MEEN 471
COMPOSITE MATERIALS

INSTRUCTOR: Dr. Ozden O. Ochoa
Mechanical Engineering Dept.
ENPH 225, [Phone: 5-2022]

TEXT: Fiber-Reinforced Composites
P. K. Mallick
Marcel-Dekker, Inc., 1988

COURSE OUTLINE

Week No		
	PRELIMINARIES	
1	Ch. 1: 1-3	General Characteristics Overview
	MATERIALS	
2	Ch. 2: 1-2	Fibers, Matrix
3	3-7	Thermoset Polymers, Density & Voids
	MECHANICS	
4	Ch. 3: 1	Unidirectional Lamina
5	Ch. 3: 2	Characteristics of Reinforced Lamina
6	3-4	Laminated Structure
	PERFORMANCE	
7	Ch. 4: 1	Static Mechanical Properties
8	2-3	Fatigue and Impact
9	4-6	Environmental Effects
10	7	Fracture Behavior & Damage Tolerance
	MANUFACTURING	
11	Ch. 5: 1-3	Molding
12	4-7	Poltrusion, Filament Winding
	DESIGN	
13	Ch. 6: 1-2	Laminate Design
14	3	Joint Design
15	4-5	Design/Application Examples

EXAMS : Weeks #5, #9, #14
December 12(Final Exam)

GRADING :

Quizzes	10 %
Projects	20 %
3 Exams	45 %
Final	25 %

ASEE Faculty Development Program: PRINCIPLES of COMPOSITES
June 12-23, 1989, Hyatt Regency Hotel, Fort Worth, Texas

PROBLEM SET #4

Design a stiffness critical support structure that is dimensionally stable[low thermal expansion].

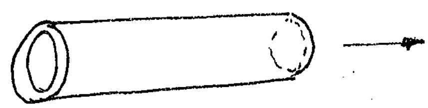
Constraints:

0 + 0.9 E-06/K at -20 to 18 C, CTE α

150 cm long, 3 cm. O.D., 0.38 cm thick walls

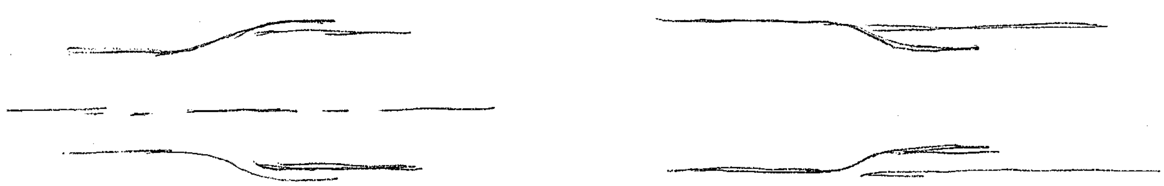
110 Gpa, axial elastic modulus

Minimum weight .848kg. 1.886 lb. $E_x = 110$



Attachments at both ends

$[0,0,0,90,0,0,0]_{2S}$



$[5\ -5\ 5\ -5\ 5]_s$ $E_x = 135$ problem w/ torsion

$[0\ 0\ 0\ 90\ 0\ 0\ 0]_{2s}$ $E_x\ 124$

$[15\ 0\ 15]$ $E_x = 118$ -

$[90\ 25\ 0\ 0]_s$ $E_x = 111$

$[0\ 45\ -45\ 0]_s$ $E_x = 104$

$[0\ 38\ 0\ -38]_s$ $E_x = 112$

shallower the angle the higher E_x
the lower the α_x

PRINCIPLES OF COMPOSITES
FUNDAMENTALS

Dr. Ozden Ochoa
Department of Mechanical Engineering
Texas A&M University

ASEE Faculty Development Program
June 12-23, 1989
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Table of Contents

OVERVIEW

- FIBER-REINFORCED COMPOSITES IS THREE DECADES OLD.
- BEGAN AS NEED FOR MORE MANEUVERABLE AIRCRAFT (LIGHTER)

- CHARACTERISTICS
  - LOW DENSITY / HIGH STRENGTH
  - HIGH STIFFNESS TO WEIGHT
  - DURABLE
  - FLEXIBILITY IN DESIGN

- USED IN MANY STRUCTURAL COMPONENTS
  - AIRCRAFT / SPACECRAFT
  - AUTOMOTIVE
  - MARINE
  - SPORTING GOODS

- NOMENCLATURE
  - LAMINA, LAYER, PLY
  - LAMINAE, LAMINAS, LAYERS, PLYS
  - LAMINATE

- FIBRE-REINFORCED COMPOSITES - CONSIST OF HIGH STRENGTH FIBERS EMBEDDED IN OR BONDED TO A MATRIX W/ DISTINCT BOUNDARIES BETWEEN THEM

- RETAIN PHYSICAL PROPERTIES OF EACH BUT PRODUCE A MATERIAL THAT HAS PROPERTIES THAT CAN'T BE ACHIEVED BY EITHER ALONE

- IN GENERAL

  FIBER- PRINCIPAL LOAD CARRYING MEMBER

  MATRIX - KEEPS FIBERS ORIENTED & FIXED

  - ACTS AS A LOAD TRANSFER MEDIUM
  - PROTECTS FIBER FROM ENVIRONMENT (TEMP/HUMIDITY)

- PRINCIPAL FIBERS - GLASS, CARBON, KEVLAR
- FIBERS ARE INCORPORATED VIA
  - CONTINUOUS STRAND ROVING
  - CONTINUOUS WEAVE ROVING
  - CHOPPED STRANDS
- MATRIX MAY BE POLYMER, METAL OR CERAMIC

- COMMON FORM USED IS A LAMINATE

  STACKS OF THIN LAYERS OF FIBER/MATRIX CONSOLIDATED TO PROPER THICKNESS

→ <u>CHARACTERISTICS</u>

- PHYSICAL & MECHANICAL PROPERTIES ARE CONTROLLED BY
  - FIBER ORIENTATION IN EACH LAYER
  - STACKING SEQUENCE

- STRENGTH TO WEIGHT RATIOS, LOW SPECIFIC GRAVITIES

  MODULUS TO WEIGHT RATIO $\Rightarrow$ COMPETITIVE W/ METALS

		SPEC GRAV	E GPa	Tensile str σ_t	Yield	E/weight	σ_t/wt
	EXAMPLE						
aged	INCO 718 NICKEL ALLOY	8.2	207	1399	1247	2.57	17.4
(unidirectional)	KEVLAR 49 FIBER EPOXY	1.38	75.8	1378	–	5.60	38

If we want same load carrying capacity ⇒ cross-section ≈ same

weight = density · Volume = density · CSA · length

$$\frac{\text{NICKEL}}{\text{KEVLAR}} = \frac{8.2}{1.38} \cong 6$$ NICKEL part weighs 6 TIMES AS much

STRENGTH

- METALS ARE NORMALLY ISOTROPIC (STRUCTURAL MATERIALS)
- COMPOSITES DEPEND ON DIRECTION OF MEASUREMENTS
  STRONGEST IN DIRECTION OF FIBER, WEAKEST ⊥ TO FIBER

- SIMILAR PROPERTY CHARACTERISTICS CAN BE DETERMINED FOR
  - COEFF OF THERMAL EXPANSION
  - CONDUCTIVITY, IMPACT STRENGTH

- CAN STACK LAMINA TO SIMULATE ISOTROPIC CONDITIONS
  - BUT YOU SACRIFICE STRENGTH OF UNIDIRECTIONAL FIBERS
  - YET SAVE ON WEIGHT

DESIGN YOUR STRUCTURE

- ANISOTROPY OF COMPOSITES ALLOWS FOR
  - TAILORING PROPERTIES FOR JOB REQUIREMENTS
  - SELECTIVE REINFORCEMENT OF STRUCTURE
  - DESIGNING A MEMBER W/ ZERO THERMAL EXPANSION
  - STABILITY OVER WIDE RANGE OF TEMPERATURES
  - HIGH INTERNAL DAMPING ⇒ REDUCED VIBRATION TRANSMISSION

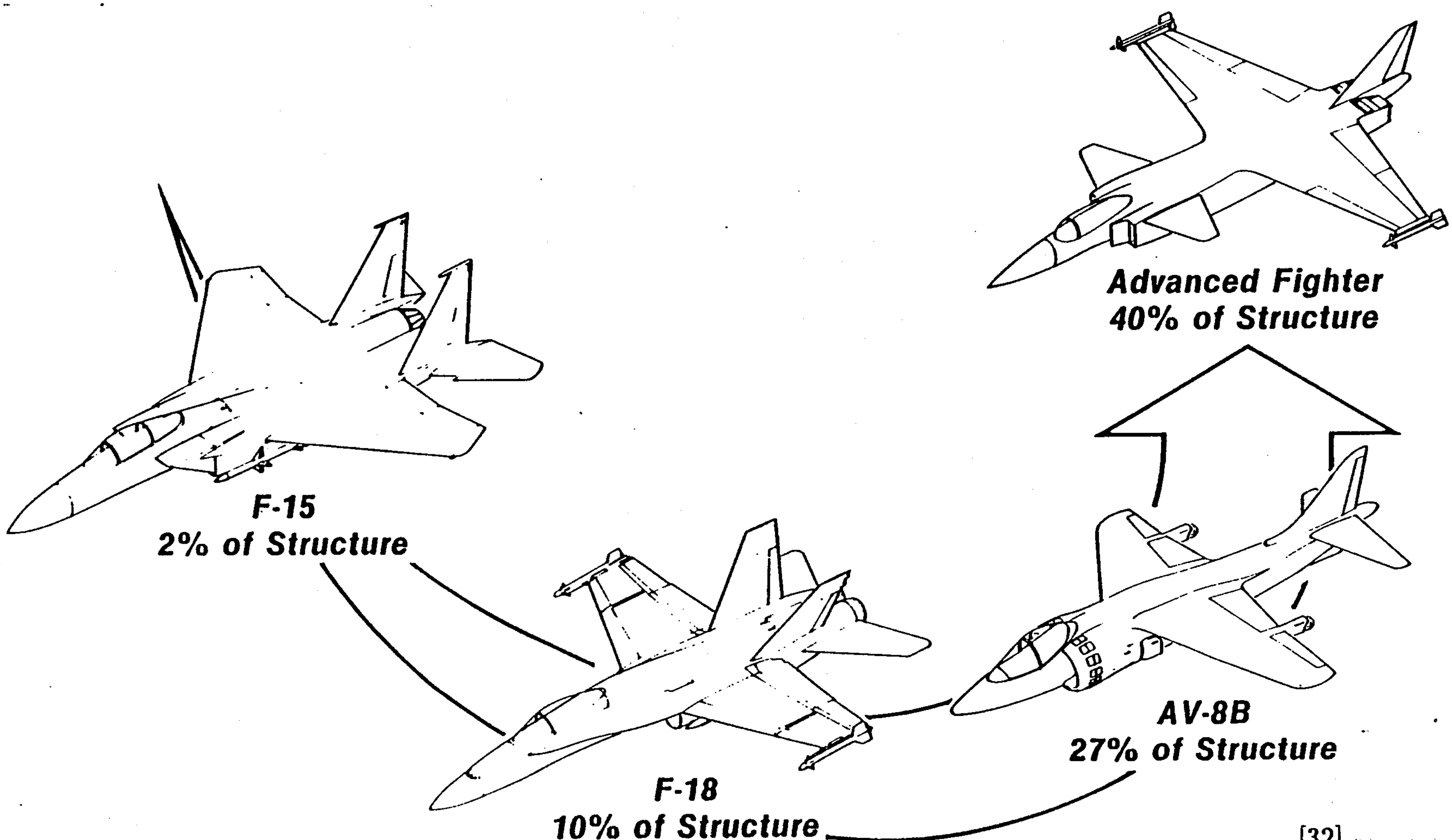
APPLICATIONS

- Aircraft
- Space
- Automotive
- Sporting Goods
- Watercraft

Applications of Fiber-Reinforced Composites in Military Aircraft [1]

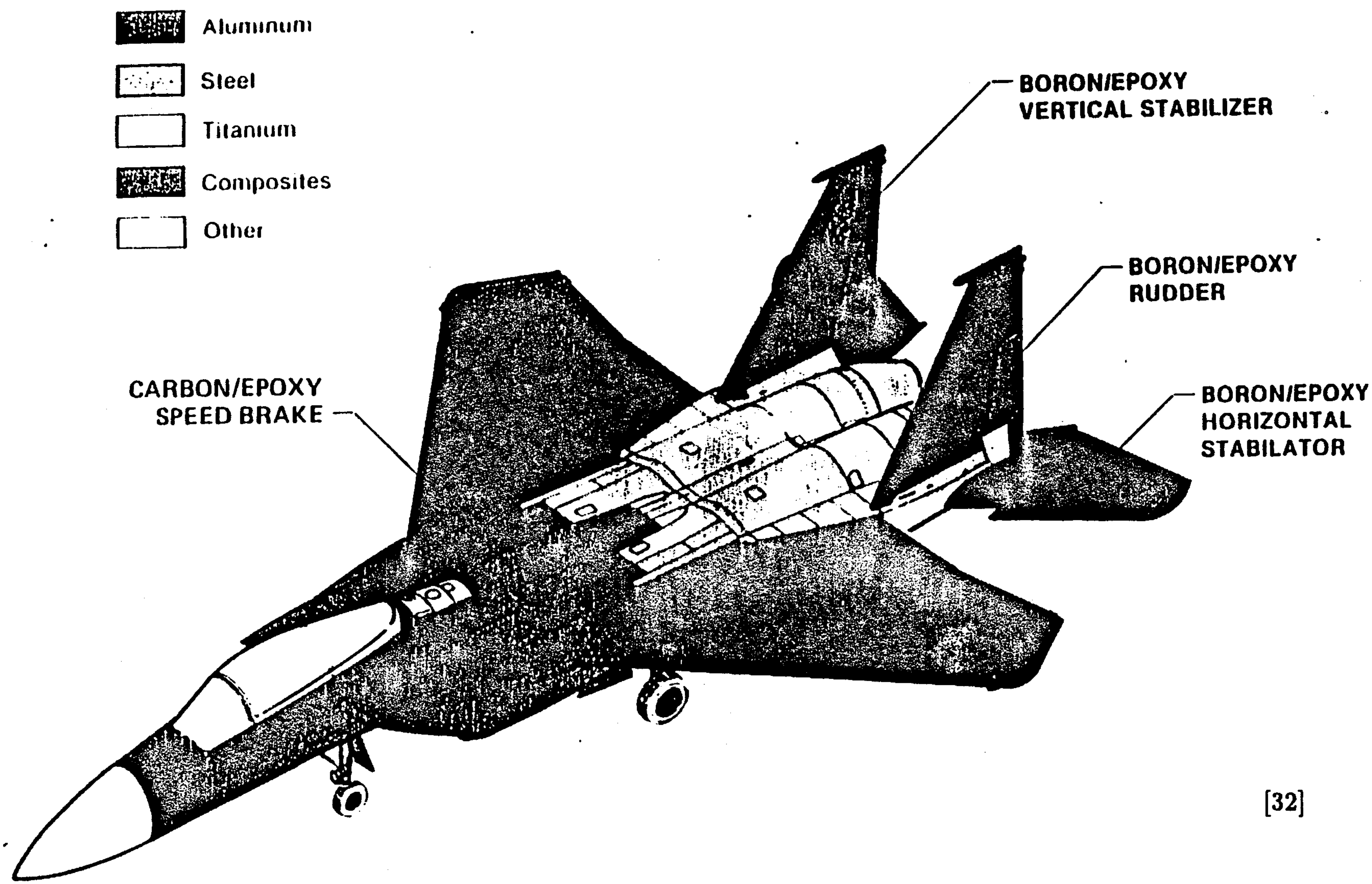
Aircraft	Component	Material	Overall weight savings over metal parts (%)
F-14 (1969)	Skin on the horizontal stabilizer box	Boron fiber-epoxy	19
F-11	Underwing fairings	Carbon fiber-epoxy	
F-15 (1975)	Fin, rudder, and stabilizer skins	Boron fiber-epoxy	25
F-16 (1977)	Skins on vertical fin box, fin leading edge	Carbon fiber-epoxy	23
F/A-18 (1978)	Wing skins, horizontal and vertical tailboxes; wing and tail control surfaces, etc.	Carbon fiber-epoxy	35
AV-8B (1982)	Wing skins and sub-structure; forward fuselage; horizontal stabilizer; flaps; ailerons	Carbon fiber-epoxy	25

Advanced Composite Structural Applications Are Continually Expanding



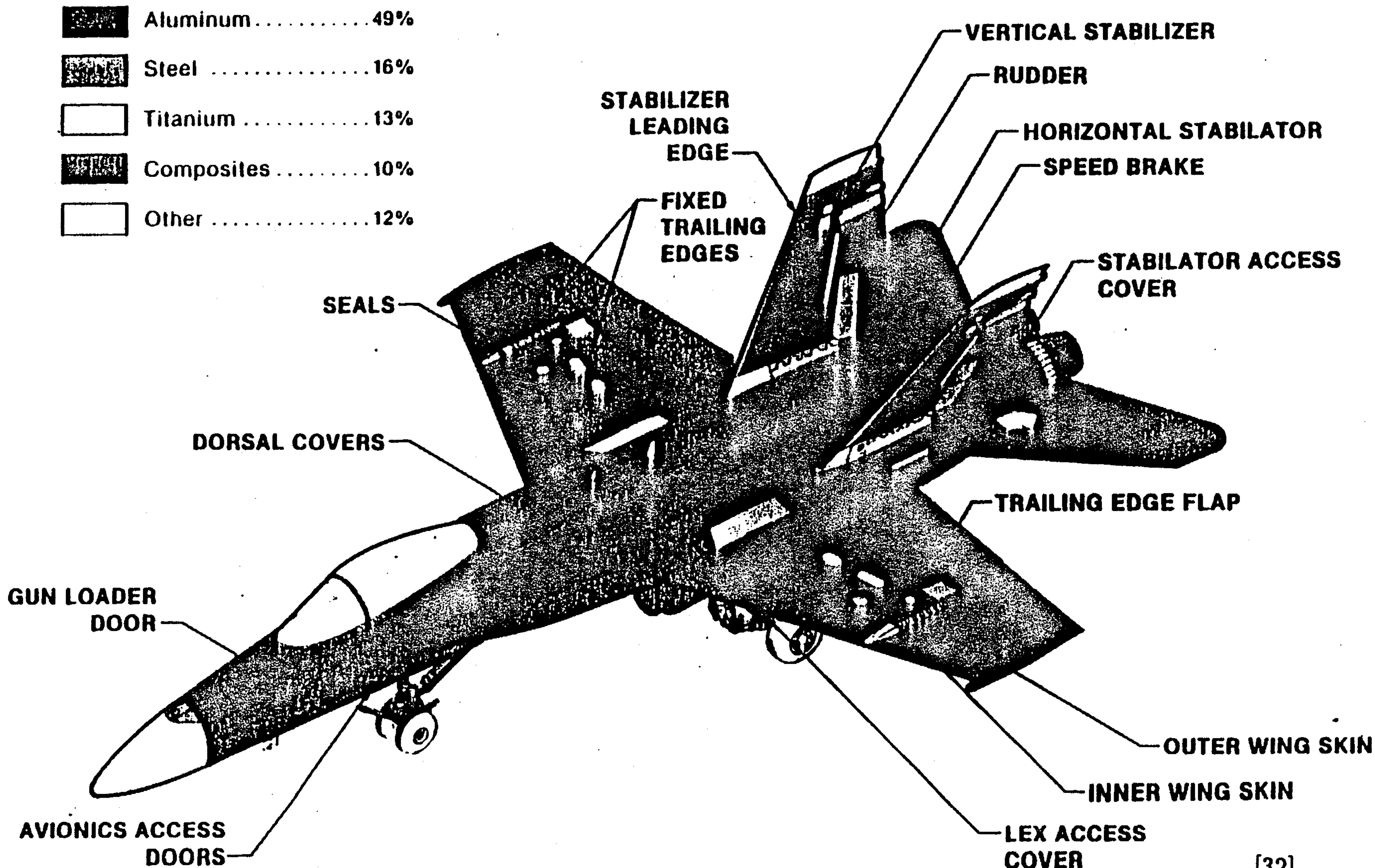
[32]

F-15 COMPOSITE APPLICATIONS



[32]

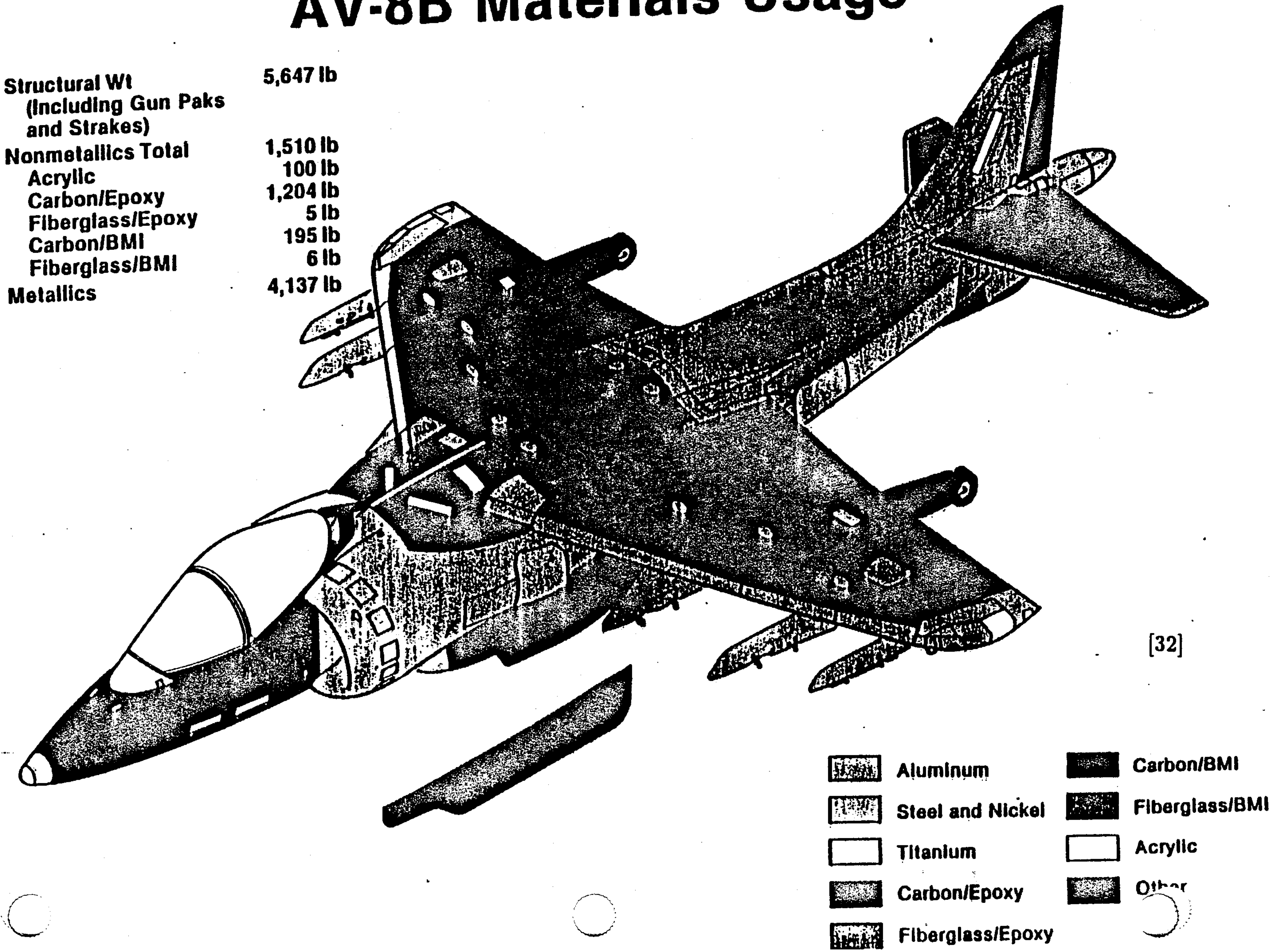
F/A-18
COMPOSITE APPLICATIONS



[32]

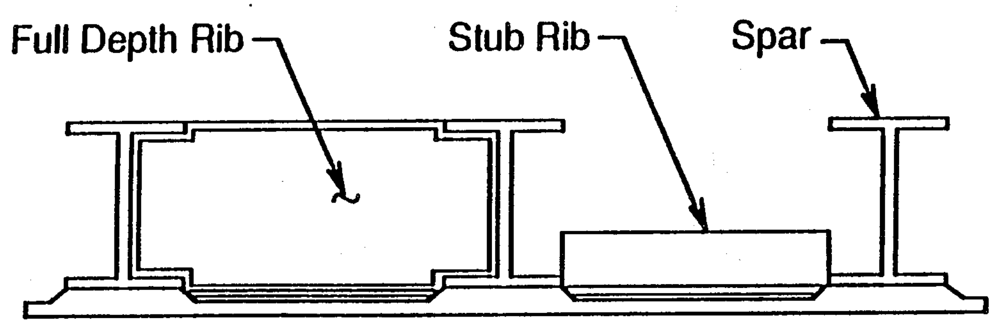
AV-8B Materials Usage

Structural Wt (Including Gun Paks and Strakes)	5,647 lb
Nonmetallics Total	1,510 lb
Acrylic	100 lb
Carbon/Epoxy	1,204 lb
Fiberglass/Epoxy	5 lb
Carbon/BMI	195 lb
Fiberglass/BMI	6 lb
Metallics	4,137 lb

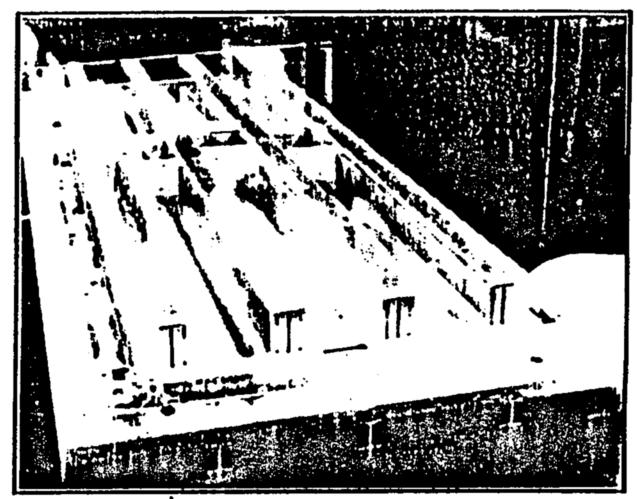


[32]

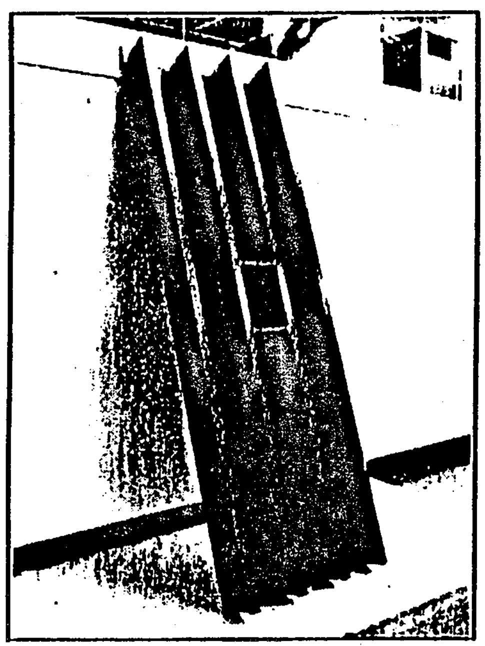
High Strain Wingbox



Design



Tooling



Lower Skin and Substructure

[32]

Integrally Cured Composite Wing Designs

Benefits

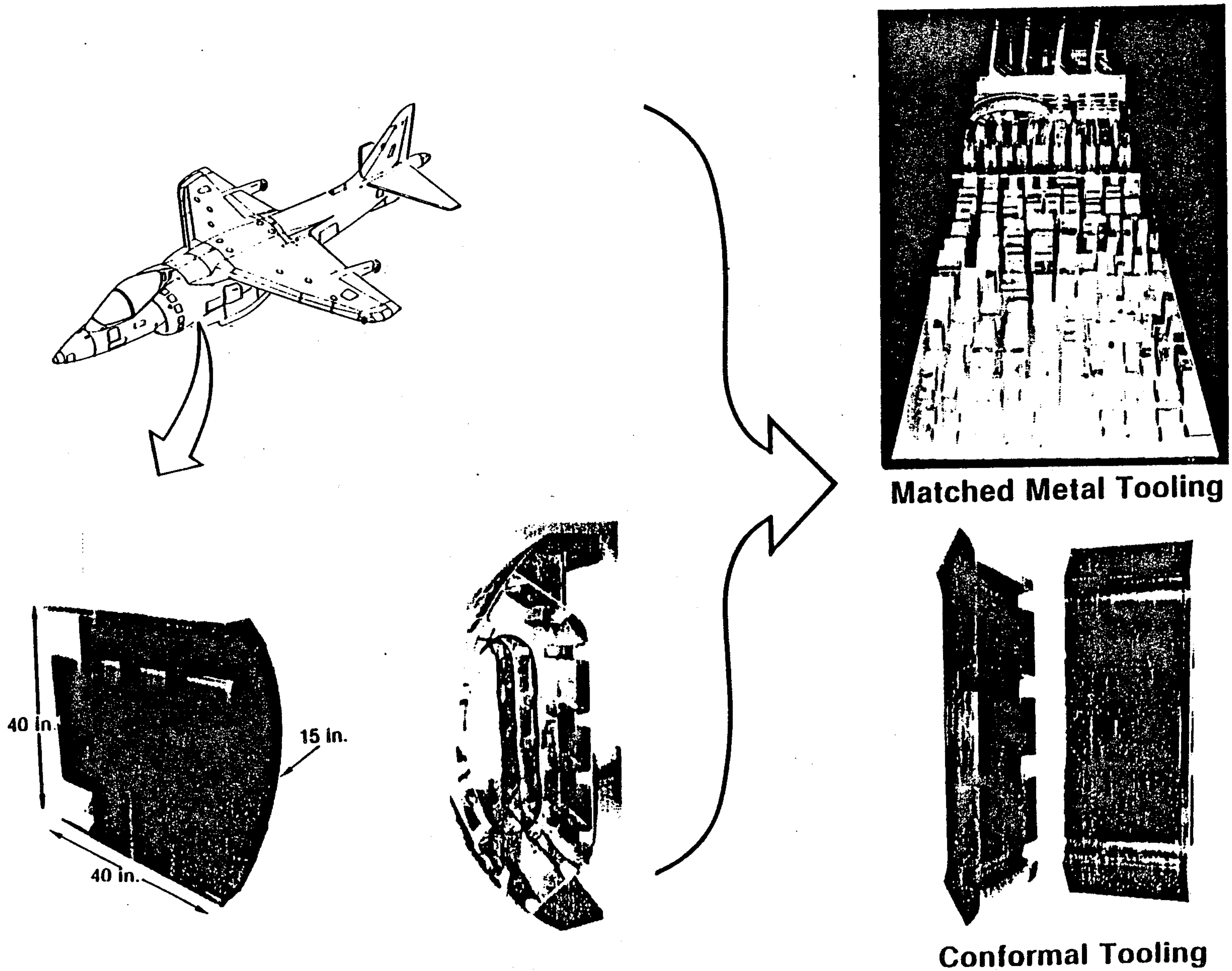
- Reduced Assembly Costs
- Reduced Assembly Fit-Up Problems
- Reduced Fuel Leakage Paths

Concerns

- Increased Tooling Costs and Span Time
- Increased Life Cycle Tooling Costs
- Increased Composite Part Fabrication Time

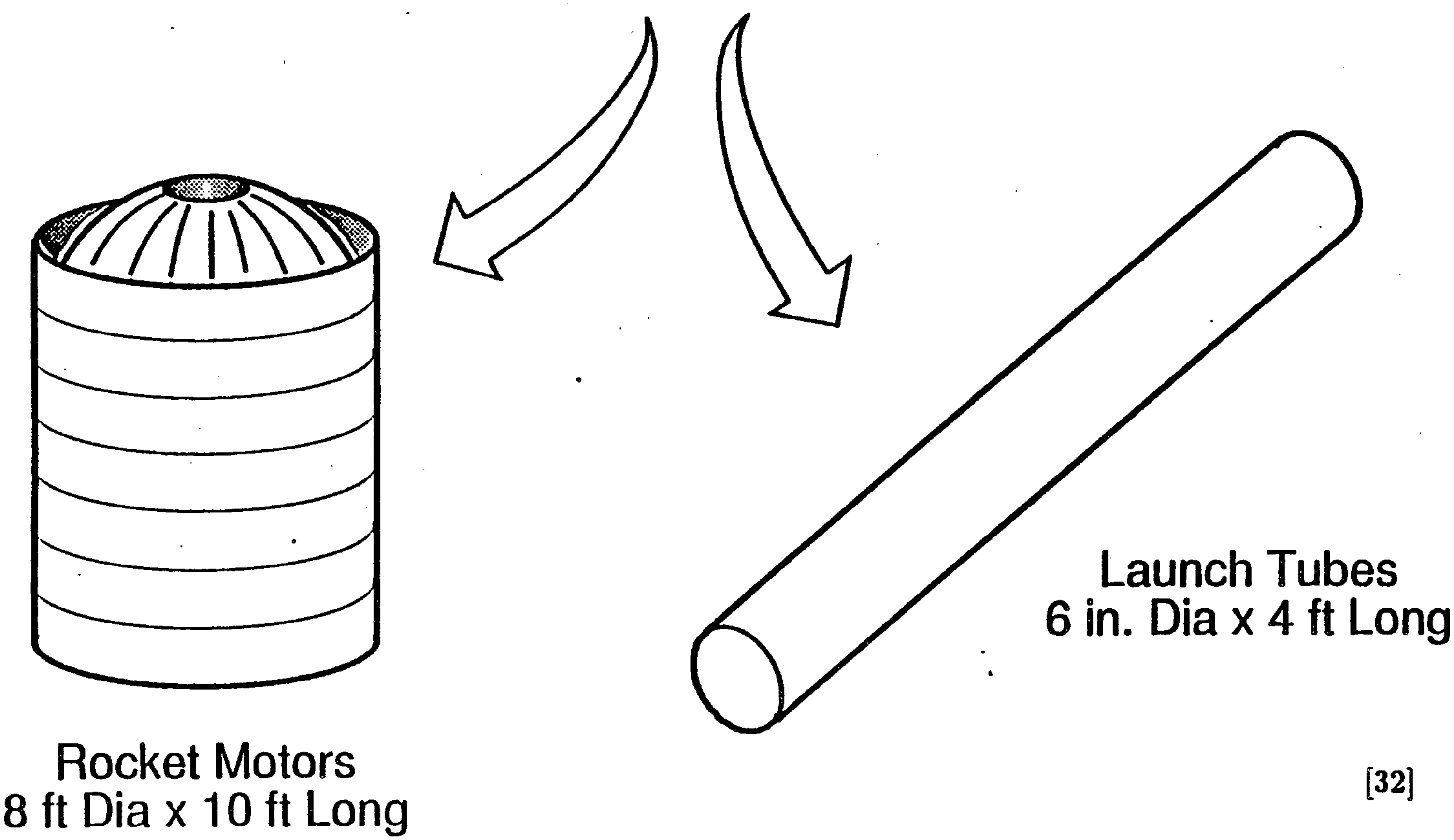
[32]

Cocured Composite Fuel Tanks

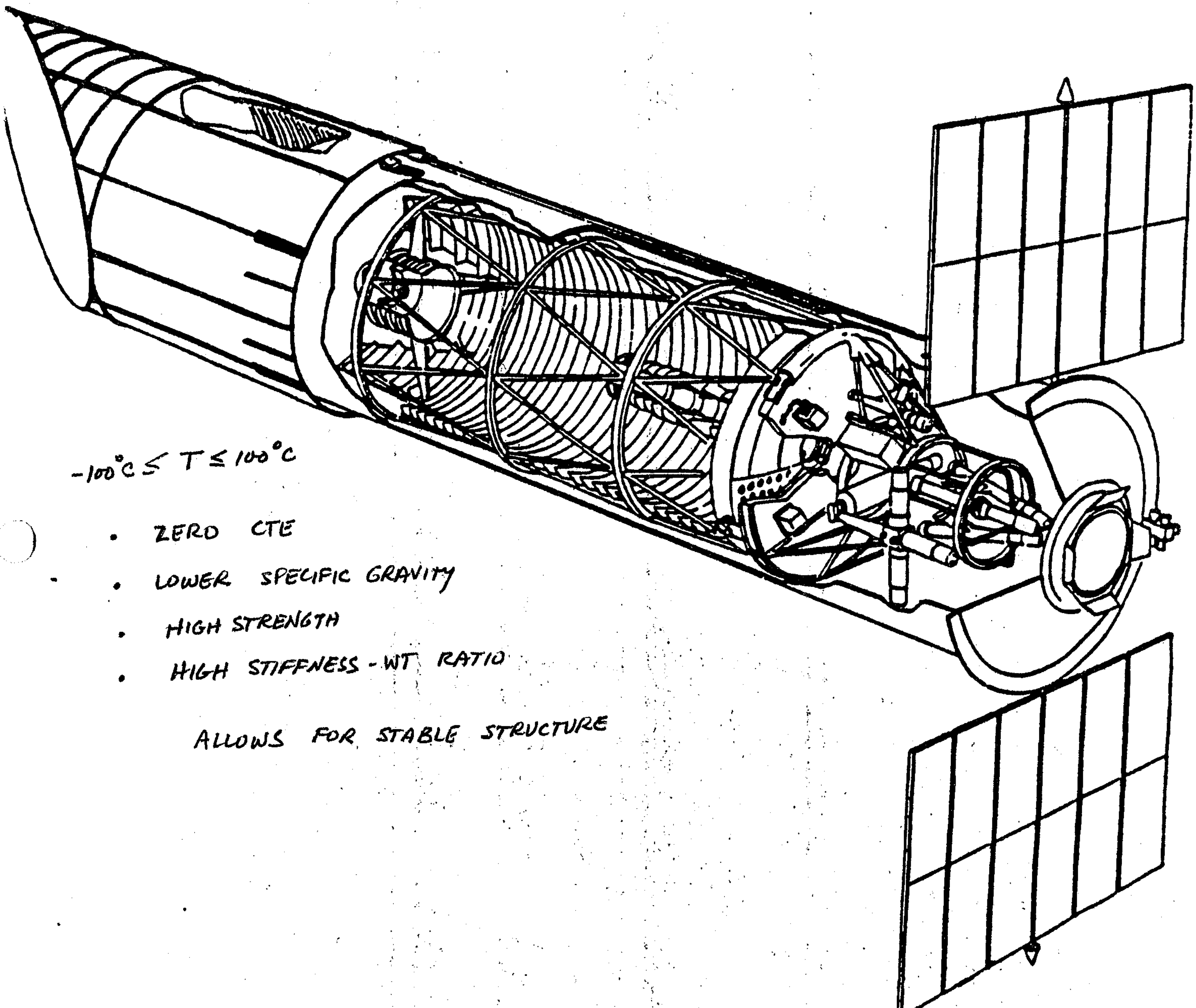


[32]

Missile Components



[32]

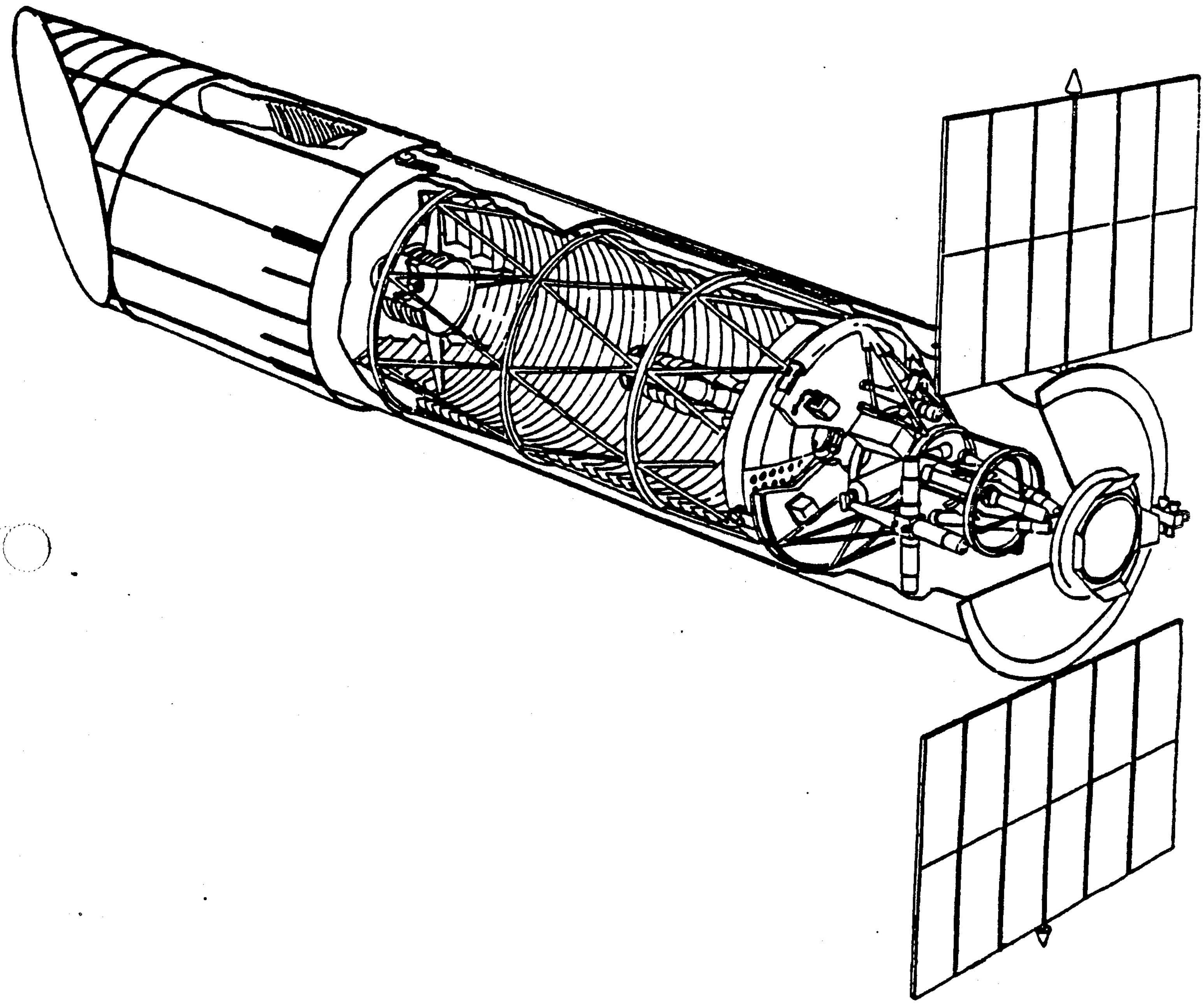


$-100°C \leq T \leq 100°C$

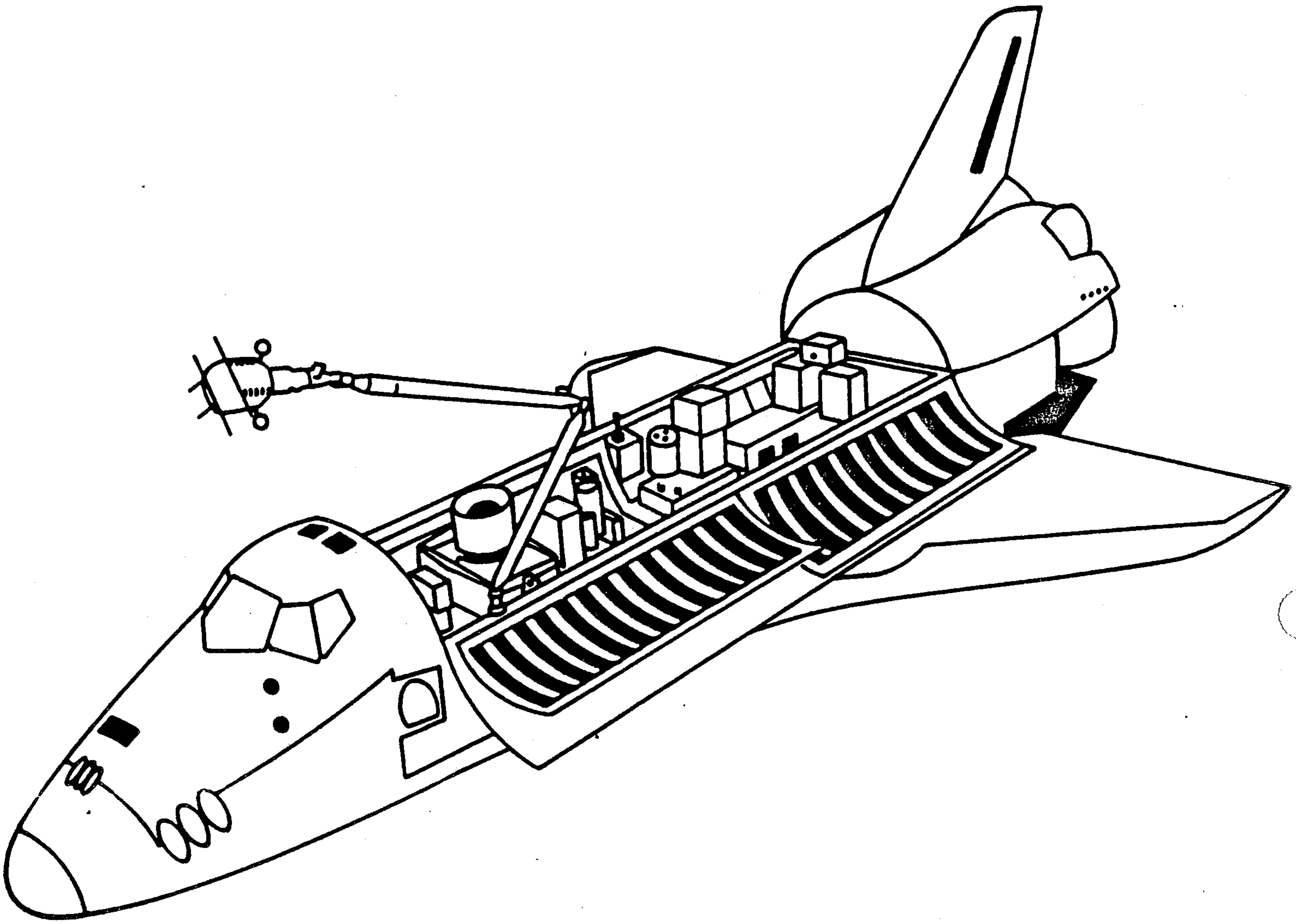
- ZERO CTE
- LOWER SPECIFIC GRAVITY
- HIGH STRENGTH
- HIGH STIFFNESS-WT RATIO

ALLOWS FOR STABLE STRUCTURE

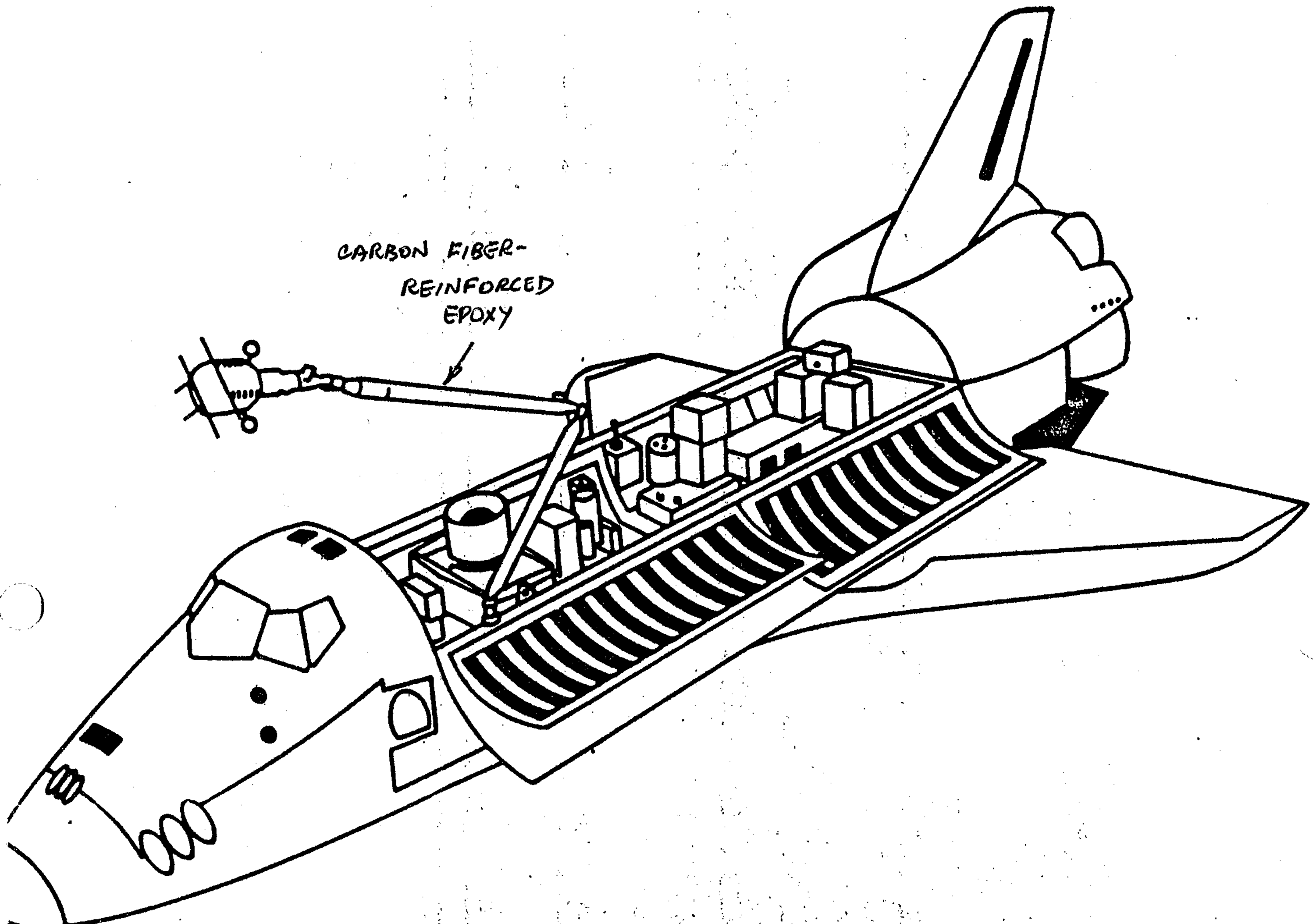
Large Space Telescope (LST). [36]



Large Space Telescope (LST). [36]



Space Shuttle cargo-bay doors. [36]



Space Shuttle cargo-bay doors. [36]

CARBON FIBER - REINF. EPOXY
&
ALUMINUM HONEYCOMB

APPLICATIONS OF FIBER-REINFORCED POLYMERS IN SPORTING GOODS [2]

- Tennis Rackets
- Racket Ball Rackets
- Golf Club Shafts
- Fishing Rods
- Bicycle Frames
- Snow and Water Skis
- Ski Poles, Pole Vault Poles
- Hockey Sticks
- Sailboats and Kayaks
- Oars, Paddles
- Canoe Hulls
- Surfboards
- Archery Bows
- Javelins
- Helmets

CONSTITUENTS

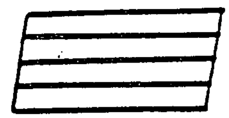
FIBERS + MATRIX + COUPLING AGENTS OR COATINGS + FILLERS

↓

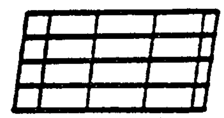
LAMINA

(PLY, LAYER, CHARGE)

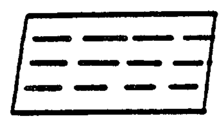
(a) UNIDIRECTIONAL CONTINUOUS



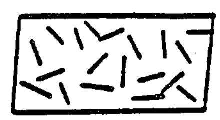
(b) BIDIRECTIONAL CONTINUOUS



(c) UNIDIRECTIONAL DISCONTINUOUS



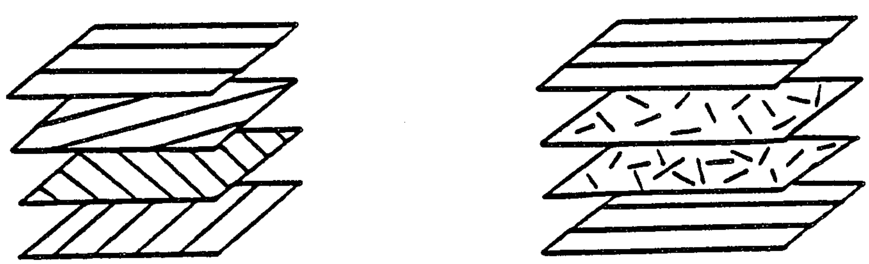
(d) RANDOM DISCONTINUOUS



SIMILAR TO ISOTROPIC

↓

(e) LAMINATE



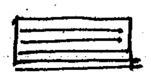
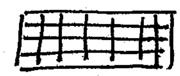
[2]

LAMINA IS MADE UP OF:

- FIBER + MATRIX
- COUPLING AGENTS / COATINGS – IMPROVE BONDING OF FIBER TO MATRIX
  – IMPROVE LOAD TRANSFER
- FILLERS → CONTROL BRITTLENESS

- LAMINA THICKNESS 0.1 – 1 mm (.004 – .04 in)

SEE PG 16

- MANUFACTURING METHODS
  - UNIDIRECTIONAL  0°
  - BIDIRECTIONAL  $0-90^\circ$
  - BALANCED – PROPERTIES IN 0° & 90° SAME

- LAMINATE : SEVERAL LAMINAS STACKED IN GIVEN SEQUENCE TO SUPPORT A REQUIRED LOAD
- NOT NECESSARY FOR SAME FIBER TO BE USED IN DIFFERENT LAMINA OR WITHIN SAME LAMINA

SEE PG 18–19

- FIBERS : DIAMETER OF FIBERS VERY SMALL – HARD TO HANDLE
  FIBERS MANUFACTURED IN BUNDLES OR STRANDS

MUST PAY ATTENTION TO THE MICROMECHANICS AS IT AFFECT THE MACROMECHANICS OF A LOAD CARRYING STRUCTURE

SINGLE FILAMENT VALUES LISTED IN THE TABLE ; STRAND VALUES NORMALLY LOWER & REPRESENT AVERAGE VALUES

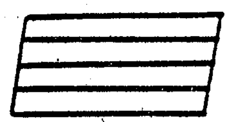
CONSTITUENTS

FIBERS + MATRIX + COUPLING AGENTS OR COATINGS + FILLERS

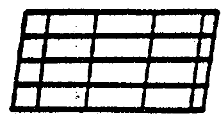
LAMINA

(PLY, LAYER, CHARGE)

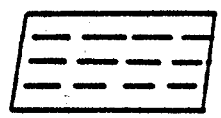
(a) UNIDIRECTIONAL CONTINUOUS



(b) BIDIRECTIONAL CONTINUOUS

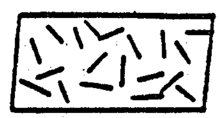


(c) UNIDIRECTIONAL DISCONTINUOUS



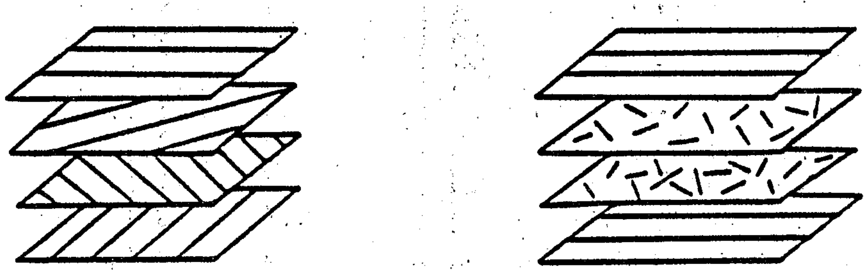
LOWER STRENGTH THAN (a)

(d) RANDOM DISCONTINUOUS



SIMILAR TO ISOTROPIC

(e) LAMINATE



[2]

• FAILURE IS NOT CATASTROPHIC IN THE NORMAL SENSE

APPLICATIONS

AIRCRAFT – weight savings ⇒ higher speeds / increased payloads
PG 7 & 8

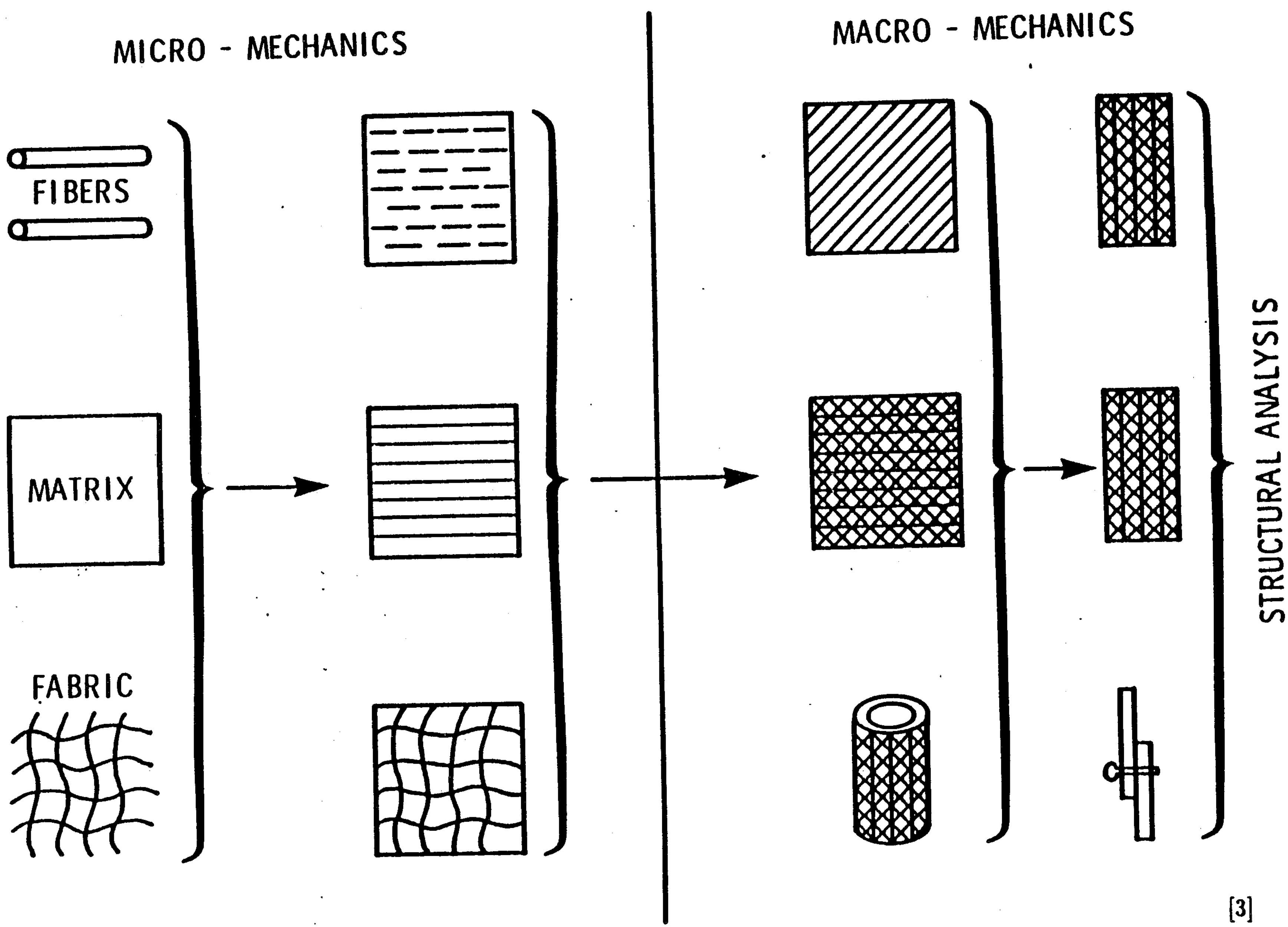
ROTOR BLADES – CONTROL VIBRATIONS BY MASS DISTRIB ⇒ CHANGE NATURAL FREQUENCIES OF BLADES
– SHAPES CAN BE GENERATED

SPACE – WEIGHT REDUCTION ⇒ INCREASED PAYLOADS
– TEMPERATURE STABILITY THROUGH 0 CTE WHILE KEEPING WEIGHT DOWN & HIGHER STIFFNESS/WT

AUTOMOTIVE – DAMAGE TOLERANCE & HIGH STIFFNESS (EXTERIOR BODY)
STIFFNESS (CHASSIS) REAR LEAF SPRING
– WEIGHT SAVINGS ⇒ FUEL ECONOMY

SPORTING GOODS – WEIGHT REDUCTION, VIBRATION DAMPING, DESIGN FLEXIBILITY
weight reduction ⇒ increased speeds, maneuverability (bikes, canoes)
vib damping ⇒ reduce shock to player (tennis)
design flex ⇒ golf shaft weight moved to club head increase impulse to ball ⇒ increased range

marine – weight reduction ⇒ reduced drag ⇒ faster speeds, maneuverabil[ity]
fuel efficiency


MICRO - MECHANICS
MACRO - MECHANICS
FIBERS
MATRIX
FABRIC
STRUCTURAL ANALYSIS

[3]

MATERIALS

- Fibers
  - Glass
  - Carbon
  - Kevlar
  - Boron
  - Alumina
- Matrix
  - Polymeric
    thermosets
    thermoplastics
  - Metallic
  - Ceramic

ADDITIVES

- Coupling Agents
- Fillers
- Tougheners
- Colorants
- Flame Retardants
- Ultraviolet Absorbers

FIBER FORMS

- Continuous Strand Roving
- Woven Roving
- Chopped Strands
- Chopped Strand Mat
- Woven Roving Mat

FIBER

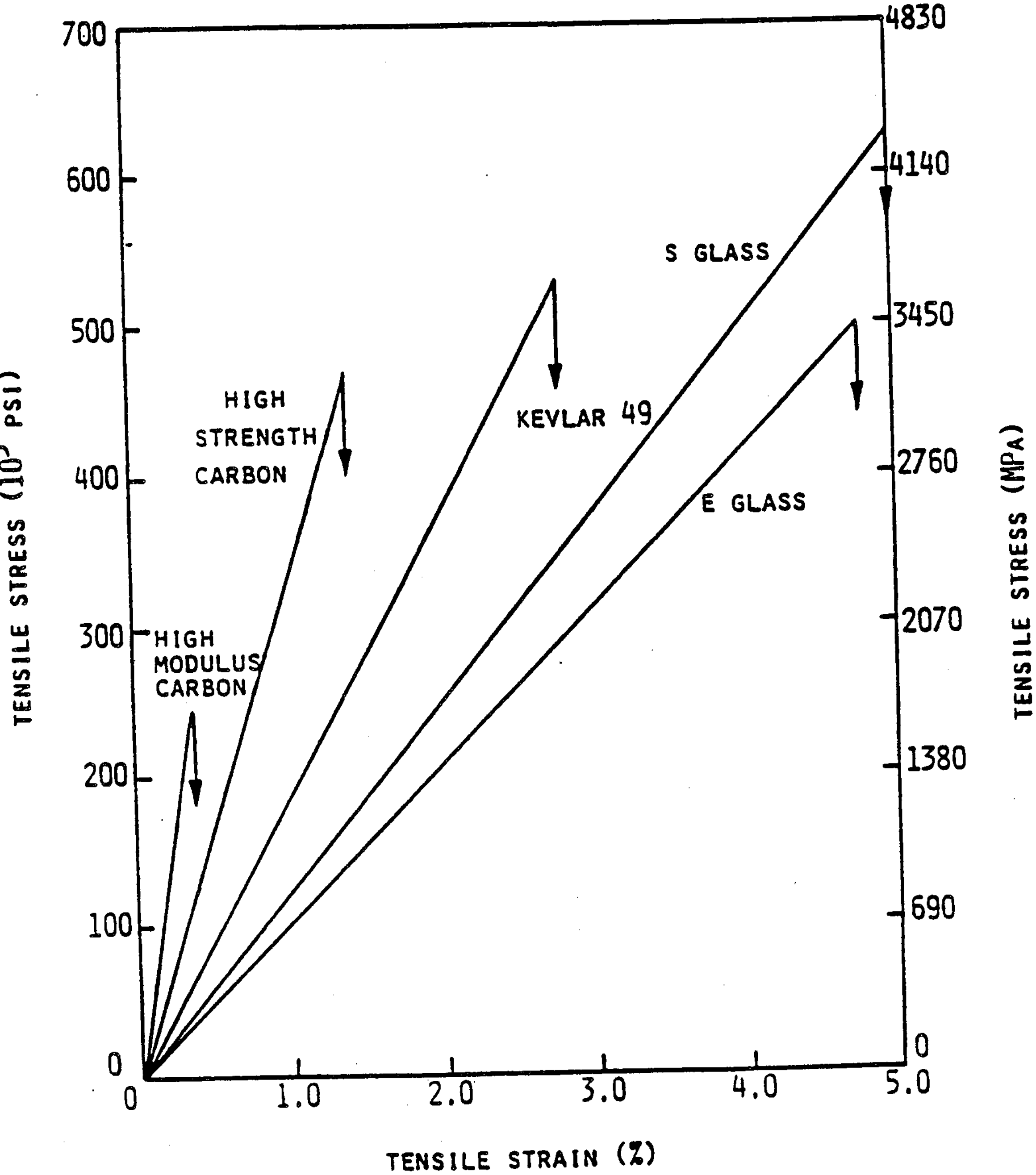
- Major Load Carrying Component
- Occupy the Largest Volume Fraction

Typical Properties of Commercial Reinforcing Fibers [2]

Fiber	Typical diameter (μm)[a]	Specific gravity	Tensile modulus, GPa (10^6 psi)	Tensile strength, GPa (10^3 psi)	Strain to failure (%)	Coefficient of thermal expansion (10^{-6} m/m per °C, 0-100°C)[b]	Poisson's ratio
Glass							
E glass	10 (round)	2.54	72.4 (10.5)	3.45 (500)	4.8	5	0.2
S glass	10 (round)	2.49	86.9 (12.6)	4.30 (625)	5.0	2.9	0.22
PAN-carbon							
T-300[c]	7 (round)	1.76	228 (33.5)	3.2 (470)	1.4	-0.1 to -0.5 (longitudinal) 7-12 (radial)	~0.2
AS[d]	7 (round)	1.77	220 (32)	3.1 (450)	1.2	-0.5 to -1.2 (longitudinal) 7-12 (radial)	
T-40[c]	6 (round)	1.81	276 (40)	5.65 (820)	2		
HMS[d]	7 (round)	1.85	344.5 (50)	2.34 (340)	0.58		
GY-70[e]	8.4 (bilobal)	1.96	483 (70)	1.52 (220)	0.38		
Pitch-carbon							
P-55[c]	10	2.0	380 (55)	1.90 (275)	0.5	-0.9 (longitudinal)	
P-100[c]	10	2.15	690 (100)	2.2 (325)	0.31	-1.6 (longitudinal)	
Kevlar 49[f]	11.9 (round)	1.45	131 (19)	3.62 (525)	2.8	-2 (longitudinal) +59 (radial)	0.35
Boron	140 (round)	2.7	393 (57)	3.1 (450)	0.79	5	0.2
SiC	133 (round)	3.08	400 (58)	3.44 (485)	0.84	1.5	—
Al_2O_3	20 (round)	3.95	379.3 (55)	1.90 (275)	0.4	8.3	

[a] 1 μm = 0.000039 in.
[b] 1 m/m per °C = 0.556 in./in. per °F.
[c] Amoco.
[d] Hercules Inc.
[e] Celanese.
[f] DuPont.

SINGLE FIBER VALUES



FIBER ONLY [2]

NOTE LINEAR BEHAVIOR ; BRITTLE FRACTURE

HIGH STRENGTH DUE TO FILAMENT - LESS # OF FLAWS

FIBERS - CARRY LOAD & HAVE LARGEST VOLUME FRACTION IN LAMINATE

GLASS FIBERS - most commonly used

ADV. - low cost, high tensile strength, excellent insulator, chemical resistance

DISADV. - low tensile modulus, high specific grav., damageable, low fatigue

causes excessive wear on tools

- E glass - lowest cost available
- S glass - higher cost, highest tensile strength

PG. 26-27 • BASIC COMMERCIAL FORM IS STRAND, ROVING (GROUP OF STRANDS), WOVEN

CARBON FIBERS - can be manufacted w/ variety of tensile modulus

ADV. - high tensile strength to weight, high E/wt, low CTE, good fatigue char.

DISADV - low impact resistance, high conductivity, high cost

- used in aircraft industry weight savings more important than cost

IN GENERAL LOW MODULUS FIBERS HAVE LOWER SPECIFIC GRAVITIES, LOWER COST,

HIGHER TENSILE STRENGTH, HIGHER TENSILE STRAINS TO FAILURE

PG 18-19 AS vs GY-70 OR HMS

- BASIC FORM IS TOW (BUNDLE OF SINGLE PARALLEL FILAMENTS)

KEVLAR FIBERS - ONLY ORGANIC FIBER ON MARKET

ADV. - LOWEST SPECIFIC GRAVITY, HIGHEST TENSILE STRENGTH/WT, DAMAGE TOL

DISADV - MACHINABILITY, LOW COMPRESSIVE STRENGTH

- BASIC COMMERCIAL FORM IS YARN, ROVING

USED IN BODY ARMOR & HELMETS

BORON FIBERS

ADV. - HIGH TENSILE MODULUS, EXCELLENT BUCKLING RESISTANCE & COMPRESSIVE STR.

DISADV. - HIGH COST (RESTICTED TO AEROSPACE APPLIC.)

ALUMINA, SILICA FIBERS

ADV. - MAINTAIN STRENGTH AT HIGH TEMP (TURBINE BLADES

STRANDS - PARALLEL FILAMENTS

ROVING - UNTWISTED PARALLEL STRANDS

CHOPPED STRANDS - SHORT LENGTH STRANDS

STRAND MATS - CHOPPED STRANDS MIXED W/ BINDER

WOVEN ROVINGS - ROVINGS WOVEN IN ⊥ DIRECTIONS

YARNS - WEAVED STRANDS

MATRIX CHARACTERISTICS

1. TRANSFER STESSES BET. FIBERS; 2. PROVIDE A BARRIER AGAINST ADVERSE ENVIRONMENT; 3. PROTECT FIBERS FROM MECHANICAL DAMAGE

- ALSO AFFECTS INTERLAMINAR SHEAR (ESPECIALLY IN BENDING)
  INPLANE SHEAR (UNDER TORSIONAL LOADS)
- ALSO VISCOSITY, MELTING POINT & CURING TEMPERATURE OF MATRIX AFFECT PROCESSIBILITY & DEPECTS IN THE COMPOSITE

- TYPES OF MATRIX
  - THERMOSET, THERMOPLATIC POLYMERS
  - METALLIC
  - CERAMIC

  } EXAMPLES P. 38

- POLYMERIC MATRIX

P. 39 POLYMER - LONG-CHAIN MOLECULE MADE UP OF ONE OR MORE REPEATING UNITS OF ATOMS JOINED TOGETHER BY STRONG COVALENT BONDS

THERMO PLATIC POLYMER - INDIVIDUAL MOLECULES, LINEAR STRUCTURE, NO CHEMICAL LINK

ADV. - CAN BE HEAT SOFTENED, MELTED, POSTFORMED, EASILY REPAIRED

- HAS SHORTER FABRICATION TIME, UNLIMITED SHELF LIFE
- HAVE HIGH IMPACT STRENGTH, FRACTURE RESISTANCE

DISADV - HIGH SOLUTION VISCOSITIES MAKE THEIR USE W/ CONTINUOUS FIBERS HARD.

THERMOSETS - THE MOLECULES ARE CHEMICALLY JOINED FORMING 3-D STRUCTURE (ALSO KNOWN AS RESINS)

ADV - THERMALLY STABLE & CHEMICALLY RESISTANT
EXHIBIT LESS CREEP & STRESS RELAXATION

DISADV - LIMITED SHELF LIFE, LONG FABRICATION TIME, LOW STRAINS/FAILU
(MUST BE STORED AT CONSTANT TEMP), COST

P. 47-56 EXAMPLES OF THERMOSET POLYMERS USED P. 47-56

- METALLIC MATRIX

ADV. USED WHERE LONG-TERM RESISTANCE TO SEVERE ENVIRONMENT (HIGH TEMP) IS NEEDED

- YIELD STRENGTH & MODULUS IS HIGHER THAN POLYMERS
- CAN BE PLASTICALLY DEFORMED & STRENGTHENED BY THERMAL& MECH. TREATMENTS

DISADV - HIGH SPECIFIC GRAVITY, HIGH PROCESSING TEMP (MELT PT HIGH)
CORROSION AT INTERFACE

MOST COMMON

ALUMINUM - MUST CHECK FOR GALVANIC CORROSION IN CARBON/AL COMPO

TITANIUM - STRONGER AT HIGHER TEMP; REDUCED THERMAL MISMATCH W/ FIBERS

- CERAMICS SUITABLE FOR HIGH-TEMPERATURE APPLICATIONS

- **ADDITIVES**

- COUPLING AGENTS – IMPROVE FIBER/MATRIX STRENGTH THROUGH PHYSICAL/CHEM. BOND
  - → PROTECTS FIBER SURFACE FROM ENVIRONMENTAL CONDITIONS

- MUST BE COMPATIBLE w/ MATRIX RESIN TO BE EFFECTIVE

  EXAMPLE: SILANES

- WITHOUT THEM – INTERFACIAL CRACKING, REDUCED MECH. PROPERTIES
  - → MOISTURE, FLUIDS CAN ACCUMULATE AT INTERFACE

- COUPLING AGENTS HELP IN SHEAR STRESS TRANSFER AT INTERFACE F/M

- FILLERS

  ADDED TO THERMOSETS TO: REDUCE COST

  INCREASE STIFFNESS ⇒

  REDUCE MOLD SHRINKAGE

  CONTROL MATRIX VISCOSITY

  PRODUCE SMOOTHER SURFACE

- TOUGHENERS – HELP IN REDUCING FIBER/MATRIX SEPARATION
- COLORANTS
- FLAME RETARDANTS
- UV ABSORBERS

- DETERMINATION OF FIBER VOLUME FRACTION, COMPOSITE DENSITY AND VOID VOLUME FRACTION

$$V_c = V_f + V_m = \frac{w_f}{\rho_f} + \frac{w_m}{\rho_m}$$

V_c – vol of composite
V_f – " " fibers
w – weight

$$v_f = \text{volume fraction of fiber} = \frac{V_f}{V_c} = \frac{w_f/\rho_f}{w_f/\rho_f + w_m/\rho_m} = \frac{w_f/\rho_f}{w_f/\rho_f + [w_c - w_f]/\rho_m}$$

if $w/w_c = \tilde{w}$ $\qquad v_f = \dfrac{\tilde{w}_f/\rho_f}{\tilde{w}_f/\rho_f + (1-\tilde{w}_f)/\rho_m}$

$\tilde{w} = \dfrac{w}{w_c}$ = weight fraction

$$V_c = \frac{w_c}{\rho_c} = V_f + V_m = \frac{w_f}{\rho_f} + \frac{w_m}{\rho_m} \Rightarrow \frac{1}{\rho_c} = \frac{\tilde{w}_f}{\rho_f} + \frac{\tilde{w}_m}{\rho_m} = \frac{\tilde{w}_f}{\rho_f} + \frac{(1-\tilde{w}_f)}{\rho_m}$$

$$\therefore \boxed{\rho_c = \left[\frac{\tilde{w}_f}{\rho_f} + \frac{(1-\tilde{w}_f)}{\rho_m} \right]^{-1}}$$

$\tilde{w}_f$ - EXPERIMENTALLY DETERMINED

- VOID VOLUME FRACTION

$$v_v = \frac{\rho_c - \rho}{\rho_c}$$

ρ - actual density of composite; determined experimentally

$v_v > .02$ indicates a lower fatigue resistance

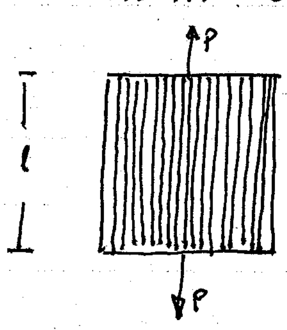
MECHANICS OF COMPOSITES

- NORMAL MATERIALS ARE ISOTROPIC (PROPERTIES INDEPENDENT OF DIRECTION)
  HOMOGENEOUS (" " " LOCATION)
- FIBER REINFORCED COMPOSITES ARE NEITHER

- CAN APPROACH COMPOSITE FROM
  - MICROMECHANICS POINT OF VIEW: UNDERSTAND THE INTERACTION OF CONSTITUENT PARTS TO DETERMINE FAILURE, ELASTIC & THERMAL
  - MACROMECHANICS: DEFINE AN EQUIVALENT SYSTEM HAVING AGGREGATE PROPERTIES & APPLY EQNS OF ORTHOTROPIC ELASTICITY

- USE MACROPROPERTIES TO DEFINE FIBER-MATRIX INTERACTION IN A COMPOSITE MATERIAL

- LOOK AT CONTINUOUS PARALLEL FIBER LAMINA UNDER TENSILE LOAD



ASSUME

1. CONTINUOUS DISTRIB OF FIBERS
2. PERFECT BONDING
3. NO VOIDS
4. P APPLIED // TO FIBER LONG. AXIS

5. LAMINA INITIAL STRESS FREE

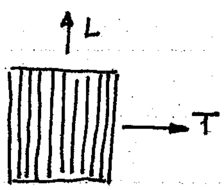
6. FIBERS/MATRIX BEHAVE LINEARLY ELASTIC

- PERFECT BOND $\Rightarrow$ $\epsilon_f = \epsilon_m = \epsilon_c$
- ALSO LINEAR ELASTICITY $\Rightarrow$ $\sigma_f = E_f \epsilon_f = E_f \epsilon_c$; $\sigma_m = E_m \epsilon_m = E_m \epsilon_c$

  note $E_f \gg E_m$ $\Rightarrow$ $\sigma_f \gg \sigma_m$
- $P_c = P_f + P_m$ AND $A_c = A_f + A_m$ $V_c = A_c \ell$, etc.

  since $v_f = V_f/V_c = A_f/A_c$ & $v_m = 1 - v_f$ (ASSUME NO ADDITIVES)

$$P_c = P_f + P_m \Rightarrow E_c = \boxed{E_f V_f + E_m (1 - V_f) = E_L} \quad \text{(Rule of MIXTURES)}$$



and $$P_f/P_c = \frac{v_f (E_f/E_m)}{v_f (E_f/E_m) + (1 - v_f)}$$

p 77 • AS VOLUME FRACTION $\uparrow$ FOR FIXED E_f/E_m P_f/P_c $\uparrow$

- THEORETICALLY v_f CAN BE 90% PRACTICAL LIMIT – 80% FOR MATRIX TO WET FIBERS

- NOTE THAT $E_L = (E_f - E_m) V_f + E_m$ LINEAR IN v_f

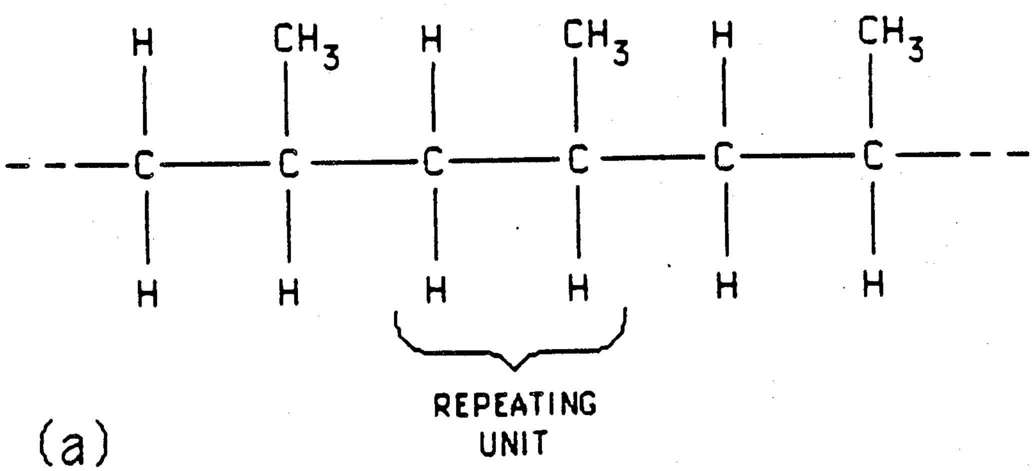
- from $P_c = \sigma_c A_c = P_f + P_m = \sigma_f A_f + \sigma_m A_m \Rightarrow \sigma_c = \sigma_f v_f + \sigma_m (1 - v_f)$
  - SINCE FIBER FAILURE STRAIN < MATRIX FAILURE STRAIN

    TENSILE RUPTURE IN FIBER CAUSES RUPTURE IN COMPOSITE
  - MUST BE CRITICAL v_f FOR THIS NOT TO HAPPEN

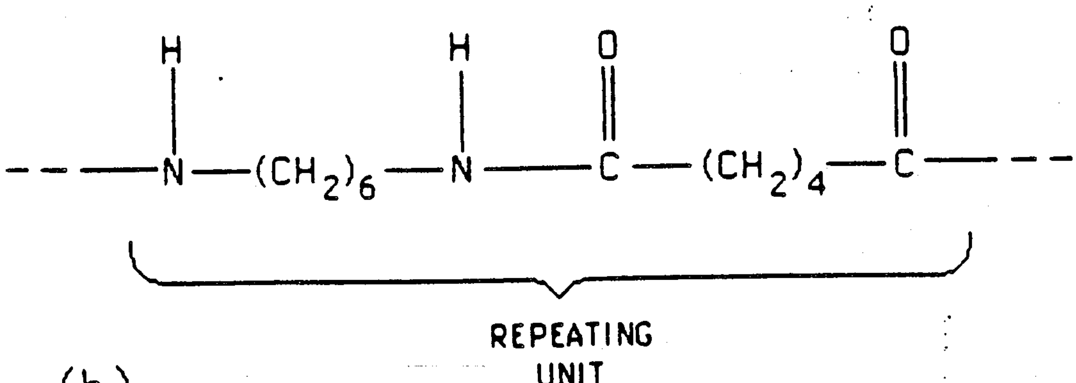
$$v_{f_{crit}} = \frac{\sigma_c - \sigma_m}{\sigma_f - \sigma_m} \qquad \text{WHERE} \quad \left.\begin{array}{l} \sigma_c = \sigma_{mu} \\ \sigma_f = \sigma_{fu} \end{array}\right\} \epsilon_m = \epsilon_{fu}$$

MATRIX

- Transfer Stresses between the Fibers
- Provide a Barrier against
  Adverse Environments
- Protect the Surface from Abrasion
- Provide Lateral Support



(a)



(b)

Polymer molecules (a) polypropylene molecule (b) nylon 6.6 molecule [2]

THERMOSET

- Thermal Stability
- Chemical Resistance
- Less Creep and Stress Relaxation

THERMOPLASTIC

- Unlimited Shelf Life
- Shorter Fabrication Time
- Postformability
- Ease of Repair
  welding or Solvent Bonding

[2]

EPOXY MATRIX - THERMOSET

- Advantages over other thermosets
  - Wide Variety of Properties
  - Absence of Volatile Matters During Cure
  - Low Shrinkage During Cure
  - Excellent Resistance to Chemicals and Solvents
  - Excellent Adhesion to Fillers, Fibers, etc.

- Disadvantages
  - High Cost
  - Long Cure Time

[2]

Matrix Materials [2]

Polymeric

Thermoset polymers (resins)

Epoxies: principally used in aerospace and aircraft applications

Polyesters, vinyl esters: commonly used in automotive, marine, chemical, and electrical applications

Phenolics: used in bulk molding compounds

Polyimides, polybenzimidazoles (PBI), polyphenylquinoxaline (PPQ): for high-temperature aerospace applications (temperature range: 250-400°C)

Thermoplastic polymers

Nylons (such as nylon 6, nylon 6,6), thermoplastic polyesters (such as PET, PBT), polycarbonate (PC), polyacetals: used with discontinuous fibers in injection molded articles

Polyamide-imide (PAI), polyether-ether ketone (PEEK), polysulfone (PSUL), Polyphenylene sulfide (PPS), polyether imide (PEI): suitable for moderately high temperature applications with continuous fibers

Metallic

Aluminum and its alloys, titanium alloys, magnesium alloys, nickel-based superalloys, stainless steel: suitable for high-temperature applications (temperature range: 300-500°C)

Ceramic

Aluminum oxide (Al_2O_3), carbon, silicon carbide (SiC), silicon nitride (Si_3N_4): suitable for high-temperature applications

FIBER/MATRIX CONTENT

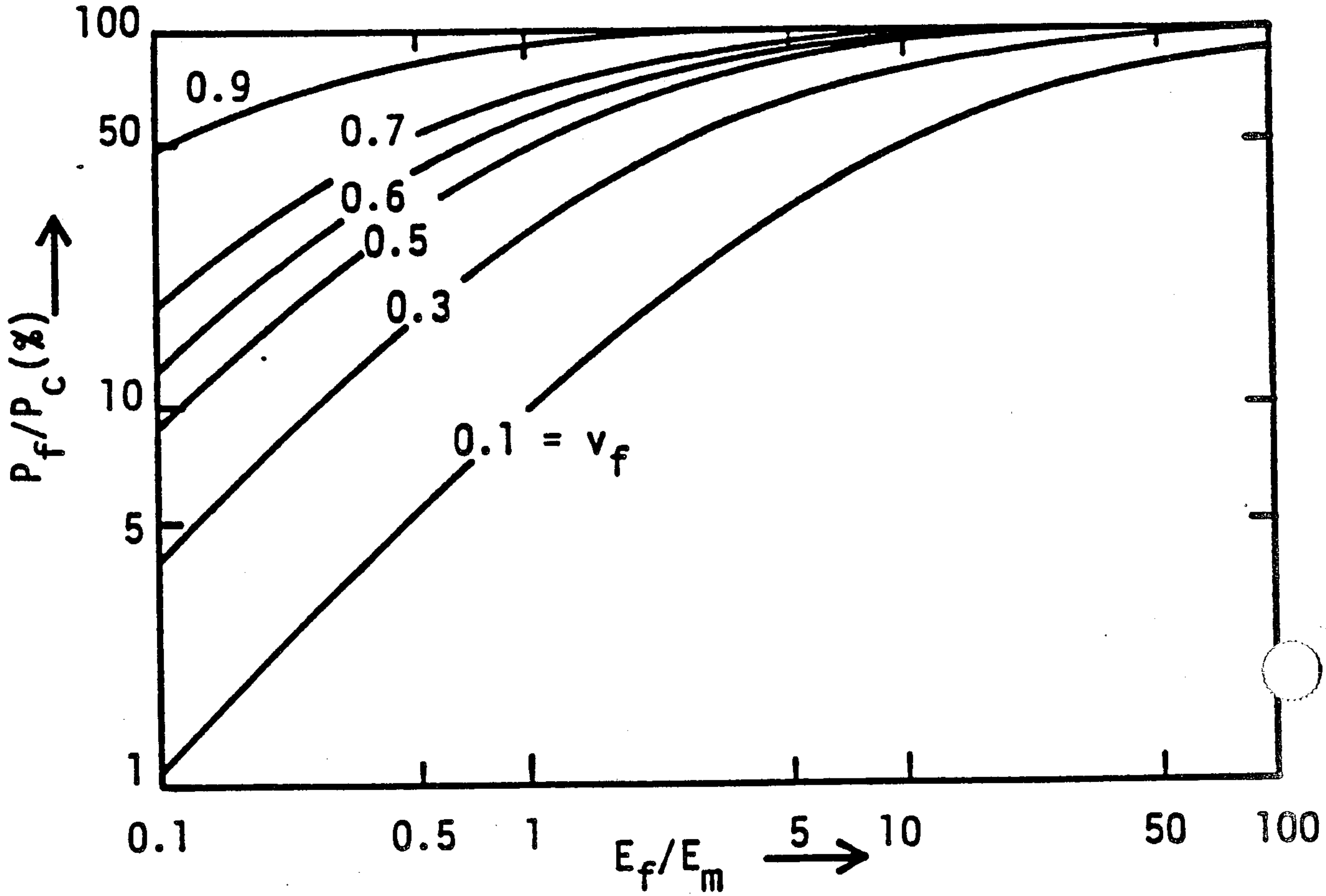
$$v_f = \frac{w_f/\rho_f}{(w_f/\rho_f) + (1 - w_f)/\rho_m}$$

$$v_m = (1 - v_f)$$

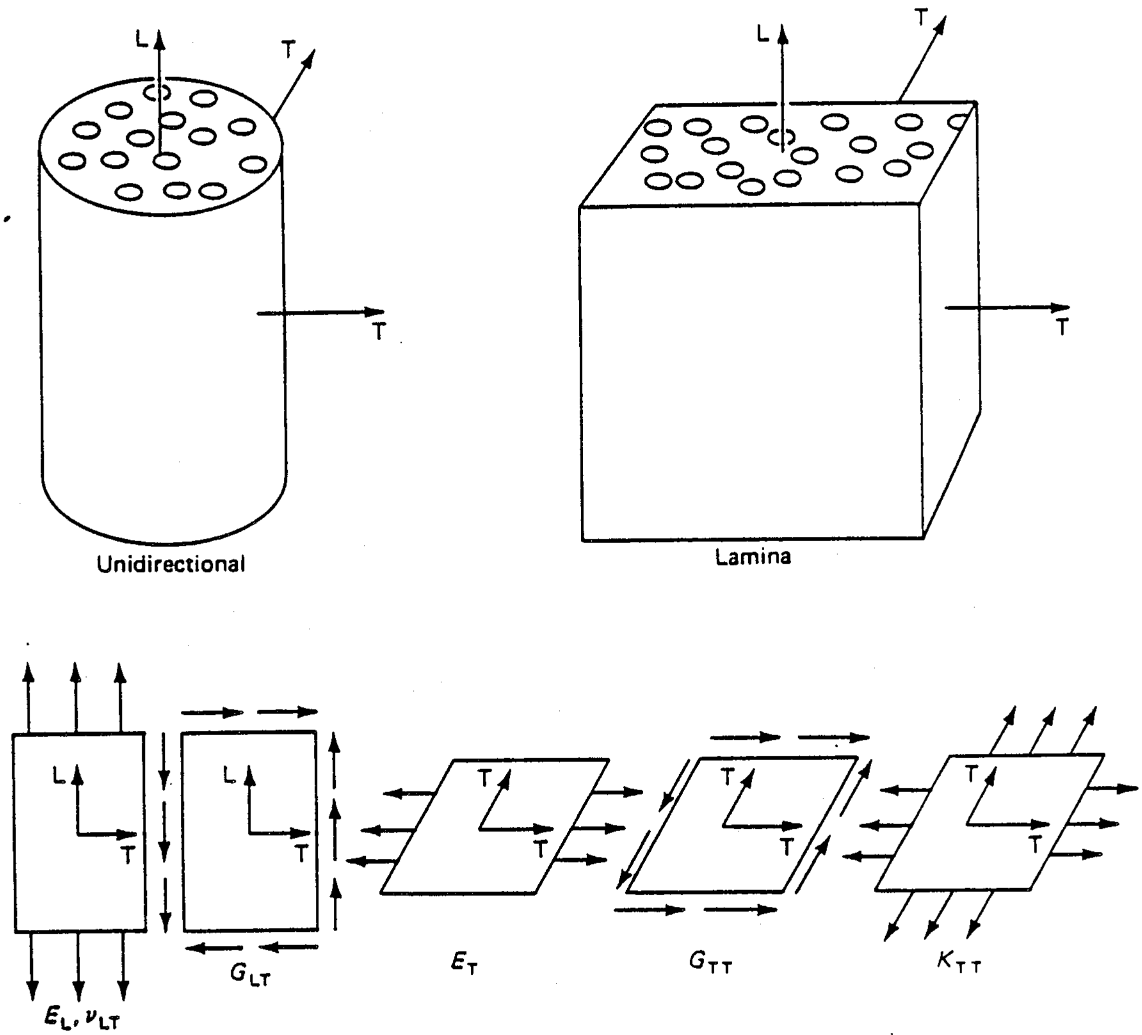
DENSITY

$$\rho_c = \frac{1}{(w_f/\rho_f) + (1 - w_f)/\rho_m}$$

$\rho_c =$ composite density
$v_f =$ fiber volume fraction
$w_f =$ fiber weight fraction
$\rho_f =$ fiber density
$\rho_m =$ matrix density
$v_m =$ matrix volume fraction [2]



Fraction of load shared by fibers in longitudinal tensile loading of a continuc parallel fiber lamina. [2]



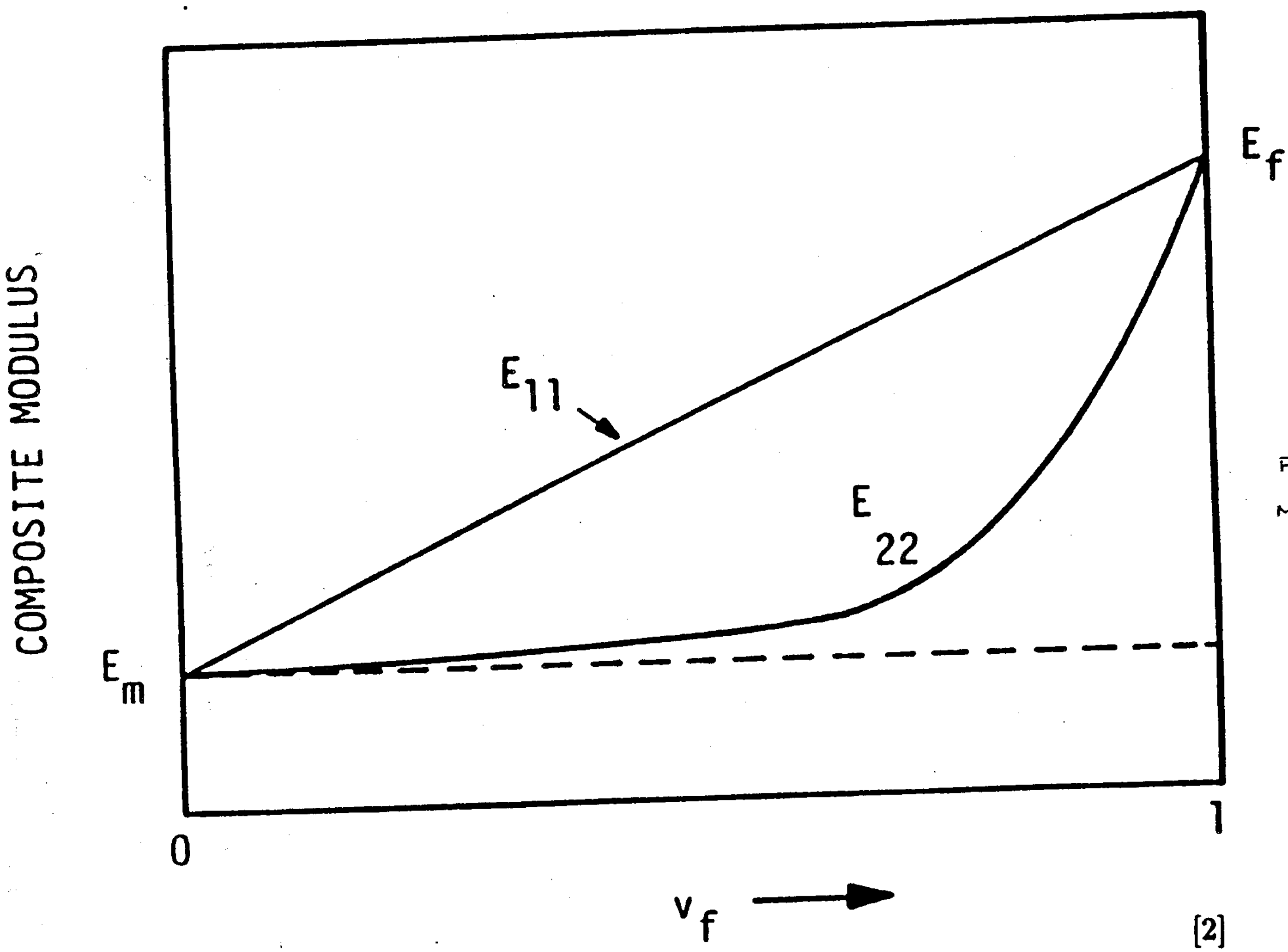
Loading conditions for the evaluation of basic elastic properties [5]

L - LONGITUDINAL ALONG FIBER

T - TRANSVERSE $\begin{cases} \text{THROUGH THE THICKNESS} \\ \text{INPLANE} \end{cases}$

Properties of typical fiber-epoxy composites and structural metals [5]

Material	Density, g/cm^3	Young's modulus of elasticity, E_L, GPa	E_L, 10^6 psi	E_T, GPa	E_T, 10^6 psi	Shear modulus, G_{LT}, GPa	G_{LT}, 10^6 psi	Poisson's ratio, ν_{LT}	Ultimate tensile strength, σ_L^{tu}, MPa	σ_L^{tu}, ksi	σ_T^{tu}, MPa	σ_T^{tu}, ksi	Ultimate compressive strength, σ_L^{cu}, MPa	σ_L^{cu}, ksi	σ_T^{cu}, MPa	σ_T^{cu}, ksi	Ultimate shear strength, τ_{LT}^{su}, MPa	τ_{LT}^{su}, ksi
Unidirectional composites ($V_f = 0.6$)																		
E-glass	1.94	45	6.5	12	1.7	4.4	0.64	0.25	1000	150	34	5	550	80	140	20	40	6
Kevlar 49 ..	1.30	76	11.0	5.5	0.8	2.1	0.3	0.34	1380	200	28	4	280	40	140	20	55	8
T-300	1.47	132	19.2	10.3	1.5	6.5	0.95	0.25	1240	180	45	6.5	830	120	140	20	62	9
VSB-32....	1.63	229	33.2	6.9	1.0	5.5	0.8	0.25	1170	170	41	6	690	100	140	20	680	11
Boron	1.86	274	39.8	15	2.2	52	7.5	0.25	1310	190	34	5	2480	360	310	45	100	15
GY-70.....	1.61	320	46.4	5.5	0.8	4.1	0.6	0.25	690	100	41	6	620	90	140	20	96	14
Metals																		
2024-T3 ...	2.77	72.3	10.5	72.3	10.5	27.6	4.0	0.31	462	67	455	66	345	50	345	50	276	40
7075-T6 ...	2.80	71.0	10.3	71.0	10.3	27.6	4.0	0.31	544	79	530	77	475	69	475	69	324	47
4130	7.84	207	30.0	207	30.0	82.7	12.0	0.25	655	95	655	95	1100	160	1100	160	380	55



[2]

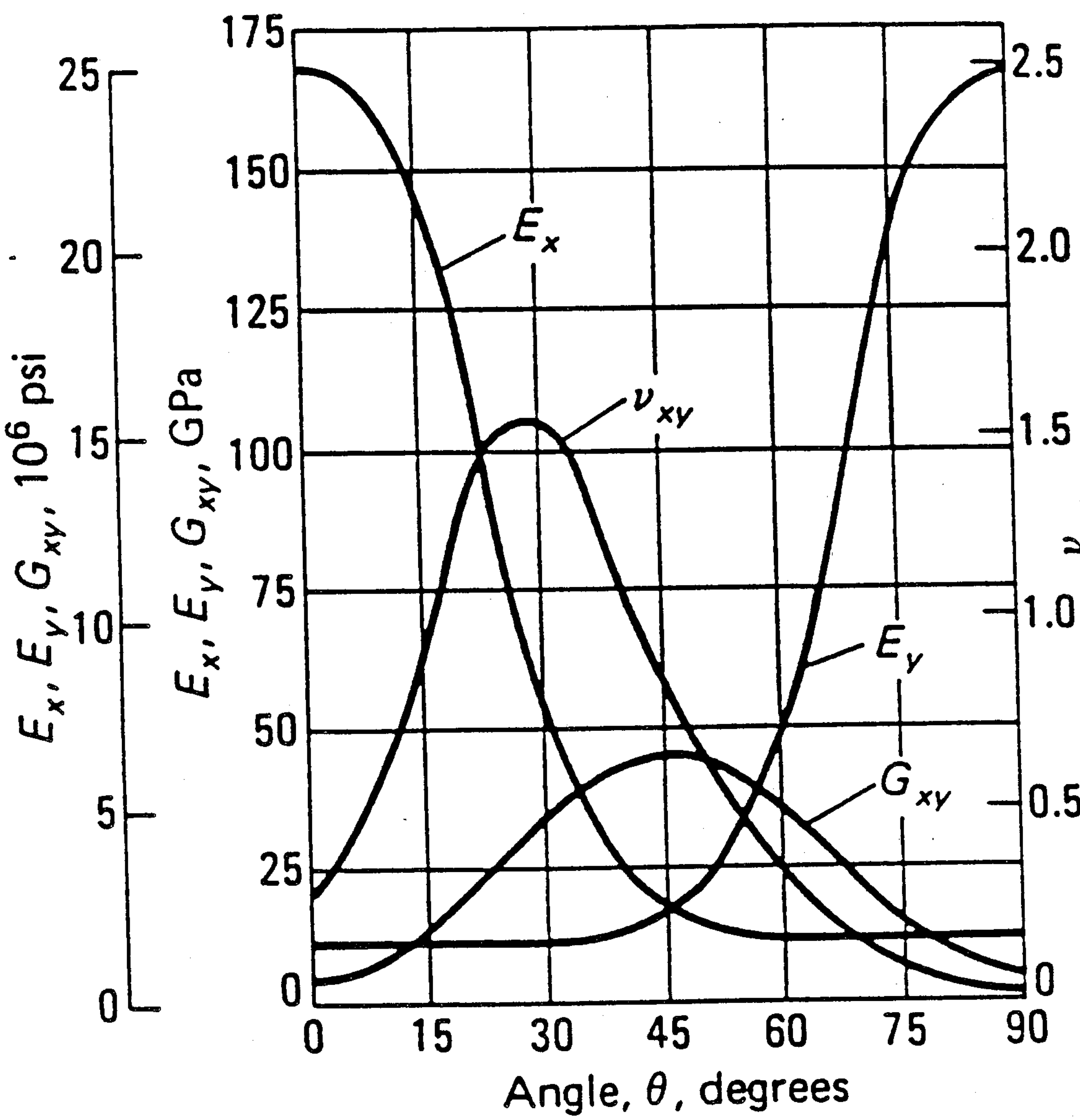
$$E_{11} = E_f v_f + E_m v_m$$

$$E_{22} = \frac{E_f E_m}{E_f v_m + E_m v_f}$$

$$v_m = 1 - v_f$$

FIBER CONTRIBUTE TO E_{11}

MATRIX CONTRIBUTE TO E_{22}

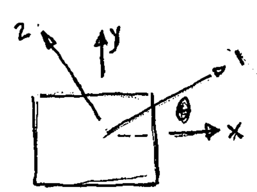


Laminate elastic constants for high-modulus graphite-epoxy system, $[+\theta/-\theta]_s$ [5]

4 LAYERS

TRANSVERSE ORTHOTROPIC : 5 INDEPENDENT CONSTANTS

E_{11}, E_{22}, G_{12}, ν_{12}, ν_{23}



$+\theta$
$-\theta$
$-\theta$
$+\theta$

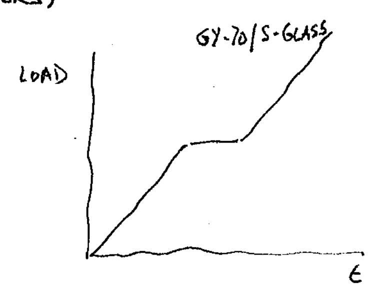
1. ν CAN BE > 1
2. MUST WORRY ABOUT POISSON RATIO MISMATCH BETWEEN LAYERS (DELAMINATION)
3. G MORE DEPEDENT ON MATRIX
4. FOR IN-PLANE BEHAVIOR OF LAMINA ONLY NEED E_{11}, E_{22}, G_{12}, ν_{12}

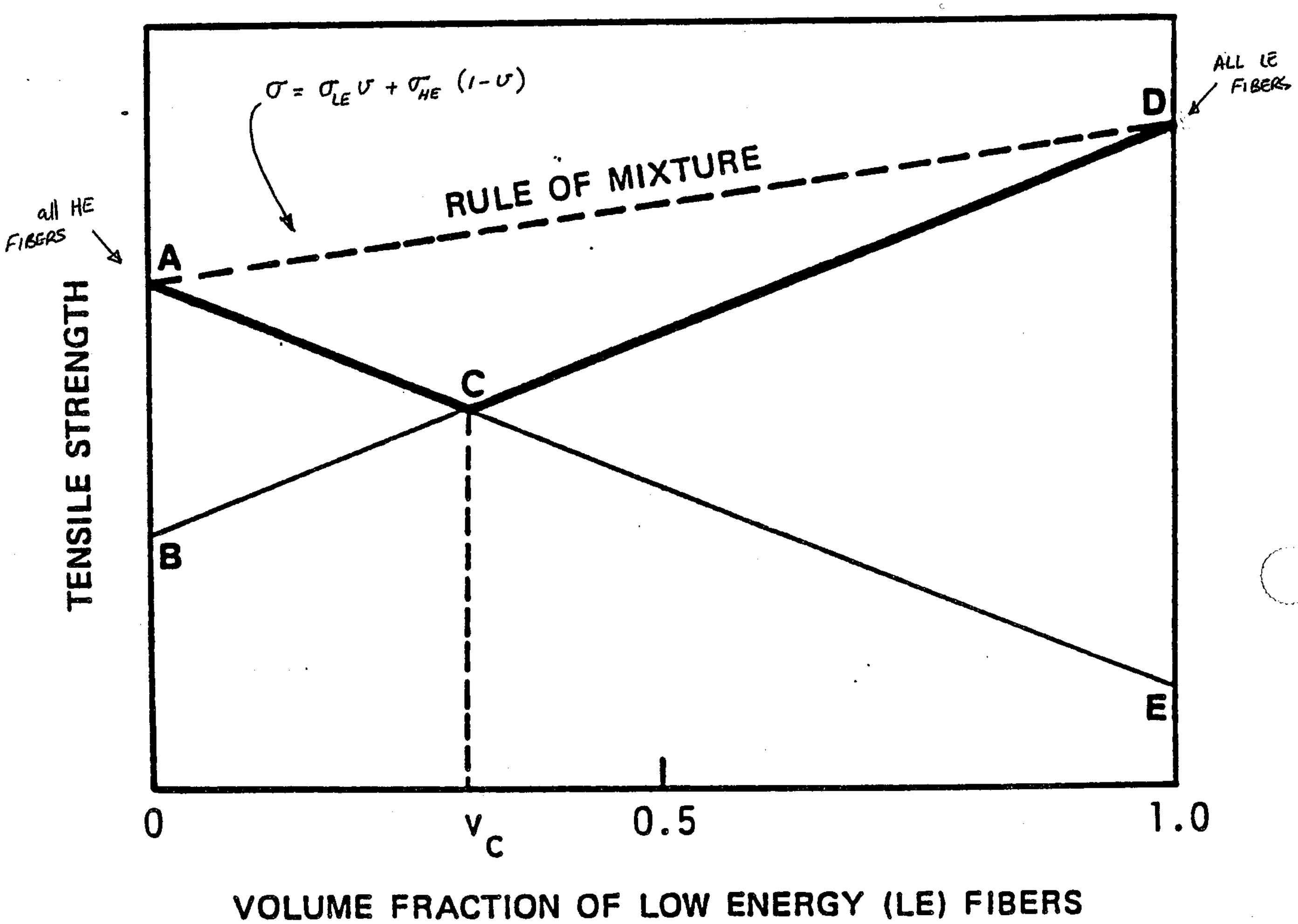
INTERPLY HYBRID LAMINATES

- Made of separate layers of LE fibers and HE fibers both in a common matrix
- Higher ultimate strain than LE fiber composites
- Strain at which LEfibers begin to fail is greater than or equal to the ultimate tensile strain of the LE fibers
- LE fibers fail in a controlled manner, producing a step or smooth inflection in the tensile stress/strain diagram
- Ultimate strength is lower than the tensile strength of the LE or HE fiber composites
- Ultimate strain is lower than that of the HE composite
- Tensile modulus falls between that of the LE and HE composites
- Advantage is enhanced strain to failure
- Strength and modulus is sacrificed

[2]

LE - LOW ELONGATION (HIGH MODULUS CARBON FIBERS)
HIGH - ELONGATION (E-GLASS, KEVLAR)



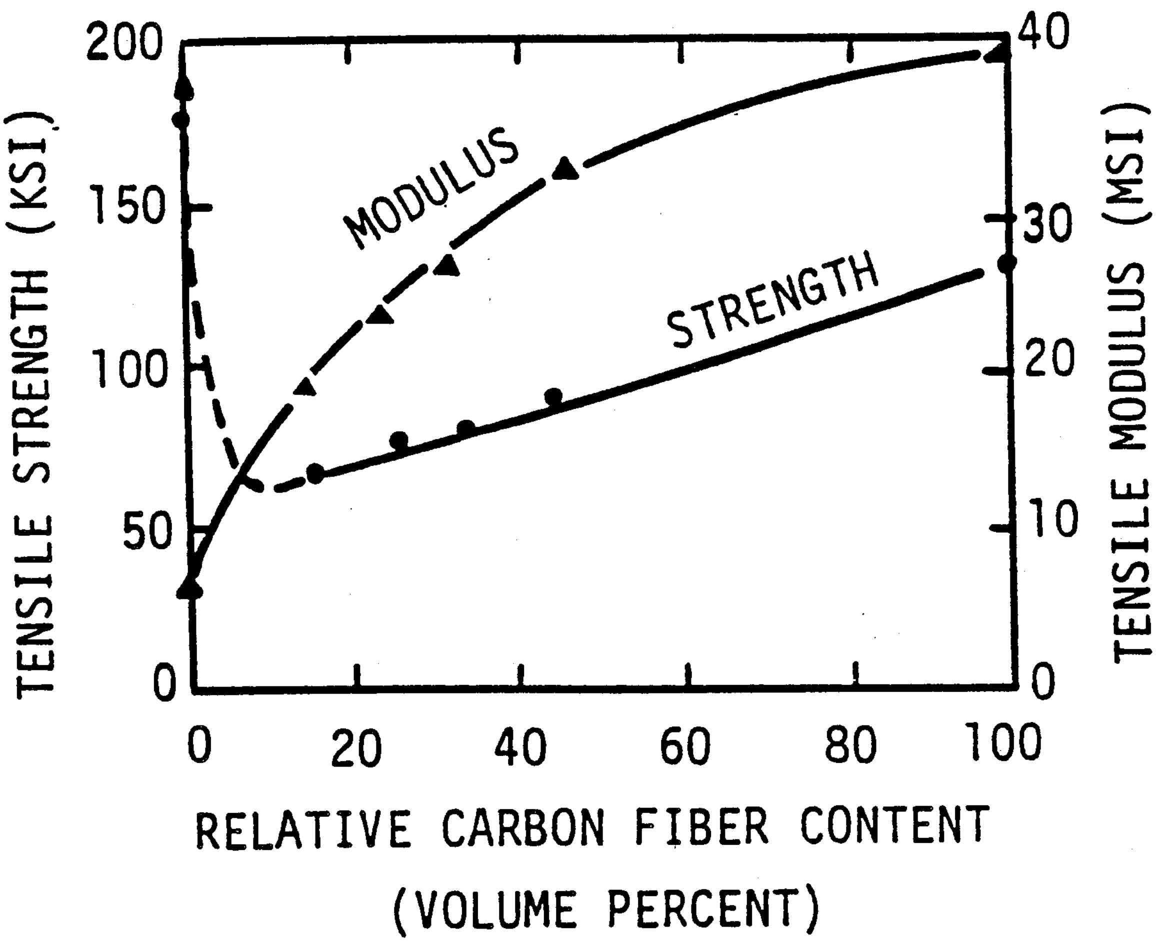


Model for tensile strength variation in interply hybrid laminates [2]

BD - STRESS IN HYBRID AT WHICH FAILURE IN LE FIBERS OCCUR (HE CARRIES NO LOAD)
ASSUME
AE - " " " " " " " " " " (LE CARRIES NO LOAD)

$\upsilon < \upsilon_c$ HE FIBERS CONTROL

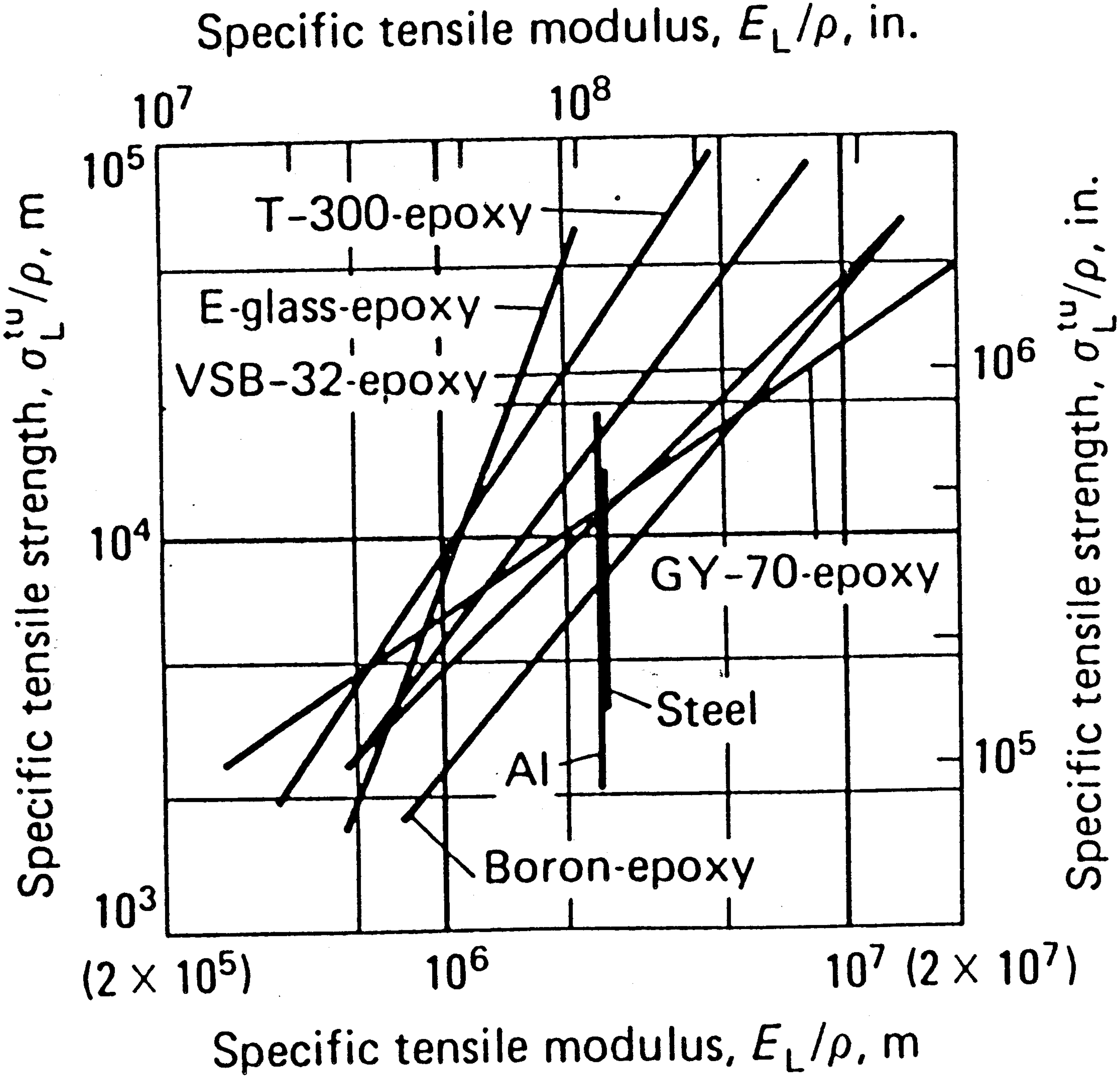
$\upsilon > \upsilon_c$ LE FIBERS CONTROL



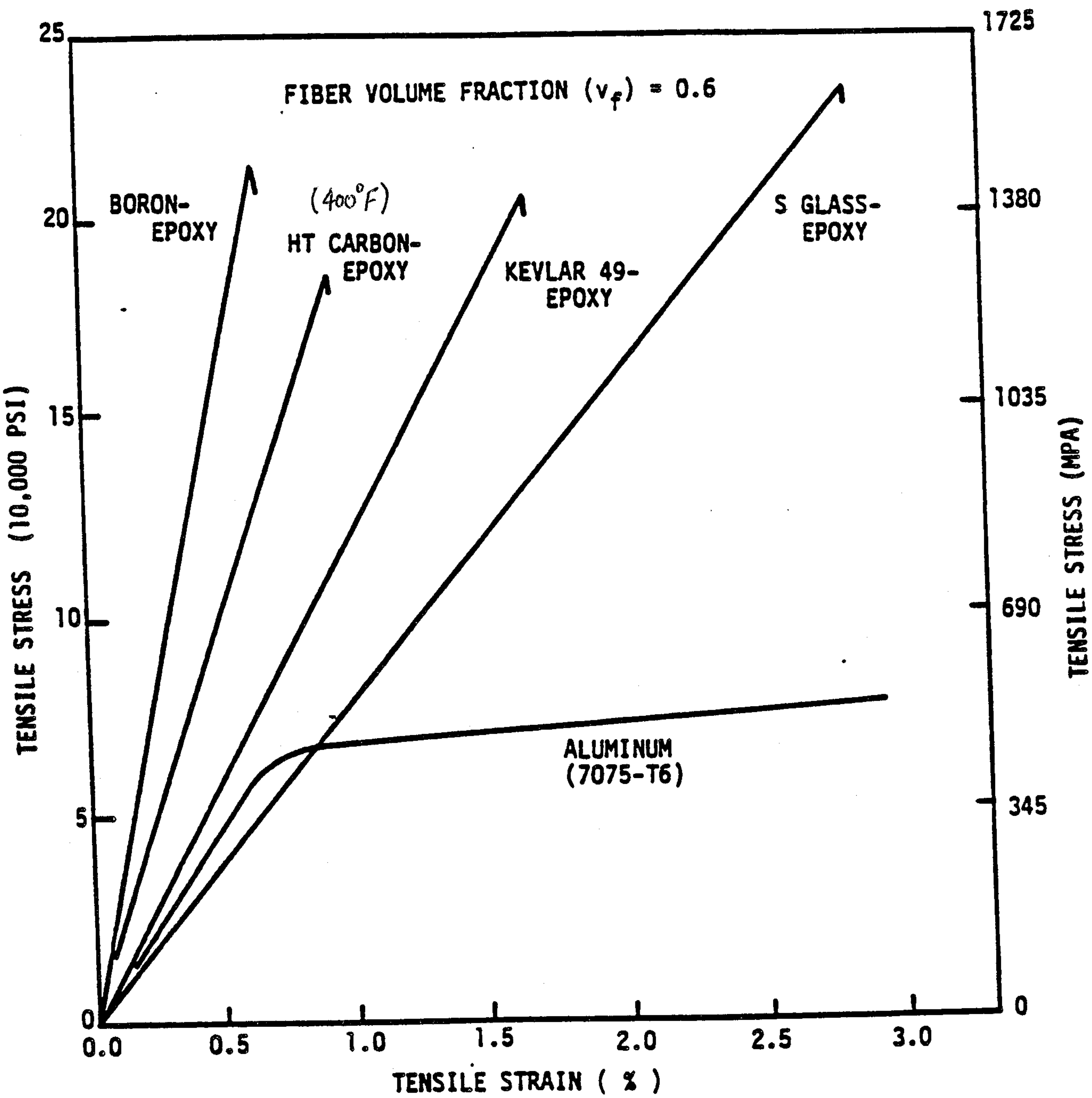
Variations of tensile strength and modulus of a carbon/glass-epoxy interply hybrid laminate with carbon fiber content [4]

MECHANICAL PROPERTIES

- Static
  - Tensile
  - Compressive
  - Flexure
  - Inplane Shear
- Fatigue
- Impact
- Environmental
- Fracture
- Creep



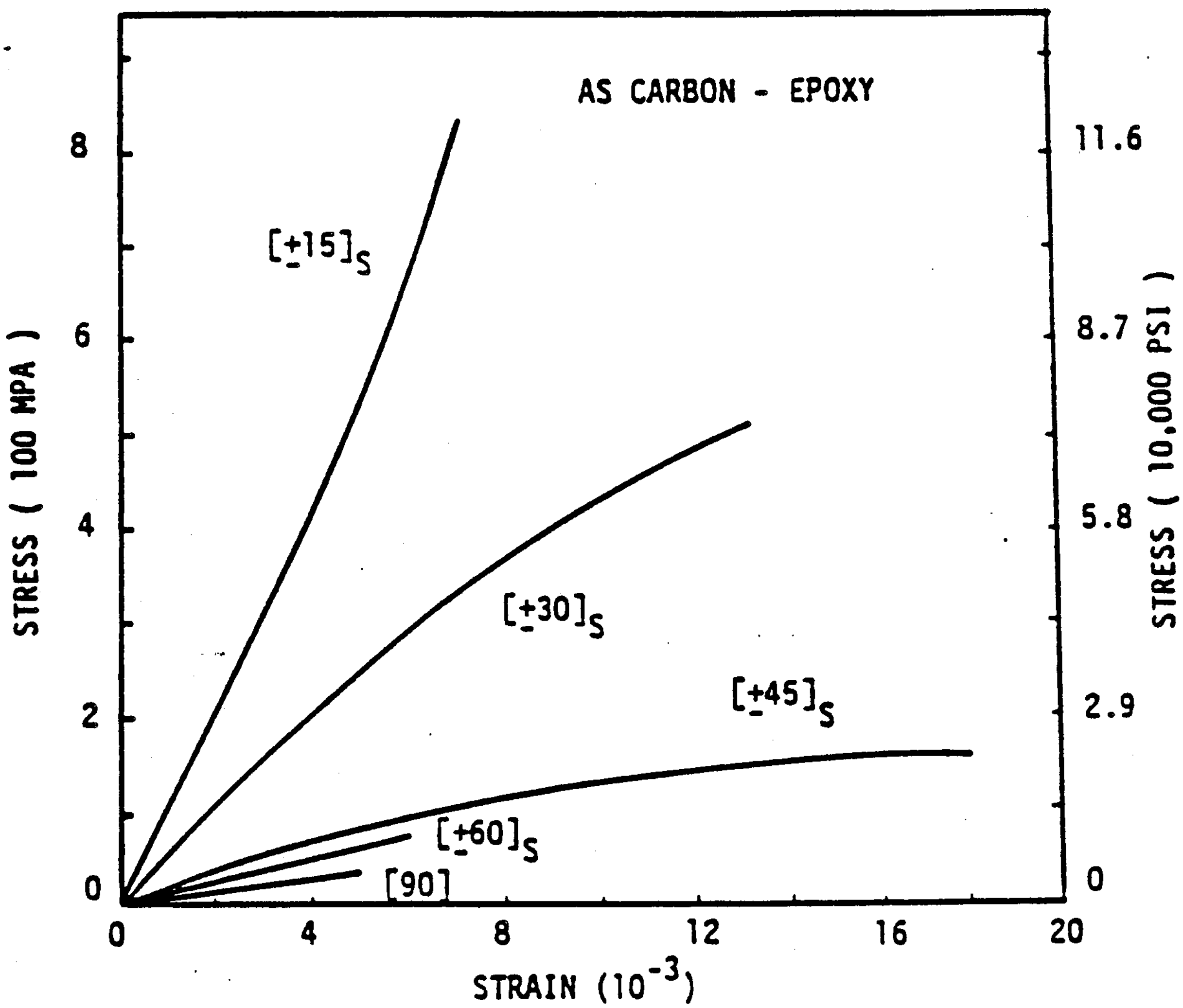
Approximate ranges of values of strength/weight and stiffness/weight ratios for structural metals and for composites of various reinforcement. [5]



Tensile stress-strain curves for 0° laminates [2]

COMPOSITES - LINEAR TO FAILURE ; TENSILE RUPTURE OF FIBERS THEN DEBOND

METALS - PLASTIC REGION

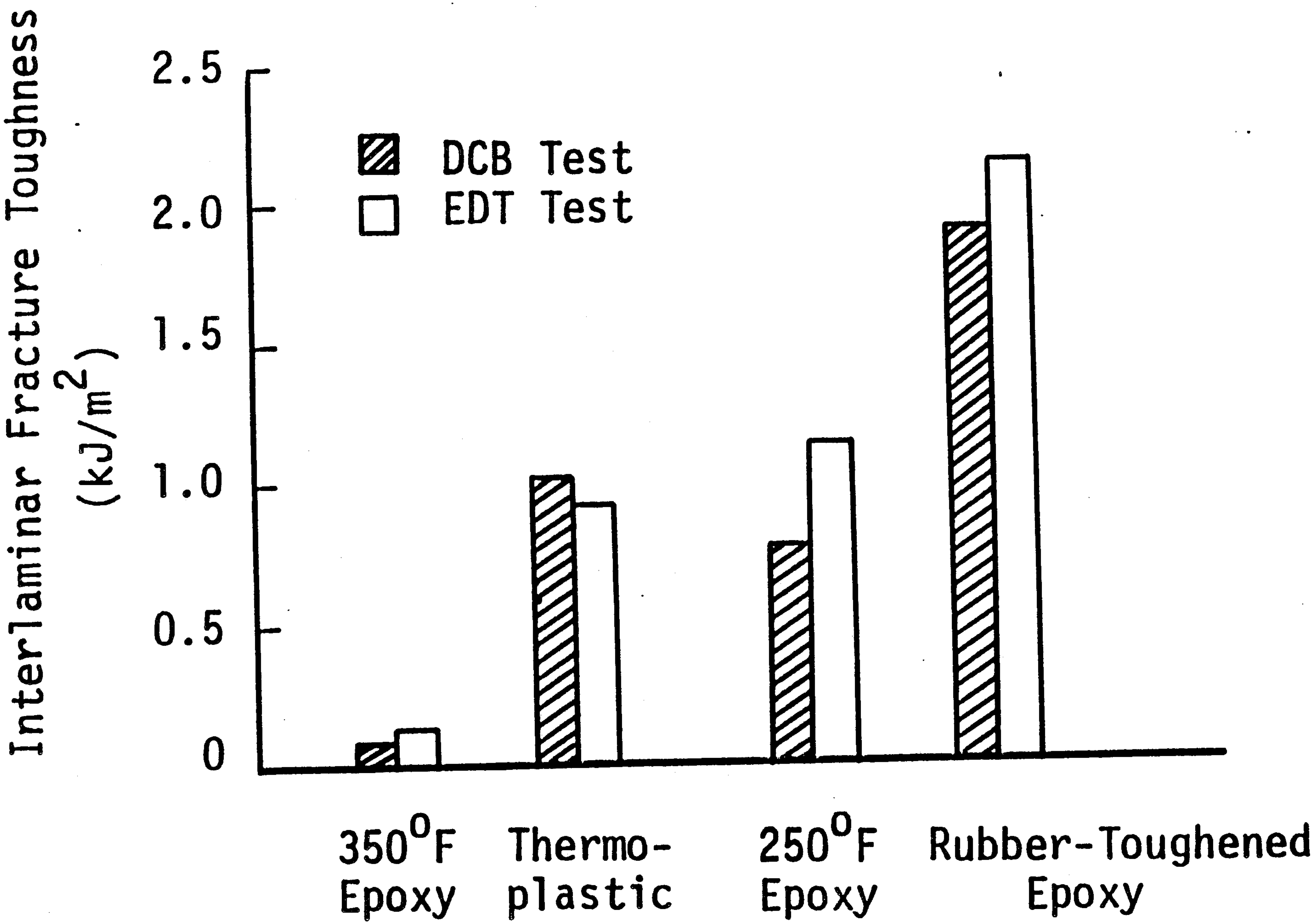


[6]

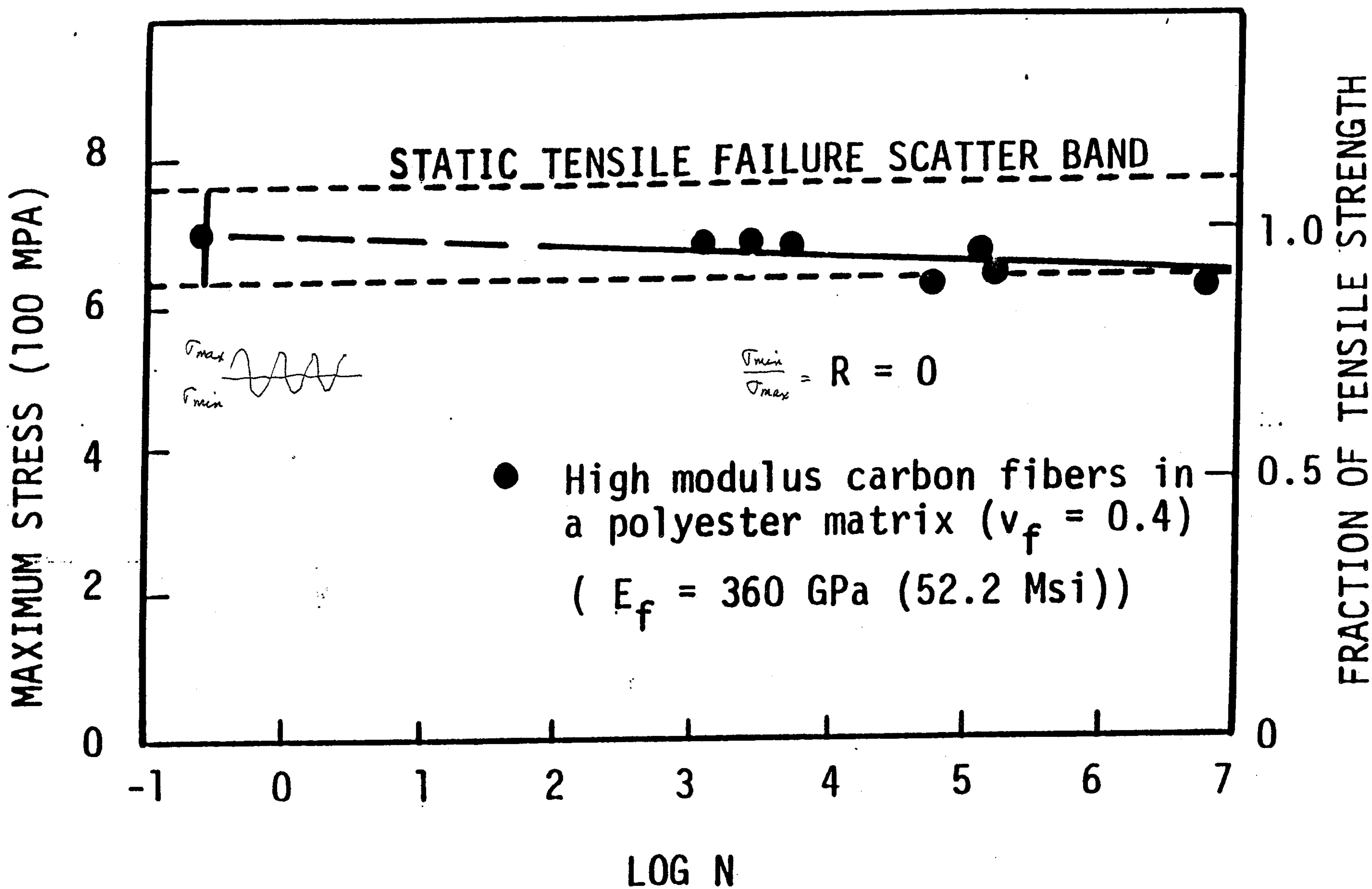
AS θ ↑ LESS STIFF (SHEAR STRESS RELATED), EXHIBITS RES STRAIN

θ ↓ MORE STIFF (TENSILE STRESS RELATED), NO RESIDUAL STRAIN

UNLIKE METALS



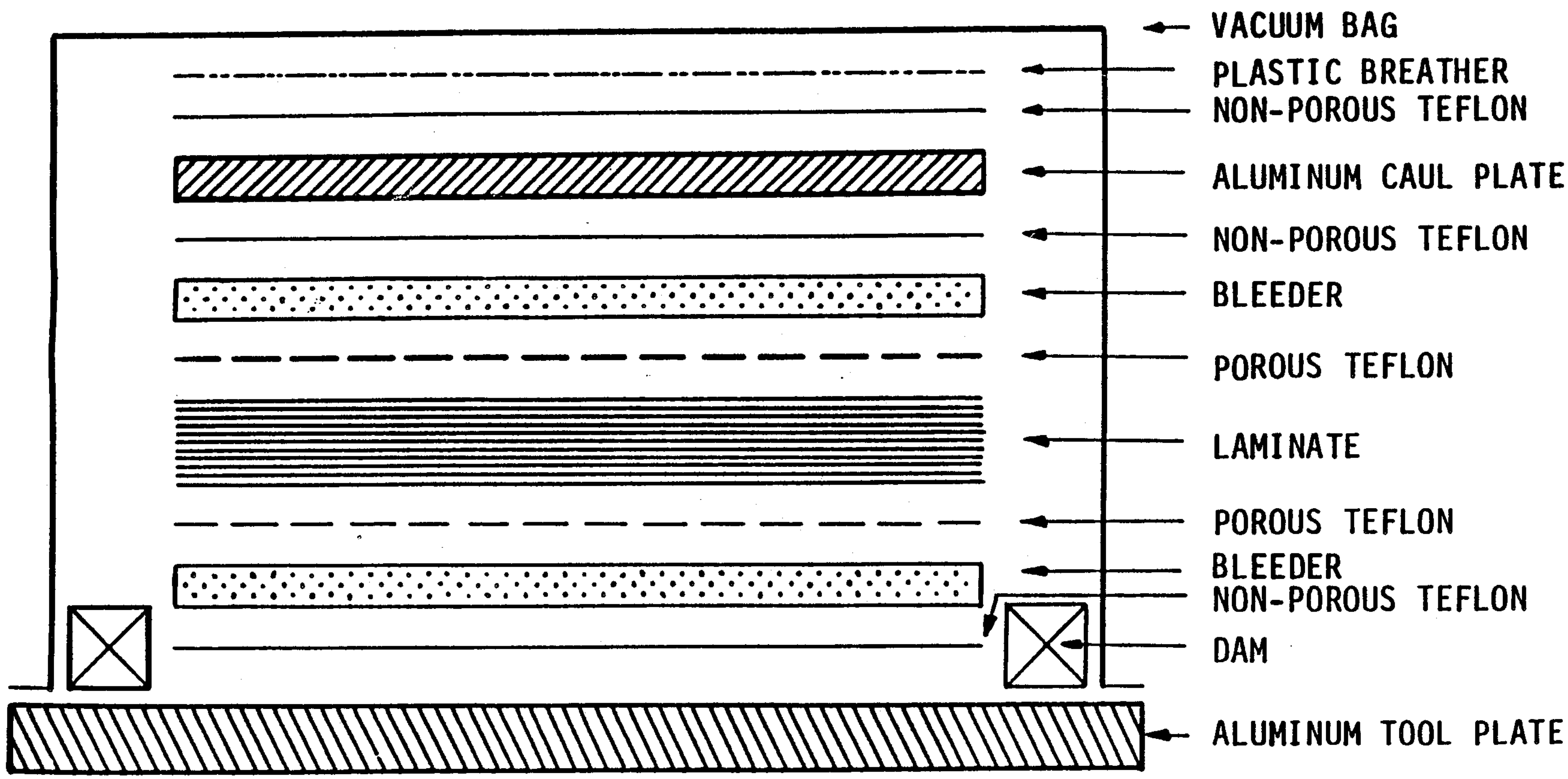
Comparison of the interlaminar fractu[illegible]ughness of various resin systems [7]



Tension-tension S-N diagram for a 0° ultra-high-modulus carbon fiber-epoxy composite [8]

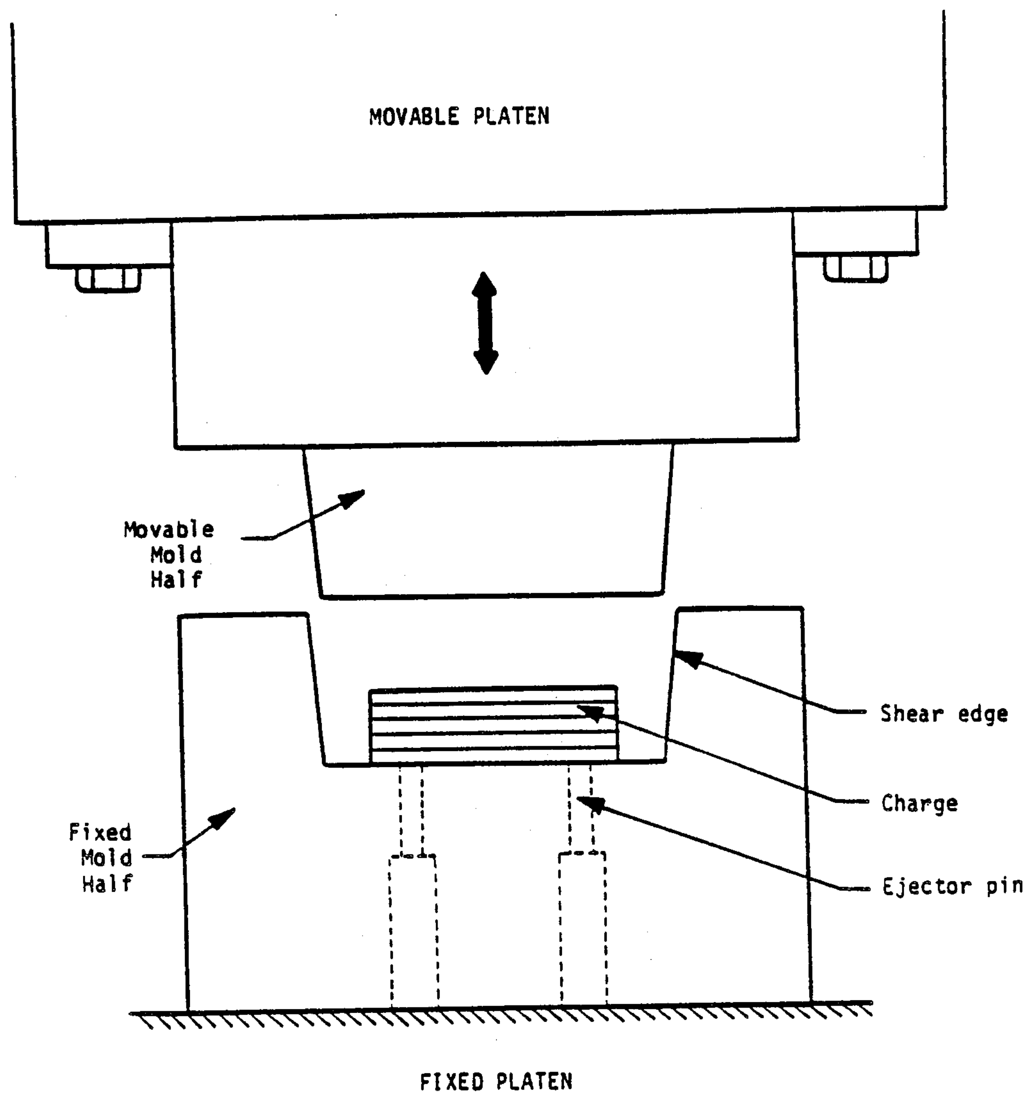
MANUFACTURING

- Bag Molding
- Compression Molding
- Pultrusion
- Filament Winding
- Resin Transfer Molding

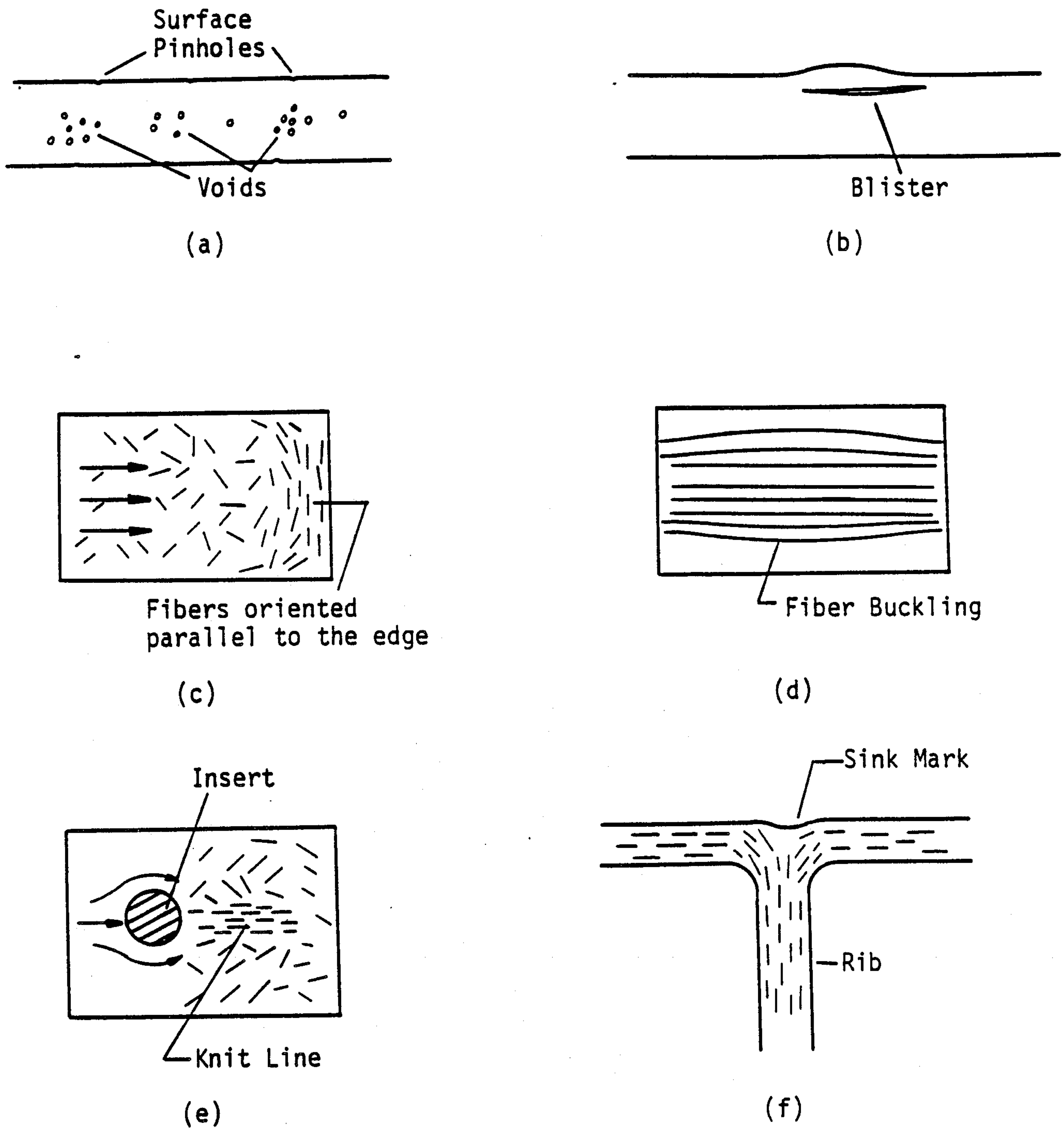


Schematic of a bag molding process [2]

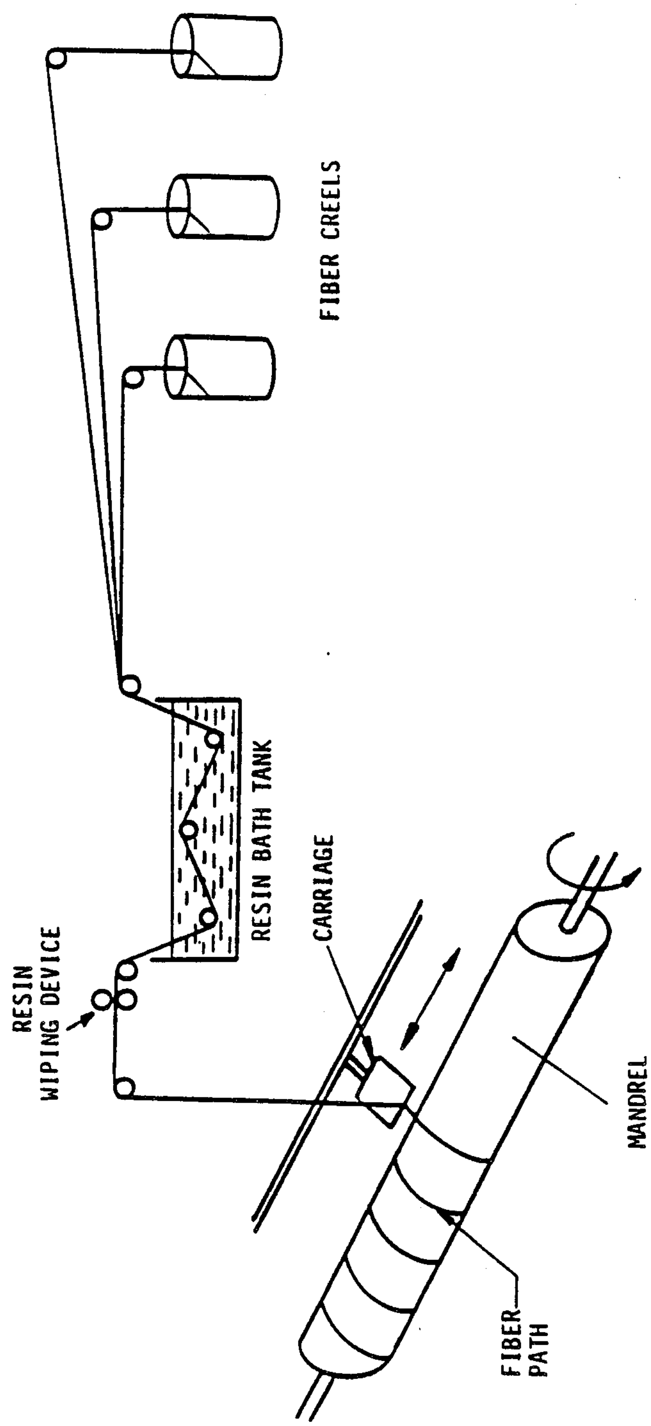
EXPENSIVE



Schematic of a compression molding process [2]



Various defects in a compression-molded SMC part [2]


FIBER CREELS
RESIN BATH TANK
RESIN WIPING DEVICE
CARRIAGE
MANDREL
FIBER PATH

[2]

DESIGN

- Design Allowables Depend On
  - Fiber/Matrix Material
  - Stacking Sequence
  - Manufacturing Technique
  - Operating Environment
  - Failure Modes/Prediction
  - Hole/Size Effect
  - Joints: Mechanical or Bonded

Unique Characteristics of Fiber-Reinforced Polymeric Composites

Nonisotropic
- Orthotropic
- Directional properties
- Four independent elastic constants instead of two
- Principal stresses and principal strains not in the same direction
- Coupling between extensional and shear deformations

Nonhomogeneous
- More than one macroscopic constituent
- Local variation in properties due to resin-rich areas, voids, fiber misorientation, etc.
- Laminated structure

Laminated structure
- Extensional-bending coupling
- Planes of weakness between layers
- Interlaminar stresses
- Properties depend on the laminate type
- Properties may depend on stacking sequence
- Properties can be tailored according to requirements
- Poisson's ratio can be greater than 0.5

Nonductile behavior
- Lack of plastic yielding
- Nearly elastic or slightly nonelastic stress-strain behavior
- Stresses are not locally redistributed around bolted or riveted holes by yielding
- Low strains to failure in tension

Noncatastrophic failure modes
- Delamination
- Localized damage (fiber breakage, matrix cracking, debonding, fiber pull-out, etc.)
- Less notch sensitivity
- Progressive loss in stiffness during cyclic loading
- Interlaminar shear failure in bending

Low coefficient of thermal expansion
- Dimensional stability
- Zero coefficient of thermal expansion possible
- Attachment problem with metals due to thermal mismatch

High internal damping: High attenuation of vibration and noise

Noncorroding

References: 1, 2, 3, 4, 5, 6, 7, 8, 32, 36

ANISOTROPIC ELASTICITY

DIFFERENT RESPONSES DUE TO ISOTROPIC, ANISOTROPIC MATERIALS

ISOTROPY - APPLIED TENSILE LOAD PRODUCES ELONGATION IS LOAD DIRECTION
CONTRACTION IN DIRECTIONS $\perp$ TO LOAD

ORTHOTROPIC - APPLIED TENSILE LOAD IN PRINCIPAL DIRECTION : SAME RESPONSE
- IN NON PRINCIPAL DIRECTION : EXTENSION & SHEAR COUPLING

ANISOTROPIC - APPLIED TENSILE LOAD IN ANY DIRECTION PRODUCES EXTENSION & SHEAR COUPLING.

SAME IS TRUE FOR APPLIED PURE SHEAR

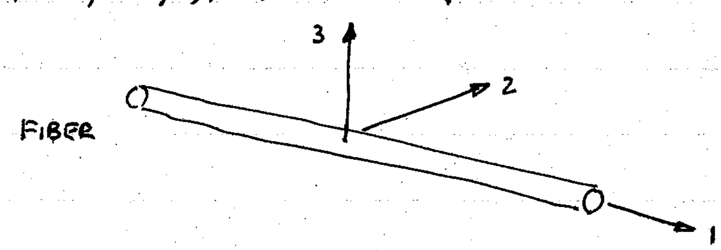
- HOW DO WE GET THESE TYPES OF MATERIALS

ANISOTROPIC - MATERIAL PROPERTIES ARE DIFFERENT IN ALL DIRECTIONS
$\Rightarrow$ NO PLANES OF SYMMETRY

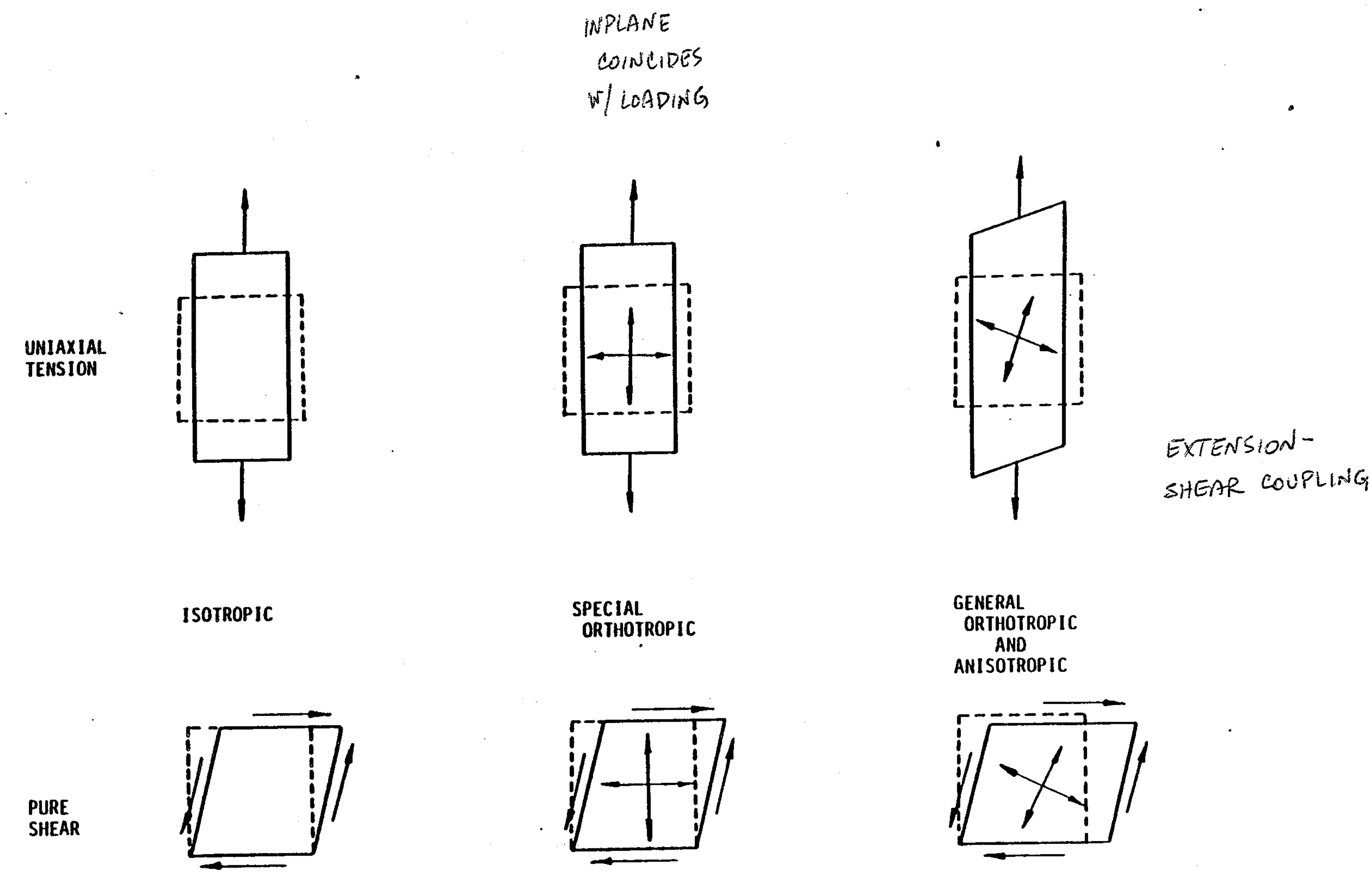
ISOTROPIC - MATERIAL PROPERTIES ARE SAME IN ALL DIRECTIONS
$\Longrightarrow$ INFINITE NO. OF PLANES OF SYMMETRY

ORTHOTROPIC - MATERIAL HAS THREE PLANES OF SYMMETRY

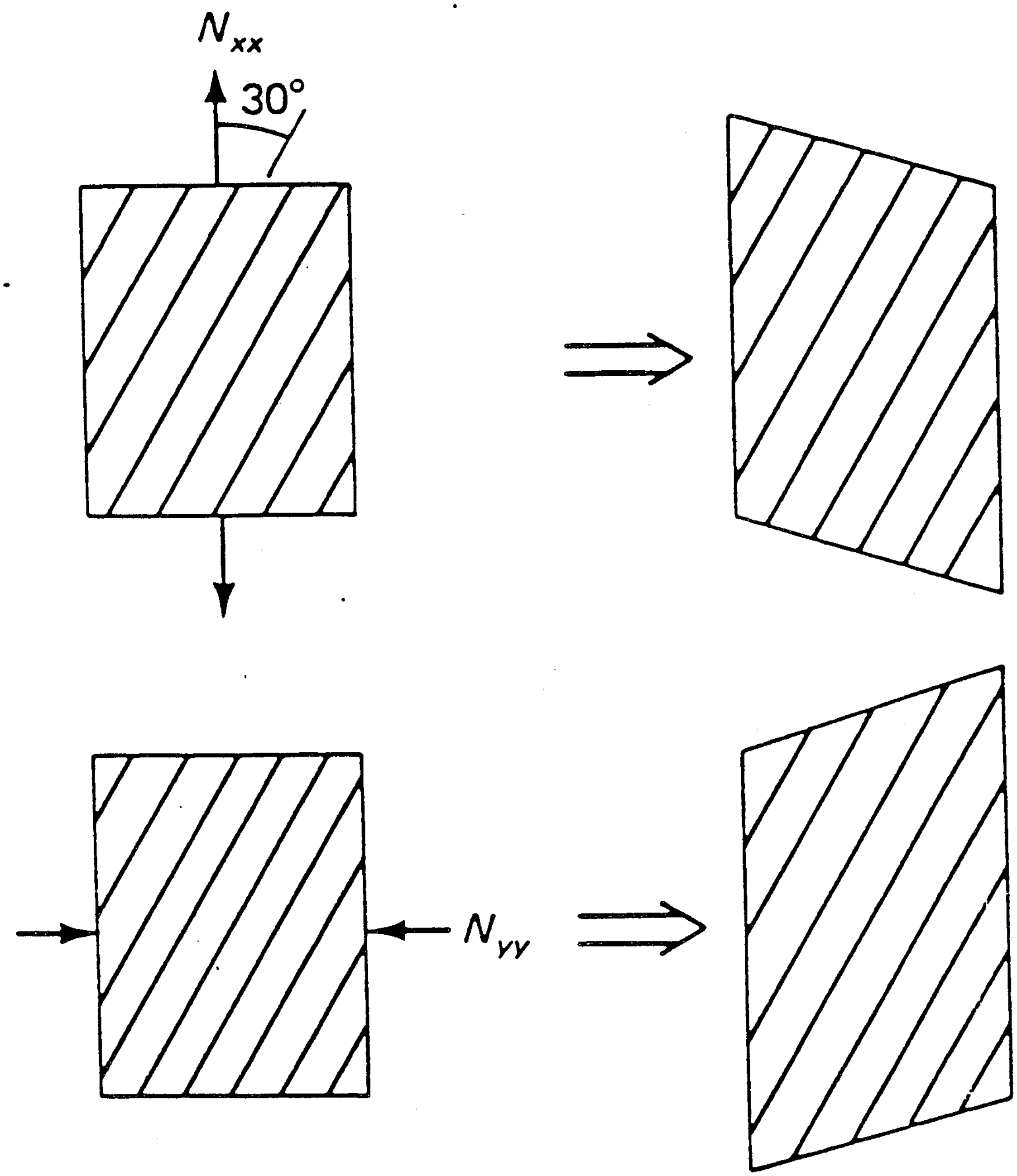
- FIBER REINFORCED COMPOSITES HAVE 3 PLANES OF SYMMETRY CONTAINING 1,2,3 AXES (PRINCIPAL MATERIAL DIRECTIONS)



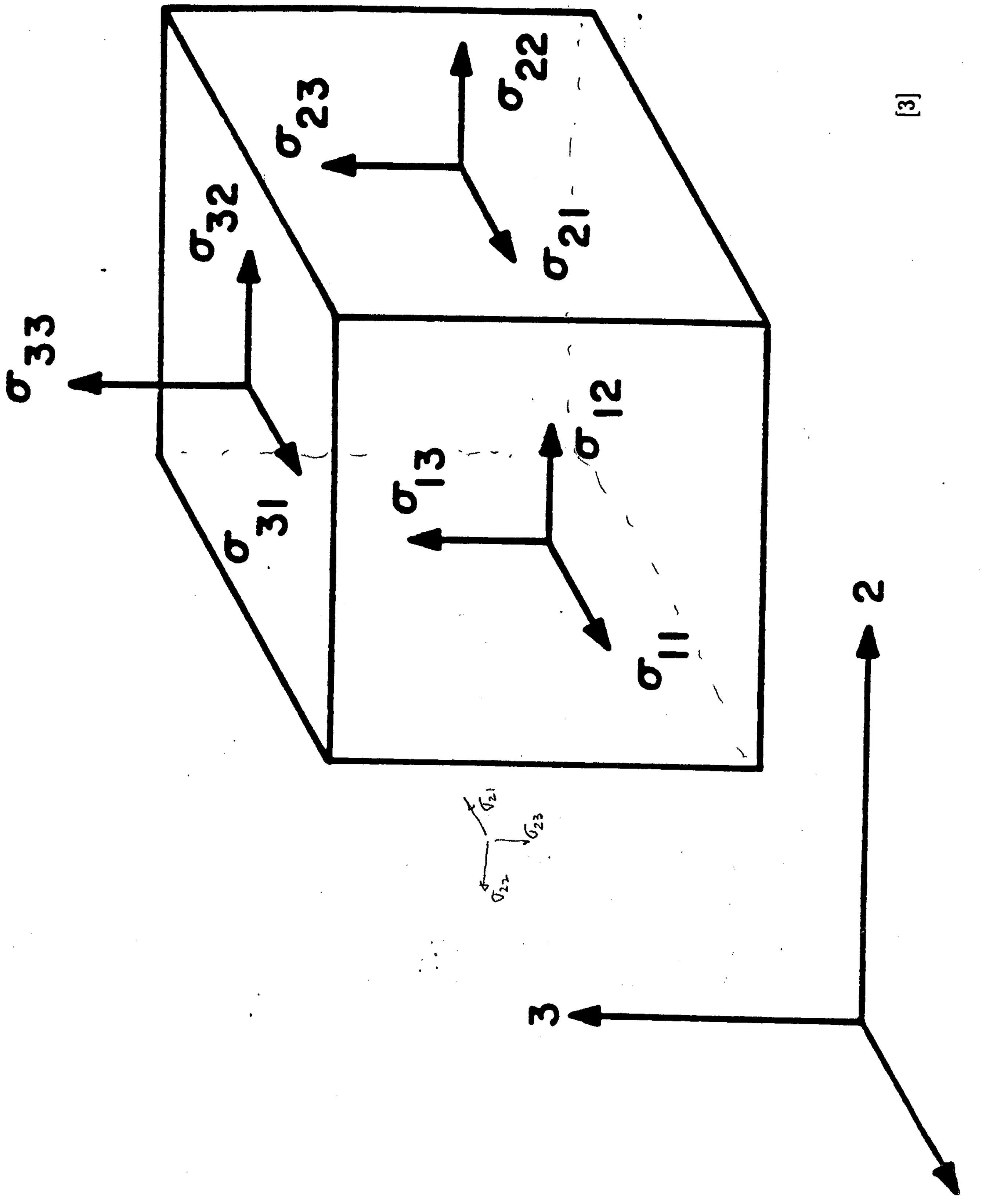
- FOR UNIDIRECTIONAL FIBERS PROPERTIES IN 2-3 PLANE ARE THE SAME



Differences in the deformations of isotropic, special orthotropic, and anisotropic materials subjected to uniaxial tension and pure shear stresses [2]



Extensional-shear coupling and Poisson's ratio [5]



[3]

COMPONENTS OF STRESS AND STRAIN

$$\sigma_{ii} = \sigma_i \ , \ \varepsilon_{ii} = \varepsilon_i \ , \ (i = 1,2,3)$$

$$\tau_{23} = \sigma_{23} = \sigma_4 \ , \ \gamma_{23} = 2\varepsilon_{23} = \varepsilon_4$$

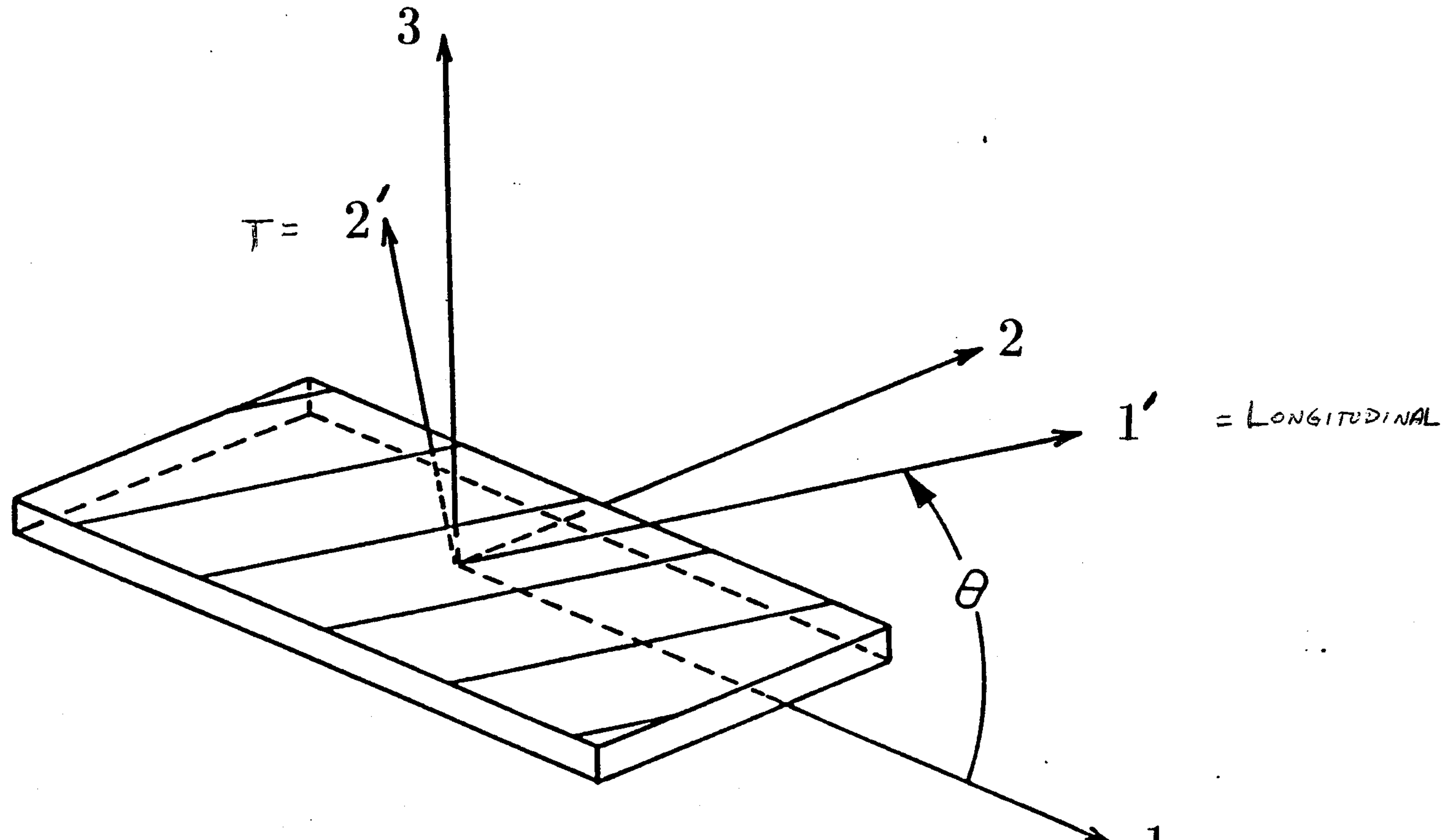
$$\tau_{13} = \sigma_{13} = \sigma_5 \ , \ \gamma_{13} = 2\varepsilon_{13} = \varepsilon_5$$

$$\tau_{12} = \sigma_{12} = \sigma_6 \ , \ \gamma_{12} = 2\varepsilon_{12} = \varepsilon_6$$

[3]

γ - ENGINEERING STRAIN

- NORMALLY 9 COMPONENTS OF STRESS : $\sigma_{11}, \sigma_{12}, \cdots, \sigma_{33}$
- FOR NO ROTATIONS $\sigma_{32} = \sigma_{23}$ etc $\sigma_{ij} = \sigma_{ji}$
- INSTEAD OF WORKING WITH A STRESS TENSOR (3x3) & REDUNDANCY (σ_{23}, σ_{32}, etc) WORK WITH A VECTOR HAVING 6 ENTRIES



1′,2′ DIRECTIONS // & ⊥ TO FIBER

1,2,3 NORMAL x,y,z DIRECTIONS OR LOADING DIRECTIONS

ROTATION OF AXES

$$\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} = \begin{bmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

m_i, n_i, AND p_i ARE DIRECTION COSINES

$$\begin{bmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_3' \\ \sigma_4' \\ \sigma_5' \\ \sigma_6' \end{bmatrix} = [T_\sigma] \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \qquad \begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \\ \varepsilon_3' \\ \varepsilon_4' \\ \varepsilon_5' \\ \varepsilon_6' \end{bmatrix} = [T_\varepsilon] \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}$$

$$[\sigma'] = [T_\sigma]\,[\sigma] \qquad [\varepsilon'] = [T_\varepsilon]\,[\varepsilon]$$

[3]

$$\begin{bmatrix} \varepsilon'_1 \\ \varepsilon'_2 \\ \varepsilon'_3 \\ \varepsilon'_4 \\ \varepsilon'_5 \\ \varepsilon'_6 \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & mn \\ n^2 & m^2 & 0 & 0 & 0 & -mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -2mn & 2mn & 0 & 0 & 0 & (m^2-n^2) \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \quad [3]$$

PRIMARILY CONCERNED WITH ROTATION IN X_1-X_2 PLANE

ABOUT X_3 AXIS

$$\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} \begin{bmatrix} m & n & 0 \\ -n & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$m = \cos\theta \quad , \quad n = \sin\theta$$

$$\begin{bmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_3' \\ \sigma_4' \\ \sigma_5' \\ \sigma_6' \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & m^2-n^2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

[3]

GENERALIZED HOOKE'S LAW

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \begin{array}{l} = \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{yz} \\ \gamma_{xz} \\ \gamma_{xy} \end{array}$$

$[\sigma] = [C][\varepsilon]$

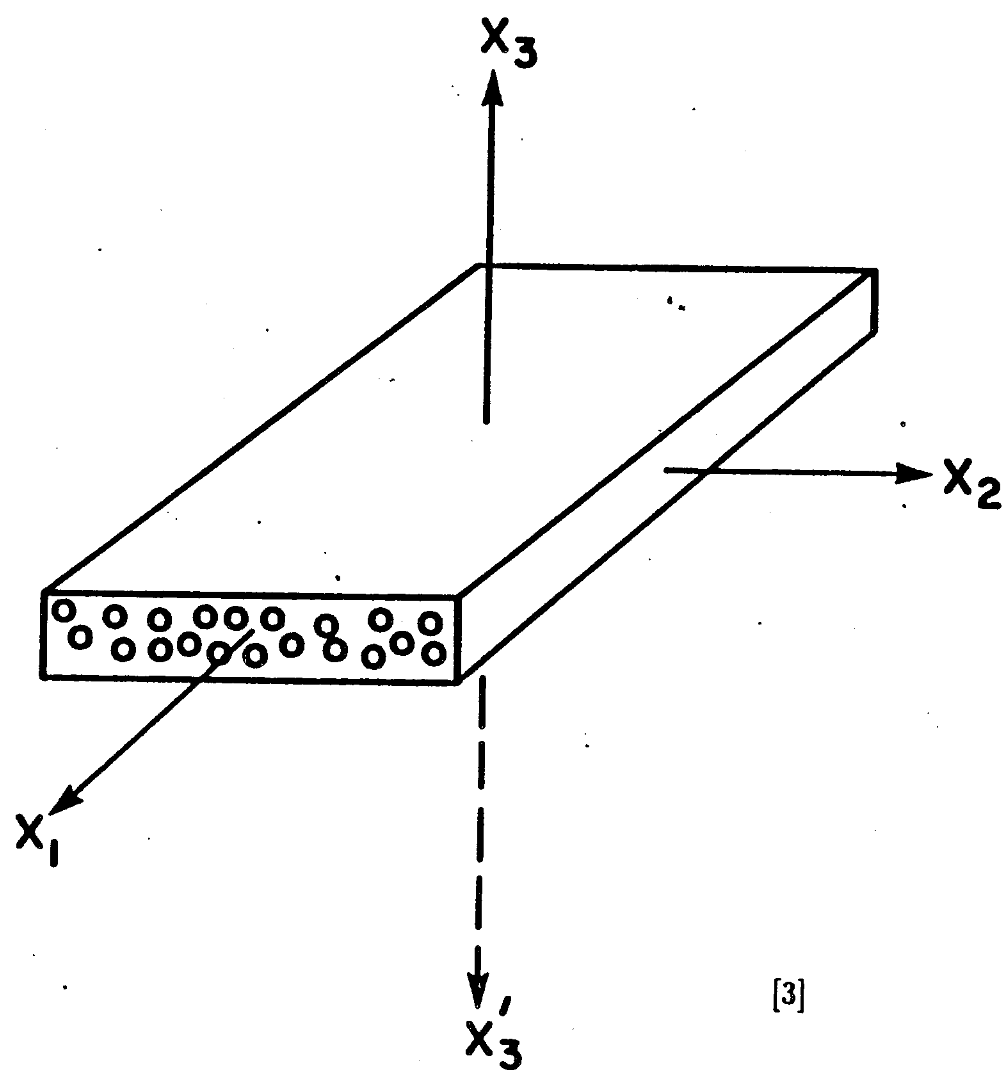
$[\sigma'] = [C][\varepsilon']$

$C_{ij} = C_{ji}$ 36 REDUCED TO 21

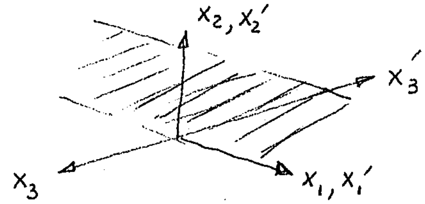
C_{ij} = STIFFNESS MATRIX – invariant under transformation

$$\sigma_i = \sum^{6} C_{ij}\varepsilon_j \, , \quad (i = 1, 2, \ldots, 6) \qquad [3]$$

X_1-X_2 PLANE A PLANE OF ELASTIC SYMMETRY



[3]



X_1-X_2 PLANE A PLANE OF ELASTIC SYMMETRY

$$\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

$$m_1 = n_2 = 1 \ , \ p_3 = -1$$

ALL OTHER DIRECTION COSINES = 0

$[T\sigma] = [T\epsilon]$ FOR THIS TRANSFORMATION

$$\begin{bmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_3' \\ \sigma_4' \\ \sigma_5' \\ \sigma_6' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}$$

note $\sigma_1' = \sigma_1$, $\sigma_2' = \sigma_2$, $\sigma_3' = \sigma_3$, $\sigma_4' = -\sigma_4$, $\sigma_5' = -\sigma_5$, $\sigma_6' = \sigma_6$ [3]

also $\varepsilon_1' = \varepsilon_1$, $\varepsilon_2' = \varepsilon_2$, $\varepsilon_3' = \varepsilon_3$, $\varepsilon_4' = -\varepsilon_4$, $\varepsilon_5' = -\varepsilon_5$, $\varepsilon_6' = \varepsilon_6$

ELASTIC STIFFNESSES ARE INVARIANT UNDER THIS TRANSFORMATION

MUST BE EQUAL

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 + C_{14}\varepsilon_4 + C_{15}\varepsilon_5 + C_{16}\varepsilon_6$$

$$\sigma_1' = C_{11}\varepsilon_1' + C_{12}\varepsilon_2' + C_{13}\varepsilon_3' + C_{14}\varepsilon_4' + C_{15}\varepsilon_5' + C_{16}\varepsilon_6'$$

$$\vdots$$

FROM TRANSFORMATION RELATIONS

$\sigma_1' = \sigma_1$
$\varepsilon_1' = \varepsilon_1$
$\varepsilon_2' = \varepsilon_2$ etc

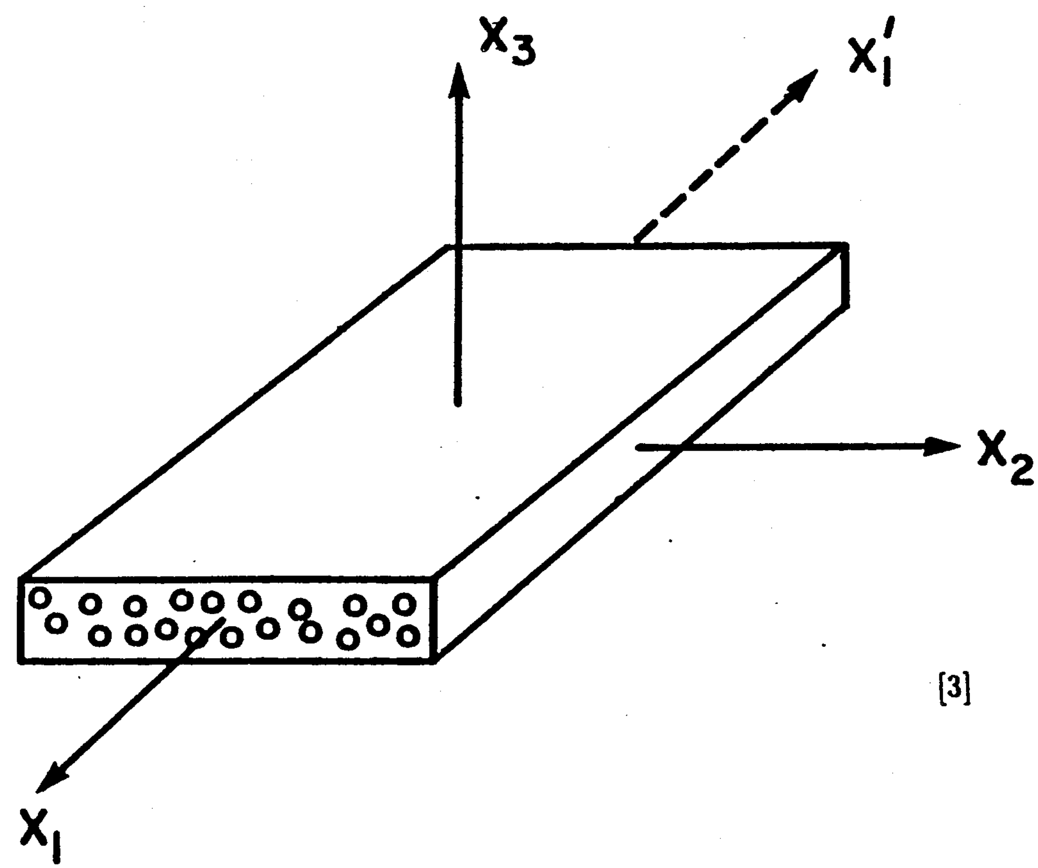
$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 - C_{14}\varepsilon_4 - C_{15}\varepsilon_5 + C_{16}\varepsilon_6$$

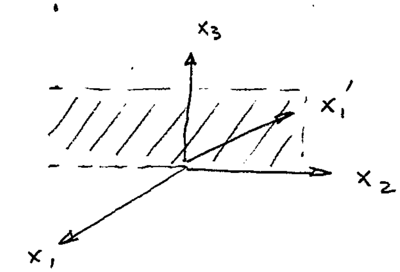
THEREFORE $C_{14} = C_{15} = 0$

SIMILAR CONSIDERATIONS OF $\sigma_2, \sigma_3, \ldots, \sigma_6$ YIELDS

$$C_{24} = C_{34} = C_{35} = C_{46} = C_{56} = 0 \qquad [3]$$

X_2-X_3 PLANE A PLANE OF ELASTIC SYMMETRY





[3]

IF X_2-X_3 PLANE IS ALSO PLANE OF ELASTIC SYMMETRY, STIFFNESSES ARE INVARIANT UNDER THE TRANSFORMATION

$$\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

SIMILAR PROCEDURE AS BEFORE YIELDS

$$C_{16} = C_{26} = C_{36} = C_{45} = 0$$

SYMMETRY WITH RESPECT TO THE X_1-X_2 AND X_2-X_3 PLANES ALSO IMPLIES SYMMETRY RELATIVE TO THE X_1-X_3 PLANE. SUCH A MATERIAL IS REFERRED TO AS ORTHOTROPIC AND IS DESCRIBED BY 9 INDEPENDENT CONSTANTS. [3]

STRESS-STRAIN RELATIONS FOR ORTHOTROPIC MATERIALS

$$\varepsilon_1 = \frac{1}{E_1}(\sigma_1 - \nu_{12}\sigma_2 - \nu_{13}\sigma_3)$$

$$\varepsilon_2 = \frac{-\nu_{12}}{E_1}\sigma_1 + \frac{1}{E_2}(\sigma_2 - \nu_{23}\sigma_3)$$

$$\varepsilon_3 = \frac{-\nu_{13}}{E_1}\sigma_1 - \frac{\nu_{23}}{E_2}\sigma_2 + \frac{1}{E_3}\sigma_3$$

$$\varepsilon_4 = \frac{\sigma_4}{G_{23}}, \quad \varepsilon_5 = \frac{\sigma_5}{G_{13}}, \quad \varepsilon_6 = \frac{\sigma_6}{G_{12}}$$

[3]

ORTHOTROPIC MATERIAL

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

[3]

TRANSVERSELY ISOTROPIC MATERIAL

THE 2-3 PLANE IS A PLANE OF ISOTROPY ELASTIC STIFFNESS TERMS INVARIANT UNDER THE TRANSFORMATION

$$\begin{bmatrix} X_1' \\ X_2' \\ X_3' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

45° rotation

WITH THE RESULT $C_{44} = (C_{22}-C_{23})/2$

THUS A TRANSVERSELY ISOTROPIC MATERIAL IS DESCRIBED BY 5 INDEPENDENT ELASTIC STIFFNESSES. [3]

TRANSVERSELY ISOTROPIC MATERIAL

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & (C_{22}-C_{23})/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

TRANSFORMATION OF [C]

$$[\sigma'] = [C'] [\epsilon']$$

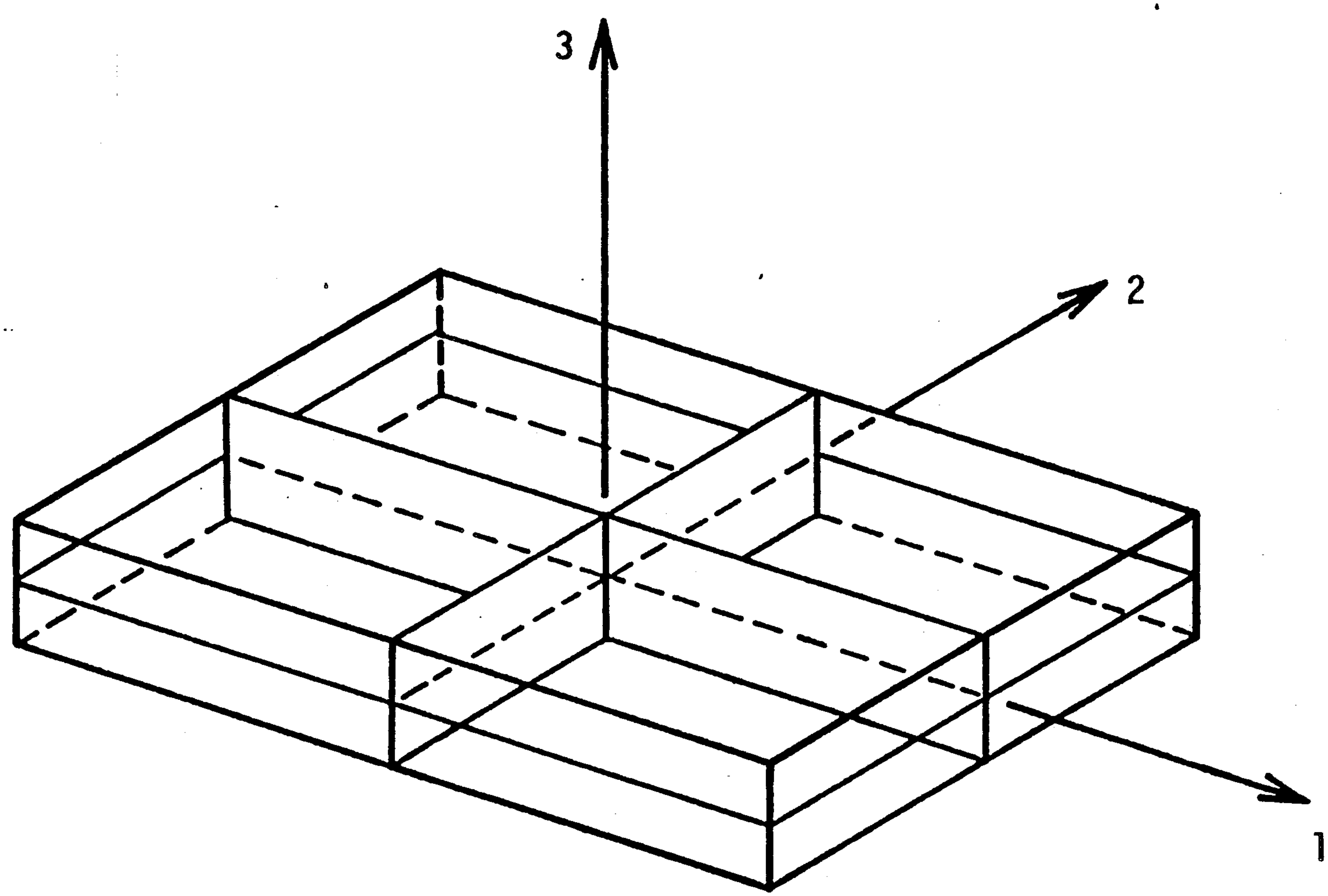
$$[T_\sigma] [\sigma] = [C'] [T_\epsilon] [\epsilon]$$

$$[\sigma] = [T_\sigma]^{-1} [C'] [T_\epsilon] [\epsilon] , \quad [C] = [T_\sigma]^{-1} [C'] [T_\epsilon]$$ [3]

MATERIAL PROPERTIES

- Isotropic
  - Properties are the Same in All Directions
  - Three Elastic Constants
- Anisotropic
  - Contains No Planes of Material Symmetry
  - Twenty-one Elastic Constants
- Orthotropic
  - Three Orthogonal Planes of Symmetry
  - Nine Elastic Constants

[2]



Three planes of symmetry in an orthotropic material [2]

FOR ORTHOTROPIC MATERIALS - 9 INDEPENDENT ELASTIC CONSTANTS: E_1, E_2, E_3, ν_{12}, ν_{23}, ν_{13}, G_{12}, G_{23}, AND G_{13}

FOR TRANSVERSELY ISOTROPIC - 5 INDEPENDENT CONSTANTS: E_1, E_2, ν_{12}, ν_{23}, G_{12},

$$G_{23} = \frac{E_2}{2(1 + \nu_{23})}$$

BULK MODULUS - ASSOCIATED WITH UNIFORM HYDROSTATIC PRESSURE $\sigma_1 = \sigma_2 = \sigma_3 = -p$

$$k = \frac{-p}{(\varepsilon_1 + \varepsilon_2 + \varepsilon_3)} = -\frac{p}{\varepsilon}$$

[3]

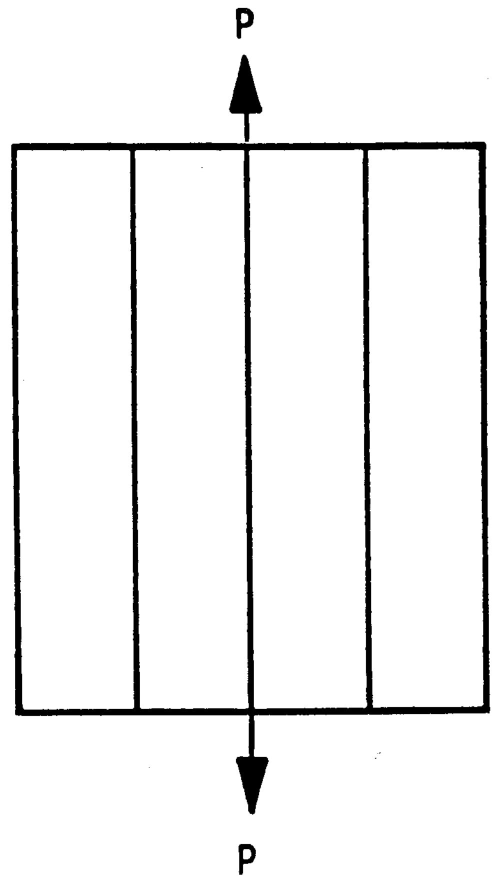
References: 2, 3, 5

MATERIAL AND LAMINA PROPERTIES

FIBER/MATRIX INTERACTIONS IN A UNIDIRECTIONAL LAMINA

- Basic Assumptions
  - Fibers are Uniformly Distributed
  - Fibers are Straight
  - * – Fibers have No Imperfections
  - Perfect Bonding
  - Matrix is Free of Voids
  - Loads are Parallel or Normal to the Fibers
  - * – No Residual Stresses are Present
  - Fibers and Matrix Behave in a Linearly Elastic Manner [2]

LONGITUDINAL TENSILE LOADING



For Perfect Bonding $\epsilon_f = \epsilon_m = \epsilon_c$

Using the 1-D Hooke's Law

$$\sigma = E\epsilon$$

Therefore

$$\sigma_f = E_f\epsilon_f = E_f\epsilon_c \qquad (1)$$

$$\sigma_m = E_m\epsilon_m = E_m\epsilon_c \qquad (2)$$

Noting that $E_f >> E_m$

$$\sigma_f > \sigma_m$$

Applying a Tensile Force

$$P = P_f + P_m$$

Since Load Equals Stress Times Area

$$\sigma_c A_c = \sigma_f A_f + \sigma_m A_m$$

Noting that

$$A_c = A_f + A_m$$

$$v_f = A_f / A_c$$

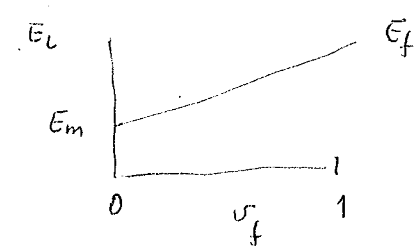
$$v_m = A_m / A_c$$

$$\sigma_c = \sigma_f v_f + \sigma_m (1 - v_f)$$

Dividing by ϵ_c and using equations 1 and 2 produces the Rule of Mixtures

$$E_L = E_f v_f + E_m (1 - v_f)$$

[2]



VOLUME FRACTION LIMITS

- Cylindrical Fibers
  - Theoretically $v_f = 0.9$
  - Practical Limit for Good Wetting of the Fibers $v_f = 0.8$
  - Critical Volume Fraction

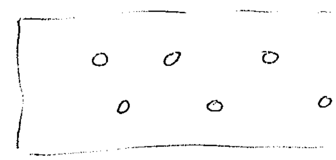
$$v_f = \frac{(\sigma_{mu} - \sigma'_m)}{(\sigma_{fu} - \sigma'_m)}$$

[2]

TRANSVERSE TENSILE LOADING

- Neglects Interaction of Neighboring Fibers

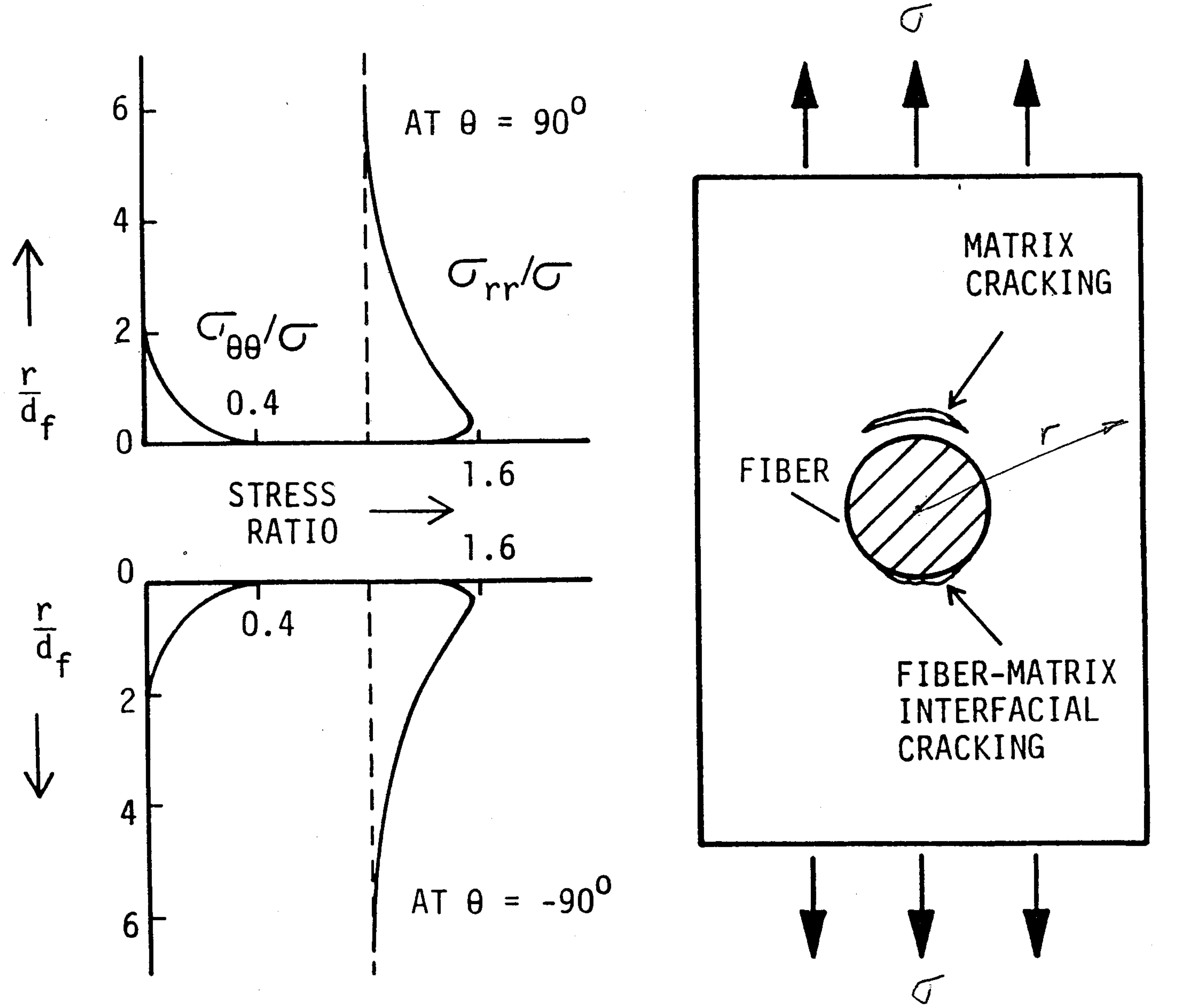
- Fibers Act as Hard Inclusions



- Local Stresses/Strains in the Matrix are Higher than the Applied Stress

- Cracks may Develop Normal to the Loading Direction at the Fiber/Matrix Interface or in the Matrix

[2]



Transverse tensile loading on a lamina containing a single cylindrical fiber (a) Stress distribution around a single fiber due to transverse tensile loading (b) Possible microfailure modes. [2]

ELASTIC PROPERTIES OF A LAMINA

- Longitudinal Modulus

$$E_{11} = E_f v_f + E_m v_m$$

$v_m = 1 - v_f$
Depends more on E_f

- Transverse Modulus

$$E_{22} = \frac{E_f E_m}{E_f v_m + E_m v_f}$$

spings in parallel

$$\frac{1}{E_{22}} = \frac{v_m}{E_m} + \frac{v_f}{E_f}$$

Depends more on E_m

- Major Poisson's Ratio

$$\nu_{12} = \nu_f v_f + \nu_m v_m$$

- Minor Poisson's Ratio

E_{11}/E_{22} measure of orthropy

$$\nu_{21} = \left(E_{22}/E_{11}\right) \nu_{12}$$

- Shear Modulus

$$G_{12} = \frac{G_f G_m}{G_f v_m + G_m v_f}$$

Depends more on G_m

[2]

E_{22} & G_{12} Give lower values than experiments

Depend on void content, v_m & FIBER ANISOTROPY

OBSERVATIONS ABOUT MODULI

- $E_{11} > E_{22}$
- Fibers Contribute More to E_{11}
- Matrix Contributes More to E_{22}
- $\nu_{12} > \nu_{21}$
- Matrix Contributes More to G_{12}
  Than the Fibers

[2]

COEFFICIENTS OF LINEAR THERMAL EXPANSION

- Unidirectional Continuous Fiber Lamina

$$\alpha_{11} = \frac{\alpha_{fl} E_f v_f + \alpha_m E_m v_m}{E_f v_f + E_m v_m}$$ ALONG FIBER DIR. (0°)

$$\alpha_{22} = (1+\nu_f)\frac{(\alpha_{fl}+\alpha_{fr})}{2}\ v_f$$ ⊥ TO FIBER DIR (90°)

$$+(1+\nu_f)\alpha_m v_m + \alpha_{11}\nu_{12}$$

Where

$\nu_{12} = \nu_f v_f + \nu_m v_m$

$\alpha_{fl} =$ The CTE of the Fiber in the Longitudinal Direction

$\alpha_{fr} =$ The CTE of the Fiber in the Radial Direction

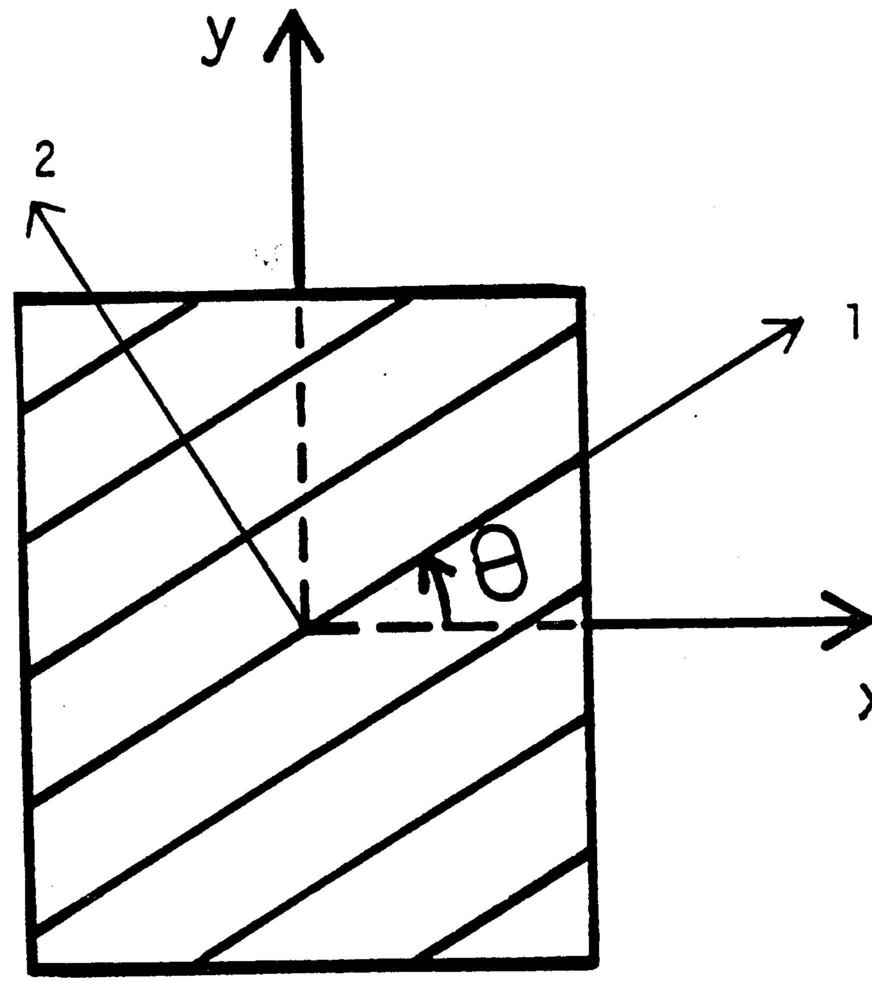
$\alpha_m =$ The CTE of the Matrix

[2]

$\alpha_{22} > \alpha_{11}$ MATRIX DOMINATED

α_{22} (Graphite-fiber epoxy) $\cong \frac{1}{2}\ \alpha$ (aluminum)

IN A CASE $\alpha_m >> \alpha_f$ $\alpha(90°) > \alpha(0°)$



1,2 MATERIAL) COOR
X,Y LOADING

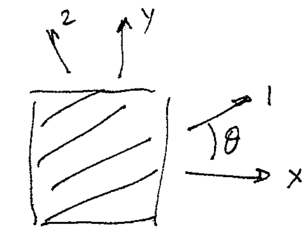
Angled unidirectional continuous fiber lamina [2]

PROPERTIES MEASURED IN X,Y PLANE (IN PLANE)
IN Z PLANE (OUT OF PLANE)

IN PLANE INTRA LAMINAR

OUT OF PLANE INTER LAMINAR

MODULI TRANSFORMATION
LAMINA



$$\frac{1}{E_{xx}} = \frac{cos^4\theta}{E_{11}} + \frac{sin^4\theta}{E_{22}} + \frac{1}{4}\left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right) sin^2 2\theta$$

$E_{11} = E_f v_f + E_m (1 - v_f)$

$E_{22} \quad \dfrac{E_f E_m}{E_f v_m + E_m v_f}$

$G_{12} = \dfrac{G_f G_m}{G_f v_m + G_m v_f}$

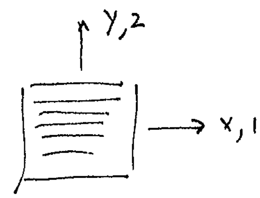
$$\frac{1}{E_{yy}} = \frac{sin^4\theta}{E_{11}} + \frac{cos^4\theta}{E_{22}} + \frac{1}{4}\left(\frac{1}{G_{12}} - \frac{2\nu_{12}}{E_{11}}\right) sin^2 2\theta$$

$$\frac{1}{G_{xy}} = \frac{1}{E_{11}} + \frac{2\nu_{12}}{E_{11}} + \frac{1}{E_{22}} - \left(\frac{1}{E_{11}} + \frac{2\nu_{12}}{E_{11}} + \frac{1}{E_{22}} - \frac{1}{G_{12}}\right) cos^2 2\theta$$

[2]

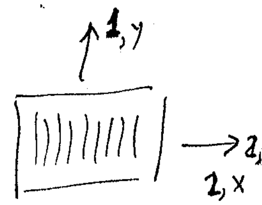
$$\nu_{xy} = E_{11}\left[\frac{\nu_{12}}{E_{11}} - \frac{1}{4}\left(\frac{1}{E_{11}} + \frac{2\nu_{12}}{E_{11}} + \frac{1}{E_{22}} - \frac{1}{G_{12}}\right) sin^2 2\theta\right.$$

$$\nu_{yx} = E_{yy}\nu_{xy}/E_{xx}$$



Note that at $\theta = 0^o$, $E_{xx} = E_{11}$

and at $\theta = 90^o$, $E_{xx} = E_{22}$



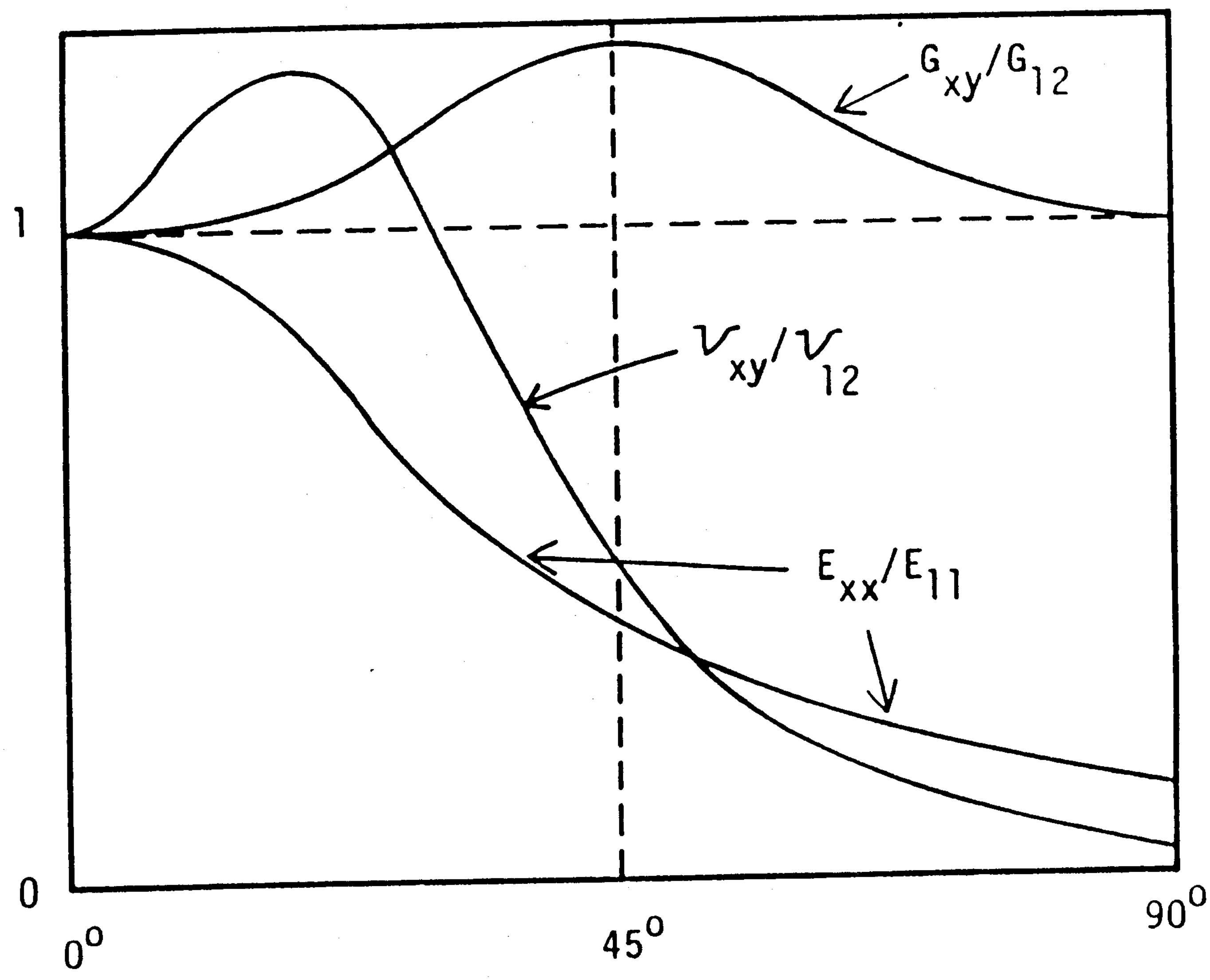
$$\begin{pmatrix} \alpha_{xx} \\ \alpha_{yy} \\ \underline{\underline{\alpha_{xy}}} \end{pmatrix} = \begin{bmatrix} cos^2\theta & sin^2\theta \\ sin^2\theta & cos^2\theta \\ 2sin\theta cos\theta & -sin\theta cos\theta \end{bmatrix} \begin{pmatrix} \alpha_{11} \\ \alpha_{22} \end{pmatrix}$$

this is due to coordinate transf not actual existence of CTE in bidirectional

α_{xx}, α_{yy} COEFF LINEAR EXPANSION [2]

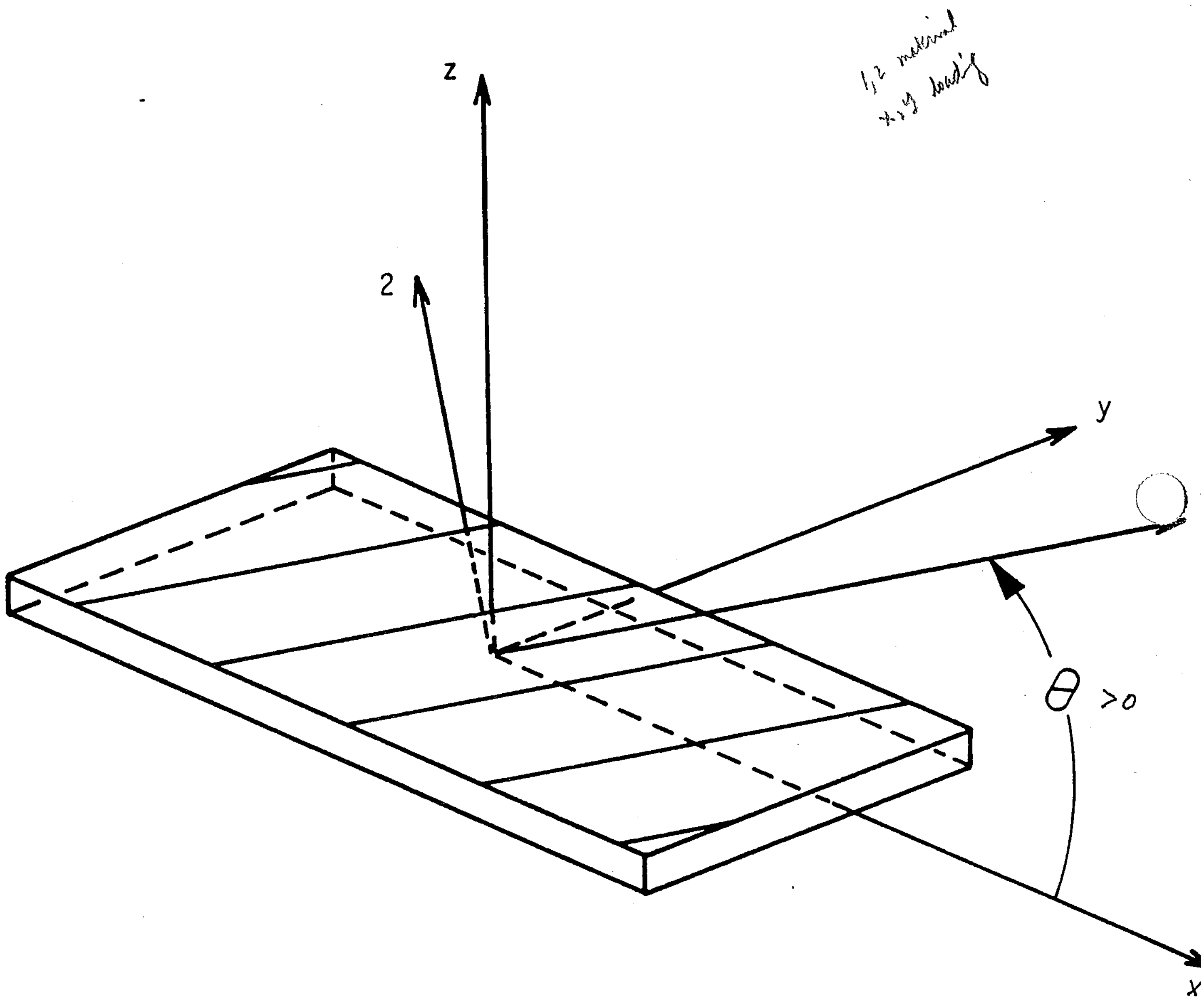
α_{xy} " SHEAR "

FIBER-REINFORCED LAMINA CHARACTERISTICS

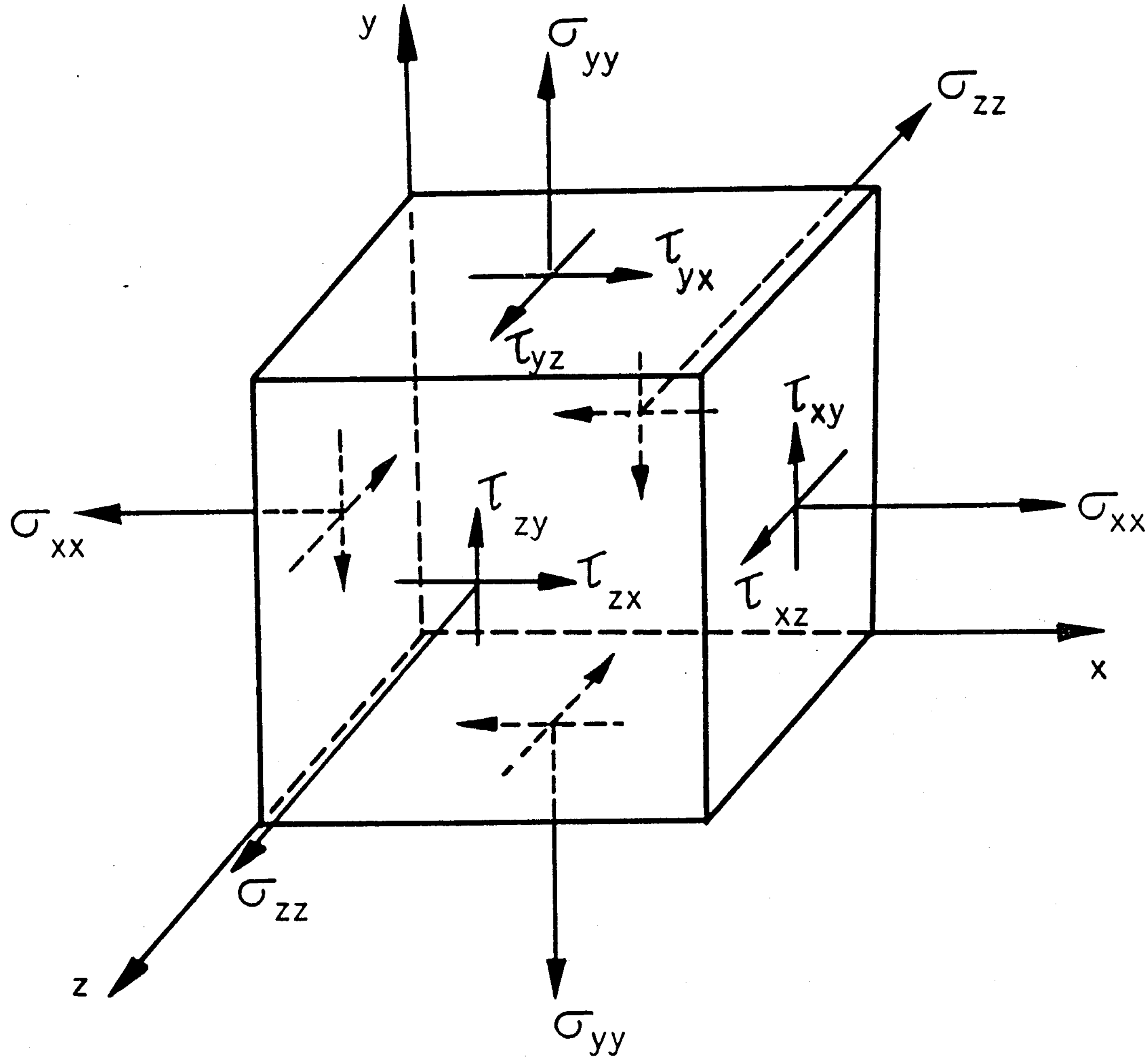


[2]

$\theta = 90^\circ$
$E_{xx} = E_{22}$
$G_{xy} = G_{12}$
$\nu_{yx} = \nu_{12}$
$\nu_{xy} = \nu_{21}$



Definition of principal material axes and loading axes for a lamina [2]



Normal stress and shear stress components [2]

ORTHOTROPIC LAMINA IN PLANE STRESS

$$(\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0)$$

coefficient of mutual influence

shear & extensional coupling

$$\epsilon_{xx} = \sigma_{xx}/E_{xx} - \nu_{yx}\sigma_{yy}/E_{yy} - m_x\tau_{xy}$$

$$\epsilon_{yy} = -\nu_{xy}\sigma_{xx}/E_{xx} + \sigma_{yy}/E_{yy} - m_y\tau_{xy}$$

$$\epsilon_{xy} = -m_x\sigma_{xx} - m_y\sigma_{yy} + \tau_{xy}/G_{xy}$$

ISOTROPIC LAMINA IN PLANE STRESS

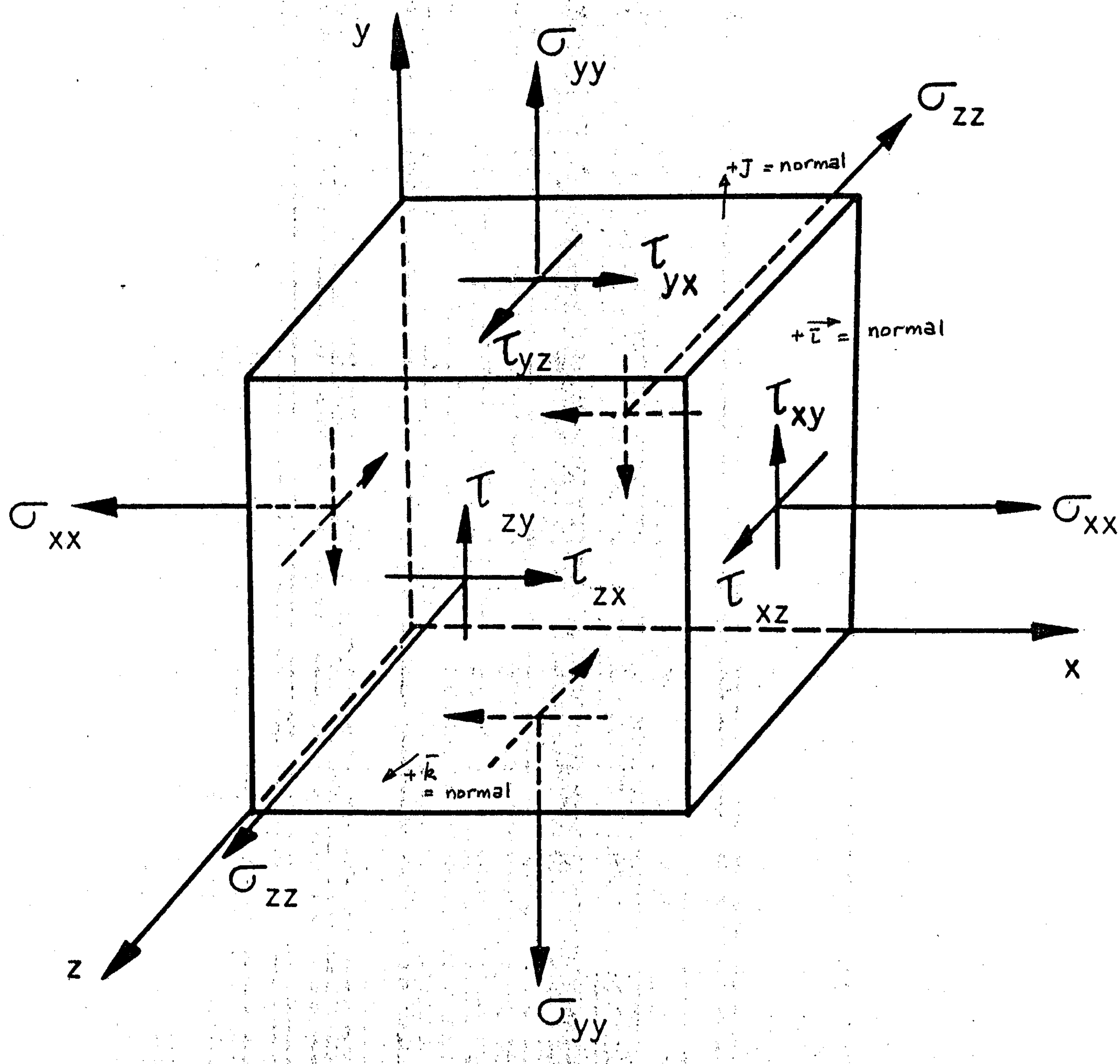
$$(\sigma_{zz} = \tau_{xz} = \tau_{yz} = 0)$$

$$\epsilon_{xx} = \frac{1}{E}\,(\sigma_{xx} - \nu\sigma_{yy})$$

$$\epsilon_{yy} = \frac{1}{E}\,(-\nu\sigma_{xx} + \sigma_{yy})$$

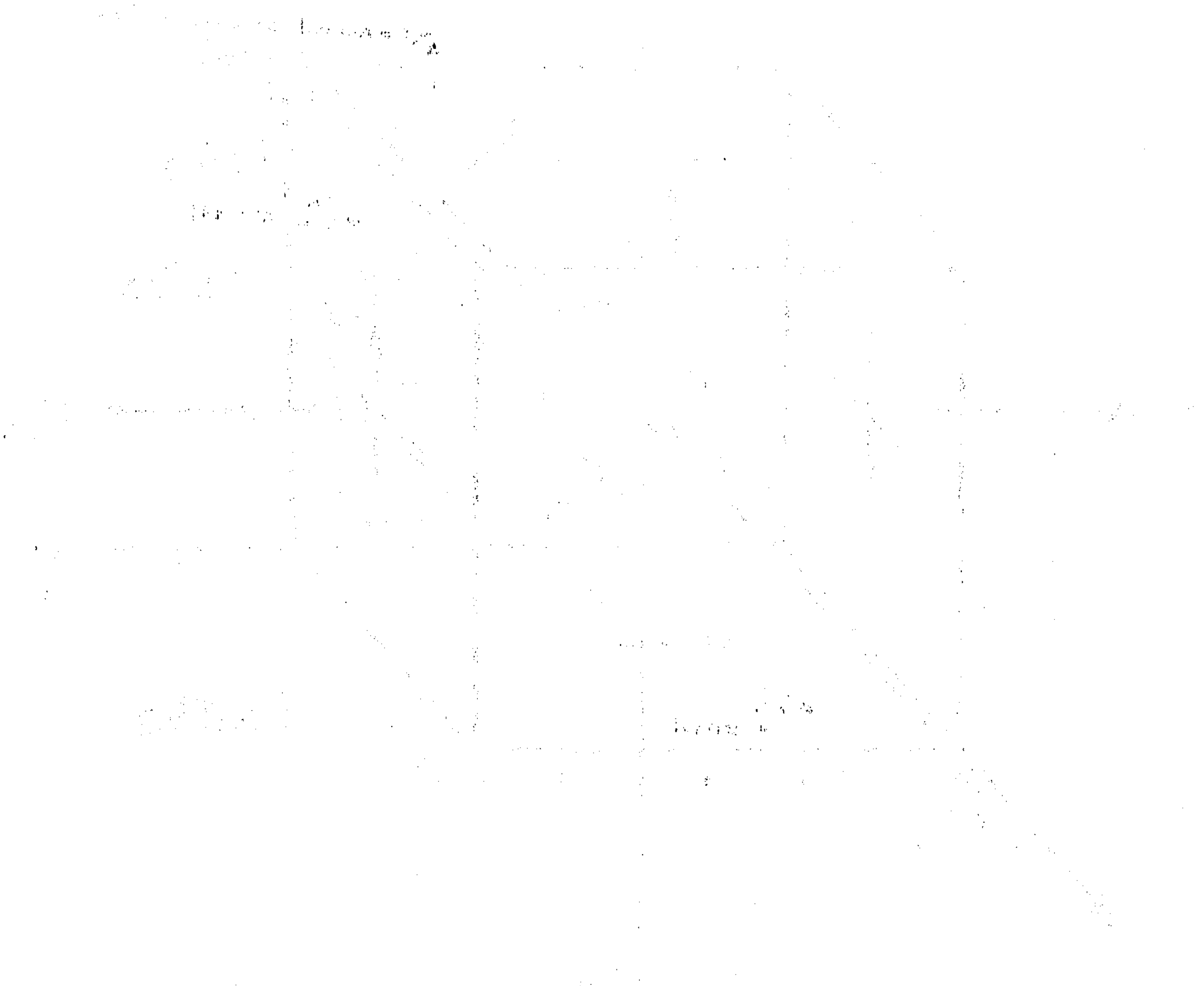
$$\epsilon_{xy} = \frac{1}{G}\,(\tau_{xy})$$

[2]



Normal stress and shear stress components [2]

- IN THE SYSTEM SHOWN ABOVE THE DIRECTION OF THE <u>OUTWARD</u> NORMAL DEFINES THE + DIRECTION, IF POINTED IN + COORDINATE DIRECTION
- THE FACE ON WHICH IT ACTS IS + FACE
- STRESS +, IF ACTS IN + DIRECTION ON + FACE <u>OR</u> IN − DIRECTION ON − FACE



Where

m_x and m_y are Coefficients of Mutual Influence

$$m_x = sin2\theta \left[\frac{\nu_{12}}{E_{11}} + \frac{1}{E_{22}} - \frac{1}{2G_{12}} \right.$$

$$\left. - \; cos^2\theta \left(\frac{1}{E_{11}} + \frac{2\nu_{12}}{E_{11}} + \frac{1}{E_{22}} - \frac{1}{G_{12}} \right) \right]$$

$$m_y = sin2\theta \left[\frac{\nu_{12}}{E_{11}} + \frac{1}{E_{22}} - \frac{1}{2G_{12}} \right.$$

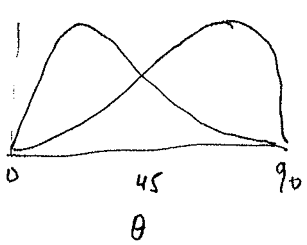
$$\left. - \; sin^2\theta \left(\frac{1}{E_{11}} + \frac{2\nu_{12}}{E_{11}} + \frac{1}{E_{22}} - \frac{1}{G_{12}} \right) \right]$$

[2]

when $\theta = 90^\circ$ $m_x = m_y = 0$ no mutual influence

$\theta = 0$ $m_x = m_y = 0$

specially orthotropic



SPECIALLY ORTHOTROPIC LAMINA

- Extensional and Shear Deformations are Not Coupled

- Principal Stresses are in the Same Direction as the Material Axes

$$\epsilon_{xx} = \epsilon_{11} = \sigma_{xx}/E_{11} - \nu_{21}\sigma_{yy}/E_{22}$$

$$\epsilon_{yy} = \epsilon_{22} = \sigma_{yy}/E_{22} - \nu_{12}\sigma_{xx}/E_{11}$$

$$\epsilon_{xy} = \epsilon_{yx} = \epsilon_{12} = \epsilon_{21} = \tau_{xy}/G_{12}$$

GENERAL ORTHOTROPIC LAMINA

- Extensional and Shear Deformations are Coupled

* - Direction of Principal Stresses and Strains Do Not Coincide [2]

geometry driven: unlike isotropy

SPECIALLY ORTHOTROPIC LAMINA

- Extensional and Shear Deformations are Not Coupled

- Principal Stresses are in the Same Direction as the Material Axes

$$\epsilon_{xx} = \epsilon_{11} = \sigma_{xx}/E_{11} - \nu_{21}\sigma_{yy}/E_{22}$$

$$\epsilon_{yy} = \epsilon_{22} = \sigma_{yy}/E_{22} - \nu_{12}\sigma_{xx}/E_{11}$$

$$\epsilon_{xy} = \epsilon_{yx} = \epsilon_{12} = \epsilon_{21} = \tau_{xy}/G_{12}$$

GENERAL ORTHOTROPIC LAMINA

- Extensional and Shear Deformations are Coupled

* - Direction of Principal Stresses and Strains Do Not Coincide [2]

geometry driven:
unlike isotropy
DEPEND ON $(E_{11}/E_{22}\ ,\ \sigma_{11}/\sigma_{22})$
E_{11}/E_{22} MEASURE OF ORTHOTROPY

STIFFNESS AND COMPLIANCE MATRICES

- Relate Stresses and Strains
- Are Functions of Lamina Properties — depend on E_f, E_m
- [S] is the Compliance Matrix
- [Q] is the Stiffness Matrix

FOR ISOTROPIC LAMINA $[Q] = [S]^{-1}$

SPECIALLY ORTHOTROPIC LAMINA
$\theta = 0^o$ or 90^o

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & S_{66} \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}$$

compliance

$$S_{11} = 1/E_{11}$$

$$S_{12} = S_{21} = -\nu_{12}/E_{11} = -\nu_{21}/E_{22}$$

$$S_{22} = 1/E_{22}$$

$$S_{66} = 1/G_{12}$$

[2]

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{pmatrix}$$

stiffness

$$Q_{11} = \frac{E_{11}}{(1 - \nu_{12}\nu_{21})}$$

$$Q_{12} = Q_{21} = \frac{\nu_{12} E_{22}}{(1 - \nu_{12}\nu_{21})} = \frac{\nu_{21} E_{11}}{(1 - \nu_{12}\nu_{21})}$$

$$Q_{22} = \frac{E_{22}}{(1 - \nu_{12}\nu_{21})}$$

$$Q_{66} = G_{12}$$

[2]

GENERAL ORTHOTROPIC LAMINA

$$\begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{pmatrix} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{16} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{26} \\ \overline{S}_{16} & \overline{S}_{26} & \overline{S}_{66} \end{bmatrix} \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = [\overline{S}] \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix}$$

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \end{pmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{pmatrix} = [\overline{Q}] \begin{pmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{pmatrix}$$

[2]

$$\overline{S}_{11} = 1/E_{xx} = S_{11}cos^4\theta + (2S_{12} + S_{66})sin^2\theta cos^2\theta + S_{22}sin^4\theta$$

$$\overline{S}_{12} = \nu_{xy}/E_{yy} = S_{12}(sin^4\theta + cos^4\theta) + (S_{11} + S_{22} + S_{66})sin^2\theta cos^2\theta$$

$$\overline{S}_{22} = 1/E_{yy} = S_{11}sin^4\theta + (2S_{12} + S_{66})sin^2\theta cos^2\theta + S_{22}cos^4\theta$$

$$\overline{S}_{16} = -m_x = (2S_{11} - 2S_{12} - S_{66})sin\theta cos^3\theta - (2S_{22} - S_{12} - S_{66})sin^3\theta cos\theta$$

$$\overline{S}_{26} = -m_y = (2S_{11} - 2S_{12} - S_{66})sin^3\theta cos\theta - (2S_{22} - S_{12} - S_{66})sin\theta cos^3\theta$$

$$\overline{S}_{66} = 1/G_{xy} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})sin^2\theta cos^2\theta + S_{66}(sin^4\theta + cos^4\theta)$$

[2]

- IT APPEARS THAT THE 4 ELASTIC CONSTANTS DEPENDS ON 6 EQNS E_{11}, E_{22}, G_{12}, ν_{12}
- BUT $\overline{S}_{16}$ & $\overline{S}_{26}$ ARE LINEAR COMBOS OF S_{11}, S_{22}, S_{12}, S_{66} & ARE NOT INDEPENDENT EQNS.

$$\overline{S}_{11} = 1/E_{xx} = S_{11}cos^4\theta + (2S_{12} + S_{66})sin^2\theta cos^2\theta + S_{22}sin^4\theta$$

$\frac{1}{4}(S_{11} + 2S_{12} + S_{66} + S_{22})$

$$\overline{S}_{12} = \nu_{xy}/E_{yy} = S_{12}(sin^4\theta + cos^4\theta) + (S_{11} + S_{22} + S_{66})sin^2\theta cos^2\theta$$

$\frac{1}{4}(2S_{12} + S_{11} + S_{22} + S_{66})$

$$\overline{S}_{22} = 1/E_{yy} = S_{11}sin^4\theta + (2S_{12} + S_{66})sin^2\theta cos^2\theta + S_{22}cos^4\theta$$

$= \frac{1}{4}(S_{11} + 2S_{12} + S_{66} + S_{22})$

$$\overline{S}_{16} = -m_x = (2S_{11} - 2S_{12} - S_{66})sin\theta cos^3\theta - (2S_{22} - S_{12} - S_{66})sin^3\theta cos\theta$$

$$\overline{S}_{26} = -m_y = (2S_{11} - 2S_{12} - S_{66})sin^3\theta cos\theta - (2S_{22} - S_{12} - S_{66})sin\theta cos^3\theta$$

$$\overline{S}_{66} = 1/G_{xy} = 2(2S_{11} + 2S_{22} - 4S_{12} - S_{66})sin^2\theta cos^2\theta + S_{66}(sin^4\theta + cos^4\theta)$$

[2]

$= \frac{2}{4}(2S_{11} + 2S_{22} - 4S_{12})$

$$\overline{Q}_{11} = Q_{11}cos^4\theta + 2(Q_{12} + 2Q_{66})sin^2\theta cos^2\theta + Q_{22}sin^4\theta$$

$$\overline{Q}_{12} = Q_{12}(sin^4\theta + cos^4\theta) + (Q_{11} + Q_{22} - 4Q_{66})sin^2\theta cos^2\theta$$

$$\overline{Q}_{22} = Q_{11}sin^4\theta + 2(Q_{12} + 2Q_{66})sin^2\theta cos^2\theta + Q_{22}cos^4\theta$$

$$\overline{Q}_{16} = (Q_{11} - Q_{12} - 2Q_{66})sin\theta cos^3\theta + (Q_{12} - Q_{22} + 2Q_{66})sin^3\theta cos\theta$$

$$\overline{Q}_{26} = (Q_{11} - Q_{12} - 2Q_{66})sin^3\theta cos\theta + (Q_{12} - Q_{22} + 2Q_{66})sin\theta cos^3\theta$$

$$\overline{Q}_{66} = (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})sin^2\theta cos^2\theta + Q_{66}(sin^4\theta + cos^4\theta)$$

$\overline{Q}_{16}$ $\overline{Q}_{26}$ NOT INDEP

[2]

Using trigonometric identities, Tsai and Pagano have shown.

$$\overline{Q}_{11} = U_1 + U_2 cos2\theta + U_3 cos4\theta$$

$$\overline{Q}_{12} = \overline{Q}_{21} = U_4 - U_3 cos4\theta$$

$$\overline{Q}_{22} = U_1 - U_2 cos2\theta + U_3 cos4\theta$$

$$\overline{Q}_{16} = \frac{1}{2}\ U_2 sin2\theta + U_3 sin4\theta$$

$$\overline{Q}_{26} = \frac{1}{2}\ U_2 sin2\theta - U_3 sin4\theta$$

$$\overline{Q}_{66} = U_5 - U_3 cos4\theta$$ [2]

U_1 through U_5 represent angle-invariant stiffness properties of a lamina.

$$U_1 = \frac{1}{8}\ (3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66})$$

$$U_2 = \frac{1}{2}\ (Q_{11} - Q_{22})$$

$$U_3 = \frac{1}{8}\ (Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66})$$

$$U_4 = \frac{1}{8}\ (Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66})$$

$$U_5 = \frac{1}{2}\ (U_1 - U_4)$$

[2]

U_1 through U_5 represent angle-invariant stiffness properties of a lamina.

$$U_1 = \frac{1}{8}\left(3Q_{11} + 3Q_{22} + 2Q_{12} + \underset{\sim}{4Q_{66}}\right)$$

$$U_2 = \frac{1}{2}\left(Q_{11} - Q_{22}\right)$$

$$U_3 = \frac{1}{8}\left(Q_{11} + Q_{22} - 2Q_{12} - \underset{\sim}{4Q_{66}}\right)$$

$$U_4 = \frac{1}{8}\left(Q_{11} + Q_{22} + 6Q_{12} - \underset{\sim}{4Q_{66}}\right)$$

$$U_5 = \frac{1}{2}\left(U_1 - U_4\right)$$

[2]

$\bar{S}$ CAN ALSO BE EXPRESSED IN TERMS OF ANGLE INVARIANT QUANTITIES $V_1 - V_5$, WHOSE FORMS ARE SAME AS ABOVE EXCEPT FOR UNDERLINED TERMS

ie.

$$V_1 = \frac{1}{8}\left(3S_{11} + 3S_{22} + 2S_{12} + S_{66}\right)$$

PG 126 MALLICK

DISCONTINUOUS PARALLEL FIBERS

UNIDIRECTIONAL DISCONTINUOUS FIBER LAMINA

- Assumptions
  - Fiber has Circular Cross-section
  - Fibers are Arranged in a Square Array
  - Fibers are Uniformly Distributed
  - Matrix is Free of Voids

The tensile load is transfered to the fibers by a shearing mechanism between the fibers and the matrix. $\Big\}\ \sigma_m = \sigma_f$

$$E_m < E_f$$

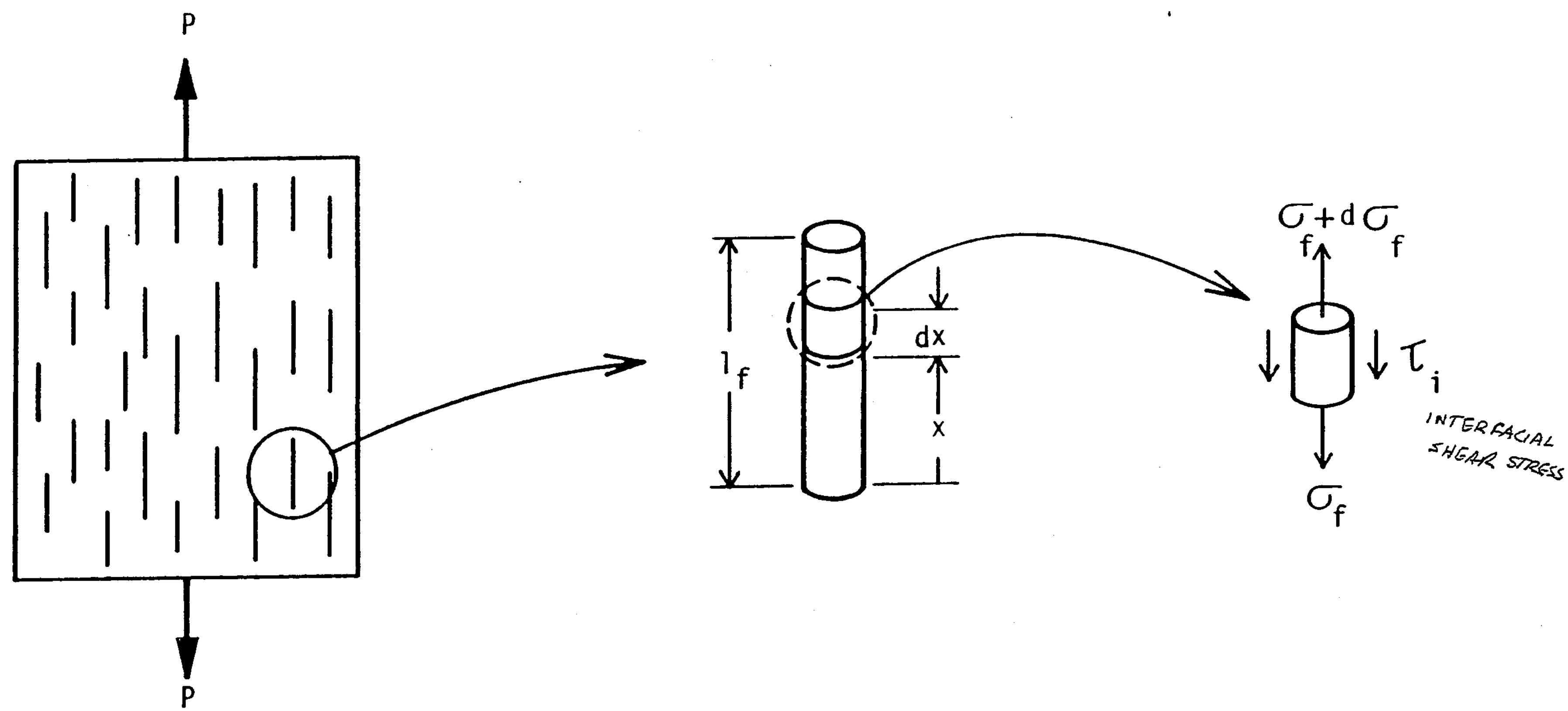
Therefore, the longitudinal strain in the matrix is higher than the longitudinal strain in the adjacent fibers.

Assuming

- Perfecting Bonding Between the Fiber and the Matrix

The difference in the longitudinal strains creates a shear stress distribution across the fiber/matrix interface.

[2]



Longitudinal tensile loading of a discontinuous parallel fiber lamina [2]

Neglecting

- The Stress Transfer at the Fiber Ends
- The Interaction Between Neighboring Fibers

Considering

- A Length dx at a distance x From the Free End

The force equilibrium equation is

$$\underbrace{(\pi/4)d_f^2}_{\text{cross-sectional area}}(\sigma_f + d\sigma_f) - (\pi/4)d_f^2\sigma_f - \underbrace{\pi d_f dx}_{\text{surface area}}\tau = 0$$

Simplifying

$$\frac{d\sigma_f}{dx} = \frac{4\tau}{d_f}$$

Where

σ_f = The Longitudinal Stress in the Fiber at a Distance x From One of its Ends

τ = The Shear Stress at the Fiber/Matrix Interface

d_f = The Fiber Diameter

Setting $\sigma_f = 0$ at $x = 0$ and integrating

if transfer add σ_o here

$$\sigma_f = \frac{4}{d_f} \int_0^x \tau \, dx$$

Assuming that the interfacial shear is constant

$$\sigma_f = \frac{4\tau_i}{d_f} x$$

[2]

DISCONTINUOUS FIBER LAMINA

- Fiber Stress
  - is Zero at the Ends of the Fiber
  - Builds to a Maximum Value at the Midpoint of the Fiber

CRITICAL FIBER LENGTH

WHEN FIBER BREAKS

The minimum fiber length required for the fiber stress to be equal to the fiber ultimate strength at its midlength is the critical fiber length.

$$l_c = \frac{\sigma_{fu}}{2\tau_i} d_f$$

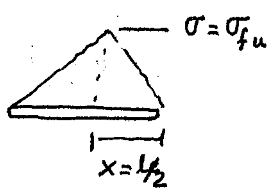
For Effective Fiber Reinforcement

$$l_f >> l_c$$

For a given fiber, the l_c can be controlled by increasing/decreasing τ_i. τ_i may be increased by using coupling agents.

CRITICAL FIBER LENGTH

The minimum fiber length required for the fiber stress to be equal to the fiber ultimate strength at its midlength is the critical fiber length.



$$l_c = \frac{\sigma_{fu}}{2\tau_i} d_f$$

For Effective Fiber Reinforcement

$$l_f >> l_c$$

$\sigma_f \approx \sigma_{fu}$ OVER MOST OF FIBER

USE FIBER TO ITS POTENTIAL TO CARRY LOAD

For a given fiber, the l_c can be controlled by increasing/decreasing τ_i. τ_i may be increased by using coupling agents.

IF $\tau_i \uparrow$ $l_c \downarrow$ - CAN ACHIEVE FIBER CARRYING POTENTIAL W/O CHANGING FIBER LENGTH

For $l_f < l_c$

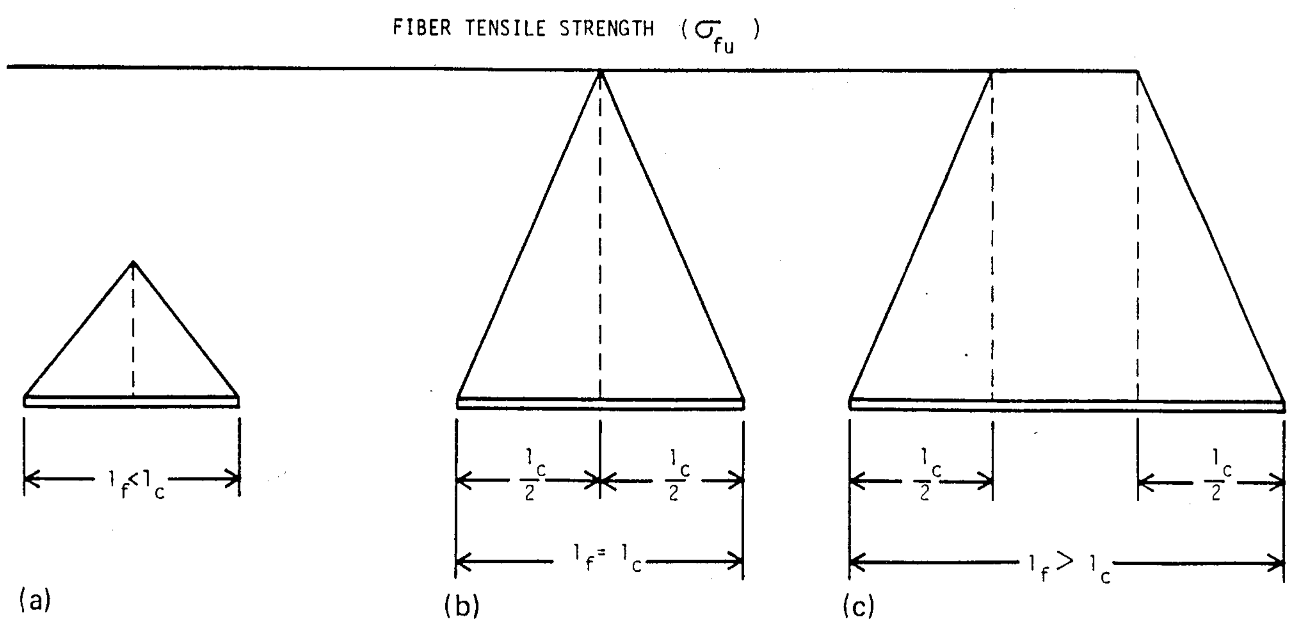
- There Will be no Fiber Failure

- Two Failure Modes are Possible

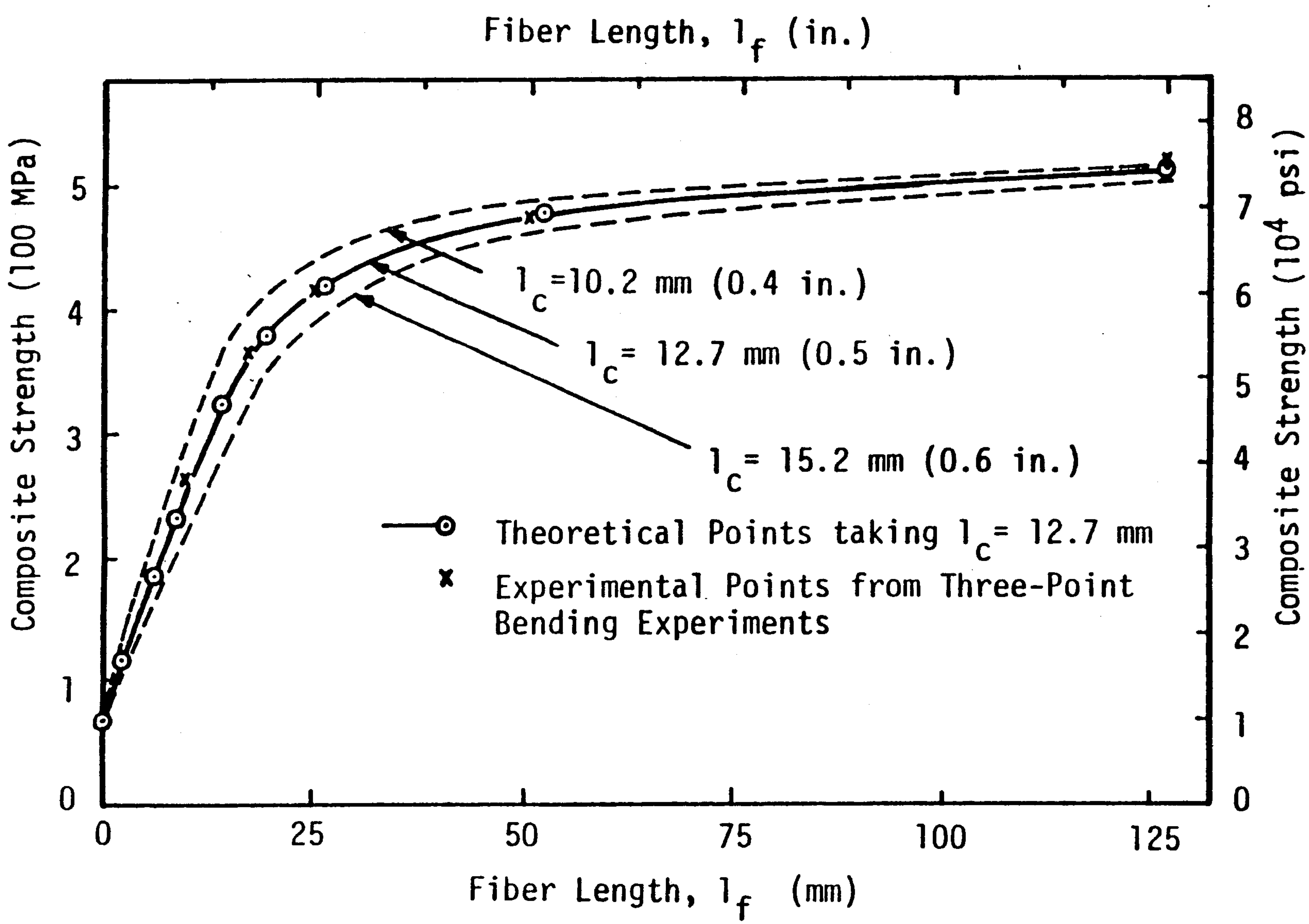
  - Interfacial Bond Failure

  - Matrix Tensile Failure

  - The Actual Failure Mode Depends on the relative values of τ_m and σ_{mu}



Significance of critical fiber length on the longitudinal stresses of a discontinuous fiber. [2]



Variation in the longitudinal strength of a discontinuous parallel fiber composite as a function of fiber length [9]

For $l_f < l_c$

- There Will be no Fiber Failure $\sigma_f < \sigma_{fu}$

- Two Failure Modes are Possible $\sigma_{Ltu} = \frac{\sigma_{max}}{2} v_f + \sigma_m (1 - v_f)$

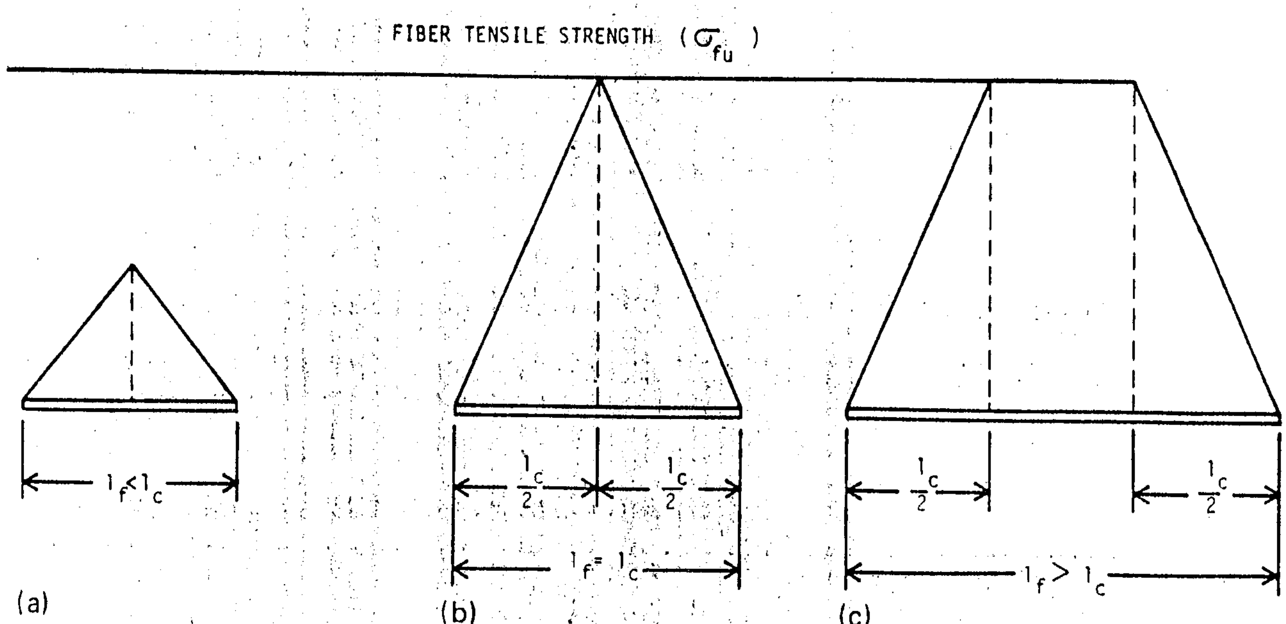
  - Interfacial Bond Failure (FIBER PULLOUT)

    $\sigma_{Ltu} = \tau_m \frac{l_f}{d_f} v_f + \sigma_m' (1 - v_f)$

  - Matrix Tensile Failure

    $\sigma_{Ltu} = \tau_i \frac{l_f}{d_f} v_f + \sigma_{mu} (1 - v_f)$

  - The Actual Failure Mode Depends on the relative values of τ_m and σ_{mu}



l_t = LOAD TRANSFER LENGTH

l_c = LOAD TRANSFER LENGTH

Significance of critical fiber length on the longitudinal stresses of a discontinuous fiber. [2]

ACTUAL LOADING SEE MALLICK PG 85

INTERFACIAL BOND FAILURE

$$\sigma_{Ltu} = \frac{\tau_m l_f v_f}{d_f} + \sigma'_m(1 - v_f)$$

τ_m = Fiber/Matrix Interfacial Shear Strength

σ'_m = Matrix Stress at the Instant of Fiber Pullout

MATRIX TENSILE FAILURE

$$\sigma_{Ltu} = \frac{\tau_i l_f v_f}{d_f} + \sigma_{mu}(1 - v_f)$$

[2]

UNIDIRECTIONAL DISCONTINUOUS FIBER LAMINA
$\theta = 0^o$

- Longitudinal Modulus

fiber aspect ratio

$$E_{11} = \frac{1+2(l_f/df_f)n_L v_f}{(1-n_L v_f)} E_m$$

- Transverse Modulus

$$E_{22} = \frac{1+2n_T v_f}{(1-n_T v_f)} E_m$$

- Shear Modulus

$$G_{12} = G_{21} = \frac{1+n_G v_f}{(1-n_G v_f)} G_m$$

- Major Poisson's Ratio

$$\nu_{12} = \nu_f v_f + \nu_m v_m$$

Where

$$n_L = \frac{(E_f/E_m)-1}{(E_f/E_m)+2(l_f/d_f)}$$

$$n_T = \frac{(E_f/E_m)-1}{(E_f/E_m)+2}$$

$$n_G = \frac{(G_f/G_m)-1}{(G_f/G_m)+1}$$

[2]

RANDOMLY ORIENTED DISCONTINUOUS FIBER LAMINA

- Lamina is Assumed Thin

$$E_{random} = (3/8)E_{11} + (5/8)E_{22}$$

$$G_{random} = (1/8)E_{11} + (1/4)E_{22}$$

$$v_{random} = \frac{E_{random}}{2G_{random}} - 1$$

[2]

MICROFAILURE MODES

- Partial/Total Debonding of the Broken Fiber

- Microcrack Initiation

- Microyielding

- Failure of Neighboring Fibers

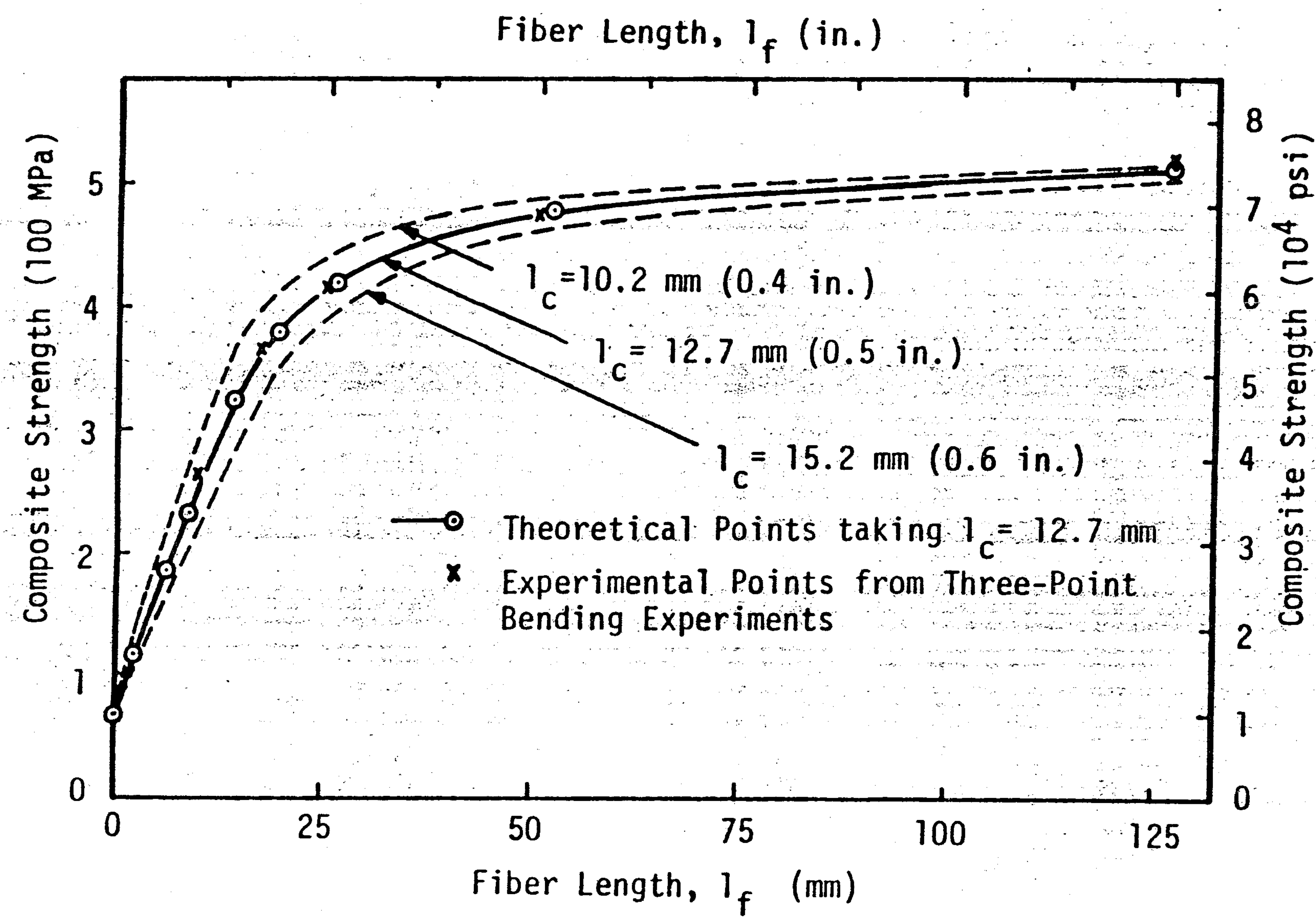
[2]

MICROFAILURE

- When a Fiber Breaks
  - The Normal Stress at each Broken End is Zero
  - Stress Concentrations at the Void are Created by the Broken Fiber
  - High Shear Stress Concentrations Occur in the Matrix Near the Fiber Ends
  - There is an Increase in the Average Normal Stress in the Adjacent Fibers

[2]

NOTE THAT FOR $l_f > 5 l_c$ 90% STRENGTH ACHIEVED

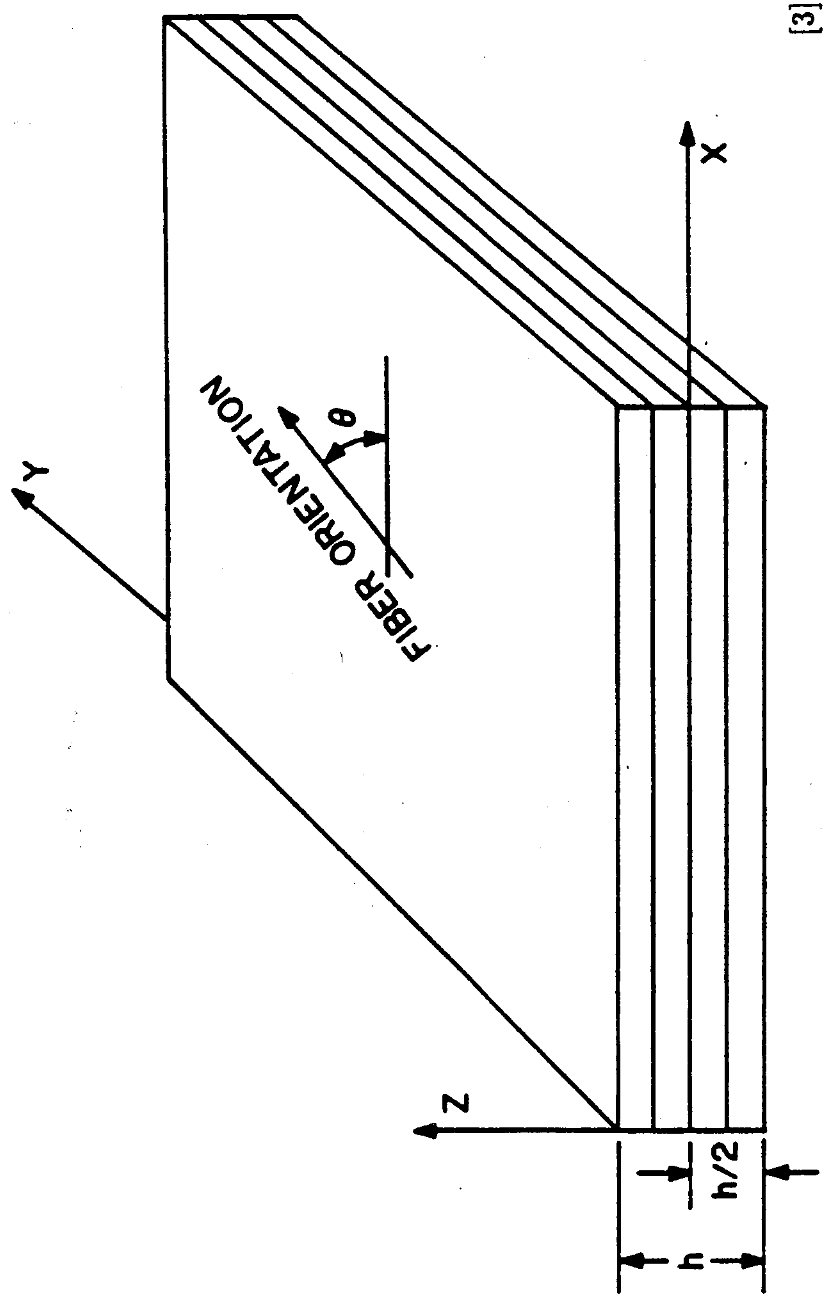


Variation in the longitudinal strength of a discontinuous parallel fiber composite as a function of fiber length [9]

References: 2, 9

LAMINATE ANALYSIS

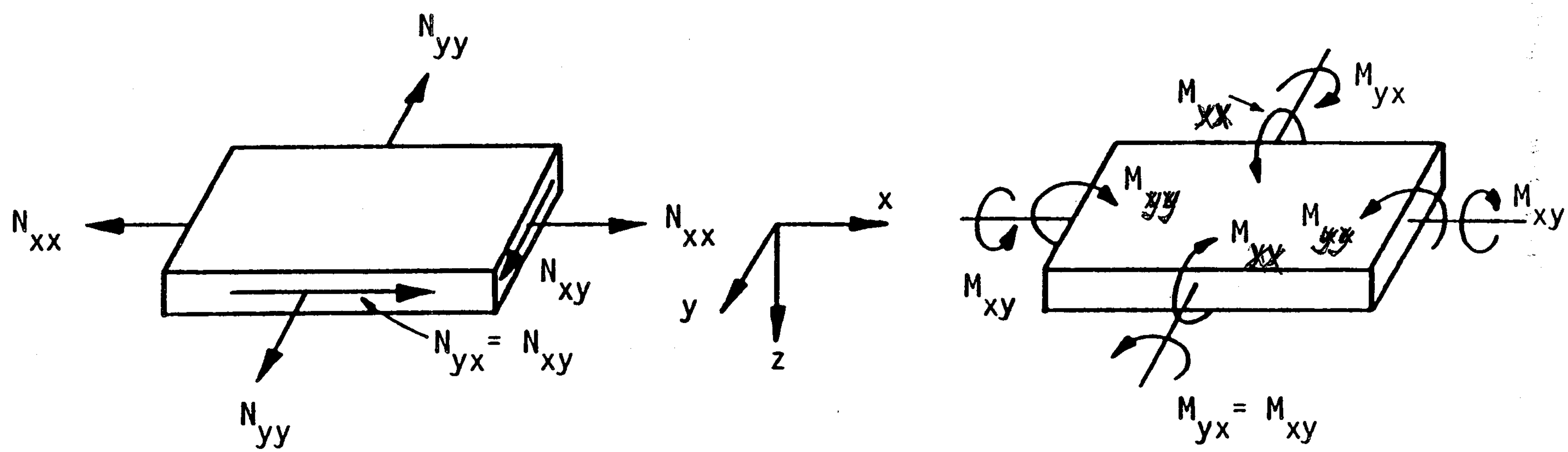
MANY PLIES = LAMINATE



[3]

WIDTH >> THICK
PERFECT BOND
ϵ, γ LIN. IN z
LAMINAS MACROSCOPICALLY HOMOG
BEHAVE ELASTICALLY

F & M ON LAMINATE



UNIT WIDTH

$$M_{xx} = \int_{-h/2}^{h/2} \sigma_{xx} \cdot z\, dz \cdot 1 \qquad [2]$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy} \cdot z \cdot dz \cdot 1$$

N_{xx} = Normal Force Resultant
in the x-direction

N_{yy} = Normal Force Resultant
in the y-direction

N_{xy} = Shear Force Resultant

M_{xx} = Bending Moment Resultant
in the yz-plane

M_{yy} = Bending Moment Resultant
in the xz-plane

M_{xy} = Twisting Moment Resultant

FORCE RESULTANTS

$$N_x = \int_{-h/2}^{h/2} \sigma_x^k \, dz$$

unit width.

$k^{\underline{th}}$ ply

$$N_y = \int_{-h/2}^{h/2} \sigma_y^k \, dz$$

$$N_{xy} = \int_{-h/2}^{h/2} \tau_{xy}^k \, dz$$

MOMENT RESULTANTS

$$M_x = \int_{-h/2}^{h/2} \sigma_x^k \, z \, dz$$

$$M_y = \int_{-h/2}^{h/2} \sigma_y^k \, z \, dz$$

$$M_{xy} = \int_{-h/2}^{h/2} \tau_{xy}^k \, z \, dz$$

[3]

LAMINATION THEORY ASSUMPTIONS

- Laminate is Thin and Wide
  plane stress. $\tau_{xz}, \tau_{yz}, \sigma_{zz}$ are 0.
  drop bc. effects

- Interlaminar Bond Between Laminae is Perfect

- Strain Distribution Through the Thickness Direction is Linear
  1st order plate theory.

- All Laminas are Macroscopically Homogeneous and Linearly Elastic Manner
  use E_L, ν_L

- Plate has Constant Thickness

- Transverse Normal and Shear Strains are Negligible
  $\epsilon_{zz}, \gamma_{zx}, \gamma_{zy} = 0$

- Small Displacements and Strains

[2] [3]

LAMINATE STRAINS

$$\epsilon_{xx} = \epsilon^{o}_{xx} + zk_{xx}$$

$$\epsilon_{yy} = \epsilon^{o}_{yy} + zk_{yy}$$

$$\epsilon_{xy} = \epsilon^{o}_{xy} + zk_{xy}$$

Where

$\epsilon^{o}_{xx}, \epsilon^{o}_{yy} =$ The Mid-plane Normal Strains
$\epsilon^{o}_{xy} =$ The Mid-plane Shear Strain
$k_{xx}, k_{yy} =$ Bending Curvatures
$k_{xy} =$ Twisting Curvature
$z =$ Distance From the Mid-plane in the Thickness Direction

[2]

$[\sigma] = [Q]\,[\epsilon]$

PLY STRESSES

$$\sigma_x^k = Q_{11}^k(\varepsilon_x^0 + z K_x) + Q_{12}^k(\varepsilon_y^0 + z K_y) + Q_{16}^k(\gamma_{xy}^0 + z K_{xy})$$

$$\sigma_y^k = Q_{12}^k(\varepsilon_x^0 + z K_x) + Q_{22}^k(\varepsilon_y^0 + z K_y) + Q_{26}^k(\gamma_{xy}^0 + z K_{xy})$$

$$\tau_{xy}^k = Q_{16}^k(\varepsilon_x^0 + z K_x) + Q_{26}^k(\varepsilon_y^0 + z K_y) + Q_{66}^k(\gamma_{xy}^0 + z K_{xy})$$

[3]

LAMINATE FORCES AND MOMENTS

$$N_x = \int Q_{11}(\epsilon_x^o + z k_{xx})dz + \int Q_{12}(\epsilon_y^o + z k_{yy})dz + \int Q_{16}(\gamma_{xy}^o + z k_{xy})dx + \cdots$$

$$M_x = \int Q_{11}(\epsilon_x^o + z k_{xx})z\,dz + \int Q_{12}(\epsilon_y^o + z k_{yy})z\,dz + \cdots$$

$$\begin{pmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{pmatrix} \epsilon_{xx}^o \\ \epsilon_{yy}^o \\ \epsilon_{xy}^o \\ k_{xx} \\ k_{yy} \\ k_{xy} \end{pmatrix}$$

$[A]$, $[B]$,and$[D]$ are functions of the elastic properties of each lamina and its location with respect to the midplane of the laminate.

$[A]$ = The Extensional Stiffness Matrix (lb/in)

$$[A] = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}$$

$[B]$ = The Coupling Stiffness Matrix (lb)

extension - bending coupling

FOR SYMMETRIC MATRIX
$[B] = [0]$

$$[B] = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix}$$

$[D]$ = The Bending Stiffness Matrix (lb-in)

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$$

[2]

EVALUATION OF ELEMENT STIFFNESS MATRICES

$$A_{mn} = \sum_{j=1}^{N} (\overline{Q}_{mn})_j (h_j - h_{j-1})$$

$$B_{mn} = 1/2 \sum_{j=1}^{N} (\overline{Q}_{mn})_j (h_j^2 - h_{j-1}^2)$$

$$D_{mn} = 1/3 \sum_{j=1}^{N} (\overline{Q}_{mn})_j (h_j^3 - h_{j-1}^3)$$

transformed values dependent on the direction of the plys.

N = The Total Number of Laminae

$(\overline{Q}_{mn})_j$ = The Element in the $[\overline{Q}]$ matrix of the jth Lamina

h_{j-1} = The Distance from the Midplane to the Top of the jth Lamina

h_j = The Distance from the Midplane to the Bottom of the jth Lamina

[2]

$\overline{Q}$ const in each ply in dir. of measurements

determine $\overline{Q}$s in same dir then take $\Sigma \overline{Q}(h_j - h_{j-1})$

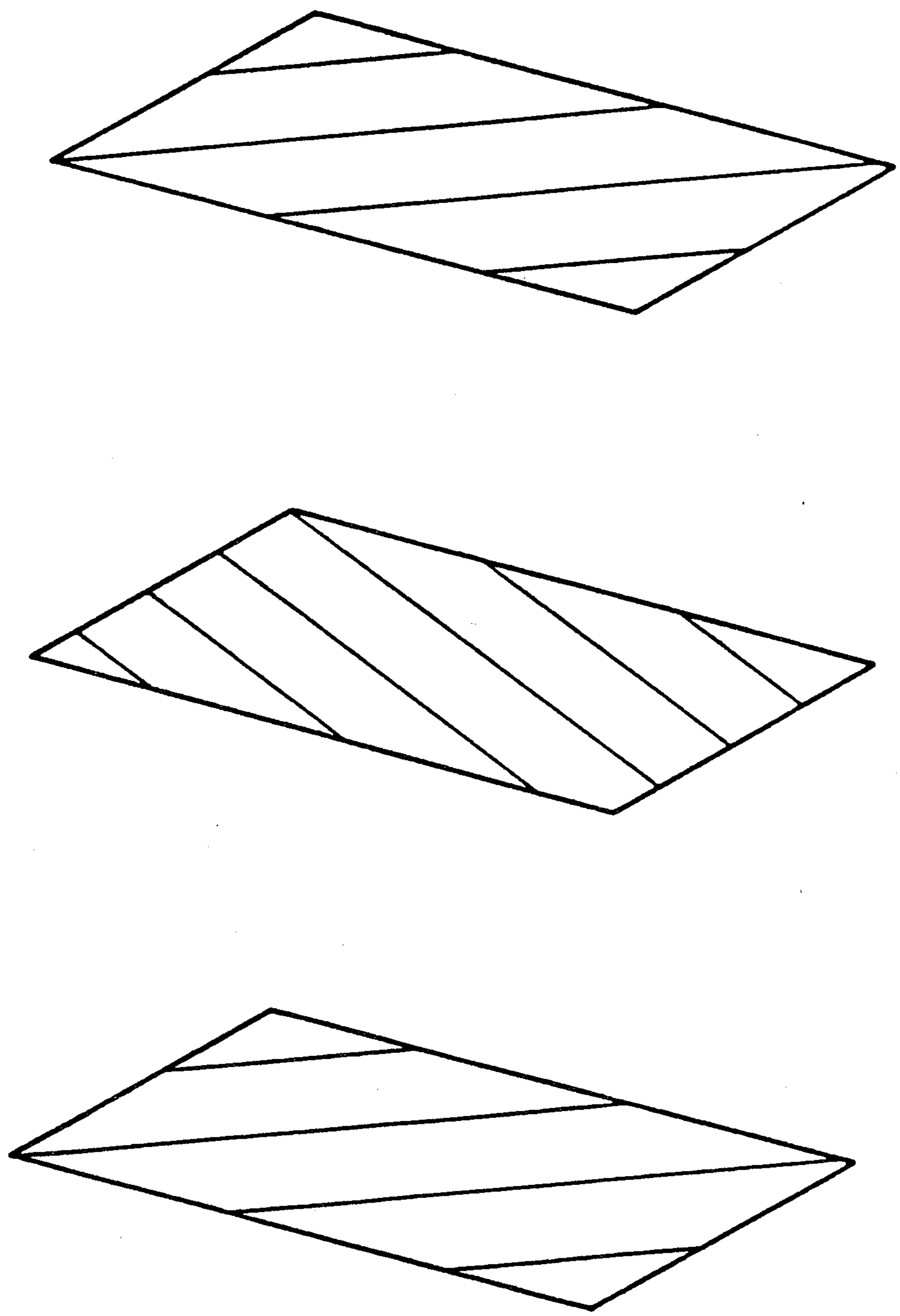
LAMINATE GEOMETRY/DEFINITIONS

- Laminate is Constructed by Stacking a Number of Laminae in the Thickness (z) Direction

- Unidirectional Laminates
  - Fiber Direction is the Same in All Laminae

- Angle-ply Laminates
  - Fiber Orientation Angles in Alternate Layers are $.../\theta/-\theta/\theta/-\theta/...$ Where $\theta \neq 0^o \text{or} 90^o$

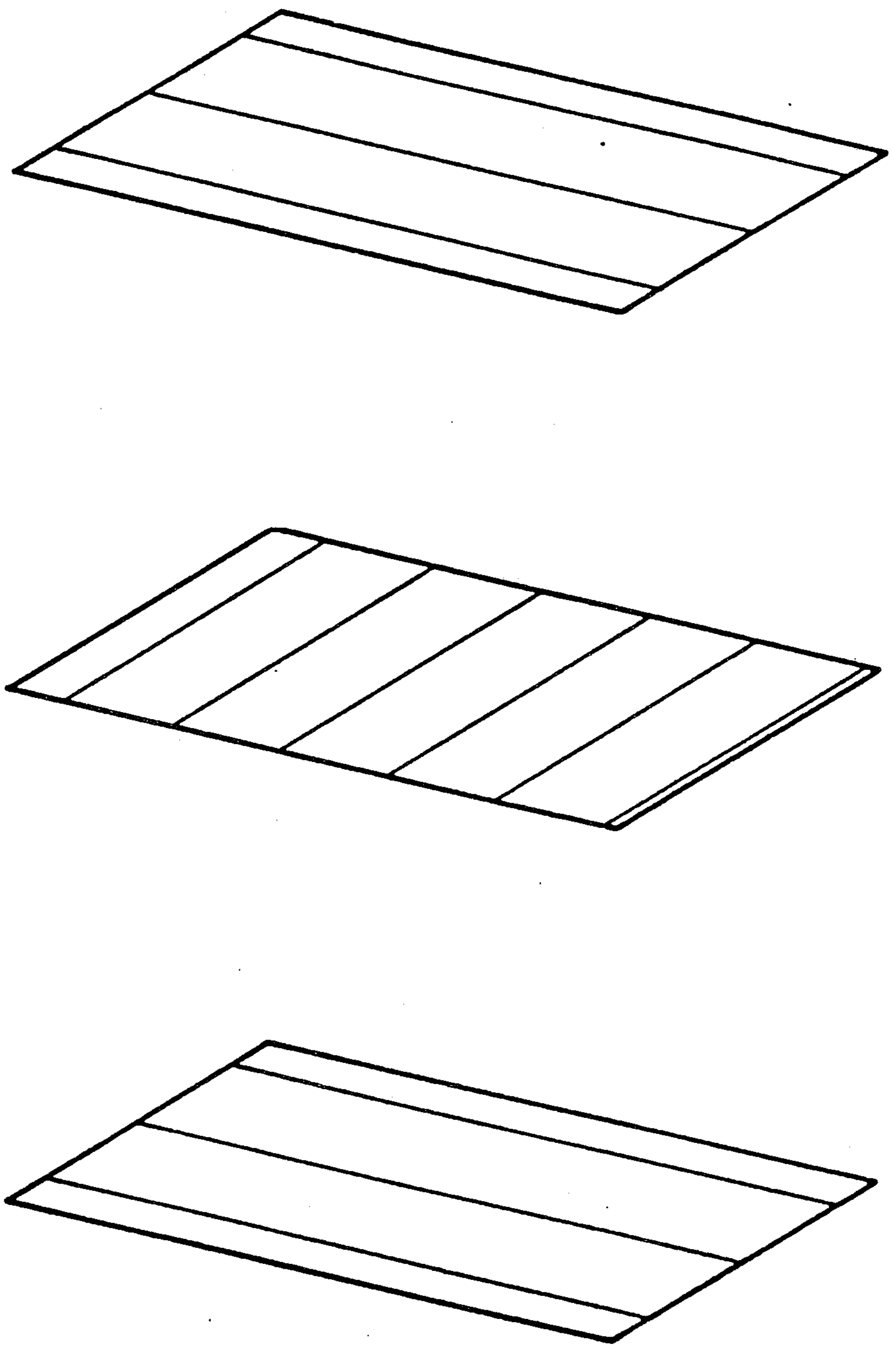
[2]



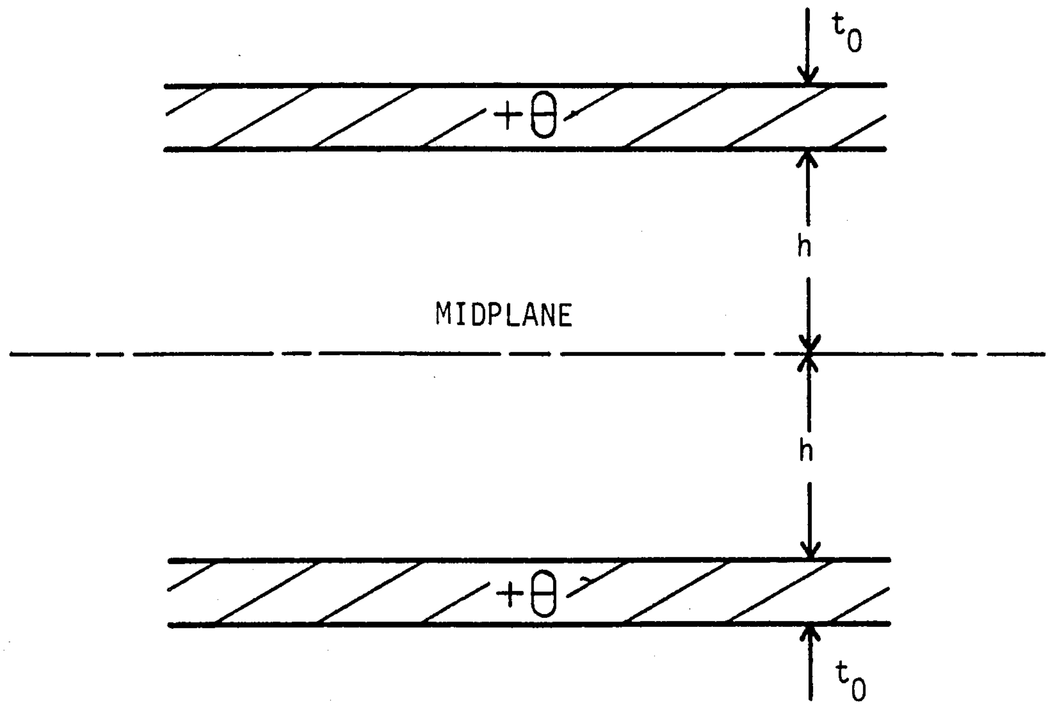
Angle-ply laminate [2]

- Cross-ply Laminates
  - Fiber Orientation Angles in Alternate Layers are $.../0^o/90^o/0^o/90^o/...$

- Symmetric Laminates
  - The Ply Orientation is Symmetrical About the Centerline of the Laminate
  - For Each Ply Above the Midplane, There is an Identical Ply at an Equal Distance Below the Midplane
  - The $[B] = 0$ [2]



Cross-ply laminate [2]



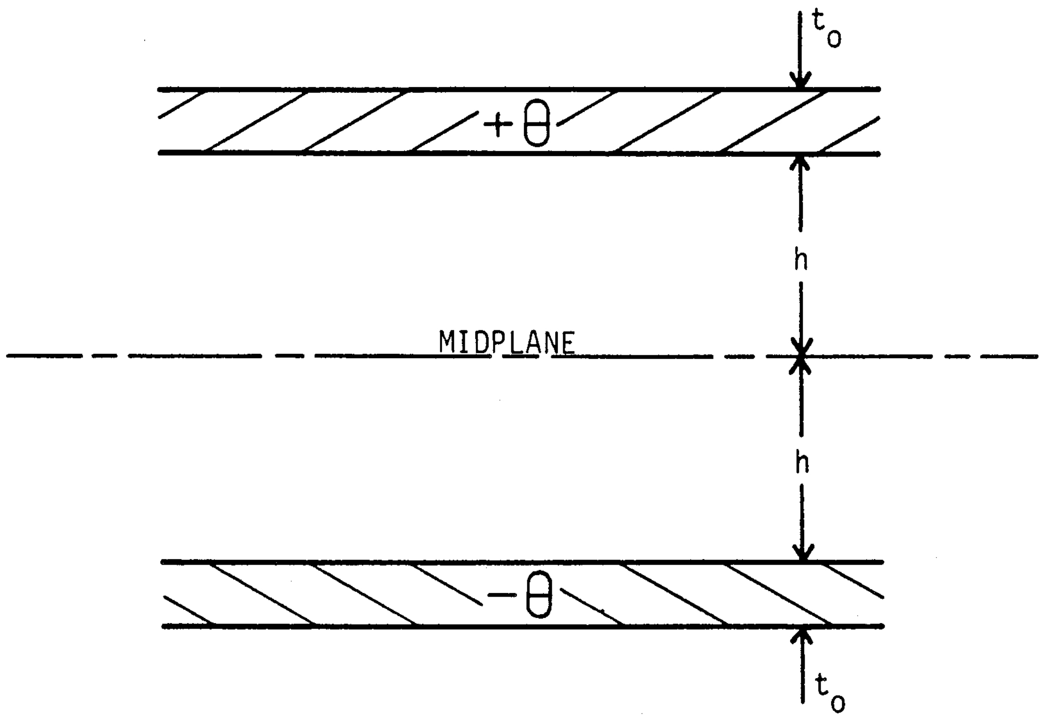
Symmetric laminate configuration [2]

balanced laminate if $+\theta, -\theta$

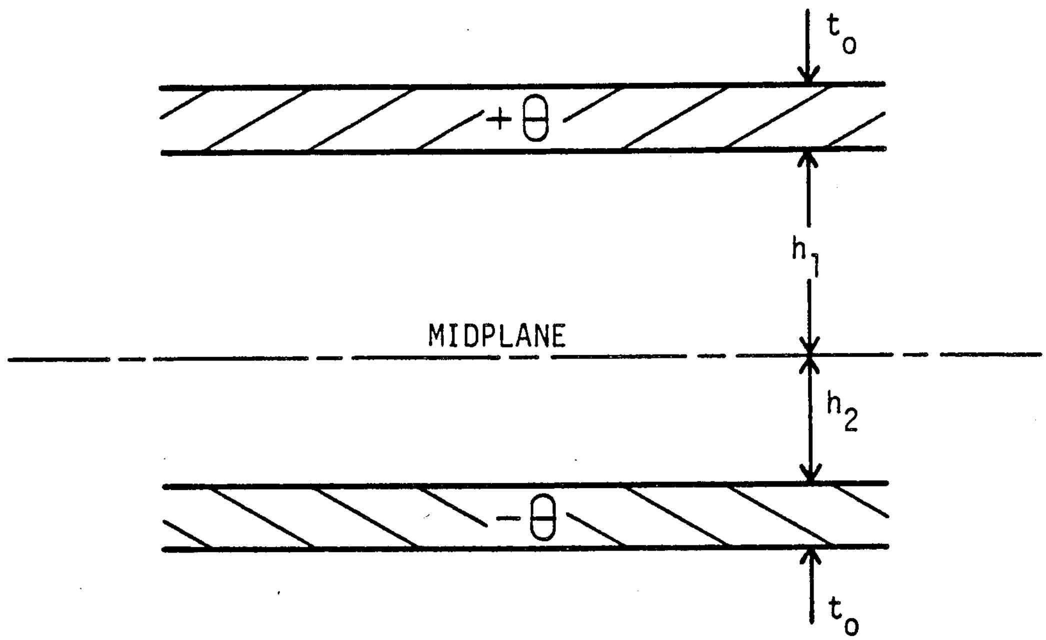
- Quasi-isotropic Laminate
  - Laminate Made of Three or More
    Laminae of Identical
    Thickness and Material
    with Equal Angles Between
    Each Adjacent Lamina

- Balanced Laminate
  - Every $+\theta$ above the midplane can be matched with a $-\theta$ below the midplane
  - A_{16} and $A_{26} = 0$, there is no normal stress-shear strain coupling
  - A Laminate can be balanced symmetric

[2]



Laminate configuration for which $D_{16} = D_{26} = 0$ [2]



Balanced laminate configuration [2]

$+\theta, -\theta$ implies balanced only

~~when $h_1 = h_2$ it becomes balanced symmetric~~

$A_{16} = A_{26} = 0$: no shear extension coupling in the plane of the laminate

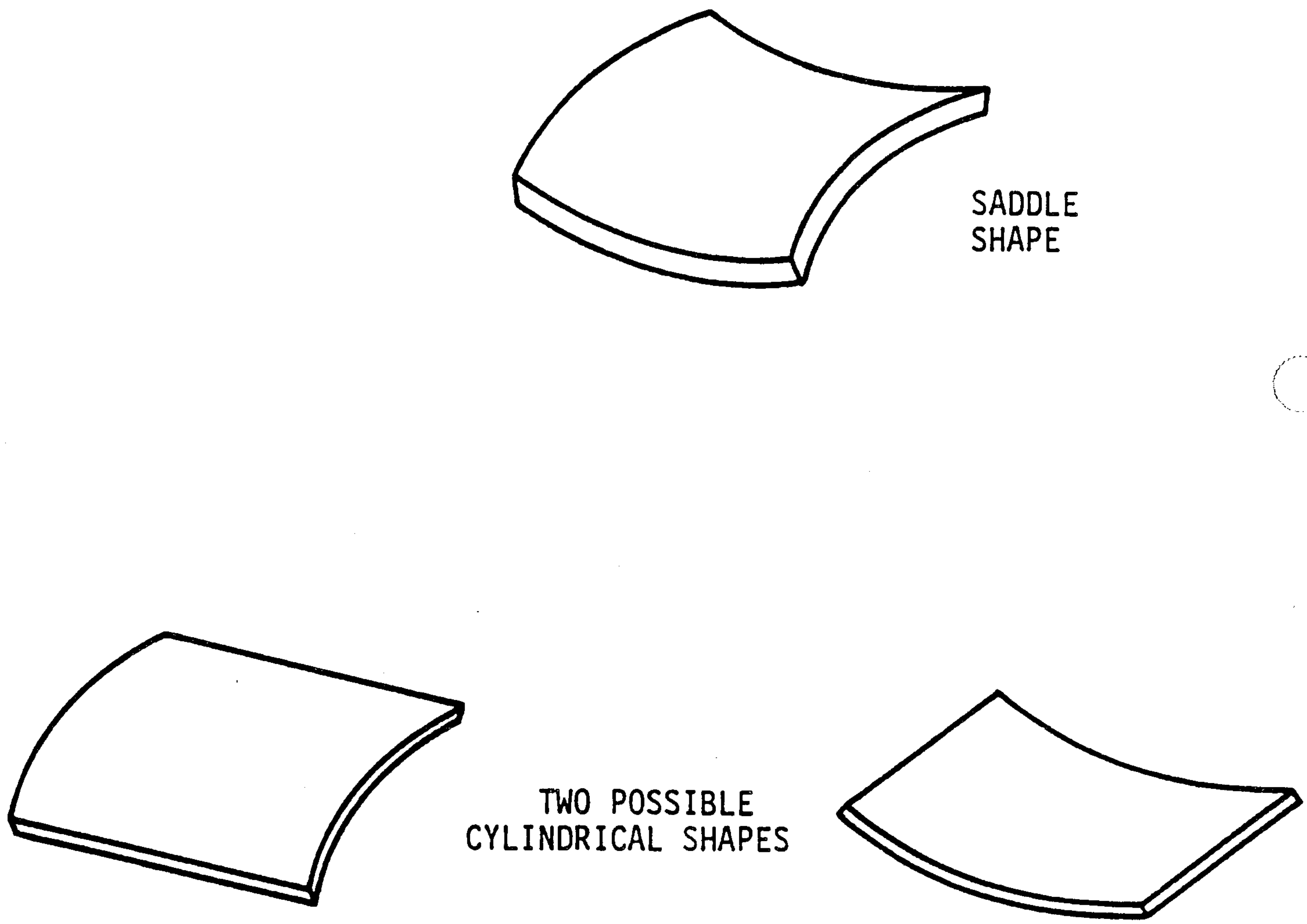
$D_{16} \neq 0$ $D_{26} \neq 0$ bending & twisting coupling

when $h_1 = h_2$ $D_{16} = D_{26} = 0$ no midplane symmetry (antisymmetric)

[B] may be nonzero.

- Unsymmetric Laminate
  - Laminate is Not Symmetric
    About the Midplane
  - Will form cylindrical or saddle shapes
  - $[B] \neq 0$, a Normal Force such as N_{xx} or a Bending such as M_{xy} will create both bending-twisting curvatures and extension-shear deformation.

- If for every $+\theta$ above the midplane there is an identical lamina of $-\theta$ at the same distance below the midplane $D_{16} = D_{26} = 0$. There is no bending moment-twisting curvature coupling. [2]



Possible configurations for unsymmetric laminates [2]

LAMINATION THEORY

- Step-by-Step Procedure
  - Evaluate Stiffness Matrices for the Laminate
  - Evaluate Midplane Strains and Curvatures for the Laminate Due to a Given Set of Applied Forces and Moments
  - Evaluate In-plane Strains for Each Lamina
  - Evaluate In-plane Stresses for Each Lamina [2]

LAMINATION CODE

Symmetric Laminates

$$[0^\circ/90^\circ_2/0^\circ] = [0^\circ/90^\circ]_s$$

Angle-Ply Laminates

$$[0^\circ/45^\circ/-45^\circ]_s = [0^\circ/\pm 45^\circ]_s$$

Repeating Sets of Plies

$$[0^\circ/\pm 45^\circ/0^\circ/\pm 45^\circ]_s = [0^\circ/\pm 45^\circ]_{2s}$$

$$[0^\circ/90^\circ_3/0^\circ]$$ [3]

EXAMPLE

Determine $[A], [B]$, and $[D]$ matrices for a $[+45/-45]$ angle-ply laminate. Each lamina is 6mm thick and contains 60 volume percent of T-300 carbon fiber in an epoxy matrix. The material properties are $E_f = 220$ GPa, $\nu_f = 0.2$, $E_m = 3.6$ GPa, $\nu_m = 0.35$.

SOLUTION:

Step 1: Calculate $E_{11}, E_{22}, \nu_{12}, \nu_{21}$, and G_{12} for $\theta = 0^o$.

$$E_{11} = E_f v_f + E_m v_m$$

$$= 220(0.6) + 3.6(1.0 - 0.6)$$

$$= 133.44 \quad \text{GPa}$$

$$E_{22} = \frac{E_f E_m}{E_f v_m + E_m v_f}$$

$$= \frac{220(3.6)}{220(1.0 - 0.6) + 3.6(0.6)}$$

$$= 8.78 \quad \text{GPa}$$

$$\nu_{12} = \nu_f v_f + \nu_m v_m$$

$$= 0.2(0.6) + 0.35(1.0 - 0.6)$$

$$= 0.26$$

$$\nu_{21} = E_{22}/E_{11}\ \nu_{12}$$

$$= 8.78/133.44\ (0.26)$$

$$= 0.017$$

Need to find G_f and G_m in order to calculate G_{12}

Assuming that both the matrix and fiber are isotropic materials

$$G_f = \frac{E_f}{2(1+v_f)}$$

$$= \frac{200}{2(1+0.2)}$$

$$= 91.7 \quad \text{GPa}$$

$$G_m = \frac{E_m}{2(1 + v_m)}$$

$$= \frac{3.6}{2(1 + 0.35)}$$

$$= 1.33 \quad \text{GPa}$$

$$G_{12} = \frac{G_f G_m}{G_f v_m + G_m v_f}$$

$$= \frac{91.7(1.33)}{91.7(1.0 - 0.6) + 1.33(0.6)}$$

$$= 3.254 \quad \text{GPa}$$

Step 2: Calculate $Q_{11}, Q_{22}, Q_{12}, Q_{21}$, and Q_{66}.

$$Q_{11} = \frac{E_{11}}{(1 - \nu_{12}\nu_{21})}$$

$$= \frac{133.44}{(1 - 0.26(0.017))}$$

$$= 134.03 \quad \text{GPa}$$

$$Q_{22} = \frac{E_{22}}{(1 - \nu_{12}\nu_{21})}$$

$$= \frac{8.78}{(1 - 0.26(0.017))}$$

$$= 8.82 \quad \text{GPa}$$

$$Q_{12} = Q_{21} = \frac{\nu_{12}E_{22}}{(1 - \nu_{12}\nu_{21})}$$

$$= \frac{0.26(8.78)}{(1 - 0.26(0.017))}$$

$$= 2.29 \quad \text{GPa}$$

$$Q_{66} = G_{12} = 3.254 \quad \text{GPa}$$

Step 3: Calculate $\overline{Q}_{11}, \overline{Q}_{12}, \overline{Q}_{22}, \overline{Q}_{16}, \overline{Q}_{26}$, and $\overline{Q}_{66}$ for the $\theta = +45^o$ lamina.

$$U_1 = \frac{1}{8}\left(3Q_{11} + 3Q_{22} + 2Q_{12} + 4Q_{66}\right)$$

$$= \frac{1}{8}\left(3(134.03) + 3(8.82) + 2(2.29) + 4(3.254)\right)$$

$$= 55.77 \quad \text{GPa}$$

$$U_2 = \frac{1}{2}\left(Q_{11} - Q_{22}\right)$$

$$= \frac{1}{2}\left(134.03 - 8.82\right)$$

$$= 62.61 \quad \text{GPa}$$

$$U_3 = \frac{1}{8}\left(Q_{11} + Q_{22} - 2Q_{12} - 4Q_{66}\right)$$

$$= \frac{1}{8}\left(134.03 + 8.82 - 2(2.29) - 4(3.254)\right)$$

$$= 15.66 \quad \text{GPa}$$

$$U_4 = \frac{1}{8}\ (Q_{11} + Q_{22} + 6Q_{12} - 4Q_{66})$$

$$= \frac{1}{8}\ (134.03 + 8.82 + 6(2.29) - 4(3.254))$$

$$= 17.95 \quad \text{GPa}$$

$$U_5 = \frac{1}{2}\ (U_1 - U_4)$$

$$= \frac{1}{2}\ (55.77 - 17.95)$$

$$= 18.91 \quad \text{GPa}$$

$$\overline{Q}_{11} = U_1 + U_2 cos2\theta + U_3 cos4\theta$$

$$= 55.77 + 62.61 cos2(45) + 15.66 cos4(45)$$

$$= 40.11 \quad \text{GPa}$$

$$\overline{Q}_{12} = \overline{Q}_{21} = U_4 - U_3 cos4\theta$$

$$= 17.95 - 15.66 cos4(45)$$

$$= 33.61 \quad \text{GPa}$$

$$\overline{Q}_{22} = U_1 - U_2 cos2\theta + U_3 cos4\theta$$

$$= 55.77 - 62.61 cos2(45) + 15.66 cos4(45)$$

$$= 40.11 \quad \text{GPa}$$

$$\overline{Q}_{16} = \frac{1}{2}\ U_2 sin2\theta + U_3 sin4\theta$$

$$= \frac{1}{2}\ (62.61) sin2(45) + 15.66 sin4(45)$$

$$= 31.31 \quad \text{GPa}$$

$$\overline{Q}_{26} = \frac{1}{2}\ U_2 sin2\theta - U_3 sin4\theta$$

$$= \frac{1}{2}\ (62.61) sin2(45) - 15.66 sin4(45)$$

$$= 31.31 \quad \text{GPa}$$

$$\overline{Q}_{66} = U_5 - U_3 cos4\theta$$

$$= 18.91 - 15.66 cos4(45)$$

$$= 34.57 \quad \text{GPa}$$

Similarly, for the $\theta = -45^o$ lamina

$$\begin{aligned}
\overline{Q}_{11} &= 40.11 \text{ GPa} \\
\overline{Q}_{12} &= 33.61 \text{ GPa} \\
\overline{Q}_{16} &= -31.31 \text{ GPa} \\
\overline{Q}_{22} &= 40.11 \text{ GPa} \\
\overline{Q}_{26} &= -31.31 \text{ GPa} \\
\overline{Q}_{66} &= 34.57 \text{ GPa}
\end{aligned}$$

Writing $[Q]$ in matrix form

$$[\overline{Q}_{0^o}] = \begin{bmatrix} 134.03 & 2.29 & 0 \\ 2.29 & 8.82 & 0 \\ 0 & 0 & 3.254 \end{bmatrix}$$

$$[\overline{Q}_{+45^o}] = \begin{bmatrix} 40.11 & 33.61 & 31.31 \\ 33.61 & 40.11 & 31.31 \\ 31.31 & 31.31 & 34.57 \end{bmatrix}$$

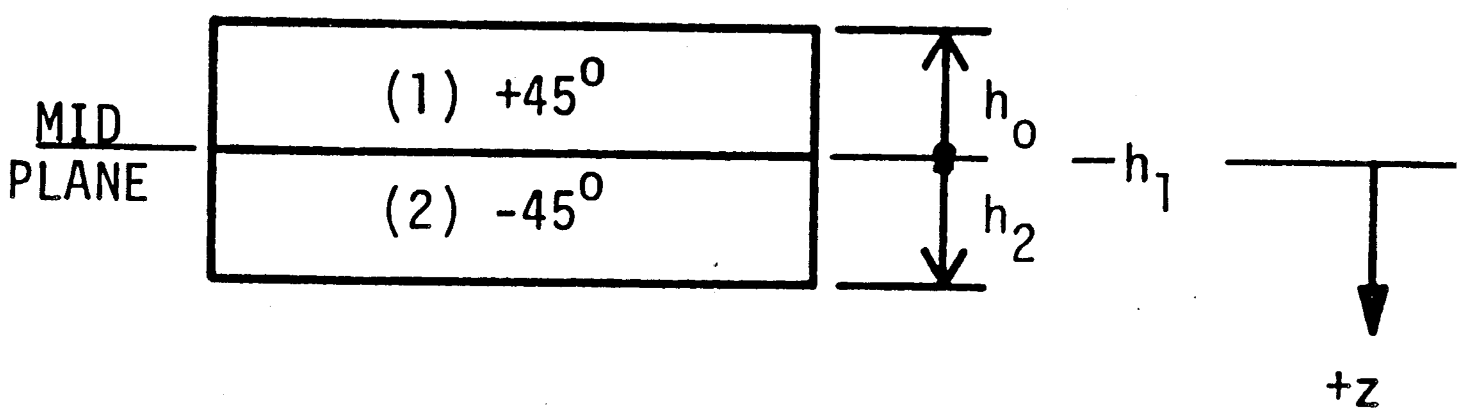
$$[\overline{Q}_{-45^o}] = \begin{bmatrix} 40.11 & 33.61 & -31.31 \\ 33.61 & 40.11 & -31.31 \\ -31.31 & -31.31 & 34.57 \end{bmatrix}$$

$$h_0 = -0.006\text{m}$$
$$h_1 = 0\text{m}$$
$$h_2 = 0.006\text{m}$$

$$A_{mn} = \sum_{j=1}^{N} (\overline{Q}_{mn})_j (h_j - h_{j-1})$$

$$B_{mn} = 1/2 \sum_{j=1}^{N} (\overline{Q}_{mn})_j (h_j^2 - h_{j-1}^2)$$

$$D_{mn} = 1/3 \sum_{j=1}^{N} (\overline{Q}_{mn})_j (h_j^3 - h_{j-1}^3)$$



$$A_{mn} = (\overline{Q}_{mn})_{45}(h_1 - h_0) + (\overline{Q}_{mn})_{-45}(h_2 - h_1)$$

$$= (\overline{Q}_{mn})_{45}(0.006) + (\overline{Q}_{mn})_{-45}(0.006)$$

$$B_{mn} = \frac{1}{2}[(\overline{Q}_{mn})_{45}(h_1^2 - h_0^2) + (\overline{Q}_{mn})_{-45}(h_2^2 - h_1^2)]$$

$$= (\overline{Q}_{mn})_{45}(0 - (-0.006)^2) + (\overline{Q}_{mn})_{-45}(0.006^2)$$

$$= (\overline{Q}_{mn})_{45}(-1.8x10^{-5}) + (\overline{Q}_{mn})_{-45}(1.8x10^{-5})$$

$$D_{mn} = \frac{1}{3}[(\overline{Q}_{mn})_{45}(h_1^3 - h_0^3) + (\overline{Q}_{mn})_{-45}(h_2^3 - h_1^3)]$$

$$= (\overline{Q}_{mn})_{45}(0 - (-0.006)^3) + (\overline{Q}_{mn})_{-45}(0.006^3)$$

$$= (\overline{Q}_{mn})_{45}(7.2x10^{-8}) + (\overline{Q}_{mn})_{-45}(7.2x10^{-8})$$

$2(6\times10^{-3}\ m \times 40.11\times10^{9}) \sim 480\times10^{6}$

$$A_{11} = 0.006(40.11) + 0.006(40.11)$$
$$= 0.4813x10^9 \text{ N/m}$$

$$A_{12} = 0.006(33.61) + 0.006(33.61)$$
$$= 0.4033x10^9 \text{ N/m}$$

$$A_{16} = 0.006(31.31) + 0.006(-31.31)$$
$$= 0 \text{ N/m}$$

$$A_{22} = 0.006(40.11) + 0.006(40.11)$$
$$= 0.4813x10^9 \text{ N/m}$$

$$A_{26} = 0.006(31.31) + 0.006(-31.31)$$
$$= 0 \text{ N/m}$$

$$A_{66} = 0.006(34.57) + 0.006(34.57)$$
$$= 0.4148x10^9 \text{ N/m}$$

$$B_{11} = -1.8x10^{-5}(40.11) + 1.8x10^{-5}(40.11)$$
$$= 0 \text{ N}$$

$$B_{12} = -1.8x10^{-5}(33.61) + 1.8x10^{-5}(33.61)$$
$$= 0 \text{ N}$$

$$B_{16} = -1.8x10^{-5}(31.31) + 1.8x10^{-5}(-31.31)$$
$$= -1.1271x10^{6} \text{ N}$$

$$B_{22} = -1.8x10^{-5}(40.11) + 1.8x10^{-5}(40.11)$$
$$= 0 \text{ N}$$

$$B_{26} = -1.8x10^{-5}(31.31) + 1.8x10^{-5}(-31.31)$$
$$= -1.1271x10^{6} \text{ N}$$

$$B_{66} = -1.8x10^{-5}(34.57) + 1.8x10^{-5}(34.57)$$
$$= 0 \text{ N}$$

$72 \times 10^{-9} \times 40 \times 10^{9}$

$$D_{11} = 7.2x10^{-8}(40.11) + 7.2x10^{-8}(40.11)$$
$$= 5.7758x10^{3} \text{ N-m}$$

$$D_{12} = 7.2x10^{-8}(33.61) + 7.2x10^{-8}(33.61)$$
$$= 4.8398x10^{3} \text{ N-m}$$

$$D_{16} = 7.2x10^{-8}(31.31) + 7.2x10^{-8}(-31.31)$$
$$= 0 \text{ N-m}$$

$$D_{22} = 7.2x10^{-8}(40.11) + 7.2x10^{-8}(40.11)$$
$$= 5.7758x10^{3} \text{ N-m}$$

$$D_{26} = 7.2x10^{-8}(31.31) + 7.2x10^{-8}(-31.31)$$
$$= 0 \text{ N-m}$$

$$D_{66} = 7.2x10^{-8}(34.57) + 7.2x10^{-8}(34.57)$$
$$= 4.9780x10^{3} \text{ N-m}$$

Writing in matrix form

$$[A] = \begin{bmatrix} 0.4813 & 0.4033 & 0 \\ 0.4033 & 0.4813 & 0 \\ 0 & 0 & 0.4148 \end{bmatrix} x10^9 \qquad \text{N/m}$$

$$[B] = \begin{bmatrix} 0 & 0 & -1.1271 \\ 0 & 0 & -1.1271 \\ -1.1271 & -1.1271 & 0 \end{bmatrix} x10^6 \qquad \text{N}$$

$$[D] = \begin{bmatrix} 5.7758 & 4.8398 & 0 \\ 4.8398 & 5.7758 & 0 \\ 0 & 0 & 4.9780 \end{bmatrix} x10^3 \qquad \text{N-m}$$

SINCE $+\theta, -\theta$ BALANCED $A_{16}, A_{26} = 0$

SINCE THICKNESS SAME $D_{16}, D_{26} = 0$

NO BENDING OR TWIST COUPLE

NO NORMAL STRESS - SHEAR EXTENSION COUPLE

EXAMPLE

Determine the midplane strains and curvatures for the [+45/ − 45] angle-ply laminate in the previous example subjected to $N_{xx} = 100$ kN/m.

Step 1: Put equation in matrix form.

$$\begin{pmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \\ M_{xx} \\ M_{yy} \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \epsilon_{xx}^o \\ \epsilon_{yy}^o \\ \epsilon_{xy}^o \\ k_{xx} \\ k_{yy} \\ k_{xy} \end{pmatrix}$$

$$\begin{pmatrix} 100000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 0.4813x10^9 & 0.4033x10^9 & 0 & 0 & 0 & -1.1271x10^6 \\ 0.4033x10^9 & 0.4813x10^9 & 0 & 0 & 0 & -1.1271x10^6 \\ 0 & 0 & 0.4148x10^9 & -1.1271x10^6 & -1.1271x10^6 & 0 \\ 0 & 0 & -1.1271x10^6 & 5775.8 & 4839.8 & 0 \\ 0 & 0 & -1.1271x10^6 & 4839.8 & 5775.8 & 0 \\ -1.1271x10^6 & -1.1271x10^6 & 0 & 0 & 0 & 4978.0 \end{bmatrix} \begin{pmatrix} \epsilon_{xx}^o \\ \epsilon_{yy}^o \\ \epsilon_{xy}^o \\ k_{xx} \\ k_{yy} \\ k_{xy} \end{pmatrix}$$

Step 2: Use matrix inversion to solve for ϵ^o_{xx}, ϵ^o_{yy}, ϵ^o_{xy}, k_{xx}, k_{yy}, and k_{xy}.

$$\epsilon^o_{xx} = 7.738x10^{-4}$$

$$\epsilon^o_{yy} = -5.067x10^{-4}$$

$$\epsilon^o_{xy} = 0$$

$$k_{xx} = 0$$

$$k_{yy} = 0$$

$$k_{xy} = 0.0604 \quad /m$$

MOST GEN CASE

$\underset{\sim}{\epsilon}^o = A_1 \underline{N} + B_1 \underline{M}$

$\underline{k} = C_1 \underline{N} + D_1 \underline{M}$

$D^* = D - BA^{-1}B$

$D_1 = (D^*)^{-1}$

$B_1 = -A^{-1}BD_1$

$C_1 = B_1^T$

$A_1 = A_*^{-1}[I - BC_1]$

IN SYM LAMINATE $B=0$

$D^o = D$

$D_1 = (D^*)^{-1} = D^{-1}$

$C_1 = 0$, $B_1 = 0$

$A_1 = A^{-1}$

ELASTIC PROPERTIES OF A SYMMETRIC LAMINATE

- For a balanced symmetric laminate

$$[A] = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}$$

- The inverse of $[A]$ is

$$[A^{-1}] = \frac{1}{A_{11}A_{22} - A_{12}^2} \begin{bmatrix} A_{22} & -A_{12} & 0 \\ -A_{12} & A_{11} & 0 \\ 0 & 0 & \frac{A_{11}A_{22}-A_{12}^2}{A_{66}} \end{bmatrix}$$

- Substituting into the following

$$\begin{pmatrix} \epsilon^o_{xx} \\ \epsilon^o_{yy} \\ \epsilon^o_{xy} \end{pmatrix} = [A^{-1}] \begin{pmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{pmatrix}$$

$$= \frac{1}{A_{11}A_{22} - A_{12}^2} \begin{bmatrix} A_{22} & -A_{12} & 0 \\ -A_{12} & A_{11} & 0 \\ 0 & 0 & \frac{A_{11}A_{22} - A_{12}^2}{A_{66}} \end{bmatrix} \begin{pmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{pmatrix}$$

- For a uniaxial tensile load $N_{xx} = h\sigma_{xx}$, and $N_{yy} = N_{xy} = 0$

$$\epsilon^o_{xx} = \frac{A_{22}}{A_{11}A_{22} - A_{12}^2} h\sigma_{xx}$$

$$\epsilon^o_{yy} = \frac{-A_{12}}{A_{11}A_{22} - A_{12}^2} h\sigma_{xx}$$

$$\epsilon^o_{xy} = 0$$

Therefore

$$E_{xx} = \frac{\sigma_{xx}}{\epsilon^o_{xx}} = \frac{A_{11}A_{22} - A_{12}^2}{hA_{22}}$$

$$\nu_{xy} = \frac{-\epsilon^o_{yy}}{\epsilon^o_{xx}} = \frac{A_{12}}{A_{22}}$$

- Similarly, by applying N_{yy} and N_{xy}

$$E_{yy} = \frac{A_{11}A_{22} - A_{12}^2}{hA_{11}}$$

$$\nu_{yx} = \frac{A_{12}}{A_{11}}$$

$$G_{xy} = \frac{A_{66}}{h}$$

- For a balanced symmetric laminate, $[B] = 0$

$$[D] = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}$$

- The inverse of $[D]$ is

$$[D^{-1}] = \frac{1}{D_0} \begin{bmatrix} D_{11}^o & D_{12}^o & D_{16}^o \\ D_{12}^o & D_{22}^o & D_{26}^o \\ D_{16}^o & D_{26}^o & D_{66}^o \end{bmatrix}.$$

$$D_0 = D_{11}(D_{22}D_{66} - D_{26}^2) - D_{12}(D_{12}D_{66} - D_{16}D_{26})$$

$$+D_{16}(D_{12}D_{26} - D_{22}D_{16})$$

$$D_{11}^o = \cancel{D_{11}}(D_{22}D_{66} - D_{26}^2)$$

$$D_{12}^o = -\cancel{D_{12}}(D_{12}D_{66} - D_{16}D_{26})$$

$$D_{16}^o = \cancel{D_{16}}(D_{12}D_{26} - D_{22}D_{16})$$

$$D_{22}^o = \cancel{D_{22}}(D_{11}D_{66} - D_{16}^2)$$

$$D_{26}^o = -\cancel{D_{26}}(D_{11}D_{26} - D_{12}D_{16})$$

$$D_{66}^o = \cancel{D_{66}}(D_{11}D_{22} - D_{12^2})$$

- Substituting into the following equation and applying M_{xx} with $M_{yy} = M_{xy} = 0$

$$\begin{pmatrix} k^o_{xx} \\ k^o_{yy} \\ k^o_{xy} \end{pmatrix} = [D^{-1}] \begin{pmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{pmatrix}$$

Yields

$$k_{xx} = \frac{D^o_{11}}{D_0} M_{xx}$$

$$k_{yy} = \frac{D^o_{12}}{D_0} M_{xx}$$

$$k_{xy} = \frac{D^o_{16}}{D_0} M_{xx}$$

[2]

BEAMS

- For all symmetric beams, excluding all 0^o, all 90^o, and $0^o/90^o$ layups for which $D_{16} = D_{26} = 0$,there will be a bending -twisting coupling

- The deflection equation for a symmetric beam

$$\frac{d^2 w}{dx^2} = \frac{M_{xx}}{E_b I}$$

Where

w = beam deflection
M_{xx} = bending moment
I = moment of inertia
E_b = effective bending modulus

[2]

- Effective Bending Modulus

$$E_b = \frac{12}{bt^3 D_{11}^*}$$

Where

$b =$ beam width
$t =$ beam thickness
$D_{11}^* =$ the first element in the inverse $[D]$ matrix

- Effective Bending Stiffness

$$E_b I = \frac{1}{D_{11}^*}$$ [2]

- For all 0^o, all 90^o, or 0^o and 90^o combinations

$$(EI)_b = \sum (E_{11})_j I_j$$

Where

$(E_{11})_j$ = longitudinal modulus of the jth layer
I_j = moment of inertia of the jth layer with respect to the midplane

- Bending Stiffness of a Sandwich Beam

$$(EI)_b = E_S \frac{bt^3}{6} + 2bE_S t \left(\frac{d+t}{2}\right)^2 + E_C \frac{bd^3}{12}$$

Where

E_S = modulus of the skin
E_C = modulus of the core
b = beam width
t = skin thickness
d = core thickness

[2]

TORSIONAL MEMBERS

- Maximum torsional stress in a thin-walled tube of balanced symmetric laminate construction.

$$\tau_{xy} = \frac{T}{2\pi r^2 t}$$

  - The angle of twist per unit length of tube

$$\phi = \frac{T}{2\pi G_{xy} r^3 t}$$

Where

T = applied torque
r = mean radius
t = wall thickness

[2]

NORMAL G (COMPOSITE) < G ST
EITHER ↑ t OR r ↑
$[\pm 45]_s$ HAS LARGEST G (USED IN TORSION
IF G_f ↑ THEN G ↑

- Critical buckling torque for symmetrically laminated tubes of moderate length

$$T_{cr} = 24.4 C D_{22}^{5/8} A_{11}^{3/8} r^{5/4} L^{-1/2}$$

Where

$$C = 0.925 \quad \text{for simply supported ends}$$
$$C = 1.030 \quad \text{for clamped ends}$$

[2]

References: 2, 3