let's define $\zeta = \frac{c}{c_{crit}}$ (ZETA)

for the overdamped case - $c^2 - 4mk > 0$ $\quad c > \sqrt{4mk} = c_{crit} \Rightarrow \zeta > 1$

for the critically damped case $c^2 - 4mk = 0$ $\quad c = \sqrt{4mk} = c_{crit} \Rightarrow \zeta = 1$

for the underdamped case $c^2 - 4mk < 0$ or $4mk - c^2 > 0 \Rightarrow \sqrt{4mk} > c \Rightarrow c_{crit} > c \Rightarrow \zeta < 1$

but for the underdamped case [$c = 0$ remember is the undamped situation] $\therefore\ 0 < \zeta < 1$

to rewrite the equations in terms of ζ & ω_n

$$\frac{c}{2m} = p = \frac{c}{c_{crit}} \cdot \frac{c_{crit}}{2m} = \zeta \cdot \frac{2m\omega_n}{2m} = \zeta\omega_n$$

Case I $$\frac{\sqrt{c^2 - 4mk}}{2m} = \sqrt{\frac{c^2}{4m^2} - \frac{4mk}{4m^2}} = \sqrt{(\zeta\omega_n)^2 - \frac{k}{m}} = \sqrt{(\zeta\omega_n)^2 - \omega_n^2} = \omega_n\sqrt{\zeta^2 - 1}$$

Case III $$q = \omega_d = \frac{\sqrt{4mk - c^2}}{2m} = \omega_n\sqrt{1 - \zeta^2}$$ so

Case I
$\zeta > 1$
overdamped

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

$x(t=0) = x_0 = C_1 + C_2$

$\dot{x}(t=0) = V_0 = C_1 s_1 + C_2 s_2$

$s_1 = (-\zeta + \sqrt{\zeta^2 - 1})\omega_n$

$s_2 = (-\zeta - \sqrt{\zeta^2 - 1})\omega_n$

Case II
$\zeta = 1$
critically damped

$c_{crit} = 2m\omega_n$ $\qquad x(t) = (C_1 + C_2 t)e^{-\omega_n t}$

$x(t=0) = x_0 = C_1$

$\dot{x}(t=0) = V_0 = C_2 - \omega_n C_1$ and $C_2 = V_0 + \omega_n C_1 = V_0 + \omega_n x_0$

Case III
$< \zeta < 1$
underdamped

$\omega_d = \omega_n\sqrt{1 - \zeta^2} = q$ $\quad p = \frac{c}{2m} = \zeta\omega_n$

$$x(t) = e^{-\zeta\omega_n t}\left[C_1 \sin\omega_d t + C_2 \cos\omega_d t\right] = X e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

$$X = \sqrt{x_0^2 + \left(\frac{V_0 + p x_0}{q}\right)^2} = \sqrt{x_0^2 + \left(\frac{V_0 + \zeta\omega_n x_0}{\omega_d}\right)^2}$$

$$\tan\phi = \frac{x_0 q}{V_0 + p x_0} = \frac{x_0\omega_d}{V_0 + \zeta\omega_n x_0}$$