\frac{P}{T} = \frac{1}{RWH}$

$\frac{Q}{\Omega} = WHR$



$Q = w \cdot h \cdot \text{velocity}$
$= w \cdot h \cdot \Omega R$

if no energy lost the $T\Omega = \mathcal{P} = Q \cdot P_s$

P_s varies between 400 – 600 psi

$P_{av} = 500 \frac{\#}{in^2} \quad \therefore \quad \mathcal{P} = Q \cdot P_s = 10 \frac{gal}{min} \cdot 500 \frac{lb}{in^2} \cdot$

$231 \frac{in^3}{gal} \cdot \frac{1\,ft}{12\,in}$

$= 96100 \frac{ft\text{-}lb}{min}$

$\frac{1\ HP}{33000 \frac{ft\text{-}lb}{min}} \cong 3\ hp$

or a spring

- FLUID INERTANCE (ANALOGOUS TO electrical INDUCTANCE)
  - LOOK AT FLOW THAT IS NONSTEADY, FRICTIONLESS, INCOMPRESSIBLE

FROM FLUIDS $\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = -\frac{1}{\rho} \nabla p$

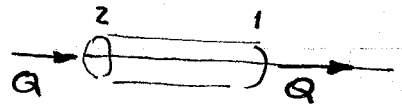
and $\frac{\partial \rho}{\partial t} + \underline{v} \cdot \nabla \rho = 0$ continuity

$\Rightarrow \frac{\partial (\rho \underline{v})}{\partial t} + \underline{v} \cdot \nabla (\rho \underline{v}) = -\nabla p$

- LOOK AT A NON ACCELERATING PIPE; UNSTEADY FLOW OF FLUID, FRICTIONLESS

FOR INCOMPRESSIBLE FLUID $\nabla (\rho \underline{v}) = 0 \Rightarrow \rho V_1 A_1 = \rho V_2 A_2 = Q = \text{const.}$

$\frac{\partial v}{\partial t} = -\frac{1}{\rho} \nabla p = -\frac{1}{\rho l} [p_1 - p_2]$



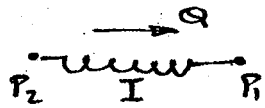
$\therefore \underbrace{\rho l \cdot A}_{\text{fluid mass}} \frac{\partial v}{\partial t} = A p_{21} = (p_2 - p_1) A = \text{force}$; now $Q = Av \Rightarrow A \frac{\partial v}{\partial t} = \frac{\partial Q}{\partial t}$

$\therefore \frac{\rho l}{A} \frac{\partial Q}{\partial t} = p_{21}$ $\qquad \underline{I = \frac{\rho l}{A}} = \text{fluid inertance}$

- define $\frac{d \Gamma_{21}}{dt} = p_{21} \Rightarrow \Gamma_{21} = I Q$ ideal fluid inertance relation

Γ_{21} = pressure momentum difference

$\underline{p_{21} = I \frac{\partial Q}{\partial t}}$ elemental equation FOR FLUID INERTANCE



in electrical system $\lambda_{21} = L i$ & $v_{21} = L \frac{di}{dt}$

in mech systems $x_{21} = \frac{1}{k} F$ $\qquad v_{21} = \frac{1}{k} \frac{dF}{dt}$

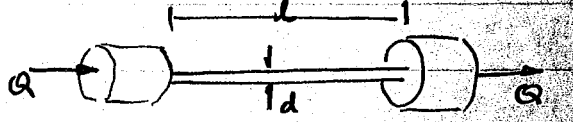
- energy stored due to fluid inertance is due to kinetic energy of flow

$\underline{\mathcal{E}_k = \int Q p_{21} dt = \int Q \, d\Gamma_{21} = \int Q \cdot I \, dQ = \frac{1}{2} I Q^2}$

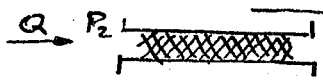
- Γ_{21} is known as the pressure momentum difference $\qquad Q$ is the state variable

FLUID RESISTANCE

- FLOW THRU PIPES NORMALLY PRODUCE FRICTION & THERE IS RESISTANCE TO FLOW. ASSUME p_{21} IS SMALL COMPARED TO $p_{Average}$
  - FLOW THROUGH POROUS MEDIUM
  - FLOW THROUGH CAPILLARY TUBE
  - FLOW THROUGH LONG PIPE
  - FLOW THRAUGH ORIFICE



IN ALL CASES $QR_f = P_{21}$ R_f = fluid resistance



THIS IS ANALOGOUS TO OHM'S LAW $V_{21} = iR$

$V_{21} = F/b$

- FOR A POROUS PLUG, R_f IS A FN. OF porosity, viscosity of FLUID
  EXPERIMENTALLY OBTAINED
- FOR CAPILLARY TUBE $R_f = \frac{128\mu l}{\pi d^4}$

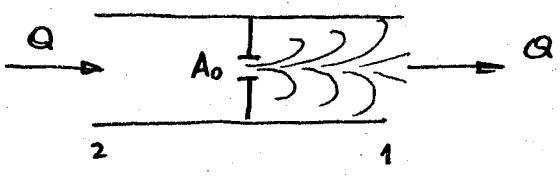
- FLOW THROUGH LONG PIPE $R_N < 2000$ $R_f = \frac{128\mu l}{\pi d^4}$ & $P_{21} = R_f Q$

$R_N = \frac{4\rho Q}{\pi d \mu}$ (REYNOLDS NO.)

$R_N > 5000$ $P_{21} = CQ|Q|^{3/4}$

C IS A FN OF d, l, ρ, μ, roughness of pipe & Q at LAMINAR-TURB TRANSITION

- FLOW THROUGH ORIFICE

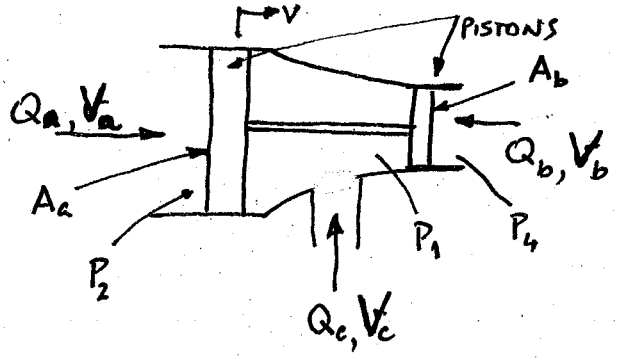


$P_{21} = \frac{\rho}{2C_d^2 A_0^2} Q|Q|$

$\frac{\partial P_{21}}{\partial Q} = R_f = \begin{cases} \frac{\rho}{C_d^2 A_0^2} Q\big|_{op} & Q > 0 \\ \frac{-\rho}{C_d^2 A_0^2} Q\big|_{op} & Q \leq 0 \end{cases}$

- AGAIN - REMEMBER FLUID IS CONSIDERED INCOMPRESSIBLE OR ALMOST INCOMPRESSIBLE

- PURE FLUID TRANSFORMER - ASSUMED ZERO NET POWER



Q - FLOW RATE

$V = \int Q dt$ = volume

a - INPUT SIDE

b - OUTPUT SIDE

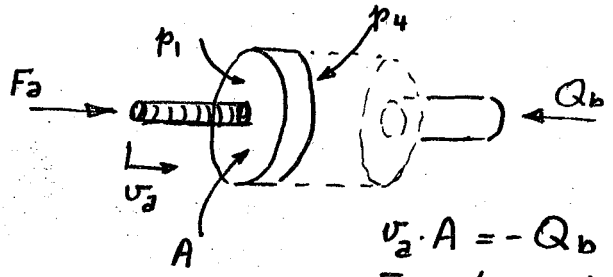
IF FORCE ON PISTON a is $(P_2-P_1)A_a$; FORCE ON PISTON b is $(P_4-P_3)A_b$. IF pistons are ideal (no mass)

$$(P_2-P_1)A_a = (P_4-P_3)A_b \quad \text{or} \quad P_{41}/P_{21} = \frac{A_a}{A_b} = n$$

• since pistons move at same velocity $\quad v_p = \frac{Q_a}{A_a} = -\frac{Q_b}{A_b}$

$$\frac{Q_b}{Q_a} = -\frac{A_b}{A_a} = +\frac{\int Q_b dt}{\int Q_a dt} = -\frac{1}{n} = \frac{V_b}{V_a}$$

• FLUID TRANSDUCER CHANGES FORCE APPLIED TO CHANGES IN PRESSURE



$$v_a \cdot A = -Q_b$$

$$F_a = -(P_1-P_4)A = (P_4-P_1)A$$

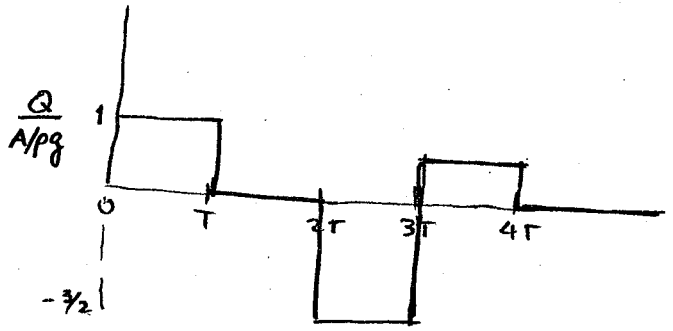
v_a — P_{41} are across variables
F_a — Q_b are through variables

Gyrating Transducer

Do 3-8
3-9
do 3-14
3-1

3-3 $\Delta p = \rho g h = p$ $\quad Q = A\frac{dH}{dt} = A\frac{d}{dt}\left(\frac{p}{\rho g}\right) = \frac{A}{\rho g}\frac{d}{dt}p$

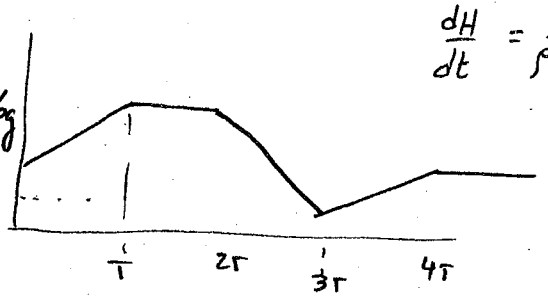
$Q \sim \frac{d}{dt}p$



$p = \rho g H$

$\therefore p/\rho g = H$

H_0



$\frac{dH}{dt} = \frac{1}{\rho g}\frac{dp}{dt} \rightarrow H - H_0 = \frac{1}{\rho g}(P - P_0$

$H = \frac{P}{\rho g} - \frac{P_0}{\rho g} + H_0$

$H_0 = \frac{P_0}{\rho g}$

3-5 THIS A PRESSURIZED TANK $\quad C_f = \frac{V}{\beta} = \frac{A \cdot H_T}{\beta} = \frac{\pi/4 \cdot 1^2 \cdot 1}{200{,}000 \frac{\#}{in^2} \cdot \frac{1.44\ in^3}{144\ in^2}}$

a) $= 5.155 \times 10^{-4}\ 2.7 \times 10^{-8}\ FT^5/lb \quad (6.786 \times 10^{-3}\ in^5/lb)$

b) for the air $\beta = nP$ $\quad n = 1$ for isothermal process

$C_{f_{air}} = \frac{V_{air}}{P_0} = \frac{\pi}{4} \cdot 1^2 \cdot \frac{H_A}{P_0}$ $\quad$ if $H_A = .1\ H_0$ $\quad H_0 = .91$ $H_A = .0$

for the oil β $\quad$ if $H_A = 0$ $\quad H_0 = 1$

$C_{f_{oil}} = \frac{V_{oil}}{\beta} = \frac{\pi}{4} \cdot 1^2 \cdot \frac{H_0}{\beta}$

$C_{f_{TOT}} = C_{f_{air}} + C_{f_{oil}} = 1360\ \frac{H_A}{P_0} + 6.786 \times 10^{-3} H_0$

@ $H_A = 0$ $\quad H_0 = 1$ $\quad C_f = 6.786 \times 10^{-3}\ in^5/lb$

@ $H_A = .1\ H_0 = .09$ ft $\quad P_0 = 200$ psi & $H_0 = .91$ ft $\quad C_f = .62\ \frac{in^5}{lb.}$

Thus air bubbles considerably increase the capacitance of containers

PROBLEMS

3–1. A simple hydraulic pump (a source of fluid power) consists of two meshing gears contained in a close-fitting housing (Fig. 3–25). As the gears rotate, each tooth space captures a nearly constant amount of fluid at the inlet and carries it around to the outlet. At the meshing region the gear teeth contact and seal the outlet. Ideally, Q = constant through the device and the pressure rises from 0 to P_s.

a) Qualitatively discuss how Q is related to the gear-tooth size, the speed Ω, and the width w of the gears.

b) If there are no energy losses, how are Q, P_s, T, and Ω related?

c) The above pump is connected to a fluid system. During operation, Q = const = 10 gal/min, and the outlet pressure P_s is found to vary sinusoidally about a mean pressure of 500 psi, $P_s = 500 + 100 \sin 20\pi t$. Determine the average power and the maximum instantaneous power delivered to the system.

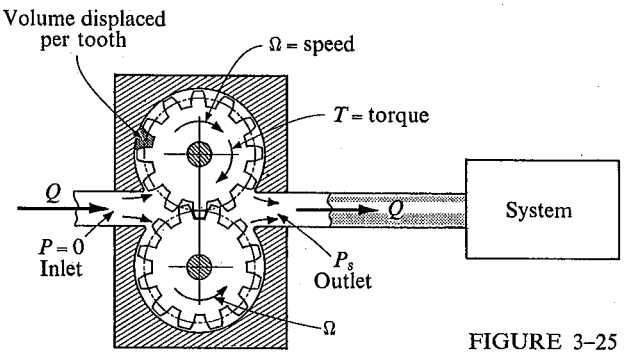


FIGURE 3–25

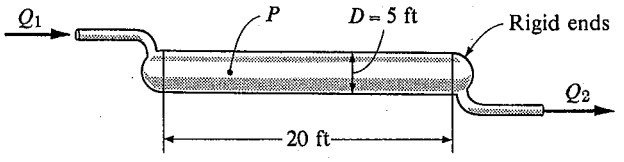


FIGURE 3–26

3–2. In a chemical-processing plant, liquid flows into a long cylinder with a stainless-steel wall 1 in. thick (Fig. 3–26). The pressure in the tank may be assumed uniform and varies from 100 to 500 psi during operation of the process. In this problem we wish to study the capacitive properties of this tank. The fluid has a bulk modulus of 150,000 psi and a density of 60 lb/ft^3; the steel has a Young's modulus of 30×10^6 psi and a Poisson's ratio of 0.3.

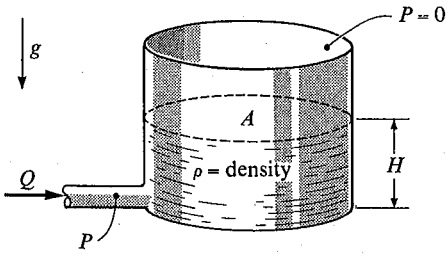
a) Assuming that the tank is rigid, compute the fluid capacitance of the tank, $C_f = (\int(Q_1 - Q_2)\,dt)/P = V/P$.

b) Compute the energy stored when P increases from 0 to 500 psi.

c) Assuming that the fluid is incompressible, compute C_f and compute the energy stored when P = 500 psi.

d) Combine the above to obtain the actual tank capacitance and the total energy stored when P = 500 psi.

e) Discuss the linearity of the V/P relations in (a), (c), and (d).



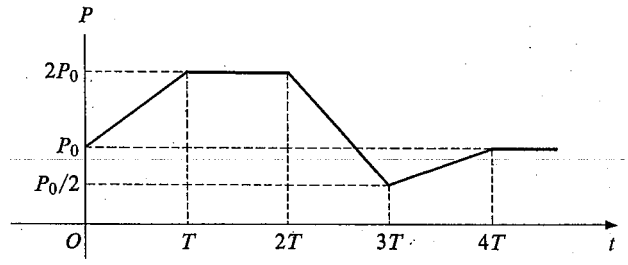


FIGURE 3–27

3–3. Consider an open reservoir being filled with water. The pressure in the tank bottom varies as shown in Fig. 3–27. Sketch curves of the flow rate Q into the tank and the height H of the water in the tank vs. time.

3–4. The flow of water into the tank of Problem 3–3 varies as shown in Fig. 3–28. Given that A = 1 ft^2, determine, as a function of time, the pressure P required at the inlet to produce this flow rate. $P = P_0$ at $t = 0$.

3–5. A cylindrical steel tank of the dimensions shown in Fig. 3–29 has a single inlet for oil flow. The tank is rigid, and the oil has a bulk modulus β = 200,000 psi and a density ρ_O = 62 lb/ft^3. The average pressure in the tank

CHAPTER 3

3-1

$Q = vA \qquad A = w \cdot h_{tooth}$

A) $Q \propto W$ SINCE VOLUME DISPLACED $\propto$ WIDTH

$Q \propto \Omega \qquad Q = vA \qquad v = \Omega \cdot r_{gear} \therefore Q \propto \Omega$

B) $T \propto P_s$, $Q \propto \Omega \qquad T = F \cdot r_{gear} = h_{tooth} \cdot w \cdot p_s \cdot r_{gear}$

$Q = \Omega r_{gear} \cdot w \cdot h_{tooth}$

C) $P = T\Omega = P_s Q$

$$P_i = (500 + 100 \sin 20\pi t)\frac{LB_f}{IN^2}\left(10\frac{GAL}{MIN}\right)\left(231\frac{IN^3}{GAL}\right)\left(\frac{FT}{12IN}\right)$$

$$P_{AVG.} = \frac{(500)(10)(231)}{12}\frac{FT\text{-}LB_f}{MIN}$$

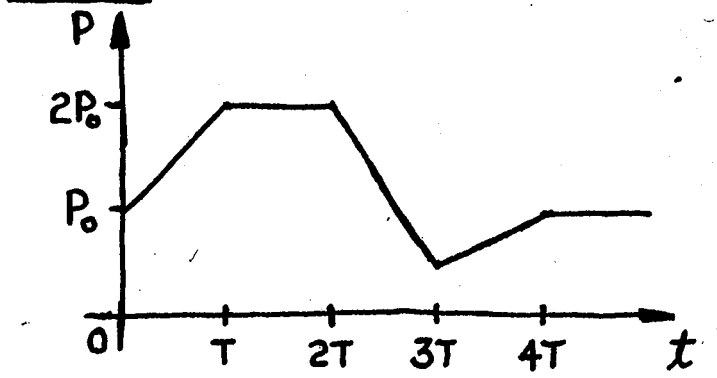
$$P_{AVG.} = 96100\frac{FT\text{-}LB_f}{MIN} \times \frac{1\,HP}{33000\frac{FT\text{-}LB_f}{MIN}}$$

$$P_{AVG} = 2.92\ HP$$

$$P_{MAX} = \frac{(500+100)(10)(231)}{12}\frac{FT\text{-}LB_f}{MIN}$$

$$P_{MAX} = 115500\frac{FT\text{-}LB_f}{MIN} \approx 3.5\ HP$$

3-3



$$P = \rho g H \; ; \quad Q = A\frac{dH}{dt} = A\frac{d}{dt}\left(\frac{P}{\rho g}\right)$$

3-3

$$\int_{H_0}^{H} \frac{dH}{dt}$$

H –

H =

∴ TO
THE P

H
$\frac{2P_0}{\rho g}$
$\frac{P_0}{\rho g}$

3-4

P = ρ

$Q\left[\frac{GAL}{MIN}\right]$(

$Q\left[\frac{FT^3}{MIN}\right]$

Q = 7.

INTEGI

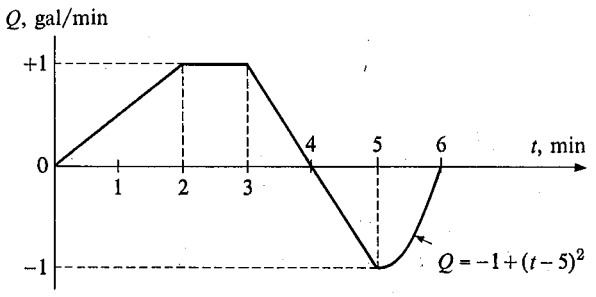


FIGURE 3-28

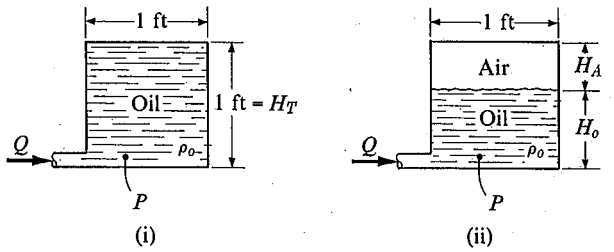


FIGURE 3-29

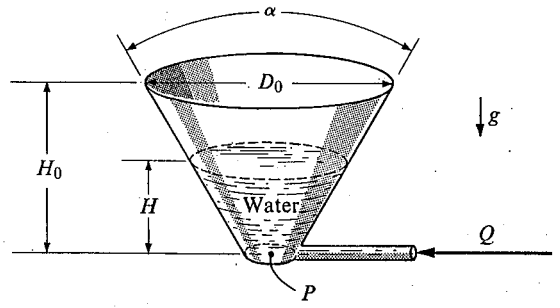


FIGURE 3-30

is P_O, and P varies slightly from this value. Gravity forces are negligible.

a) Assuming that the tank is entirely full of oil as in Fig. 3-29(i), compute the fluid capacitance.

b) In Fig. 3-29(ii) a quantity of air was accidentally left in the tank when it was filled with oil. Derive an expression for the capacitance when $P = P_O + \Delta P$, where ΔP is small in terms of H_O, H_A, P_O, etc., assuming that the air is isothermal and acts like a perfect gas.

c) If $H_A = 0{,}1\ H_O$ and $P_O = 200$ psi, compute the capacitances of the tanks in Fig. 3-29(i) and (ii). Comment on the effect of entrained gas bubbles on the capacitance of liquid-filled tanks.

3-6. A water storage tank has a conical shape (Fig. 3—30). Assuming that $\alpha = 30°$, $H_0 = D_0$, and the tank is rigid, compute and sketch a curve of stored volume V vs. pressure P for $0 < H < H_0$. Sketch a curve of stored energy vs. P. For small changes of P about a mean value P_0 the tank is approximated by an ideal capacitance. Sketch a curve of capacitance vs. P_0.

3-7. A fluid accumulator (Fig. 3-9) is required to have a capacitance of 20 lb/in^5 and a volume of 50 in^3. Design the device; i.e., specify piston area, spring stiffness, length and wall thickness of cylinder, and any other important parameters. Discuss qualitatively the conditions on Q, P_2, and their respective integrals and derivatives which must be met for the device to behave approximately like an ideal capacitance.

3-8. A conduit, called a *penstock*, which leads water from a dam to the inlet of a hydraulic turbine driving an electric generator has a diameter of 5 ft and a length of 700 ft. Assuming that the conduit is rigid, compute the inertance of the fluid. During full-load operation the average flow velocity in the pipe is 50 ft/sec. For this condition, find the energy stored in the fluid inertance and determine how long a 100-watt light bulb could operate from this amount of energy.

3-9. In Problem 3-8, an attempt is made to reduce the flow in the conduit to zero as a linear function of time in 30 sec, starting with full-load conditions. This is done by closing a gate valve near the turbine inlet. Estimate the pressure rise at the valve if the upstream pressure at the inlet to the conduit remains constant.

3-10. A circular tube has a gradual uniform taper, so that its internal diameter decreases linearly from D_2 at the left end to D_1 at the right. The tube is rigid and has length L.

a) Assuming that an incompressible fluid of density ρ flows from left to right in the tube, compute the fluid inertance.

b) If the fluid flows from right to left, how does the inertance change?

c) If the taper is nonuniform, how will this affect the inertance, assuming that D_2 and D_1 do not change?

3-4 CONT.

FOR $3 \le t \le 5$,

$$p_3 - p_2 = 0.14 \int^t (4-t)\,dt$$

$$p_3 = 0.14\,(4t - t^2/2) + p_2$$

FOR $5 \le t \le 6$,

$$p_4 - p_3 = 0.14 \int^t [-1 + (t-5)^2]\,dt$$

$$p_4 = 0.14\left[-t + (t-5)^3/3\right] + p_3$$

3-5

A)
$$C_f = \frac{V}{\beta} = \frac{\frac{\pi D^2}{4} H}{\beta} = \frac{\pi D^2 H}{4\beta}$$

$$C_f = \frac{\pi \left(1728 \frac{IN^3}{FT^3}\right)(1\,FT)^2(1\,FT)}{4\,(2\times 10^5)\,LB_f/IN^2}$$

$$C_f = 6.8 \times 10^{-3}\ IN^5/LB$$

B) FOR AIR, $\beta = nP$; WHERE $n = 1$ FOR ISOTHERMAL PROCESS

$$C_{f_{AIR}} = \frac{V_{AIR}}{\beta} = \frac{V_{AIR}}{P} = \frac{\pi D^2 H_a}{4P_o}$$

$$C_{f\,AIR} = \frac{\pi (12\,IN)^2 (H_a)(12\,IN/FT)}{4P_o}, \quad (H_a \text{ IN FT})$$

$$C_{f\,AIR} = 1360 \frac{H_a}{P_o} \frac{IN^5}{LB}, \quad (H_a \text{ IN FEET})$$

$$C_{f\,OIL} = \frac{V_{OIL}}{\beta} = \frac{\pi (12\,IN)^2 (H_o)(12\,IN/FT)}{4(2\times 10^5)}$$

$$C_{f\,OIL} = (6.8 \times 10^{-3})\, H_o \frac{IN^5}{LB}, \quad (H_o \text{ IN FEET})$$

$$C_f = C_{f\,AIR} + C_{f\,OIL}$$

$$C_f = 1360 \frac{H_a}{P_o} + (6.8 \times 10^{-3})\, H_o$$

C) FOR NO AIR, $H_o = 1$ & $H_a = 0$, HENCE

$$C_f = 6.8 \times 10^{-3} \frac{IN^5}{LB}$$

3-5 CONT.

C) CONT. IF $H_a = 0.1 H_o$ & $P_o = 200$ PSI,

$$C_f = \frac{1360}{200}(.1H_o) + (6.8 \times 10^{-3})\, H_o$$

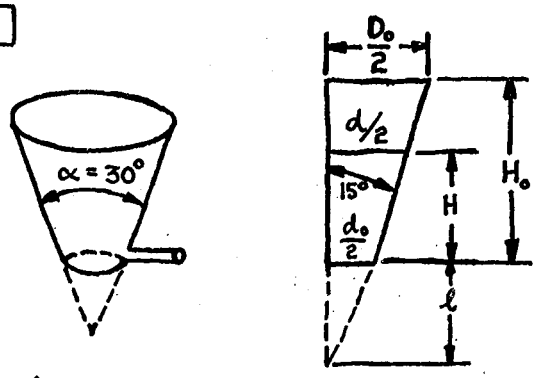
$$C_f = (0.68 + 0.0068)\, H_o \frac{IN^5}{LB}$$

BUT $H_o = 0.9$ FT, $\therefore$ $C_f = (0.69)(.9) \frac{IN^5}{LB}$

$$C_f = 0.62\ IN^5/LB$$

SMALL ENTRAINED GAS BUBBLES SIGNIFICANTLY INCREASE THE CAPACITANCE OF LIQUID-FILLED CONTAINERS.

3-6



$$\frac{d_o}{2} = \ell \tan 15^\circ = 0.286\,\ell$$

$$\frac{d}{2} = (H+\ell)\tan 15^\circ = 0.286\,(H+\ell)$$

$$D_o/2 = (H_o+\ell)\tan 15^\circ = 0.286\,(H_o+\ell)$$

$$D_o/2 = 0.286\,H_o + d_o/2$$

VOLUME OF COVE $= \frac{\pi}{3}\frac{D^2}{4} H = \frac{\pi D^2}{12} H$

VOL. OF WATER $= V = \frac{\pi}{12}\left[d^2(H+\ell) - d_o^2 \ell\right]$

$$V = \frac{\pi}{12}\left[(4)(0.286)^2(H+\ell)^3 - (0.286)^2 \ell^2 \ell\right]$$

$$V = \frac{\pi}{12}(0.286)^2\left[4(H+\ell)^3 - \ell^3\right]$$

$$V = \frac{\pi}{12}(0.286)^2\left[4H^3 + 8H^2\ell + 8H\ell^2 + 3\ell^3\right]$$

$P = \rho g H$; $H = P/\rho g$

$$V = \frac{\pi}{12}(.286)^2\left[\frac{4}{\rho^3 g^3}P^3 + \frac{8\ell}{\rho^2 g^2}P^2 + \frac{8\ell^2}{\rho g}P + 3\ell^3\right]$$

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$$C_{f\,OIL} = (6.8\times 10^{-3})\, H_o \frac{IN^5}{LB}, \quad (H_o \text{ IN FEET})$$

$$C_f = C_{f\,AIR} + C_{f\,OIL}$$

$$C_f = 1360 \frac{H_a}{P_o} + (6.8\times 10^{-3}) H_o$$

C) FOR NO AIR, $H_o = 1$ & $H_a = 0$, HENCE

$$C_f = 6.8 \times 10^{-3} \frac{IN^5}{LB}$$

3-5 CONT.

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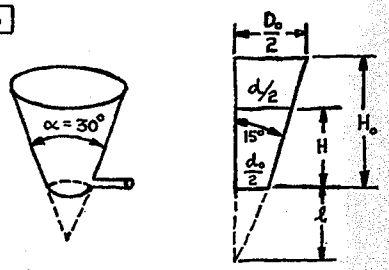
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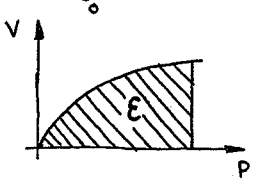
3-6 CONT.

THE PHYSICAL DIMENSIONS OF THE TANK ARE NOT GIVEN, BUT $\ell, \rho,$ & g ARE ALL CONSTANTS, THEREFORE THE EQUATION CAN BE REWITTEN WITH ARBITRARY CONSTANTS TO GIVE THE FORM:

$$V = A_1P^3 + A_2P^2 + A_3P + A_4$$

SINCE $V = f(P)$, THIS IS A PURE CAPACITANCE.

ENERGY, $\mathcal{E} = \int_0^P V\,dP$



FOR SMALL CHANGES,

$$V = V_o + \Delta V$$

$$P = P_o + \Delta P$$

NEGLECTING HIGHER ORDER TERMS OF ΔP

$$\Delta V + V_o = A_1(P_o^3 + 3P_o^3\Delta P) + A_2(P_o^2 + 2P_o\Delta P) + A_3(P_o + \Delta P) + A_4$$

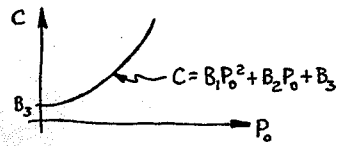
$$V_o = A_1P_o^3 + A_2P_o^2 + A_3P_o + A_4$$

SUBTRACTING,

$$\Delta V = 3A_1P_o^2\Delta P + 2A_2P_o\Delta P + A_3\Delta P$$

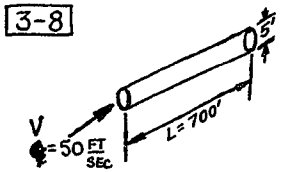
$$\Delta V = (B_1P_o^2 + B_2P_o + B_3)\Delta P = C\Delta P$$

WHERE: $C = B_1P_o^2 + B_2P_o + P_3$



THE CAPACITANCE IS A FUNCTION OF P_o, $\therefore$ C INCREASES WITH INCREASING HEIGHT, H.

3-8



$$I = \frac{\rho L}{gA} = \frac{(62.4\,\frac{LB}{FT^3})(700\,FT)}{(32.2\,\frac{FT}{SEC^2})\pi(\frac{5}{2}\,FT)^2} =$$

$\mathcal{E} = \frac{1}{2}IQ^2$ FOR IDEAL INERTANC

$$\mathcal{E} = \frac{1}{2}(69\,\tfrac{LB\text{-}SEC^2}{FT^5})[(50\,\tfrac{FT}{SEC})(\tfrac{25}{4}\pi\,FT$$

$$\mathcal{E} = 3.36\times 10^{8}\ FT\text{-}LB_f$$

$$\mathcal{E} = 3.36\times 10^{6}\ FT\text{-}LB_f\left(\frac{1\,HP}{550\,\frac{FT\text{-}LB_f}{SEC}} \times \frac{74}{1}\right.$$

$$\mathcal{E} = 4.45\times 10^{3}\ WATT\text{-}SEC$$

$$t = \frac{\mathcal{E}}{P} = \frac{4.45\times 10^{3}\ WATT\text{-}SEC}{10^2\ WATT} \times \frac{1}{36}$$

$$t = 12.4\ HR.$$

3-9

$$P_{21} = I\frac{dQ}{dt} = I\frac{\Delta Q}{\Delta t}$$

$$\Delta Q = \left(\frac{25\pi}{4} \times 50\right)\frac{FT^3}{SEC}; \quad \Delta t = 30\,s$$

$$P_{21} = \frac{69\ LB\text{-}SEC^2}{FT^5}\left(\frac{25(50)\pi}{4}\,\frac{FT^3}{SEC}\right)\left(\frac{1}{30\,SEC}\right)($$

$$P_{21} = 15.7\ PSI$$

3-11

A) $$R_f = \frac{128\,\mu\ell}{\pi d^4} = \frac{128(5\times 10^{-5}\,\frac{LB\text{-}SE}{IN^2})}{\pi(3.2\times 10^{-4}}$$

$$R_f = 23.3\times 10^3\ LB\text{-}SEC/IN^5$$

B) FOR ABOVE TO BE VALID,

REYNOLDS NO. < 2000

REYNOLDS NO. $= R_{EY} = \frac{4\rho Q}{\pi d\mu} < 2$

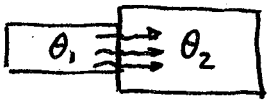
$$Q < \frac{2000\pi d\mu}{4\rho}$$

THERMAL SYSTEMS

- HEAT FLOW RELATED TO TEMPERATURE IN SAME WAY AS i TO v IN ELECTRICAL SYSTEM or F TO v (MECH.) or Q TO p

- LIMIT OUR DISCUSSION TO SIMPLE THERMAL SYSTEMS

- SINCE HEAT FLOW WILL BE RELATED TO TEMPERATURE DIFFERENCE θ_{21} THEN θ IS MEASURED IN ANY TEMPERATURE SCALE °C, °K, °F, °R

- 

$\theta_1 > \theta_2$ 2 BODIES IN CONTACT — HEAT FLOWS FROM HOTTER TO COLDER

- TEMPERATURE IS A MEASURE OF THE MEAN ENERGY OF THE MOLECULES OF BODY $\Rightarrow$ HEAT IS A FORM OF ENERGY

- $\mathcal{H}$ IS DEFINED AS A QUANTITY OF HEAT (Btu, kg-cal)
  1 Btu = 0.252 kg-cal
  acts like p, q, h, V (integ thru var)

- q_H HEAT FLOW RATE is $= \frac{d\mathcal{H}}{dt}$ — q_H is A THROUGH VAR.
  θ is AN ACROSS VAR.

- THERMAL SYSTEMS WILL HAVE SEVERAL IMPORTANT DIFFERENCES

- FROM FIRST LAW OF THERMODYNAMICS HEAT IS RELATED TO WORK $d\mathcal{H} = dW$
  THUS $\frac{d\mathcal{H}}{dt} = \frac{dW}{dt} = \mathcal{P}$ (power) $\Rightarrow$ $q_H = \frac{d\mathcal{H}}{dt}$ is THE POWER
  $p \cdot Q$
  $v \cdot F_i$
  THIS IS UNLIKE PREVIOUS SYSTEMS ($\mathcal{P}$ = across · through)

- $\mathcal{P} = q_H$ (kg-cal/sec or BTU/sec)
  $1 \frac{Btu}{sec} = 778.26 \frac{ft\text{-}lb}{sec}$
  $1 \frac{kg\text{-}cal}{sec} = 3.96 \frac{Btu}{sec} = 4200 \frac{N}{sec}$

$$R_t = \frac{l}{\sigma_c A}$$

$$dR_t = \frac{dx}{\sigma_c \cdot A} = \frac{dR}{\sigma_c \cdot 2\pi R W} = \frac{\ln R}{\sigma_c \cdot 2\pi W}\Bigg|_{R_i}^{R_o} = \frac{1}{\sigma_c \cdot 2\pi W} \ln\left(\frac{R_o}{R_i}\right)$$

- FIRST LAW : HEAT TRANSFERRED THROUGH BOUNDARIES OF SYSTEM INTO SYSTEM = ALL ENERGY TRANSFERRED OUT OF SYSTEM (IN ALL FORMS) + ENERGY STORED IN SYSTEM

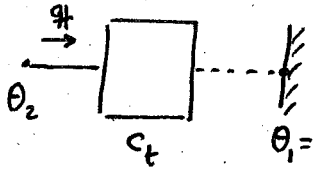
- IN THERMAL SYSTEMS THERE IS CAPACITANCE & RESISTANCE BUT NO INDUCTANCE AT NORMAL OPERATING TEMPERATURES

- THERMAL CAPACITANCE (q_H RELATED TO $\frac{d\theta_{21}}{dt}$) like $F = m\frac{dv}{dt}$, $i = C\frac{dv}{dt}$, $Q = C_f\frac{dp}{dt}$

$$q_H = C_p m \frac{d\theta_{21}}{dt}$$ C_p is specific heat m is mass

AN IDEAL THERMAL CAPACITANCE $\mathcal{H} = C_t\,\theta_{21} \Rightarrow q_H = C_t\frac{d\theta_{21}}{dt}$

$C_p m = C_t$



SYMBOLIC REPRESENTATION

may use θ_1 as a ref. temp (absolute zero)

E_t (THERMAL ENERGY STORED IF NO WORK EFFECTS) $= \mathcal{H}$

- EXAMPLES - WELL STIRRED TANK OF FLUID ; MASS OF METAL
- IN SUCH AN IDEAL ELEMENT ASSUME NO THERMAL RESISTANCE
  $\Rightarrow$ UNIFORM TEMPERATURE THROUGHOUT MASS

- THERMAL RESISTANCE (q_H RELATED TO θ_{21}) LIKE $i = \frac{1}{R}v_{21}$, $F = bv_{21}$, $Q = \frac{1}{R_f}P_{21}$

RELATION IS (FOR A CONTINUOUS & A LUMPED PARAMETER)

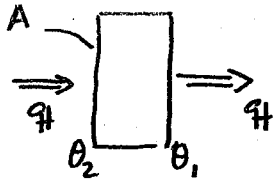
FOURIER LAW $$q_H = -kA\frac{\partial T}{\partial x} = \frac{\sigma_c A}{l}(\theta_2 - \theta_1) = \frac{1}{R_t}\theta_{21}$$

- A IS THE CROSS-SECTIONAL AREA



- ALL MATERIALS EXHIBIT THIS EVEN IN SPACE

$$R_t = \frac{l}{\sigma_c A} \quad \left[\frac{°C\text{-sec}}{\text{kg-cal}}\right] \text{ or } \left[\frac{°F\text{-sec}}{\text{Btu}}\right]$$

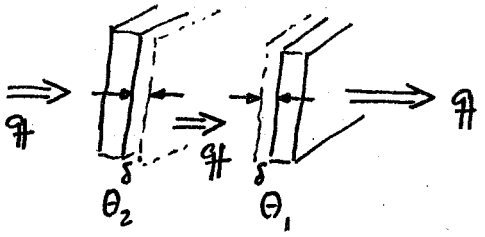


- IF HEAT FLOWS THROUGH <u>MOTION LESS</u> LIQUID : COMPUTE RESISTANCE IN SAME MANNER
- IF HEAT FLOWS THROUGH FLUID IN MOTION (CONVECTION, FORCED FLOW)

$$q_t = C_h A\,\theta_{21} = \frac{\theta_{21}}{R_t} \qquad R_t = \frac{1}{C_h A}$$

C_h $\frac{\text{kg-cal}}{^\circ C \cdot \text{sec} \cdot m^2}$ or $\frac{\text{Btu}}{^\circ F \cdot \text{sec} \cdot ft^2}$

- IN FLOWS THAT ARE TURBULENT HEAT TRANSFER OCCURS IN A VERY THIN LAYER



δ produces the thermal resistance

- DETERMINATION OF C_h is normally obtained experimentally
- FOR THERMAL RADIATION q_t IS <u>NOT</u> REPRESENTED BY A PURE THERMAL RESISTANCE ($q_t = f(\theta_{21})$ for pure thermal resist)

$$q_t \neq f(\theta_{21}) = C_r(\theta_2^4 - \theta_1^4) = C_r(\theta_2 - \theta_1)\left[\theta_2^3 + \theta_2^2\theta_1 + \theta_2\theta_1^2 + \theta_1^3\right]$$

if θ_2 & θ_1 are close; we can linearize this so that

$$q_t = C_r(\theta_2 - \theta_1)\left[4\theta_1^3\right] = \frac{1}{R_{eq}}\theta_{21} \quad \therefore \quad 4C_r\theta_1^3 = \frac{1}{R_{eq}}$$

- NOTE FOR RADIATION PROBLEMS θ must be in $^\circ R$, $^\circ K$

$$^\circ R = 459.6 + ^\circ F \qquad ^\circ K = 273.2 + ^\circ C$$

- also $C_r = \sigma A$ A is the surface area radiating the heat

σ is the Stefan-Boltzmann constant 1.37×10^{-5} kg-cal/m^2·sec·$^\circ K^4$

or 3.34×10^{-13} Btu/ft^2·sec·$^\circ R^4$

$R_i\left(1+\frac{\Delta R}{R_i}\right)$

$\ln(R_i+\Delta R) - \ln R_i$

$\ln R_i + \ln\left(1+\frac{\Delta R}{R_i}\right) - \ln R_i = \ln\left(1+\frac{\Delta R}{R_i}\right) = \frac{\Delta R}{R_i}$ if $\frac{R_o - R_i}{} \ll R_i$

$\ln(1+x) = \ln 1 + 1\cdot x - \frac{x^2}{2} + \cdots \cong x = \frac{\Delta R}{R_i}$

$\left.\frac{1}{1+x}\right|_{x=0} \qquad \left.-\frac{1}{(1+x)^2}\right|_{x=0}$

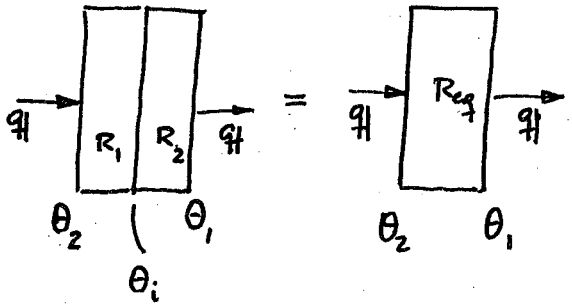
$R_i + \frac{\Delta R}{2} \qquad \frac{R_i}{\Delta R}$

ΔR

- A PURE TRANSFORMER CHANGES magnitudes of <u>through</u> and <u>across</u> variables while keeping power unchanged. In thermal system no such thing is possible since $\mathcal{P} = q$ (through variable)
  $\Rightarrow$ no transformers / transducers for <u>simple</u> systems.

<u>Problem [redacted]</u>

What is the equivalent resistance? assume <u>no</u> heat storage

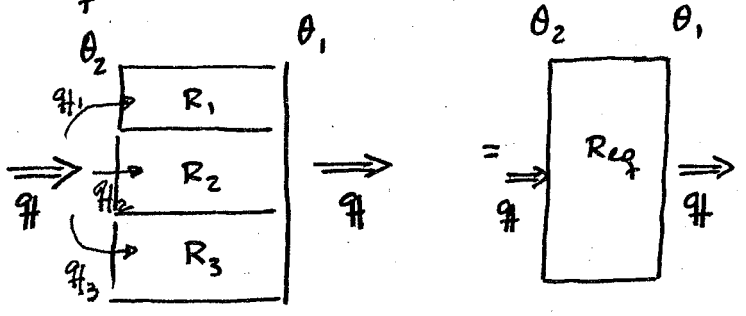


$$q = \frac{\theta_{2i}}{R_1} \qquad q = \frac{\theta_{i1}}{R_2}$$

$$q = \frac{1}{R_{eq}}\theta_{21}$$

$$(R_1 + R_2)q = \theta_{2i} + \theta_{i1} = \theta_{21} = R_{eq}\cdot q$$

$R_{eq} = R_1 + R_2$ (series)

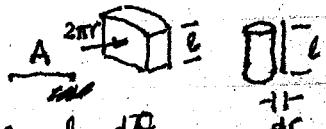


$$q_1 = \frac{1}{R_1}\theta_{21}$$
$$q_2 = \frac{1}{R_2}\theta_{21}$$
$$q_3 = \frac{1}{R_3}\theta_{21}$$
$$\Big\} \equiv q = \frac{1}{R_{eq}}\theta_{21}$$

$$q = q_1 + q_2 + q_3 = \theta_{21}\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right] = \frac{1}{R_{eq}}\theta_{21}$$

$$1/R_{eq} = \sum_{i=1}^{3} \frac{1}{R_i} \quad \text{(parallel)}$$

Do. 3-19, 3-16



What about a circular conductor $q = k \cdot 2\pi r l \frac{d\theta}{dr}$

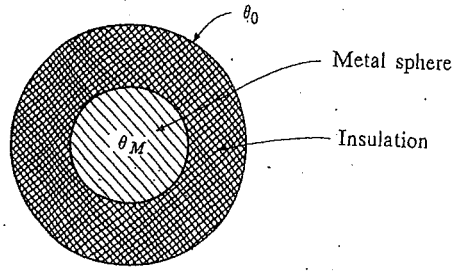
$$q\frac{dr}{r} = k2\pi l\, d\theta$$

$$q \ln R\Big|_{R_i}^{R_o} = k2\pi l\,\theta_{21} \qquad q \ln\frac{R_o}{R_i} = k2\pi\theta_{21} l$$



$$q = \theta_{21}\cdot\frac{2\pi R l}{\Delta R}\cdot k$$

surface area



Overall thermal resistance R_t of insulation = 1/3 °F·hr/Btu
Thermal capacitance C_t = 2 Btu/°F of metal

FIGURE 3–32

3–19. An electric hot-water heater for home use is cylindrical in shape and is insulated by a 2 in.-thick layer of glass wool having a conductivity of 0.25 Btu/hr·ft·°F. The heat-transfer coefficient C_h between the outside of the tank and the air is 1 Btu/hr·ft^2·°F. The specific heat of water is 1 Btu/lb·mass·°F. The tank is 5 ft high by 2 ft in diameter.

a) Assuming that the water temperature is uniformly equal to 170°F and the average room temperature is 60°F, estimate the electrical energy in kw-hr/day required to maintain the water at constant temperature.

b) Assuming that there are 100 gal/day of hot water used and replaced by cold water at 50°F, estimate the electrical energy required per day to heat the water.

c) By adding the results of (a) and (b), estimate the total energy per day required. Is this estimate high or low? Why?

3–20. The thermal system in Fig. 3–32 consists of insulation wrapped around a large mass of metal. The system is placed in an oven and heated so that the temperature of the metal (which had been constant at room temperature) increases linearly by 100°F in 1 hr and then remains constant. Determine θ_o, the temperature of the outside of the insulation vs. time, and sketch.

3–21. A block of copper whose mass is 2 slugs is surrounded with insulation. The property of this insulation is that the temperature difference across it is proportional to the heat flow through it, so that a temperature difference of 10°F produces a heat flow of 10 Btu/hr. The system is initially at room temperature and then the outside of the insulation is suddenly raised 100 F° above room temperature.

Discuss this situation in detail. Make sketches of the change of temperature of the copper with time as well as the amount of heat flow into the system. Explain your sketches and *make some estimate of the time it takes* for the copper to heat up.

Springs in parallel $k_{eq} = \Sigma k_i$ (linear mechanical / rotational systems)

Springs in series $\frac{1}{k_{eq}} = \Sigma \frac{1}{k_i}$ (" / ")

Fluid reservoir $C_f = \frac{A}{\rho g}$

Pressurized tank $C_f = \frac{V}{\beta}$ (fluid) $C_f = \frac{V}{nP}$ (gases) $n = \frac{c_p}{c_v}$

Fluid Resistance - flow through capillary tube $R_f = \frac{128 \mu l}{\pi d^4}$

- flow through pipe / through orifice $Q = f(P_{21})$ $R_f = \frac{1}{df/dP_{21}}$

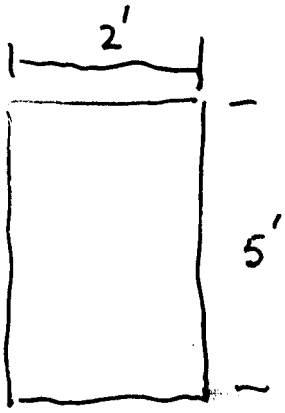
Thermal Resistance $R_t = \frac{l}{\sigma_c A}$ conduction ; $R_t = \frac{1}{c_h A}$ convection;

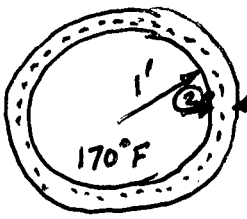
$R_t = \frac{1}{4 C_r \theta^3}$ radiation

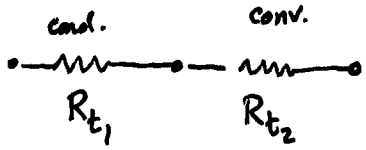
Thermal Capacitance $C_t = c_p \cdot m$ m = mass

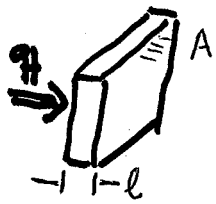
Resistances in series (electrical / fluid / thermal) $R_{eq} = \Sigma R_i$

Resistances in parallel (" / " / ") $\frac{1}{R_{eq}} = \Sigma \frac{1}{R_i}$









$$q_f = \frac{\theta_{21}}{R_{t_{eq}}}$$

$$R_{t_1} = \frac{\ell}{\sigma_c A_{av}} \qquad R_{t_2} = \frac{1}{C_h \cdot A_o}$$

$$A_{av} = 2\pi r_{av} \cdot h = 2\pi\left(1' + \frac{t}{2}\right)(5') = 34.1\ ft^2$$

$$R_{t_1} = \frac{t}{\sigma_c \cdot A_{av}} = \frac{.167}{.25(34.1)} = 1.95 \times 10^{-2}\ \frac{hr\ ^\circ F}{BTU}$$

$$R_{t_2} = \frac{1}{C_h \cdot A_o} \qquad A_o = 2\pi(1+t)\cdot h = 2\pi(1.167)\cdot 5 = \frac{70\pi}{6} = 36.65\ \frac{hr\ ^\circ F}{BTU}$$

$$R_t = 1.95\times10^{-2} + 0.0273 = 0.468\ \frac{hr\ ^\circ F}{BTU}$$

$$q_f = \frac{\theta_{21}}{R_t} = \frac{170-60}{0.0468} = \frac{110}{0.0468} \cong 2351.55\ \frac{BTU}{hr}$$

$$P = q_f = 2351.55\ \frac{BTU}{hr} \cdot \left(0.292\ \frac{watt\text{-}hr}{BTU}\right) \cong 686.65\ watts$$

$$E_{per\,day} = 686.65 \times 24\,\frac{hr}{day}$$

EML 3222

SYSTEM DYNAMICS

LECTURE FOR 5/23/07

$1\ gal = 231 in^3$

$100\ gal = 23100\ in^3 \times \left(\frac{1 ft}{12 in}\right)^3 = 13.38\ ft^3$

$\rho = 1.94 \frac{slug}{ft^3} \times 32.2 = 62.4 \frac{lb_m}{ft^3}$

$m = \rho V = (62.4)(13.38) = 836 \frac{lb_m}{day}$

$q_H = m C_p \Delta T = (836)(1)(170-50)$

$\approx 10^5 \frac{BTU}{day} \times \frac{1\ day}{24\ hrs} = 4.16 \times 10^3 \frac{BTU}{hr}$

$P = q_H \times \left(0.292 \frac{watt-hr}{BTU}\right)$

$= 1.21\ kW$

$E_{per\ day} = 24(1.21) = 28.8 \frac{kW-hr}{day}$

3-14 CONT.

CASE I : FLUID RESISTANCE, R, CONNECTED

$$\frac{P_{41}}{P_{21}} = \frac{A_a}{A_b} = N \quad , \quad \frac{Q_b}{Q_a} = -\frac{1}{N}$$

$$Q_b = \frac{1}{R} P_{41} = \frac{N}{R} P_{21}$$

$$Q_b = -\frac{1}{N} Q_a = \frac{N}{R} P_{21}$$

$$Q_a = -\frac{N^2}{R} P_{21} = \frac{1}{R_f} P_{21}$$

$\therefore$ SIDE "A" SEES $R_f = R/N^2$

CASE II : FLUID CAPACITANCE, C,

$$Q_b = C \frac{dP_{41}}{dt}$$

$$Q_b = -1/N\, Q_a \quad , \quad P_{41} = NP_{21}$$

$$-\frac{1}{N} Q_a = C \frac{d(NP_{21})}{dt}$$

$$Q_a = -N^2 C \frac{dP_{21}}{dt} = C_f \frac{dP_{21}}{dt}$$

$\therefore$ SIDE "A" SEES $C_f = N^2 C$

CASE III : FLUID INERTANCE, I,

$$P_{41} = I \frac{dQ_b}{dt}$$

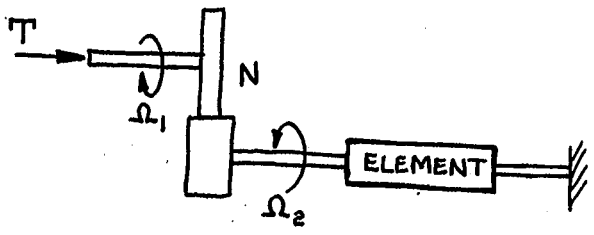
$$P_{41} = NP_{21} \quad , \quad Q_b = -1/N\, Q_a$$

$$NP_{21} = I \frac{d}{dt}(-1/N\, Q_a)$$

$$P_{21} = -\frac{I}{N^2} \frac{dQ_a}{dt} = I_f \frac{dQ_a}{dt}$$

$\therefore$ SIDE "A" SEES $I_f = I/N^2$

FOR MECHANICAL ROTATIONAL SYSTEM REFER TO PROBLEM 2-33



3-14 CONT.

CASE I : DAMPER RESISTANCE, B,

$$B_{EQ} = N^2 B$$

CASE II : INERIA, J,

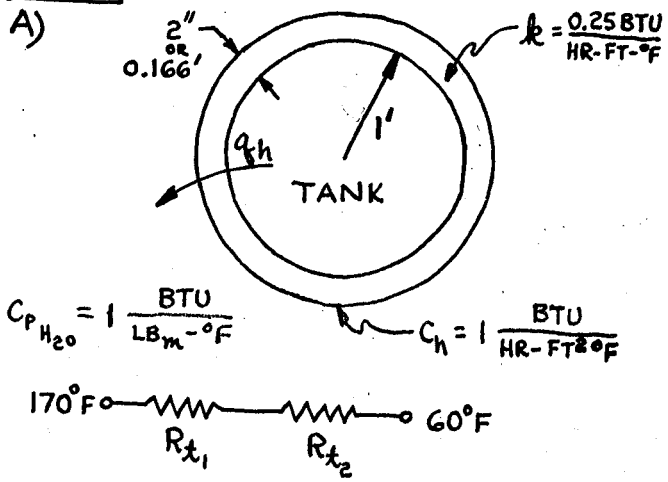
$$J_{EQ} = N^2 J$$

CASE III : SPRING, K,

$$K_{EQ} = N^2 K$$

3-19

A)



$$\frac{L}{kA} = R_{t_1} = \frac{.166\ FT}{kA} \quad , \quad R_{t_2} = \frac{1}{C_h}$$

FOR R_{t_1}, USE EFFECTIVE RADIUS OF INSULATION TO CALCULATE A

$$A = 2\pi r h = 2\pi (1\ FT + .083\ FT)(5\ FT)$$

$$A = 34.1\ FT^2$$

$$R_{t_1} = \frac{.166\ FT}{kA} = \frac{.166}{.25(34.1)} = 1.95 \times 10^{-2} \frac{HR-°F}{BTU}$$

$$R_{t_2} = \frac{1}{C_h} = 1 \frac{HR-°F}{BTU}$$

$$R_T = R_{t_1} + R_{t_2} = 1.02\ HR-°F/BTU$$

$$q_h = \frac{T_{TANK} - T_{AIR}}{R_T} = 108\ BTU/HR$$

$$HP = 108 \frac{BTU}{HR} (.292 \frac{W-HR}{BTU}) = 0.0315\ KW$$

3-19 CONT.

A) CONT.

$\mathcal{E} = P \cdot \text{TIME}$

$\mathcal{E} = (0.0315\,KW)\left(24\,\frac{HR}{DAY}\right) = 0.757\,\frac{KW\text{-}HR}{DAY}$

B) $1\ \text{GAL.} = 231\ IN^3$

$100\ \text{GAL.} = 23100\ IN^3 = 13.38\ FT^3$

FOR WATER: $\rho = 1.94\,\frac{SLUG}{FT^3} = 62.4\,\frac{LB_m}{FT^3}$

$m = \rho V = (62.4\,\frac{LB_m}{FT^3})(13.38\,FT^3/DAY)$

$m = 836.0\ LB_m/DAY$

$q = m C_p \Delta T$

$q = (836\,\frac{LB_m}{DAY})(1\,\frac{BTU}{LB_m\text{-}^\circ F})(170^\circ F - 50^\circ F)$

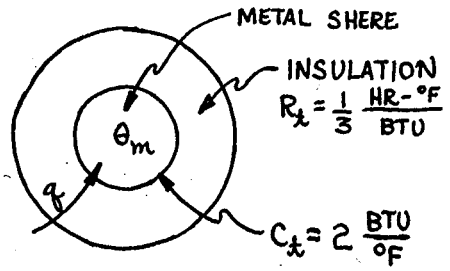
$q = 10^5\ BTU/DAY = 4.16 \times 10^3\ BTU/HR$

$P = (4.16\times10^3\,\frac{BTU}{HR})(0.292\,\frac{W\text{-}HR}{BTU})$

$P = 1.21\ KW$

$\mathcal{E} = 28.8\,\frac{KW\text{-}HR}{DAY}$

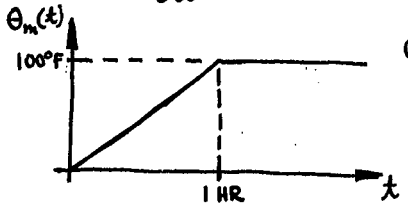
3-20



$q = \frac{\theta_o - \theta_m}{R_t}$ THROUGH INSULATION

$q = C_t \frac{d\theta_m}{dt}$, $C_t \frac{d\theta_m}{dt} = \frac{\theta_o - \theta_m}{R_t}$

$R_t C_t \frac{d\theta_m}{dt} + \theta_m = \theta_o$



$0 < t \le 1\ HR$, $\theta_m = 100t$

$t > 1\ HR$ $\theta_m = 100^\circ F$

3-20 CONT.

$R_t C_t = 2/3\ HR$

$\frac{2}{3}\frac{d\theta_m}{dt} + \theta_m = \theta_o$

FOR $0 < t < 1\ HR$

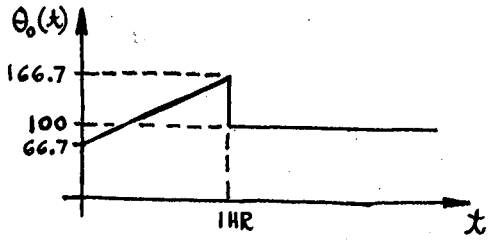
$\theta_o = \frac{2}{3}\frac{d(100t)}{dt} + 100t$

$\theta_o = 200/3 + 100t$

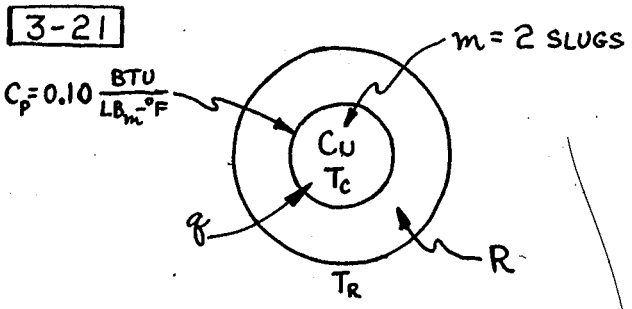
FOR $t > 1\ HR$

$\theta_o = \frac{2}{3}\frac{d(100)}{dt} + 100$

$\theta_o = 100$



3-21



T_R = ROOM TEMP. (ARBITRARY)

$q = R\Delta T$

$10\,\frac{BTU}{HR} = R\,10^\circ F$, $R = 1\,\frac{BTU}{HR\text{-}^\circ F}$

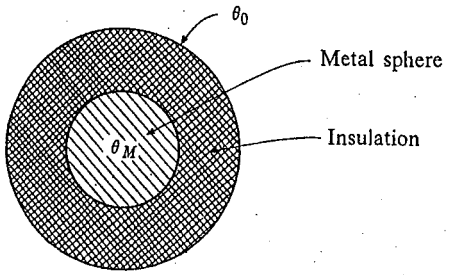
$q = R(T_1 - T_2) = 1\,[T_R - (T_R + 100^\circ F)]$

$q = 100\ BTU/HR$

$q_i = \frac{1}{R}(T_a - T_c)$

$q_c = C_t \frac{dT_c}{dt}$

$C_t = C_p m = (0.10)(2)(32.2) = 6.44\,\frac{BTU}{^\circ F}$



Overall thermal resistance R_t of insulation = 1/3 °F·hr/Btu
Thermal capacitance C_t = 2 Btu/°F of metal

FIGURE 3-32

3–19. An electric hot-water heater for home use is cylindrical in shape and is insulated by a 2 in.-thick layer of glass wool having a conductivity of 0.25 Btu/hr·ft·°F. The heat-transfer coefficient C_h between the outside of the tank and the air is 1 Btu/hr·ft²·°F. The specific heat of water is 1 Btu/lb·mass·°F. The tank is 5 ft high by 2 ft in diameter.

a) Assuming that the water temperature is uniformly equal to 170°F and the average room temperature is 60°F, estimate the electrical energy in kw-hr/day required to maintain the water at constant temperature.

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c) By adding the results of (a) and (b), estimate the total energy per day required. Is this estimate high or low? Why?

3–20. The thermal system in Fig. 3–32 consists of insulation wrapped around a large mass of metal. The system is placed in an oven and heated so that the temperature of the metal (which had been constant at room temperature) increases linearly by 100°F in 1 hr and then remains constant. Determine θ_o, the temperature of the outside of the insulation vs. time, and sketch.

3–21. A block of copper whose mass is 2 slugs is surrounded with insulation. The property of this insulation is that the temperature difference across it is proportional to the heat flow through it, so that a temperature difference of 10°F produces a heat flow of 10 Btu/hr. The system is initially at room temperature and then the outside of the insulation is suddenly raised 100 F° above room temperature.

Discuss this situation in detail. Make sketches of the change of temperature of the copper with time as well as the amount of heat flow into the system. Explain your sketches and *make some estimate of the time it takes* for the copper to heat up.

3–11. A fluid having density $\rho = 0.04$ lb/in^3 and viscosity $\mu = 5 \times 10^{-5}$ lb·sec/in^2 flows through a tube of 0.032-in. internal diameter and 1-ft length.

a) Compute the fluid resistance, assuming Hagen-Poiseuille flow.

b) Estimate the range of steady flow rates Q where (a) is valid.

c) Compute the fluid inertance.

3–12. Assuming that the flow varies sinusoidally in the tube of Problem 3–11, $Q = Q_0 \sin \omega t$, estimate the range of frequency ω over which the tube can be represented by an ideal resistance. Note that the pressure drop due to inertance must be small.

3–13. The pressure-flow relation for an orifice is given by Eqs. (3–35) and (3–36). If $P_{21} = P_0 + \Delta P_{21}$, and if $Q = Q_0 + \Delta Q$, where ΔP_{21} and ΔQ are small, there will be a nearly linear relation between ΔP_{21} and ΔQ, that is,

$$\Delta P_{21} = R_f \Delta Q.$$

This is called the *linearized* or *local* resistance near $P_{21} = P_0$ and $Q = Q_0$, and is approximately the local slope of the P_{21}-Q curve (Fig. 2–17). Determine R_f as a function of P_0 and sketch qualitatively. Comment on the resistance of an orifice at very small flow rates.

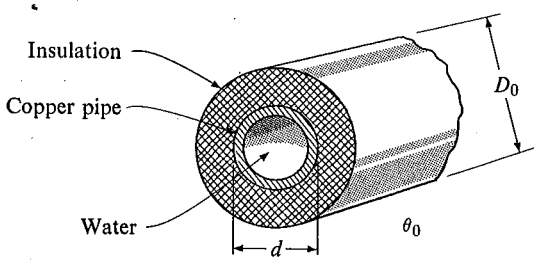
3–14. The fluid transformer (side *b*) of Fig. 3–18(a) is connected in turn to an ideal resistance, an ideal inertance, and an ideal capacitance. Find the equivalent ideal elements as observed from side *a* of the transformer. Compare these results with the results obtained when a gear train is connected successively to an ideal damper, a spring, and an inertance.

3–15. The gear pump described in Problem 3–1 may be considered to be a transducer between (fluid flow and pressure) and (rotary torque and speed).

a) Determine an appropriate relation between these four variables. State your assumptions clearly.

b) Is the device a transforming or gyrating transducer? Define and compute an appropriate transduction ratio.

c) A fluid capacitance is connected to the output of the gear pump, $Q = C_f(dPs/dt)$. Determine the relationship between torque T and speed Ω at the drive shaft of the pump. What ideal element represents this relationship?



$D_0 = 2$ in.
$d = 1$ in.
Pipe wall thickness = 0.125 in.

FIGURE 3–31

3–16. Figure 3–31 shows the cross section of an insulated water pipe. This pipe passes near the outside wall of a poorly heated building, and the object of the insulation is to prevent the water from freezing. The specific heats of the insulation, copper pipe, and water are 0.2, 0.093, and 1 Btu/lb·mass, and their thermal conductivities are 0.26, 242, and 0.34 Btu/hr·ft·°F, respectively.

a) Estimate the thermal resistances per unit length of the insulation and the copper.

b) Estimate the thermal capacitances per unit length of the water, the copper, and the insulation, assuming each to be at uniform temperature.

c) Compare the resistances and capacitances of the various components, and describe a simplified model of this configuration.

3–17. A thick hollow cylinder of insulation has outer diameter D_o and inner diameter D_i. If the material has conductivity σ_c, compute the thermal resistance

a) for heat flow outward, and

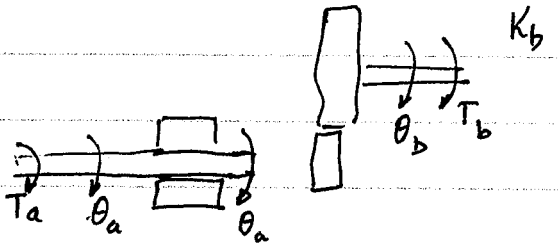
b) for heat flow inward.

3–18. The inner surface of a thick hollow cylinder is at temperature θ_i and has diameter D_i. The outside, of diameter D_o, is in contact with air, and there is an over-all coefficient of heat transfer C_h between this surface and the surroundings, which are at temperature θ_o.

a) Determine an expression for the total thermal resistance between θ_i and θ_o.

b) Does the resistance R_t continue to increase as more insulation is added (D_o increases)? If not, find the optimum D_o which maximizes R_t.

2-18



$$\frac{T_b}{T_a} = \frac{N_B}{N_A} = -\frac{1}{n}$$

$$\frac{\dot\theta_b}{\dot\theta_a} = \frac{\Omega_4}{\Omega_2} = -\frac{N_a}{N_B} = +n$$

$$T_b = -\frac{1}{n} T_a$$

$$\Omega_4 = \dot\theta_b = n\Omega_2$$

$$K_a = \frac{T_a}{\theta_a} = \frac{-nT_B}{\theta_b/n} = n^2 T_B$$

let $\frac{N_2}{N_1} = \frac{1}{n} = \frac{T_B}{T_a}$

$$-\frac{\dot\theta_b}{\dot\theta_a} = +\frac{N_1}{N_2} = n$$

$$T_B = \frac{T_A}{n}$$

$$\Omega_4 = \dot\theta_b = n\dot\theta_a = -n\Omega_2$$

$$\therefore K_a = \frac{T_a}{\theta_a} = \frac{nT_B}{-\theta_b/n} = n^2\frac{T_B}{-\Omega_b}$$

let $\frac{T_B}{-\theta_b} = K_B$

$$\therefore K_a = n^2 K_B = \left(\frac{N_1}{N_2}\right)^2 K_B$$

~~$E = \frac{1}{2} K_b \theta_b^2 = \frac{1}{2} K_B \theta_B \quad E_p = \frac{1}{2} T_B \theta_B = \frac{1}{2}\left(\frac{K_a}{n^2}\frac{\theta_a}{n}\right) = T_a$~~

$$E_p = \frac{1}{2} T_a \theta_a = \frac{1}{2} T_a \cdot \frac{T_a}{K_a} = \frac{1}{2}\frac{T_a^2}{K_B}\cdot\left(\frac{N_2}{N_1}\right)^2 = \frac{1}{2}\left(\frac{100}{100}\right)^2 (4)^2 = 800 \text{ in-l}$$

since no power but energy is transfered

if $\frac{N_2}{N_1}$ is doubled; energy is quadrupled.



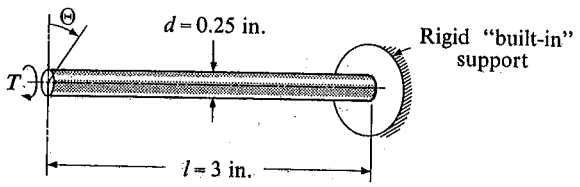


FIGURE 2–52

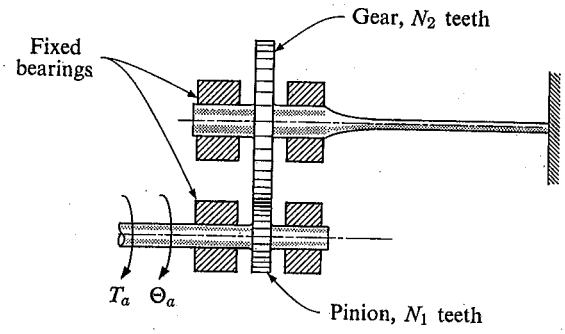


FIGURE 2–53

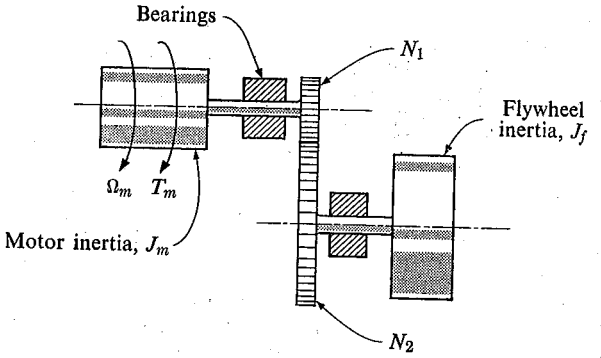


FIGURE 2–54

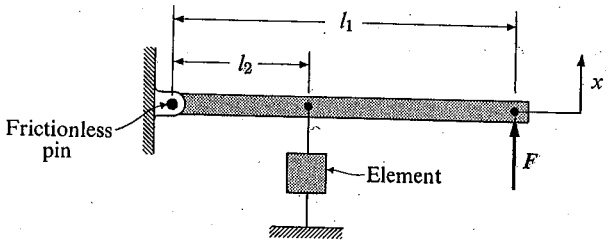


FIGURE 2–55

a) Compute the numerical value of the torsional stiffness K_Θ.

b) Compute the inertia J of the shaft.

2–17. The end of the bar in Problem 2–16 is oscillated sinusoidally:

$$\Theta = \Theta_0 \sin \omega t.$$

Estimate the range of frequency ω over which the bar will act approximately like an ideal spring.

2–18. An ideal torsional spring is attached to the shaft of a gear which is driven by a smaller pinion gear (Fig. 2–53). A torque T_a is applied to the shaft of the pinion, causing both gears to rotate, thus twisting the spring.

a) Find the torsional spring stiffness that would be observed at the pinion shaft. This is called the stiffness "reflected" through the gear train.

b) Assuming that $K = 100$ in·lb/rad, $N_1 = 5$, $N_2 = 20$, compute the energy stored in the spring when $T_a = 100$ in·lb.

c) If the gear ratio (transformer ratio) is doubled while everything else remains the same, how does the stored energy change?

2–19. An electric motor drives a flywheel through a gear train (Fig. 2–54). The motor and flywheel are supported in bearings. If the stiffnesses of the various parts of the system are very high and friction is negligible, what is the relationship between the motor speed Ω_m and the motor torque T_m? What single ideal element will represent this system?

2–20. A lever (a translational transformer) is used to apply a force to an element (Fig. 2–55). Assume that the lever is massless and perfectly rigid and that only small angular deflections are allowed.

a) Determine an equivalent element which represents this system to the force F for each of the three cases (Fig. 2–56) where the element is attached to the lever.

b) Consider an interchange of the locations of applied force and element connection. Describe the difference.

c) Compare these situations to that of a rotational system where a torque is applied to each of the three ideal rotational elements through a set of gears. Define a lever ratio n, by analogy with the gear ratio.

2–21. A torsional element is constructed by combining an ideal linear spring with a pure rotational-to-translational transducer (Fig. 2–57). The unstressed length of

2-16

$$K_\theta = \frac{GI_p}{\ell} = \frac{(13\times10^6)\frac{\pi d^4}{32}}{3} = 1660 \ \frac{\text{IN-LB}}{\text{RAD}}$$

$$J = \frac{1}{2}mr^2 \qquad m = \rho V = \rho\frac{\pi d^2}{4}\ell$$
$$= \frac{1}{2}(.108\times10^{-3})(.125)^2$$

$$J = 8.45\times10^{-7} \ \text{IN-LB-SEC}^2$$

2-17 THE BAR WILL BEHAVE AS AN IDEAL SPRING WHEN $\omega << \omega_m$, WHERE

$$\omega_m = \sqrt{K_\theta/J}$$

$$\omega_m = \sqrt{\frac{1660}{8.45\times10^{-7}}} = 4.43\times10^4 \ \frac{\text{RAD}}{\text{SEC}}$$

2-18 a) LET $\eta = \frac{N_2}{N_1}$, THUS $\frac{\theta}{\theta_a} = \frac{1}{\eta}$ & $\frac{T}{T_a} = \eta$
IF K_a = "REFLECTED" STIFFNESS

$$K_a = \frac{T_a}{\theta_a} = \frac{T/\eta}{\theta\eta} = \frac{T}{\theta}\left(\frac{1}{\eta}\right)^2 = K_\theta\left(\frac{N_1}{N_2}\right)^2$$

b) $E_p = \frac{1}{2}T_a\theta_a = \frac{1}{2}T_a\frac{T_a}{K_a} = \frac{1}{2}\frac{T_a^2}{K_a} = \frac{1}{2}\frac{T_a^2}{K_\theta}\eta^2$

$$= \frac{1}{2}\left(\frac{(100)^2}{100}\right)\left(\frac{20}{5}\right)^2 = 800 \ \text{IN-LB}$$

c) IF η IS DOUBLED WHILE T_a AND K_θ REMAIN CONSTANT, THE E_p IS QUADRUPLED

2-19 LET $\eta = N_2/N_1$

$$\Sigma T = T_m = J_{EQ}\dot\Omega_M$$
$$= J_M\dot\Omega_M + J_{F_{EQ}}\frac{\dot\Omega_M}{\eta}$$
$$= J_M\dot\Omega_M + \frac{J_F}{\eta^3}\frac{\dot\Omega_M}{\eta}$$

$$T_m = \left[J_M + \frac{J_F}{\eta^3}\right]\dot\Omega_M$$

η^3 ?

THE COMPLETE ARRANGEMENT IS EQUIVALENT TO A SIMPLE ROTATIONAL ELEMENT HAVING AN INERTIA $\left[J_M + \frac{1}{\eta^3}J_F\right]$

2-20 a) i) FOR A MASS:

$$\Sigma M_o = 0 \ \text{OR} \ F\ell_1 = m\ell_2\frac{d^2}{dt^2}\left(\frac{\ell_2}{\ell_1}x\right)$$

$$F = m\left(\frac{\ell_2}{\ell_1}\right)^2\ddot x \qquad m_{EQ} = m\left(\frac{\ell_2}{\ell_1}\right)^2$$

a) ii) FOR A SPRING:

$$F\ell_1 = K\left(\frac{\ell_2}{\ell_1}\right)x\,\ell_2$$

$$F = K\left(\frac{\ell_2}{\ell_1}\right)^2 x \qquad K_{EQ} = K\left(\frac{\ell_2}{\ell_1}\right)^2$$

a) iii) FOR A DAMPER:

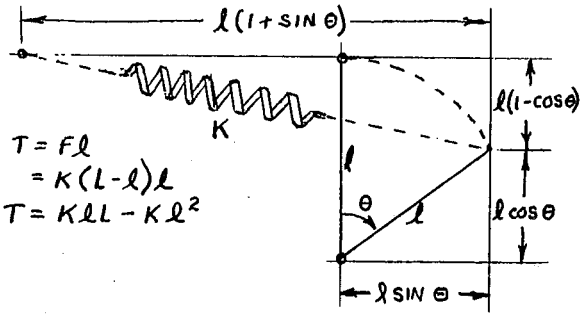
$$F\ell_1 = b\ell_2\frac{d}{dt}\left(\frac{\ell_2}{\ell_1}x\right)$$

$$F = b\left(\frac{\ell_2}{\ell_1}\right)^2\frac{dx}{dt} \qquad b_{EQ} = b\left(\frac{\ell_2}{\ell_1}\right)^2$$

b) THE RATIOS $\left(\frac{\ell_2}{\ell_1}\right)^2$ ARE INVERTED

c) THE EQUIVALENT RATIO IS η^2 INSTEAD OF $\left(\frac{\ell_2}{\ell_1}\right)^2$, WHERE $\eta = N_1/N_2$

2-21

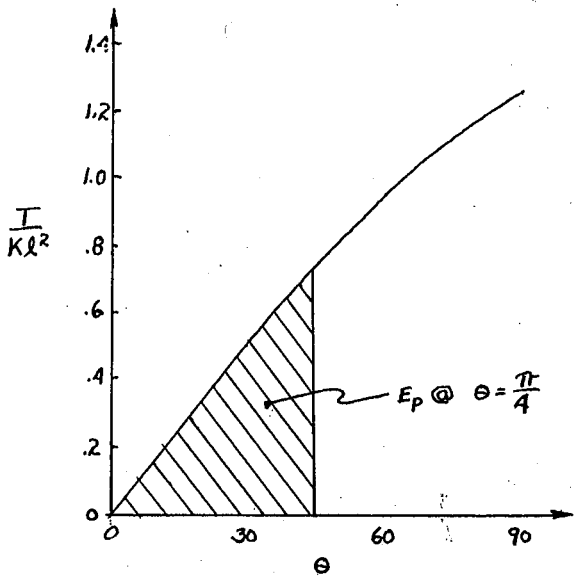


$$T = F\ell$$
$$= K(L-\ell)\ell$$
$$T = KL\ell - K\ell^2$$

$$L^2 = [\ell(1+\sin\theta)]^2 + [\ell(1-\cos\theta)]^2$$

$$L = \ell\sqrt{3+2(\sin\theta-\cos\theta)}$$

$$T = K\ell^2\left[\sqrt{3+2(\sin\theta-\cos\theta)}-1\right]$$



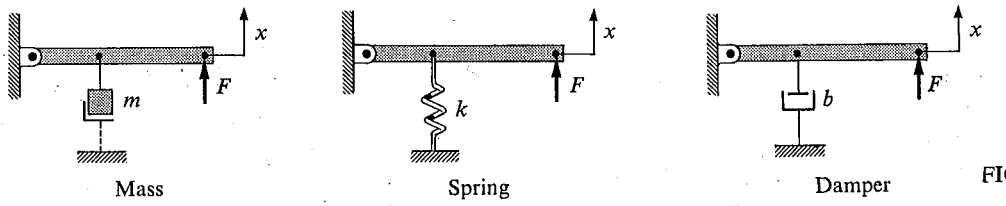


FIGURE 2–56

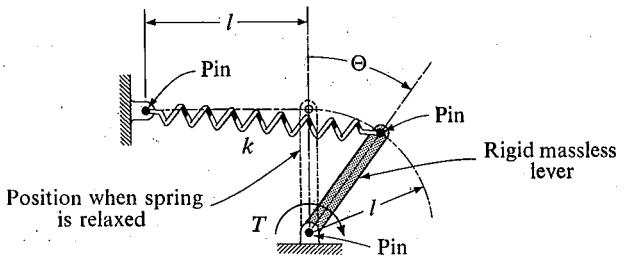


FIGURE 2–57

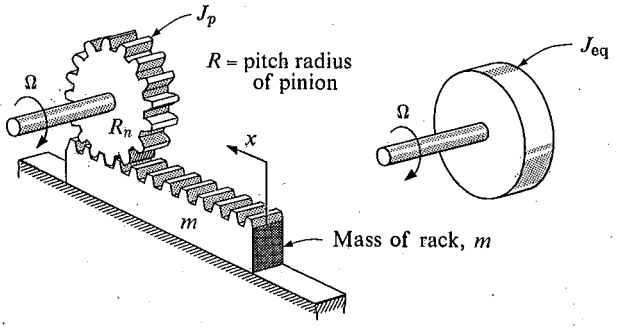


FIGURE 2–58

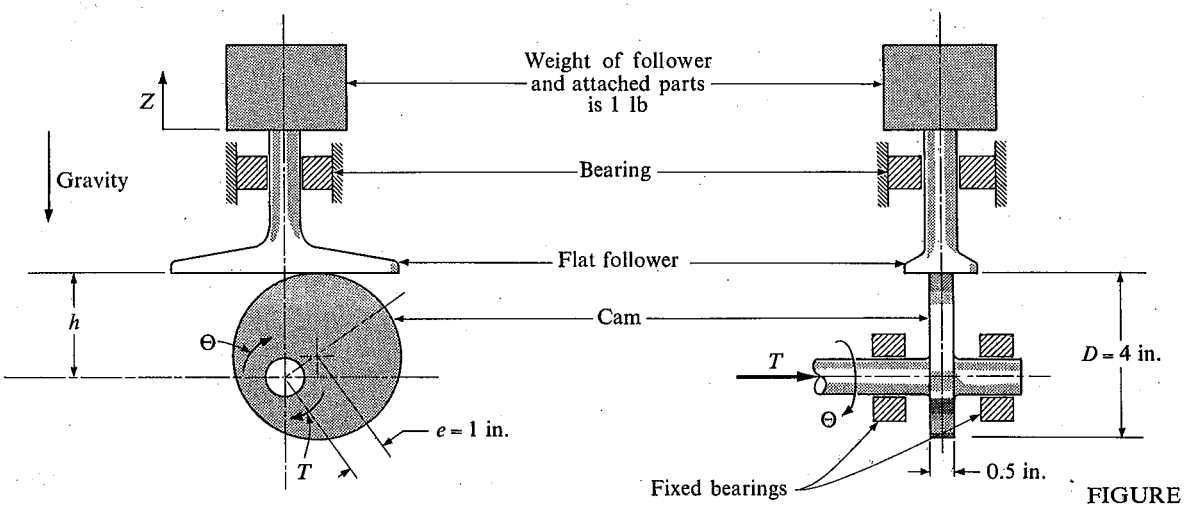


FIGURE 2–59

the spring is l, and the lever radius is also l. The lever is vertical and the spring axis is horizontal when the torque T is zero.

a) Determine and sketch an expression for T vs. Θ for $0 < \Theta < \pi/2$. What element represents this function?

b) On your sketch show graphically the area representing the energy stored when $\Theta = \pi/4$.

2–22. For each of the three ideal mechanical elements (mass, damper, and spring), determine how the velocity varies if the force is $F_0 \sin \omega t$.

2–23. What conditions must be satisfied for the rack and pinion system shown in Fig. 2–58 to be suitably modeled by an equivalent inertia J_{eq} as shown at the right? Express the equivalent inertia J_{eq} in terms of J_p and m (and anything else you think may be important).

2–24. To perform a certain function in a beer-bottle capping machine, it is necessary to achieve a reciprocating motion of amplitude Z of ± 1 in. at a cyclic rate of 1 cps. The designer of the machine has decided to use the so-called flat-follower cam system illustrated in Fig. 2–59.

2-22 SPRING: $F = Kx$

$F_0 \sin\omega t = Kx$

$\omega F_0 \cos\omega t = K v$ OR $v = \frac{\omega F_0}{K}\cos\omega t$

DAMPER: $F = bv$ OR $v = \frac{F_0}{b}\sin\omega t$

MASS: $\int_0^t v = \frac{1}{m}\int_0^t F_0 \sin\omega t$

$v(t) = -\frac{F_0}{\omega m}\cos\omega t\Big|_0^t = -\frac{F_0}{\omega M}[\cos\omega t - 1]$

2-23 $KE_{EQ} = J_{EQ}\frac{\Omega^2}{2}$

FOR PINION: $KE_P = J_P\frac{\Omega^2}{2}$ FOR RACK: $KE_R = m\frac{\dot{x}^2}{2}$

BUT $\dot{x} = R\Omega$

$KE_{EQ} = KE_P + KE_R = J_P\frac{\Omega^2}{2} + m R^2\frac{\Omega^2}{2}$

$KE_{EQ} = [J_P + mR^2]\frac{\Omega^2}{2}$ $\therefore J_{EQ} = J_P + mR^2$

2-24 $h = 2'' + 1''\sin\theta$ OR $z = \sin\theta$ (IN)

IF $\dot{\theta} = 2\pi$ RAD/SEC, THEN $\theta = 2\pi t$

$z = \sin 2\pi t$

$\dot{z} = 2\pi\cos 2\pi t$

$\ddot{z} = -4\pi^2 \sin 2\pi t$

$F_i = m\ddot{z} = -\frac{W}{g}4\pi^2\sin 2\pi t$

$|F_{i_{MAX}}| = \frac{W}{g}4\pi^2 = \frac{1}{386}4\pi^2 = 0.102$ LB

$F_{i_{MAX}}$ OCCURS AT $\theta = 0$ & π WHERE THE LENGTH OF THE EQUIVALENT LEVER ARM IS 1 IN., THUS

$T_{MAX} = (1'')(F_{i_{MAX}} + W) = 1.102$ IN-LB

2-25

$C = \frac{\varepsilon A}{\quad} = \frac{6(8.85\times10^{-12})[(1)(1)]}{\quad}$

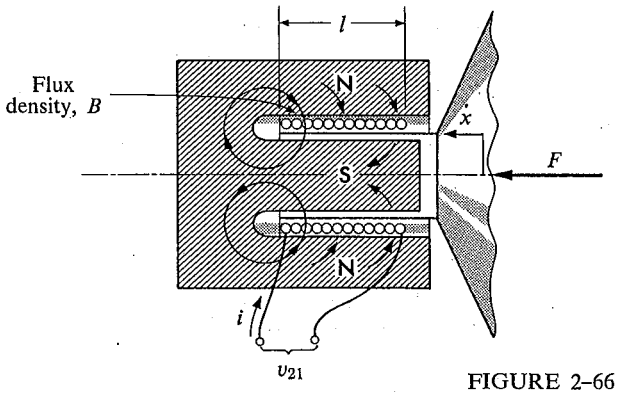


FIGURE 2–66

a) Cable open at end A and annular region occupied by a perfect vacuum. (Neglect end effects.)

b) Inner and outer cylinders connected by a perfect conducting plate at end A and annular region occupied by a perfect vacuum.

c) Annular region filled with material of conductivity σ.

d) What conditions are necessary for the system to act nearly like an ideal element in each of these cases?

2–33. Find the equivalent elements which result when each of the ideal electric elements is connected across the terminals of an ideal transformer (Fig. 2–64). Compare these results with those obtained when the ideal mechanical elements are connected to an ideal gear train (Fig. 2–65).

2–34. The terminals of a galvanometer movement such as the one described in Section 2–6.5 are connected through an ideal resistance R. Assuming that the galvanometer is represented by an ideal transducer with transduction ratio n:

a) Find the relationship between the mechanical torque and the angular velocity of the movement.

b) What ideal rotary mechanical element represents this electromechanical system?

c) Repeat (a) and (b) with a capacitance connected across the terminals of the coil.

2–35. Figure 2–66 shows a "voice-coil" movement used in most loudspeakers. A current i flowing through the coil interacts with a constant radial flux of density B webers/m^2 set up in the air gap by a permanent magnet. The coil has N turns, length l, and radius R.

Assuming that all masses are negligible and that the current i does not affect the flux density B, derive an idealized relation between the two electrical variables v_{21} and i and between the two mechanical variables $\dot{x}$ and F. Determine the transduction ratio n.

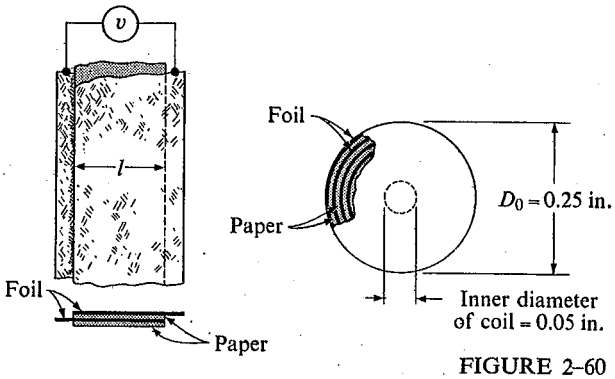


FIGURE 2–60

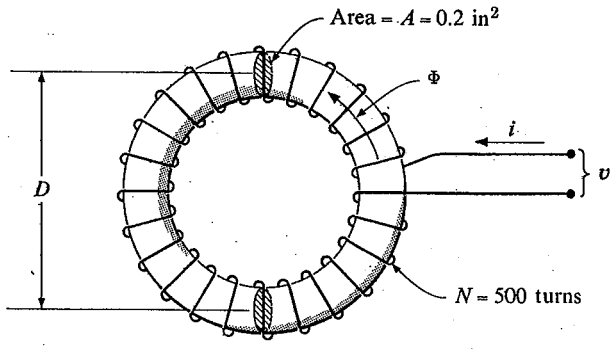


FIGURE 2–61

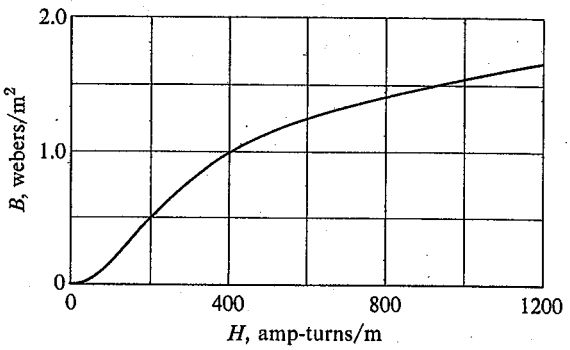


FIGURE 2–62

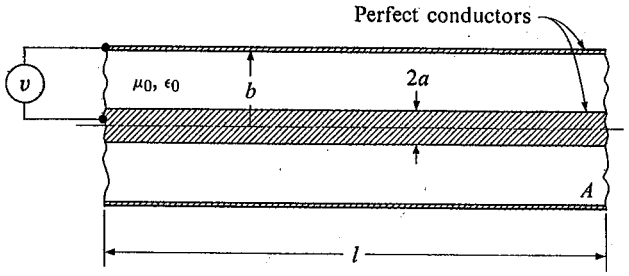


FIGURE 2–63

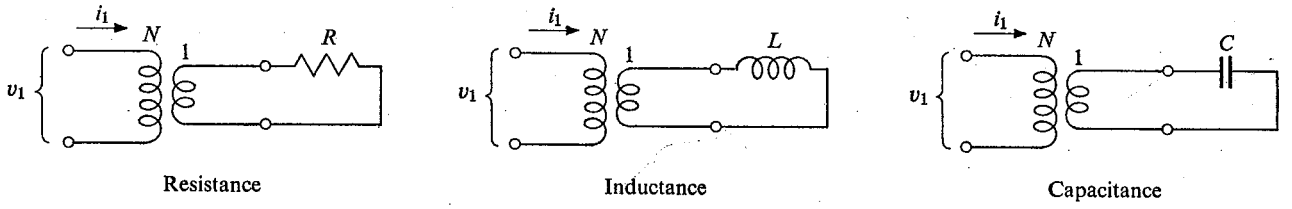


FIGURE 2–64

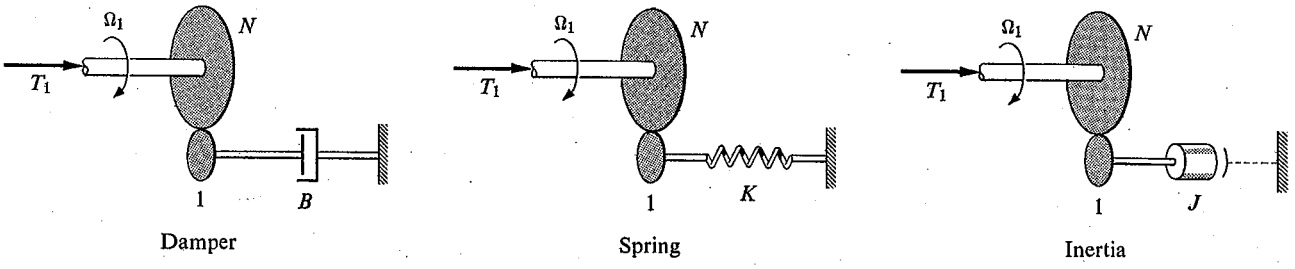


FIGURE 2–65

2-31

a) $V = N\frac{d\Phi}{dt} = \frac{d\lambda}{dt}$

$B = \frac{\Phi}{A}$; $H = \frac{Ni}{\ell}$

$B \sim \frac{\text{WEBERS}}{m^2}$

$H \sim \frac{\text{AMP-TURNS}}{m}$

$V = NA\frac{dB}{dt} = NA\frac{d[F(H)]}{dt}$

$V = \frac{N^2A}{\ell}\frac{d[F(i)]}{dt}$

b) $E = \int_0^t Vi\,dt = \int_0^t \frac{H\ell}{N} NA\frac{dB}{dt}\,dt$

$E = A\ell\int_0^B H\,dB$

IF THE B-H CURVE IS APPROXIMATED AS LINEAR IN THE REGION FROM $H=0$ TO $H = \frac{Ni}{\ell} = 335$, THEN:

$E = A\ell\int_0^B \frac{B}{C}\,dB$

WHERE $C \equiv \frac{\Delta B}{\Delta H} \approx \frac{1}{400}$

$E = \frac{A\ell}{2C}B^2$

$E = \frac{(0.2)(3\pi)(400)(0.8)^2}{2(39.4)^3}$

$E = 4.0 \times 10^{-3}$ joules

2-32

a) $C = \frac{2\pi\epsilon_0\ell}{\ln b/a}$, REQUIRES $\frac{di}{dt}$ BE SMALL

b) $L = \frac{\ell\mu_0}{2\pi}\ln\frac{b}{a}$

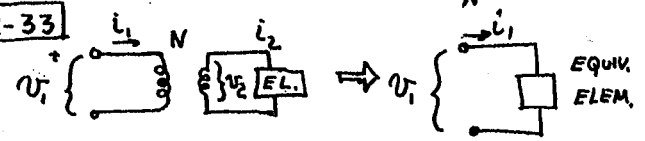
REQUIRES $\frac{dv}{dt}$ BE SMALL

c) $R = \frac{1}{2\pi\sigma\ell}\ln\frac{b}{a}$

REQUIRES LOW FREQUENCY

$v_2 = i_2R = v_1\frac{1}{N} = i_1NR \Rightarrow v_1 = i_1N^2R$

2-33



$v_1 = i_1 \cdot R_{eq}$

FOR A RESISTOR

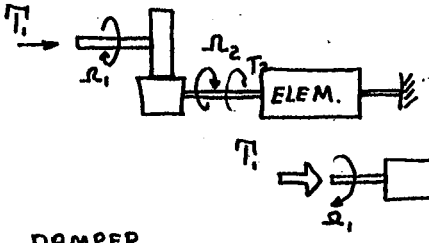
$R_{eq} = N^2R$

FOR AN INDUCTOR

$L_{eq} = N^2L$

FOR A CAPACITOR

$C_{eq} = \frac{1}{N^2}C$



$T_2 = \Omega_2B = \frac{T_1}{N} = \Omega_1NB$

$T_1 = \Omega_1N^2B$

$T_1 = \Omega_1B_{eq}$

DAMPER

$B_{eq} = N^2B$

SPRING

$K_{eq} = N^2K$

INERTIA

$J_{eq} = N^2J$

2-34

FOR THE METER (SEE EQ. 2-96)

$T_b = 2NB\ell r_c i_a = -\frac{1}{m}i_a$

$v_{21} = \frac{1}{m}\Omega_4$

FOR A RESISTOR:

$i_a = \frac{v_{21}}{R} = \frac{1}{m}\frac{\Omega_4}{R}$

$T_b = -\frac{1}{Rm^2}\Omega_4$, SIMILIAR TO A ROTARY DAMPER

FOR A CAPACITOR:

$i_a = C\frac{dv_{21}}{dt} = \frac{C}{m}\frac{d\Omega_4}{dt}$

$T_b = -\frac{C}{m^2}\frac{d\Omega_4}{dt}$, SIMILAR TO A ROTARY MASS

2-35 $F = BLi$, $L = 2\pi RN$

$v_{21} = BL\dot{x}$

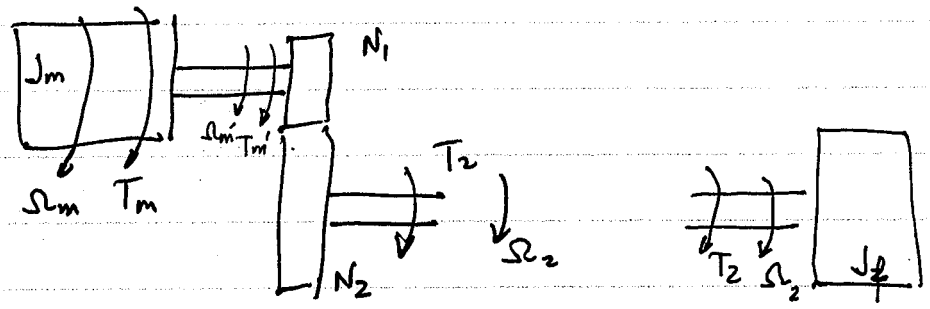
$F = 2\pi RNBi = \frac{1}{m}i$

$v_{21} = 2\pi RNB\dot{x} = \frac{1}{m}\dot{x}$

$F\dot{x} = v_{21}i$

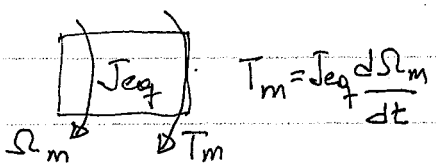
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2-19
also 6.1
in your books



$$T_m' + T_m = J_m \frac{d\Omega_m}{dt} \qquad \frac{T_2}{T_m'} = \frac{1}{n} = \left(\frac{N_2}{N_1}\right)$$

$$\frac{\Omega_2}{\Omega_m'} = -n = -\frac{N_1}{N_2}$$



$$T_m = J_{eq} \frac{d\Omega_m}{dt}$$

$$T_2 = J_f \frac{d\Omega_2}{dt} \qquad \Omega_2 = -n\Omega_m'$$

$$T_2 = -J_f n \frac{d\Omega_m'}{dt} \qquad T_2 = \frac{T_m'}{n}$$

$$T_2 = \frac{T_m'}{n} = -J_f n \frac{d\Omega_m'}{dt} \qquad \text{so} \quad T_m' + T_m = J_m \frac{d\Omega_m}{dt} \Rightarrow T_m = J_m \frac{d\Omega_m}{dt} + J_f n^2 \frac{d\Omega_m'}{dt}$$

$$\text{total KE} = \frac{1}{2} J_m \dot{\Omega}_m^2 + \frac{1}{2} J_f \dot{\Omega}_2^2 = \frac{1}{2} J_m \dot{\Omega}_m^2 + \frac{1}{2} J_f n^2 \dot{\Omega}_m^2$$

$$= \frac{1}{2} \left(J_m + J_f n^2\right) \dot{\Omega}_m^2$$

$$\text{Total KE} = \frac{1}{2} J_{eq} \dot{\Omega}_m^2 \qquad J_{eq} = J_m + J_f n^2 = J_m + J_f \left(\frac{N_1}{N_2}\right)^2$$

- WE HAVE NOTED SIMILARITIES IN ELECTRICAL / MECHANICAL ELEMENT EQS
- BASIS FOR COMPARISON IDEAS OF THROUGH & ACROSS VARIABLES

- IN OUR LUMPED PARAMETER ANALYSES FOR MASS, SPRING, DAMPER, INDUCTANCE CAPACITANCE, RESISTANCE, ETC.

**

- NOTE RELATIONSHIPS BETWEEN A THROUGH VARIABLE (WHOSE VALUE DOESN'T CHANGE AT THE TERMINAL ENDS OF AN IDEAL ELEMENT) AND AN ACROSS VARIABLE (WHOSE VALUE IS GIVEN AS A RELATIVE VALUE OR DIFFERENCE BETWEEN TERMINALS)

  THROUGH VARIABLE - FORCE, TORQUE, CURRENT, VOLUMETRIC FLOWRATE,

  ACROSS VARIABLE - VELOCITY, ANGULAR VELOCITY, VOLTAGE DROP, PRES. DROP

- IN A PHYSICAL SYSTEMS TO MEASURE THROUGH VARIABLE MUST SEVER THE SYSTEM AT SOME POINT TO MEASURE FORCE, TORQUE, CURRENT
  - REMEMBER AMMETER MUST BE PLACED IN SERIES
  - FORCES & TORQUES - FREE BODY DIAGRAMS

- FOR ACROSS VARIABLES NO NEED TO SEVER OR TO ENTER SYSTEM
  - REMEMBER VOLTMETER PLACED IN PARALLEL

- THERMAL & FLUID SYSTEMS

  THROUGH - FLUID FLOW, HEAT FLOW $Q = vA$, $q = k \Delta \theta$

  ACROSS - PRESSURE DROP, TEMPERATURE DIFFERENCE

- A CONVENIENT METHOD TO RELATE THROUGH & ACROSS VARIABLES IS THE LINEAR GRAPH REPRESENTATION.

- TERM LINEAR MEANS GRAPH IS DEFINED BY LINE SEGMENT
- NOT TO BE CONFUSED W/ LINEAR RELATIONSHIP OF IDEAL ELEMENT

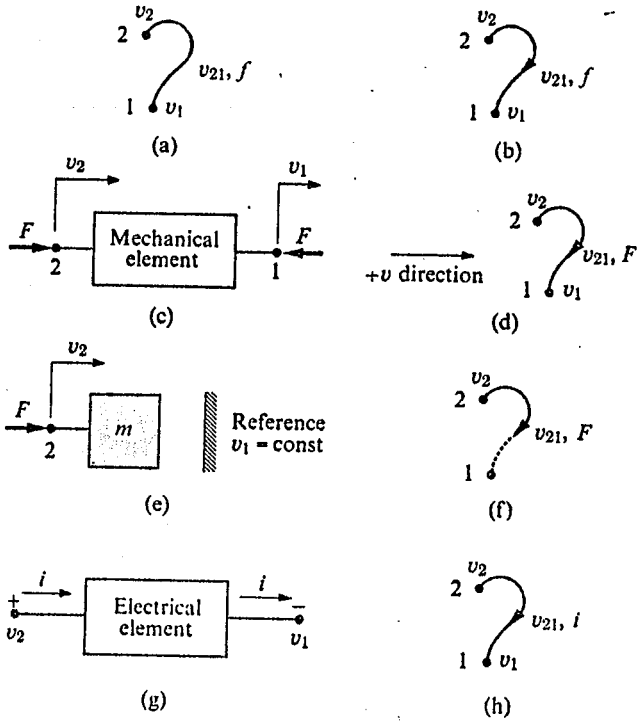


FIG. 2-39. Linear graph representation of pure mechanical and electrical system elements. (a) Linear graph. (b) Oriented linear graph. (c) Nonmass mechanical element. (d) Oriented graph. (e) Translational mass. (f) Oriented graph. (g) Electrical element. (h) Oriented graph.

- THE NODES INDICATE ACROSS-VARIABLE OF ELEMENT
- LINE INDICATES THE CONTINUITY OF THE THROUGH VARIABLE
- ARROW INDICATES ORIENTATION OF BOTH POSITIVE ACROSS & THROUGH VARIABLE

  $+V_{21}$ CAUSES $+f$
- $\Rightarrow$ POWER FLOWS INTO ELEMENT WHEN V_{21} & f ARE + $(\bar{\mathcal{P}} = \bar{f} \cdot \bar{V}_{21})$
- MASS IS SPECIAL SINCE ONE TERMINAL IS THE FIXED REF. FRAME AND FORCE NOT PHYSICALLY TRANSMITTED TO THAT TERMINAL

POWER & ENERGY

POWER IS THE PRODUCT OF THRU VARIABLE & ACROSS VARIABLE

$$\bar{\mathcal{P}} = \bar{f} \cdot \bar{V}_{21}$$

$$E = \int \bar{\mathcal{P}}\, dt = \int \bar{f} \cdot \bar{V}_{21}\, dt$$

		EXAMPLES
MASS, INERTIA (J), CAPACITANCE	A - TYPE ELEMENTS	$E = \frac{1}{2} m V_2^2$
SPRINGS & INDUCTANCE	T - TYPE ELEMENTS	$E = \frac{1}{2} \frac{F^2}{k}$
DAMPERS & RESISTANCE	D - TYPE ELEMENTS	$\mathcal{P} = b V_{21}^2$

A-TYPE ELEMENTS STORE ENERGY BY MEANS OF THEIR ACROSS VARIABLES : V_{21}, ω_{21}

T-TYPE " " " " " " " " THROUGH " : F, T, i

TABLE 2-2
Power and Energy Transferred at a Connecting Point

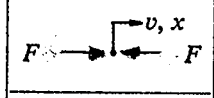
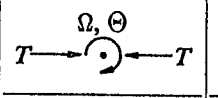
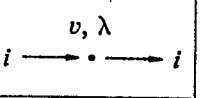
Mechanical-translational	Mechanical-rotational	Electrical
		
$\mathcal{P} = Fv$	$\mathcal{P} = T\Omega$	$\mathcal{P} = iv$
$\mathcal{E} = \int_{t_a}^{t_b} Fv\,dt$	$\mathcal{E} = \int_{t_a}^{t_b} T\Omega\,dt$	$\mathcal{E} = \int_{t_a}^{t_b} iv\,dt$

TABLE 2-3
Summary of Energy Storage and Dissipation for Ideal Elements

A-type storage	*T*-type storage	*D*-type dissipation
Mass $\mathcal{E}_k = \frac{1}{2}mv_2^2$	Spring $\mathcal{E}_p = \frac{1}{2}\frac{F^2}{k}$	Damper $\mathcal{P} = bv_{21}^2 = \frac{F^2}{b}$
Inertia $\mathcal{E}_k = \frac{1}{2}J\Omega_2^2$	Spring $\mathcal{E}_p = \frac{1}{2}\frac{T^2}{K}$	Damper $\mathcal{P} = B\Omega_{21}^2 = \frac{T^2}{B}$
Capacitance $\mathcal{E}_e = \frac{1}{2}C(v_{21})^2$	Inductance $\mathcal{E}_m = \frac{1}{2}Li^2$	Resistance $\mathcal{P} = \frac{(v_{21})^2}{R} = Ri^2$

DEFINE f = GENERALIZED THROUGH VARIABLE ex: (FORCE, TORQUE, CURRENT)

v = GENERALIZED ACROSS VARIABLE ex: (VELOC, ω, VOLTAGE)

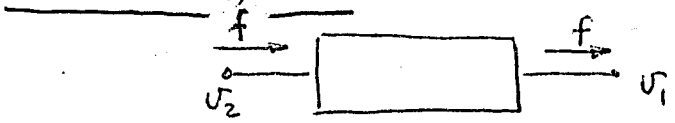
h = INTEGRATED THROUGH VARIABLE $f = dh/dt$ (mom., h, q) ← examples

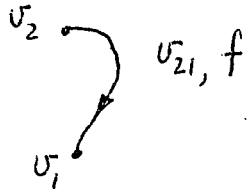
x = INTEGRATED ACROSS VARIABLE $v = \frac{dx}{dt}$ ex: (disp, θ, FLUX LINKAGE)

POWER $\mathbb{P}$ INTO AN ELEMENT $= f \cdot v_{21}$

ENERGY $E = \int \mathbb{P}\,dt = \int f \cdot v_{21}\,dt$

AN ELEMENT WHICH RELATES f TO v_{21} IS CALLED <u>A TWO TERMINAL SINGLE ENERGY PORT</u>





<u>GENERALIZED INDUCTANCE (T-TYPE STORAGE)</u>

FORM: $x_{21} = Lf$ $\qquad v_{21} = L\frac{df}{dt}$

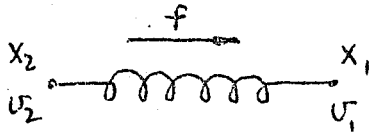
SPRING $L = \frac{1}{k}$, f - FORCE

T. SPRING $L = \frac{1}{K}$; f - TORQUE

ELECTRICAL L - INDUCT; f - CURRENT

$$E = \frac{1}{2}\frac{1}{L}(x_{21})^2 = \frac{1}{2}Lf^2$$

COIL SYMBOL REPRESENTS T-TYPE ELEMENT



GENERALIZED CAPACITANCE (A-TYPE ENERGY STORAGE)

$$f = C \frac{dv_{21}}{dt}$$

LINEAR MECH C = mass; f = force; V_{21} = veloc.

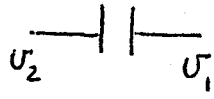
ROTATIONAL $C = J$; $f \Rightarrow$ torque; $v_{21} \Rightarrow \omega_{21}$

ELECTRICAL C = CAPACITANCE; $f \Rightarrow$ CURRENT; $v_{21} \Rightarrow$ VOLT.

FLUID C_f = FLUID CAP.; $f \Rightarrow$ VOL. FLOW RATE; $v_{21} \Rightarrow P_{21}$

$$E = \frac{1}{2} C (v_{21})^2$$

SYMBOL FOR GENERALIZED CAPACITANCE IS SAME AS THAT FOR ELECT.



GENERALIZED RESISTANCE (D-TYPE DISSIPATORS)

$$f = \frac{1}{R} V_{21}$$

MECH - $\frac{1}{R} = b$ f = force

TORS. - $\frac{1}{R} = B$ $f \Rightarrow$ TORQUE $V_{21} \Rightarrow \omega$

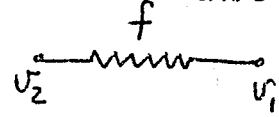
ELECT. - R RESISTANCE $f \Rightarrow$ CURRENT $V_{21} \Rightarrow$ VOLT.

FLUID - R_f $f \Rightarrow Q$ $V_{21} \Rightarrow P_{21}$

THERMAL - R_t $f \Rightarrow q$ $V_{21} \Rightarrow \theta_{21}$

$$P = \frac{1}{R} V_{21}^2$$ **EXCEPT THERMAL**

ELECTRICAL RESISTANCE SYMBOL



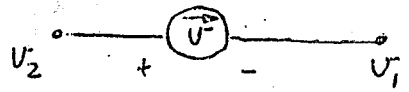
IS SYMBOL FOR GEN. RESISTANCE

DFN - TWO TERMINAL SOURCE IS A DEVICE CAPABLE OF DELIVERING ENERGY CONTINUOUSLY TO A SYSTEM.

TWO LIMITING CASES - IF SOURCE'S ACROSS-VARIABLE IS A SPECIFIED FN OF TIME (A-TYPE SOURCE) OR IF SOURCE'S THROUGH VARIABLE IS A SPECIFIED FN OF TIME (T-TYPE SOURCE)

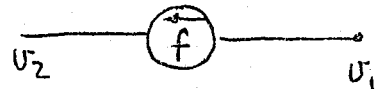
FOR IDEAL SOURCES

$v_{21} = f(t)$



ARROW INDICATES DIRECTION OF ACROSS VARIABLE DROP

$f = f(t)$



f is a fn of time; independent of THE VALUE OF v_{21} ACROSS THE TERMINALS, AND IN DIRECTION OF ARROW

$$\frac{dp}{dt} = F$$

$F = m\dot{v}$	$p = mv$	$v = \frac{dx}{dt}$	$x = \int v\,dt$
$T = J\dot{\omega}$ $\frac{dh}{dt} = T$	$h = J\omega$	$\omega = \frac{d\theta}{dt}$	$\theta = \int \omega\,dt$
$i = C\dot{v}$	$q = Cv$	$v = \frac{d\lambda}{dt}$	$\lambda = \int v\,dt$
$Q = C_f \dot{p}$	$V = C_f p$	$p = \frac{d\Gamma}{dt}$	$\Gamma = \int p\,dt$
$q_H = C_t \dot{\Theta}$	$H = C_t \Theta$		

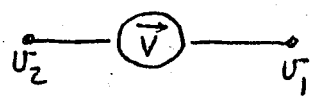
A TYPE SOURCES — EXAMPLES — BATTERY (OCEAN DEPTH PRODUCES ΔP)

T TYPE SOURCES — TORQUE, FORCE (HYDRAULIC PUMP - FLUID FLOW)

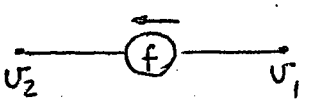
FOR AN A-TYPE SOURCE : f IS KNOWN AS CONJUGATE VARIABLE

T-TYPE SOURCE : v_{21} IS KNOWN AS CONJUGATE VARIABLE

** SIGN OF CONJUGATE VARIABLE IS POSITIVE FOR A SOURCE WHEN POWER FLOWS INTO THE SOURCE.



($-f_v$ is conj variable)

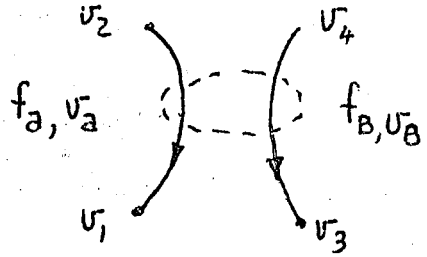


($-v_f$ is conj variable)

NOTE IF f & V_{21} are of opposite signs $P = f \cdot V_{21} < 0$ & $E < 0$

LINEAR GRAPHS FOR TRANSFORMERS

TRANSFORM ACROSS VARIABLES INTO ACROSS VARIABLES, THRU VARIABLES INTO THRU VARIABLES



$v_B = n v_a$ $\quad v_a = v_{21}$ $\quad v_B = v_{43}$

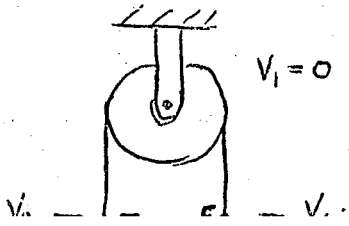
$f_B = -\frac{1}{n} f_a$ $\quad f_a$ & f_B ARE + IN DIRECTION OF ARROWS

v_a, v_B ARE + IF $V_2 > V_1$, $V_4 > V_3$

n IS TRANSFORMATION RATIO

THIS IS AN EXAMPLE OF A FOUR TERMINAL - TWO ENERGY PORT SYSTEM

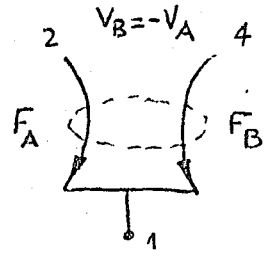
EXAMPLE OF MECHANICAL TRANSFORMER



INEXTENSIBLE CORD

FRICTIONLESS PULLEY NO INERTIA

$V_B = V_{41} = -V_{21} = -V_A$



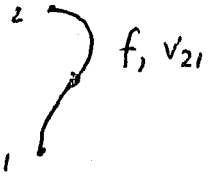
SYSTEMS ANALOGY

- WE HAVE STUDIED TRANSLATIONAL, ROTATIONAL, ELECTRICAL SYSTEMS, FLUID & THERMAL BECAUSE OF SIMILARITY IN BEHAVIOR
- SIMILARITY IS DUE TO TWO FACTORS IN LUMPED ELEMENT SYSTEM!

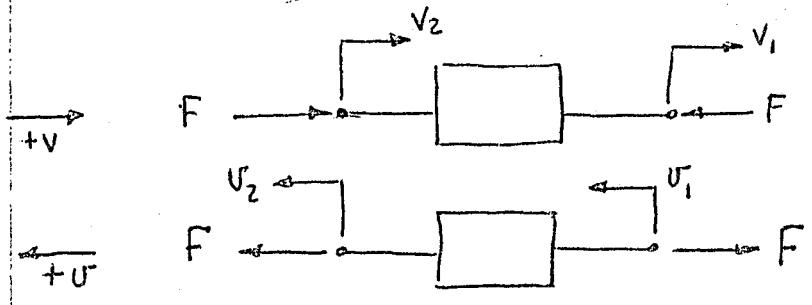
(1) • ELEMENTAL EQUATIONS

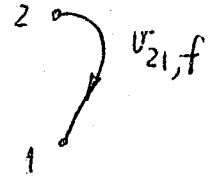
(2) • SIMILARITY OF PHYSICAL LAWS GOVERNING THE RELATIONSHIPS OF VARIABLE
  • SIMILAR COMPATIBILITY & CONTINUITY RELATIONSHIPS

- HAVE USED THIS TO DEFINE GENERALIZED APPROACH TO THE ANALYSIS
- (2) HAS LED TO THE LINEAR GRAPHS WHICH ARE AN AID TO VISUALIZATION AND UNDERSTANDING OF PHYSICAL SYSTEMS
- LINEAR GRAPHS - GENERAL METHOD OF DIAGRAMMING LUMPED PHYSICAL SYSTEMS
  - SHOWS SYSTEM TOPOLOGY - INTERCONNECTEDNESS OF SYSTEM ELEMENTS
  - MAY BE USED FOR ANY SYSTEM WHETHER OR NOT COMPOSED OF IDEAL ELEMENTS
  - LINEAR GRAPH IS ORIENTED (DIRECTED GRAPH); LINES OF GRAPH ARE CALLED BRANCHES, WHICH HAVE DIRECTION TO INDICATE REFERENCE CONDITION



- ARROW SHOWS ACROSS VARIABLE DROP & POSITIVE THRU VARIABLE FLOW
- IF ARROW IS REVERSED SIGN OF f, V_{21} ARE REVERSED
- $P = f \cdot V_{21}$ is + WHEN IT FLOWS INTO ELEMENT
- NOTE ARROW DOES NOT CONTAIN ANY INFO ABOUT DIRECTION OF ACTUAL SYSTEM. ONLY DEFINES ACROSS VARIABLE & THROUGH VARIABLE TO BE USED

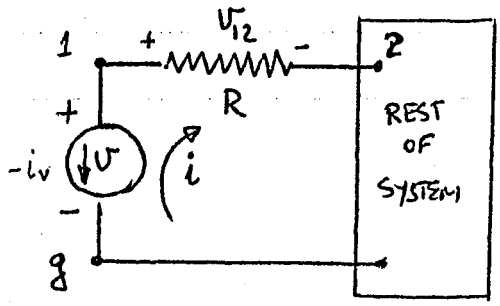


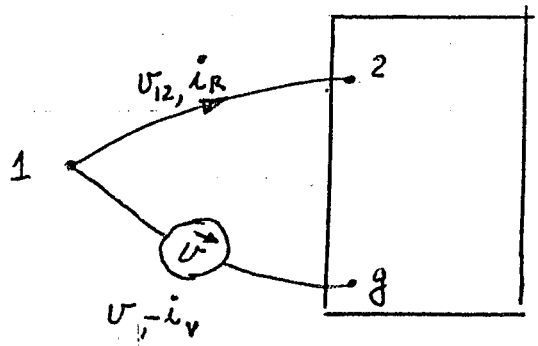


- DOES NOT INDICATE TENSION OR COMPRESSION EITHER

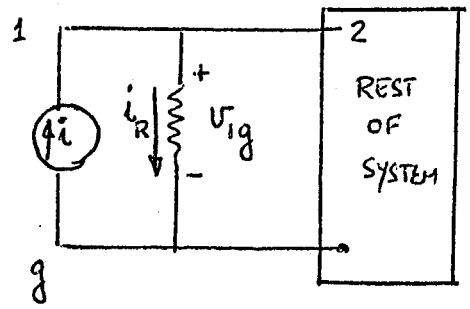
NOTE: IN A COMPLEX SYSTEM W/ SEVERAL MASSES, INERTIAS, CAPACITANCES ALL VELOCITY MUST BE MADE WRT A COMMON REFERENCE (GROUND)

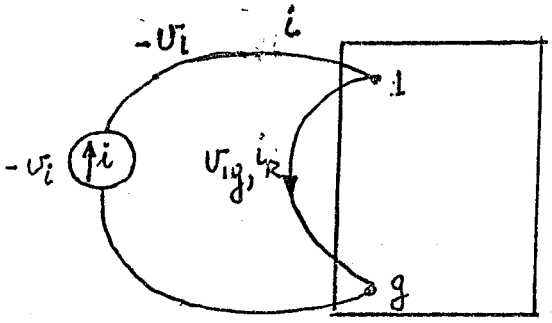
MODELLING OF SOURCES - ALSO KNOWN AS ACTIVE ELEMENTS, ALL OTHER ELEMENTS ARE PASSIVE





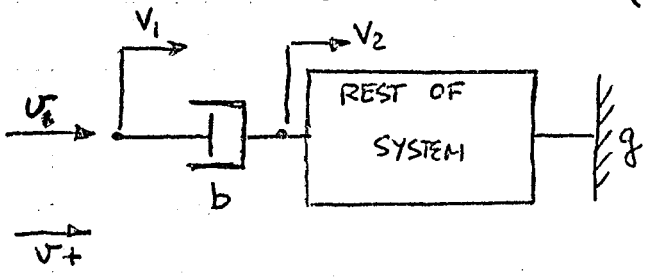
A-TYPE SOURCE : NOTE VOLTAGE DROP v GOES FROM + TO - ; CONJUGATE VARIABLE IS $-i$

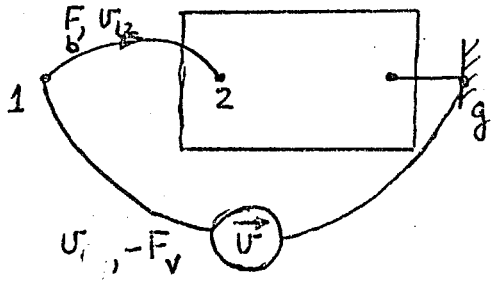




T-TYPE SOURCE : NOTE VOLTAGE DROP GOES FROM + TO - ; CONJUGATE VARIABLE IS $-V_{1g}$

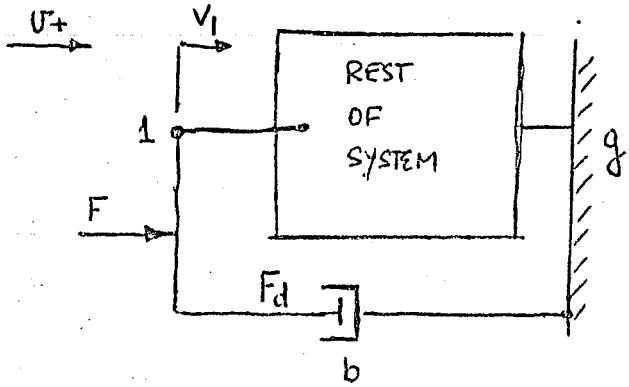
SAME AS IN RESISTOR (SINCE PARALLEL CIRCUIT)

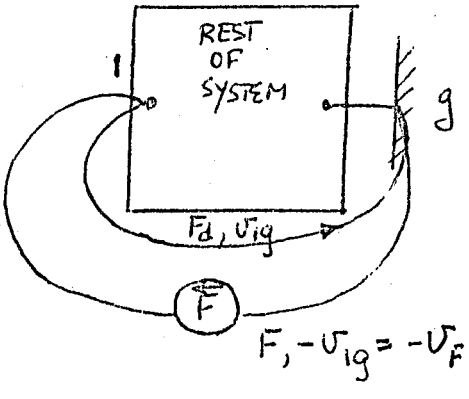




VELOCITY SOURCE

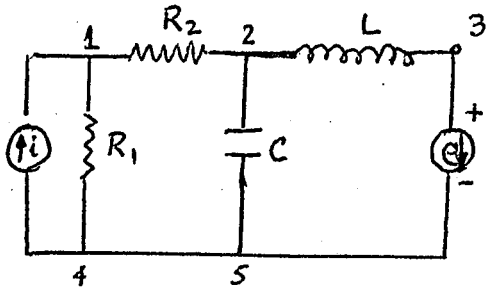
A-TYPE SOURCE : NOTE RELATIVE VELOCITY BETWEEN 1 & g ; CONJUGATE VARIABLE IS $-F$

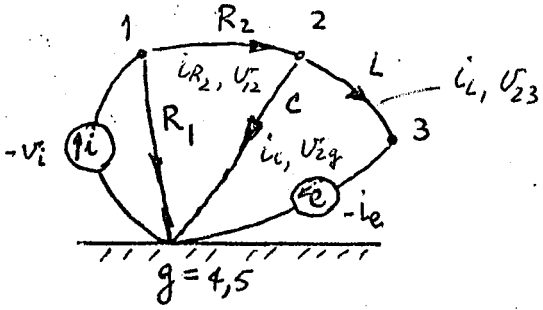


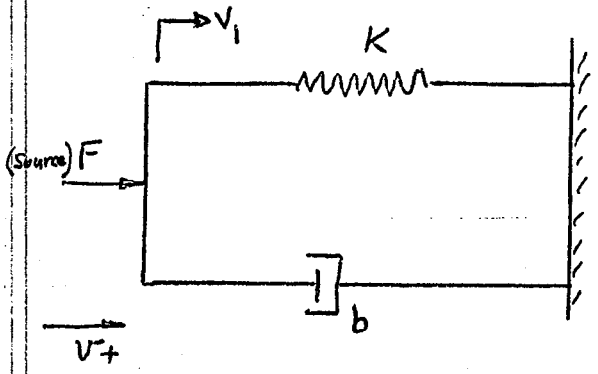


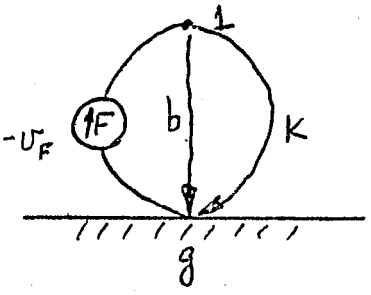
T-TYPE SOURCE :

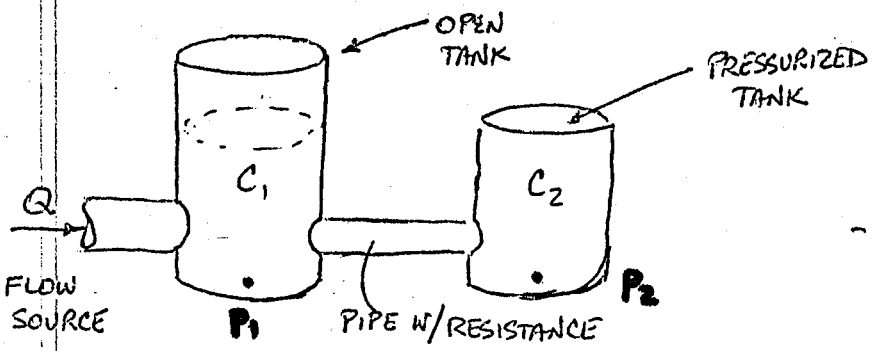
DEFN A LINEAR GRAPH OF A COMPLETE SYSTEM IS A SYSTEM GRAPH

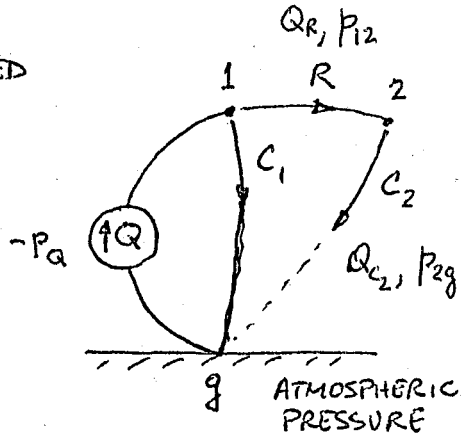












HOW TO DRAW LINEAR GRAPH

[1] IDENTIFY LUMPED ELEMENTS (WILL REQUIRE MODELLING)

NUMBER OF ELEMENTS INCLUDING SOURCES = NO. OF BRANCHES IN GRAPH

[2] ESTABLISH A REFERENCE PT (GROUND) AS A REFERENCE FOR MEASUREMENT OF ALL ACROSS VARIABLES

- IN TRANSLATIONAL, ROTATIONAL, FLUID FLOW SYSTEMS - INERTIAL FRAME OF REFERENCE

[3] AT EACH 2-TERMINAL ELEMENT AN ACROSS VARIABLE EXISTS

EACH ACROSS VARIABLE IS MEASURED WRT THE REFERENCE PT

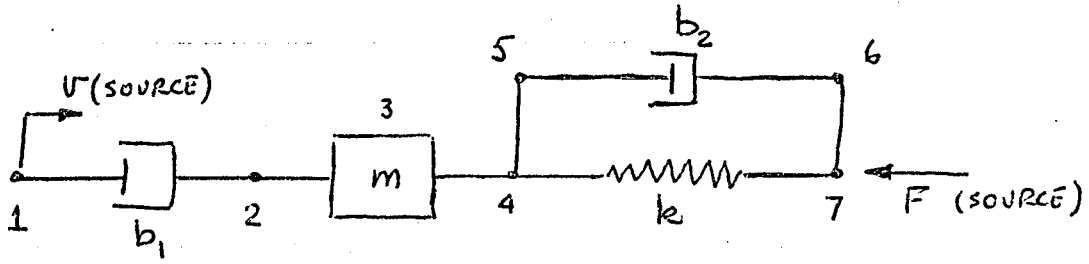
$\therefore V_{12} = V_1 - V_2$ AT EACH NODE (OR VERTEX) - REPRESENTS

[4] FOR EACH 2-TERMINAL ELEMENT, A BRANCH IS INSERTED. ALL INERTIA* BRANCHES ARE INSERTED BETWEEN THEIR NODE & THE REFERENCE VERTEX

[5] EACH SOURCE GETS A BRANCH — CONNECT ONE END TO PT WHERE SOURCE APPLIED, ONE END TO INERTIAL FRAME

[6] ORIENT THE SYSTEM GRAPH. THIS ESTABLISHES REFERENCE CONDITIONS OF THE THRU AND ACROSS VARIABLES

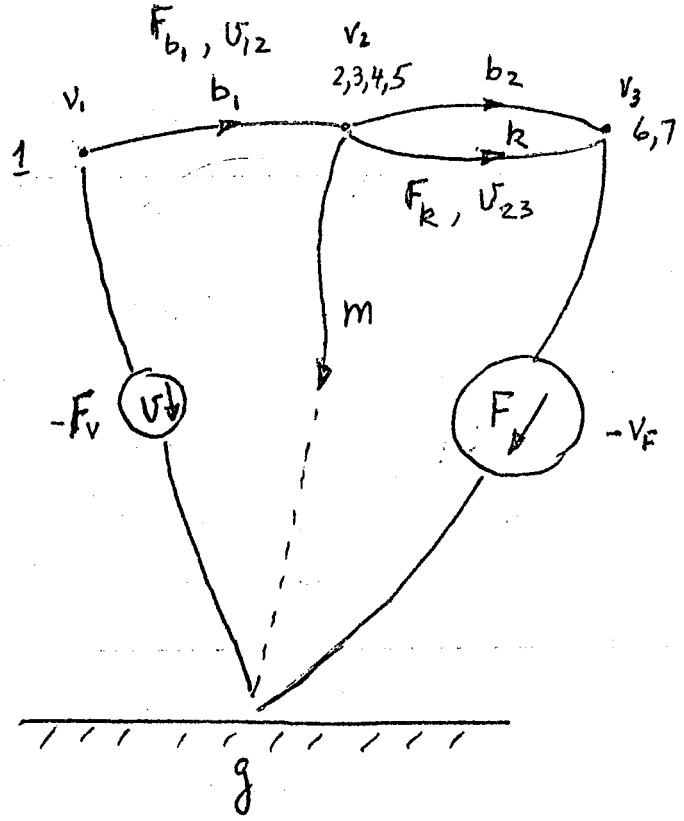
EXAMPLE



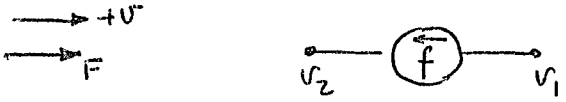
→ + v

- HOW MANY ELEMENTS & WHAT ARE THEY ?
- WHAT IS THE REFERENCE PT ?
- HOW MANY DISTINCT VELOCITIES ARE THERE ?
- WHERE DO YOU INSERT BRANCHES ?
- WHAT DO YOU DO WITH THE MASS ?
- WHAT ABOUT THE SOURCES ?
- WHAT ABOUT ORIENTATION ?

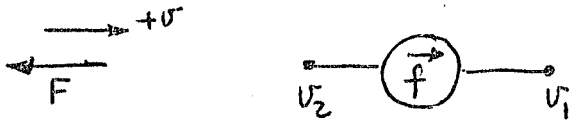
REMEMBER - ASSIGNMENT OF ORIENTATION TO THE BRANCHES OF THE GRAPH (WHEN RELATED TO THE CHOSEN POSITIVE DIRECTION FOR VELOCITY) IMPLIES THE CHOICE OF REFERENCE CONDITIONS FOR THE VARIOUS FORCES AND VELOCITIES IN THE SYSTEM


F_{b_1}, v_{12}
v_2
2,3,4,5
b_2
v_3
6,7
v_1
b_1
1
k
F_k, v_{23}
m
$-F_V$
V
F
$-v_F$
g

FOR T-TYPE SOURCES : IF FORCE IS IN DIRECTION OF + VELOCITY



IF FORCE IS IN OPPOSITE DIRECTION OF + VELOCITY



- GENERALIZED CONTINUITY AND COMPATIBILITY

  THE CONTINUITY AND COMPATIBILITY REQ'TS ARE STATEMENTS OF PHYSICAL RELATIONSHIPS THAT MUST HOLD FOR VARIABLES OF A CONNECTED SET OF LUMPED ELEMENTS BECAUSE OF THE WAY THEY ARE CONNECTED. THESE RELATIONS DEAL W/ THE STRUCTURE OF SYSTEM.

- CONTINUITY - VERTEX (OR NODE) LAW

  FOR AN ORIENTED SYSTEM GRAPH THE ALGEBRAIC SUM OF THE THRU VARIABLES ENTERING A NODE IS ZERO

  - $\sum$ THRU VARIABLES ENTERING NODE = $\sum$ THRU VARIABLES LEAVING

  ELECTRICAL SYSTEMS - CONSERVATION OF CHARGE OR KIRCHOFF'S CURRENT LAW

  MECHANICAL SYSTEMS - CONSERVATION OF MOMENTUM OR NEWTON'S 1 & 3rd LAWS

  FLUID SYTEMS - CONSERVATION OF MATTER

  THERMAL SYSTEMS - CONSERVATION OF ENERGY

  - APPLIES TO A CLOSED VOLUME WHICH CUTS THE SYSTEM GRAPH
  - CAN ALSO BE APPLIED AT EACH VERTEX WITHIN THIS CLOSED VOLUME HOWEVER THIS IS NOT NECESSARY
  - GIVEN A SYSTEM GRAPH OF 2-TERMINAL ELEMENTS WITH n VERTICES ONLY (n-1) VERTEX EQNS ARE LINEARLY INDEPENDENT.

- FOR SYSTEMS W/ MULTITERMINAL ELEMENTS HAVING p parts NOT CONNECTED THERE ARE $(n-p)$ LINEAR INDEPENDENT VERTEX EQNS

EXAMPLE: IF A TRANSFORMER ELEMENT IS IN THE SYSTEM IT WILL SPLIT THE SYSTEM GRAPH INTO 2 PARTS $\Rightarrow (n-2)$ VERTEX EQNS

COMPATIBILITY OR PATH LAW

ELECTRICAL - POTENTIAL IS A SCALAR (KIRCHHOFF'S VOLTAGE LAW)

MECHANICAL - DISTANCE IS A SCALAR

FLUID - PRESSURE IS A SCALAR

THERMAL - TEMPERATURE IS SCALAR

} CAN ADD AS YOU MOVE FROM PT-PT

- ALGEBRAIC SUM OF THE ACROSS VARIABLES AROUND ANY CLOSED PATH IS ZERO.
- GIVEN A SYSTEM GRAPH OF 2-TERMINAL ELEMENTS WITH n VERTICES AND b branches; ONLY $b-(n-1)$ OF THE PATH EQNS ARE LINEARLY INDEPENDENT. IF THERE ARE P PARTS; THERE ARE $b-(n-p)$ LINEARLY INDEPENDENT EQNS

- FORMULATION OF SYSTEM EQNS
- A SUFFICIENT SET OF EQNS FOR DETERMINING THE SYSTEM EQUATION FOR ANY OUTPUT OF ANY SYSTEM (LINEAR OR NONLINEAR) IS OBTAINED BY USING A SET OF

$(n-1)$ LINEARLY INDEPENDENT VERTEX EQNS

$(b-n+1)$ LINEARLY INDENDENT PATH EQNS

$b-s$ ELEMENTAL EQNS

n IS NO OF VERTICES, b IS NO OF BRANCHES, S IS NO OF SOURCES

- FOR A SYSTEM GRAPH WITH b branches OF WHICH S BRANCHES ARE SOURCES $\Rightarrow (2b-s)$ UNKNOWNS. WHY?

FOR EACH NONSOURCE BRANCH : 2 UNKNOWNS - THRU & ACROSS VARIABLE

FOR EACH SOURCE : 1 UNKNOWN - CONJUGATE VARIABLE

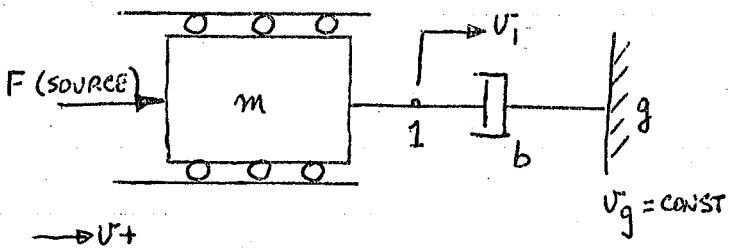
NOTE $\underset{\text{VERTEX}}{(n-1)} + \underset{\text{PATH}}{(b-n+1)} + \underset{\text{ELEMENTAL}}{(b-s)} = 2b-s$

- THUS THIS FORMS A NECESSARY & SUFFICIENT SET ALLOWING US TO ELIMINATE ALL BUT ONE PARTICULAR VARIABLE.

NONLINEAR SYSTEMS

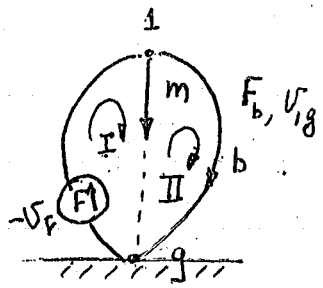
- THIS FORMS A SUFFICIENT SET FOR NONLINEAR SYSTEMS, BUT IT IS NOT ALWAYS POSSIBLE TO ELIMINATE ALL BUT ONE VARIABLE.
- THE ENTIRE SET OF EQNS MAY NOT BE NECESSARY TO DETERMINE RESPONSE

EXAMPLE



NOTE: MASS & DAMPER HAVE SAME VELOCITY

- APPLIED FORCE DOESN'T ACCELERATE MASS, BUT $F-F_b$ DOES.



ELEMENTAL EQNS $(b-s) = 3-1 = 2$

$$F_m = m\frac{dV_{1g}}{dt} = m\frac{dV_1}{dt} \qquad (1)$$

$$F_b = bV_{1g} = bV_1 \qquad (2)$$

CONTINUITY (VERTEX LAW) $n-1 = 2-1 = 1$

$$F - F_b - F_m = 0 \Rightarrow F = F_m + F_b \qquad (3)$$

COMPATIBILITY (PATH LAW) $b-n+1 = 3-2+1 = 2$

REMEMBER $-V_F$ IS CONJ. VAR. OF F: $-V_F + V_{1g} = 0$ (PATH ALONG F & m) (4)

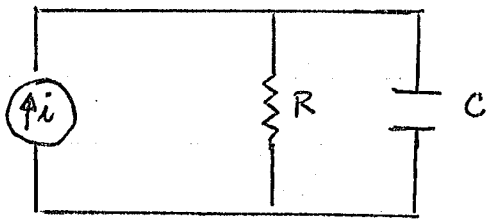
$-V_{1g} + V_{1g} = 0$ (PATH ALONG m & b) (5)

FROM (1) & (2) INTO (3)

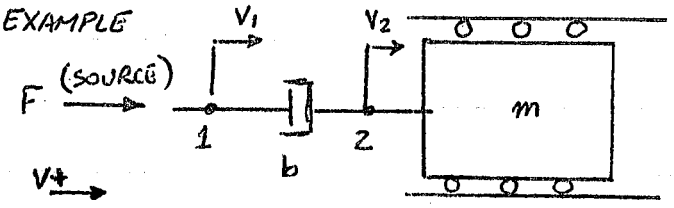
$$F = F_m + F_b = m\frac{dV_1}{dt} + bV_1$$

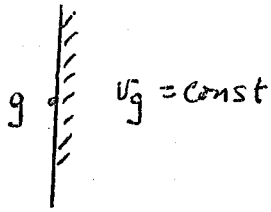
THIS ALLOWS USE TO FIND V_1 AND $x_1 = \int^t V_1 d\tilde{t} + x_1(t=0)$

EXERCISE: COMPARE THIS TO:

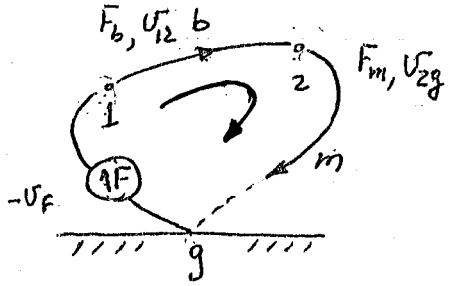


EXAMPLE





NOTE: FORCE IS TRANSMITTED THRU DAMPER OR SERIES CONNECTION



ELEMENTAL EQNS. $(b-s) = 3-1=2$

$$F_m = m\frac{dv_{2g}}{dt} = m\frac{dv_2}{dt}$$

$$F_b = b v_{12}$$

CONTINUITY (VERTEX) $n-1 = 3-1=2$

@ 1: $F - F_b = 0$

@ 2: $F_b - F_m = 0$

$\Big\}\ F = F_b = F_m$

COMPATIBILITY (PATH LAW) $b-n+1 = 3-3+1$

REMEMBER: $-v_F$ IS CONJ VAR OF F:

$$-v_F + v_{12} + v_{2g} = 0$$

DIFFERENTIATE COMPATIBILITY:

$$-\frac{dv_F}{dt} + \frac{dv_{12}}{dt} + \frac{dv_{2g}}{dt} = 0$$

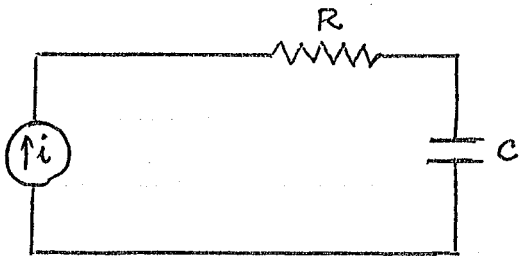
$$-\frac{dv_F}{dt} + \frac{1}{b}\frac{dF_b}{dt} + \frac{F_m}{m} = 0$$

$$\frac{dv_F}{dt} = \frac{1}{b}\frac{dF}{dt} + \frac{F}{m}$$

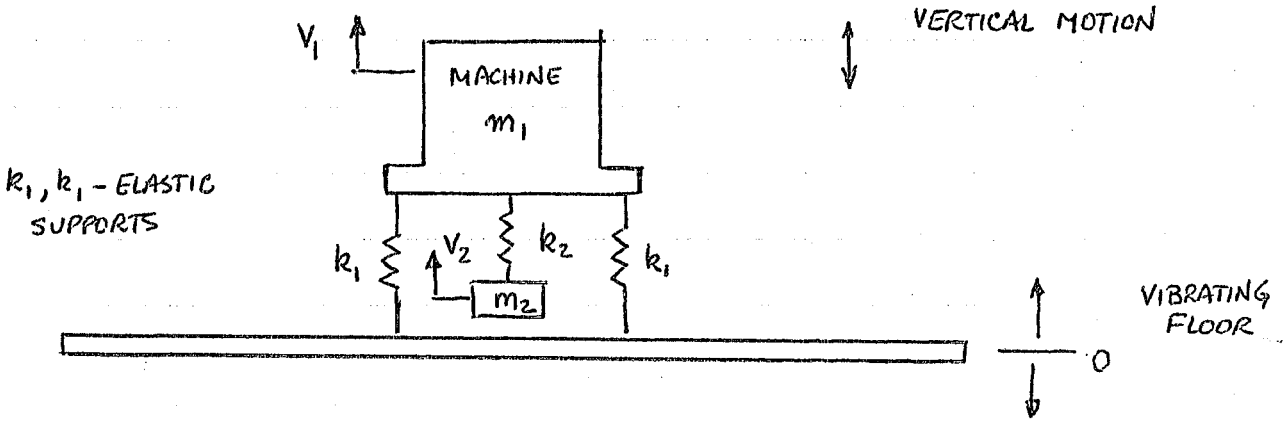
KNOWING F AS FN OF t, WE CAN FIND v_F AS FN OF t

IF WE WANT $V_{2g} \Rightarrow F = F_m = m\frac{dv_{2g}}{dt} \Rightarrow v_{2g} = \int_0^t \frac{F}{m}\,dt$. HERE $F = F(t)$ KNOWN FN.

EXERCISE COMPARE THIS TO

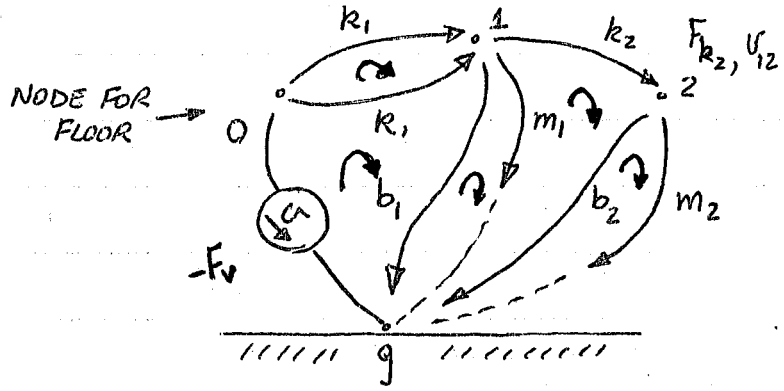


VIBRATION ABSORBER



- SYSTEM IS EXCITED BY THE VIBRATING FLOOR - WHICH IS MODELLED AS AN IDEAL VELOCITY SOURCE
- VIBRATION OF MACHINE IS EXCESSIVE - AS DESIGNERS OUR JOB IS TO DESIGN A VIBRATION ABSORBER (k_2, m_2) TO REDUCE AMPLITUDE OF VIBRATION (X_1) OF THE SYSTEM

CHOOSE (GROUND) AS INERTIAL FRAME NOT FLOOR!!



MISSING FROM VIBRATION ABSORBER

- HERE - b_1, b_2 are AIR DAMPING (ASSUMED LINEAR)
- CHOOSE INPUT AS VELOCITY OF FLOOR V, OUTPUT V_1

ELEMENTAL EQNS $(b-s) = 8-1 = 7$

$$k_1 V_{01} = \frac{dF_{K1}}{dt} \quad (2) \qquad F_{b_1} = b_1 V_{1g} = b_1 V_1$$

$$k_2 V_{12} = \frac{dF_{K2}}{dt} \qquad F_{b_2} = b_2 V_{2g} = b_2 V_2$$

$$F_{m_1} = m_1 \frac{dV_{1g}}{dt} = m_1 \frac{dV_1}{dt} \qquad F_{m_2} = m_2 \frac{dV_{2g}}{dt} = m_2 \frac{dV_2}{dt}$$

CONTINUITY (VERTEX) $n-1 = 4-1 = 3$

node 0 : $-F_v + 2F_{K1} = 0$

1 : $2F_{K1} = F_{b_1} + F_{m_1} + F_{K2}$ (1)

2 : $F_{K2} = F_{b_2} + F_{m_2}$ (2)

COMPATIBILITY EQNS (PATH) $b-n+1 = 8-4+1 = 5$

$$-V + V_{01} + V_{1g} = 0 \implies V = V_{01} + V_1 \text{ or } V_{01} = V - V_1 \quad (3)$$

$V_{01} - V_{01} = 0$ IDENTITY (VELOCITY IN SPRINGS K_1 ARE SAME)

$-V_{1g} + V_{1g} = 0$ IDENTITY (VELOCITY IN DAMPER$_1$ & MASS$_1$ ARE SAME)

$$-V_{1g} + V_{12} + V_{2g} = 0 \implies V_{12} = V_1 - V_2 \quad (4)$$

$-V_{2g} + V_{2g} = 0$ IDENTITY (VELOCITY IN DAMPER$_2$ & MASS$_2$ ARE SAME)

WANT V_{1g} AS FUNCTION OF FLOOR VELOCITY, V:

- IF WE DIFFERENTIATE (1): $2\frac{dF_{K1}}{dt} = \frac{dF_{b_1}}{dt} + \frac{dF_{m_1}}{dt} + \frac{dF_{K2}}{dt}$

$$2k_1 V_{01} = b_1 \frac{dV_1}{dt} + m_1 \frac{d^2 V_1}{dt^2} + k_2 V_{12} \quad (5)$$

- USE (3) & (4) IN (5) TO FIND V_2

$$V_2 = \frac{1}{K_2}\left[(2k_1 + k_2) V_1 + b_1 \dot{V}_1 + m_1 \ddot{V}_1 - 2k_1 V\right] \quad (6)$$

IF WE DIFFERENTIATE (2)

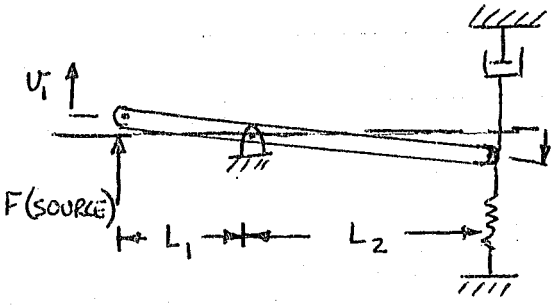
$$dF_{K2/12} = dF_{b2/12} + dF_{m2} \implies k_2 V_{12} = b_2 \frac{dV_2}{dt} + m_2 \frac{d^2 V_2}{dt^2} \quad (7)$$

• KNOWING (4) & (6) PLACED INTO (7) LEADS TO

$$m_1 m_2 \frac{d^4 V_1}{dt^4} + (m_1 b_2 + m_2 b_1)\frac{d^3 V_1}{dt^3} + [m_1 k_2 + b_1 b_2 + m_2 (2k_1 + k_2)]\frac{d^2 V_1}{dt^2}$$

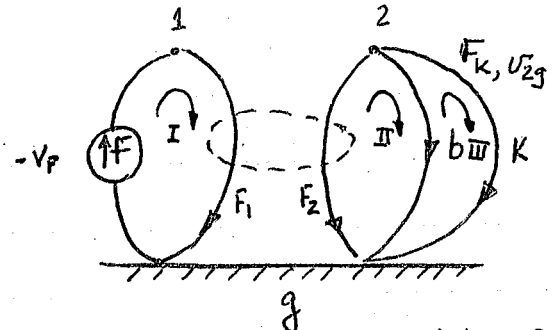
$$+ [b_1 k_2 + b_2 (2k_1 + k_2)]\frac{dV_1}{dt} + 2k_1 k_2 V_1 =$$

$$2m_2 k_1 \frac{d^2 V}{dt^2} + 2k_1 b_2 \frac{dV}{dt} + 2k_1 k_2 V$$



FOR LEVERED SYSTEM

F is input, θ output



ELEMENTAL EQNS $b-s = 5-1 = 4$

$\frac{d}{dt}F_K = K V_{2g} = K V_2$ (3)

$F_b = b V_{2g} = b V_2$ (3')

$F_2/F_1 = +\frac{1}{n}$

$-V_{2g}/V_{1g} = n$ $\quad n = l_2/l_1$ LEVER RATIO

LET F_1 & F_2 BE THE THRU VARIABLES AND 1 & 2 RESPECTIVELY OF THE LEVER

CONTINUITY (VERTEX) $n-p = 4-2 = 2$

@1: $F - F_1 = 0 \Rightarrow F = F_1$ (1)

@2: $F_2 + F_b + F_K = 0 \Rightarrow F_b + F_K = -F_2$ (2)

COMPATIBILITY OR PATH $b-n+p = 5-4+2 = 3$

@ I: $-V_F + V_{1g} = 0 \Rightarrow V_F = V_{1g}$

@ II, III: $-V_{2g} + V_{2g} = 0$ (2 TIMES) IDENTITIES

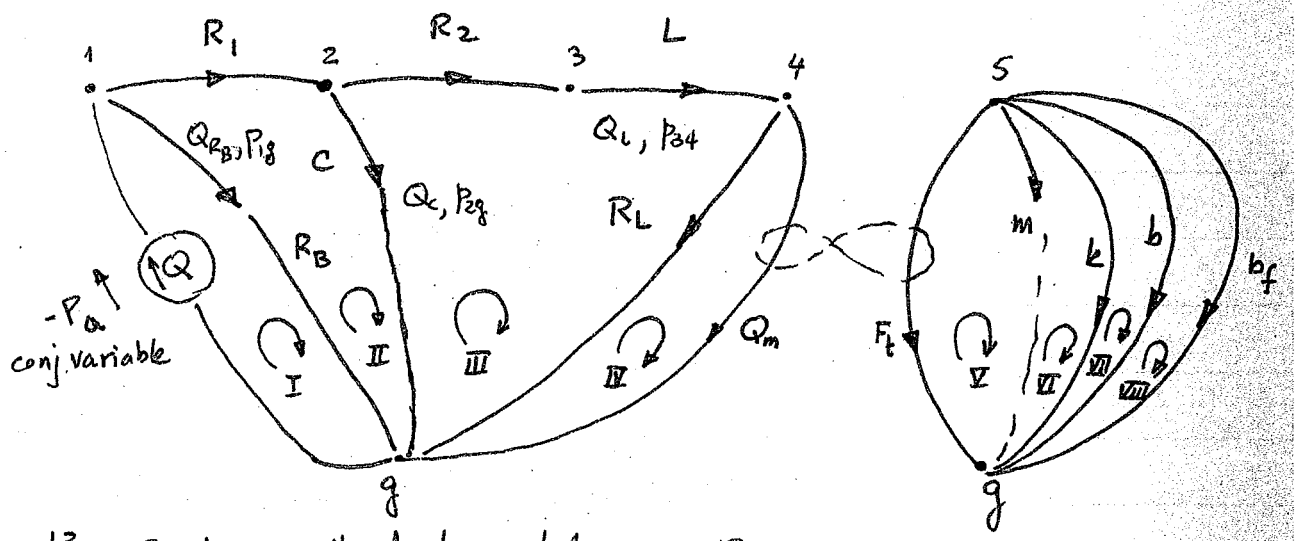
FROM (1) & (2) $(F_b + F_K)/F = -F_2/F_1 = +\frac{1}{n} \Rightarrow F_b + F_K = +\frac{1}{n}F$ (4)

BUT $n = +L_2/L_1$ and $x_2 = -L_2\theta \Rightarrow V_2 = -L_2\dot{\theta}$ (5)

THUS FROM (3), (3') INTO (4) $\Rightarrow K x_2 + b V_2 = -\frac{L_1}{L_2}F$, WITH (5)

$$K L_2^2 \theta + b L_2^2 \dot{\theta} = L_1 F$$

THIS IS THE WORKED OUT SOLUTION FOR THE LAST PROBLEM



$b = 13$ $\quad s = 1$ $\quad$ # of elemental eqns = 12

$$Q_{R_B} = \frac{P_{1g}}{R_B} \qquad Q_{R_1} = \frac{P_{12}}{R_1} \qquad Q_c = C\frac{dP_{2g}}{dt} \qquad Q_{R_2} = \frac{P_{23}}{R_2} \qquad P_{34} = L\frac{dQ_L}{dt}$$

$$Q_{R_L} = \frac{P_{4g}}{R_L} \qquad F_m = m\frac{dV_{5g}}{dt} \qquad \frac{dF_k}{dt} = kV_{5g} \qquad F_b = bV_{5g} \qquad F_{b_f} = b_f V_{5g}$$

Transducer equations: $P_{4g} \cdot A = F_t$

$$-Q_m = V_{5g} \cdot A$$

$n-p$ = # nodal equations $\quad n = 7$ (remember 2 nodes for g) $\quad p = 2$

$\therefore$ 5 nodal equations

@1: $Q = Q_{R_B} + Q_{R_1}$ $\quad$ @2: $Q_{R_1} = Q_C + Q_{R_2}$ $\quad$ @3: $Q_{R_2} = Q_L$ $\quad$ @4: $Q_L = Q_{R_L} + Q_m$

@5: $F_t + F_m + F_k + F_b + F_{b_f} = 0$

path equations is $b-(n-p) = 13-5 = 8$

$\circlearrowright$ I : $-P_a + P_{1g} = 0$

$\circlearrowright$ II : $-P_{1g} + P_{12} + P_{2g} = 0$

$\circlearrowright$ III : $-P_{2g} + P_{23} + P_{34} + P_{4g} = 0$

$\circlearrowright$ IV : $-P_{4g} + P_{4g} = 0$

$\circlearrowright$ V : $-V_{5g} + V_{5g} = 0$

$\circlearrowright$ VI, $\circlearrowright$ VII, $\circlearrowright$ VIII : $-V_{5g} + V_{5g} = 0$

Depending on what is required the differential equation may be as high as 4th order (since 4 storage elements C, L, m, k).

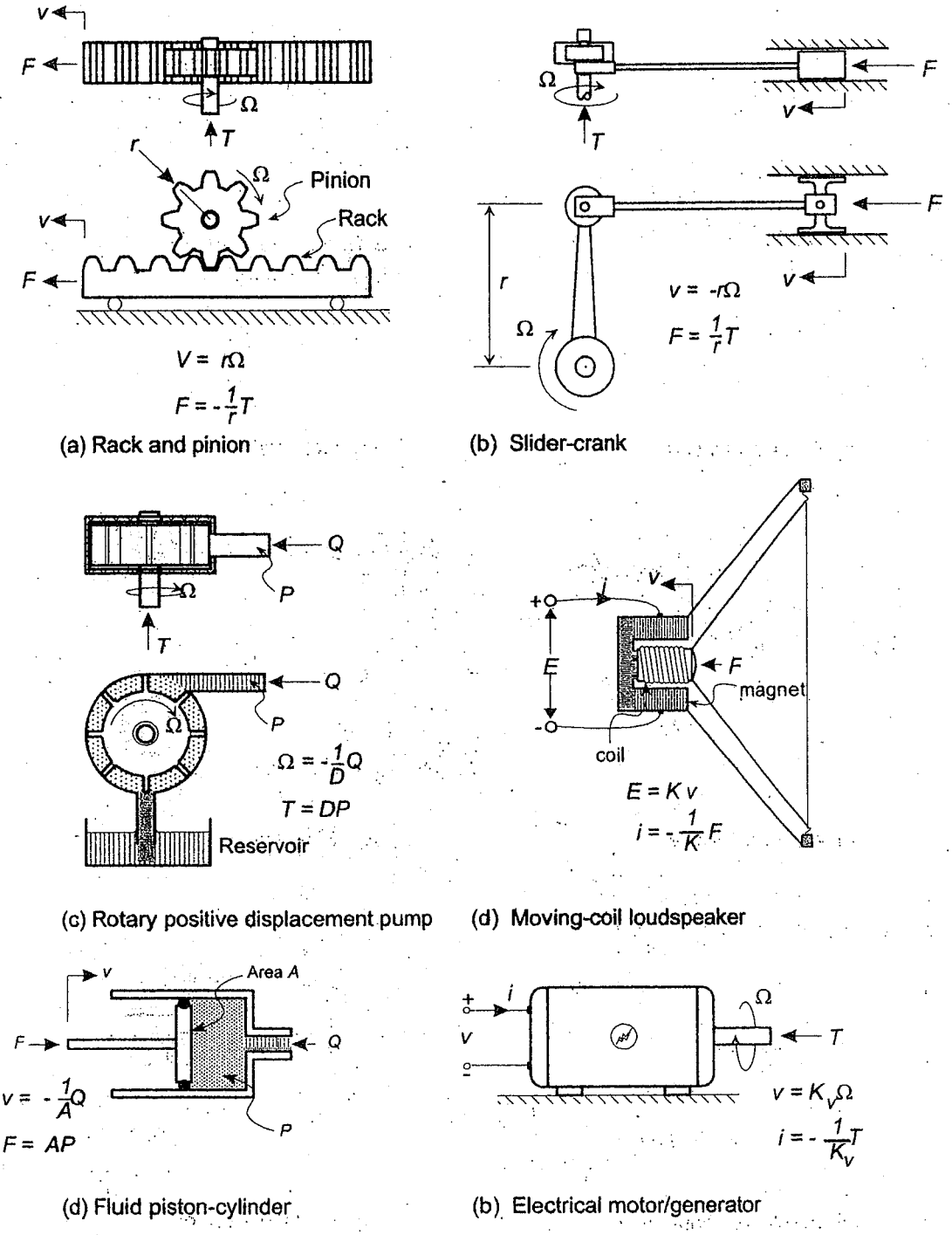


Figure 1: Examples of systems and devices using two-port transducers between different energy domains.

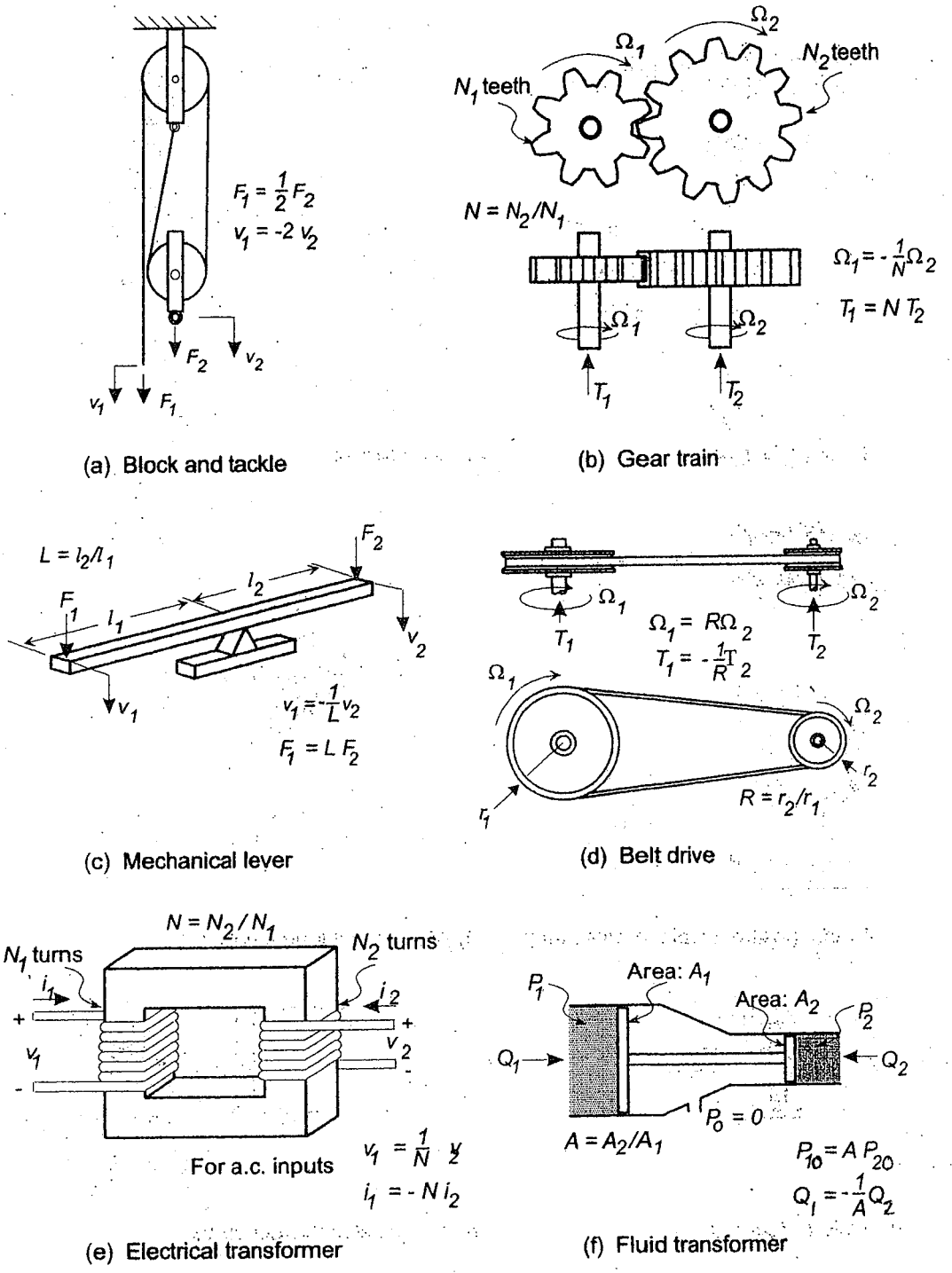
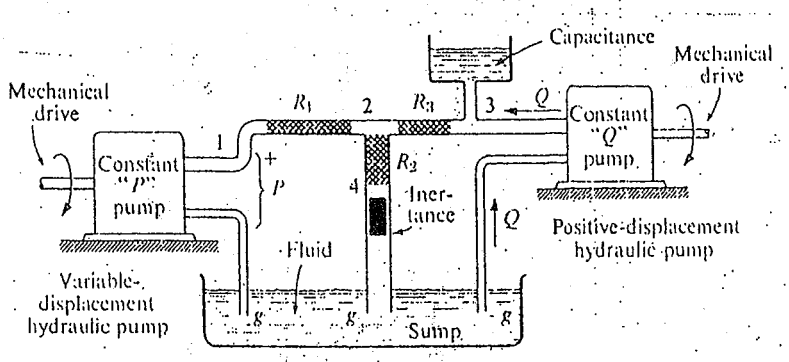
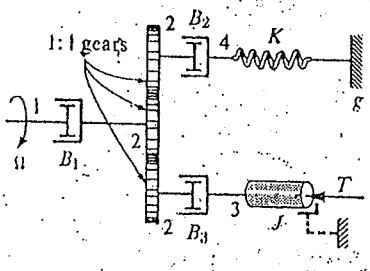


Figure 2: Examples of two-port transducers within a single energy domain.

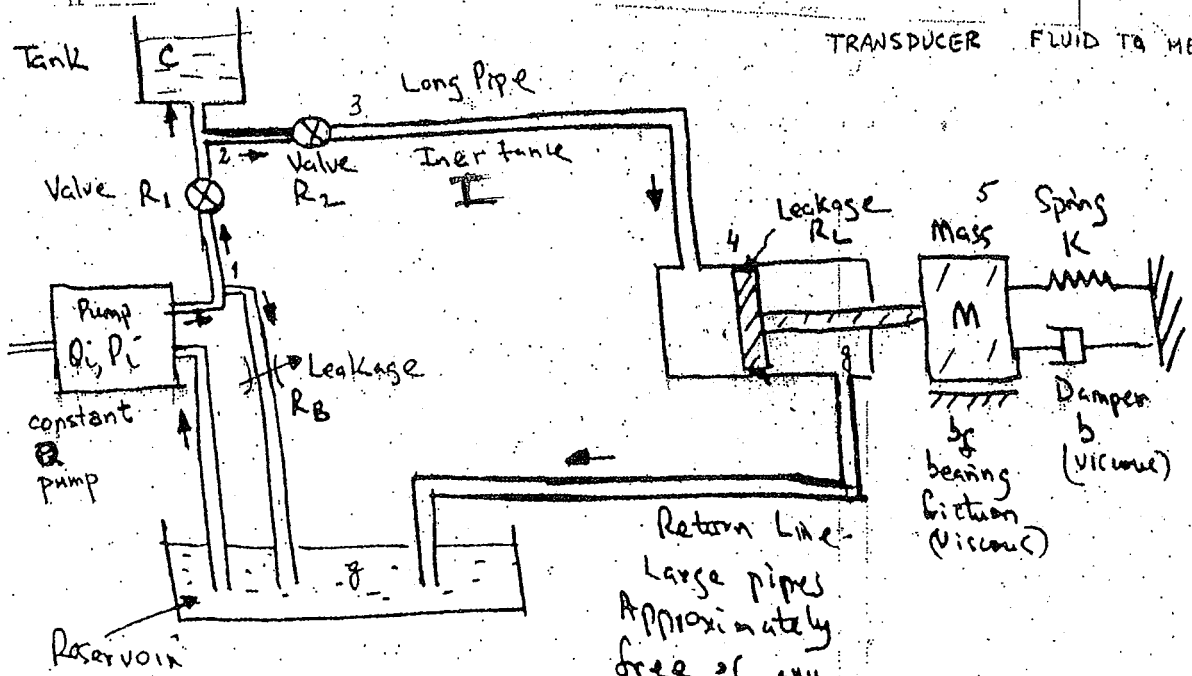


TREAT PUMPS LIKE SOURCES



TRANFORMER GEARS

TRANSDUCER FLUID TO MECHANICAL



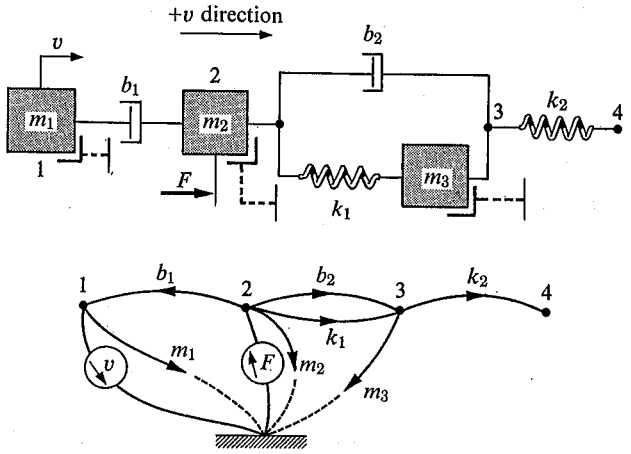


FIG. 8–10. Mechanical translational system and graph.

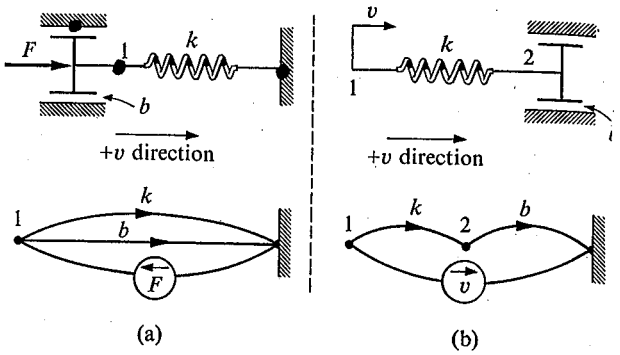


FIG. 8–11. Illustration of different structure of seemingly similar systems.

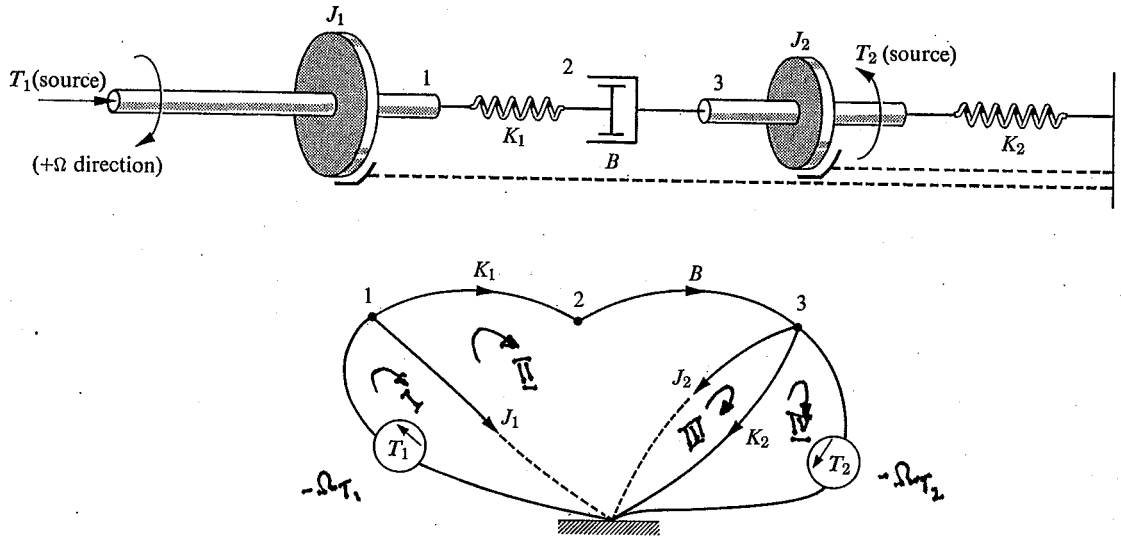
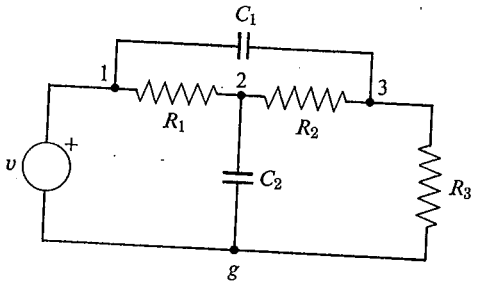


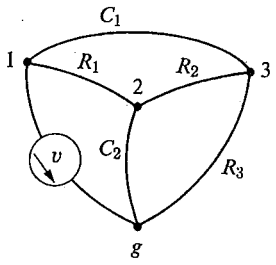
FIG. 8–12. Mechanical rotational system and graph.

Additional examples of graph construction are given in Figs. 8–10, 8–11, 8–12, 8–13, 8–14, 8–15, 8–16, and 8–17. The reader should, by a procedure similar to the previous example, determine the graphs of the systems and check with the result shown in the figures.

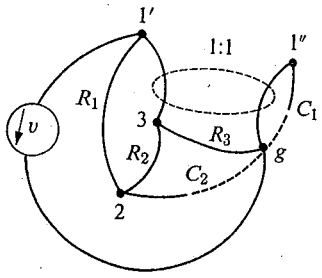
Note that element k_2 in Fig. 8–10 does not really affect the system performance since there is no way for a force to exist in the spring with one end free. This clearly shows in the graph since element k_2 is not in any closed path of the graph. Also the velocity source is applied directly to the mass, m_1, and thus



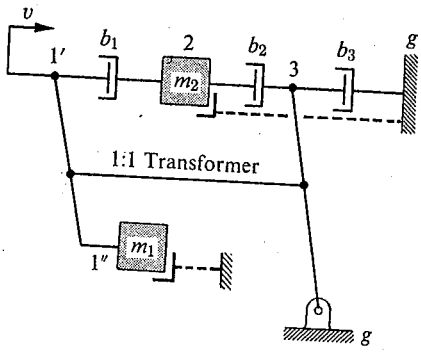
(a) System



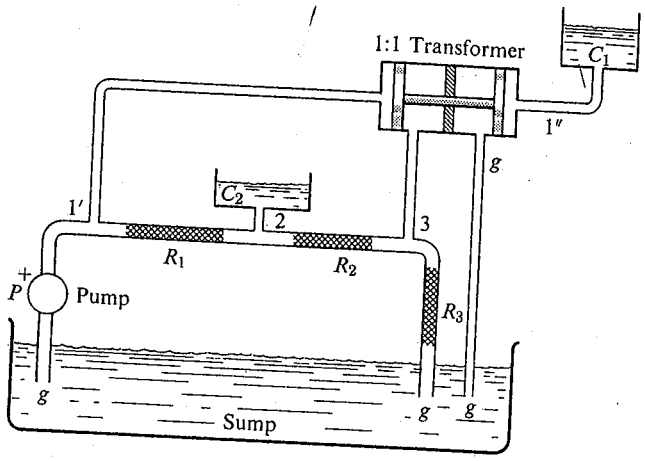
(b) Graph



(c) System graph with added transformer



(d) Mechanical analog



(e) Fluid analog

FIG. 8–43. An electric circuit without a directly physically realizable analog.

inductance) element will not be physically realizable since the analogous thermal element does not exist. Additionally, not all electrical systems will possess physically realizable analogs in the various other systems. In an electrical circuit, all capacitors are not restricted to having one terminal in common, whereas this condition must be met with the capacitance elements in the other systems. Figure 8–43(a) and (b) shows an electrical circuit which does not possess a directly physically realizable mechanical, fluid, or thermal analog.

There is a scheme whereby a mechanical, or fluid, analog of any electrical system may be constructed regardless of the location of capacitors in the electrical system. This involves inserting ideal 1 : 1 transformers in the electrical system to isolate the various capacitors from one another. Then one terminal of each capacitor may be connected to a common point without changing the circuit performance. The ideal electrical transformers are analogous to four-terminal mechanical or fluid transformers. In this way, "floating" or ungrounded masses or fluid capacitances may be realized. Figure 8–43(c), (d), and (e) shows such a realization for the analogs of Fig. 8–43(a). A gyrator connected to a spring can also be used in place of mass, but this is stretching the analogy.

Figure P7.29

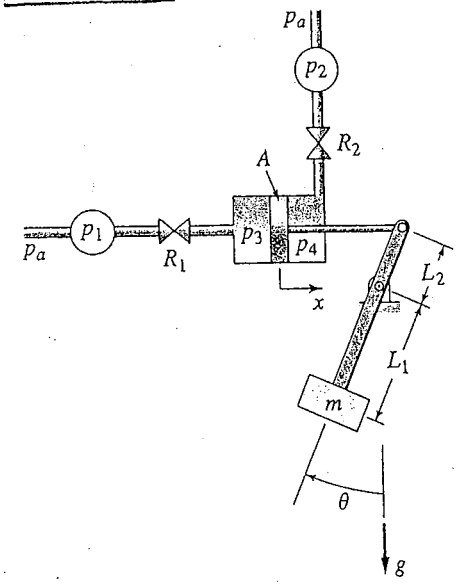


Figure P7.30

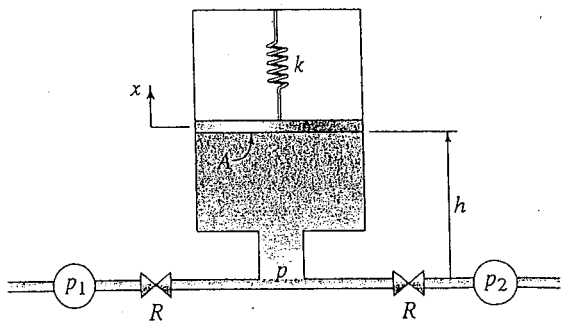
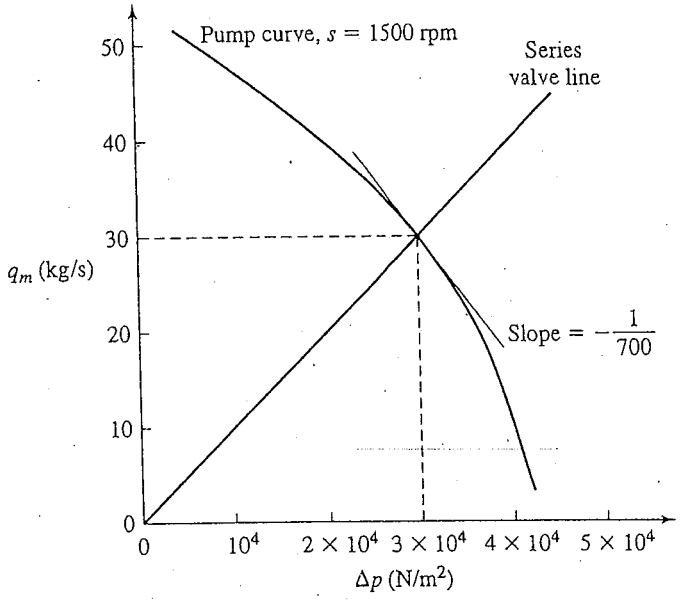


Figure P7.32



7.33 Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.33. The outlet resistance is linear. Derive the dynamic model of the height h.

7.34 Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.34. The outlet is an orifice of area A_o and discharge coefficient C_d. Derive the dynamic model of the height h.

7.35 Consider the cylindrical container treated in Problem 7.8. In Figure P7.35 the tank is shown with a valve outlet at the bottom of the tank. Assume that the flow through the valve is turbulent with a resistance R. Derive the dynamic model of the height h.

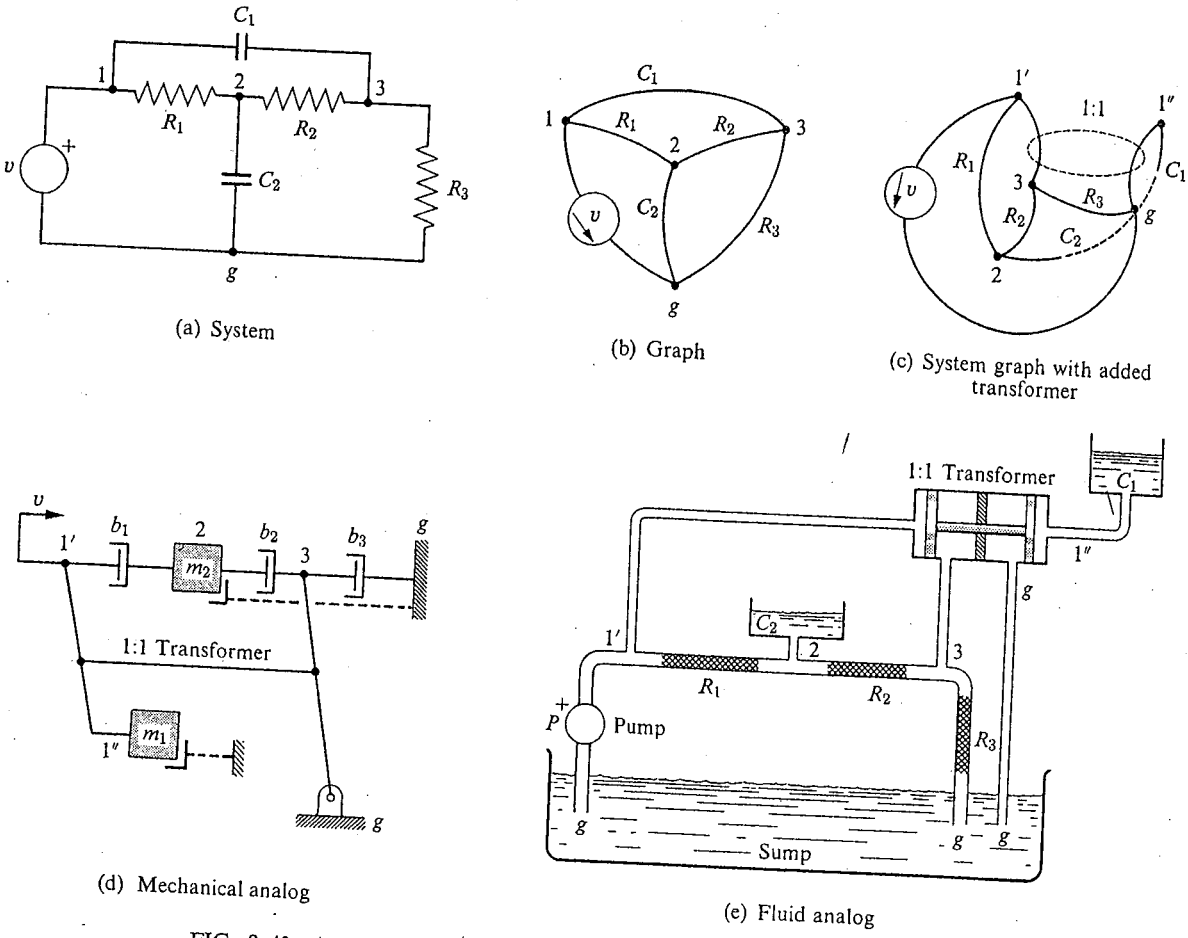


FIG. 8–43. An electric circuit without a directly physically realizable analog.

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state variables [$\frac{d}{dt}$ term of the storage elements]

$$\frac{d}{dt}Q_I = (\quad)P_{3g} + (\quad)Q_I + (\quad)Q + (\quad)P$$

$$\frac{d}{dt}P_{3g} = (\quad)P_{3g} + (\quad)Q_I + (\quad)Q + (\quad)P$$

$$\frac{dQ_I}{dt} = \frac{P_{4g}}{I} = \frac{-P_{24} + P_{23} + P_{3g}}{I} = \frac{P_{3g}}{I} + \frac{-P_{2g}}{I} + \frac{Q_{R3} \cdot R_3}{I}$$

$$= \frac{P_{3g}}{I} \quad \frac{-P_{2g}}{I} + \frac{R_3}{I}(Q_{R_1} - Q_{R_2})$$

$$= \frac{P_{3g}}{I} - \frac{R_3}{I}Q_I - \frac{P_{2g}}{I} + Q_{R_1}\frac{R_3}{I}$$

$$\vdots$$

$$\frac{dQ_I}{dt} = \frac{P_{4g}}{I} = \frac{P - P_{12} + P_{24}}{I} = \frac{P - Q_{R_1}R_1 - Q_{R_2}R_2}{I} = \frac{P}{I} - \frac{(Q_{R_3} + Q_{R_2})R_1}{I} - \frac{Q_I R_2}{I}$$

$$= \frac{P}{I} - \frac{[Q_c - Q]R_1}{I} - \frac{Q_I R_1}{I} - \frac{Q_I R_2}{I}$$

$$\boxed{\frac{dQ_I}{dt} = \frac{P}{I} + \frac{QR_1}{I} - \frac{(R_1+R_2)Q_I}{I} - \frac{R_1}{I} \cdot C\frac{dP_{3g}}{dt}}$$

$$\frac{dP_{3g}}{dt} = \frac{Q_c}{C} = \frac{Q + Q_{R_3}}{C} = \frac{Q}{C} + \frac{(Q_{R_1} - Q_{R_2})}{C} = \frac{Q}{C} + \frac{Q_I}{C} + \frac{P_{12}}{R_1 C} = \frac{Q}{C} - \frac{Q_I}{C} + \frac{1}{R_1 C}[P - P_{24} - P_{4g}]$$

$$+ \frac{1}{R_1 C}P \quad - \frac{1}{R_1 C}Q_{R_2}R_2 - \frac{1}{R_1 C}I\frac{dQ_I}{dt}$$

$$\frac{dP_{3g}}{dt} = \frac{Q}{C} - \frac{Q_I}{C} + \frac{1}{R_1 C}P - \frac{1}{R_1 C}Q_I R_2 - \frac{1}{R_1 C}I\frac{dQ_I}{dt}$$

$$= \frac{Q}{C} - \frac{1}{R_1 C}(R_1 + R_2)Q_I + \frac{1}{R_1 C}P - \frac{I}{R_1 C}\frac{dQ_I}{dt}$$

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Figure P7.29

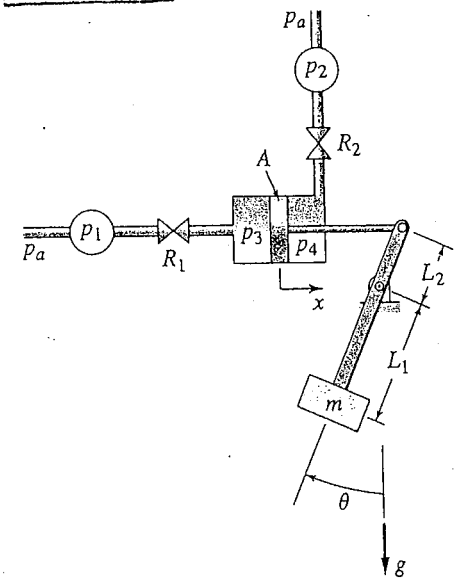


Figure P7.30

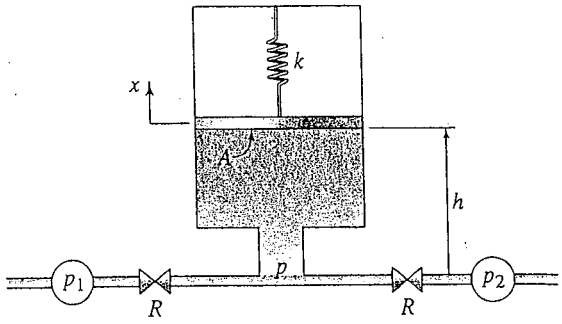
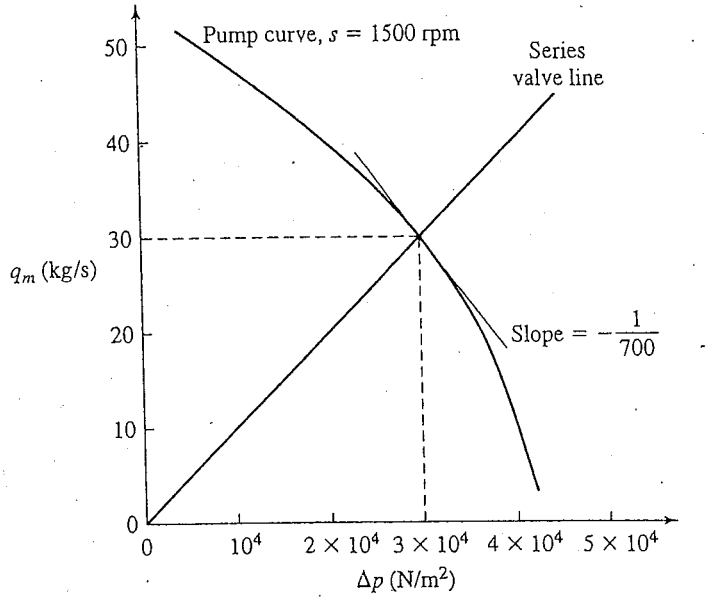


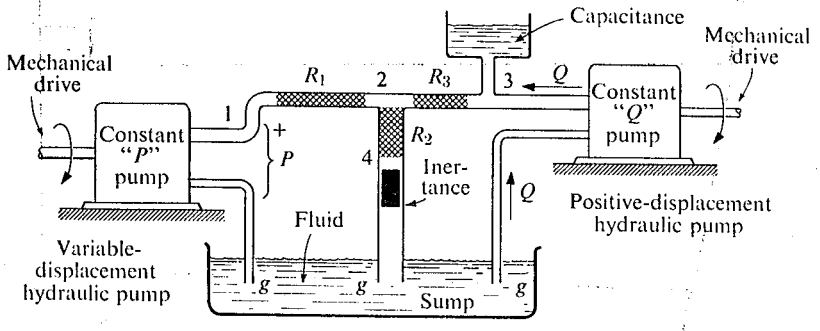
Figure P7.32



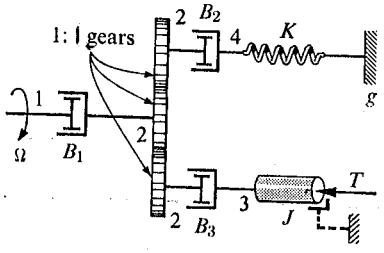
7.33 Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.33. The outlet resistance is linear. Derive the dynamic model of the height h.

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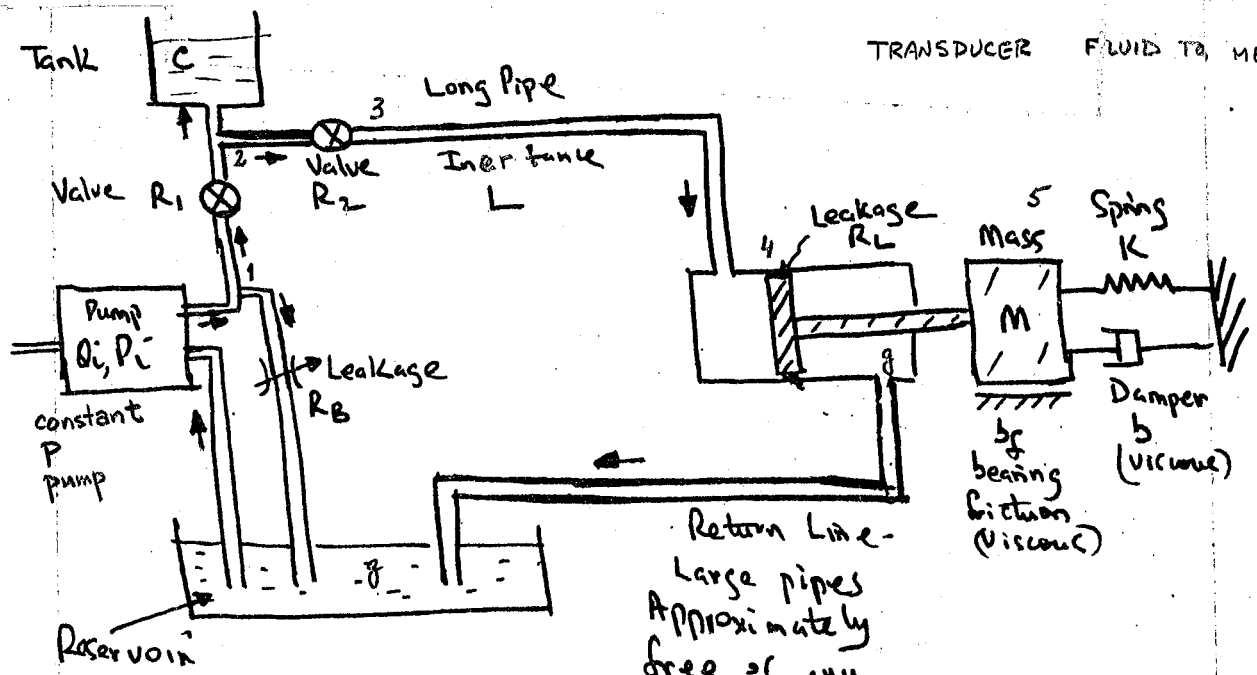
7.35 Consider the cylindrical container treated in Problem 7.8. In Figure P7.35 the tank is shown with a valve outlet at the bottom of the tank. Assume that the flow through the valve is turbulent with a resistance R. Derive the dynamic model of the height h.



TREAT PUMPS LIKE SOURCES



TRANFORMER GEARS



TRANSDUCER FLUID TO MECHANICAL

mentance & capacitance

spring → stores energy by flow. Q_I

mass → stores energy by across var. P_{3g}.

$$\frac{dQ_I}{dt} = P_{3g} \quad \cdots \quad Q_I \quad P \quad Q$$

$$\frac{dP_{3g}}{dt} = \quad P_{3g} \quad \cdots \quad Q_I \quad P \quad Q$$

$$-P + P_{12} + P_{24} + I\frac{dQ_I}{dt} = 0$$

$$-P + Q_{R_1}\cdot R_1 + Q_{R_2}\cdot R_2 +$$

$$-P + R_1(Q_{R_3} + Q_{R_2}) +$$

$$-P + R_1(Q_{R_3} + Q_I) + Q_I \cdot R_2$$

$$Q_{R_3} = Q_C - Q$$

$$-P + P_{12} + P_{23} + P_{3g} = 0 \qquad \Rightarrow P_{4g} + P_{2g} = P_{23} + P_{3g}$$

$$-P + P_{12} + P_{23} + P_{3g} = 0$$

$$Q_{R_1}\cdot R_1 \quad Q_{R_3}\cdot R_3$$

$$-P + (Q_C - Q + Q_I)R_1 + (Q_C - Q)R_3 + P_{3g} = 0$$

$$-\dot{P} + (C\dot{P}_{3g} - \dot{Q} + \dot{Q}_I)R_1 + (C\dot{P}_{3g} - \dot{Q})R_3 + \dot{P}_{3g} = 0$$

$$-\dot{P} - Q(R_1+R_3) + C(R_1+R_3)\dot{P}_{3g} + Q_I R_1 + \dot{P}_{3g} = 0$$

$$\dot{P}_{3g} = -\frac{1}{C(R_1+R_3)}P_{3g} + \frac{Q_I R_1}{C(R_1+R_3)} + \frac{Q}{C} + \frac{\dot{P}}{C(R_1+R_3)}$$

$$-P_{4g} + P \qquad -P + P_{12} + P_{24} + P_{4g} = 0$$

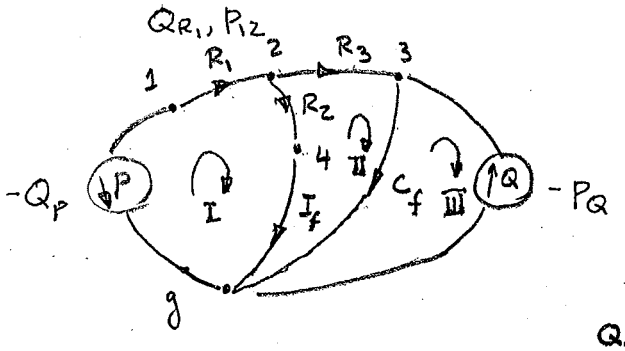
$$-P + Q_{R_1}R_1 + Q_{R_2}R_2 + I\frac{dQ_I}{dt} = 0$$

$$-P + R_1[Q_C - Q + Q_I] + Q_I R_2 + I\dot{Q}_I = 0$$

$$-P + Q_I(R_1+R_2) - QR_1 + R_1 C\dot{P}_{3g}$$

$$+ -\frac{R_1}{R_1+R_3}P_{3g} - Q_I\frac{R_1^2}{(R_1+R_2)_I} + QR_1 + \frac{PR_1}{R_1+R_3} + I\dot{Q}_I$$

$$-\frac{PR_3}{R_1+R_3} + Q_I\left(\frac{2R_2R_1+R_2^2}{R_1+R_2}\right) - \frac{R_1}{R_1+R_3}P_{3g} + I\dot{Q}_I = 0$$



P & Q are inputs

nodes = 5

branches = 7

b-s = 5 elemental eqs.

$$Q_{R_1} = \frac{P_{12}}{R_1} \qquad Q_{R_3} = \frac{P_{23}}{R_3}$$

$$Q_{R_2} = \frac{P_{24}}{R_2} \qquad P_{4g} = I_f \frac{dQ_I}{dt}$$

$$Q_{C_f} = C_f \frac{dP_{3g}}{dt}$$

Continuity

Node eqs $n-1 = 5-1 = 4$

$$+ Q_P = Q_{R_1}$$

$$Q_{R_1} = Q_{R_3} + Q_{R_2}$$

$$Q_{R_2} = Q_{I_f}$$

$$Q_{R_3} + Q = Q_{C_f}$$

Compatibility

$b-(n-1) = 7-5+1 = 3$

$$-P + P_{12} + P_{24} + P_{4g} = 0$$

$$-P_{4g} - P_{2g} + P_{23} + P_{3g} = 0$$

$$-P_{3g} + P_{3g} = 0 = -P_{3g} - (-P_Q)$$

$P_{24} = P_{23} + P_{3g} + P_{4g}$
$= P_{23} + P_{3g}$

$$-Q_P = \frac{P_{12}}{R_1}$$

$$\frac{P_{12}}{R_1} = \frac{P_{23}}{R_3} + \frac{P_{24}}{R_2}$$

$$\frac{dP_{24}/dt}{R_2} = \frac{P_{4g}}{I}$$

$$P = P_{12} + P_{24} + P_{4g}$$

$$P = P_{12} + \frac{I}{R_2}\frac{dP_{24}}{dt}$$

$$= R_1\left(\frac{P_{23}}{R_3} + \frac{P_{24}}{R_2}\right)$$

$$= \frac{R_1}{R_3}P_{23} + \frac{R_1}{R_2}P_{24} + \frac{I}{R_2}\frac{dP_{24}}{dt}$$

$$R_1\left(\frac{P_{23}}{R_3} + \frac{P_{24}}{R_2}\right)$$

$$-P + P_{12} + P_{23} + P_{3g} = 0$$

$$-\frac{dP}{dt} + \frac{dP_{12}}{dt} + \frac{dP_{23}}{dt} + \frac{1}{C}\left[\frac{P_{23}}{R_3} + \right.$$

$$-\frac{dP}{dt} + \frac{Q}{C} + \frac{dP_{23}}{dt} + \frac{P_{23}}{CR_3} + \frac{dP_{12}}{dt}$$

$$-\frac{dP}{dt} + \frac{Q}{C} + \frac{dP_{23}}{dt} + \frac{P_{23}}{CR_3} + \frac{dP_{12}}{dt}$$

$$-P + P_{12} + P_{24} + P_{4g} = 0$$

$$-P_{4g} - P_{2g} + P_{23} + P_{3g} = 0$$

$$-P_{3g} + P_{3g} = 0 = -P_{3g} - (-P_Q)$$

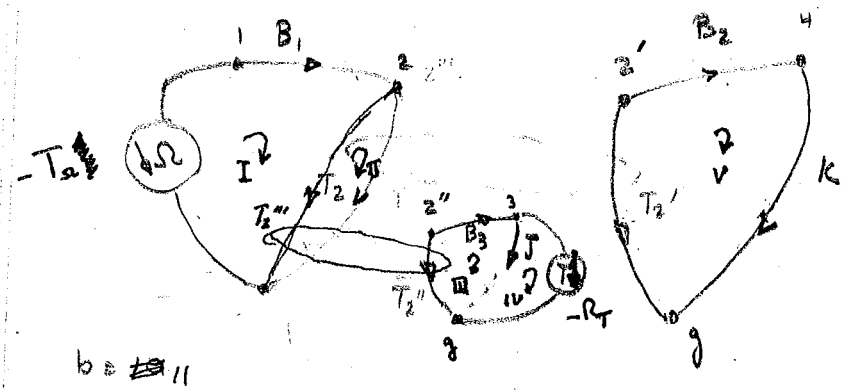
$$-P + P_{12} + P_{23} + P_{3g} = 0$$

$$\frac{dQ_I}{dt} + \frac{Q_I R_2}{I} + \frac{R_3 Q}{I} = \frac{P_{3g}}{I} = + R_3 C\left\{-\frac{1}{C(R_1+R_3)}P_{3g}\right.$$

$$\left. - \frac{Q_I R_1}{C(R_1+R_3)} + \frac{Q}{C} - \frac{P}{C(R_1+R_3)}\right\}$$

$$\frac{dQ_I}{dt} = P_{3g}\left\{\frac{\frac{R_1}{I(R_1+R_3)}}{\frac{1}{I} - \frac{R_2}{I(R_1+R_3)}}\right\} + Q_I\left\{\frac{-R_2}{I} - \frac{R_3 R_1}{I R_1+R_3}\right\}$$

$$+ Q\left\{\frac{R_3}{I} + \frac{R_3}{I}\right\} + \frac{P}{I}\frac{R_3}{R_1+R_3}$$



b = ~~11~~

s = 2

p = 3

b - s = ~~9~~ 7

$T_{B_1} = B_1 \Omega_{12}$

$T_{B_2} = B_2 \Omega_{2'4}$

$\frac{d}{dt} T_K = K \Omega_{4g}$

$T_{B_3} = B_3 \Omega_{2''3}$

$T_J = J \frac{d\Omega_{3g}}{dt}$

4 eqs

$\frac{\Omega_{2g}}{\Omega_{2'g}} = -\frac{T_2'}{T_2} = n_1 = 1$

$\frac{\Omega_{2g}}{\Omega_{2''g}} = -\frac{T_2''}{T_2'''} = n_2 = 1$

b - (n-p) = ~~10~~ - 9 + 3 = ~~5~~ 4

I: $-\Omega + \Omega_{12} + \Omega_{2g} = 0$; $\Omega_{2g} - \Omega_{2g} = 0$ II

III: $-\Omega_{3''g} + \Omega_{2''3} + \Omega_{3g} = 0$

IV $-\Omega_{3g} + (-\Omega_T) = 0$

V $-\Omega_{2'g} + \Omega_{2'4} + \Omega_{4g} = 0$

n - p = 9 - 3 = 6

@1 $T_\Omega = T_{B_1}$

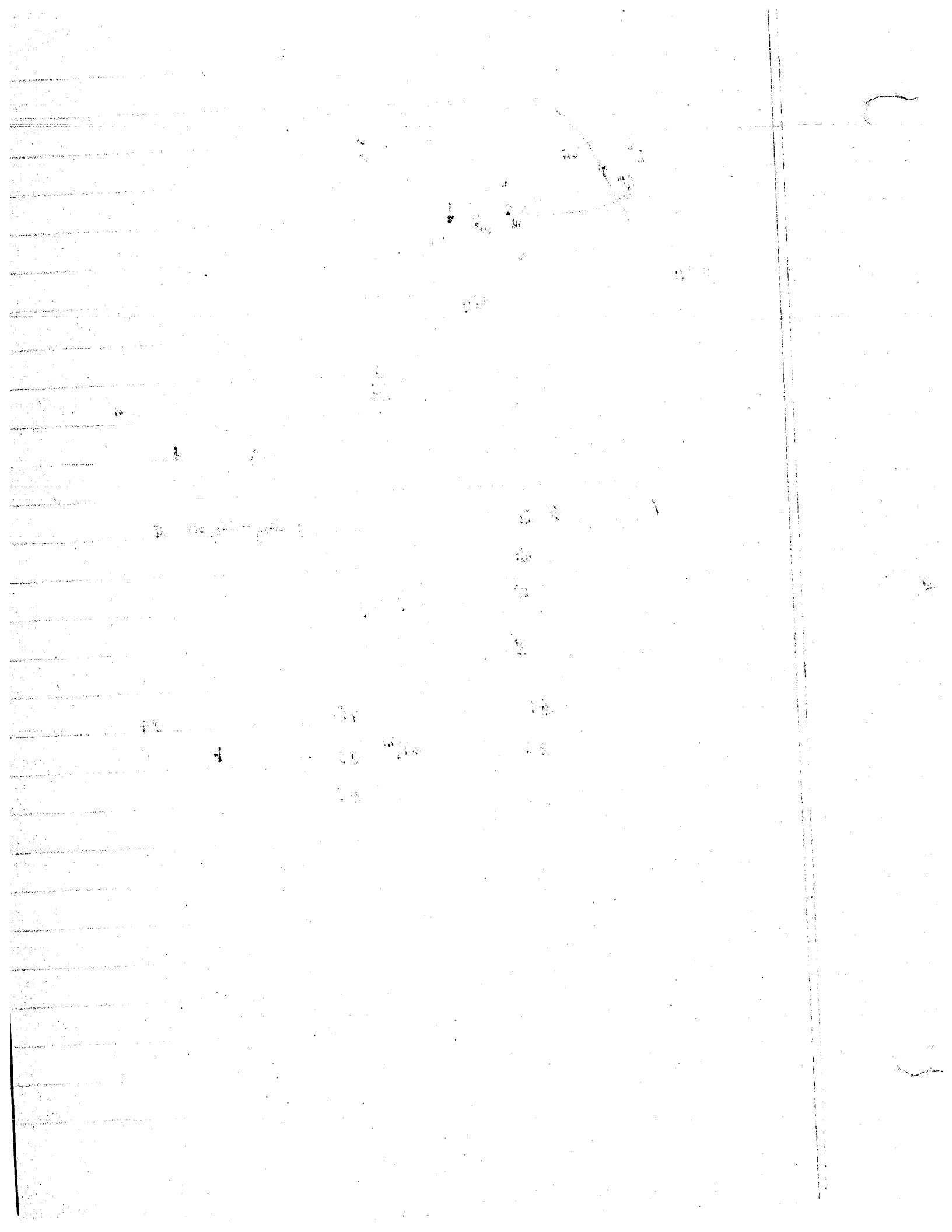
@2 $T_{B_1} = T_2 + T_2'''$

@2'' $T_2'' + T_{B_3} = 0$

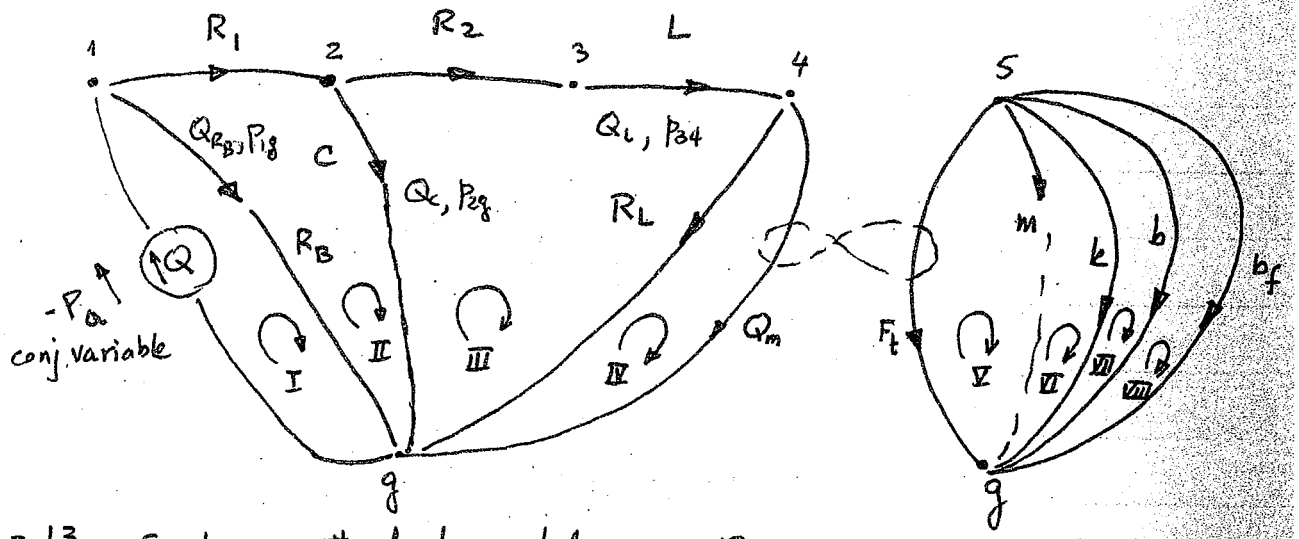
@3 $T_{B_3} = T_J + T$

@2' $T_2' + T_{B_2} = 0$

@4 $T_{B_2} = T_K$



THIS IS THE WORKED OUT SOLUTION FOR THE LAST PROBLEM



$b = 13$ $\quad s = 1$ $\quad$ # of elemental eqns = 12

$$Q_{R_B} = \frac{P_{1g}}{R_B} \qquad Q_{R_1} = \frac{P_{12}}{R_1} \qquad Q_c = C\frac{dP_{2g}}{dt} \qquad Q_{R_2} = \frac{P_{23}}{R_2} \qquad P_{34} = L\frac{dQ_L}{dt}$$

$$Q_{R_L} = \frac{P_{4g}}{R_L} \qquad F_m = m\frac{dV_{5g}}{dt} \qquad \frac{dF_k}{dt} = kV_{5g} \qquad F_b = bV_{5g} \qquad F_{b_f} = b_f V_{5g}$$

Transducer equations: $P_{4g} \cdot A = F_t$

$-Q_m = V_{5g} \cdot A$

$n - p =$ # nodal equations $\quad n = 7$ (remember 2 nodes for g) $\quad p = 2$

$\therefore$ 5 nodal equations

@1: $Q = Q_{R_B} + Q_{R_1}$ $\quad$ @2: $Q_{R_1} = Q_C + Q_{R_2}$ $\quad$ @3: $Q_{R_2} = Q_L$ $\quad$ @4: $Q_L = Q_{R_L} + Q_m$

@5: $F_t + F_m + F_k + F_b + F_{b_f} = 0$

path equations is $b - (n-p) = 13 - 5 = 8$

$\circlearrowright$ I : $-P_a + P_{1g} = 0$

$\circlearrowright$ II : $-P_{1g} + P_{12} + P_{2g} = 0$

$\circlearrowright$ III : $-P_{2g} + P_{23} + P_{34} + P_{4g} = 0$

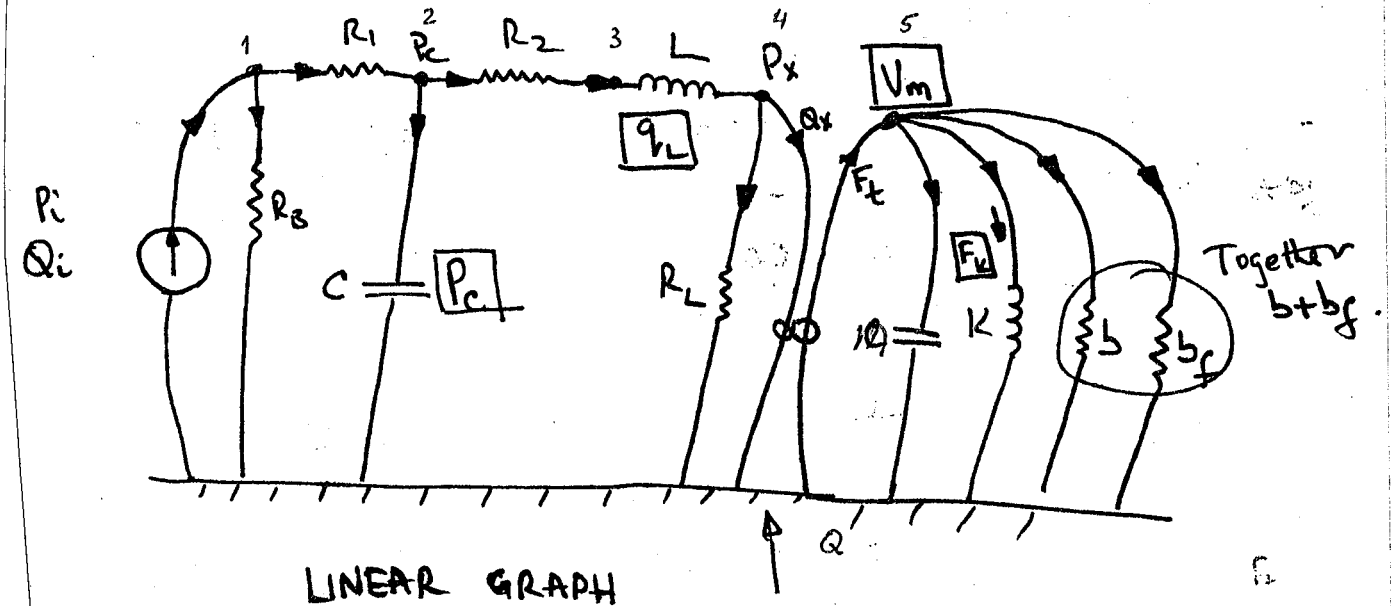
$\circlearrowleft$ IV : $-P_{4g} + P_{4g} = 0$

$\circlearrowright$ V : $-V_{5g} + V_{5g} = 0$

$\circlearrowright$ VI, $\circlearrowright$ VII, $\circlearrowright$ VIII : $-V_{5g} + V_{5g} = 0$

Depending on what is required the differential equation may be as high as 4th order (since 4 storage elements C, L, m, k).

SYSTEM (PHYSICAL)



Note gyrating transducer $Q_x/A = V_m$, i.e. $(q_L - q_{R_L})/A = V_m$

$P_x = P_{R_L} = f_t/A$.

$b = 13 \quad s = 1 \quad b - s = 12$

$n = 7 \quad p = 2 \quad n - p = 5$

$Q_i - Q_{R_1} - Q_{R_B} = 0$

$Q_{R_1} - Q_{R_2} - Q_C = 0$

$Q_{R_2} = Q_L$

$Q_L = Q_{R_L} + Q_x$

$-F_t = F_m + F_K + F_b + F_{b_F}$

$b - (n-p) = 13 - 7 + 2 = 8$

$-P_i + P_{1g} = 0$

$-P_{1g} + P_{12} + P_{2g} = 0$

$-P_{2g} + P_{23} + P_{34} + P_{4g} = 0$

$-P_{4g} + P_{4g} = 0$

$-V_{5g} + V_{5g} = 0$

$-V_{5g} + V_{5g} = 0$

$-V_{5g} + V_{5g} = 0$

$-V_{5g} + V_{5g} = 0$

$Q_{R_B} = \frac{P_{1g}}{R_B}$

$Q_{R_1} = \frac{P_{12}}{R_1}$

$Q_{R_L} = \frac{P_{4g}}{R_L}$

$F_{kin} = m \frac{dV_5}{dt}$

$F_K = k x_{5g}$

$F_b = b V_{5g}$

$Q_{R_2} = \frac{P_{23}}{R_2}$

$P_{3g} = L \frac{dQ_L}{dt}$

$Q_C = C dP_{2g}/dt$

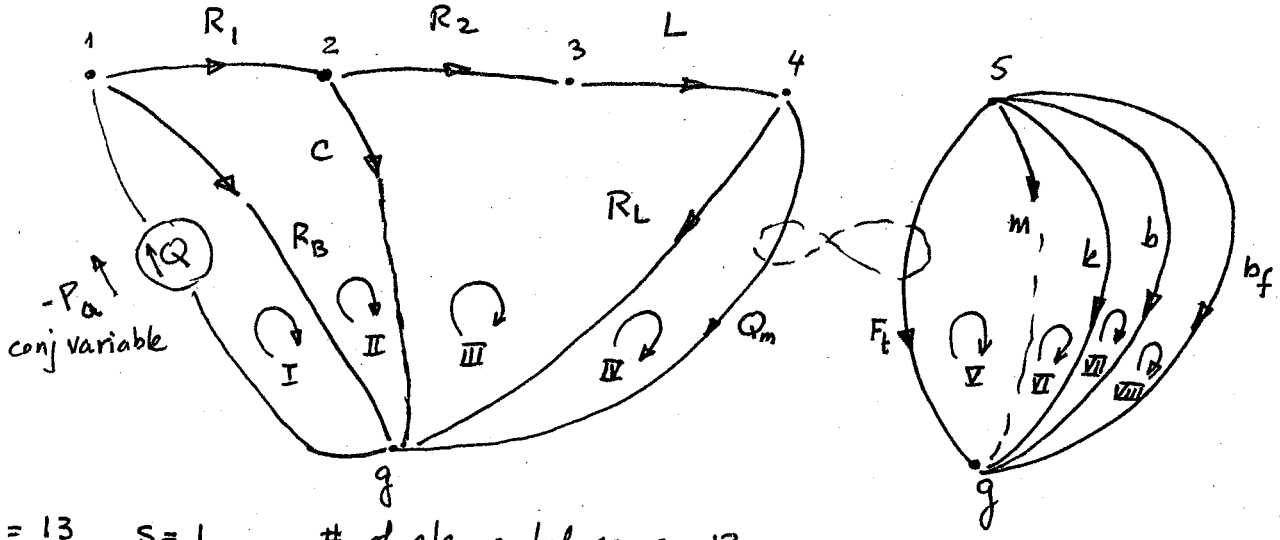
2 eqs due to gyrat'g transducers.

$Q_x/A = V_5$

$P_4 = F_4/A$

$F_{b_f} = b_f V_{5g}$

THIS IS THE WORKED OUT SOLUTION FOR THE LAST PROBLEM OF WED. JULY 5, 1989.



$b = 13 \quad s = 1 \quad$ # of elemental eqns = 12

$$Q_{R_B} = \frac{P_{1g}}{R_B} \qquad Q_{R_1} = \frac{P_{12}}{R_1} \qquad Q_C = C\frac{dP_{2g}}{dt} \qquad Q_{R_2} = \frac{P_{23}}{R_2} \qquad P_{34} = L\frac{dQ_L}{dt}$$

$$Q_{R_L} = \frac{P_{4g}}{R_L} \qquad F_m = m\frac{dV_{5g}}{dt} \qquad \frac{dF_k}{dt} = kV_{5g} \qquad F_b = bV_{5g} \qquad F_{b_f} = b_f V_{5g}$$

Transducer equations: $P_{4g} \cdot A = F_t$

$-Q_m = V_{5g} \cdot A$

$n-p =$ # nodal equations $\qquad n = 7$ (remember 2 nodes for g) $\qquad p = 2$

$\therefore$ 5 nodal equations

$$Q = Q_{R_B} + Q_{R_1} \qquad Q_{R_1} = Q_C + Q_{R_2} \qquad Q_{R_2} = Q_L \qquad Q_L = Q_{R_L} + Q_m$$

$$F_t + F_m + F_k + F_b + F_{b_f} = 0$$

path equations is $b-(n-p) = 13-5 = 8$

$\circlearrowright$ I : $-P_a + P_{1g} = 0$

$\circlearrowright$ II : $-P_{1g} + P_{12} + P_{2g} = 0$

$\circlearrowright$ III : $-P_{2g} + P_{23} + P_{34} + P_{4g} = 0$

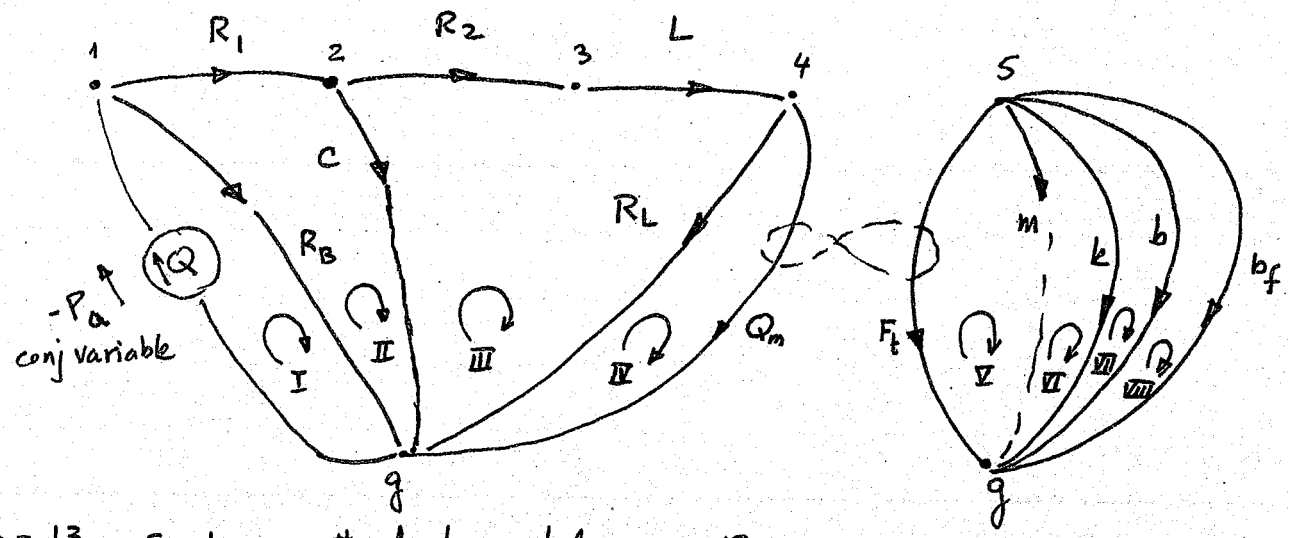
$\circlearrowleft$ IV : $-P_{4g} + P_{4g} = 0$

$\circlearrowright$ V : $-V_{5g} + V_{5g} = 0$

$\circlearrowright$ VI, $\circlearrowright$ VII, $\circlearrowright$ VIII : $-V_{5g} + V_{5g} = 0$

Depending on what is required the differential equation may be as high as 4th order (since 4 storage elements C, L, m, k).

THIS IS THE WORKED OUT SOLUTION FOR THE LAST PROBLEM OF WED. JULY 5, 1989.



$b = 13 \quad s = 1 \quad$ # of elemental eqns = 12

$$Q_{R_B} = \frac{P_{1g}}{R_B} \qquad Q_{R_1} = \frac{P_{12}}{R_1} \qquad Q_C = C\frac{dP_{2g}}{dt} \qquad Q_{R_2} = \frac{P_{23}}{R_2} \qquad P_{34} = L\frac{dQ_L}{dt}$$

$$Q_{R_L} = \frac{P_{4g}}{R_L} \qquad F_m = m\frac{dV_{5g}}{dt} \qquad \frac{dF_k}{dt} = kV_{5g} \qquad F_b = bV_{5g} \qquad F_{b_f} = b_f V_{5g}$$

Transducer equations: $P_{4g} \cdot A = F_t$

$$-Q_m = V_{5g} \cdot A$$

$n-p =$ # nodal equations $\qquad n = 7$ (remember 2 nodes for g) $\qquad p = 2$

$\therefore$ 5 nodal equations

$$Q = Q_{R_B} + Q_{R_1} \qquad Q_{R_1} = Q_C + Q_{R_2} \qquad Q_{R_2} = Q_L \qquad Q_L = Q_{R_L} + Q_m$$

$$F_t + F_m + F_k + F_b + F_{b_f} = 0$$

path equations is $b-(n-p) = 13-5 = 8$

$\circlearrowright$ I : $-P_a + P_{1g} = 0$

$\circlearrowright$ II : $-P_{1g} + P_{12} + P_{2g} = 0$

$\circlearrowright$ III : $-P_{2g} + P_{23} + P_{34} + P_{4g} = 0$

$\circlearrowleft$ IV : $-P_{4g} + P_{4g} = 0$

$\circlearrowright$ V : $-V_{5g} + V_{5g} = 0$

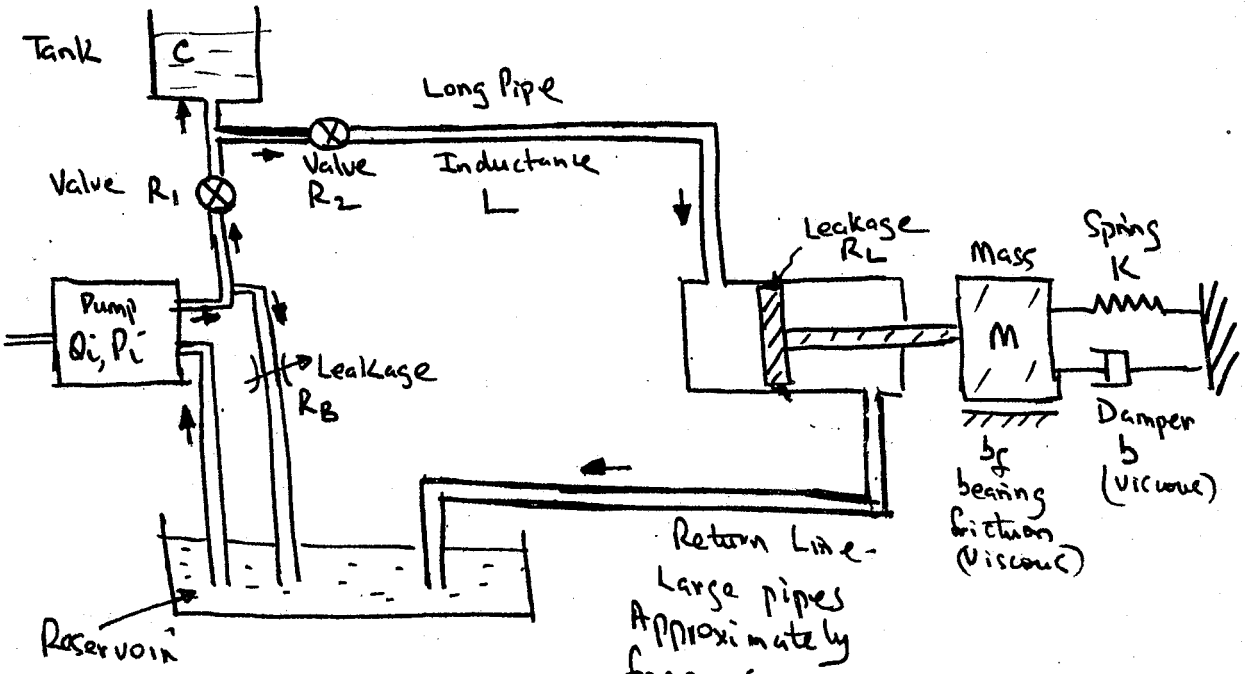
$\circlearrowright$ VI, $\circlearrowright$ VII, $\circlearrowright$ VIII : $-V_{5g} + V_{5g} = 0$

Depending on what is required the differential equation may be as high as 4th order (since 4 storage elements C, L, m, k).

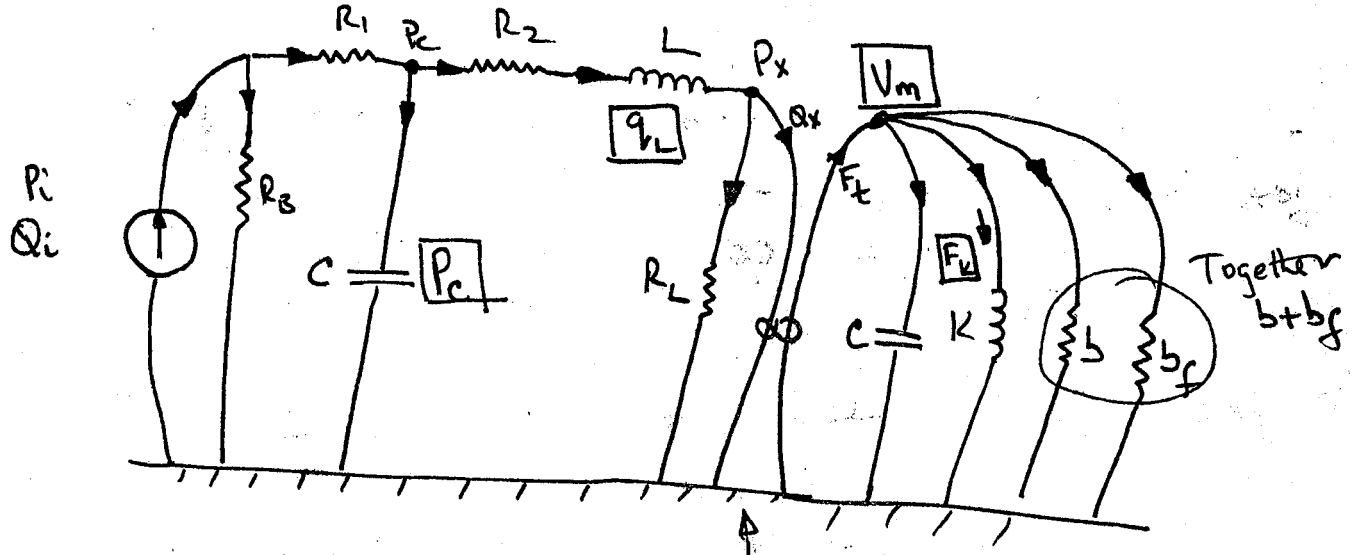
LINEAR GRAPH MODELING

DR. S.O. AJUMU

Consider the system shown below, derive the systems equations using the linear graph technique.



SYSTEM (PHYSICAL)



LINEAR GRAPH

Note gyrating transducer

$Q_x/A = V_m$, i.e. $(q_L - q_{RL})/A = V_m$

$P_x = P_{RL} = f_t/A$.

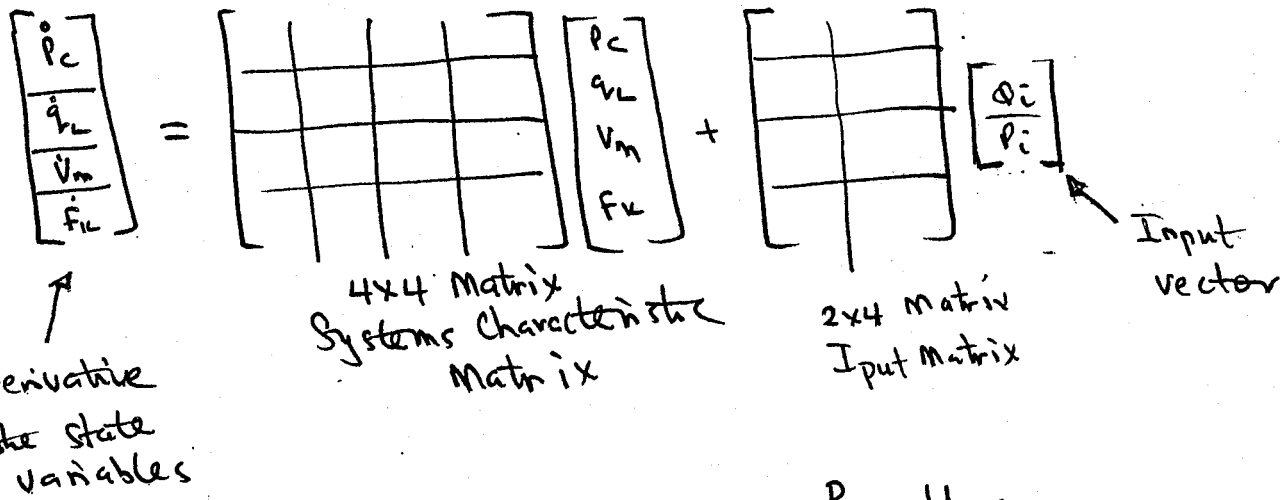
Ind. Four Energy Storage elements : C, L, M, K

I expect four first order differential equations

No. of State variables = no of independent storage elements.

$= P_c, q_L, V_m, F_K$.

I expect equations in linearized form to look like.



$\equiv \quad \dot{X} = AX + BU.$

NOW LET US WRITE THE EQUATIONS FROM THE LINEAR GRAPH.

Eqn. ① Around the capacitance C: $q_c = C\, dP_c/dt \quad \therefore \dot{P}_c = \frac{1}{C}(q_L)$

$q_c = Q_i - q_{R_B} - q_{R_2} = Q_i - \frac{P_i}{R_B} - q_L$

$$\dot{P}_c = \frac{1}{C}\left[Q_i - \frac{P_i}{R_B} - q_L\right] \quad \text{— ①}$$

Eqn. ② Around inductance L. $L = \frac{di}{dt} \quad = \frac{1}{L}$

$L = \frac{1}{L}(\quad)$

$= P_c - P_{R_2} - P_{R_L} = P_c - R_2 q_L - R_L(q_L - Q_x)$

$= P_c - R_2 q_L - R_L(q_L - AV_m)$

$= \frac{1}{L}[P_c - R_2 q_L - R_L(q_L - AV_m)$

$$\dot{i}_L = \frac{1}{L}\left[P_c - (R_2 + R_L) q_L + R_L A V_m\right] \quad \text{—— (2)}$$

Equation 3 around M, $F_m = m\frac{dV_m}{dt} \therefore \dot{V}_m = \frac{1}{m} F_m$

$$F_m = F_t - F_K - b_{(b+b_L)} = F_t - F_K - (b+b_L) V_m$$

$$f_t = A P_{RL} = A R_L (q_L - Q_x) = A R_L (q_L - A V_m)$$

$$F_m = A R_L (q_L - A V_m) - F_K - (b+b_L) V_m$$

$$\dot{V}_m = \frac{1}{m}\left[A R_L (q_L - A V_m) - F_K - (b+b_L) V_m\right]$$

$$\dot{V}_m = \frac{1}{m}\left[A R_L q_L - F_K - (b + b_L + A^2 R_L) V_m\right] \quad \text{—— (3)}$$

Equation 4 around K, $V_K = \frac{1}{K} df_K/dt \quad \therefore \dot{f}_K = K V_K$

$V_K = V_m$

$$\dot{F}_K = K V_m \quad \text{—— (4)}$$

$$\begin{bmatrix} \dot{P}_c \\ \dot{q}_L \\ \dot{V}_m \\ \dot{F}_K \end{bmatrix} = \left[\begin{array}{c|c|c|c} 0 & -1/C & 0 & 0 \\ \hline 1/L & -\frac{(R_2+R_L)}{L} & \frac{R_L A}{L} & 0 \\ \hline 0 & \frac{A R_L}{m} & -\frac{(b+b_L+A^2R_L)}{m} & -\frac{1}{m} \\ \hline 0 & 0 & K & 0 \end{array}\right] \begin{bmatrix} P_c \\ q_L \\ V_m \\ F_K \end{bmatrix} + \left[\begin{array}{c|c} 1/C & -\frac{1}{R_1 C} \\ \hline 0 & 0 \\ \hline 0 & 0 \\ \hline 0 & 0 \end{array}\right] \begin{bmatrix} Q_i \\ P_i \end{bmatrix}$$

Figure P7.29

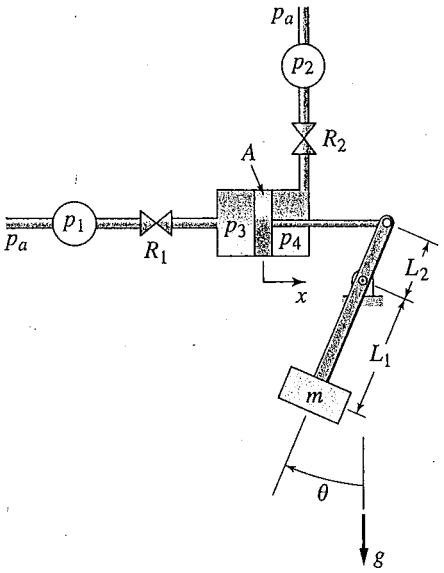


Figure P7.30

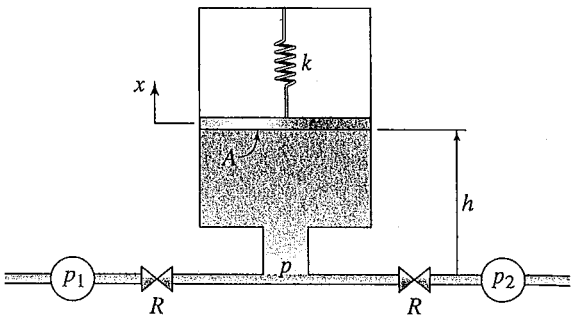
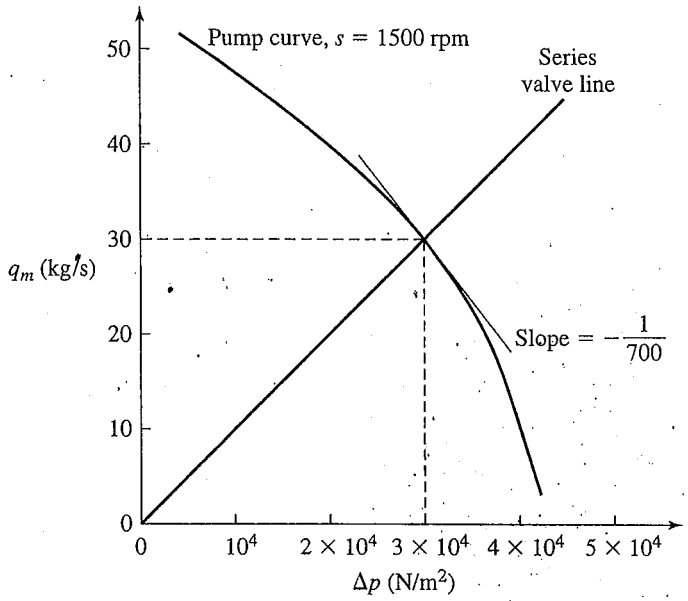


Figure P7.32



7.33 Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.33. The outlet resistance is linear. Derive the dynamic model of the height h.

7.34 Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.34. The outlet is an orifice of area A_o and discharge coefficient C_d. Derive the dynamic model of the height h.

7.35 Consider the cylindrical container treated in Problem 7.8. In Figure P7.35 the tank is shown with a valve outlet at the bottom of the tank. Assume that the flow through the valve is turbulent with a resistance R. Derive the dynamic model of the height h.

Figure P7.29

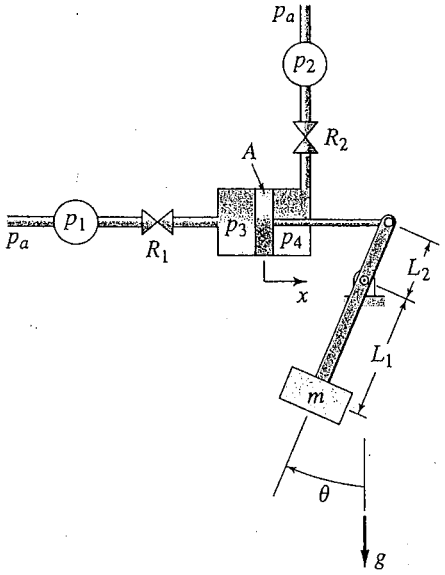
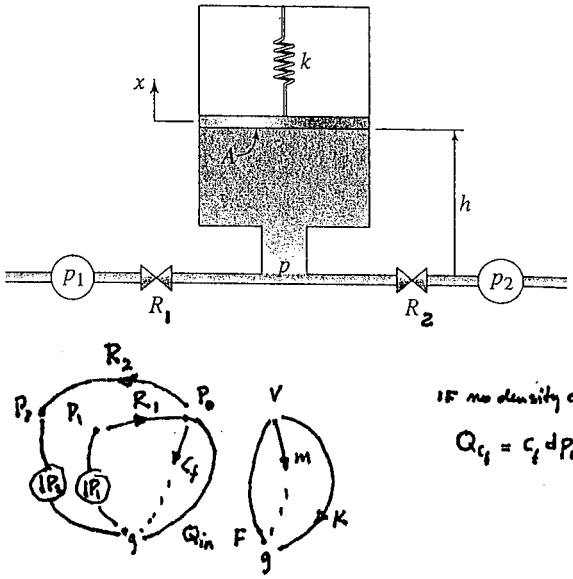


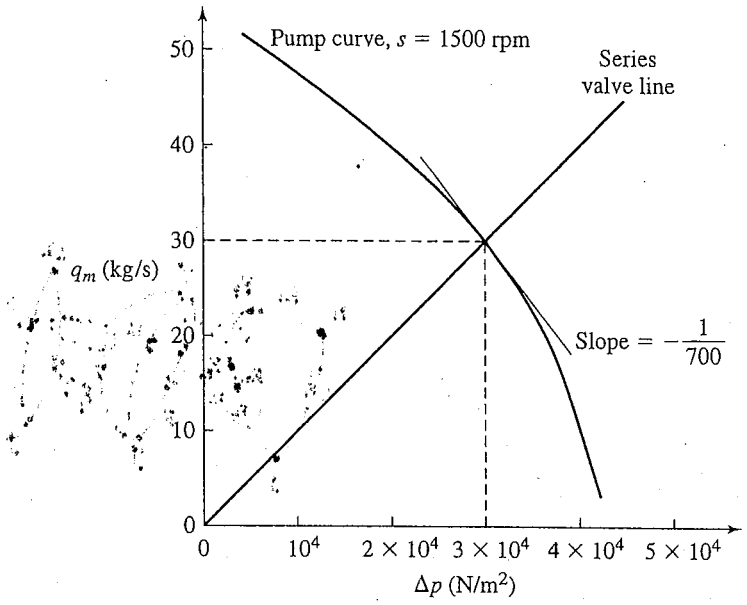
Figure P7.30



IF no density ch.

$Q_{c_f} = c_f \, dP_{og}$

Figure P7.32



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7.35 Consider the cylindrical container treated in Problem 7.8. In Figure P7.35 the tank is shown with a valve outlet at the bottom of the tank. Assume that the flow through the valve is turbulent with a resistance R. Derive the dynamic model of the height h.

7.27 An electric motor is sometimes used to move the spool valve of a hydraulic motor. In Figure P7.27 the force f is due to an electric motor acting through a rack-and-pinion gear. Develop a model of the system with the load displacement y as the output and the force f as the input. Consider two cases: (a) $m_1 = 0$ and (b) $m_1 \neq 0$.

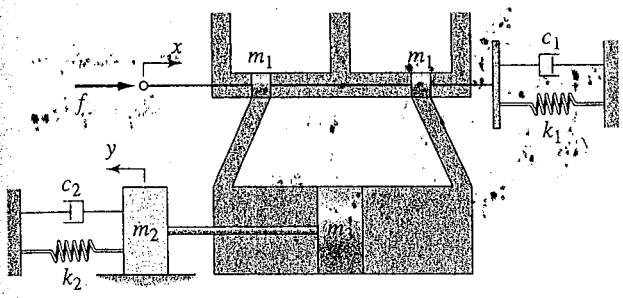


Figure P7.27

7.28 In Figure P7.28 the piston of area A is connected to the axle of the cylinder of radius R, mass m, and inertia I about its center. Develop a dynamic model of the axle's translation x, with the pressures p_1 and p_2 as the inputs.

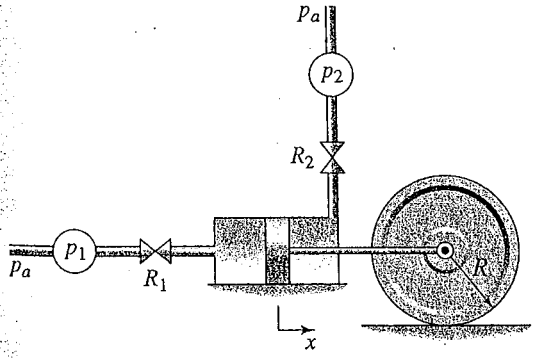
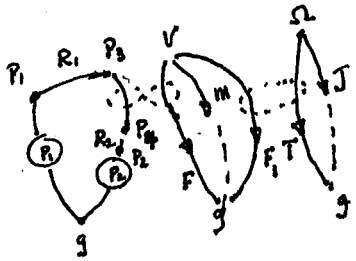


Figure P7.28



$A p_{34} = F$
$-Q = Av$
$\Omega R = v$
$F_1 R = T$

7.29 Figure P7.29 shows a pendulum driven by a hydraulic piston. Assuming small angles θ and a concentrated mass m a distance L_1 from the pivot, derive the equation of motion with the pressures p_1 and p_2 as inputs.

7.30 Figure P7.30 shows an example of a *hydraulic accumulator,* which is a device for reducing pressure fluctuations in a hydraulic line or pipe. The fluid density is ρ, the plate mass is m, and the plate area is A. Develop a dynamic model of the pressure p with the pressures p_1 and p_2 as the given inputs. Assume that $m\ddot{x}$ of the plate is small, and that the hydrostatic pressure ρgh is small.

7.31 Design a hydraulic accumulator of the type shown in Figure P7.30. The liquid volume in the accumulator should increase by 30 in.^{3} when the pressure p increases by 1.5 lb/in.^{2}. Determine suitable values for the plate area A and the spring constant k.

7.32 Consider the liquid-level system treated in Example 7.4.8 and shown in Figure 7.4.9. The pump curve and the line for the steady-state flow through both valves are shown in Figure P7.32. It is known that the bottom area of the tank is 2 m^2 and the outlet resistance is $R_2 = 400$ 1/(m · s). (a) Compute the pump resistance R_1 and the steady-state height. (b) Derive a linearized dynamic model of the height deviation δh in the tank.

In some cases they are also useful in visualizing how to combine the system equations to eliminate unwanted variables; and they can be helpful in checking the accuracy of terms in a set of state-variable equations, including the discovery of terms that "accidentally disappear" during the complex algebraic manipulations often needed to achieve desired input-output relationships.

Problems

3.1 **(a)** Draw the simulation block diagram for the spring-damper system shown in Figure 2.9, using separate blocks for the system parameters b and k, with x_2 and F_i as system inputs.
(b) Rearrange the diagram so that b and k are shown together as a quotient in the feedback block.

3.2 Draw a simulation block diagram for the mass-damper system shown in Figure 2.10, with F_i as the input. Compare this diagram with those drawn for Problem 3.1.

3.3 **(a)** Draw the simulation block diagram for the mass-spring-damper system shown in Figure 2.11, showing the parameters m, b, and k in separate blocks and showing F_i as the input.
(b) Rearrange the diagram so that the outer feedback block has unity gain and by adding a coefficient block after the input F_i.
(c) Compare your diagrams for this system with your diagram for Problem 3.3.

3.4 **(a)** Draw a simulation block diagram for the nonlinear mechanical system shown in Figure 2.12, using a block labelled "$f_{NL}(v_{3g})$" to obtain the damper force F_{NLD} from the velocity v_{3g}.
(b) Prepare a similar diagram for the set of linearized equations developed in Chapter 2 for this system.

3.5 **(a)** Draw a simulation block diagram for the nonlinear ship propulsion system modeled in Figure 2.16, employing a block labelled "$f_{NL}(T_K, \Omega_{1g})$" to obtain the shaft speed Ω_{2g} from the drive shaft torque T_K and the engine speed Ω_{1g}.
(b) Prepare a similar diagram for the set of linearized equations developed in Chapter 2 for this system.
(c) Show how you would implement "$f_{NL}(T_K, \Omega_{1g})$," used in part (a), with a coefficient block, a summer, a multiplier block, and a signed square-root block. (An SSR block provides an output that has the same sign as its input and a magnitude that is the square root of the magnitude of the input.)

4

Mathematical Models

Dynamic Modeling & Control of Engineering Systems
Shearer & Kulakowski
Macmillan Publ. 1990

4.1 Introduction

In almost all areas of engineering, and certainly in all those areas where new processes or devices are being developed, considerable efforts are directed towards acquiring information on various aspects of system performance. This process is generally referred to as a system analysis. Traditionally, the system analysis was carried out by investigating the performance of an existing physical object subjected to selected test input signals. Although there is no doubt that such an experimental approach provides extremely valuable and most reliable information about the system characteristics, experimenting with an actual full-scale system is not always feasible and is very often practically impossible, especially in the early stages of the system analysis. There are several reasons: First, an actual system must be available for testing or a new test object must be constructed, which may involve high cost in terms of time and money. Second, the extent to which the parameters of an existing engineering system can be varied in order to observe their effects on system performance is usually very limited. Third, the experimental results are always "object specific," since they represent only a particular system under investigation, and may be difficult to generalize.

In order to overcome these problems, researchers develop simplified representations of actual systems called system models. The system models may be physical in nature (down-scaled actual systems), or they may be developed in a form of abstract descriptions of the relationships existing among the system variables. In the latter case a dynamic system can be described using verbal text, plots and graphs, tables of relevant numerical data, or mathematical equations. The language of mathematics is preferred in modeling engineering systems because of its superior precision and generality of expression. Mathematical methods of system modeling lead to better ability to generalize the results and to apply them to solving control

and design problems. A representation of the relationships existing among the system variables in a form of mathematical equations is a mathematical model of the system.

Two types of mathematical models are introduced in this chapter, input-output models and state models. Both types of models carry essentially the same information about the system dynamics, but the sets of model differential equations are different from each other in several respects. These differences carry serious implications on practical usefulness and applicability of the types of models in various engineering problems.

4.2 Input-Output Models

Consider the single-input single-output dynamic system shown in Figure 4.1. In general, the relationship between the input and output signals of the system can be represented by a nth-order differential equation of the following form

$$a_n\left(\frac{d^n y}{dt^n}\right) + a_{n-1}\left(\frac{d^{n-1}y}{dt^{n-1}}\right) + \cdots + a_1\left(\frac{dy}{dt}\right) + a_0 y = f\left(\frac{d^m u}{dt^m}, \frac{d^{m-1}u}{dt^{m-1}}, \ldots, \frac{du}{dt}, u, t\right) \quad (4.1)$$

where $m \leq n$ for existing and realizable engineering systems because of inherent inertia of those systems. Also, having $m > n$ is physically impossible because it would imply the ability to "predict the future" of the system input. A set of n initial conditions, $y(0^+), \dot{y}(0^+), \ldots, y^{(n-1)}(0^+)$, must be known in order to solve the equation for a given input $u(t)$, $t \geq 0$.

The model parameters, $a_0, a_1, \ldots, a_n$, may be functions of y, u, and/or t, in which case the system is nonlinear. If the system is stationary, the model parameters are constant and the model differential equation can then be written as

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = f\left(\frac{d^m u}{dt^m}, \frac{d^{m-1}u}{dt^{m-1}}, \ldots, \frac{du}{dt}, u\right) \quad (4.2)$$

where f may be a nonlinear function.

For a stationary linear model, the function f on the right hand side of Equation (4.2) is a sum of terms linear with respect to u and its derivatives.

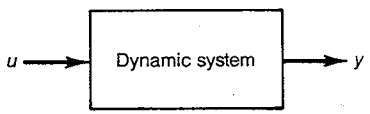


FIGURE 4.1 Single-input single-output dynamic system.

$$a_n \frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_1\frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_1\frac{du}{dt} + b_0 u \quad (4.3)$$

where $b_0, b_1, \ldots, b_m$ are constants. As before, the order of the highest derivative of the input variable cannot be greater than the order of the highest derivative of the output variable, $m \leq n$.

Equations (4.1), (4.2), and (4.3) represent general forms of input-output models for single-input single-output dynamic systems.

EXAMPLE 4.1

Derive input-output equations for the mechanical system shown in Figure 4.2 using force $F_1(t)$ as the input variable and displacements $x_1(t)$ and $x_2(t)$ as the output variables. The symbols r_1 and r_2 represent spring relaxed positions in the gravity field.

Solution

The equation of motion for mass m_1 is

$$m_1\ddot{x}_1 + b_1\dot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = F_1(t)$$

The equation of motion for mass m_2 is

$$m_2\ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$

Combining the equations for the two masses and eliminating x_1 yields the input-output equation for the system.

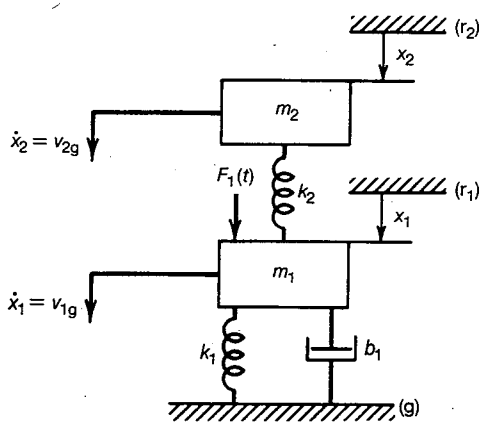


FIGURE 4.2 Mechanical system considered in Example 4.1.

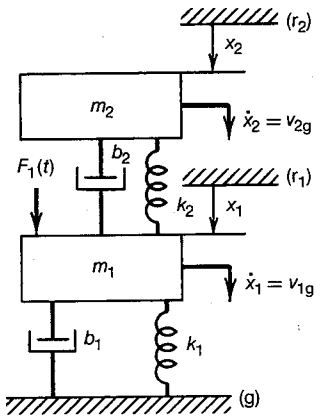


FIGURE 4.3 Mechanical system considered in Example 4.2.

$$\frac{d^4x_2}{dt^4} + \left(\frac{b_1}{m_1}\right)\frac{d^3x_2}{dt^3} + \left(\frac{k_2}{m_2} + \frac{k_1}{m_1} + \frac{k_2}{m_1}\right)\frac{d^2x_2}{dt^2} + \left(\frac{b_1k_2}{m_1m_2}\right)\frac{dx_2}{dt} + \left(\frac{k_1k_2}{m_1m_2}\right)x_2 = \left(\frac{k_2}{m_1m_2}\right)F_1(t)$$

The preceding fourth-order differential equation can be solved provided the initial conditions, $x_2(0)$, $(dx_2/dt)|_{t=0}$, $(d^2x_2/dt^2)|_{t=0}$, $(d^3x_2/dt^3)|_{t=0}$, and the input variable, $F_1(t)$ for $t \geq 0$, are known.

Similarly, the input-output equation relating x_1 to $F_1(t)$ is

$$\frac{d^4x_1}{dt^4} + \left(\frac{b_1}{m_1}\right)\frac{d^3x_1}{dt^3} + \left(\frac{k_2}{m_2} + \frac{k_1}{m_1} + \frac{k_2}{m_1}\right)\frac{d^2x_1}{dt^2} + \left(\frac{b_1k_2}{m_1m_2}\right)\frac{dx_1}{dt} + \left(\frac{k_1k_2}{m_1m_2}\right)x_1 = \left(\frac{1}{m_1}\right)\frac{d^2F_1}{dt^2} + \left(\frac{k_2}{m_1m_2}\right)F_1(t)$$

■

The process of deriving the input-output equation in Example 4.1 will become considerably more complicated if an additional damper b_2 is included between the two masses, as shown in Figure 4.3.

EXAMPLE 4.2

Derive an input-output equation for the mechanical system shown in Figure 4.3, using x_2 as the output variable.

Solution

The equations of motion for masses m_1 and m_2 now take the form

$$m_1\frac{d^2x_1}{dt^2} + (b_1 + b_2)\frac{dx_1}{dt} + (k_1 + k_2)x_1 - b_2\frac{dx_2}{dt} - k_2x_2 = F_1(t)$$

$$m_2\frac{d^2x_2}{dt^2} + b_2\frac{dx_2}{dt} + k_2x_2 - b_2\frac{dx_1}{dt} - k_2x_1 = 0$$

The unwanted variable x_1 cannot be eliminated from these equations using simple substitutions as in Example 4.1 because the derivatives of both x_1 and x_2 are present in each equation. In such case an operator D can be introduced, defined as

$$D^kx(t) = \frac{d^kx(t)}{dt^k}$$

Using the D operator, the differential equations of motion can be rearranged into the following form

$$m_1D^2x_1 + (b_1 + b_2)Dx_1 + (k_1 + k_2)x_1 - b_2Dx_2 - k_2x_2 = F_1(t)$$

$$m_2D^2x_2 + b_2Dx_2 + k_2x_2 - b_2Dx_1 - k_2x_1 = 0$$

From the last equation, x_1 can be expressed as

$$x_1 = \left(\frac{m_2D^2 + b_2D + k_2}{b_2D + k_2}\right)x_2$$

Substituting into the operator equation for mass m_1 yields

$$m_1m_2D^4x_2 + (m_2b_1 + m_2b_2 + m_1b_2)D^3x_2 + (m_1k_2 + m_2k_1 + m_2k_2 + b_1b_2)D^2x_2 + (b_1k_2 + b_2k_1)Dx_2 + k_1k_2x_2 = b_2DF_1 + k_2F_1$$

Using the inverse of the definition of the D operator to transform this equation back to the time domain gives the input-output equation for the system.

$$(m_1m_2)\frac{d^4x_2}{dt^4} + (m_2b_1 + m_2b_2 + m_1b_2)\frac{d^3x_2}{dt^3} + (m_1k_2 + m_2k_1 + m_2k_2 + b_1b_2)\frac{d^2x_2}{dt^2} + (b_1k_2 + b_2k_1)\left(\frac{dx_2}{dt}\right) + k_1k_2x_2 = b_2\frac{dF_1}{dt} + k_2F_1$$

■

Example 4.2 demonstrates that by inserting a damper between the two masses the process of deriving the input-output equation becomes considerably more complicated. In the next example a two-input, two-output system will be considered. Two separate input-output equations, one for each output variable, will have to be derived.

EXAMPLE 4.3

Consider again the mechanical system shown in Figure 4.2. The system is now subjected to two external forces, $F_1(t)$ and $F_2(t)$. The displacements of both

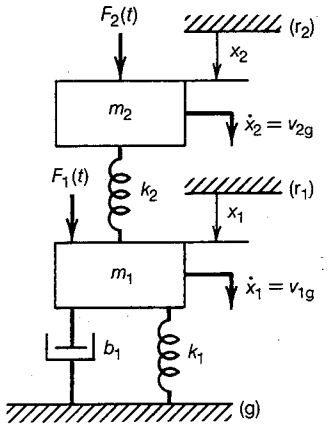


FIGURE 4.4 Two-input two-output system considered in Example 4.3.

masses are of interest and therefore $x_1(t)$ and $x_2(t)$ will be the two output variables of this system. The system is shown in Figure 4.4. The pair of differential equations of motion for the two masses are

$$m_1 \frac{d^2x_1}{dt^2} + b_1 \frac{dx_1}{dt} + (k_1 + k_2)x_1 - k_2x_2 = F_1$$

$$m_2 \frac{d^2x_2}{dt^2} + k_2x_2 - k_2x_1 = F_2$$

By combining the above two equations the separate input-output equations for $x_1(t)$ and $x_2(t)$ are obtained

$$m_1m_2 \frac{d^4x_1}{dt^4} + m_2b_1 \frac{d^3x_1}{dt^3} + (m_1k_2 + m_2k_1 + m_2k_2) \frac{d^2x_1}{dt^2} + b_1k_2 \frac{dx_1}{dt} + k_1k_2x_1 = m_2 \frac{d^2F_1}{dt^2} + k_2F_1 + k_2F_2$$

and

$$m_1m_2 \frac{d^4x_2}{dt^4} + m_2b_1 \frac{d^3x_2}{dt^3} + (m_1k_2 + m_2k_1 + m_2k_2) \frac{d^2x_2}{dt^2} + b_1k_2 \frac{dx_2}{dt} + k_1k_2x_2 = k_2F_1 + m_1 \frac{d^2F_2}{dt^2} + b_1 \frac{dF_2}{dt} + (k_1 + k_2)F_2$$

Note that the two input-output equations are independent of each other and can be solved separately. *On the other hand, the coefficients of each of the terms on the left-hand sides are the same, regardless of which system variable is chosen as the output.*

■

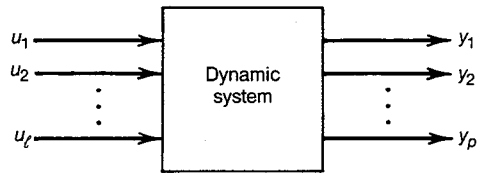


FIGURE 4.5 Multi-input multi-output system.

In general, a linear system with l inputs and p outputs shown schematically in Figure 4.5 is described by p independent input-output equations given below.

$$\begin{aligned}
a_{1n}y_1^{(n)} + \cdots + a_{11}\dot{y}_1 + a_{10}y_1 &= f_1(u_1^{(m)}, \ldots, \dot{u}_1, u_1, u_2^{(m)}, \ldots, \dot{u}_2, u_2, \ldots, u_l^{(m)}, \ldots, \dot{u}_l, u_l, t) \\
a_{2n}y_2^{(n)} + \cdots + a_{21}\dot{y}_2 + a_{20}y_2 &= f_2(u_1^{(m)}, \ldots, \dot{u}_1, u_1, u_2^{(m)}, \ldots, \dot{u}_2, u_2, \ldots, u_l^{(m)}, \ldots, \dot{u}_l, u_l, t) \\
&\vdots \\
a_{pn}y_p^{(n)} + \cdots + a_{p1}\dot{y}_p + a_{p0}y_p &= f_p(u_1^{(m)}, \ldots, \dot{u}_1, u_1, u_2^{(m)}, \ldots, \dot{u}_2, u_2, \ldots, u_l^{(m)}, \ldots, \dot{u}_l, u_l, t)
\end{aligned} \tag{4.4}$$

where $m \leq n$ and a superscript enclosed in parentheses denotes the order of a derivative. If a system is assumed to be stationary, time t does not appear explicitly on the right hand side of Equation (4.4).

The functions $f_1, f_2, \ldots, f_p$ are linear combinations of terms involving the system inputs and their derivatives. The input-output model for a linear, stationary, multi-input multi-output system can be presented in a more compact form

$$\begin{aligned}
\sum_{i=0}^{n} a_{1i}y_1^{(i)} &= \sum_{j=1}^{l}\sum_{k=0}^{m} b_{1jk}u_j^{(m)} \\
\sum_{i=0}^{n} a_{2i}y_2^{(i)} &= \sum_{j=1}^{l}\sum_{k=0}^{m} b_{2jk}u_j^{(m)} \\
&\vdots \\
\sum_{i=0}^{n} a_{pi}y_p^{(i)} &= \sum_{j=1}^{l}\sum_{k=0}^{m} b_{pjk}u_j^{(m)}
\end{aligned} \tag{4.5}$$

Note that some of the a and b coefficients may be equal to zero; also that $a_{1i} = a_{2i} = \cdots a_{pi}$, $i = 1, 2, \ldots, n$.

The input-output equations, even for relatively simple multi-input multi-output models become extremely complicated. Moreover, as will be shown in Chapter 5, analytical methods for solving input-output equations are practical only for low-order, single-input single-output models. In fact, most numerical methods for solv-

ing high-order differential equations require that those equations be replaced by an equivalent set of first-order equations. Also, many quite powerful concepts and methodologies based on input-output models, such as the transfer function, are applicable to linear, stationary models only. The conceptual simplicity of using the input-output representation of a dynamic system is lost in the complexity of the mathematical forms with models that are nonlinear, have many inputs and/or outputs, or simply are of higher order than three. Moreover, it is not even possible to obtain an input-output differential equation for most nonlinear systems since the presence of nonlinearities inhibits combination of model equations to eliminate unwanted variables.

4.3 State Models

A concept of a state of a dynamic system was promoted in the 1950s. Its significance has grown since then, and today the state approach to modeling is the most powerful and dominant technique used in analysis of engineering systems.

Several basic terms associated with state models will first be defined.

- *State* of a dynamic system is defined by the smallest set of variables such that the knowledge of these variables at time $t = t_0$ together with the knowledge of the input for $t > t_0$ completely determines the behavior of the system for time $t \geq t_0$.
- *State variables*, $q_1, q_2, \ldots, q_n$, are the elements of the smallest set of variables required to completely describe the state of the system. One important implication of this definition is that state variables are independent of each other. If it were possible to express any of the state variables in terms of others, those variables would not be necessary to uniquely describe the system dynamics, and such a set of variables would not constitute the smallest set of variables as required by the definition of the state of a dynamic system.
- *State vector* of a dynamic system is the column vector $\mathbf{q}$ whose components are the state variables, $q_1, q_2, \ldots, q_n$.
- *State space* is an n-dimensional space containing the n system state variables. The state of a dynamic system at any instant of time t is represented by a single point in the state space.
- *State trajectory* is the path over time of the point representing the state of the system in a state space. Figure 4.6 shows a state trajectory for a rock thrown vertically with the initial velocity v_0 at the location x_0, reaching the maximum height, x_f, with velocity v_f equal to zero and falling down to the initial position x_0 with the final velocity equal to $-v_0$ if losses of energy due to air resistance are negligible. The position of the rock and its velocity are the state variables and the plane determined by the coordinate system (x, v) is the two-dimensional state space.

Mathematically, state models take the form of sets of first-order differential equations as follows

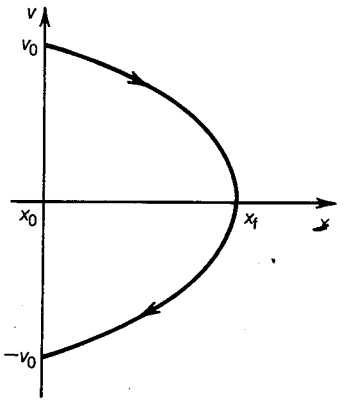


FIGURE 4.6 State trajectory of a rock thrown vertically from x_0.

$$\begin{aligned}
\dot{q}_1 &= f_1(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_l, t) \\
\dot{q}_2 &= f_2(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_l, t) \\
&\vdots \\
\dot{q}_n &= f_n(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_l, t)
\end{aligned} \tag{4.6}$$

where $q_1, q_2, \ldots, q_n$ are the system state variables and $u_1, u_2, \ldots, u_l$ are the input variables. If the model is nonlinear, at least one of the functions f_i, $i = 1, 2, \ldots, n$, is nonlinear.

Although the state vector completely represents the system dynamics, the selected state variables are not necessarily the same as the system outputs. However, each output variable or, in general any system variable that is of interest, can be expressed mathematically in terms of the system state variables and the input variables. A block diagram of a state representation of a multi-input multi-output dynamic system is shown in Figure 4.7.

In general, the system output equations can be written in the following form.

$$\begin{aligned}
y_1 &= g_1(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_l, t) \\
y_2 &= g_2(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_l, t) \\
&\vdots \\
y_p &= g_p(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_l, t)
\end{aligned} \tag{4.7}$$

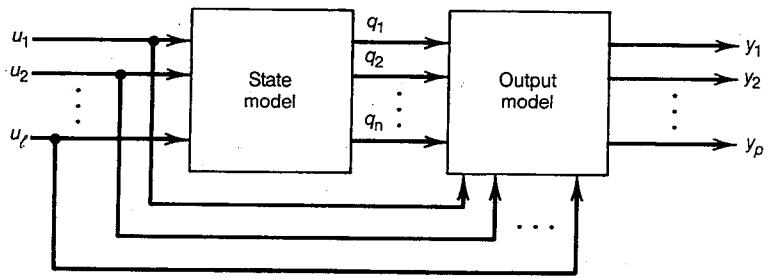


FIGURE 4.7 Block diagram of a state model.

If a system model is linear, all functions on the right-hand side of the state equations (4.6), f_i for $i = 1, 2, \ldots, n$, and all functions on the right-hand side of the output equations (4.7), g_j for $j = 1, 2, \ldots, p$ are linear. Also, in stationary model equations system parameters do not vary with time so that linear, stationary-state model equations take the form

$$\begin{aligned}
\dot{q}_1 &= a_{11}q_1 + a_{12}q_2 + \cdots + a_{1n}q_n + b_{11}u_1 + b_{12}u_2 + \cdots + b_{1l}u_l \\
\dot{q}_2 &= a_{21}q_1 + a_{22}q_2 + \cdots + a_{2n}q_n + b_{21}u_1 + b_{22}u_2 + \cdots + b_{2l}u_l \\
&\vdots \\
\dot{q}_n &= a_{n1}q_1 + a_{n2}q_2 + \cdots + a_{nn}q_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nl}u_l
\end{aligned} \tag{4.8}$$

The linear output equations are

$$\begin{aligned}
y_1 &= c_{11}q_1 + c_{12}q_2 + \cdots + c_{1n}q_n + d_{11}u_1 + d_{12}u_2 + \cdots + d_{1l}u_l \\
y_2 &= c_{21}q_2 + c_{22}q_2 + \cdots + c_{2n}q_n + d_{21}u_1 + d_{22}u_2 + \cdots + d_{2l}u_l \\
&\vdots \\
y_p &= c_{p1}q_1 + c_{p2}q_2 + \cdots + c_{pn}q_n + d_{p1}u_1 + d_{p2}u_2 + \cdots + d_{pl}u_l
\end{aligned} \tag{4.9}$$

Equations (4.8) and (4.9) can be written using matrix-vector notation

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \vdots \\ \dot{q}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1l} \\ b_{21} & b_{22} & \cdots & b_{2l} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nl} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_l \end{bmatrix} \tag{4.10}$$

and

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \cdots & c_{pn} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1l} \\ d_{21} & d_{22} & \cdots & d_{2l} \\ \vdots & & & \\ d_{p1} & d_{p2} & \cdots & d_{pl} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_l \end{bmatrix} \tag{4.11}$$

Rewriting these equations in a more compact form

$$\dot{\mathbf{q}} = \mathbb{A}\mathbf{q} + \mathbb{B}\mathbf{u} \tag{4.10'}$$

$$\mathbf{y} = \mathbb{C}\mathbf{q} + \mathbb{D}\mathbf{u} \tag{4.11'}$$

where $\mathbb{A}$ is a $n \times n$ state matrix, $\mathbb{B}$ is a $n \times l$ input matrix, $\mathbb{C}$ is a $p \times n$ output matrix, $\mathbb{D}$ is a $p \times l$ direct transmission matrix, $\mathbf{q}$ is a state vector, $\mathbf{u}$ is an input vector, and $\mathbf{y}$ is an output vector.

As mentioned before, the selection of state variables constitutes a nontrivial problem since for each system there are usually many different sets of variables that uniquely represent the system dynamics. The following sets of variables are most commonly used as state variables:

(1) sets of T-type and A-type variables associated with T and A energy-storing elements of the system
(2) sets including one variable and its successive derivatives
(3) sets including two or more variables and their derivatives
(4) sets including an auxiliary variable and its successive derivatives

In addition, there are still other, relatively less common types of state variables, such as the variables associated with the roots of the system characteristic equation obtained by means of manipulation of the system matrix or sets of variables obtained as nonredundant algebraic combinations of other state variables.[1]

In general, a state of a dynamic system evolves from the process of storing energy in those system components which are capable of it. In fact, the number of state variables is always equal to the number of independent energy-storing elements in the system, regardless of the type of the state-variables employed.

In Example 4.4 the first three most common types of the state models listed above will be derived. The use of an auxiliary variable and its derivatives as the state variables will be demonstrated in Section 4.4.

EXAMPLE 4.4

Derive state models of the types (1), (2), and (3) for the mechanical system shown in Figure 4.2.

Type (1): T-type and A-type variables

Solution

There are four independent energy-storing elements in the system shown in Figure 4.2: masses m_1 and m_2 and springs k_1 and k_2. Masses in mechanical systems are A-type elements, which can store kinetic energy, whereas springs are T-type elements, capable of storing potential energy. The respective A-type and T-type variables are the velocities of the two masses, v_{1g} and v_{2g}, and the forces exerted by the springs, F_{k1} and F_{k2}. Hence, the four variables selected to represent the state of the system are $q_1 = F_{k1}$, $q_2 = v_{1g}$, $q_3 = F_{k2}$, and $q_4 = v_{2g}$. To derive the state equations, first consider the forces exerted by the springs F_{k1} and F_{k2}. The equations defining these forces are

$$F_{k1} = k_1 x_1$$

$$F_{k2} = k_2(x_2 - x_1)$$

Differentiating both sides of the above equations with respect to time gives the first two state-variable equations

$$\dot{F}_{k1} = (k_1)v_{1g}$$

$$\dot{F}_{k2} = (-k_2)v_{1g} + (k_2)v_{2g}$$

[1] Y. Takahashi, M. J. Rabins, and D. M. Auslander, *Control and Dynamic Systems*, Addison-Wesley Publishing Company, Reading, Mass., 1972.

The equations of motion for masses m_1 and m_2 derived in Example 4.1 were

$$m_1 \underbrace{\frac{d^2x_1}{dt^2}}_{dv_{1g}\,dt} + b_1 \underbrace{\frac{dx_1}{dt}}_{v_{1g}} + \underbrace{k_1x_1}_{F_{k1}} - \underbrace{k_2(x_2 - x_1)}_{F_{k2}} = F_1(t)$$

and

$$m_2 \underbrace{\frac{d^2x_2}{dt^2}}_{dv_{2g}\,dt} + \underbrace{k_2(x_2 - x_1)}_{F_{k2}} = 0$$

which give the other two state variable equations

$$\dot{v}_{1g} = -\frac{1}{m_1}F_{k1} - \frac{b_1}{m_1}v_{1g} + \frac{1}{m_1}F_{k2} + \frac{1}{m_1}F_1(t)$$

$$\dot{v}_{2g} = -\frac{1}{m_2}F_{k2}$$

Rewriting the state variable equations in a vector-matrix form yields

$$\begin{bmatrix} \dot{F}_{k1} \\ \dot{v}_{1g} \\ \dot{F}_{k2} \\ \dot{v}_{2g} \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ -1/m_1 & -b_1/m_1 & 1/m_1 & 0 \\ 0 & -k_2 & 0 & k_2 \\ 0 & 0 & -1/m_2 & 0 \end{bmatrix} \begin{bmatrix} F_{k1} \\ v_{1g} \\ F_{k2} \\ v_{2g} \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m_1 \\ 0 \\ 0 \end{bmatrix} F_1(t)$$

The output equations for the displacements x_1 and x_2 are

$$x_1 = \frac{1}{k_1}F_{k1}$$

$$x_2 = \frac{1}{k_1}F_{k1} + \frac{1}{k_2}F_{k2}$$

In a vector-matrix form the output model equations are

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/k_1 & 0 & 0 & 0 \\ 1/k_1 & 0 & 1/k_2 & 0 \end{bmatrix} \begin{bmatrix} F_{k1} \\ v_{1g} \\ F_{k2} \\ v_{2g} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} F_1(t)$$

Type (2): One variable and its successive derivatives This type of state variables is particularly convenient when an input-output equation is available. Very often in such cases the output variable and successive derivatives are selected as the state variables. In Example 4.1 the input-output equations for the system shown in Figure 4.2 were derived relating displacements x_1 and x_2 to the input force $F_1(t)$. Both equations are of fourth order, so four state variables are needed to uniquely represent the system dynamics. Let x_2 and its first three derivatives be selected as the state variables, i.e., $q_1 = x_2$, $q_2 = dx_2/dt$, $q_3 = d^2x_2/dt^2$, $q_4 = d^3x_2/dt^3$.

The input-output equation relating x_2 to the input force, $F_1(t)$ was

$$\frac{d^4x_2}{dt^4} + \left(\frac{b_1}{m_1}\right)\frac{d^3x_2}{dt^3} + \left(\frac{k_2}{m_2} + \frac{k_1}{m_1} + \frac{k_2}{m_1}\right)\frac{d^2x_2}{dt^2} + \left(\frac{b_1k_2}{m_1m_2}\right)\frac{dx_2}{dt} + \left(\frac{k_1k_2}{m_1m_2}\right)x_2 = \left(\frac{k_2}{m_1m_2}\right)F_1(t)$$

Using the preceding equation and the definitions of the selected state variables, the following state equations are formed

$$\dot{q}_1 = q_2$$

$$\dot{q}_2 = q_3$$

$$\dot{q}_3 = q_4$$

$$\dot{q}_4 = -\left(\frac{k_1k_2}{m_1m_2}\right)q_1 - \left(\frac{b_1k_2}{m_1m_2}\right)q_2 - \left(\frac{k_1}{m_1} + \frac{k_2}{m_2} + \frac{k_2}{m_1}\right)q_3 - \left(\frac{b_1}{m_1}\right)q_4 + \left(\frac{k_2}{m_1m_2}\right)F_1(t)$$

Recognizing that $x_1 = x_2 - (x_2 - x_1) = x_2 - F_{k2}/k_2 = x_2 + (m_2/k_2)(d^2x_2/dt^2)$, then if x_1 and x_2 are selected as the output variables, y_1, and y_2 respectively, the output equations become

$$y_1 = q_1 + \left(\frac{m_2}{k_2}\right)q_3 = x_1$$

$$y_2 = q_1 = x_2$$

Type (3): Two variables and their derivatives This type of state variables is used most often in modeling mechanical systems. For the system considered in this example, x_1 and x_2 and their derivatives, $\dot{x}_1$, $\dot{x}_2$, are selected as the four state variables. Noting that $\dot{x}_1 = v_{1g}$ and $\dot{x}_2 = v_{2g}$ and using the equations of motion for masses m_1, m_2, derived in Example 4.1, the following state equations are obtained

$$\dot{q}_1 = q_3$$

$$\dot{q}_2 = q_4$$

$$\dot{q}_3 = -\left(\frac{k_1}{m_1} + \frac{k_2}{m_1}\right) q_1 + \left(\frac{k_2}{m_1}\right) q_2 - \left(\frac{b_1}{m_1}\right) q_3 + \left(\frac{1}{m_1}\right) F_1(t)$$

$$\dot{q}_4 = \left(\frac{k_2}{m_2}\right) q_1 - \left(\frac{k_2}{m_2}\right) q_2$$

where $q_1 = x_1$, $q_2 = x_2$, $q_3 = v_{1g}$, and $q_4 = v_{2g}$.

The output equations are simply

$$y_1 = q_1$$
$$y_2 = q_2$$

■

4.4 Transition Between Input-Output and State Models

At the beginning of this chapter both input-output and state models were said to be equivalent in the sense that each form completely represents the dynamics of the same system. It is, therefore, natural to expect that there is a corresponding state model involving successive derivatives of one state variable for each input-output model and vice versa. The transition between the two forms of models is indeed possible, although it is not always straightforward.

Consider first a simple case of a single-input single-output system model with no derivatives of input variable present in the input-output equation

$$a_n \frac{d^n y}{dt^n} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_0 u \tag{4.12}$$

If the following state variables are selected [as in Type (2) Sect. 4.3]

$$q_1 = y, \quad q_2 = \frac{dy}{dt}, \quad \ldots, \quad q_n = \frac{d^{n-1}y}{dt^{n-1}} \tag{4.13}$$

the equivalent set of state model equations is

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= q_3 \\ &\vdots \\ \dot{q}_{n-1} &= q_n \\ \dot{q}_n &= -\left(\frac{a_0}{a_n}\right) q_1 - \left(\frac{a_1}{a_n}\right) q_2 - \cdots - \left(\frac{a_{n-1}}{a_n}\right) q_n + \left(\frac{b_0}{a_n}\right) u \end{aligned} \tag{4.14}$$

The transition procedure between an input-output model and an equivalent state model becomes more complicated if derivatives of input variables are present on the right-hand side of the input-output equation. The input-output equation of a single-input single-output model in this case is

$$a_n \frac{d^n y}{dt^n} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_m \frac{d^m u}{dt^m} + \cdots + b_1 \frac{du}{dt} + b_0 u \tag{4.15}$$

where $m \leq n$ for physically realizable systems. An equivalent state model will in this case consist of a set of state variable equations based on a state variable and $(n - 1)$ of its derivatives together with the output equations, as illustrated in Figure 4.8.

In order to obtain an equivalent state model, first the higher derivatives of the input variable in Equation (4.15) are ignored to yield an auxiliary input-output differential equation

$$a_n \frac{d^n x}{dt^n} + \cdots + a_1 \frac{dx}{dt} + a_0 x = u \tag{4.16}$$

where x is an auxiliary state variable. The set of equivalent state equations for the auxiliary input-output Equation (4.16) can be obtained as before, using the auxiliary variable and its successive derivatives as the state variables [type (4) of state variables discussed in Section 4.3]

$$q_1 = x, \quad q_2 = \dot{x}, \quad \ldots, \quad q_n = \frac{d^{n-1}x}{dt^{n-1}} \tag{4.17}$$

The state equations become

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= q_3 \\ &\vdots \\ \dot{q}_{n-1} &= q_n \\ \dot{q}_n &= -\frac{a_0}{a_n} q_1 - \frac{a_1}{a_n} q_2 - \cdots - \frac{a_{n-1}}{a_n} q_n + \frac{1}{a_n} u \end{aligned} \tag{4.18}$$

Applying the differentiation operator introduced in Section 4.2 to Equations (4.15) and (4.16) gives

$$y(a_n D^n + \cdots + a_1 D + a_0) = u(b_m D^m + \cdots + b_1 D + b_0) \tag{4.19}$$

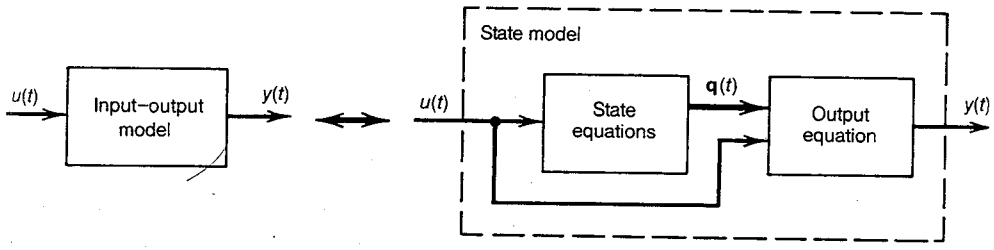


FIGURE 4.8 Equivalent input-output and state models.

and

$$x(a_nD^n + \cdots + a_1D + a_0) = u \tag{4.20}$$

Substitution of the expression on the left-hand side of Equation (4.20) for u in Equation (4.19) yields

$$y = x(b_mD^m + \cdots + b_1D + b_0) \tag{4.21}$$

Now use the definition of the D operator to transform Equation (4.21) back to the time domain.

$$y = b_m\frac{d^mx}{dt^m} + \cdots + b_1\frac{dx}{dt} + b_0x \tag{4.22}$$

The auxiliary variable x and its derivatives can be replaced by the state variables defined by (4.17) to produce the following output equation.

$$y = b_0q_1 + b_1q_2 + \cdots + b_mq_{m+1} \quad \text{for } m < n \tag{4.23}$$

Equation (4.23) holds for $m < n$. If both sides of the input-output equation are of the same order, $m = n$, then q_{m+1} in Equation (4.23) is replaced by the expression for $\dot{q}_n$ given by the last of Equations (4:18) and the output equation becomes

$$\begin{aligned} y = &\left(b_0 - \frac{b_ma_0}{a_n}\right)q_1 + \left(b_1 - \frac{b_ma_1}{a_n}\right)q_2 + \cdots \\ &+ \left(b_{m-1} - \frac{b_ma_{n-1}}{a_n}\right)q_n + \left(\frac{b_m}{a_n}\right)u \quad \text{for } m = n \end{aligned} \tag{4.24}$$

The input-output Equation (4.15) involving derivatives of the input variable is therefore equivalent to the state model, consisting of the auxiliary state variable Equations (4.18) and output Equations (4.23) or (4.24) for $m < n$ and $m = n$, respectively.

The procedure for transforming input-output equations into an equivalent state model is illustrated in Example 4.5.

EXAMPLE 4.5

Consider again the mechanical system shown in Figure 4.3. The input-output equation for this system, derived in Example 4.2, was

$$\begin{aligned} m_1m_2\frac{d^4x_2}{dt^4} &+ (m_2b_1 + m_2b_2 + m_1b_2)\frac{d^3x_2}{dt^3} \\ &+ (m_1k_2 + m_2k_1 + m_2k_2 + b_1b_2)\frac{d^2x_2}{dt^2} \\ &+ (b_1k_2 + b_2k_1)\frac{dx_2}{dt} \\ &+ k_1k_2x_2 = b_2\frac{dF_1}{dt} + k_2F_1 \end{aligned}$$

Derive an equivalent state model for this system.

Solution

First, the derivative of F_1 is ignored and a simplified input-output equation is obtained using the auxiliary output variable x.

$$\begin{aligned} m_1m_2\frac{d^4x}{dt^4} &+ (m_2b_1 + m_2b_2 + m_1b_2)\frac{d^3x}{dt^3} \\ &+ (m_1k_2 + m_2k_1 + m_2k_2 + b_1b_2)\frac{d^2x}{dt^2} \\ &+ (b_1k_2 + b_2k_1)\frac{dx}{dt} + k_1k_2x = F_1 \end{aligned}$$

The state variables are

$$q_1 = x, \quad q_2 = \frac{dx}{dt}, \quad q_3 = \frac{d^2x}{dt^2}, \quad q_4 = \frac{d^3x}{dt^3}$$

and the state variable equations are

$$\begin{aligned} \dot{q}_1 &= q_2 \\ \dot{q}_2 &= q_3 \\ \dot{q}_3 &= q_4 \\ \dot{q}_4 &= -\left(\frac{k_1k_2}{m_1m_2}\right)q_1 - \left[\frac{(b_1k_2 + b_2k_1)}{m_1m_2}\right]q_2 \\ &\quad - \left[\frac{(m_1k_2 + m_2k_1 + m_2k_2 + b_1b_2)}{m_1m_2}\right]q_3 \\ &\quad - \left[\frac{(m_2b_1 + m_2b_2 + m_1b_2)}{m_1m_2}\right]q_4 + \left(\frac{1}{m_1m_2}\right)F_1 \end{aligned}$$

The output equation is

$$y_2 = k_2q_1 + b_2q_2$$

which completes the system mathematical model in a state form. ■

4.5 Nonlinearities in Input-Output and State Models

Very often in modeling dynamic systems, nonlinear characteristics of some of the system elements cannot be linearized either because the linearization error is not acceptable or because a particular nonlinearity may be essential for the system performance and must not be replaced with a linear approximation. The superiority

of state models over input-output models in such cases is particularly pronounced. Derivation of input-output differential equations for systems in which nonlinearities are to be modeled without linearization is usually very cumbersome or even impossible, to say nothing about solving those equations. The derivation of state models, on the other hand, is barely affected by the presence of nonlinearities in the system. Furthermore, most modern computer programs for solving sets of state variable equations are capable of handling both linear and nonlinear models.

The effect of nonlinearities on the process of derivation of the two forms of mathematical models is illustrated in Example 4.6.

EXAMPLE 4.6

The mechanical system shown in Figure 4.9 is similar to the system considered earlier in Examples 4.1 and 4.4, except that the linear viscous friction element b_1 is here replaced with a nonlinear damper NLD.

The force generated by the damper, F_{NLD} is

$$F_{NLD} = f_{NL}(v_{1g}) = a|v_{1g}|v_{1g}$$

The square law expression embodied in the function f_{NL} is a fairly common type of nonlinearity, although the specific form of the nonlinear function is of no significance here.

Derive an input-output equation using x_2 as the output variable and a set of state equations for this system.

Solution

The equations of motion for masses m_1 and m_2 are

$$m_1 \frac{d^2x_1}{dt^2} + a\left|\frac{dx_1}{dt}\right|\frac{dx_1}{dt} + k_1x_1 - k_2(x_2 - x_1) = F_1(t)$$

and

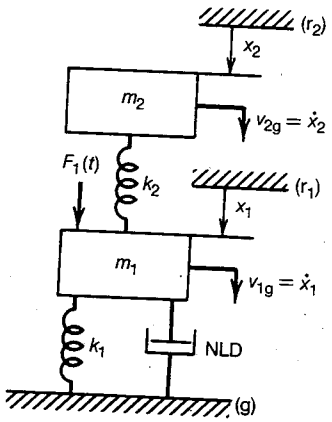


FIGURE 4.9 Mechanical system with nonlinear damper.

$$m_2 \frac{d^2x_2}{dt^2} + k_2(x_2 - x_1) = 0$$

From the equation for mass m_2

$$x_1 = \left(\frac{m_2}{k_2}\right)\frac{d^2x_2}{dt^2} + x_2$$

Taking derivatives of x_1 and substituting into the equation for mass m_1 gives the following input-output equation

$$\frac{d^4x_2}{dt^4} + \frac{k_2a}{m_1m_2}\left|\frac{m_2}{k_2}\frac{d^3x_2}{dt^3} + \frac{dx_2}{dt}\right|\left(\frac{m_2}{k_2}\frac{d^3x_2}{dt^3} + \frac{dx_2}{dt}\right) + \left(\frac{k_1 + k_2}{m_1}\right)\frac{d^2x_2}{dt^2} + \frac{k_1k_2}{m_1m_2}x_2 = F_1(t)$$

Only those readers not yet convinced about the superiority of state models over input-output models are encouraged to attempt to solve this equation.

To obtain an equivalent state model, select displacements x_1 and x_2 and their derivatives v_{1g} and v_{2g} as the state variables. The state variable equations are

$$\dot{q}_1 = q_2$$

$$\dot{q}_2 = -\left[\frac{(k_1 + k_2)}{m_1}\right]q_1 - \left(\frac{a}{m_1}\right)|q_2|q_2 + \left(\frac{k_2}{m_1}\right)q_3 + \left(\frac{1}{m_1}\right)F_1(t)$$

$$\dot{q}_3 = q_4$$

$$\dot{q}_4 = \left(\frac{k_2}{m_2}\right)q_1 - \left(\frac{k_2}{m_2}\right)q_3$$

where $q_1 = x_1$, $q_2 = v_{1g}$, $q_3 = x_2$, and $q_4 = v_{2g}$.

The preceding state variable equations are simpler and can be solved numerically using one of the methods described in Chapter 6. ■

The examples presented in this chapter illustrate the effectiveness of the state approach to system modeling. The state equations are much simpler, and the entire process of derivation is less vulnerable to so-called "stupid mistakes," which may often be made when the input-output model is developed. Although, as the next chapter will show, there are certain advantages associated with using input-output models in analysis of low-order linear systems, the state-variable approach is in general superior and will be used as much as possible in modeling systems throughout the rest of this book.

4.6 Synopsis

Two types of mathematical models of dynamic systems were presented, input-output models and state models. Although both models are essentially equivalent

OBTAINING THE EQUATIONS NEEDED TO SOLVE THE DYNAMICS OF THE SYSTEM

ONE WAY IS THE METHOD SHOWN FOR THE VIBRATION ABSORBER

- GET ONE EQUATION OF HIGH ORDER
  - TEDIOUS
  - FORM IS $$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = f\left(\frac{d^m u}{dt^m}, \frac{d^{m-1} u}{dt^{m-1}}, \dots, \frac{du}{dt}, u\right)$$

- where y is the OUTPUT REQUIRED DUE TO INPUT u & $m \le n$
- f may be a non-linear function that also involves t

- IF THE PROBLEM IS LINEAR THEN

$$\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = \sum_{j=0}^{m} b_j \frac{d^j u}{dt^j} \qquad \text{AND } \frac{d^0}{dt^0} \text{ means } 1$$

FOR EXAMPLE WE SHOWED THAT FOR THE VIBRATION ABSORBER

$$m_1 m_2 \frac{d^4 V_1}{dt^4} + (m_1 b_2 + m_2 b_1) \frac{d^3 V_1}{dt^3} + \left[m_1 k_2 + b_1 b_2 + m_2 (2k_1 + k_2)\right] \frac{d^2 V_1}{dt^2} + \left[b_1 k_2 + b_2 (2k_1 + k_2)\right] \frac{dV_1}{dt}$$

$$+ 2k_1 k_2 V_1 = 2m_2 k_1 \frac{d^2 V}{dt^2} + 2k_1 b_2 \frac{dV}{dt} + 2k_1 k_2 V$$

HERE $y \Rightarrow V_1$ $u \Rightarrow V$ $m \Rightarrow 2$ $n \Rightarrow 4$ AND V IS INPUT V_1 IS OUTPUT

- ANOTHER METHOD IS THE STATE EQUATION METHOD
  - EASIER YET MORE COMPLICATED
  - EQUATIONS LEAD TO MATRIX SET OF EQUATIONS

$$\dot{\underline{\zeta}} = A\underline{\zeta} + B\underline{u}$$

- HERE $\underline{\zeta}$ IS THE STATE VARIABLE VECTOR ; $\underline{u}$ IS VECTOR OF INPUTS; A & B MATRICES

① FIND ALL ELEMENTS THAT ARE T-TYPE (SPRINGS, INDUCTANCE, INERTANCE) & A-TYPE (MASSES, MOMENTS OF INERTIA, CAPACITANCES). NO. OF SUCH ELEMENTS = NO. OF S.V. EQS

OBTAINING THE EQUATIONS NEEDED TO SOLVE THE DYNAMICS OF THE SYSTEM

ONE WAY IS THE METHOD SHOWN FOR THE VIBRATION ABSORBER

- GET ONE EQUATION OF HIGH ORDER
  - TEDIOUS
  - FORM IS $$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = f\left(\frac{d^m u}{dt^m}, \frac{d^{m-1} u}{dt^{m-1}}, \ldots, \frac{du}{dt}, u\right)$$

- where y is the OUTPUT REQUIRED DUE TO INPUT u & $m \le n$
- f may be a non-linear function that also involves t

- IF THE PROBLEM IS LINEAR THEN
$$\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = \sum_{j=0}^{m} b_j \frac{d^j u}{dt^j}$$ AND $\frac{d^0}{dt^0}$ means 1

FOR EXAMPLE WE SHOWED THAT FOR THE VIBRATION ABSORBER

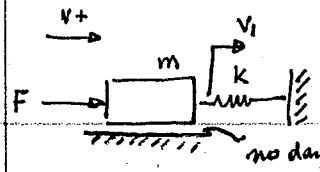
$$m_1 m_2 \frac{d^4 V_1}{dt^4} + (m_1 b_2 + m_2 b_1) \frac{d^3 V_1}{dt^3} + \left[m_1 k_2 + b_1 b_2 + m_2 (2k_1 + k_2)\right] \frac{d^2 V_1}{dt^2} + \left[b_1 k_2 + b_2 (2k_1 + k_2)\right] \frac{dV_1}{dt}$$

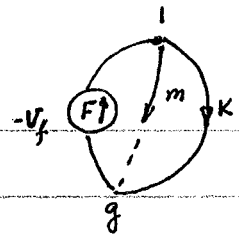
$$+ 2k_1 k_2 V_1 = 2m_2 k_1 \frac{d^2 V}{dt^2} + 2k_1 b_2 \frac{dV}{dt} + 2k_1 k_2 V$$

HERE $y \Rightarrow V_1$ $u \Rightarrow V$ $m \Rightarrow 2$ $n \Rightarrow 4$ AND V IS INPUT V_1 IS OUTPUT

- ANOTHER METHOD IS THE STATE EQUATION METHOD
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$$\dot{\underline{\zeta}} = A\underline{\zeta} + B\underline{u}$$
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(1) FIND ALL ELEMENTS THAT ARE T-TYPE (SPRINGS, INDUCTANCE, INERTANCE) & A-TYPE (MASSES, MOMENTS OF INERTIA, CAPACITANCES). NO. OF SUCH ELEMENTS $\geq$ NO. OF S.V. EQS





$b=3 \quad s=1 \quad b-s=2$

$F_m = m\frac{dv_{1g}}{dt}$

$\frac{dF_k}{dt} = kV_{1g}$

NODES $n-1 = 2-1 = 1$

$$F = F_m + F_k \Rightarrow \dot{F} = \dot{F}_m + \dot{F}_k \Rightarrow \dot{F} = m\ddot{v}_{1g} + k v_{1g}$$

PATH $b-(n-p) = 3-(2-1) = 2$

$$-v_F + v_{1g} = 0$$

$$-v_{1g} + v_{1g} = 0$$

since one A type (mass) equation must be written in terms of $\dot{v}_{1g}$

since one T type (spring) " " " " " " " $\dot{F}_k$

$$\begin{matrix} (1) & \dot{F}_k = k v_{1g} \\ (2) & \dot{v}_{1g} = \frac{F_m}{m} = \frac{F - F_k}{m} \end{matrix} \Bigg\} \quad \begin{pmatrix} \dot{F}_k \\ \dot{v}_{1g} \end{pmatrix} = \begin{bmatrix} 0 & k \\ -\frac{1}{m} & 0 \end{bmatrix} \begin{pmatrix} F_k \\ v_{1g} \end{pmatrix} + \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix} F$$

here $\dot{\underline{x}} = \begin{pmatrix} \dot{F}_k \\ \dot{v}_{1g} \end{pmatrix}$; $A = \begin{bmatrix} 0 & k \\ -\frac{1}{m} & 0 \end{bmatrix}$; $\underline{x} = \begin{pmatrix} F_k \\ v_{1g} \end{pmatrix}$; $B = \begin{pmatrix} 0 \\ \frac{1}{m} \end{pmatrix}$; $\underline{u} = F$

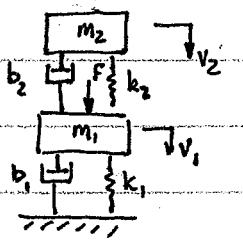
- NOTE THE FORM OF THE STATE VARIABLE EQS

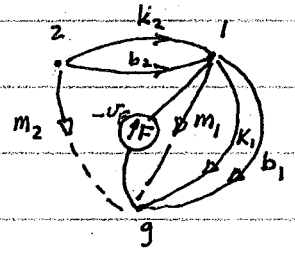
$$\dot{q}_1 = f_1(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_m, t)$$

$$\dot{q}_2 = f_2(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_m, t)$$

$$\vdots$$

$$\dot{q}_n = f_n(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_m, t)$$



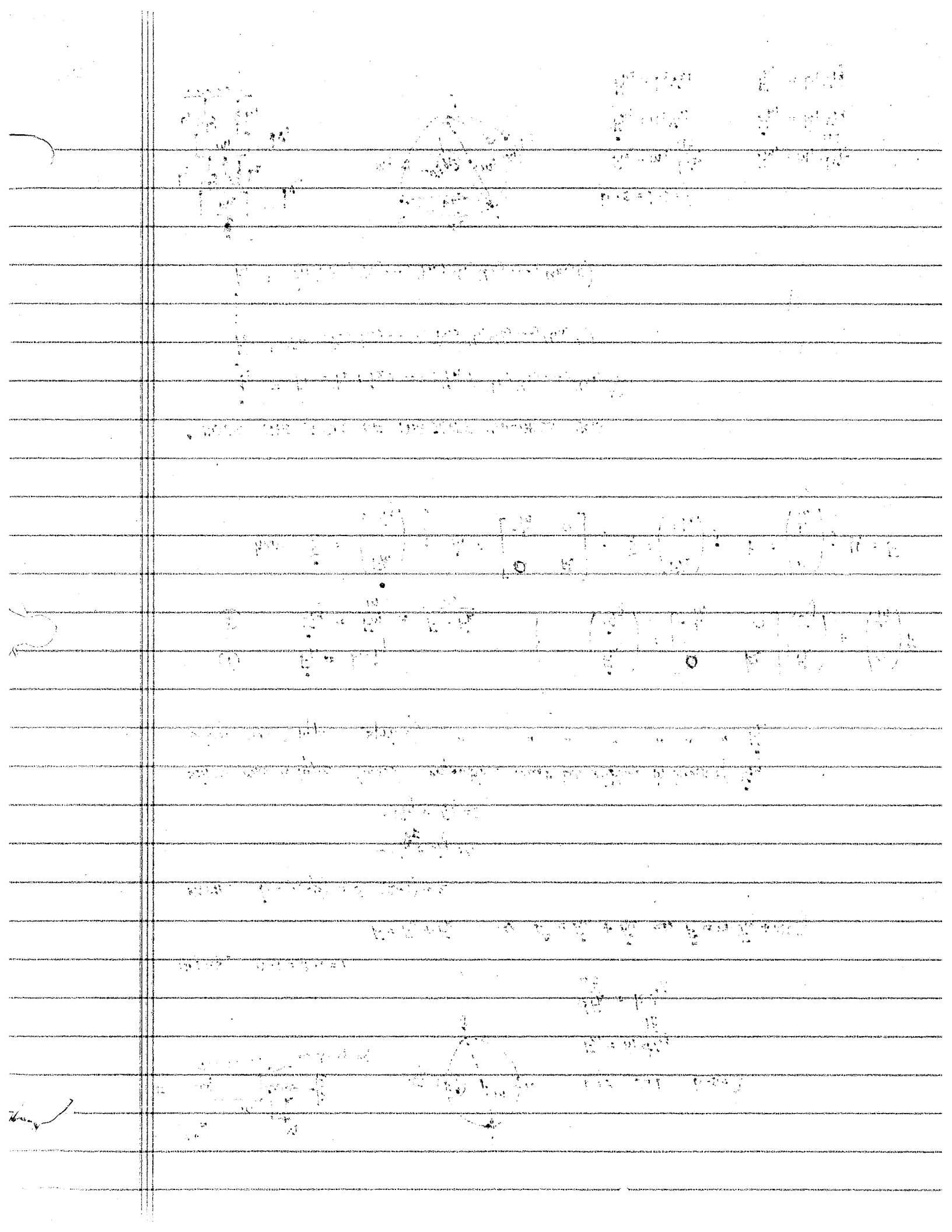


$b-s = 7-1 = 6$

$F_{m_2} = m_2\frac{dv_{2g}}{dt}$ $\quad F_{m_1} = m_1\frac{dv_{1g}}{dt}$

$\dot{F}_{k_2} = k_2 v_{21}$ $\quad \dot{F}_{k_1} = k_1 v_{1g}$

$F_{b_2} = b_2 v_{21}$ $\quad F_{b_1} = b_1 v_{1g}$



$n-p = 3-1 = 2$ $\quad F_{m_2} + F_{k_2} + F_{b_2} = 0$ @ node 2

$F_{k_2} + F_{b_2} + F = F_{m_1} + F_{k_1} + F_{b_1}$ @ node 1

$b-(n-p) = 7-(3-1) = 5$ $\quad -v_{2g} + v_{21} + v_F = 0$ $\quad -(-v_F)$

$-v_{2g} + v_{2g} = 0$

$-v_F + v_{1g} = 0$

$-v_{1g} + v_{1g} = 0 \quad (\times 2)$

2 masses + 2 springs $\quad \underline{\xi} = \begin{pmatrix} v_{1g} \\ v_{2g} \\ F_{k_1} \\ F_{k_2} \end{pmatrix}$

$$\dot{F}_{k_2} = k_2 v_{21} = k_2 (v_2 - v_1) = k_2 (v_{2g} - v_{1g}) \quad + 0F$$

$$\dot{F}_{k_1} = k_1 v_{1g} = k_1 v_1 \quad + 0F$$

$$\dot{v}_{1g} = \frac{F_{m_1}}{m_1} = \frac{1}{m_1}\left[F_{k_2} + F_{b_2} + F - F_{k_1} - F_{b_1}\right] = \frac{1}{m_1}\left[F_{k_2} + b_2(v_2 - v_1) + F - F_{k_1} + b_1 v_1\right]$$

$$= \frac{1}{m_1}\left[-(b_2+b_1)v_{1g} + b_2 v_{2g} - F_{k_1} + F_{k_2} + F\right]$$

$$\dot{v}_{2g} = \frac{F_{m_2}}{m_2} = -\frac{(F_{k_2} + F_{b_2})}{m_2} = -\frac{1}{m_2}\left[F_{k_2} + b_2(v_2 - v_1)\right] + 0F$$

$$= -\frac{1}{m_2}\left[-b_2 v_{1g} + b_2 v_{2g} + 0F_{k_1} + F_{k_2}\right] + 0F$$

PUT IN FORM $\quad \dot{\underline{\xi}} = A\underline{\xi} + B\underline{u}$

$$\begin{pmatrix} \dot{v}_{1g} \\ \dot{v}_{2g} \\ \dot{F}_{k_1} \\ \dot{F}_{k_2} \end{pmatrix} = \begin{bmatrix} -(b_2+b_1)/m_1 & b_2/m_1 & -1/m_1 & +1/m_1 \\ +b_2/m_2 & -b_2/m_2 & 0 & -1/m_2 \\ k_1 & 0 & 0 & 0 \\ -k_2 & k_2 & 0 & 0 \end{bmatrix} \begin{pmatrix} v_{1g} \\ v_{2g} \\ F_{k_1} \\ F_{k_2} \end{pmatrix} + \begin{pmatrix} 1/m_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} F$$

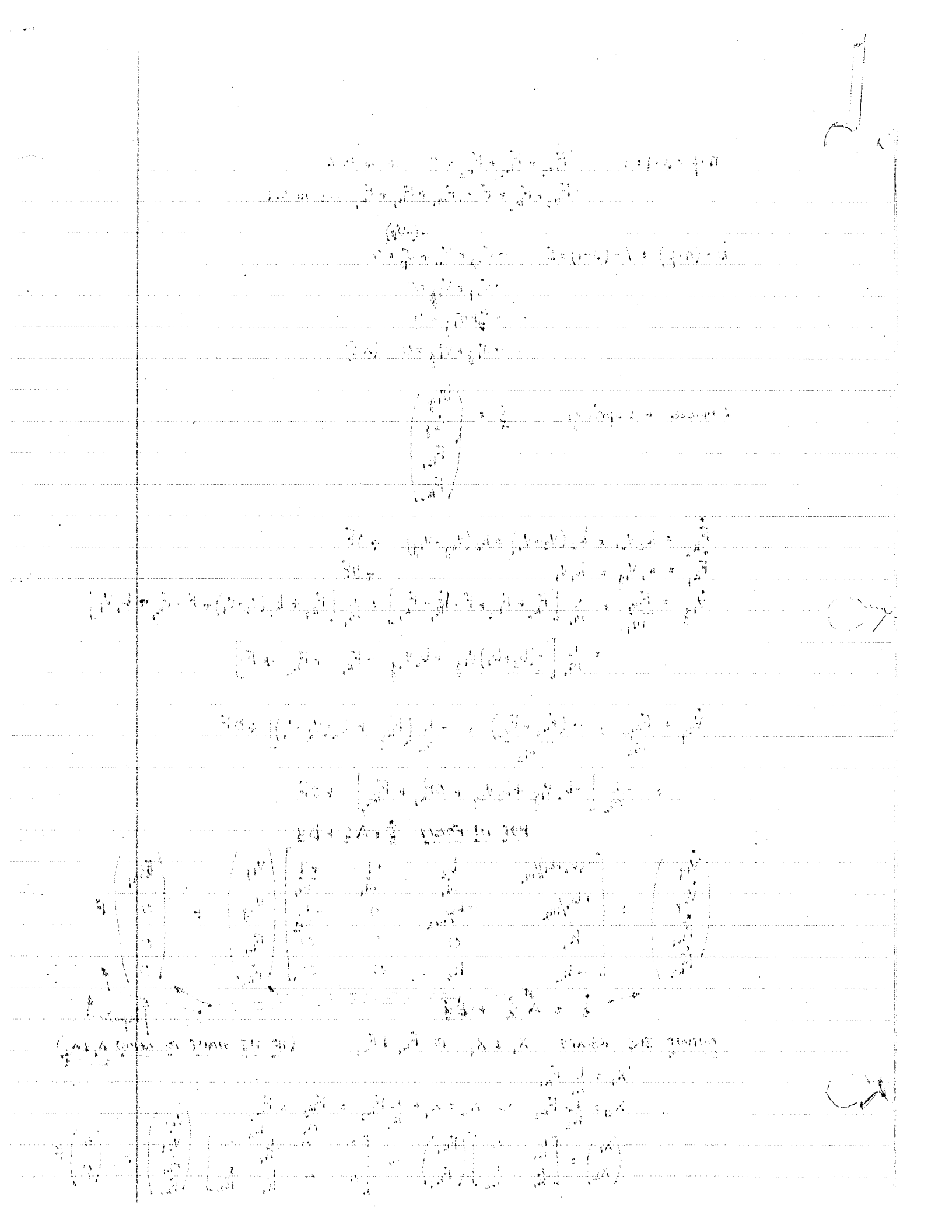
$\dot{\underline{\xi}} = A\underline{\xi} + B\underline{u}$ (input)

OUTPUT EQS RELATE X_1 & X_2 TO F_{k_1} & F_{k_2} (IF WE WANT TO KNOW X_1 & X_2)

$$X_1 = \frac{1}{k_1} F_{k_1}$$

$$X_{21} = \frac{1}{k_2} F_{k_2} \quad \text{or} \quad X_2 = X_1 + \frac{1}{k_2} F_{k_2} = \frac{F_{k_1}}{k_1} + \frac{F_{k_2}}{k_2}$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} \frac{1}{k_1} & 0 \\ \frac{1}{k_1} & \frac{1}{k_2} \end{bmatrix} \begin{pmatrix} F_{k_1} \\ F_{k_2} \end{pmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & \frac{1}{k_1} & 0 \\ 0 & 0 & \frac{1}{k_1} & \frac{1}{k_2} \end{bmatrix} \begin{pmatrix} v_{1g} \\ v_{2g} \\ F_{k_1} \\ F_{k_2} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} F$$



$n-p = 3-1 = 2$ $\quad F_{m_2} + F_{k_2} + F_{b_2} = 0$ @ node 2

$F_{k_2} + F_{b_2} + F = F_{m_1} + F_{k_1} + F_{b_1}$ @ node 1

$b-(n-p) = 7-(3-1) = 5$ $\quad -v_{2g} + v_{21} + v = 0$

$-v_{2g} + v_{2g} = 0$

$-v + v_{1g} = 0$

$-v_{1g} + v_{1g} = 0 \quad (x2)$

2 masses + 2 springs $\quad \underline{S} = \begin{pmatrix} v_{1g} \\ v_{2g} \\ F_{k_1} \\ F_{k_2} \end{pmatrix}$

$$\dot{F}_{k_2} = k_2 v_{21} = k_2 (v_2 - v_1) = k_2 (v_{2g} - v_{1g})$$

$$\dot{F}_{k_1} = k_1 v_{1g} = k_1 v_1$$

$$\dot{v}_{1g} = \frac{F_{m_1}}{m_1} = \frac{1}{m_1}\left[F_{k_2} + F_{b_2} + F - F_{k_1} - F_{b_1}\right] = \frac{1}{m_1}\left[F_{k_2} + b_2(v_2 - v_1) + F - F_{k_1} + b_1 v_1\right]$$

$$= \frac{1}{m_1}\left[-(b_2+b_1)v_{1g} + b_2 v_{2g} - F_{k_1} + F_{k_2} + F\right]$$

$$\dot{v}_{2g} = \frac{F_{m_2}}{m_2} = -\frac{(F_{k_2} + F_{b_2})}{m_2} = -\frac{1}{m_2}\left[F_{k_2} + b_2(v_2 - v_1)\right]$$

$$= -\frac{1}{m_2}\left[-b_2 v_{1g} + b_2 v_{2g} + 0F_{k_1} + F_{k_2}\right]$$

PUT IN FORM $\quad \dot{\underline{S}} = A\underline{S} + B\underline{u}$

$$\begin{pmatrix} \dot{v}_{1g} \\ \dot{v}_{2g} \\ \dot{F}_{k_1} \\ \dot{F}_{k_2} \end{pmatrix} = \begin{bmatrix} -(b_2+b_1)/m_1 & b_2/m_1 & -1/m_1 & +1/m_1 \\ +b_2/m_2 & -b_2/m_2 & 0 & -1/m_2 \\ k_1 & 0 & 0 & 0 \\ -k_2 & k_2 & 0 & 0 \end{bmatrix} \begin{pmatrix} v_{1g} \\ v_{2g} \\ F_{k_1} \\ F_{k_2} \end{pmatrix} + \begin{pmatrix} 1/m_1 \\ 0 \\ 0 \\ 0 \end{pmatrix} F$$

$\dot{\underline{S}} = A\underline{S} + B\underline{u}$ (input)

OUTPUT EQS RELATE X_1 & X_2 TO F_{k_1} & F_{k_2} (IF WE WANT TO KNOW X_1 & X_2)

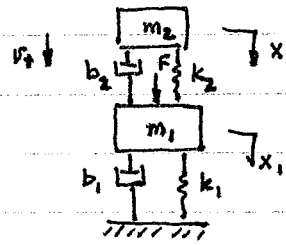
$$X_1 = \frac{1}{k_1} F_{k_1}$$

$$X_{21} = \frac{1}{k_2} F_{k_2} \quad \text{or} \quad X_2 = X_1 + \frac{1}{k_2} F_{k_2} = \frac{F_{k_1}}{k_1} + \frac{F_{k_2}}{k_2}$$

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{bmatrix} \frac{1}{k_1} & 0 \\ \frac{1}{k_1} & \frac{1}{k_2} \end{bmatrix} \begin{pmatrix} F_{k_1} \\ F_{k_2} \end{pmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 0 & \frac{1}{k_1} & 0 \\ 0 & 0 & \frac{1}{k_1} & \frac{1}{k_2} \end{bmatrix} \begin{pmatrix} v_{1g} \\ v_{2g} \\ F_{k_1} \\ F_{k_2} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} F$$

* ANOTHER MANNER OF WRITING EQUATIONS - 2 VARIABLES & THEIR DERIVS.

SUPPOSE WE KNOW THE GOVERNING ODE'S



For $m_2 \frac{d^2x_2}{dt^2} = -b_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) \Rightarrow \frac{dv_{2g}}{dt} = -\frac{b_2}{m_2}(v_{2g} - v_{1g}) - \frac{k_2}{m_2}(x_2 - x_1)$

For $m_1 \frac{d^2x_1}{dt^2} = +b_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) + F - b_1\dot{x}_1 - k_1 x_1 \Rightarrow \frac{dv_{1g}}{dt} = +\frac{b_2}{m_1}(v_{2g} - v_{1g}) - \left(\frac{k_1+k_2}{m_1}\right)x_1 + \frac{k_2}{m_1}x_2 + \frac{F}{m_1} - \frac{b_1}{m_1}v_{1g}$

let $q_1 = x_1 \quad q_2 = x_2 \quad q_3 = \dot{x}_1 = v_{1g} \quad q_4 = \dot{x}_2 = v_{2g}$

so: $\dot{q}_1 = q_3 \Rightarrow (\dot{x}_1 = v_{1g})$

$\dot{q}_2 = q_4 \Rightarrow (\dot{x}_2 = v_{2g})$

$(\dot{v}_{1g}) \Rightarrow \dot{q}_3 = \frac{b_2}{m_1}(q_4 - q_3) - \left(\frac{k_1+k_2}{m_1}\right)q_1 + \frac{k_2}{m_1}q_2 + \frac{F}{m_1} - \frac{b_1}{m_1}q_3$

$(\dot{v}_{2g}) \Rightarrow \dot{q}_4 = -\frac{b_2}{m_2}(q_4 - q_3) - \frac{k_2}{m_2}(q_2 - q_1)$

$$\begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{(k_1+k_2)}{m_1} & \frac{k_2}{m_2} & -\frac{(b_2+b_1)}{m_1} & \frac{b_2}{m_1} \\ \frac{k_2}{m_2} & -\frac{k_2}{m_2} & +\frac{b_2}{m_2} & -\frac{b_2}{m_2} \end{bmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{pmatrix} F$$

TO FIND THE OUTPUTS y_1 & y_2 IS VERY SIMPLE HERE

$y_1 = x_1 = q_1$

$y_2 = x_2 = q_2$

OBTAINING THE EQUATIONS NEEDED TO SOLVE THE DYNAMICS OF THE SYSTEM

ONE WAY IS THE METHOD SHOWN FOR THE VIBRATION ABSORBER

- GET ONE EQUATION OF HIGH ORDER
  - TEDIOUS
  - FORM IS $$a_n \frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \dots + a_1 \frac{dy}{dt} + a_0 y = f\left(\frac{d^m u}{dt^m}, \frac{d^{m-1} u}{dt^{m-1}}, \dots, \frac{du}{dt}, u\right)$$

- where y is the OUTPUT REQUIRED DUE TO INPUT u & $m \le n$
- f may be a non-linear function that also involves t

- IF THE PROBLEM IS LINEAR THEN
$$\sum_{i=0}^{n} a_i \frac{d^i y}{dt^i} = \sum_{j=0}^{m} b_j \frac{d^j u}{dt^j} \qquad \text{AND } \frac{d^0}{dt^0} \text{ means } 1$$

FOR EXAMPLE WE SHOWED THAT FOR THE VIBRATION ABSORBER

$$m_1 m_2 \frac{d^4 V_1}{dt^4} + (m_1 b_2 + m_2 b_1) \frac{d^3 V_1}{dt^3} + \left[m_1 k_2 + b_1 b_2 + m_2 (2k_1 + k_2)\right] \frac{d^2 V_1}{dt^2} + \left[b_1 k_2 + b_2 (2k_1 + k_2)\right] \frac{dV_1}{dt}$$

$$+ 2k_1 k_2 V_1 = 2m_2 k_1 \frac{d^2 V}{dt^2} + 2k_1 b_2 \frac{dV}{dt} + 2k_1 k_2 V$$

HERE $y \Rightarrow V_1$ $\quad u \Rightarrow V$ $\quad m \Rightarrow 2$ $\quad n \Rightarrow 4$ AND V IS INPUT $\quad V_1$ IS OUTPUT

- ANOTHER METHOD IS THE STATE EQUATION METHOD
  - EASIER YET MORE COMPLICATED
  - EQUATIONS LEAD TO MATRIX SET OF EQUATIONS
$$\dot{\underline{\zeta}} = A\underline{\zeta} + B\underline{u}$$
  - HERE $\underline{\zeta}$ IS THE STATE VARIABLE VECTOR ; $\underline{u}$ IS VECTOR OF INPUTS; A & B MATRICES

① FIND ALL ELEMENTS THAT ARE T-TYPE (SPRINGS, INDUCTANCE, INERTANCE) & A-TYPE (MASSES, MOMENTS OF INERTIA, CAPACITANCES). NO. OF SUCH ELEMENTS ≥ NO. OF S.V. EQS

LET'S LOOK AT THE SOLUTION OF $\dot{\underline{\xi}} = A\underline{\xi} + \underline{g}$ WHERE $\underline{g} = B\underline{u}$ IS DUE TO THE SOURCE TERMS

LET'S DEFINE $\frac{d}{dt}\underline{\xi} = \frac{d}{dt}\xi_i \quad i=1,\dots,N$ THE ENTRIES OF THE VECTOR $\underline{\xi}$

$\frac{d}{dt}A = \frac{d}{dt}a_{ij} \quad i=1,\dots,N \;\&\; j=1,\dots,m$ THE ENTRIES OF MATRIX A

(1) SOLVE HOMOGENEOUS EQN $\dot{\underline{\xi}} = A\underline{\xi} \quad (\dot{\underline{\xi}} - A\underline{\xi} = \underline{0})$

a) FORM MATRIX $A - rI$ (CHARACTERISTIC VALUES)

b) SET DETERMINANT OF $A - rI = 0$ THIS GIVES THE EIGENVALUES OF A

$\Rightarrow A\underline{y} = r\underline{y}$ WHERE $\underline{y}$ IS THE EIGENVECTOR

c) FIND $\underline{y}_i$ THAT SOLVES $A\underline{y}_i = r_i\underline{y}_i \quad i=1,\dots,N$. FOR EACH r VALUE THERE IS A UNIQUE $\underline{y}$ VECTOR

d) THEN A SOLUTION TO $\dot{\underline{\xi}} = A\underline{\xi}$ IS $\underline{\xi} = \underline{y}e^{rt}$

e) SINCE THERE ARE N SUCH SOLUTIONS, THE MOST GENERAL IS

$$\underline{\xi} = c_1\underbrace{\underline{y}_1 e^{r_1 t}}_{\underline{\xi}_1} + c_2\underbrace{\underline{y}_2 e^{r_2 t}}_{\underline{\xi}_2} + \dots + c_N\underbrace{\underline{y}_N e^{r_N t}}_{\underline{\xi}_N} = \sum_{n=1}^{N} c_i\underline{\xi}_i$$

VECTORS $\underline{y}_1, \underline{y}_2, \dots, \underline{y}_N$ ARE LINEARLY INDEPENDENT

f) WE CAN CREATE THE FUNDAMENTAL MATRIX Ψ BY HAVING THE COLUMNS OF Ψ BE

$$\Psi = [\underline{\xi}_1, \underline{\xi}_2, \dots, \underline{\xi}_N]$$

g) AND $[\underline{y}_1, \underline{y}_2, \dots, \underline{y}_N] = T$ (T SHALL HAVE AS ITS COLUMNS THE $\underline{y}$ VECTORS)

h) AND $Q = \begin{pmatrix} e^{r_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{r_2 t} & 0 & \dots & 0 \\ 0 & & \ddots & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & 0 & \dots & e^{r_N t} \end{pmatrix}$ IS A MATRIX SO THAT $\Psi = TQ$

i) NOW $\dot{\Psi} = [\dot{\underline{\xi}}_1, \dot{\underline{\xi}}_2, \dot{\underline{\xi}}_3, \dots, \dot{\underline{\xi}}_N] = [A\underline{\xi}_1, A\underline{\xi}_2, \dots, A\underline{\xi}_N] = A[\underline{\xi}_1, \underline{\xi}_2, \dots, \underline{\xi}_N] = A\Psi$

LET'S LOOK AT THE SOLUTION OF $\dot{\underline{\xi}} = A\underline{\xi} + \underline{g}$ ($\underline{g} = B\underline{u}$) WHERE $\underline{g}$ IS DUE TO THE SOURCE TERMS

LET'S DEFINE $\frac{d}{dt}\underline{\xi} = \frac{d}{dt}\xi_i \quad i=1,\dots,N$ THE ENTRIES OF THE VECTOR $\underline{\xi}$

$\frac{d}{dt}A = \frac{d}{dt}a_{ij} \quad i=1,\dots,N \;\&\; j=1,\dots,m$ THE ENTRIES OF MATRIX A

① SOLVE HOMOGENEOUS EQN $\dot{\underline{\xi}} = A\underline{\xi}$ $\quad(\dot{\underline{\xi}} - A\underline{\xi} = \underline{0})$

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d) THEN A SOLUTION TO $\dot{\underline{\xi}} = A\underline{\xi}$ IS $\underline{\xi} = \underline{y}e^{rt}$

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$$\underline{\xi} = c_1\underbrace{\underline{y}_1 e^{r_1 t}}_{\underline{\xi}_1} + c_2\underbrace{\underline{y}_2 e^{r_2 t}}_{\underline{\xi}_2} + \dots + c_N\underbrace{\underline{y}_N e^{r_N t}}_{\underline{\xi}_N} = \sum_{n=1}^{N} c_i\underline{\xi}_i$$

VECTORS $\underline{y}_1, \underline{y}_2, \dots, \underline{y}_N$ ARE LINEARLY INDEPENDENT

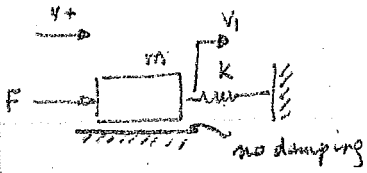
f) WE CAN CREATE THE FUNDAMENTAL MATRIX Ψ BY HAVING THE COLUMNS OF Ψ BE

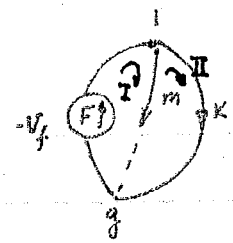
$$\Psi = [\underline{\xi}_1, \underline{\xi}_2, \dots, \underline{\xi}_N]$$

g) AND $[\underline{y}_1, \underline{y}_2, \dots, \underline{y}_N] = T$ (T SHALL HAVE AS ITS COLUMNS THE $\underline{y}$ VECTORS)

h) AND $Q = \begin{pmatrix} e^{r_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{r_2 t} & 0 & \dots & 0 \\ 0 & & \ddots & & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & e^{r_N t} \end{pmatrix}$ IS A MATRIX SO THAT $\Psi = TQ$

i) NOW $\dot{\Psi} = [\dot{\underline{\xi}}_1, \dot{\underline{\xi}}_2, \dot{\underline{\xi}}_3, \dots, \dot{\underline{\xi}}_N] = [A\underline{\xi}_1, A\underline{\xi}_2, \dots, A\underline{\xi}_N] = A[\underline{\xi}_1, \underline{\xi}_2, \dots, \underline{\xi}_N] = A\Psi$





$b=3 \quad s=1 \quad b-s=2$

$F_m = m\frac{dV_{1g}}{dt}$

$\frac{dF_k}{dt} = kV_{1g}$

NODES $n-1 = 2-1 = 1$

$$F = F_m + F_K \Rightarrow \dot{F} = \dot{F}_m + \dot{F}_k \Rightarrow \dot{F} = m\dot{V}_{1g} + kV_{1g}$$

PATH $b-(n-p) = 3-(2-1) = 2$

$$-V_F + V_{1g} = 0$$

$$-V_{1g} + V_{1g} = 0$$

since one A type (mass) equation must be written in terms of $\dot{V}_{1g}$

since one T type (spring) " " " " " " " " $\dot{F}_k$

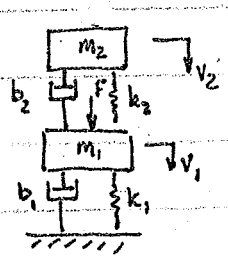
$$\textcircled{1} \quad \dot{F}_k = kV_{1g} = 0F_k + kV_{1g} + 0F \qquad \begin{pmatrix}\dot{F}_k \\ \dot{V}_{1g}\end{pmatrix} = \begin{bmatrix}0 & k \\ -\frac{1}{m} & 0\end{bmatrix}\begin{pmatrix}F_k \\ V_{1g}\end{pmatrix} + \begin{pmatrix}0 \\ \frac{1}{m}\end{pmatrix}F$$

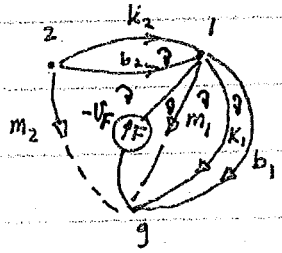
$$\textcircled{2} \quad \dot{V}_{1g} = \frac{F_m}{m} = \frac{F - F_k}{m} = -\frac{F_k}{m} + 0V_{1g} + \frac{F}{m}$$

here $\dot{\underline{x}} = \begin{pmatrix}\dot{F}_k \\ \dot{V}_{1g}\end{pmatrix}$; $A = \begin{bmatrix}0 & k \\ -\frac{1}{m} & 0\end{bmatrix}$; $\underline{x} = \begin{pmatrix}F_k \\ V_{1g}\end{pmatrix}$; $B = \begin{pmatrix}0 \\ \frac{1}{m}\end{pmatrix}$; $\underline{u} = F$

* NOTE THE FORM OF THE STATE VARIABLE EQS

$$\dot{q}_1 = f_1(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_m, t)$$
$$\dot{q}_2 = f_2(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_m, t)$$
$$\vdots$$
$$\dot{q}_n = f_n(q_1, q_2, \ldots, q_n, u_1, u_2, \ldots, u_m, t)$$





$b-s = 7-1 = 6$

$F_{m_2} = m_2\frac{dV_{2g}}{dt}$ $\quad F_{m_1} = m_1\frac{dV_{1g}}{dt}$

$\dot{F}_{k_2} = k_2V_{21}$ $\quad \dot{F}_{k_1} = k_1V_{1g}$

$F_{b_2} = b_2V_{21}$ $\quad F_{b_1} = b_1V_{1g}$

EXAMPLE: $\dot{\underline{\zeta}} = A\underline{\zeta}$ WHERE $A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}$

a) FORM $A - rI = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - r\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} - \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix}$

b) FIND DETERMINANT $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\cdot d - b\cdot c = 0$ or $(3-r)(-2-r) - 2\cdot(-2) = r^2 - r - 2 = 0$

$r = 2,\ r = -1$

c) FIND $\underline{\gamma}_1$ & $\underline{\gamma}_2$

for $r_1 = 2$ $\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}$ & $\begin{pmatrix} 1 & -2 \\ 2 & -4 \end{pmatrix}\begin{pmatrix} \gamma_{11} \\ \gamma_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

LET $\gamma_{21} = 1 \Rightarrow \gamma_{11} = 2$ $\therefore \begin{pmatrix} \gamma_{11} \\ \gamma_{21} \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \underline{\gamma}_1$

FOR $r_2 = -1$ $\begin{pmatrix} 3-r & -2 \\ 2 & -2-r \end{pmatrix} = \begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}$ & $\begin{pmatrix} 4 & -2 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} \gamma_{12} \\ \gamma_{22} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow 2\gamma_{12} - 1\cdot\gamma_{22} = 0$

LET $\gamma_{12} = 1 \Rightarrow \gamma_{22} = 2$ $\therefore \begin{pmatrix} \gamma_{21} \\ \gamma_{22} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \underline{\gamma}_2$

d) $\underline{\zeta}_1 = \underline{\gamma}_1 e^{r_1 t} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$; $\underline{\zeta}_2 = \underline{\gamma}_2 e^{r_2 t} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t}$

e) $\underline{\zeta} = c_1\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t} + c_2\begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-t} = \begin{bmatrix} 2c_1 e^{2t} + c_2 e^{-t} \\ c_1 e^{2t} + 2c_2 e^{-t} \end{bmatrix}$

f) $\Rightarrow \Psi = \begin{pmatrix} 2e^{2t} & 1\cdot e^{-t} \\ 1\cdot e^{2t} & 2\cdot e^{-t} \end{pmatrix} = [\underline{\zeta}_1, \underline{\zeta}_2]$; ALSO $\Psi = TQ$

g) $\Rightarrow T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = [\underline{\gamma}_1, \underline{\gamma}_2]$

h) $\Rightarrow Q = \begin{pmatrix} e^{r_1 t} & 0 \\ 0 & e^{r_2 t} \end{pmatrix} = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}$

CHECK i) $\dot{\Psi} = \frac{d\Psi}{dt} = \begin{pmatrix} 4e^{2t} & -1\cdot e^{-t} \\ 2\cdot 1e^{2t} & -1\cdot 2e^{-t} \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} 2e^{2t} & 1\cdot e^{-t} \\ 1\cdot e^{2t} & 2e^{-t} \end{pmatrix} = \begin{pmatrix} \overbrace{3\cdot 2e^{2t} - 2\cdot 1\cdot e^{2t}}^{4e^{2t}} & \overbrace{3\cdot 1\cdot e^{-t} - 2\cdot 2e^{-t}}^{-e^{-t}} \\ \underbrace{2\cdot 2e^{2t} - 2\cdot 1\cdot e^{2t}}_{2e^{2t}} & \underbrace{2\cdot 1\cdot e^{-t} - 2\cdot 2e^{-t}}_{-2e^{-t}} \end{pmatrix}$

$= A\Psi$ ✓

(2) WHAT ABOUT THE INHOMOGENEOUS DIFF. EQN. $\dot{\underline{\xi}} = A\underline{\xi} + \underline{g}$

FROM VARIATION OF PARAMETERS, IF WE KNOW SOLUTION TO HOMOGENEOUS EQN, Ψ, THEN TO FIND THE PARTICULAR SOLUTION LET $\underline{\xi}_p = \Psi \underline{d}$

NOW $\underline{\xi}_p$ MUST SATISFY $\dot{\underline{\xi}}_p = A\underline{\xi}_p + \underline{g} \Rightarrow \dot{\underline{\xi}}_p = (\dot{\Psi}\underline{d}) = \dot{\Psi}\underline{d} + \Psi\dot{\underline{d}} = A\underline{\xi}_p + \underline{g} = A\Psi\underline{d} + \underline{g}$

$$\dot{\Psi}\underline{d} + \Psi\dot{\underline{d}} = A\Psi\underline{d} + \underline{g} \Rightarrow (\dot{\Psi} - A\Psi)\underline{d} + \Psi\dot{\underline{d}} = \underline{g}$$

NOW $\dot{\Psi} - A\Psi = 0$ FROM i) AND $\dot{\underline{d}} = \Psi^{-1}\underline{g}$ SO $\underline{d} = \int_0^t \Psi^{-1}\underline{g}\, d\bar{t}$

AND $\underline{\xi}_p = \Psi \int_0^t \Psi^{-1}\underline{g}\, d\bar{t}$

j) FIND $\Psi^{-1} = \frac{1}{\det\Psi}\begin{bmatrix} \Psi_{22} & -\Psi_{12} \\ -\Psi_{21} & \Psi_{11} \end{bmatrix}$; $\det\Psi = 2e^{2t}\cdot 2e^{-t} - 1\cdot e^{2t}\cdot 1\cdot e^{-t} = 3e^{t}$

$$\Psi^{-1} = \frac{1}{3e^t}\begin{bmatrix} 2e^{-t} & -e^{-t} \\ -e^{2t} & 2e^{2t} \end{bmatrix} = \begin{bmatrix} \frac{2}{3}e^{-2t} & -\frac{1}{3}e^{2t} \\ -\frac{1}{3}e^{t} & \frac{2}{3}e^{t} \end{bmatrix}$$

k) SUPPOSE $\underline{g} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ THEN $\Psi^{-1}\underline{g} = \begin{pmatrix} \frac{2}{3}e^{-2t}\cdot 0 - \frac{1}{3}e^{-2t}\cdot 1 \\ -\frac{1}{3}e^{t}\cdot 0 + \frac{2}{3}e^{t}\cdot 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}e^{-2t} \\ \frac{2}{3}e^{t} \end{pmatrix}$

NOW $\int_0^t \Psi^{-1}\underline{g}\, d\bar{t} = \int_0^t \begin{pmatrix} -\frac{1}{3}e^{-2\bar{t}} \\ \frac{2}{3}e^{\bar{t}} \end{pmatrix} d\bar{t} = \begin{pmatrix} \int_0^t -\frac{1}{3}e^{-2\bar{t}}\, d\bar{t} \\ \int_0^t \frac{2}{3}e^{\bar{t}}\, d\bar{t} \end{pmatrix} = \begin{pmatrix} +\frac{1}{6}e^{-2t} + \frac{1}{6} \\ \frac{2}{3}e^{t} - \frac{2}{3} \end{pmatrix} = \underline{d}$

l) NOW $\underline{\xi}_p = \Psi\underline{d} = \begin{bmatrix} 2e^{2t} & 1e^{-t} \\ 1e^{2t} & 2e^{-t} \end{bmatrix}\begin{pmatrix} +\frac{1}{6}e^{-2t} + \frac{1}{6} \\ \frac{2}{3}e^{t} - \frac{2}{3} \end{pmatrix} = \begin{pmatrix} 1 + \frac{1}{3}e^{2t} - \frac{2}{3}e^{t} \\ \frac{3}{2} + \frac{1}{3}e^{2t} - 2e^{t} \end{pmatrix}$

AND THE TOTAL SOL'N IS $\Psi\underline{c} + \underline{\xi}_p$ where $\underline{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$

$\Psi\underline{c}$ = (e) ; $\underline{\xi}_p$ = (l)

LET'S LOOK AT $\dot{\underline{x}} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}\underline{x} \Rightarrow A - rI = \begin{pmatrix} 1-r & -1 \\ 1 & 3-r \end{pmatrix}$

DETERMINANT $= (1-r)(3-r) - 1\cdot(-1) = r^2 - 4r + 3 + 1 = r^2 - 4r + 4 = (r-2)^2$

OR THE ROOTS ARE MULTIPLES $r_1 = 2 \quad r_2 = r_1$

- $A - r_1 I = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}$ & $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow y_{11} = -y_{21}$ or $\begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

or $\begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$. SO ONE SOLUTION IS $\underline{x}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} = \underline{y}_1 e^{2t}$

- CANNOT DO THIS FOR SECOND ROOT: WE ASSUME $\underline{x}_2 = \underline{y}_1 t e^{2t} + \underline{y}_2 e^{2t}$

$\Rightarrow$ HOWEVER IF YOU SOLVE $(A - r_2 I)\underline{y}_2 = \underline{y}_1$ or $\begin{pmatrix} -1 & -1 \\ 1 & +1 \end{pmatrix}\begin{pmatrix} y_{12} \\ y_{22} \end{pmatrix} = \begin{pmatrix} y_{11} \\ y_{21} \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

ONE SOLUTION IS FOUND FROM $-1\cdot y_{12} - 1\cdot y_{22} = y_{11} = 1$

SO $y_{12} = -1 - y_{22}$

AND $\begin{pmatrix} y_{12} \\ y_{22} \end{pmatrix} = \begin{pmatrix} -1 - y_{22} \\ y_{22} + 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} + y_{22}\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix} - y_{22}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

IF YOU USE SECOND EQ. $y_{12} + y_{22} = -1 \quad y_{12} = -y_{22} - 1$ SAME AS BEFORE

SINCE $\underline{x}_2 = \underline{y}_1 t e^{2t} + \underline{y}_2 e^{2t} \Rightarrow \underline{x}_2 = \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + k\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] e^{2t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t}$

SO $\underline{x} = c_2\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t} + \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} + k\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] e^{2t}\right\} + c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t}$ WITHOUT LOSS OF GENERALITY TAKE $k \equiv 0$

$$\underline{x} = c_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{2t} + c_2\left\{\begin{pmatrix} -1 \\ 0 \end{pmatrix} e^{2t} + \begin{pmatrix} 1 \\ -1 \end{pmatrix} t e^{2t}\right\}$$

$$\underline{\Psi} = \begin{bmatrix} e^{2t} & -e^{2t} + t e^{2t} \\ -e^{2t} & 0e^{2t} - t e^{2t} \end{bmatrix} = \begin{bmatrix} 1 & (-1+t) \\ -1 & (0-t) \end{bmatrix}\begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix}$$

$\underline{\Psi}$: $\underline{x}_1$, $\underline{x}_2$ — T — Q

The four state variables are $\omega_{1g}, \omega_{3g}, T_{K1}, T_{K2}$

Write equations in terms of $\omega_{1g}, \omega_{3g}, T_{K1}, T_{K2}, T_1$ and T_2

$$\frac{d\omega_{1g}}{dt} = \frac{T_{J_1}}{J_1} = \frac{T_1 - T_{K_1}}{J_1}$$

$$\frac{d\omega_{3g}}{dt} = \frac{T_{J_2}}{J_2} = \frac{T_B - T_{K_2} - T_2}{J_2} = -\frac{(T_{K_2} + T_2)}{J_2} + \frac{T_{K_1}}{J_2}$$

$$\frac{dT_{K_1}}{dt} = K_1 \omega_{12} = K_1\left[\omega_{1g} - \omega_{23} - \omega_{3g}\right] = K_1(\omega_{1g} - \omega_{3g}) - K_1\frac{T_B}{B} = K_1(\omega_{1g} - \omega_{3g}) - K_1\frac{T_{K_1}}{B}$$

$$\frac{dT_{K_2}}{dt} = K_2 \omega_{3g}$$

$$\begin{bmatrix} \dot{\omega}_{1g} \\ \dot{\omega}_{3g} \\ \dot{T}_{K_1} \\ \dot{T}_{K_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{J_1} & 0 \\ 0 & 0 & \frac{1}{J_2} & -\frac{1}{J_2} \\ K_1 & -K_1 & -K_1/B & 0 \\ 0 & K_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_{1g} \\ \omega_{3g} \\ T_{K_1} \\ T_{K_2} \end{bmatrix} + \begin{bmatrix} \frac{1}{J_1} & 0 \\ 0 & -\frac{1}{J_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

The five state variables are $v_{1g}, v_{2g}, F_{K_1}, v_{3g}, F_{K_2}$

[reduced by one since one of state var is the source (known)]

$$\frac{dv_{1g}}{dt} = \frac{F_{m_1}}{m_1} = \frac{F_{b_1} - F_{b_1}}{m_1} = \frac{F_{b_1} - b_1(v_{12})}{m_1} = \frac{F_{b_1} - b_1(v_{1g} - v_{2g})}{m_1}$$

[reduced by one since one of state variable is known]

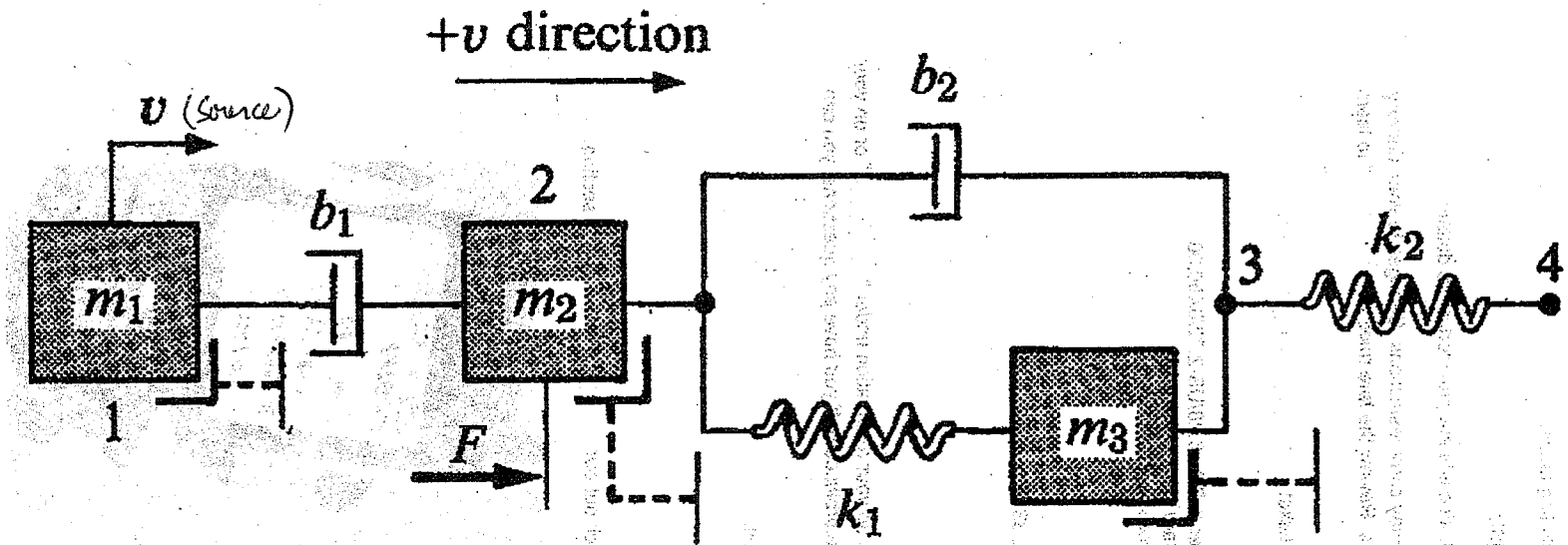
$$\frac{dv_{2g}}{dt} = \frac{F_{m_2}}{m_2} = \frac{1}{m_2}\left[F_{b_1} + F - F_{b_2} - F_{K_1}\right] = \frac{1}{m_2}\left[F - F_{K_1} + b_1(v_{1g} - v_{2g}) - b_2(v_{2g} - v_{3g})\right]$$

$$\frac{dv_{3g}}{dt} = \frac{F_{m_3}}{m_3} = \frac{1}{m_3}\left[F_{b_2} + F_{K_1} - F_{K_2}\right] = \frac{1}{m_3}\left[b_2(v_{2g} - v_{3g}) + F_{K_1} - F_{K_2}\right]$$

$$\frac{dF_{K_1}}{dt} = k_2(v_{2g} - v_{3g})$$

$$\frac{dF_{K_2}}{dt} = k_2(v_{34}) = k_2(v_{3g} - v_{4g}) \text{ ; since no reason for } k_2 \text{ to } \Longleftrightarrow \Rightarrow v_{3g} = v_{4g}$$

$$\begin{bmatrix} \dot{v}_{1g} \\ \dot{v}_{2g} \\ \dot{v}_{3g} \\ \dot{F}_{K_1} \\ \dot{F}_{K_2} \end{bmatrix} = \begin{bmatrix} -\frac{b_1}{m_1} & \frac{b_1}{m_1} & 0 & 0 & 0 \\ \frac{b_1}{m_2} & -\frac{(b_1+b_2)}{m_2} & \frac{b_2}{m_2} & -\frac{1}{m_2} & 0 \\ 0 & \frac{b_2}{m_3} & -\frac{b_2}{m_3} & \frac{1}{m_2} & -\frac{1}{m_2} \\ 0 & k_2 & -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{1g} \\ v_{2g} \\ v_{3g} \\ F_{K_1} \\ F_{K_2} \end{bmatrix} + \begin{bmatrix} \frac{1}{m_1} & 0 \\ \frac{b_1}{m_2} & \frac{1}{m_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v \\ F \end{pmatrix}$$



From b-s=7

$F_{m_1} = m_1 \frac{dv_{1g}}{dt}$

$F_{b_1} = b_1 v_{12}$

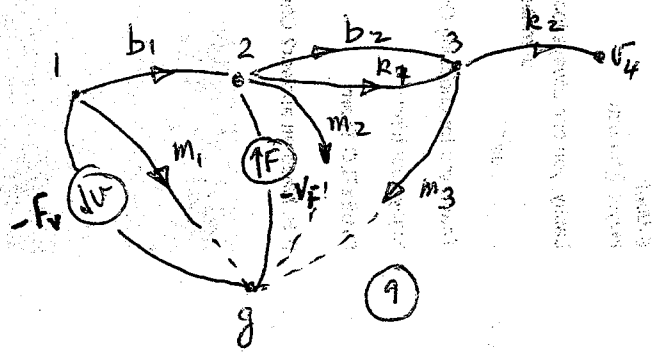
$F_{b_2} = b_2 v_{23}$

$\frac{dF_{k_1}}{dt} = k_1 v_{23}$

$F_{m_2} = m_2 \frac{dv_{2g}}{dt}$

$F_{m_3} = m_3 \frac{dv_{3g}}{dt}$

$\frac{dF_{k_2}}{dt} = k_2 v_{34}$



b = 9
s = 2
n = 5
p = 1

b-s = 7

b-(n-p) = 5

n-p = 4

Continuity

$-F_v + F_{m_1} + F_{b_1} = 0$

$-F_{b_1} - F + F_{m_2} + F_{b_2} + F_{k_1} = 0$

$F_{b_2} + F_{k_1} - F_{m_3} - F_{k_2} = 0$

$F_{k_2} = 0$ (4)

Compat

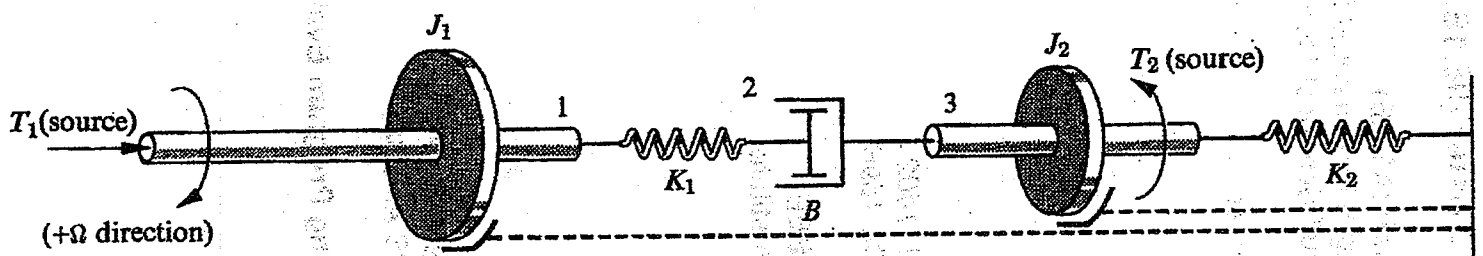
$-v + v_{1g} = 0$

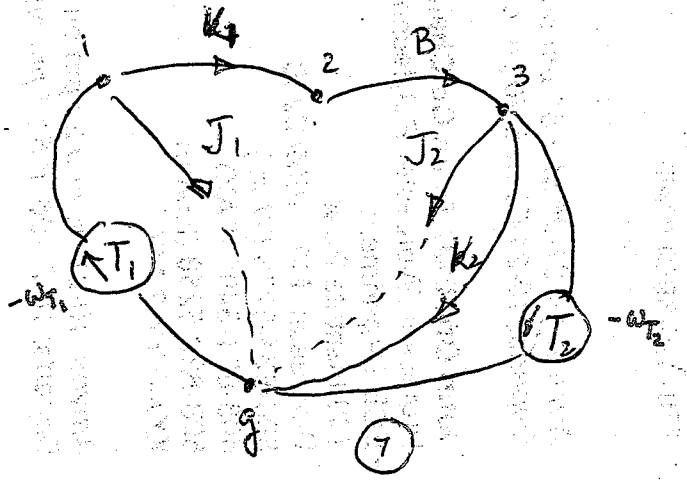
$-v_{1g} + v_{12} + v_{2g} = 0$

$-v_F + v_{2g} = 0$ (5)

$-v_{2g} + v_{23} + v_{3g} = 0$

$-v_{23} + v_{23} = 0$





from b-s=5

$T_{J_1} = J_1 \frac{d\omega_{1g}}{dt}$

$\frac{dT_{K1}}{dt} = K_1 \omega_{12}$

$T_B = B\omega_{23}$

$T_{J_2} = J_2 \frac{d\omega_{3g}}{dt}$

$\frac{dT_{K_2}}{dt} = K_2 \omega_{3g}$

$b = 7$
$s = 2$
$n = 4$
$p = 1$

$b-s=5$
$n-p=3$
$b-(n-p)=4$

$T_1 = T_{K_1} + T_{J_1}$

$T_{K_1} = T_B$

$T_B = T_{J_2} + T_{K_2} + T_2$

(3)

$-\omega_{T_1} + \omega_{1g} = 0$

$-\omega_{1g} + \omega_{12} + \omega_{23} + \omega_{3g} =$

$-\omega_{3g} + \omega_{3g} = 0$

$-\omega_{3g} + (-\omega_{T_2}) = 0$

(4)

7.27 An electric motor is sometimes used to move the spool valve of a hydraulic motor. In Figure P7.27 the force f is due to an electric motor acting through a rack-and-pinion gear. Develop a model of the system with the load displacement y as the output and the force f as the input. Consider two cases: (a) $m_1 = 0$ and (b) $m_1 \neq 0$.

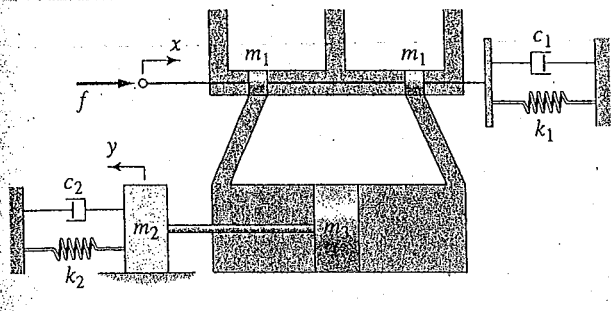


Figure P7.27

7.28 In Figure P7.28 the piston of area A is connected to the axle of the cylinder of radius R, mass m, and inertia I about its center. Develop a dynamic model of the axle's translation x, with the pressures p_1 and p_2 as the inputs.

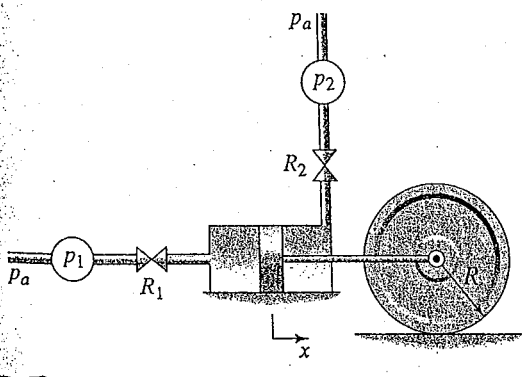


Figure P7.28

7.29 Figure P7.29 shows a pendulum driven by a hydraulic piston. Assuming small angles θ and a concentrated mass m a distance L_1 from the pivot, derive the equation of motion with the pressures p_1 and p_2 as inputs.

7.30 Figure P7.30 shows an example of a *hydraulic accumulator,* which is a device for reducing pressure fluctuations in a hydraulic line or pipe. The fluid density is ρ, the plate mass is m, and the plate area is A. Develop a dynamic model of the pressure p with the pressures p_1 and p_2 as the given inputs. Assume that $m\ddot{x}$ of the plate is small, and that the hydrostatic pressure $\rho g h$ is small.

7.31 Design a hydraulic accumulator of the type shown in Figure P7.30. The liquid volume in the accumulator should increase by 30 in.^{3} when the pressure p increases by 1.5 lb/in.^{2}. Determine suitable values for the plate area A and the spring constant k.

7.32 Consider the liquid-level system treated in Example 7.4.8 and shown in Figure 7.4.9. The pump curve and the line for the steady-state flow through both valves are shown in Figure P7.32. It is known that the bottom area of the tank is 2 m^2 and the outlet resistance is $R_2 = 400$ l/(m · s). (a) Compute the pump resistance R_1 and the steady-state height. (b) Derive a linearized dynamic model of the height deviation δh in the tank.

Figure P7.29

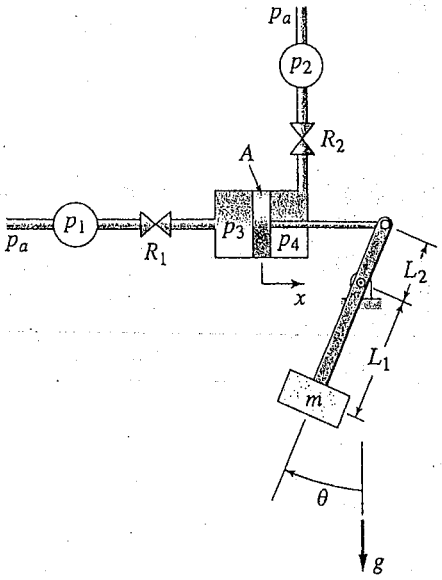


Figure P7.30

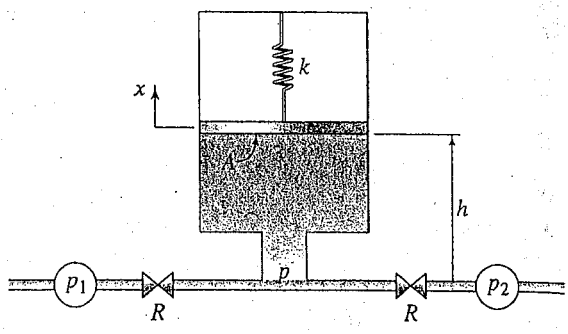
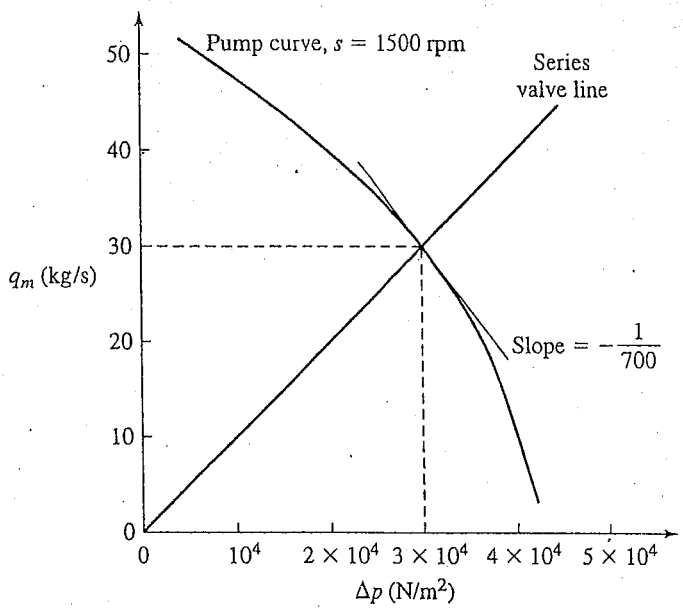


Figure P7.32



7.33 Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.33. The outlet resistance is linear. Derive the dynamic model of the height h.

7.34 Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.34. The outlet is an orifice of area A_o and discharge coefficient C_d. Derive the dynamic model of the height h.

7.35 Consider the cylindrical container treated in Problem 7.8. In Figure P7.35 the tank is shown with a valve outlet at the bottom of the tank. Assume that the flow through the valve is turbulent with a resistance R. Derive the dynamic model of the height h.

with regard to the information about the system behavior incorporated in the model equations, the techniques employed in their derivation and solution are principally different. In most cases the state models, consisting of sets of first order differential equations, are much easier to derive and to solve by computer simulation than the input-output equations. This is especially true when the mathematical models involve nonlinearities or when systems modeled have many inputs and outputs and are to be simulated on the computer.

The state models are based on the concept of state variables. The choice of state variables for a given system is not unique. Four different types of state variables were used in example problems presented in this chapter. It was shown that the number of state variables necessary to describe the system dynamics is always equal to the number of independent energy-storing elements regardless of the type of state variables used.

Problems

4.1. Derive a complete set of state model equations for the mechanical rotational system shown in Figure P4.1. Select the following state variables:
(a) T-type and A-type variables
(b) one variable and its derivative
Use torque $T_i(t)$ as the input variable and angular displacement $\Theta_{1g}(t)$ as the output variable in each model.

4.2. Derive an input-output model equation for the system shown in Figure P4.1, using torque $T_i(t)$ as the input variable and angular displacement $\Theta_1(t)$ as the output variable.

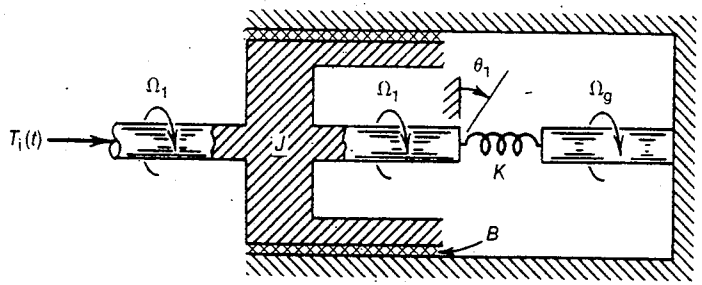


FIGURE P4.1 Mechanical system considered in Problem 4.1.

4.3. Derive complete sets of state-model equations for the system shown in Figure P4.3 using three different sets of state variables. The input variable is torque $T_m(t)$ and the output variables are displacements $x_1(t)$ and $x_2(t)$. Present the state models in a matrix form.

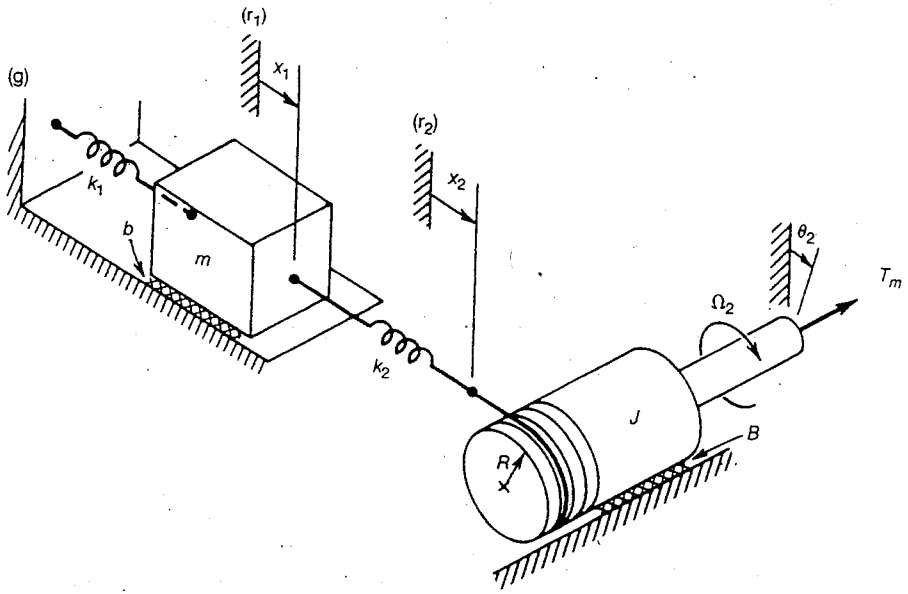


FIGURE P4.3 Mechanical system considered in Problem 4.3.

4.4. A nonlinear dynamic system has been modeled by the following state-variable equations:

$$\dot{q}_1 = -4q_1 + 10q_2 + 4u_1$$
$$\dot{q}_2 = 0.5q_1 - 2|q_2|q_2$$

where q_2 is also the output variable and u_1 is the input variable.
(a) Linearize the state-variable equations.
(b) Find the input-output equation for the linearized model.

4.5. The state-model matrices of a single-input single-output linear dynamic system are

$$\mathbb{A} = \begin{bmatrix} -3 & -19 \\ 1 & -2 \end{bmatrix} \quad \mathbb{B} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$
$$\mathbb{C} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \mathbb{D} = 0$$

The column vector $\mathbf{q}$ of the system state variables contains q_1, q_2. Find the input-output model for this system.

4.6. A linear dynamic system is described by the following differential equations:

$$\ddot{y} + 4\dot{y} + 4y = 2\dot{x} + 2x$$
$$2\dot{x} + x - y = u(t)$$

(a) Derive the input-output model for this system using $y(t)$ as the output and $u(t)$ as the input variable.
(b) Derive a state model and present it in a matrix form.

4.7. Derive a set of state variable equations for the mechanical system shown in Figure P4.7.

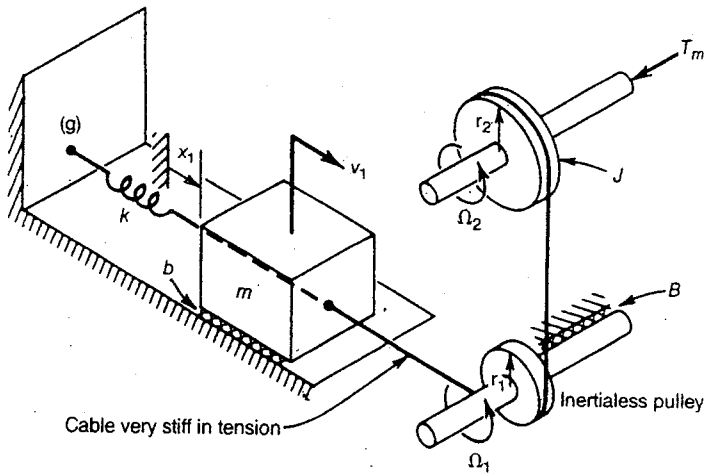


FIGURE P4.7 Mechanical systems considered in Problems 4.7 and 4.8.

4.8. Derive the input-output model equation for the system shown in Figure P4.7 using torque $T_m(t)$ as the input variable and angular velocity Ω_{2g} as the output variable.

4.9. A lumped model of a machine-tool drive system is shown in Figure P4.9. The system parameters are $J_m = 0.2$ N-m-sec^2/rad, $B = 30$ N-m-sec/rad, $m = 16$ kg, $k = 20$ N/m, and $R = 0.5$ m. The stiffness of the shaft is represented by a nonlinear torque, $T_{NLK} = 2|\Theta_1 - \Theta_2|(\Theta_1 - \Theta_2)$. The force of the nonlinear friction device NLD is $F_{NLD} = f_{NL}(v_{1g}) = 2v_{1g}^3 + 4v_{1g}$. The system is driven by a torque consisting of a constant and an incremental component, $T_i(t) = 0.8 + 0.02 \sin(0.1t)$ N-m.
(a) Select state variables and derive nonlinear state-variable equations.
(b) Find the normal operating point values for all state variables.
(c) Linearize the state-model equations in the vicinity of the normal operating point. Present the linearized state model in a matrix form.
(d) Derive the input-output model equation for the linearized system using the incremental torque, $\hat{T}_i(t)$, as the input variable and incremental displacement, $\hat{x}_1(t)$, as the output variable.
(e) Draw a simulation block diagram for this system.

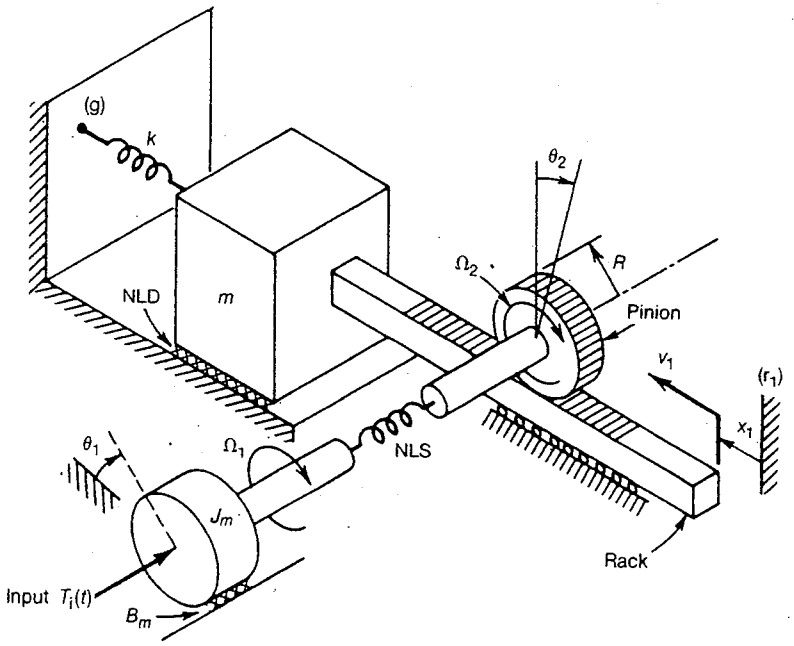


FIGURE P4.9 Lumped model of a machine-tool drive system.

4.10. Derive state-variable equations for the system shown in Figure 2.15. Use velocities v_{1g} and v_{2g} and the spring force F_{k1} as the state variables. Note that the two springs in this system are not independent as long as the lever is massless. Combine the state-variable equations to obtain an input-output equation relating velocity v_{2g} to force $F_s(t)$.

The four state variables are $\omega_{1g}, \omega_{3g}, T_{K1}, T_{K2}$

Write equations in terms of $\omega_{1g}, \omega_{3g}, T_{K1}, T_{K2}, T_1$ and T_2

$$\frac{d\omega_{1g}}{dt} = \frac{T_{J_1}}{J_1} = \frac{T_1 - T_{K_1}}{J_1}$$

$$\frac{d\omega_{3g}}{dt} = \frac{T_{J_2}}{J_2} = \frac{T_B - T_{K_2} - T_2}{J_2} = -\frac{(T_{K_2} + T_2)}{J_2} + \frac{T_{K_1}}{J_2}$$

$$\frac{dT_{K_1}}{dt} = K_1 \omega_{12} = K_1\left[\omega_{1g} - \omega_{23} - \omega_{3g}\right] = K_1(\omega_{1g} - \omega_{3g}) - K_1\frac{T_B}{B} = K_1(\omega_{1g} - \omega_{3g}) - K_1\frac{T_{K_1}}{B}$$

$$\frac{dT_{K_2}}{dt} = K_2 \omega_{3g}$$

$$\begin{bmatrix} \dot{\omega}_{1g} \\ \dot{\omega}_{3g} \\ \dot{T}_{K_1} \\ \dot{T}_{K_2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{J_1} & 0 \\ 0 & 0 & \frac{1}{J_2} & -\frac{1}{J_2} \\ K_1 & -K_1 & -K_1/B & 0 \\ 0 & K_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_{1g} \\ \omega_{3g} \\ T_{K_1} \\ T_{K_2} \end{bmatrix} + \begin{bmatrix} \frac{1}{J_1} & 0 \\ 0 & -\frac{1}{J_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$$

The five state variables are $v_{1g}, v_{2g}, F_{K_1}, v_{3g}, F_{K_2}$ [reduced by one since one of state var is the source (known)]

$$\frac{dv_{1g}}{dt} = \frac{F_{m_1}}{m_1} = \frac{F_v - F_{b_1}}{m_1} = \frac{F_v - b_1(v_{12})}{m_1} = \frac{F_v - b_1(v_{1g} - v_{2g})}{m_1}$$

[reduced by one since one of state variable is known]

$$\frac{dv_{2g}}{dt} = \frac{F_{m_2}}{m_2} = \frac{1}{m_2}\left[F_{b_1} + F - F_{k_2} - F_{K_1}\right] = \frac{1}{m_2}\left[F - F_{K_1} + b_1(v_{1g} - v_{2g}) - b_2(v_{2g} - v_{3g})\right]$$

$$\frac{dv_{3g}}{dt} = \frac{F_{m_3}}{m_3} = \frac{1}{m_3}\left[F_{b_2} + F_{K_1} - F_{K_2}\right] = \frac{1}{m_3}\left[b_2(v_{2g} - v_{3g}) + F_{K_1} - F_{K_2}\right]$$

$$\frac{dF_{K_1}}{dt} = k_2(v_{2g} - v_{3g})$$

$$\frac{dF_{K_2}}{dt} = k_2(v_{34}) = k_2(v_{3g} - v_{4g}) \text{ ; since no reason for } k_2 \text{ to } \rightleftarrows \Rightarrow v_{3g} = v_{4g}$$

$$\begin{bmatrix} \dot{v}_{1g} \\ \dot{v}_{2g} \\ \dot{v}_{3g} \\ \dot{F}_{K_1} \\ \dot{F}_{K_2} \end{bmatrix} = \begin{bmatrix} -\frac{b_1}{m_1} & \frac{b_1}{m_1} & 0 & 0 & 0 \\ \frac{b_1}{m_2} & -\frac{(b_1+b_2)}{m_2} & +\frac{b_2}{m_2} & -\frac{1}{m_2} & 0 \\ 0 & \frac{b_2}{m_3} & -\frac{b_2}{m_3} & \frac{1}{m_2} & -\frac{1}{m_2} \\ 0 & k_2 & -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{1g} \\ v_{2g} \\ v_{3g} \\ F_{K_1} \\ F_{K_2} \end{bmatrix} + \begin{bmatrix} \frac{1}{m_1} & 0 \\ -\frac{b_1}{m_2} & \frac{1}{m_2} \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v \\ F \end{pmatrix}$$

EML 3222

Oct 27, 2008

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$$\frac{d\omega_{1g}}{dt} = 0\,\omega_{1g} + 0\cdot\omega_{3g} - \frac{1}{J_1}T_{K_1} + 0\,T_{K_2} + \frac{1}{J_1}T_1 + 0\,T_2$$

$$\frac{d\omega_{3g}}{dt} = 0\,\omega_{1g} + 0\,\omega_{3g} + \frac{1}{J_2}T_{K_1} - \frac{1}{J_2}T_{K_2} + 0\,T_1 - \frac{1}{J_2}T_2$$

$$\frac{dT_{K_1}}{dt} = K_1\omega_{1g} - K_1\omega_{3g} - \frac{K_1}{B}\overset{*}{T}_{K_1} + 0\,T_{K_2} + 0\,T_1 + 0\,T_2$$

$$\frac{dT_{K_2}}{dt} = 0\omega_{1g} + K_2\omega_{3g} + 0\,T_{K_1} + 0\,T_{K_2} + 0\,T_1 + 0\,T_2$$

$$\dot{\varsigma} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix} \varsigma$$

■ Example

Determine the eigenvalues and corresponding eigenvectors associated with a system having an **A** matrix:

$$\mathbf{A} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}.$$

Solution: The characteristic equation is $\det[\lambda\mathbf{I} - \mathbf{A}] = 0$ or

$$\begin{aligned} \det\begin{bmatrix} \lambda+2 & -1 \\ -2 & \lambda+3 \end{bmatrix} &= 0 \\ \lambda^2 + 5\lambda + 4 &= 0 \\ (\lambda+4)(\lambda+1) &= 0. \end{aligned} \tag{63}$$

The two eigenvalues are therefore $\lambda_1 = -1$, and $\lambda_2 = -4$. To find the eigenvectors these values are substituted into the equation $[\lambda_i\mathbf{I} - \mathbf{A}]\underline{y}_i = 0$. For the case $\lambda_1 = -1$ this gives

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} y_{11} \\ y_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{64}$$

Both of these equations give the same result; $y_{11} = y_{21}$. The eigenvector cannot be further defined, and although one particular solution is

$$\underline{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

the general solution must be defined in terms of an unknown scalar multiplier α_1

$$\underline{y}_1 = \begin{bmatrix} \alpha_1 \\ \alpha_1 \end{bmatrix} \tag{65}$$

provided $\alpha_1 \neq 0$.

Similarly, for the second eigenvalue, $\lambda_2 = -4$, the equations are

$$\begin{bmatrix} -2 & -1 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} y_{12} \\ y_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{66}$$

which both state that $-2y_{12} = y_{22}$. The general solution is

$$\underline{y}_2 = \begin{bmatrix} \alpha_2 \\ -2\alpha_2 \end{bmatrix} \tag{67}$$

for $\alpha_2 \neq 0$. The following are all eigenvectors corresponding to $\lambda_2 = -4$:

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \begin{bmatrix} 15 \\ -30 \end{bmatrix} \qquad \begin{bmatrix} -7 \\ 14 \end{bmatrix}.$$

so now we have $\varsigma_1 = c_1 \underline{y_1} e^{-t}$ where $\underline{y_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\varsigma_2 = c_2 \underline{y_2} e^{-4t}$ where $\underline{y_2} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\begin{bmatrix} \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} \\ \\ \end{bmatrix}$$

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$\underline{\varsigma}_1 = \underline{y}_1 e^{\lambda_1 t}$ $\qquad$ $\underline{\varsigma}_2 = \underline{y}_2 e^{\lambda_2 t}$

so now we have $\varsigma_1 = \underline{y}_1 e^{-t}$ where $\underline{y}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\varsigma_2 = \underline{y}_2 e^{-4t}$ where $\underline{y}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\underline{\varsigma}_1 = \begin{pmatrix} 1 \cdot e^{-t} \\ 1 \cdot e^{-t} \end{pmatrix}$ $\qquad$ $\underline{\varsigma}_2 = \begin{pmatrix} 1 \cdot e^{-4t} \\ -2 \cdot e^{-4t} \end{pmatrix}$

and the Ψ matrix is $\begin{bmatrix} e^{-t} & e^{-4t} \\ e^{-t} & -2e^{-4t} \end{bmatrix}$. Its determinant is $(e^{-t})(-2e^{-4t})-(e^{-t})(e^{-4t})=-3e^{-5t}$

Its inverse Ψ^{-1} is $\frac{1}{-3e^{-5t}}\begin{bmatrix} -2e^{-4t} & -e^{-4t} \\ -e^{-t} & e^{-t} \end{bmatrix}$

Note: The Ψ matrix can be written as the product of the T and Q matrices ONLY if the roots of the eigenvalues equation are distinct

Suppose you wanted to solve $\dot{\varsigma} = \begin{bmatrix} -2 & 1 \\ 2 & -3 \end{bmatrix}\varsigma + \begin{pmatrix} e^{t} \\ e^{-2t} \end{pmatrix}$

Since we have already solved for the Ψ matrix and found the inverse, namely, Ψ^{-1} is $\frac{1}{-3e^{-5t}}\begin{bmatrix} -2e^{-4t} & -e^{-4t} \\ -e^{-t} & e^{-t} \end{bmatrix}$, we have to find

$$\dot{\underline{d}} = \Psi^{-1}g = \frac{1}{-3e^{-5t}}\begin{bmatrix} -2e^{-4t} & -e^{-4t} \\ -e^{-t} & e^{-t} \end{bmatrix}\begin{pmatrix} e^{t} \\ e^{-2t} \end{pmatrix} = \frac{1}{-3e^{-5t}}\begin{pmatrix} -2e^{-3t} - e^{-6t} \\ -1+e^{-3t} \end{pmatrix} = \begin{pmatrix} 2/3e^{2t} + 1/3e^{-t} \\ 1/3e^{5t} - 1/3e^{2t} \end{pmatrix}$$

integrate this to get $\underline{d} = \begin{pmatrix} 1/3e^{2t} - 1/3e^{-t} + 0 \\ 1/15e^{5t} - 1/6e^{2t} + 1/10 \end{pmatrix}$ and this must be multiplied by Ψ to get ζ_p

that is $\zeta_p = \frac{1}{-3e^{-5t}}\begin{bmatrix} -2e^{-4t} & -e^{-4t} \\ -e^{-t} & e^{-t} \end{bmatrix} \text{X} \begin{pmatrix} 1/3e^{2t} - 1/3e^{-t} + 0 \\ 1/15e^{5t} - 1/6e^{2t} + 1/10 \end{pmatrix} =$ $\underline{\varsigma}_p = \Psi\underline{d}$

$$\frac{1}{-3e^{-5t}}\begin{bmatrix} -2/3e^{-2t} + 2/3e^{-5t} - 1/15e^{t} + 1/6e^{-2t} - 1/10e^{-4t} \\ -1/3e^{t} + 1/3e^{-2t} + 1/15e^{4t} - 1/6e^{t} + 1/10e^{-t} \end{bmatrix} =$$

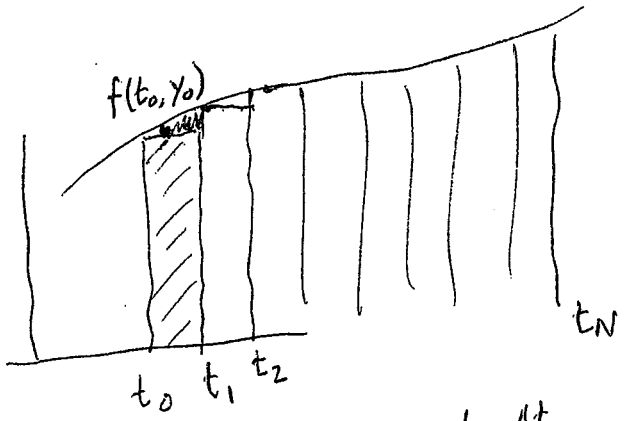
$$\frac{1}{-3e^{-5t}}\begin{bmatrix} -1/2e^{-2t} + 2/3e^{-5t} - 1/15e^{t} - 1/10e^{-4t} \\ -1/2e^{t} + 1/3e^{-2t} + 1/15e^{4t} + 1/10e^{-t} \end{bmatrix} = \begin{bmatrix} 1/6e^{3t} - 2/9 + 1/45e^{6t} + 1/30e^{t} \\ 1/6e^{6t} - 1/9e^{3t} - 1/45e^{9t} - 1/30e^{4t} \end{bmatrix}$$

Solve S.V. equation

$\frac{dy}{dt} = f(t,y) \quad y = y(t)$

given $y(t=t_0) = y_0$

$\int_{y_0}^{y} d\bar{y} \Leftarrow \int \frac{dy}{dt}\,dt = \int_{t_0}^{t} f(\bar{t},y)\,d\bar{t}$



$t_1 = t_0 + \Delta t, \quad t_2 = t_1 + \Delta t$

$$y - y_0 = \bar{y}\Big|_{y_0}^{y} = \int_{y_0}^{y} d\bar{y} = \int_{t_0}^{t} f(\bar{t},y)\,dt \cong f(t_0,y_0)\Delta t$$

Euler Method.

$$y(t_0+\Delta t) = y(t_1) \cong y_1 = y_0 + f(t_0,y_0)\Delta t$$

Given $t_1, y_1 \Rightarrow$ find y_2 using same method.

$$y_2 = y_1 + f(t_1,y_1)\Delta t$$

algorithm

GIVEN $t_0, y_0, f(t,y), \Delta t$

$\frac{t_N - t_0}{\Delta t} = N$

DO $I = 1, 2, \ldots, N \Rightarrow$

$y_I = y_{I-1} + f(t_{I-1}, y_{I-1})\Delta t$

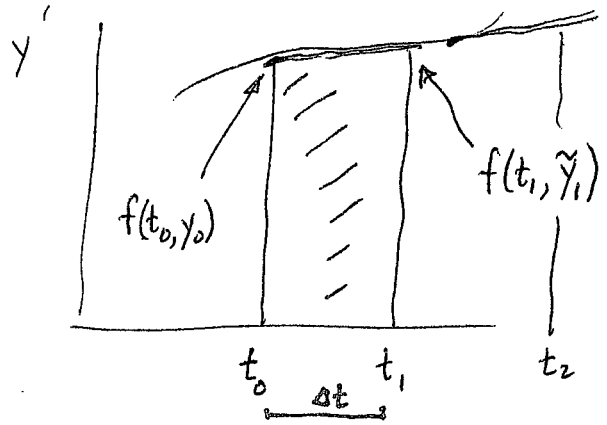
$t_I = t_{I-1} + \Delta t$

print t_I, y_I

$$y(t_0+\Delta t) = y(t_0) + \frac{dy}{dt}\Big|_{t=t_0}\cdot\Delta t + \frac{d^2y}{dt^2}\Big|_{t=t_0}\frac{\Delta t^2}{2!} + \frac{d^3y}{dt^3}\Big|_{t=t_0}\frac{\Delta t^3}{3!}$$

$$= y(t_0) + f(t_0,y_0)\Delta t \;\Big|\Big| \longrightarrow \text{local error per step error} \quad C\Delta t^2$$

for many steps error accumulates $\Rightarrow$ global $D\Delta t$



area under graph ▨ using trapezoidal rule

$$y(t=t_1) \cong y_1 = y_0 + \frac{f(t_0,y_0) + f(t_1,\tilde{y}_1)}{2}\Delta t$$

$$\tilde{y}_1 = y_0 + f(t_0,y_0)\Delta t$$

$$t_1 = t_0 + \Delta t$$

GIVEN $t_0, y_0, f(t,y), \Delta t$ MOD EULER

Do $I = 1, 2, \ldots, N$

$$Y_{IP} = Y_{I-1} + f(t_{I-1}, Y_{I-1})\Delta t$$

$$t_I = t_{I-1} + \Delta t$$

$$Y_I = Y_{I-1} + \left[f(t_{I-1}, Y_{I-1}) + f(t_I, Y_{IP})\right]\frac{\Delta t}{2}$$

print t_I, Y_I

local error $C\Delta t^3$ (per step)

global error $D\Delta t^2$ (over many steps)

---

for 2 s.v. eqs.

$$\frac{dy}{dt} = f(t,y,z) \qquad @\ t=t_0\ \ y=y_0\ \ z=z_0 \qquad (x=x_0)$$

$$\frac{dz}{dt} = g(t,y,z)$$

$$\frac{dx}{dt} = h(t,y,z,x)$$

$$y_1 = y_0 + f(t_0,y_0,z_0)\Delta t$$

$$z_1 = z_0 + g(t_0,y_0,z_0)\Delta t$$

$$t_1 = t_0 + \Delta t$$

3 state variables

$$x_1 = x_0 + h(t_0,y_0,z_0,x_0)\Delta t$$

GIVEN t_0, y_0, z_0, $f(t,y,z)$, $g(t,y,z)$, Δt

DO $I = 1, 2, \ldots, N$ $\qquad N = \frac{t_N - t_0}{\Delta t}$

$y_I = y_{I-1} + f(t_{I-1}, y_{I-1}, z_{I-1})\Delta t$

$z_I = z_{I-1} + g(t_{I-1}, y_{I-1}, z_{I-1})\Delta t$

$t_I = t_{I-1} + \Delta t$

print t_I, y_I, z_I

Euler Method for 2 state variable eqs.

$x_I = x_{I-1} + h(t_{I-1}, y_{I-1}, z_{I-1}, x_{I-1})\Delta t$ ← For 3 state variables

for example

$f(t,y,z) = e^t + \sin y + 2z$ $\qquad f(t_{I-1}, y_{I-1}, z_{I-1}) = e^{t_{I-1}} + \sin y_{I-1} + 2z_{I-1}$

$g(t,y,z) = t^2 + 4yz$ $\qquad g(t_{I-1}, y_{I-1}, z_{I-1}) = t_{I-1}^2 + 4y_{I-1} \cdot z_{I-1}$

MOD EULER 2 STATE VARIABLES

Given t_0, y_0, z_0, $f(t,y,z)$, $g(t,y,z)$, Δt

DO $I = 1, 2, \ldots, N$

$y_{IP} = y_{I-1} + f(t_{I-1}, y_{I-1}, z_{I-1})\Delta t$

$z_{IP} = z_{I-1} + g(t_{I-1}, y_{I-1}, z_{I-1})\Delta t$

3 S.V. EQS.

$x_{IP} = x_I + h(t_{I-1}, y_{I-1}, z_{I-1}, x_{I-1})\Delta t$

$t_I = t_{I-1} + \Delta t$

$y_I = y_{I-1} + \left[f(t_{I-1}, y_{I-1}, z_{I-1}) + f(t_I, y_{IP}, z_{IP})\right]\frac{\Delta t}{2}$

$z_I = z_{I-1} + \left[g(t_{I-1}, y_{I-1}, z_{I-1}) + g(t_I, y_{IP}, z_{IP})\right]\frac{\Delta t}{2}$

$x_I = x_{I-1} + [h(\quad) + h(\quad)]\frac{\Delta t}{2}$

print t_I, y_I, z_I

Example: given $y' = f(t,y) = 5y^2t + 3t$ with $y=1$ at $t=0$, use Euler & Mod Euler

euler

$$\frac{dy}{dt} = 5y^2t + 3t = f(t,y)$$

Given: $y_0 = 1$ when $t_0 = 0$

$f(t_0,y_0) = 0$ let $\Delta t = 0.1$ sec (our choice of step)

$y_1 = y_0 + f(t_0,y_0)\Delta t = 1 + 0(.1) = 1$

$t_1 = t_0 + \Delta t = 0.1$

$f(t_1,y_1) = 5(1)^2(0.1) + 3(.1) = .8$

$y_2 = y_1 + f(t_1,y_1)\Delta t = 1 + 0.8(.1) = 1.08$

$t_2 = t_1 + \Delta t = .1 + .1 = .2$

$f(t_2,y_2) = 5(1.08)^2(.2) + 3(.2) = 1.7664$

$y_3 = y_2 + f(t_2,y_2)\Delta t = 1.08 + 1.7664(.1) = 1.25664$

$t_3 = t_2 + \Delta t = 0.2 + 0.1 = .3$

t	y
0	1
.1	1
.2	1.08
.3	1.2566

---

mod euler

$$\frac{dy}{dt} = 5y^2t + 3t = f(t,y)$$

let $y_0 = 1$ when $t_0 = 0$

$f(t_0,y_0) = 5(1)^2(0) + 3(0) = 0$ let $\Delta t = 0.1$ sec

predicted $\tilde{y}_1 = y_0 + f(t_0,y_0)\Delta t = 1 + 0(.1) = 1$

$t_1 = t_0 + \Delta t = 0 + 0.1 = 0.1$

$f(t_1,\tilde{y}_1) = 5\tilde{y}_1^2 t_1 + 3t_1 = 5(1)^2(.1) + 3(.1) = 0.8$

corrected $y_1 = y_0 + \left[f(t_0,y_0) + f(t_1,\tilde{y}_1)\right]\frac{\Delta t}{2} = 1 + \left[0 + 0.8\right]\frac{0.1}{2} = 1.04$

$t_1 = t_0 + \Delta t = 0.1$

$f(t_1,y_1) = 5(1.04)^2(.1) + 3(.1) = .8408$

predicted $\tilde{y}_2 = y_1 + f(t_1,y_1)\Delta t = 1.04 + .8408(.1) = 1.1241$

$t_2 = t_1 + \Delta t = 0.1 + 0.1 = 0.2$

$f(t_2,\tilde{y}_2) = 5\tilde{y}_2^2 t_2 + 3t_2 = 5(1.1241)^2(.2) + 3(.2) = 1.8636$

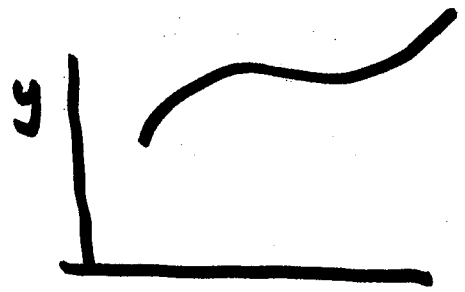
corrected $y_2 = y_1 + \left[f(t_1,y_1) + f(t_2,\tilde{y}_2)\right]\frac{\Delta t}{2} = 1.04 + \left[.8408 + 1.8636\right]\frac{.1}{2} = 1.1752$

$t_2 = t_1 + \Delta t = 0.2$

t	y
0	1
.1	1.04
.2	1.1752

$$y' = \frac{dy}{dx} = f(x,y)$$

$$y = y(x)$$



$$\int dy = \int \frac{dy}{dx} dx = \int f(x,y)\, dx$$

$$y_i' = f(x_i, y_i)$$



$$\frac{y_{i+1} - y_i}{\Delta x} = y_i' = f(x_i, y_i)$$

$$y(x_{i+1}) \cong y_{i+1} = y_i + f(x_i, y_i)\,\Delta x \qquad \text{rect. approx}$$

$$x_{i+1} = x_i + \Delta x \qquad \text{Euler's Method.}$$

EXAMPLE: $y' = 5xy + 3y^2 - 2x = f(x,y)$

$$y(x = x_0) = y_0$$

LOCAL ERROR $O(\Delta x^2)$

GLOBAL $O(\Delta x)$

$$y(x_{i+1}) = y_{i+1} = y_i + y_i'\Delta x + y_i'' \frac{\Delta x^2}{2!} + \ldots$$

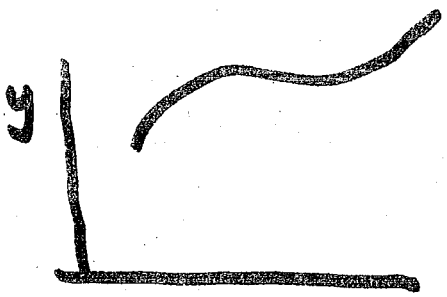
$$y_{i+1} = y_i + f(x_i, y_i)\Delta x + \underline{\qquad}$$

EXAMPLE:

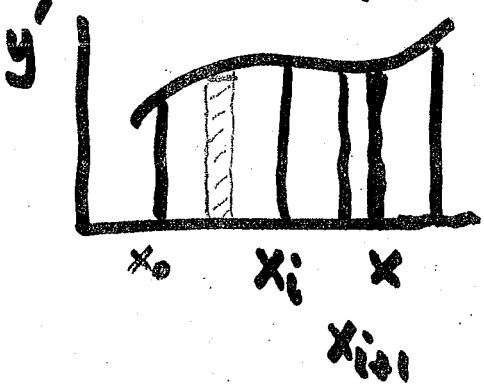
$$\int_A = \int_{x_0}^{x_1} dx = \int f(x,y)\,dA$$

$$y' = \frac{dy}{dx} = f(x,y)$$

$$y = y(x)$$



$$\int dy = \int \frac{dy}{dx} dx = \int f(x,y)\, dx$$



$$y_i' = f(x_i, y_i)$$

$$\frac{y_{i+1} - y_i}{\Delta x} = y_i' = f(x_i, y_i)$$

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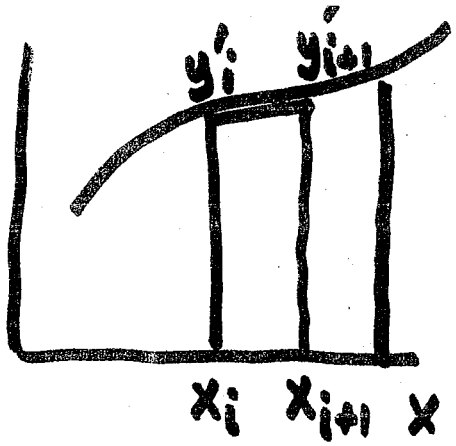
LOCAL ERROR $O(\Delta x^2)$ ERROR PER STEP

GLOBAL ERROR $O(\Delta x)$ ERROR OVER MANY STEPS

$$y(x_{i+1}) = y_{i+1} = y_i + y_i'\Delta x + y_i'' \frac{\Delta x^2}{2!} + \dots$$

$$y_{i+1} = y_i + f(x_i, y_i)\Delta x + \underline{\quad\quad}$$

MODIFIED EULER
TRAPEZ. RULE



$$y_{i+1} = y_i + \frac{y_i' + y_{i+1}'}{2}\Delta x$$

$$y_{i+1}' = f(x_{i+1}, y_{i+1})$$

M.E.

$$\tilde{y}_{i+1} = y_i + f(x_i, y_i)\Delta x \qquad \text{EULER}$$

$$\tilde{y}_{i+1}' = f(x_{i+1}, \tilde{y}_{i+1})$$

$$y_{i+1} = y_i + \frac{y_i' + \tilde{y}_{i+1}'}{2}\Delta x$$

LOCAL ERROR $O(\Delta x^3)$

$$x_{i+1} = x_i + \Delta x$$

GLOBAL $O(\Delta x^2)$

$$y_i' = f(x_i, y_i)$$

$$y_{i+1} = y_i + \left[f(x_i, y_i) + f(x_{i+1}, \tilde{y}_{i+1})\right]\frac{\Delta x}{2}$$

$$y_{i+1} = y_i + y_i'\Delta x + y_i''\frac{\Delta x^2}{2!} + \dots$$

$$y_i'' = (y_i')' = \frac{d}{dx}y_i' = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$y' = 5xy + 3y^2 - 2x = f(x,y)$$

$$\partial f/\partial x = 5y - 2 \qquad \frac{\partial f}{\partial y} = 5x + 6y$$

$$y_{i+1} = y_i + f(x_i, y_i)\Delta x + \left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} f\right)\frac{\Delta x^2}{2!} +$$

$$y'' = f(x, y', y)$$

$$\underline{y' = p} \qquad y(x_0) = y_0$$

$$y'' = \underline{p' = f(x, y, p)} \qquad p(x_0) = p_0$$

EULER METHOD

$$\frac{y_{i+1} - y_i}{\Delta x} = p_i \Rightarrow y_{i+1} = y_i + p_i \Delta x$$

$$\frac{p_{i+1} - p_i}{\Delta x} = f(x_i, y_i, p_i) \Rightarrow p_{i+1} = p_i + f(x_i, y_i, p_i)\Delta x$$

$$x_{i+1} = x_i + \Delta x$$

MOD. EULER : GIVEN y_i, x_i, p_i

$$\textcircled{3}\ y_{i+1} = y_i + \frac{p_i + \tilde{p}_{i+1}}{2}\Delta x$$

$$\textcircled{4}\ p_{i+1} = p_i + \frac{f(x_i, y_i, p_i) + f(x_{i+1}, \tilde{y}_{i+1}, \tilde{p}_{i+1})}{2}\Delta x$$

$$\textcircled{1}\ \tilde{y}_{i+1} = y_i + p_i \Delta x$$

$$\textcircled{2}\ \tilde{p}_{i+1} = p_i + f(x_i, y_i, p_i)\Delta x$$

Given y_i, x_i, p_i

(a)

(b)

(c)

$$y_{n+1} = y_n + a_1 k_1 + a_2 k_2$$ Runge-Kutta

where $k_1 = h f(x_n, y_n) = h f_n$

$$k_2 = h f(x_n + \alpha h, y_n + \beta k_1)$$

$f(x+\Delta x, y+\Delta y) = f(x,y) + \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \text{hot}$

$\Delta x = \alpha h \quad \Delta y = \beta k_1$

(1) $$y_{n+1} = y_n + a_1 [h f_n] + a_2 h \left\{ f_n + \left.\frac{\partial f}{\partial x}\right|_n \alpha h + \left.\frac{\partial f}{\partial y}\right|_n \beta k_1 \right\}$$

$$= y_n + h(a_1 + a_2) f_n + a_2 \alpha h^2 f_x + a_2 \beta h^2 f_n \frac{\partial f}{\partial y} + \cdots$$

error is $O(\text{stepsize}^3)$

TAYLOR SERIES EXPANSION (2) $$y_{n+1} = y_n + \frac{dy}{dx} h + \frac{d}{dx}\left(\frac{dy}{dx}\right) h^2/2 + \cdots$$

$\frac{dy}{dx} = f$

$\frac{d}{dx} f(x,y)$

$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx}$ ($dy/dx = f$)

now $\frac{d^3 y}{dx^3} = \frac{d}{dx}\left\{ f_x + f_y f \right\} = \left(\frac{\partial}{\partial x} + y'\frac{\partial}{\partial y}\right)\{\}$

$= f_{xx} + 2 f_{xy} f + f_{yy} f^2 + f_y f_x + \cdots$

COMPARE (1) & (2):

$a_1 + a_2 = 1 \qquad a_2 \alpha = 1/2 \qquad a_2 \beta = 1/2$

could have picked $a_1 = 3/4 \quad a_2 = 1/4 \quad \alpha = \beta = 2$

min bound to error: $\beta = \alpha = 3/2 \quad a_2 = 1/3 \quad a_1 = 2/3$

$a_2 = 2 \quad a_1 = -1 \quad \beta = 1/4 \quad \alpha = 1/4$

midpoint method: $a_1 = 0 \quad a_2 = 1 \quad \alpha = 1/2 \quad \beta = 1/2$

if $a_1 = a_2 = 1/2 \quad \alpha = \beta = 1$

THEN WE get modified Euler

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = h f_n = h y_n'$$

$$k_2 = h f(x_n + h, y_n + k_1)$$

$$= h f(\underbrace{x_n + h}_{x_{n+1}}, \underbrace{y_n + h f_n}_{\tilde{y}_{n+1}}) = h \tilde{y}_{n+1}'$$

in general $$y_{n+1} = y_n + \sum_{i=1}^{m} a_i k_i$$

$$k_i = h f\left(x_n + \alpha_{i-1} h, y_n + \sum_{j=1}^{i-1} \beta_{ij} k_j\right)$$

Pick a_i's α_i's & β_{ij}'s in such a manner to meet some $O(h^p)$

The lumped fluid capacitance C_f arises from the compressibility of the fluid under pressure between the control valve and the working parts (pistons, vanes, or gears) of the hydraulic motor. The valve is basically a variable nonlinear fluid resistor in the form of a variable area orifice and is described by the equation

$$Q_v = A(t)C_d\left(\frac{2}{\rho}\right)^{.5}\frac{P_{12}}{|P_{12}|^{.5}}$$

The shaft is very stiff so that the motor inertia J_m may be lumped together with the load inertia J_ℓ, and the motor damper B_m may be lumped together with the load damper B_ℓ. Leakage in the motor is negligible.

(a) Draw a free-body diagram of the mechanical part of the system showing the inertialess ideal motor shaft, the lumped inertias, the lumped dampers, and the load torque.

(b) Write the necessary and sufficient set of describing equations for this system.

(c) Rearrange the equations developed in part (b) to form the set of state-variable equations employing P_{2r} and Ω_{1g} as the state variables.

(d) Linearize and combine the state-variable equations to produce the system differential equation relating $\hat{\Omega}_{1g}$ to the input $\hat{A}(t)$.

10.4. A variable-displacement hydraulic pump, driven at constant speed Ω_1, is being used in a hydraulic power supply system for a variable resistance load R_ℓ as shown in Figure P10.4.

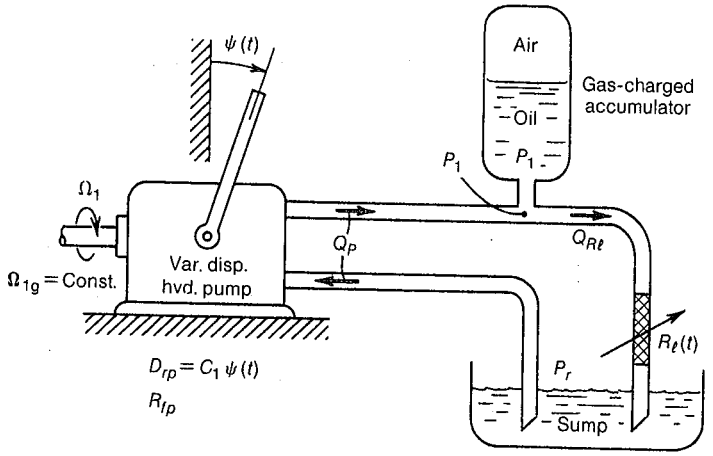
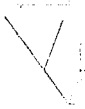


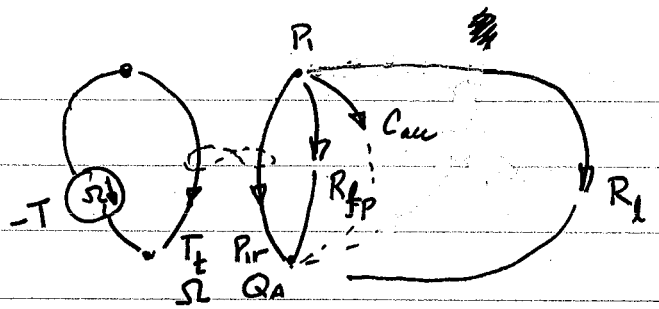
FIGURE P10.4 Schematic diagram of variable-displacement hydraulic power supply.

The displacement of the pump $D_{rp} = C_1\psi(t)$ where $\psi(t)$ is the pump stroke input to the system. The internal leakage in the pump is $Q_{\ell p} = (1/R_{fp})P_{1r}$. The air in the gas-charged accumulator occupies volume $\overline{V}$ when





10-4



$D_r = C_1 \psi$

$T_t = D_r (P_1 - P_r)$

$\Omega = \frac{-1}{D_r} Q_A$

$C_{air} = \frac{V_{air}}{k P_1}$

$Q_c = C_{air} \frac{dP_{ir}}{dt}$

$Q_{R_l} = \frac{1}{R_l} P_{ir}$

$Q_{R_{lp}} = \frac{1}{R_{fp}} P_{ir}$

b-s = 5

$n = 4$

$n-p = 2$ node eq.

$k = \frac{c_p}{c_v}$

$\begin{cases} -T + T_t = 0 \\ Q_A + Q_{R_{lp}} + Q_c + Q_{R_l} = 0 \end{cases}$

$Q_A + Q_{R_{fp}} + Q_p = 0$

~~$Q_p = Q_c + Q_{R_l}$~~

$b-(n-p) = 4$

Sources 1
Cicc is only

$-\Omega_1 + \Omega = 0$

$-P_{ir} + P_{ir} = 0$

$-P_{ir} + P_{ir} = 0$

$-P_{ir} + P_{ir} = 0$

$Q_c = C_{air} \frac{d P_{ir}}{dt} = -Q_{R_l} - Q_{R_{lp}} - Q_A$

$= -\frac{1}{R_{fp}} P_{ir} - \frac{1}{R_l} P_{ir} + C_1 \psi \Omega$

$C_{air} = \frac{V_{air}}{\gamma P_1}$

the system is at its normal operating point, and the absolute temperature of the gas is approximately the same as the atmosphere.

(a) Develop the system differential equation relating the pressure P_{1r} to the two inputs $\psi(t)$ and $R_\ell(t)$, using only symbols.

(b) Determine the normal operating point values $\overline{\psi}$ and $\overline{Q}_p$, given the following data:

$$\begin{aligned} \overline{\Omega}_{1g} &= 180 \text{ rad/sec} \\ C_1 &= 4.4 \text{ in.}^3\text{/rad per stroke radian} \\ \overline{R}_\ell &= 40 \text{ lb-sec/in.}^5 \\ \overline{P}_{1r} &= 1000 \text{ lb/in.}^2 \\ R_{fp} &= 1200 \text{ lb-sec/in.}^5 \end{aligned}$$

(c) Linearize the system differential equation for small perturbations of all variables about the normal operating point, and find the system time constant τ given the data which follows (assume 'fast' changes of pressure in the accumulator).

$$\begin{aligned} \overline{\mathcal{V}} &= 300 \text{ in.}^3 \\ T_{atm} &= 530 \text{ deg R} \end{aligned}$$

(d) Sketch the response to a small step change $\hat{R}_\ell(t) = -2$ lb-sec/in.^{5} at $t = 0$, followed by a small step change $\hat{\psi}(t) = \overline{\psi}/20$ at $t = \tau/2$.

10.5. A variable-speed hydraulic transmission used to drive the spindle of a large turret lathe is shown schematically together with constant speed drive motor and associated output gearing in Figure P10.5a.

The variable-displacement pump has displacement $D_{rp} = C_1\psi(t)$, and the internal leakage of the pump is proportional to the pressure P_{1r}, modeled by a leakage resistance R_{fp}. The fixed-displacement motor has displacement D_{rm},

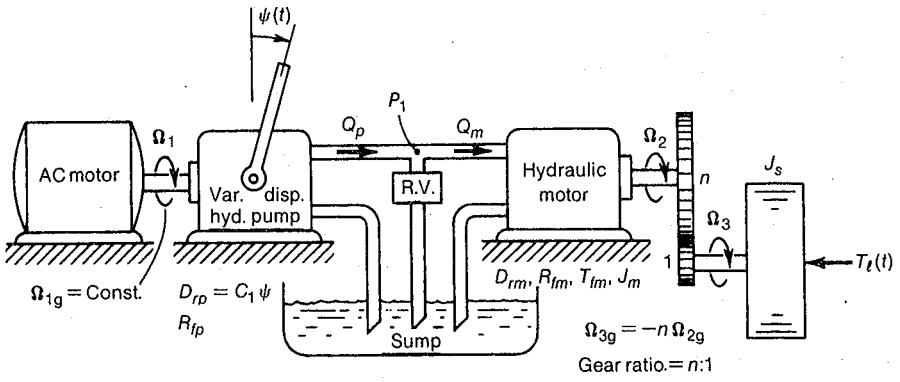
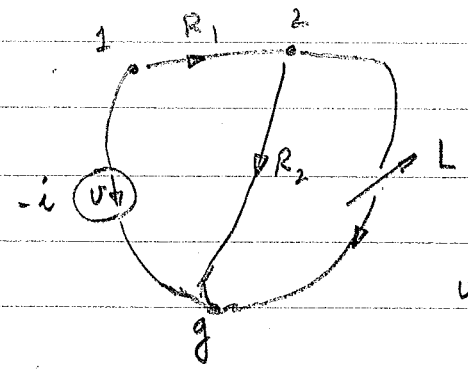


FIGURE P10.5a Schematic diagram of variable-speed drive system.

Problem 7.12



4 branches 3 nodes 1 part

b - s = 3 elem eq
n - 1 = 2 node eq
b - (n-1) = 2 loop eq

$v_{12}/R_1 = i_{R_1}$ $v_{2g}/R_2 = i_{R_2}$ $\lambda_{2g} = L(t)\, i_L$

$v_{2g} = \frac{d\lambda_{2g}}{dt}$

$E = \frac{1}{2}kx^2 = \frac{1}{2}\frac{F^2}{k}$

$E = \frac{Li^2}{2}$

node $\begin{cases} -i - i_{R_1} = 0 \\ i_{R_1} = i_{R_2} + i_L \end{cases}$

loop $\begin{cases} -v + v_{12} + v_{2g} = 0 \\ -v_{2g} + v_{2g} = 0 \end{cases}$

$$\frac{d\lambda_{2g}}{dt} = v_{2g} = R_2 i_{R_2} = R_2(i_{R_1} - i_L) = R_2\left(\frac{v_{12}}{R_1} - \lambda_{2g}/L(t)\right)$$

$$= R_2\left(\frac{v - v_{2g}}{R_1} - \lambda_{2g}/L\right) = \frac{R_2}{R_1}v - \frac{R_2}{R_1}\frac{d\lambda_{2g}}{dt} - \frac{R_2}{L}\lambda_{2g}$$

$$\therefore \frac{d\lambda_{2g}}{dt}\left(1 + R_2/R_1\right) = \left[-\frac{R_2}{L}\lambda_{2g} + R_2/R_1\, v\right]$$

$$\frac{d\lambda_{2g}}{dt} = \frac{-R_2/L}{1 + (R_2/R_1)}\lambda_{2g} + \frac{R_2}{R_1 + R_2}v$$

• normal operating pt is when system is in equil. i.e. $\frac{d\lambda_{2g}}{dt} = v_{2g} = 0$

$\therefore \bar{\lambda}_{2g} = \bar{L}v/R_1$ $\bar{L} = 1$

$= 1 \cdot 6/12000 = .0005$ webers

• if $\lambda_{2g} = \bar{\lambda}_{2g} + \hat{\lambda}_{2g}$ $\frac{d\lambda_{2g}}{dt} = \frac{d\hat{\lambda}_{2g}}{dt}$; $L = \bar{L} + \hat{L}$

$$\frac{d(\bar{\lambda} + \hat{\lambda})}{dt} = -\frac{R_2}{1 + R_2/R_1}\frac{\bar{\lambda} + \hat{\lambda}}{\bar{L} + \hat{L}} + \frac{R_2}{R_1 + R_2}v$$

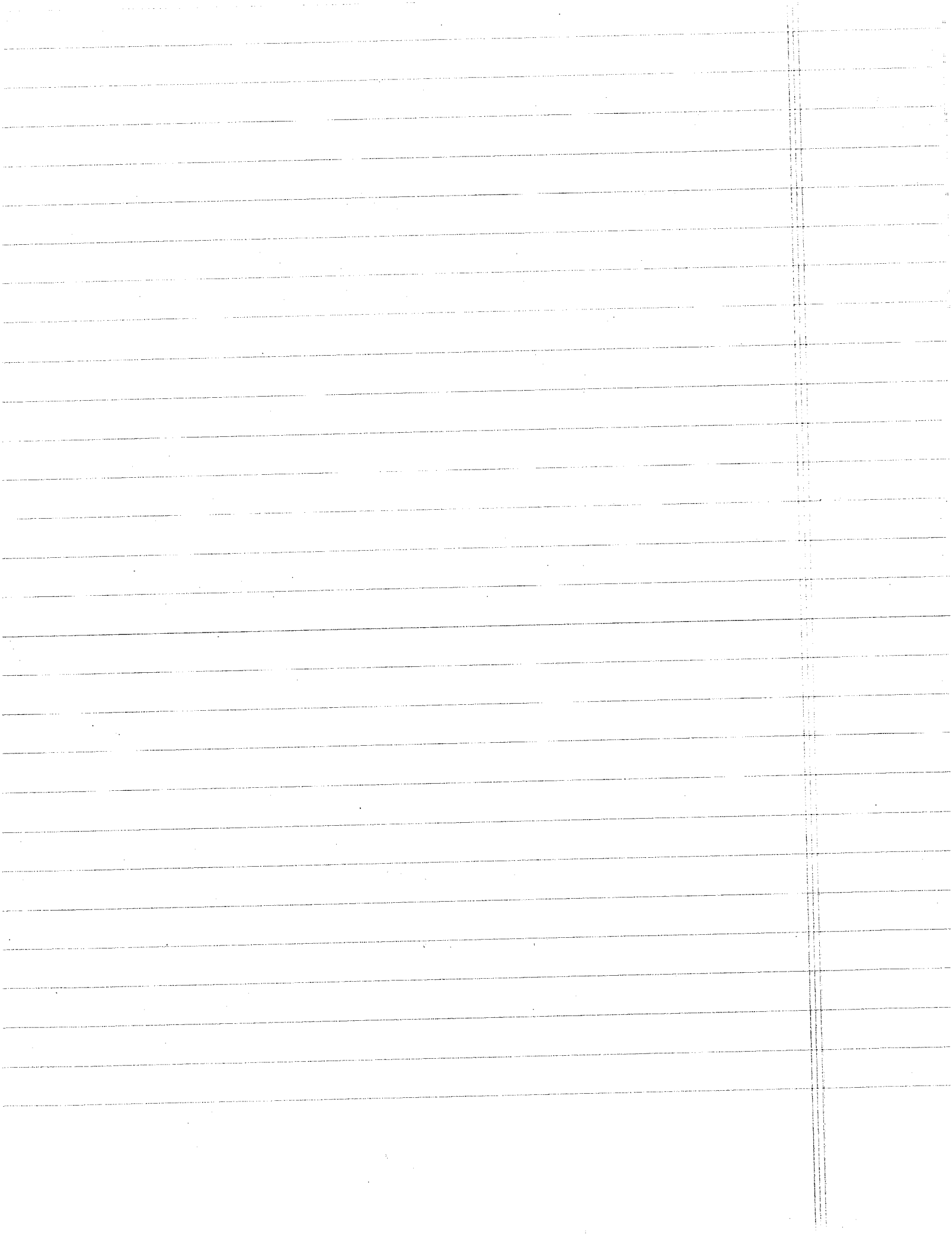
$$= \frac{-R_2}{1 + R_2/R_1}\frac{\bar{\lambda}\left(1 + \hat{\lambda}/\bar{\lambda}\right)}{\bar{L}\left(1 + \hat{L}/\bar{L}\right)} + \frac{R_2}{R_1 + R_2}v = \frac{-R_2}{(1 + R_2/R_1)\bar{L}}\bar{\lambda}\left(1 + \hat{\lambda}/\bar{\lambda} - \hat{L}/\bar{L} + \ldots\right) + \frac{R_2}{R_1 + R_2}$$

$$\frac{d\hat{\lambda}}{dt} = -\frac{R_2R_1}{R_1+R_2}\left[\frac{\hat{\lambda}}{\bar{L}} - \frac{\bar{\lambda}\hat{L}}{\bar{L}^2}\right]$$

$$\hat{v}_{2g} = -\frac{R_2R_1}{R_1+R_2}\left[\frac{\hat{\lambda}_{2g}}{\bar{L}}\right] + \frac{R_2R_1}{R_1+R_2}\frac{\bar{\lambda}}{\bar{L}^2}\hat{L}$$

$$\frac{d}{dt}\hat{v}_{2g} = -\frac{R_2R_1}{R_1+R_2}\frac{\hat{v}_{2g}}{\bar{L}} + \frac{R_2R_1}{R_1+R_2}\frac{\bar{\lambda}}{\bar{L}^2}\frac{d}{dt}\hat{L}$$

$$\frac{d}{dt}\hat{v}_{2g} \quad -\frac{2.88}{3.6}\frac{\times 10^3}{1}\hat{v}_{2g} + \frac{2.88}{3.6}\frac{\times 10^3}{1}\cdot 5\times 10^{-4}(.5\cos 5t)$$



EML 3222

CLASS FOR

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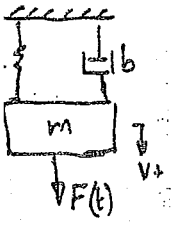
SYSTEM DYNAMICS

11/24/08

DR. LEVY

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As we have shown in class the governing equation for a second order linear ordinary differential equation made up of a mass-spring-damper is $m\frac{d^2v}{dt^2} + b\frac{dv}{dt} + kv = \frac{df}{dt}$ (*)





Homogeneous solution V_h: satisfies $m\ddot{v} + b\dot{v} + kv = 0$ and solution depends on $b^2 - 4mk$

① $b^2 - 4mk > 0$ $\quad V_h = C_1 e^{s_1 t} + C_2 e^{s_2 t}$ $\quad s_1 = \frac{-b + \sqrt{b^2-4mk}}{2m} = -\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right) < 0$

$s_2 = \frac{-b - \sqrt{b^2-4mk}}{2m} = -\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right) < 0$

where $\omega_n = \sqrt{\frac{k}{m}}$ $\quad \zeta = \frac{b}{c_{cr}}$ $\quad c_{cr} = 2\sqrt{mk} = 2m\omega_n$. Here $\underline{\zeta > 1}$ overdamped

② $b^2 - 4mk = 0$ $\quad V_h = (C_1 + C_2 t)e^{\frac{-b}{2m}t} = (C_1 + C_2 t)e^{-\omega_n t}$ since $b = c_{cr}$ and $\zeta = 1$ critically damped

③ $b^2 - 4mk < 0$ $\quad V_h = C_1 e^{-\zeta\omega_n t}\cos(\omega_d t) + C_2 e^{-\zeta\omega_n t}\sin(\omega_d t) = \Sigma e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$

with $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency); $\Sigma\cos\phi = C_2$ $\Sigma\sin\phi = C_1$ underdamp

and $\Sigma = \sqrt{C_1^2 + C_2^2}$; $\phi = \tan^{-1}(C_1/C_2)$ rad

Only in case ③ is there a frequency of oscillation: ω_d. Cases ① and ② are aperiodic

Take (*) and divide by m: $\ddot{v} + \frac{b}{m}\dot{v} + \frac{k}{m}v = \frac{1}{m}\frac{df}{dt} = \ddot{v} + 2\zeta\omega_n\dot{v} + \omega_n^2 v = \frac{1}{m}\frac{df}{dt} = g(t)$

Finding particular solution, $V_p(t)$

For all the cases there are 2 homogeneous solutions V_{h_1}, V_{h_2}

case ① $V_{h_1} = e^{s_1 t}$ $\quad V_{h_2} = e^{s_2 t}$ $\quad \zeta > 1$

② $V_{h_1} = e^{-\omega_n t}$ $\quad V_{h_2} = te^{-\omega_n t}$ $\quad \zeta = 1$

③ $V_{h_1} = e^{-\zeta\omega_n t}\cos\omega_d t$ $\quad V_{h_2} = e^{-\zeta\omega_n t}\sin\omega_d t$ $\quad \zeta < 1$

From Differential Equations, we can use variation of parameters to find V_p

let $V_p = u_1(t)V_{h_1}(t) + u_2(t)V_{h_2}(t)$

This leads to: $u_1'V_{h_1} + u_2'V_{h_2} = 0$ and $u_1'V_{h_1}' + u_2'V_{h_2}' = g(t)$

Solving for u_1' & u_2' and integrating gives u_1 & u_2

Then $u_1(t) = \int^t \frac{-g(\bar{t})\cdot V_{h_2}(\bar{t})}{\mathcal{W}(V_{h_1}, V_{h_2})}\,d\bar{t}$ and $u_2(t) = \int^t \frac{g(\bar{t})\,V_{h_1}(\bar{t})}{\mathcal{W}(V_{h_1}, V_{h_2})}\,d\bar{t}$

where

Wronsky $\mathcal{W}(V_{h_1}, V_{h_2}) = \det\begin{bmatrix} V_{h_1} & V_{h_2} \\ V_{h_1}' & V_{h_2}' \end{bmatrix}$. $\mathcal{W} \neq 0$ for a solution to exist

For case ① $\mathcal{W} = \det\begin{pmatrix} e^{s_1 t} & e^{s_2 t} \\ s_1 e^{s_1 t} & s_2 e^{s_2 t} \end{pmatrix} = (s_2 - s_1)e^{(s_1+s_2)t} = -2\omega_n\sqrt{\zeta^2-1}\,e^{-2\omega_n\zeta t}$

and

$$V_p(t) = e^{s_1 t}\cdot\int^t \frac{-g(\bar{t})\cdot e^{+s_2\bar{t}}}{-2\omega_n\sqrt{\zeta^2-1}\cdot e^{-2\omega_n\zeta\bar{t}}}\,d\bar{t} + e^{s_2 t}\cdot\int^t \frac{g(\bar{t})\cdot e^{s_1\bar{t}}}{-2\omega_n\sqrt{\zeta^2-1}\,e^{-2\omega_n\zeta\bar{t}}}\,d\bar{t}$$

For case ② $\mathcal{W} = \det\begin{pmatrix} e^{-\omega_n t} & te^{-\omega_n t} \\ -\omega_n e^{-\omega_n t} & e^{-\omega_n t}[1-\omega_n t] \end{pmatrix} = e^{-2\omega_n t}$

and

$$V_p(t) = e^{-\omega_n t}\cdot\int^t \frac{-g(\bar{t})\cdot\bar{t}e^{-\omega_n\bar{t}}}{e^{-2\omega_n\bar{t}}}\,d\bar{t} + te^{-\omega_n t}\cdot\int^t \frac{g(\bar{t})\cdot e^{-\omega_n\bar{t}}}{e^{-2\omega_n\bar{t}}}\,d\bar{t}$$

$$= \int^t g(\bar{t})\,[t-\bar{t}]\,e^{-\omega_n(t-\bar{t})}\,d\bar{t} \quad \blacktriangleleft$$

For case ③ $\mathcal{W} = \det\begin{pmatrix} e^{-\zeta\omega_n t}\cos\omega_d t & e^{-\zeta\omega_n t}\sin\omega_d t \\ e^{-\zeta\omega_n t}[-\zeta\omega_n\cos\omega_d t - \omega_d\sin\omega_d t] & e^{-\zeta\omega_n t}[-\zeta\omega_n\sin\omega_d t + \omega_d\cos\omega_d t] \end{pmatrix}$

$$= e^{-2\zeta\omega_n t}\cdot\omega_d$$

and $V_p(t) = e^{-\zeta\omega_n t}\cos\omega_d t\cdot\int^t \frac{-g(\bar{t})e^{-\zeta\omega_n\bar{t}}\sin\omega_d\bar{t}}{e^{-2\zeta\omega_n\bar{t}}\cdot\omega_d}\,d\bar{t} + e^{-\zeta\omega_n t}\sin\omega_d t\cdot\int^t \frac{g(t)e^{-\zeta\omega_n\bar{t}}\cos\omega_d\bar{t}}{e^{-2\zeta\omega_n\bar{t}}\cdot\omega_d}\,d\bar{t}$

$$V_p(t) = \int^t g(\bar{t})\cdot\frac{e^{-\zeta\omega_n(t-\bar{t})}}{\omega_d}\sin[\omega_d(t-\bar{t})]\,d\bar{t} \quad \blacktriangleleft$$

Case ① can be consolidated to

$$V_p(t) = \int^t g(\bar{t})\cdot\frac{e^{-\zeta\omega_n(t-\bar{t})}}{\omega_n\sqrt{\zeta^2-1}}\sinh[\omega_n\sqrt{\zeta^2-1}\,(t-\bar{t})]\,d\bar{t} \quad \blacktriangleleft$$

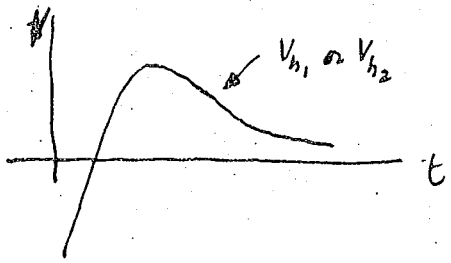
So, finally,

$$V_{TOT}(t) = C_1 V_{h_1}(t) + C_2 V_{h_2}(t) + V_p(t)$$

Note that the homogeneous part of the solution involves dying exponentials and is called the TRANSIENT part of the solution.

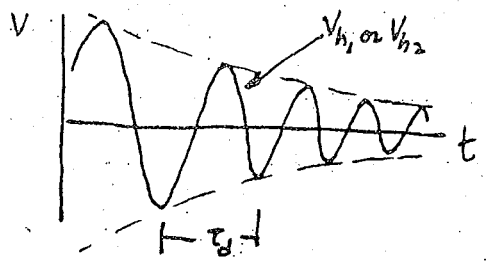
$V_p(t)$ is normally called the steady state part of the solution

Case (1) & Case (2)



no period of oscillation

Case (3)



τ_d - damped period of oscillation

$\omega_d = \frac{2\pi}{\tau_d}$

Now C_1 & C_2 are obtained knowing some initial information

i.e., $v(t=t_0) = v_0$ and $\frac{dv}{dt}(t=t_0) = a_0$ for example

Example: let $k = 1000\ N/m$ let $m = 10\ kg$ and let $b = 50\ \frac{N\text{-}s}{m}$

$\therefore\ \omega_n = \sqrt{\frac{k}{m}} = 10\ \frac{rad}{s}$ $\quad C_{cr} = 2m\omega_n = 200\ \frac{N\text{-}s}{m}$ $\quad \zeta = \frac{b}{C_{cr}} = 0.25 < 1$

so the system is underdamped $\omega_d = \omega_n \cdot \sqrt{1-\zeta^2} = 10(0.9683) = 9.683\ rad/s$

$$V_h = C_1 e^{-\zeta\omega_n t}\cos\omega_d t + C_2 e^{-\zeta\omega_n t}\sin\omega_d t = C_1 e^{-2.5t}\cos 9.683t + C_2 e^{-2.5t}\sin 9.683t$$

Now let $F(t) = 10t + 5\ N$, for example. $\frac{1}{m}\frac{dF}{dt} = \frac{1}{10}\cdot 10 = 1\ \frac{1}{kg}\cdot\frac{N}{s} = 1\ \frac{m}{s^2}\cdot\frac{1}{s}$ ✓

$= g(t)$

$$V_p = \int_0^t \frac{g(\bar{t})\, e^{-\zeta\omega_n(t-\bar{t})}}{\omega_d}\sin[\omega_d(t-\bar{t})]\,d\bar{t} = \int_0^t \frac{1\cdot e^{-2.5(t-\bar{t})}}{9.683}\sin[9.683(t-\bar{t})]\,d\bar{t}$$

Now let $t-\bar{t}=z$ $\quad d\bar{t}=-dz$ $\quad$ so $\quad$ when $\bar{t}=t \Rightarrow z=0$

$$V_p = \int^{0} \frac{1\cdot e^{-2.5z}\sin[9.683z]}{9.683}(-dz)$$

$$\int e^{az}\sin bz\,dz = \frac{e^{az}}{a^2+b^2}(a\sin bz - b\cos bz) \quad \text{using integration by parts twice}$$

$$\therefore \quad V_p = \frac{-1}{9.683}\left[\frac{e^{-2.5z}}{(2.5)^2+(9.683)^2}\left(-2.5\sin 9.683z - 9.683\cos 9.683z\right)\right]^{z=0}$$

$$= +\frac{1}{(2.5)^2+(9.683)^2} = \frac{1}{100} = .01$$

So $\quad V_{TOT} = C_1 e^{-2.5t}\cos 9.683t + C_2 e^{-2.5t}\sin 9.683t + .01$ ◀

Suppose $V(t=0)=5$ m/s $\quad$ and $\quad \frac{dV}{dt}(t=0)=0$ m/s^2 $\quad$ (initial conditions)

► Must apply these conditions on total solution.

$$\therefore V(t=0)=5 = C_1\cdot e^0\cos 0 + C_2\cdot e^0\sin 0 + .01 = C_1 + .01 \Rightarrow C_1 = 4.99 \text{ m/s} \blacktriangleleft$$

$$\frac{dV}{dt} = C_1 e^{-2.5t}\left[-2.5\cos 9.683t + 9.683\sin 9.683t\right] + C_2 e^{-2.5t}\left[-2.5\sin 9.683t + 9.683\cos 9.683t\right]$$

$$\frac{dV}{dt}(t=0) = C_1 e^0\left[-2.5\cos 0 - 9.683\sin 0\right] + C_2 e^0\left[-2.5\sin 0 + 9.683\cos 0\right] = -2.5C_1 + 9.683C_2 = 0$$

$$C_2 = \frac{2.5C_1}{9.683} = 1.288 \text{ m/s} \blacktriangleleft$$

So $\quad V(t) = 4.99e^{-2.5t}\cos 9.683t + 1.288e^{-2.5t}\sin 9.683t + .01$ m/s ◀

or $\quad = \Im e^{-2.5t}\sin(\omega_1 t+\phi) + 5.154e^{-2.5t}\sin(9.683t+1.3181) + .01$

Other helpful integrals: $\int^{t}(t-\bar{t})e^{-\omega_n(t-\bar{t})}d\bar{t} = \int^{0} -z e^{-\omega_n z}dz = +e^{-\omega_n z}\left[\frac{z}{\omega_n}+\frac{1}{\omega_n^2}\right]^{0} = \frac{1}{\omega_n^2}$

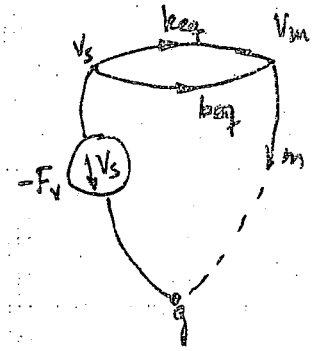
► $\int e^{az}\cdot bz\,dz = be^{az}\left[\frac{z}{a}-\frac{1}{a^2}\right]$

► $\int e^{az}\sinh bz\,dz = \int e^{az}\left[\frac{e^{bz}-e^{-bz}}{2}\right]dz = \frac{1}{2}\left[\frac{e^{(a+b)z}}{a+b} - \frac{e^{(a-b)z}}{a-b}\right]$ if $a\neq b,-b$

$= \frac{1}{2}\left[\frac{e^{2az}}{2a} - z\right]$ if $a=b$

$= \frac{1}{2}\left[z - \frac{e^{2az}}{2a}\right]$ if $a=-b$

9.23



$$-F_v + F_b + F_k = 0 \qquad F_k + F_b - F_m = 0 \qquad \Big\} \; F_m = +F_v$$

$$-m\frac{dV_m}{dt} + k\int(V_s - V_m)dt + b(V_s - V_m) = 0 \quad (*)$$

$$\frac{d}{dt}(*) \Rightarrow m\frac{d^2V_m}{dt^2} + b(\dot{V}_m) + kV_m = b\dot{V}_s + kV_s$$

$$\text{or} \quad \ddot{V}_m + \frac{b}{m}\dot{V}_m + \frac{k}{m}V_m = \frac{1}{m}\left[b\dot{V}_s + kV_s\right] = g(t)$$

$$\frac{b}{m} = 2\zeta\omega_n = 3.75 \qquad \frac{k}{m} = \omega_n^2 = \frac{16000}{320} = 50 \qquad \omega_n = \sqrt{50} = 7.07 \ \frac{\text{rad}}{\text{s}}$$

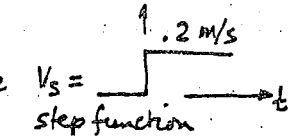
$$\zeta = \frac{1}{2\omega_n}\cdot\frac{b}{m} = \frac{b}{bc_r} = .2652 \qquad \underline{\text{underdamped}}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 6.82\ \frac{\text{rad}}{\text{s}} \qquad \zeta\omega_n = .2652(7.07) = 1.875$$

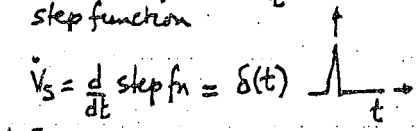
homogeneous solution

$$V_{m_h} = C_1 e^{-\zeta\omega_n t}\cos\omega_d t + C_2 e^{-\zeta\omega_n t}\sin\omega_d t = C_1 e^{-1.875t}\cos 6.82t + C_2 e^{-1.875t}\sin 6.82t$$

$$g(t) = \frac{1}{m}\left[b\dot{V}_s + kV_s\right] = 3.75\dot{V}_s + 50V_s \qquad \text{where } V_s = \text{step function}$$



$$\dot{V}_s = \frac{d}{dt}\,\text{step fn} = \delta(t)$$



$$\therefore V_{m_p} = \frac{b}{m}\int_0^t \frac{.2\,\delta(\bar{t})\,e^{-\zeta\omega_n(t-\bar{t})}\sin[\omega_d(t-\bar{t})]}{\omega_d}d\bar{t} + \frac{k}{m}\int_0^t \frac{.2\cdot u_s(\bar{t})e^{-\zeta\omega_n(t-\bar{t})}\sin\omega_d(t-\bar{t})}{\omega_d}d\bar{t}$$

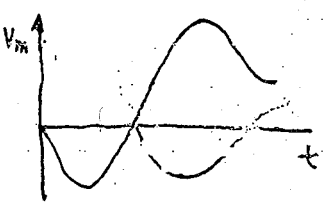
$$= \frac{b}{m}(.2)\frac{e^{-\zeta\omega_n t}\sin\omega_d t}{\omega_d} + \frac{k}{m}\frac{(.2)}{\omega_d}\left[\frac{e^{-1.875\bar{t}}\left(-1.875\sin 6.82\bar{t} - 6.82\cos 6.82\bar{t}\right)}{(-1.875)^2 + (6.82)^2}\right]_0^t$$

$$= \frac{.75e^{-\zeta\omega_n t}\sin\omega_d t}{\omega_d} + \frac{k}{m}\frac{(.2)}{\omega_d}\cdot\frac{1}{\omega_n^2} = \frac{.75e^{-\zeta\omega_n t}\sin\omega_d t + .2}{\omega_d}$$

$$V_{m_{TOT}} = C_1 e^{-1.875t}\cos\omega_d t + C_2 e^{-1.875t}\sin\omega_d t + \left(.75e^{-\zeta\omega_n t}\sin\omega_d t + .2\right)\frac{1}{\omega_d}$$

(combine)

$$= C_1 e^{-1.875t}\cos 6.82t + \bar{C}_2 e^{-1.875t}\sin\omega_d t + .2/\omega_d$$



IC $\quad V_m(t=0) = 0 \; ; \; \dfrac{dV_m}{dt}(t=0) = 0 \qquad \bar{C}_2 = C_2 + .75/\omega_d$

$$C_1 + .2/\omega_d = 0 \Rightarrow C_1 = -.2/\omega_d \ \text{m/s}$$

$$C_1(-\zeta\omega_n) + \bar{C}_2\cdot\omega_d = 0 \Rightarrow \bar{C}_2 = C_1\frac{\zeta}{\sqrt{1-\zeta^2}} = -.055 \ \text{m/s}$$

$a y' + by = h(t)$ $\qquad y' = \frac{dy}{dt} \quad y = f(t)$

$y' + \frac{b}{a} y = \frac{1}{a} h(t) = g(t)$

$(\mu y)' = \mu' y + \mu y' = \mu y' + \frac{b}{a} y \mu = \mu g$

$\therefore$

$\frac{d\mu}{\mu} = \frac{b}{a} dt \Leftarrow \mu' = \frac{b}{a}\mu$ & $\mu(t) = e^{\int \frac{b}{a} dt}$; IF $\frac{b}{a}$ are const, $\mu(t) = e^{b/a t}$

$(\mu y)' = \mu \cdot g$

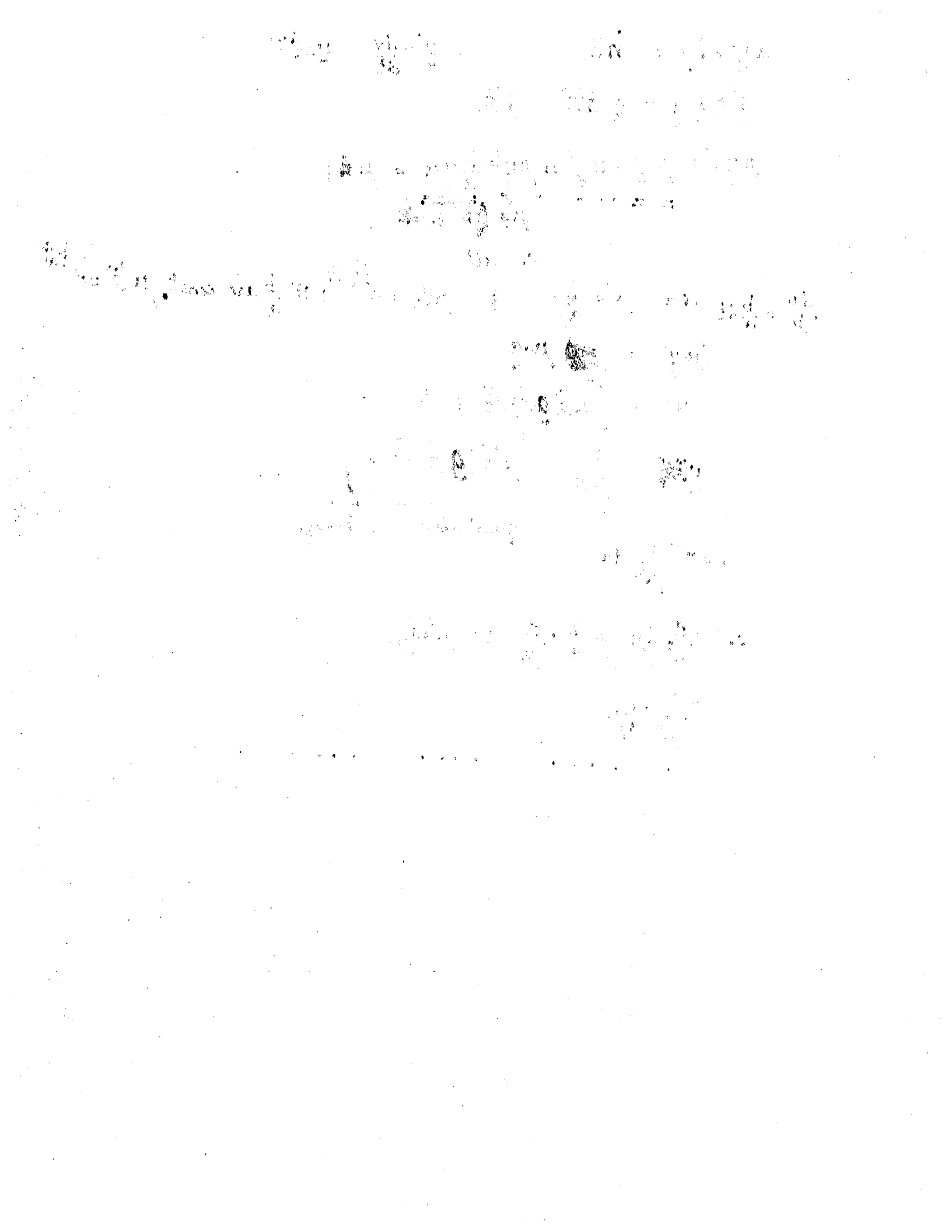
$\mu y = \int^t \mu(\bar{t}) g(\bar{t}) d\bar{t} + c$

$y(t) = \underbrace{\frac{1}{\mu(t)} \int^t \mu(\bar{t}) g(\bar{t}) d\bar{t}}_{\text{particular}} + \underbrace{\frac{c}{\mu}}_{\text{homog.}}$

$y' = -c \frac{1}{\mu^2} d\mu$

$\therefore -\frac{c}{\mu^2} d\mu + \frac{b}{a} \cdot \frac{c}{\mu} = 0 \quad \mu' = \frac{b}{a}\mu$

$-\frac{c}{\mu^2} \cdot \frac{b}{a} \mu$



homog. DE $y' = -\frac{b}{a}y$ $\quad y = Ce^{st}$

$y' = Cse^{st}$

$$Ce^{st}\left[s = -\frac{b}{a}\right] \Rightarrow y = Ce^{-b/a\,t}$$

---

like prob. 8-11

$$2\frac{dy}{dt} + 3y = 2e^{-2t}\cos 4t$$

$$\frac{dy}{dt} + \frac{3}{2}y = \frac{e^{-2t}\cos 4t}{} = g(t)$$

$$\mu(t) = e^{\int b/a\,dt} = e^{3/2\,t}$$

$$y(t) = e^{-3/2\,t}\int^{t} e^{3/2\,\bar{t}}\cdot e^{-2\bar{t}}\cos 4\bar{t}\,d\bar{t} + Ce^{-3/2\,t}$$

$$y(t) = e^{-3/2\,t}\int^{t} e^{-1/2\,\bar{t}}\cos 4\bar{t}\,d\bar{t} + Ce^{-3/2\,t}$$

$$y(t=0) = 1$$

$$y(t=0) = 1 = 1\cdot\int^{0} e^{-1/2\,\bar{t}}\cos 4\bar{t}\,d\bar{t} + C\cdot 1$$

$$C = 1 - \int^{0} e^{-1/2\,\bar{t}}\cos 4\bar{t}\,d\bar{t}$$

$$y(t) = e^{-3/2\,t}\int^{t} e^{-1/2\,\bar{t}}\cos 4\bar{t}\,d\bar{t} + e^{-3/2\,t} - e^{-3/2\,t}\int^{0} e^{-1/2\,\bar{t}}\cos 4\bar{t}\,d\bar{t}$$

$$+ e^{-3/2\,t}\int_{0} e^{-1/2\,\bar{t}}\cos 4\bar{t}\,d\bar{t}$$

$$\boxed{y(t) = e^{-3/2\,t}\int_{0}^{t} e^{-1/2\,\bar{t}}\cos 4\bar{t}\,d\bar{t} + e^{-3/2\,t}}$$

Figure P7.29

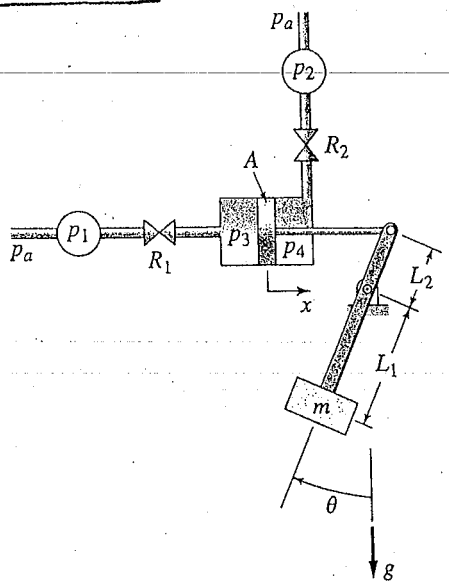


Figure P7.30

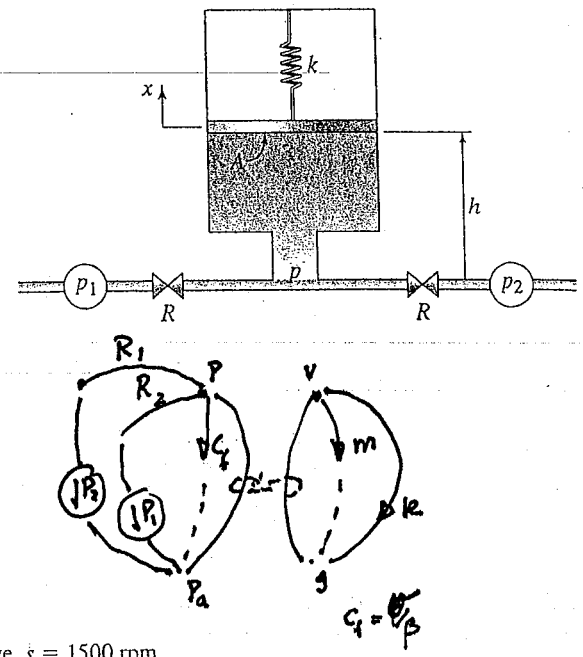
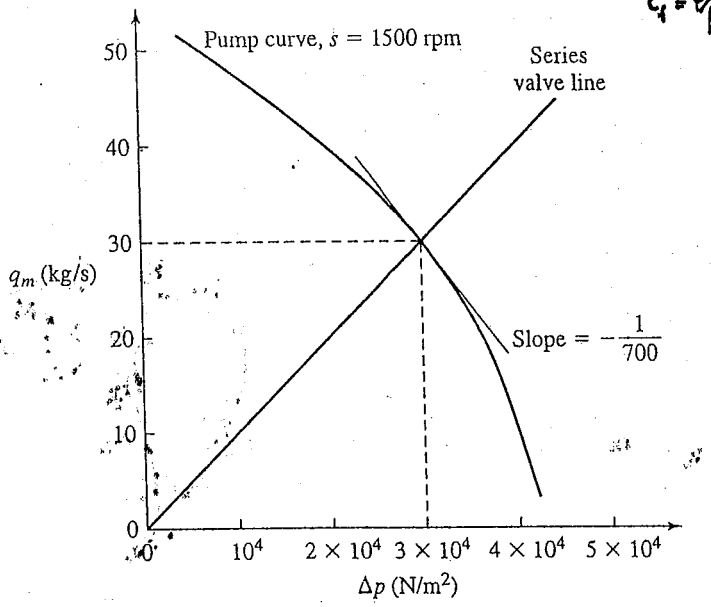


Figure P7.32



7.33 Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.33. The outlet resistance is linear. Derive the dynamic model of the height h.

7.34 Consider the V-shaped container treated in Example 7.2.2, whose cross section is shown in Figure P7.34. The outlet is an orifice of area A_o and discharge coefficient C_d. Derive the dynamic model of the height h.

7.35 Consider the cylindrical container treated in Problem 7.8. In Figure P7.35 the tank is shown with a valve outlet at the bottom of the tank. Assume that the flow through the valve is turbulent with a resistance R. Derive the dynamic model of the height h.

7.27 An electric motor is sometimes used to move the spool valve of a hydraulic motor. In Figure P7.27 the force f is due to an electric motor acting through a rack-and-pinion gear. Develop a model of the system with the load displacement y as the output and the force f as the input. Consider two cases: (a) $m_1 = 0$ and (b) $m_1 \neq 0$.

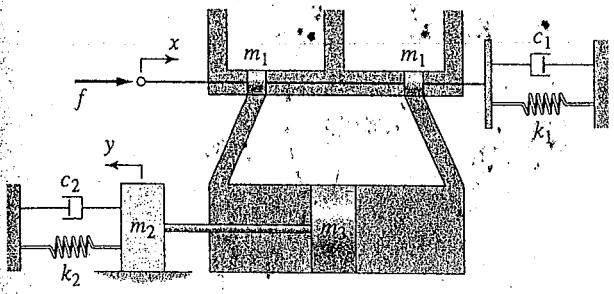


Figure P7.27

7.28 In Figure P7.28 the piston of area A is connected to the axle of the cylinder of radius R, mass m, and inertia I about its center. Develop a dynamic model of the axle's translation x, with the pressures p_1 and p_2 as the inputs.

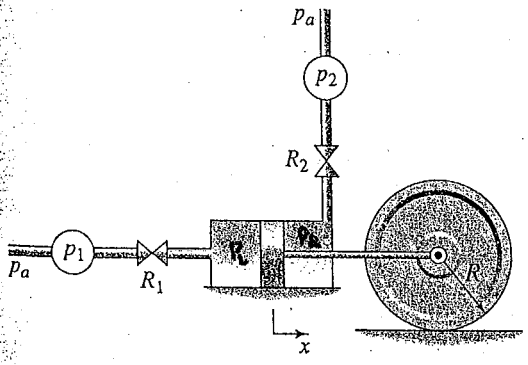
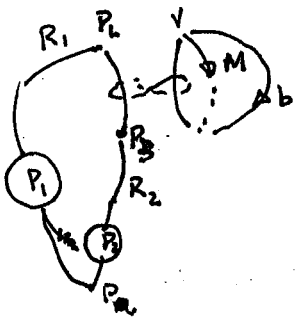


Figure P7.28



where $M = m + \frac{I}{R^2}$

7.29 Figure P7.29 shows a pendulum driven by a hydraulic piston. Assuming small angles θ and a concentrated mass m a distance L_1 from the pivot, derive the equation of motion with the pressures p_1 and p_2 as inputs.

7.30 Figure P7.30 shows an example of a *hydraulic accumulator,* which is a device for reducing pressure fluctuations in a hydraulic line or pipe. The fluid density is ρ, the plate mass is m, and the plate area is A. Develop a dynamic model of the pressure p with the pressures p_1 and p_2 as the given inputs. Assume that $m\ddot{x}$ of the plate is small, and that the hydrostatic pressure $\rho g h$ is small.

7.31 Design a hydraulic accumulator of the type shown in Figure P7.30. The liquid volume in the accumulator should increase by 30 in.^3 when the pressure p increases by 1.5 lb/in.^2. Determine suitable values for the plate area A and the spring constant k.

7.32 Consider the liquid-level system treated in Example 7.4.8 and shown in Figure 7.4.9. The pump curve and the line for the steady-state flow through both valves are shown in Figure P7.32. It is known that the bottom area of the tank is 2 m^2 and the outlet resistance is $R_2 = 400$ 1/(m · s). (a) Compute the pump resistance R_1 and the steady-state height. (b) Derive a linearized dynamic model of the height deviation δh in the tank.

(a) Derive the input-output model for this system using $y(t)$ as the output and $u(t)$ as the input variable.
(b) Derive a state model and present it in a matrix form.

4.7. Derive a set of state variable equations for the mechanical system shown in Figure P4.7.

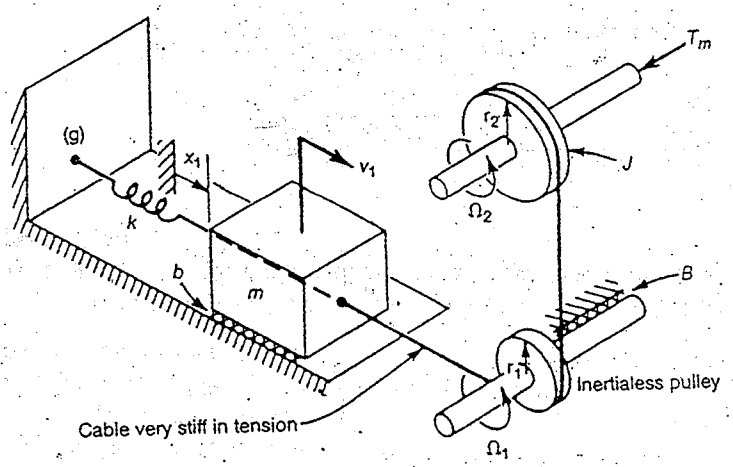


FIGURE P4.7 Mechanical systems considered in Problems 4.7 and 4.8.

4.8. Derive the input-output model equation for the system shown in Figure P4.7 using torque $T_m(t)$ as the input variable and angular velocity Ω_{2g} as the output variable.

4.9. A lumped model of a machine-tool drive system is shown in Figure P4.9. The system parameters are $J_m = 0.2$ N-m-sec^2/rad, $B = 30$ N-m-sec/rad, $m = 16$ kg, $k = 20$ N/m, and $R = 0.5$ m. The stiffness of the shaft is represented by a nonlinear torque, $T_{NLK} = 2|\Theta_1 - \Theta_2|(\Theta_1 - \Theta_2)$. The force of the nonlinear friction device NLD is $F_{NLD} = f_{NL}(v_{1g}) = 2v_{1g}^3 + 4v_{1g}$. The system is driven by a torque consisting of a constant and an incremental component, $T_i(t) = 0.8 + 0.02 \sin(0.1t)$ N-m.
(a) Select state variables and derive nonlinear state-variable equations.
(b) Find the normal operating point values for all state variables.
(c) Linearize the state-model equations in the vicinity of the normal operating point. Present the linearized state model in a matrix form.
(d) Derive the input-output model equation for the linearized system using the incremental torque, $\hat{T}_i(t)$, as the input variable and incremental displacement, $\hat{x}_1(t)$, as the output variable.
(e) Draw a simulation block diagram for this system.

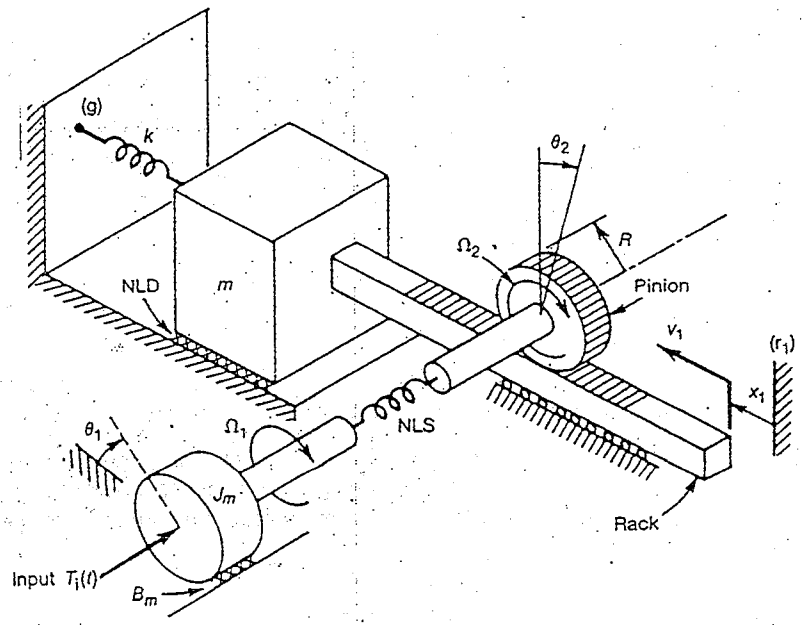


FIGURE P4.9 Lumped model of a machine-tool drive system.

4.10. Derive state-variable equations for the system shown in Figure 2.15. Use velocities v_{1g} and v_{2g} and the spring force F_{k1} as the state variables. Note that the two springs in this system are not independent as long as the lever is massless. Combine the state-variable equations to obtain an input-output equation relating velocity v_{2g} to force $F_s(t)$.

with regard to the information about the system behavior incorporated in the model equations, the techniques employed in their derivation and solution are principally different. In most cases the state models, consisting of sets of first order differential equations, are much easier to derive and to solve by computer simulation than the input-output equations. This is especially true when the mathematical models involve nonlinearities or when systems modeled have many inputs and outputs and are to be simulated on the computer.

The state models are based on the concept of state variables. The choice of state variables for a given system is not unique. Four different types of state variables were used in example problems presented in this chapter. It was shown that the number of state variables necessary to describe the system dynamics is always equal to the number of independent energy-storing elements regardless of the type of state variables used.

Problems

4.1. Derive a complete set of state model equations for the mechanical rotational system shown in Figure P4.1. Select the following state variables:
(a) T-type and A-type variables
(b) one variable and its derivative
Use torque $T_i(t)$ as the input variable and angular displacement $\Theta_{1g}(t)$ as the output variable in each model.

4.2. Derive an input-output model equation for the system shown in Figure P4.1, using torque $T_i(t)$ as the input variable and angular displacement $\Theta_1(t)$ as the output variable.

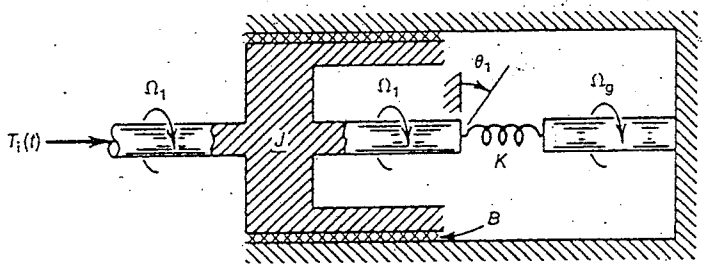


FIGURE P4.1 Mechanical system considered in Problem 4.1.

4.3. Derive complete sets of state-model equations for the system shown in Figure P4.3 using three different sets of state variables. The input variable is torque $T_m(t)$ and the output variables are displacements $x_1(t)$ and $x_2(t)$. Present the state models in a matrix form.

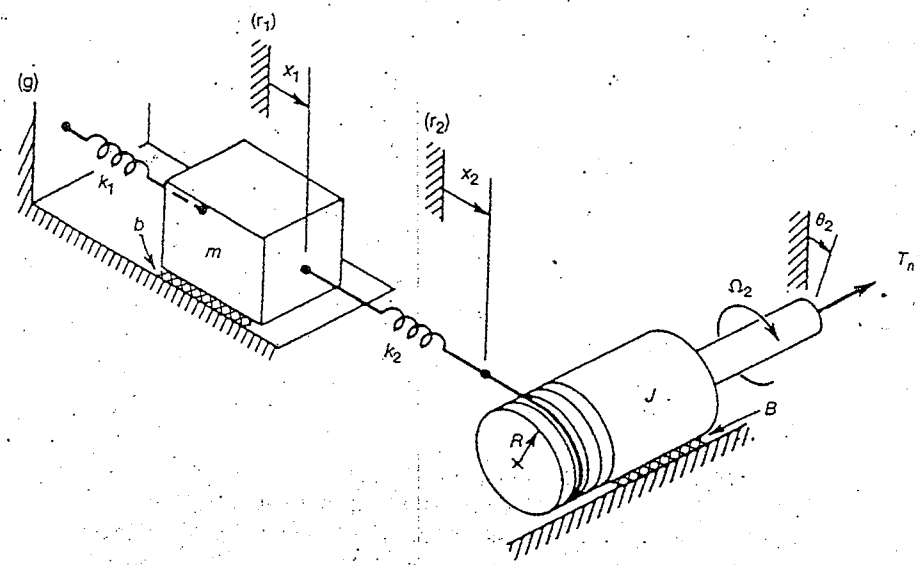


FIGURE P4.3 Mechanical system considered in Problem 4.3.

4.4. A nonlinear dynamic system has been modeled by the following state-variable equations:

$$\dot{q}_1 = -4q_1 + 10q_2 + 4u_1$$
$$\dot{q}_2 = 0.5q_1 - 2|q_2|q_2$$

where q_2 is also the output variable and u_1 is the input variable.
(a) Linearize the state-variable equations.
(b) Find the input-output equation for the linearized model.

4.5. The state-model matrices of a single-input single-output linear dynamic system are

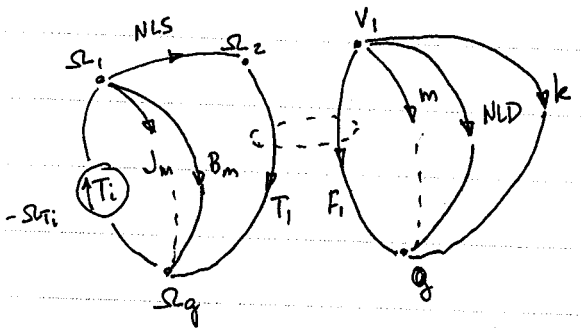
$$\mathbb{A} = \begin{bmatrix} -3 & -19 \\ 1 & -2 \end{bmatrix} \quad \mathbb{B} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

$$\mathbb{C} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} \qquad \mathbb{D} = 0$$

The column vector $\mathbf{q}$ of the system state variables contains q_1, q_2. Find the input-output model for this system.

4.6. A linear dynamic system is described by the following differential equations:

$$\ddot{y} + 4\dot{y} + 4y = 2\dot{x} + 2x$$
$$2\dot{x} + x - y = u(t)$$



$$T_{NLS} = 2(\theta_1 - \theta_2)^2 \qquad \frac{dT}{d\theta_{12}} = 4\theta_{12}\Big|_{\text{operating pt.}} = K \qquad \frac{dF}{dt} = K\Omega_{12}$$

$$F_{NLD} = 2V_{1g}^3 + 4V_{1g} \qquad \frac{dF}{dV_{1g}} = \left(6V_{1g}^2 + 4\right)\Big|_{V_{1g}\text{ operating pt.}} = b \qquad F = bV_{1g}$$

$n-p = 5-2 = 3$

$$T_i = T_{J_m} + T_{NLS} + T_{B_m}$$

$$T_{NLS} = T_1$$

$$F_1 + F_m + F_{NLD} + F_K = 0$$

$b-s = 9-1 = 8$

$b-s$

$$+V_1 = R\Omega_2$$

$$T_1 = F_1 R$$

$$J_m \frac{d\Omega_1}{dt} = T_{J_m}$$

$$\frac{dT_{NLS}}{dt} = K\Omega_{12} = K(\Omega_{1g} - \Omega_{2g})$$

$$m\frac{dV_{1g}}{dt} = F_m$$

$$bV_{1g} = F_{NLD}$$

$$\frac{dF_K}{dt} = kV_{1g}$$

$$T_{B_m} = B_m\Omega_{1g}$$

$b-(n-p) = 9-3 = 6$

$$-\Omega_{T_i} + \Omega_{1g} = 0$$

$$-\Omega_{1g} + \Omega_{1g} = 0$$

$$-\Omega_{1g} + \Omega_{12} + \Omega_{2g} = 0$$

$$-V_{1g} + V_{1g} = 0 \quad (x3)$$

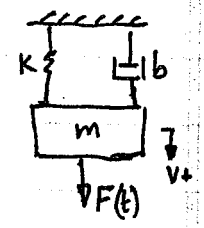
$SV \Rightarrow 4$: Ω_{1g}, T_{NLS}, V_{1g}, F_K

$$\frac{d\Omega_{1g}}{dt} = \frac{T_{B_m}}{B_m} = \frac{1}{B_m}\left[T_i - T_{B_m} - T_{NLS}\right] = \frac{1}{J_m}\left[T_i - B_m\Omega_{1g} - T_{NLS}\right]$$

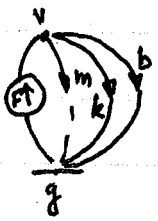
$$\frac{dT_{NLS}}{dt} = K\Omega_{12} = K(\Omega_{1g} - \Omega_{2g}) = K\left(\Omega_{1g} - \frac{V_{1g}}{R}\right)$$

$$\frac{dV_{1g}}{dt} = \frac{F_m}{m} = \frac{1}{m}\left[-F_{NLD} - F_K - F_1\right] = \frac{1}{m}\left[-bV_{1g} - F_K - \frac{T_1}{R}\right] = \frac{1}{m}\left[-bV_{1g} - F_K - \frac{T_{NLS}}{R}\right]$$

$$\frac{dF_K}{dt} = kV_{1g}$$



As we have shown in class the governing equation for a second order linear ordinary differential equation made up of a mass-spring-damper is $m\frac{d^2v}{dt^2} + b\frac{dv}{dt} + kv = \frac{dF}{dt}$ ⊛



Homogeneous solution V_h: satisfies $m\ddot{v} + b\dot{v} + kv = 0$ and solution depends on $b^2 - 4mk$

① $b^2 - 4mk > 0$ $\quad V_h = C_1 e^{s_1 t} + C_2 e^{s_2 t}$ $\quad s_1 = \frac{-b + \sqrt{b^2-4mk}}{2m} = -\omega_n(\zeta + \sqrt{\zeta^2-1}) < 0$

$s_2 = \frac{-b - \sqrt{b^2-4mk}}{2m} = -\omega_n(\zeta + \sqrt{\zeta^2-1}) < 0$

where $\omega_n = \sqrt{\frac{k}{m}}$ $\quad \zeta = \frac{b}{c_{cr}}$ $\quad c_{cr} = 2\sqrt{mk} = 2m\omega_n$. Here $\underline{\zeta > 1}$ overdamped

② $b^2 - 4mk = 0$ $\quad V_h = (C_1 + C_2 t)e^{-\frac{b}{2m}t} = (C_1 + C_2 t)e^{-\omega_n t}$ since $b = c_{cr}$ and $\zeta = 1$ critically damped

③ $b^2 - 4mk < 0$ $\quad V_h = C_1 e^{-\zeta\omega_n t}\cos(\omega_d t) + C_2 e^{-\zeta\omega_n t}\sin(\omega_d t) = \Sigma e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$

with $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (damped frequency); $\Sigma\cos\phi = C_2$ $\Sigma\sin\phi = C_1$ underdamp

and $\Sigma = \sqrt{C_1^2 + C_2^2}$; $\phi = \tan^{-1}(C_1/C_2)$ rad

Only in case ③ is there a frequency of oscillation: ω_d. Cases ① and ② are aperiodic

Take ⊛ and divide by m: $\ddot{v} + \frac{b}{m}\dot{v} + \frac{k}{m}v = \frac{1}{m}\frac{dF}{dt} = \ddot{v} + 2\zeta\omega_n\dot{v} + \omega_n^2 v = \frac{1}{m}\frac{dF}{dt} = g(t)$

Finding particular solution, $V_p(t)$

For all the cases there are 2 homogeneous solutions V_{h_1}, V_{h_2}

case ① $V_{h_1} = e^{s_1 t}$ $\quad V_{h_2} = e^{s_2 t}$ $\quad \zeta > 1$

② $V_{h_1} = e^{-\omega_n t}$ $\quad V_{h_2} = te^{-\omega_n t}$ $\quad \zeta = 1$

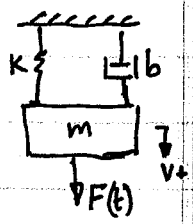
③ $V_{h_1} = e^{-\zeta\omega_n t}\cos\omega_d t$ $\quad V_{h_2} = e^{-\zeta\omega_n t}\sin\omega_d t$ $\quad \zeta < 1$

From Differential Equations, we can use variation of parameters to find V_p

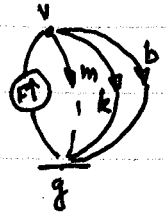
let $V_p = u_1(t)V_{h_1}(t) + u_2(t)V_{h_2}(t)$

This leads to: $u_1'V_{h_1} + u_2'V_{h_2} = 0$ and $u_1'V_{h_1}' + u_2'V_{h_2}' = g(t)$

Solving for u_1' & u_2' and integrating gives u_1 & u_2



As we have shown in class the governing equation for a second order linear ordinary differential equation made up of a mass-spring-damper is $m\frac{d^2v}{dt^2} + b\frac{dv}{dt} + kv = \frac{dF}{dt}$ (*)



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where $\omega_n = \sqrt{\frac{k}{m}}$ $\quad \zeta = \frac{b}{c_{cr}}$ $\quad c_{cr} = 2\sqrt{mk} = 2m\omega_n$. Here $\underline{\zeta > 1}$ overdamped

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From Differential Equations, we can use variation of parameters to find V_p

let $V_p = u_1(t)V_{h_1}(t) + u_2(t)V_{h_2}(t)$

This leads to: $u_1'V_{h_1} + u_2'V_{h_2} = 0$ and $u_1'V_{h_1}' + u_2'V_{h_2}' = g(t)$

Solving for u_1' & u_2' and integrating gives u_1 & u_2

Then $u_1(t) = \int^t \frac{-g(\bar{t}) \cdot V_{h_2}(\bar{t})}{WJ(V_{h_1}, V_{h_2})} d\bar{t}$ and $u_2(t) = \int^t \frac{g(\bar{t}) V_{h_1}(\bar{t})}{WJ(V_{h_1}, V_{h_2})} d\bar{t}$

where

Wronskey $WJ(V_{h_1}, V_{h_2}) = \det \begin{bmatrix} V_{h_1} & V_{h_2} \\ V_{h_1}' & V_{h_2}' \end{bmatrix}$. $WJ \neq 0$ for a solution to exist

For case ① $WJ = \det \begin{pmatrix} e^{s_1 t} & e^{s_2 t} \\ s_1 e^{s_1 t} & s_2 e^{s_2 t} \end{pmatrix} = (s_2 - s_1) e^{(s_1+s_2)t} = -2\omega_n \sqrt{\zeta^2 - 1}\, e^{-2\omega_n \zeta t}$

and

$$V_p(t) = e^{s_1 t} \cdot \int^t \frac{-g(\bar{t}) \cdot e^{+s_2 \bar{t}}}{-2\omega_n \sqrt{\zeta^2-1} \cdot e^{-2\omega_n \zeta \bar{t}}} d\bar{t} + e^{s_2 t} \cdot \int^t \frac{g(\bar{t}) \cdot e^{s_1 \bar{t}}}{-2\omega_n \sqrt{\zeta^2-1}\, e^{-2\omega_n \zeta \bar{t}}} d\bar{t}$$

For case ② $WJ = \det \begin{pmatrix} e^{-\omega_n t} & t e^{-\omega_n t} \\ -\omega_n e^{-\omega_n t} & e^{-\omega_n t}[1 - \omega_n t] \end{pmatrix} = e^{-2\omega_n t}$

and

$$V_p(t) = e^{-\omega_n t} \cdot \int^t \frac{-g(\bar{t}) \cdot \bar{t} e^{-\omega_n \bar{t}}}{e^{-2\omega_n t}} d\bar{t} + t e^{-\omega_n t} \cdot \int^t \frac{g(\bar{t}) \cdot e^{-\omega_n \bar{t}}}{e^{-2\omega_n \bar{t}}} d\bar{t}$$

$$= \int^t g(\bar{t}) [t - \bar{t}] e^{-\omega_n (t-\bar{t})} d\bar{t} \quad \blacktriangleleft$$

For case ③ $WJ = \det \begin{pmatrix} e^{-\zeta\omega_n t}\cos\omega_d t & e^{-\zeta\omega_n t}\sin\omega_d t \\ e^{-\zeta\omega_n t}[-\zeta\omega_n \cos\omega_d t - \omega_d \sin\omega_d t] & e^{-\zeta\omega_n t}[-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t] \end{pmatrix}$

$$= e^{-2\zeta\omega_n t} \cdot \omega_d$$

and $V_p(t) = e^{-\zeta\omega_n t}\cos\omega_d t \cdot \int^t \frac{-g(\bar{t}) e^{-\zeta\omega_n \bar{t}} \sin\omega_d \bar{t}}{e^{-2\zeta\omega_n \bar{t}} \cdot \omega_d} d\bar{t} + e^{-\zeta\omega_n t}\sin\omega_d t \cdot \int^t \frac{g(t) e^{-\zeta\omega_n \bar{t}} \cos\omega_d \bar{t}}{e^{-2\zeta\omega_n \bar{t}} \cdot \omega_d} d\bar{t}$

$$V_p(t) = \int^t g(\bar{t}) \cdot \frac{e^{-\zeta\omega_n (t-\bar{t})}}{\omega_d} \sin[\omega_d (t-\bar{t})] d\bar{t} \quad \blacktriangleleft$$

Case ① can be consolidated to

$$V_p(t) = \int^t g(\bar{t}) \cdot \frac{e^{-\zeta\omega_n (t-\bar{t})}}{\omega_n \sqrt{\zeta^2-1}} \sinh[\omega_n \sqrt{\zeta^2-1}\,(t-\bar{t})] d\bar{t} \quad \blacktriangleleft$$

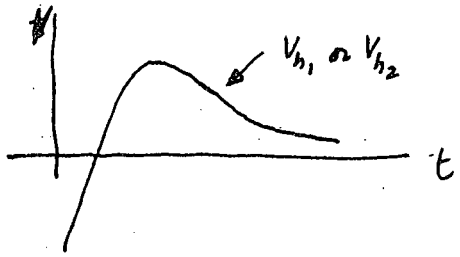
So, finally,

$$V_{TOT}(t) = C_1 V_{h_1}(t) + C_2 V_{h_2}(t) + V_p(t)$$

Note that the homogeneous part of the solution involves dying exponentials and is called the TRANSIENT part of the solution.

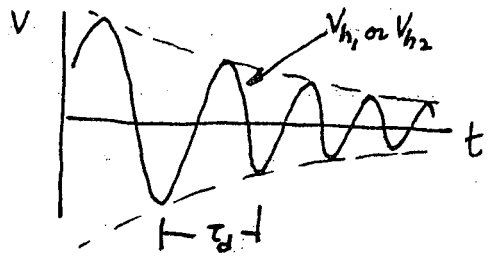
$V_p(t)$ is normally called the steady state part of the solution

Case (1) & Case (2)



no period of oscillation

Case (3)



τ_d - damped period of oscillation

$\omega_d = \frac{2\pi}{\tau_d}$

Now C_1 & C_2 are obtained knowing some initial information

i.e., $v(t=t_0)=v_0$ and $\frac{dv}{dt}(t=t_0)=a_0$ for example

Example: let $k = 1000\ N/m$ let $m = 10\ kg$ and let $b = 50\ \frac{N\text{-}s}{m}$

$\therefore\ \omega_n = \sqrt{\frac{k}{m}} = 10\ \frac{rad}{s}$ $\quad C_{cr} = 2m\omega_n = 200\ \frac{N\text{-}s}{m}$ $\quad \zeta = \frac{b}{C_{cr}} = 0.25 < 1$

so the system is underdamped $\quad \omega_d = \omega_n \cdot \sqrt{1-\zeta^2} = 10(0.9683) = 9.683\ rad/s$

$$V_h = C_1 e^{-\zeta\omega_n t}\cos\omega_d t + C_2 e^{-\zeta\omega_n t}\sin\omega_d t = C_1 e^{-2.5t}\cos 9.683t + C_2 e^{-2.5t}\sin 9.683t$$

Now let $F(t) = 10t + 5\ N$, for example. $\frac{1}{m}\frac{dF}{dt} = \frac{1}{10}\cdot 10 = 1\ \frac{1}{kg}\cdot\frac{N}{s} = 1\ \frac{m}{s^2}\cdot\frac{1}{s}$ ✓

$$V_p = \int_0^t \underbrace{g(\bar{t})}\ \frac{e^{-\zeta\omega_n(t-\bar{t})}}{\omega_d}\sin[\omega_d(t-\bar{t})]\,d\bar{t} = \int_0^t \frac{1\cdot e^{-2.5(t-\bar{t})}}{9.683}\sin[9.683(t-\bar{t})]\,d\bar{t}$$

$= g(t)$

Now let $t-\bar{t}=z$ $\quad d\bar{t}=-dz$ $\quad$ so $\quad$ when $\bar{t}=t \Rightarrow z=0$

$$V_p = \int \frac{1\cdot e^{-2.5z}\sin[9.683z]}{9.683}\,(-dz)$$

$$\int e^{az}\sin bz\,dz = \frac{e^{az}}{a^2+b^2}(a\sin bz - b\cos bz) \quad \text{using integration by parts twice}$$

$$\therefore \quad V_p = \frac{-1}{9.683}\left[\frac{e^{-2.5z}}{(2.5)^2+(9.683)^2}\left(-2.5\sin 9.683z - 9.683\cos 9.683z\right)\right]^{z=0}$$

$$= +\frac{1}{(2.5)^2+(9.683)^2} = \frac{1}{100} = .01$$

So $\quad V_{TOT} = C_1 e^{-2.5t}\cos 9.683t + C_2 e^{-2.5t}\sin 9.683t + .01$ ◀

Suppose $V(t=0)=5$ m/s $\quad$ and $\quad \frac{dV}{dt}(t=0)=0$ m/s^2 $\quad$ (initial conditions)

► Must apply these conditions on total solution.

$\therefore V(t=0)=5 = C_1\cdot e^0\cos 0 + C_2\cdot e^0\sin 0 + .01 = C_1 + .01 \Rightarrow C_1 = 4.99$ m/s ◀

$$\frac{dV}{dt} = C_1 e^{-2.5t}\left[-2.5\cos 9.683t + 9.683\sin 9.683t\right] + C_2 e^{-2.5t}\left[-2.5\sin 9.683t + 9.683\cos 9.683t\right]$$

$$\frac{dV}{dt}(t=0) = C_1 e^0\left[-2.5\cos 0 - 9.683\sin 0\right] + C_2 e^0\left[-2.5\sin 0 + 9.683\cos 0\right] = -2.5C_1 + 9.683C_2 = 0$$

$$C_2 = \frac{2.5C_1}{9.683} = 1.288 \text{ m/s} \quad ◀$$

So $\quad V(t) = 4.99e^{-2.5t}\cos 9.683t + 1.288e^{-2.5t}\sin 9.683t + .01$ m/s ◀

or $\quad = \Sigma e^{-2.5t}\sin(\omega_3 t+\phi)_A^{+0.01} = 5.154e^{-2.5t}\sin(9.683t+1.3181) + .01$

Other helpful integrals: $\int_0^t (t-\bar{t})e^{-\omega_n(t-\bar{t})}d\bar{t} = \int_0^0 -z e^{-\omega_n z}dz = +e^{-\omega_n z}\left[\frac{z}{\omega_n}+\frac{1}{\omega_n^2}\right]_0^0 = \frac{1}{\omega_n^2}$

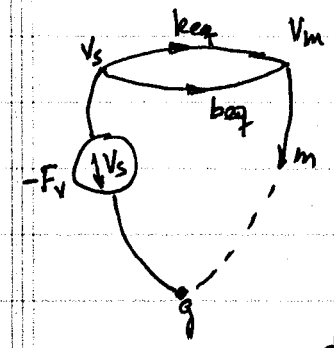
► $\int e^{az}\cdot bz\,dz = be^{az}\left[\frac{z}{a}-\frac{1}{a^2}\right]$

► $\int e^{az}\sinh bz\,dz = \int e^{az}\left[\frac{e^{bz}-e^{-bz}}{2}\right]dz = \frac{1}{2}\left[\frac{e^{(a+b)z}}{a+b} - \frac{e^{(a-b)z}}{a-b}\right]$ if $a\neq b,-b$

$= \frac{1}{2}\left[\frac{e^{2az}}{2a} - z\right]$ if $a=b$

$= \frac{1}{2}\left[z - \frac{e^{2az}}{2a}\right]$ if $a=-b$

9.23



$$-F_v + F_b + F_k = 0$$
$$F_k + F_b - F_m = 0 \quad \Big\} \; F_m = +F_v$$

$$-m\frac{dV_m}{dt} + k\int(V_s - V_m)dt + b(V_s - V_m) = 0 \quad (*)$$

$$\frac{d}{dt}(*) \Rightarrow m\frac{d^2V_m}{dt^2} + b(\dot{V}_m) + kV_m = b\dot{V}_s + kV_s$$

$$\text{or} \quad \ddot{V}_m + \frac{b}{m}\dot{V}_m + \frac{k}{m}V_m = \frac{1}{m}\left[b\dot{V}_s + kV_s\right] = g(t)$$

$$\frac{b}{m} = 2\zeta\omega_n = 3.75 \qquad \frac{k}{m} = \omega_n^2 = \frac{16000}{320} = 50 \qquad \omega_n = \sqrt{50} = 7.07 \ \frac{rad}{s}$$

$$\zeta = \frac{1}{2\omega_n}\cdot\frac{b}{m} = \frac{b}{bc_r} = .2652 \qquad \underline{\text{underdamped}}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 6.82\frac{rad}{s} \qquad \zeta\omega_n = .2652(7.07) = 1.875$$

homogeneous solution $V_{m_h} = C_1 e^{-\zeta\omega_n t}\cos\omega_d t + C_2 e^{-\zeta\omega_n t}\sin\omega_d t = C_1 e^{-1.875t}\cos 6.82t + C_2 e^{-1.875t}\sin 6.82t$

$$g(t) = \frac{1}{m}\left[b\dot{V}_s + kV_s\right] = 3.75\dot{V}_s + 50V_s \qquad \text{where } V_s = \text{step function (.2 m/s)}$$

$$\dot{V}_s = \frac{d}{dt}\,\text{step fn} = \delta(t)$$

$$\therefore V_{m_p} = \frac{b}{m}\int \frac{.2\,\delta(\bar{t})\,e^{-\zeta\omega_n(t-\bar{t})}\sin[\omega_d(t-\bar{t})]}{\omega_d}d\bar{t} + \frac{k}{m}\int \frac{.2\cdot u_s(\bar{t})\,e^{-\zeta\omega_n(t-\bar{t})}\sin\omega_d(t-\bar{t})}{\omega_d}d\bar{t}$$

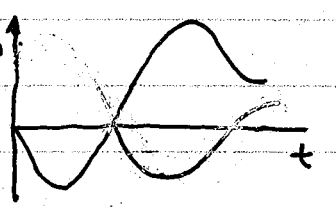
$$= \frac{b}{m}(.2)\frac{e^{-\zeta\omega_n t}\sin\omega_d t}{\omega_d} + \frac{k}{m}\frac{(.2)}{\omega_d}\left[\frac{e^{-1.875t}\left(-1.875\sin 6.82t - 6.82\cos 6.82t\right)}{(-1.875)^2 + (6.82)^2}\right]$$

$$= \frac{.75\,e^{-\zeta\omega_n t}\sin\omega_d t}{\omega_d} + \frac{k}{m}\frac{(.2)}{\omega_d}\cdot\frac{1}{\omega_n^2} = \frac{.75e^{-\zeta\omega_n t}\sin\omega_d t + .2}{\omega_d}$$

$$V_{m_{TOT}} = C_1 e^{-1.875t}\cos\omega_d t + C_2 e^{-1.875t}\sin\omega_d t + \left(.75e^{-\zeta\omega_n t}\sin\omega_d t + .2\right)\frac{1}{\omega_d}$$

(combine)

$$= C_1 e^{-1.875t}\cos 6.82t + \bar{C}_2 e^{-1.875t}\sin\omega_d t + .2/\omega_d$$

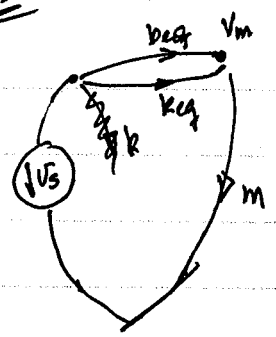


IC $V_m(t=0) = 0$; $\frac{dV_m}{dt}(t=0) = 0$ $\qquad \bar{C}_2 = C_2 + .75/\omega_d$

$$C_1 + .2/\omega_d = 0 \Rightarrow C_1 = -.2/\omega_d \ m/s$$

$$C_1(-\zeta\omega_n) + \bar{C}_2\cdot\omega_d = 0 \Rightarrow \bar{C}_2 = C_1\cdot\frac{\zeta}{\sqrt{1-\zeta^2}} = -.055 \ m/s$$

9.23



$-F_v - F_b - F_k = 0$ $\quad F_v = F_m$

$F_b + F_k + F_m = 0$

$m\frac{dv}{dt} + bv + k\int v\,dt = 0$

$m\dot{v} + b\dot{v} + kv = 0$

$m\frac{dv}{dt} = b(v_s - v_m) + k\int(v_s - v_m)dt$

$m\frac{d^2v}{dt} + \frac{b}{m}\dot{v}_m + \frac{k}{m}v_m = b\dot{v}_s + kv_s$

$= \frac{1}{m}(b\dot{v}_s + kv_s) = g(t)$

$\frac{b}{m} = 2\zeta\omega_n = 3.75 \quad \frac{k}{m} = \omega_n^2 = \frac{16000}{320} = 50 \quad \omega_n \approx 7.0$

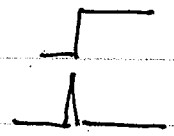
$49+1 = \omega_n^2$

$49\left(1+\frac{1}{49}\right) = 7\left(1+\frac{1}{98}\right) = 7.0$

$\frac{1200}{2(320)7.07} \approx \zeta = .2652$

$g(t) = \omega_n^2 \cdot v_s$

if v_s is a unit step

$\dot{v}_s$ 

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 7.07\sqrt{1-.2652^2} = 6.82$

$\zeta\omega_n = .2652(7.07) = 1.875$

$c_1 e^{-1.875t}\cos 6.82t + c_2 e^{-1.875t}\sin 6.82t + 50(v_s)$

$v(t=0)=0 \quad 0 = c_1 + 0$

$v_m = c_1 e^{-1.875}\cos 6.82t + c_2 e^{-1.875t}\sin 6.82t + v_p$

$v_p = \int \frac{b}{m}\dot{v}_s e^{-\zeta\omega_n(t-\bar{t})}\sin\omega_d(t-\bar{t})dt + \frac{k}{m}v_s$ ...

$v(t=0)=0$

$\frac{dv}{dt}(t=0)=0$

for nonoscillatory behavior $\zeta \geq 1 = \frac{b}{c_{cr}} = \frac{b}{2m\omega_n} = \frac{b}{2\cdot 320(7.07)}$

$b_{TOT} \geq 2(320)(7.07) \quad \frac{b_{TOT}}{4} = b = \frac{2(320)}{4}(7.07)$

$160(7.07) = 1120 + 11.2 = 1131.2$

$\frac{b}{m}\int_0^t \delta(\bar{t})\cdot\,d\bar{t} + \frac{kv_s}{m}\int_0^t 1\cdot\,d\bar{t}$

$c_1 e^{-\zeta\omega_n t}\cos\omega_d t + c_2 e^{-\zeta\omega_n t}\sin\omega_d t \quad \frac{b}{m}\left[e^{-\zeta\omega_n t}\sin\omega_d t\right] + v_s\frac{k}{m}\left[\frac{1}{\omega_n^2}\right] = v_s$

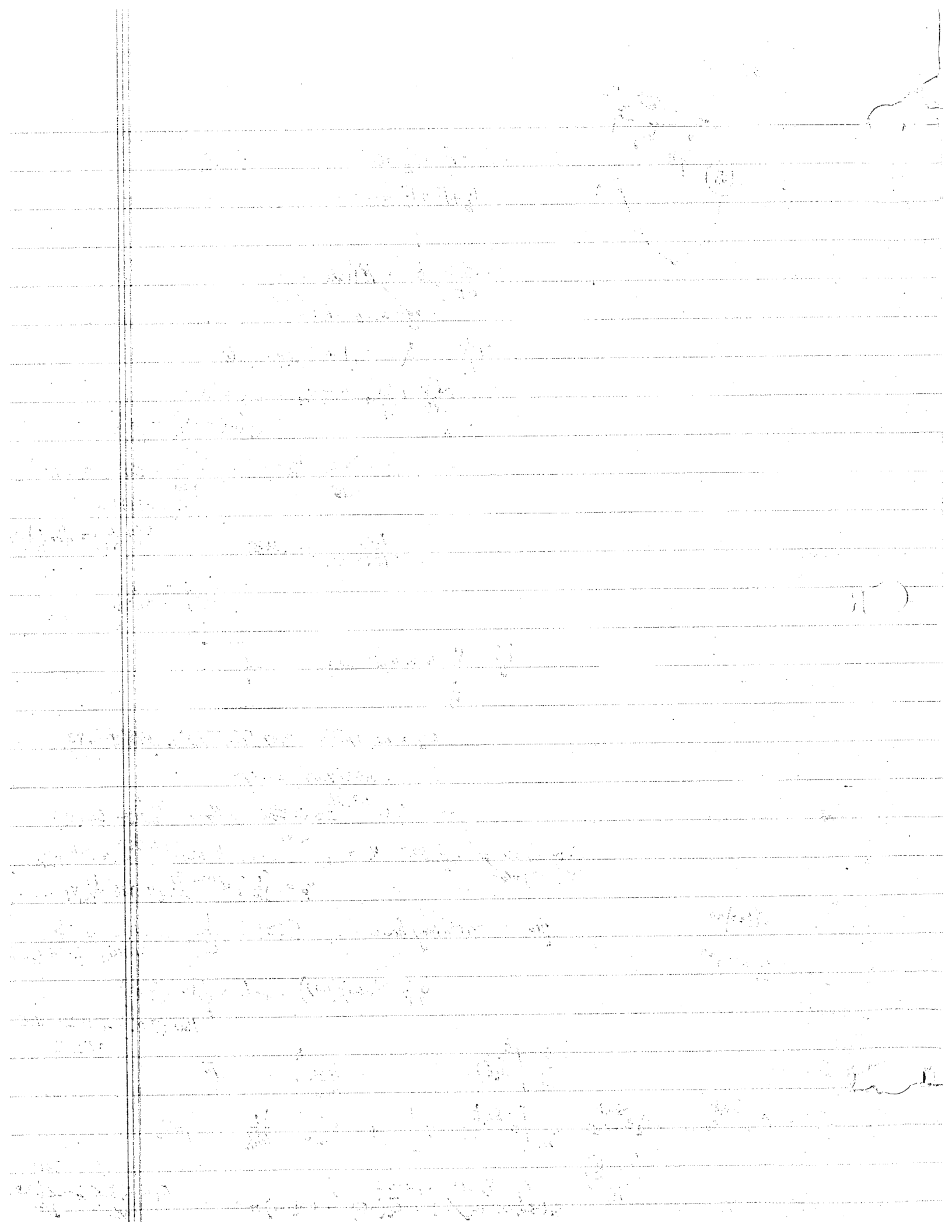
$\left(c_2 + \frac{b}{m}\right)$

$\bar{c}_2$

$c_1 + v_s = 0 \quad c_1 = -.2$

$c_1(-\zeta\omega_n\cos - \omega_d\sin) + \bar{c}_2(-\zeta\omega_n\sin + \omega_d\cos) = 0$

$c_2 = \frac{c_1\zeta\omega_n}{\omega_d} = -.2\frac{1.875}{6.82} = -.055$



VIBRATION AND TIME RESPONSE 8

8/1 INTRODUCTION

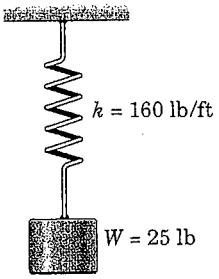
An important and special class of problems in dynamics deals with the linear and angular motions of bodies which oscillate or respond to applied disturbances in the presence of restoring forces. A few examples of this class of dynamics problems are the response of an engineering structure to earthquakes, the vibration of an unbalanced rotating machine, the time response of the plucked string of a musical instrument, the wind-induced vibration of power lines, and the flutter of aircraft wings. In many cases, excessive vibration levels must be reduced to accommodate material limitations or human factors.

In the analysis of every engineering problem, the system under scrutiny must be represented by a physical model. It is often permissible to represent a *continuous* or *distributed-parameter system* (one in which the mass and spring elements are continuously spread over space) by a *discrete* or *lumped-parameter model* (one in which the mass and spring elements are separate and concentrated). Such a modeling scheme is especially desirable when some portions of a continuous system are relatively massive in comparison with other portions. For example, the physical model of a ship propeller shaft is often assumed to be a massless but twistable rod with a disk rigidly attached to each end—one disk representing the turbine and the other representing the propeller. As a second example, we observe that the mass of springs may often be neglected in comparison with that of attached bodies. It should be noted that not every system is reducible to a discrete model. For example, the transverse vibration of a diving board after the departure of the diver is a somewhat difficult problem of distributed-parameter vibration. In this chapter, we shall begin the study of discrete systems, limiting our discussion to those whose configurations may be described with one displacement variable. Such systems are said to possess *one degree of freedom*. For a more detailed study which includes the treatment of two or more degrees of freedom and continuous systems, the student should consult one of the many textbooks devoted solely to the subject of vibrations.

Sample Problem 8/1

A body weighing 25 lb is suspended from a spring of constant $k = 160$ lb/ft. At time $t = 0$, it has a downward velocity of 2 ft/sec as it passes through the position of static equilibrium. Determine

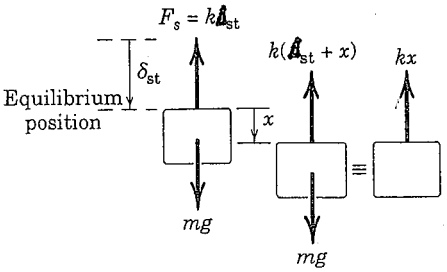
(a) the static spring deflection δ_{st}
(b) the natural frequency of the system in both rad/sec (ω_n) and cycles/sec (f_n)
(c) the system period τ_n
(d) the displacement x as a function of time, where x is measured from the position of static equilibrium
(e) the maximum velocity v_{max} attained by the mass
(f) the maximum acceleration a_{max} attained by the mass.



Solution. ***(a)*** From the spring relationship $F_s = kx$, we see that at equilibrium,

$$mg = k\delta_{st} \qquad \delta_{st} = \frac{mg}{k} = \frac{25}{160} = 0.1562 \text{ ft or } 1.875 \text{ in.} \quad \textit{Ans.}$$

(1)



(b)
$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{160}{25/32.2}} = 14.36 \text{ rad/sec} \quad \textit{Ans.}$$

$$f_n = (14.36)\left(\frac{1}{2\pi}\right) = 2.28 \text{ cycles/sec} \quad \textit{Ans.}$$

(c)
$$\tau_n = \frac{1}{f_n} = \frac{1}{2.28} = 0.438 \text{ sec} \quad \textit{Ans.}$$

(2)

(d) From Eq. 8/6:

$$x = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t$$

$$= (0) \cos 14.36t + \frac{2}{14.36} \sin 14.36t$$

$$= 0.1393 \sin 14.36t \quad \textit{Ans.}$$

As an exercise, let us determine x from the alternative Eq. (8/7):

$$x = \sqrt{x_0^2 + (\dot{x}_0/\omega_n)^2} \sin\left[\omega_n t + \tan^{-1}\left(\frac{x_0\omega_n}{\dot{x}_0}\right)\right] = A\sin(\omega_n t + \phi)$$

$$= \sqrt{0^2 + \left(\frac{2}{14.36}\right)^2} \sin\left\{14.36t + \tan^{-1}\left[\frac{(0)(14.36)}{2}\right]\right\}$$

$$= 0.1393 \sin 14.36t$$

$\phi = \tan^{-1}\left(\frac{x_0\omega_n}{v_0}\right)$

$A = \sqrt{x_0^2 + (v_0/\omega_n)^2}$

$v_{max} = A\,\omega_n$

$a_{max} = v_{max}\,\omega_n = A\,\omega_n^2$

(e) The velocity is $\dot{x} = 14.36(0.1393) \cos 14.36t = 2 \cos 14.36t$. Since the cosine function cannot be greater than 1 or less than -1, the maximum velocity v_{max} is 2 ft/sec, which, in this case, is the initial velocity. *Ans.*

(f) The acceleration is

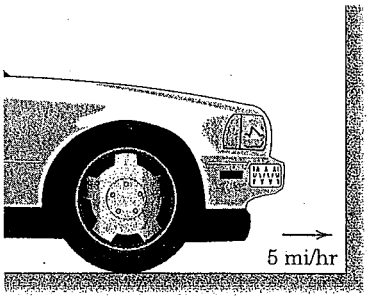
$$\ddot{x} = -14.36(2) \sin 14.36t = -28.7 \sin 14.36t$$

The maximum acceleration a_{max} is 28.7 ft/sec^2. *Ans.*

(1) The student should always exercise extreme caution in the matter of units. In the subject of vibrations, it is quite easy to commit errors due to mixing of feet and inches, cycles and radians, and other pairs that frequently enter the calculations.

(2) Recall that when we refer the motion to the position of static equilibrium, the equation of motion, and therefore its solution, for the present system is identical to that for the horizontally vibrating system.

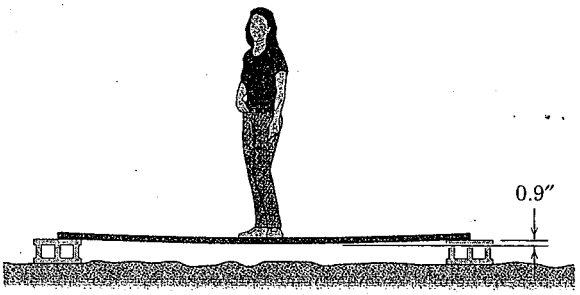
8/16 An energy-absorbing car bumper with its springs initially undeformed has an equivalent spring constant of 3000 lb/in. If the 2500-lb car approaches a massive wall with a speed of 5 mi/hr, determine (*a*) the velocity v of the car as a function of time during contact with the wall, where $t = 0$ is the beginning of the impact, and (*b*) the maximum deflection x_{max} of the bumper.



Problem 8/16

8/17 A 120-lb woman stands in the center of an end-supported board and causes a midspan deflection of 0.9 in. If she flexes her knees slightly in order to cause a vertical vibration, what is the frequency f_n of the motion? Assume elastic response of the board and neglect its relatively small mass.

Ans. $f_n =$ [redacted] Hz

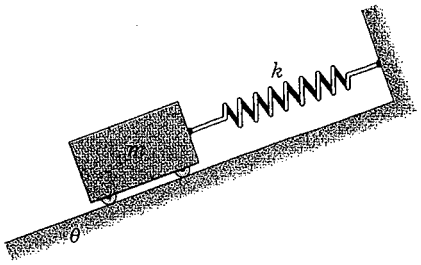


Problem 8/17

$\Delta_{st} = 0.9\,in = 3/40\,ft$ $\omega_n = \sqrt{g/\Delta_{st}} = 20.72\ rad/s$

$f_n = \frac{\omega_n}{2\pi} = 3.29$ [redacted] Hz

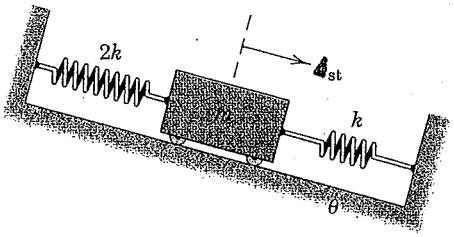
8/18 Prove that the natural frequency f_n of oscillation for the mass m is independent of θ.



Problem 8/18

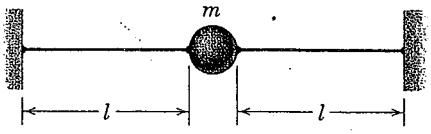
8/19 If both springs are unstretched when the mass is in the central position shown, determine the static deflection δ_{st} of the mass. What is the period of oscillatory motion about the position of static equilibrium?

Ans. $\delta_{st} = \dfrac{mg \sin\theta}{3k}$, $\tau_n = 2\pi\sqrt{\dfrac{m}{3k}}$



Problem 8/19

8/20 A small particle of mass m is attached to two highly tensioned wires as shown. Determine the system natural frequency ω_n for small vertical oscillations if the tension T in both wires is assumed to be constant. Is the calculation of the small static deflection of the particle necessary?



Problem 8/20

Replacing sin θ with θ, which is valid for small oscillations, the equation of motion is

$$\ddot{\theta}+\frac{3}{2}\frac{g}{l}\theta=0$$

This is again similar to equation 2.2, with θ replacing x and $\frac{3g}{2l}$ replacing k/m.

The natural circular frequency is

$$\omega_n=\sqrt{\frac{3}{2}\frac{g}{l}}$$

and the natural frequency, measured in cycles per second, is

$$f_n=\frac{1}{2\pi}\sqrt{\frac{3}{2}\frac{g}{l}}$$

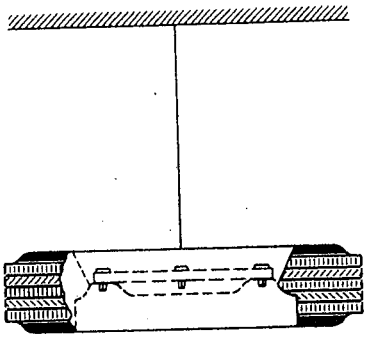
The radius of gyration is

$$I_0=mk_0^2=\frac{1}{3}ml^2$$

$$k_0=\frac{l}{\sqrt{3}}$$

and the distance to the center of percussion is

$$q_0=\frac{k_0^2}{r}=\frac{2}{3}l$$



PROBLEM 2.30 A device designed to determine the moment of inertia of a wheel-tire assembly consists of 2-mm steel suspension wire, 2 m long, and a mounting plate, to which is attached the wheel-tire assembly. The suspension wire is fixed at its upper end and hung vertically. When the system oscillates as a torsional pendulum, the period of oscillation without the wheel-tire assembly is 4 s. With the wheel-tire mounted to the mounting plate, the period of oscillation is 25 s. Determine the moment of inertia of wheel-tire assembly.

Answer: [redacted] kg·m²

Replacing $\sin\theta$ with θ, which is valid for small oscillations, the equation of motion is

$$\ddot{\theta}+\frac{3}{2}\frac{g}{l}\theta=0$$

This is again similar to equation 2.2, with θ replacing x and $\frac{3g}{2l}$ replacing k/m.

The natural circular frequency is

$$\omega_n=\sqrt{\frac{3}{2}\frac{g}{l}}$$

and the natural frequency, measured in cycles per second, is

$$f_n=\frac{1}{2\pi}\sqrt{\frac{3}{2}\frac{g}{l}}$$

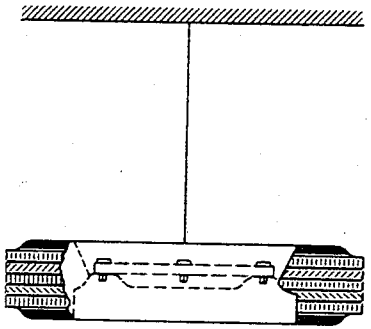
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$$k_0=\frac{l}{\sqrt{3}}$$

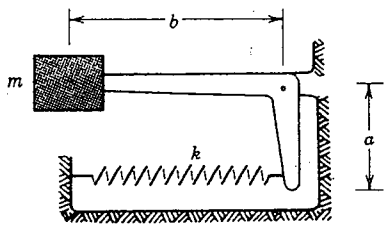
and the distance to the center of percussion is

$$q_0=\frac{k_0^2}{r}=\frac{2}{3}l$$



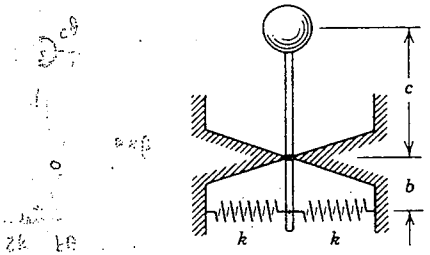
PROBLEM 2.30 A device designed to determine the moment of inertia of a wheel-tire assembly consists of 2-mm steel suspension wire, 2 m long, and a mounting plate, to which is attached the wheel-tire assembly. The suspension wire is fixed at its upper end and hung vertically. When the system oscillates as a torsional pendulum, the period of oscillation without the wheel-tire assembly is 4 s. With the wheel-tire mounted to the mounting plate, the period of oscillation is 25 s. Determine the moment of inertia of wheel-tire assembly.

Answer: [redacted] kg·m²



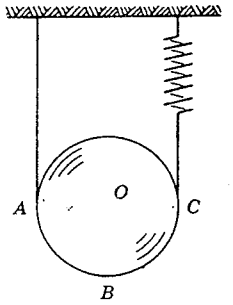
PROBLEM 2.31 Determine the natural frequency for the horizontal pendulum shown. Neglect the mass of the arm.

Answer: $f_n=\frac{1}{2\pi}\frac{a}{b}\sqrt{\frac{k}{m}}$



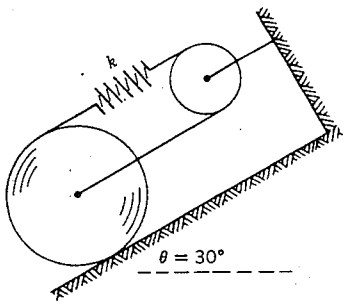
PROBLEM 2.32 A mass m is fixed to one end of a weightless rod which is pivoted a distance c from the weight. What is the natural frequency of vibration for small amplitudes of motion? This is called a vertical pendulum.

Answer: $f_n=\frac{1}{2\pi}\sqrt{\frac{2kb^2}{mc^2}-\frac{g}{c}}$



PROBLEM 2.33 The uniform wheel of mass m is supported in the vertical plane by the light flexible band ABC and the spring, which has a stiffness k. The wheel has a moment of inertia I_0 about the geometric center O, and rolls without slipping on the band ABC. Determine the natural frequency of the system.

Answer: $f_n=\frac{1}{2\pi}\sqrt{\frac{4kr^2}{I_0+mr^2}}$



PROBLEM 2.34 A solid circular cylinder is held in place by a cable and a pulley. The flexibility of the cable can be represented by a spring with a modulus k. The cable is wrapped around the cylinder. Friction is sufficient to prevent slipping. Determine the natural frequency of small oscillations. The mass of the pulley may be neglected.

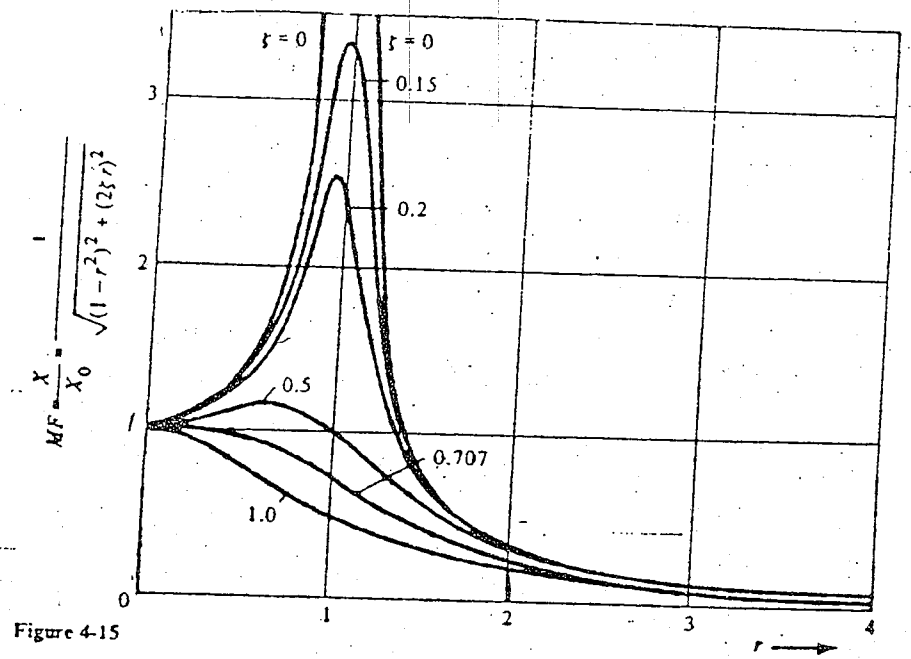


Figure 4-15

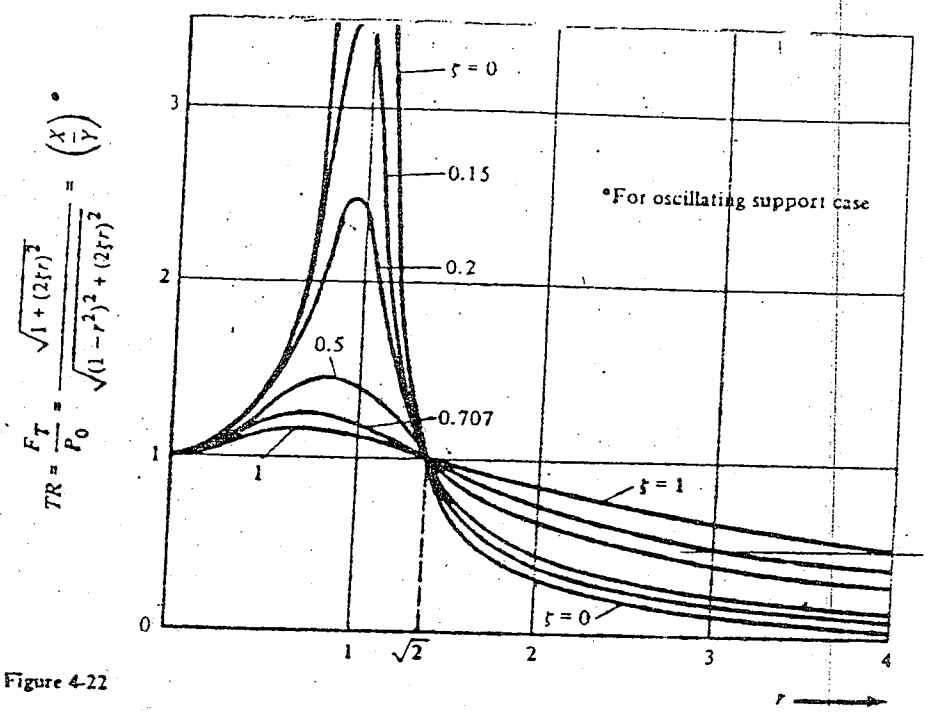


Figure 4-22

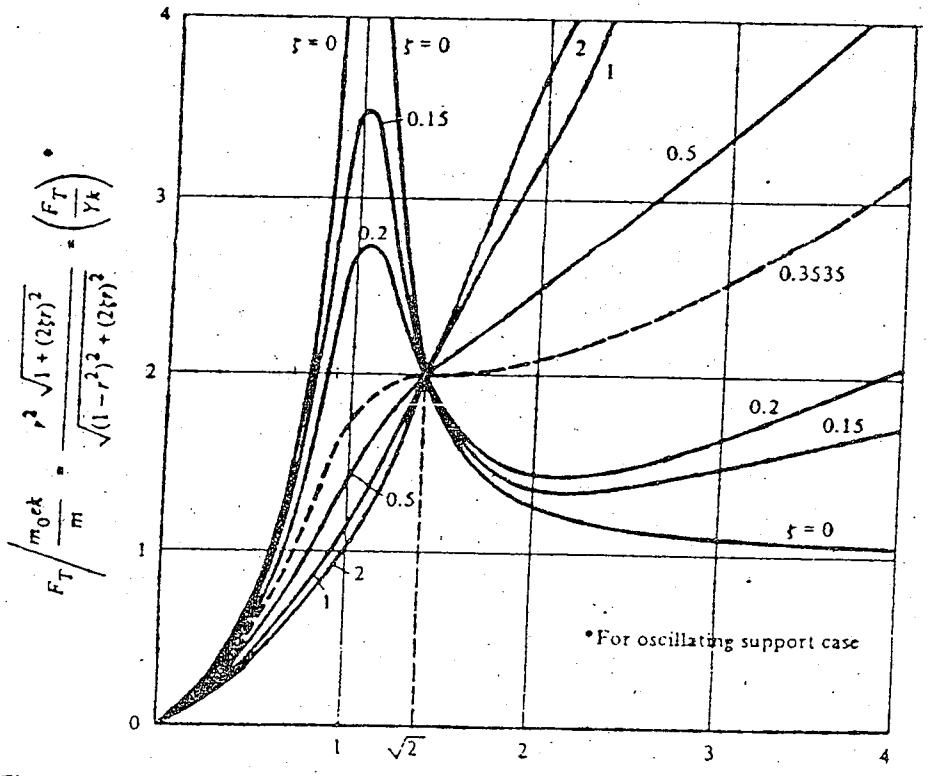
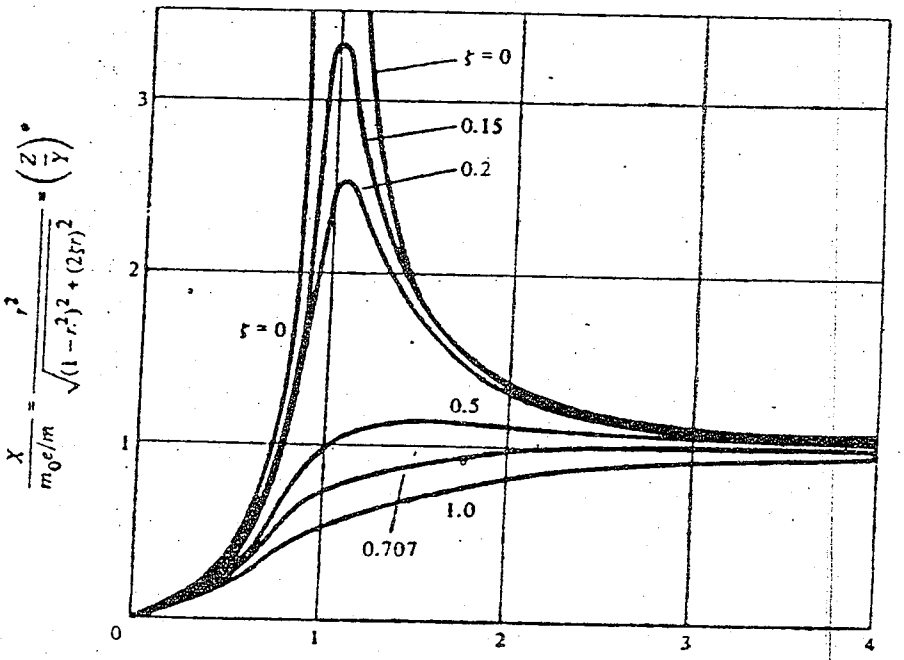


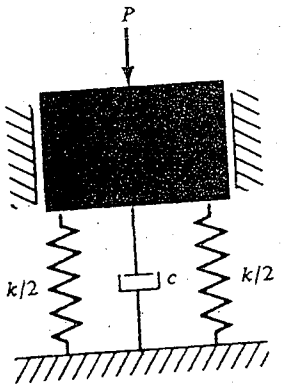
Figure 4-25



*For oscillating support case.

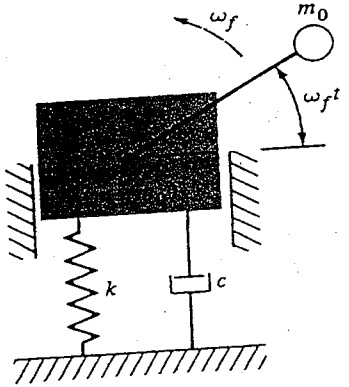
Figure 4-20

4-31. A machine having a mass of 70 kg is mounted as shown on springs having a total stiffness of 33 880 N/m. The damping factor is $\zeta = 0.20$. A harmonic force $P = 450 \sin 13.2t$ (where P is in newtons and t is in seconds) acts on the mass. For the sustained or steady-state vibration, determine (a) the amplitude of the motion of the machine, (b) its phase with respect to the exciting force, (c) the transmissibility, (d) the maximum dynamic force transmitted to the foundation, and (e) the maximum velocity of the motion.



Problem 4-31

4-32. A machine having a total weight of 96.5 lb is mounted as shown on a spring having a modulus of 900 lb/in. and is connected to a dashpot with a damping *factor* of 0.25. The machine contains a rotating unbalance ($w_0 e$) of 5 lb in. If the speed of rotation is 401.1 rpm, obtain (a) the amplitude of the steady-state motion, (b) the maximum dynamic force transmitted to the foundation, and (c) the angular position of the arm when the structure goes upward through its neutral position.

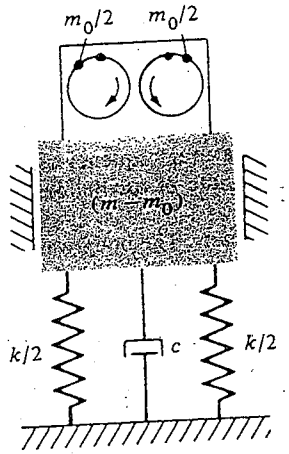


Problem 4-32

• **4-33.** A machine having a total weight of 128.67 lb contains a part that rotates at 859.44 rpm. The machine is supported equally on four identical springs and is connected to a dashpot for which the damping constant is 10 lb sec/in. operation the machine has a steady amplitude of 0.116 in. and transmits maximum dynamic force of 174 lb to the supporting base. (a) Determine modulus for the springs. (b) Calculate the unbalanced moment ($w_0 e$) for rotating part.

4-34. A machine with a total mass of 50 kg contains a shaft mechanism that rota at 1800 rpm. The machine is supported by springs for which the equival modulus is 20 000 N/m but the damping constant is unknown. Due to rotating unbalance, the steady-state amplitude is 1.32 cm and the maxim dynamic force carried by the base is 278 N. Determine (a) the damping c stant and (b) the unbalanced moment $m_0 e$.

• **4-35.** A typical vibration exciter is composed of two eccentric masses that rot oppositely, as represented in the top of the accompanying figure. Thus oscillation is produced in the vertical direction only, since the horizontal efl cancels. This device is used to determine the vibrational characteristics of structure to which it is attached. The unbalance ($ew_0/2$) of each exciter whee 3 lb in. The total arrangement has a weight $mg = 200$ lb. The exciter speed eccentric rotation) was adjusted until a stroboscope showed the structure tc moving upward through its equilibrium position at the instant the eccen weights were at their top position. The exciter speed then was 840 rpm, and steady amplitude was 0.75 in. Determine (a) the natural frequency of the str ture and the damping factor for the structure. If the speed were changed 1260 rpm, (b) obtain the steady-state amplitude of the structure and (c) angular position of the eccentrics as the structure moves upward through equilibrium position.



Problem 4-35

• **4-36.** A small compressor weighs 69 lb and runs at 875 rpm. It is to be supporte four springs (equally) and no damping is provided. Design the springs, determining k, so that only 15% of the shaking force is transmitted to supporting foundation or structure.

4-37. Solve Prob. 4-36 considering that damping is also to be included such that the damping factor will be 0.4 for the system. Equal damping at each spring will be provided by a plug or block of energy-absorbing material. Determine the required damping constant as well as the spring constant.

4-38. A refrigerator compressor unit weighs 179 lb and operates at 590 rpm. The maximum dynamic force imposed on the compressor is 20 lb. The unit is to be supported equally by three springs and is to be undamped. (a) Determine the required spring constant if 10% of the dynamic force is to be transmitted to the supporting base. (b) Determine the clearance that must be provided for the unit.

4-39. Solve Prob. 4-38, considering that *each* spring contains a viscous damping absorber having a damping constant of 0.8333 lb sec/in.

4-40. A small machine is known to contain a rotating unbalance on its main shaft. The machine weighs 65 lb and when placed on elastic vibration isolators (which serve as a viscous damping member as well as the elastic support), the static equilibrium displacement is 2.166 in. Also, when the machine is displaced further and released, the subsequent vibration diminishes from an amplitude of 2.350 in. to 0.135 in. in exactly 3 cycles. Operating the machine at resonance produces a sustained amplitude of 0.0193 in. (a) Determine the damping constant for the system. (b) Considering that the shaft diameter is 3 in. so that the eccentric moment arm is 1.5 in., calculate the unbalanced weight. (c) Determine the steady amplitude at an operation speed of twice the resonant speed.

4-41. A damped system with a rotating unbalance is composed of a 96.5-lb body and a spring having a constant of 80 lb/in. but the damping constant is unknown. When operated at resonance, the sustained amplitude is 2.143 in. If the speed of operation is greatly increased, the steady amplitude eventually approaches the value of 0.3572 in. Determine the damping constant for the system.

4-42. The support for a damped system (see Fig. 4-26) is oscillated harmonically with a frequency of 75 cycles/min, causing the mass to vibrate with a steady-state amplitude of 0.6 in. and a maximum dynamic force of 48.75 lb to be transmitted to the support. For the system, the spring modulus is 13 lb/in. and the damping constant is 0.454 lb sec/in. Determine (a) the natural frequency of the system, (b) the mass of the system, and (c) the amplitude of oscillation of the support.

4-43. The support for a damped system is oscillated harmonically. The body weighs 20 lb and the spring constant is 14 lb/in. When the support is oscillated at a rate of 300 cycles/min, the sustained amplitude of the body is 0.793 in. (a) Determine the maximum dynamic force carried by the support. (b) If the support is oscillated at resonance with an amplitude of 1.121 in., the body has a steady amplitude of 3.069 in. Obtain the damping constant for the system.

4-44. A mass-spring system, arranged horizontally, is damped only by Coulomb friction (refer to Fig. 4-27). The system is driven by a harmonic force having a frequency of 54.356 cycles/min and a maximum value of 30 lb, causing the body to oscillate with a steady amplitude of 3.09 in. The spring constant is 12 lb/in. and the body weighs 28.95 lb. Determine the average coefficient of sliding friction for the surfaces involved.

4-45. A machine is designed to produce an abrasive action on a horizontal surface in order to wear and smoothen the surface. It consists of a sliding member weighing 20 lb fixed to an elastic member having a stiffness coefficient of 8 lb/in. The average coefficient of sliding friction of the body against the surface is 0.3. A harmonic force having a maximum value of 15 lb actuates the body at a frequency of 0.9 of the resonant value. Determine the total excursion or stroke of the member.

4-46. A certain mass-spring system exhibits hysteresis damping only. The spring modulus is 310 lb/in. When excited harmonically at resonance, the steady amplitude is 1.540 in. for an energy input of 32 in. lb. When the resonant energy input is increased to 87 in. lb, the sustained amplitude changes to 2.464 in. Determine the hysteresis coefficient β and the exponent γ.

4-47. A small built-up structure shows solid damping characteristics with $\gamma = 2$. A 1000-lb load on the structure causes a static displacement of 2.500 in., and a supported machine that is subjected to a harmonic force having a constant amplitude of 75 lb produces a resonant amplitude of 6.818 in. (a) Determine the hysteresis damping coefficient β and (b) the energy dissipated per cycle by hysteresis at resonance. (c) Calculate the steady amplitude at twice the resonant frequency and (d) at one-half the resonant frequency.

7-27 @ r=1 $X_{ss} = \frac{P_o/k}{2\zeta}$ $\zeta = \frac{P_o}{2kX_{ss}} = \frac{15}{2\cdot24\cdot2.8409} = .11$

$c/c_c = \zeta \Rightarrow c_c = c/\zeta = \frac{.88}{.11} = 8 \frac{lb\text{-}s}{in} = 2m\omega_n$

$c_c = 2m\omega_n = \frac{2m}{K}\cdot\omega_n K = 2\frac{K}{\omega_n}$

$\omega_n = \frac{2K}{c_c} = \frac{2\times24}{8} = 6 \frac{rad}{sec}$

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 5.9636 \text{ rad/s}$

4-29 $\Theta_{ss} = 3.368\cdot\frac{\pi}{180} = .0588 \text{ rad}$

$\Theta_{ss} = \frac{T_o/K_T}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ $\omega_n = \sqrt{\frac{K_T}{I}} = \sqrt{\frac{60000}{24}} = 50 \text{ rad/s}$

$c_{c_T} = 2I\omega_n = 2\times24\times50 = 2400 \text{ lb-in-sec}^2/\text{rad}$

$\zeta = c/c_{c_T} = \frac{840}{2400} = .35$

$\Theta_{ss} = .0588 = \frac{2700/60000}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$ solve for r

$r^4 - 2r^2 + 1 + 4\zeta^2 r^2 = \left[\Theta_{ss}/(T_o/K_T)\right]^{-2} = .586$

$r^4 - 1.51r^2 + .414 = 0$

$r = .6$ & $r = 1.072$

$\omega_f = r\omega_n = 30$

$= 53.62$

$T_{max,trans} = \frac{T_o\sqrt{1+(2\zeta r)^2}}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

@ $r=.6$ $T_{max,tr} = 2700\ (1.4169) = 3825.5 \text{ lb in}$

@ $r=1.072$ $T_{max,tr} = 2700\ (1.6334) = 4410.1 \text{ lb-in}$

4-41 $X_{ss,RV} = \frac{m_o e}{m}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}$

@ $r=1$

$X_{ss,RV} = \frac{m_o e}{m}\cdot\frac{1}{2\zeta} = 2.143 \text{ in}$

@ $r=\infty$

$X_{ss,RV} = \frac{m_o e}{m} = .3572$

$\therefore \quad \zeta = \frac{.3572}{2.143\cdot2} = .0834$

$\zeta = c/c_c = \frac{c}{2m\omega_n} = \frac{c}{2\left(\frac{96.5}{32.2}\right)\omega_n}$ $\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{80\times12}{96.5/32.2}}$

$= 17.9 \text{ rad/s}$

$\zeta = \frac{c}{107.33}$ so $c = 8.944 \frac{lb\text{-}s}{ft}$

$= .7454 \frac{lb\text{-}s}{in}$