let's define $\zeta = \frac{c}{c_{crit}}$ (ZETA)

for the overdamped case - $c^2 - 4mk > 0$ $\quad c > \sqrt{4mk} = c_{crit} \Rightarrow \zeta > 1$

for the critically damped case $c^2 - 4mk = 0$ $\quad c = \sqrt{4mk} = c_{crit} \Rightarrow \zeta = 1$

for the underdamped case $c^2 - 4mk < 0$ or $4mk - c^2 > 0 \Rightarrow \sqrt{4mk} > c \Rightarrow c_{crit} > c \Rightarrow \zeta < 1$

but for the underdamped case [remember $c = 0$ is the undamped situation] $\therefore 0 < \zeta < 1$

to rewrite the equations in terms of ζ & ω_n

$$\frac{c}{2m} = p = \frac{c}{c_{crit}} \cdot \frac{c_{crit}}{2m} = \zeta \cdot \frac{2m\omega_n}{2m} = \zeta\omega_n$$

Case I $\quad \frac{\sqrt{c^2 - 4mk}}{2m} = \sqrt{\frac{c^2}{4m^2} - \frac{4mk}{4m^2}} = \sqrt{(\zeta\omega_n)^2 - \frac{k}{m}} = \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = \omega_n\sqrt{\zeta^2 - 1}$

Case III $\quad q = \omega_d = \frac{\sqrt{4mk - c^2}}{2m} = \omega_n\sqrt{1 - \zeta^2}$ $\quad$ so

Case I $\quad x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$

$\zeta > 1$

overdamped

$x(t=0) = X_0 = C_1 + C_2$

$\dot{x}(t=0) = V_0 = C_1 s_1 + C_2 s_2$

$s_1 = (-\zeta + \sqrt{\zeta^2 - 1})\omega_n$

$s_2 = (-\zeta - \sqrt{\zeta^2 - 1})\omega_n$

Case II $\quad c_{crit} = 2m\omega_n \quad x(t) = (C_1 + C_2 t)e^{-\omega_n t}$

$\zeta = 1$

critically damped

$x(t=0) = X_0 = C_1$

$\dot{x}(t=0) = V_0 = C_2 - \omega_n C_1$ and $C_2 = V_0 + \omega_n C_1 = V_0 + \omega_n X_0$

Case III $\quad \omega_d = \omega_n\sqrt{1 - \zeta^2} = q \quad p = \frac{c}{2m} = \zeta\omega_n$

$0 < \zeta < 1$

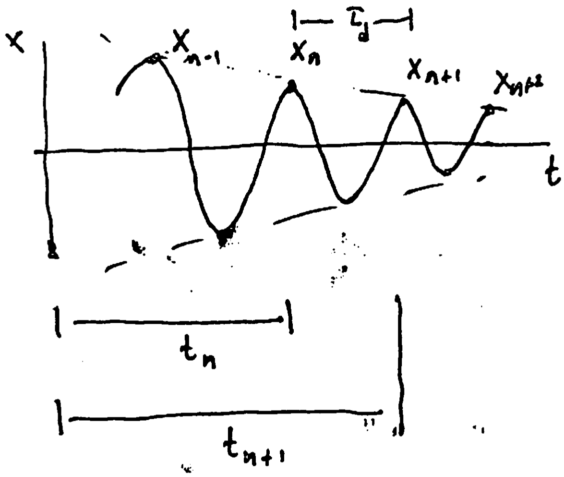
underdamped $x(t) = e^{-\zeta\omega_n t}[C_1 \sin\omega_d t + C_2 \cos\omega_d t] = X e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$

$$X = \sqrt{X_0^2 + \left(\frac{V_0 + pX_0}{q}\right)^2} = \sqrt{X_0^2 + \left(\frac{V_0 + \zeta\omega_n X_0}{\omega_d}\right)^2}$$

$$\tan\phi = \frac{X_0 q}{V_0 + pX_0} = \frac{X_0\omega_d}{V_0 + \zeta\omega_n X_0}$$

if we compare case III to friction damping

1. Viscous damping takes an ∞ amount of time to come to a stop



$t_{n+1} = t_n + \tau_d$

$x_n = X e^{-\zeta\omega_n t_n} \sin(\omega_d t_n + \phi)$

$x_{n+1} = X e^{-\zeta\omega_n (t_n + \tau_d)} \sin(\omega_d [t_n + \tau_d] + \phi)$

$\dfrac{x_n}{x_{n+1}} = \dfrac{1}{e^{-\zeta\omega_n \tau_d}}$ and the two sine function are equal

$\sin(\omega_d (t_n + \tau_d) + \phi) = \sin(\omega_d t_n + 2\pi + \phi)$
$= \sin(\omega_d t_n + \phi)$

$$\therefore \ln\left(\frac{x_n}{x_{n+1}}\right) = \ln\left(e^{\zeta\omega_n \tau_d}\right) = \zeta\omega_n \tau_d = \zeta\omega_n \cdot \frac{2\pi}{\omega_d} = \frac{\zeta\omega_n 2\pi}{\omega_n\sqrt{1-\zeta^2}}$$

LOGARITHMIC DECREMENT $= \delta = \ln\left(\dfrac{x_n}{x_{n+1}}\right) = \dfrac{2\pi\zeta}{\sqrt{1-\zeta^2}}$ CAN GET DAMPING RATIO FROM GRAPH

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}}$$

IF $\zeta < 0.3$ $\quad \zeta \cong \dfrac{\delta}{2\pi}$

2. Viscous damping is not linear but logarithmic $\delta = \ln\left(\dfrac{x_n}{x_{n+1}}\right)$ $\quad x_n, x_{n+1}$ 2 adjacent points in phase

3. $x_n - x_{n+1} \neq x_{n+1} - x_{n+2}$

Lastly $\quad m\ddot{x} + c\dot{x} + kx = m\left[\ddot{x} + \dfrac{c}{m}\dot{x} + \dfrac{k}{m}x\right] = m\left[\ddot{x} + 2\cdot\underbrace{\dfrac{c}{2m}}_{\zeta\omega_n}\dot{x} + \underbrace{\dfrac{k}{m}}_{\omega_n^2}x\right] = m\left[\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x\right] = 0$

$\Downarrow$

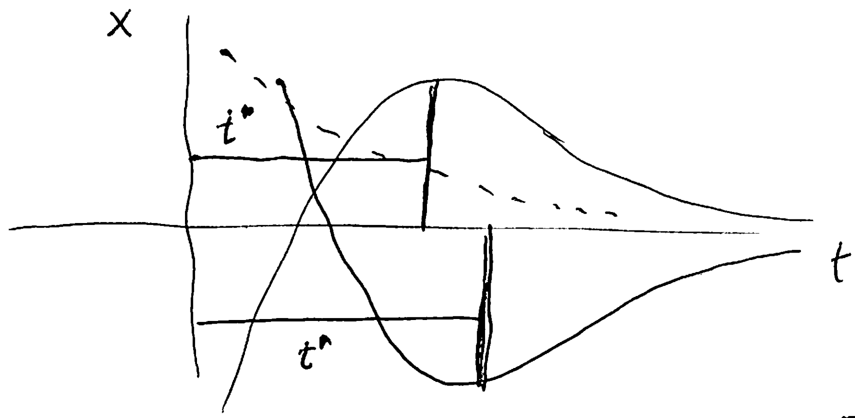
$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$ is also the governing equation for viscous systems.

4. what about $\ln\left(\dfrac{x_0}{x_n}\right) = \ln\left(\dfrac{x_0}{x_1}\cdot\dfrac{x_1}{x_2}\cdot\dfrac{x_2}{x_3}\cdots\dfrac{x_{n-1}}{x_n}\right) = n\ln\left(\dfrac{x_0}{x_1}\right) = n\delta \quad \therefore \delta = \dfrac{1}{n}\ln\left(\dfrac{x_0}{x_n}\right)$

so if you know x_0 & x_n and the no. of cycles between them, you can find δ

5.

overdamped $\quad x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$



$$\dot{x}(t = t^*) = 0$$

$$\dot{x} = C_1 s_1 e^{s_1 t^*} + C_2 s_2 e^{s_2 t^*} = 0$$

$$\frac{C_1 s_1}{-C_2 s_2} = \frac{e^{s_2 t^*}}{e^{s_1 t^*}} = e^{(s_2 - s_1) t^*}$$

$$\frac{1}{s_2 - s_1} \ln\left(\frac{C_1 s_1}{-C_2 s_2}\right) = t^*$$

$$O.S. = x(t^*) = C_1 e^{s_1 t^*} + C_2 e^{s_2 t^*}$$

critically damped case $\quad x(t) = (C_1 + C_2 t) e^{-\omega_n t}$

$$\dot{x}(t) = \left[C_2 + (C_1 + C_2 t)(-\omega_n)\right] e^{-\omega_n t}$$

$$\Rightarrow \quad C_2 + (C_1 + C_2 t^*)(-\omega_n) = 0$$

$$t^* = \frac{1}{\omega_n} - \frac{C_1}{C_2}$$

and $\quad O.S. = x(t^*) = \left[C_1 + C_2 t^*\right] e^{-\omega_n t^*}$