AE 201 - DYNAMICS I

TEXT: Fowles: "Analytical Mechanics" 2nd Edition

READING ASSIGNMENTS

Assignment No.	Chapter	Pages	Topic
1	1	1-20	Review of Fundamental Concepts
2	2	21-35	Vector Calculus; Kinematics
3	3	41-58	Rectilinear Motion
4	4	78-86	Conservative Forces; Potential Energy
5	4	86-114	Projectile; Pendulum Motion
6	6	139-160	Central Forces and Celestial Mechanics
7	7	171-185	Systems of Particles
8	8	190-217	Plane Motion of Rigid Bodies
9	5	118-134	Moving Reference Systems
10	9	221-250	General Motion of a Rigid Body

AE 211 Dynamics I

TEXT: Fowles : "Analytical Mechanics"

Holt, Rinehart, Winston

OUTLINE OF TOPICS

I. Review of Kinematics

II. Review of Equations of Motion for a Particle, Energy Principles, Angular Momentum

III. Kinetics of Particles

- Motion of a Projectile in a Resisting Medium
- Pendulum Problems
- Harmonic Oscillators
- Gravitation; Center of Gravity
- Central Force Motion

IV. Momentum and Energy Principles for Systems of Particles

V. Plane Motion of a Rigid Body

VI. Impulsive Motion

VII. Moments and Products of Inertia; Kinetic Energy; Angular Momentum

VIII. Kinematics of Rotation and of Rigid Bodies

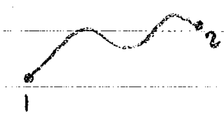
IX. Moving Frames of Reference; Effects of Earth's Rotation

X. Motion of a Rigid Body

AE 211

Analytical Mechanics - Fowles Second Edit.

A. Work done moving particle from 1 to 2



$U_{1-2} = T_2 - T_1$ $\quad$ T- kinetic energy

If conservative force field:

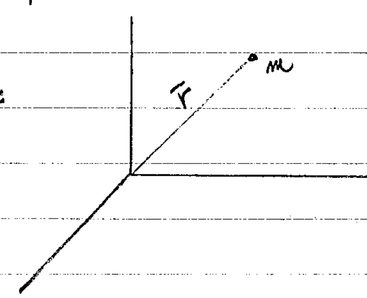
$T_1 + V_1 = T_2 + V_2$ $\quad$ V- potential energy

B. Impulse - Momentum relations

$$m\bar{v}_i + \int \bar{F}\,dt = m\bar{v}_f$$

C. Review of Kinematics

Definition:

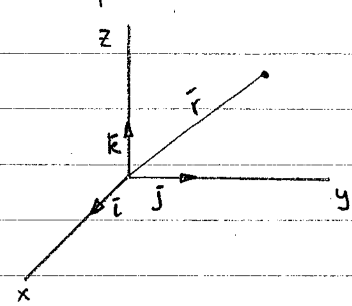


$\bar{r} = \bar{r}(t)$, $\bar{v} = \frac{d\bar{r}}{dt}$, $\bar{a} = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2}$

1. <u>Rectangular Coordinates</u>

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

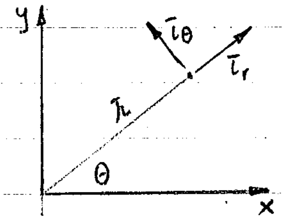
Inertial frame - frame in which $\bar{F} = m\bar{a}$ holds



$$\bar{v} = \frac{d\bar{r}}{dt} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k}$$

$$\bar{a} = \frac{d\bar{v}}{dt} = \ddot{x}\bar{i} + \ddot{y}\bar{j} + \ddot{z}\bar{k}$$

2. Polar Coordinates (Plane)

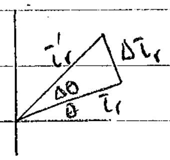


$\bar{r} = r\bar{i}_r$

$\bar{v} = \dfrac{d\bar{r}}{dt} = \dot{r}\bar{i}_r + r\dfrac{d\bar{i}_r}{dt}$

$= \dot{r}\bar{i}_r + r\dot{\theta}\dfrac{d\bar{i}_r}{d\theta}$

$= \dot{r}\bar{i}_r + r\dot{\theta}\bar{i}_\theta$



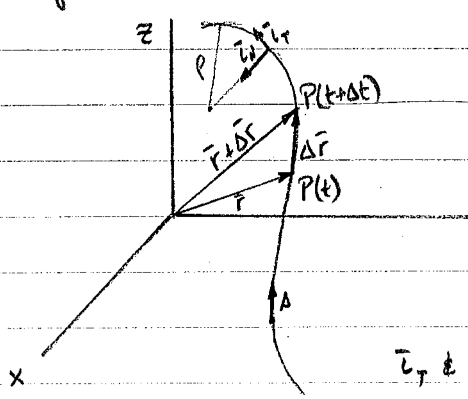
$\lim\limits_{\Delta\theta\to 0} \dfrac{\Delta\bar{i}_r}{\Delta\theta} = \lim\limits_{\Delta\theta\to 0} \dfrac{2|\bar{i}_r| \sin\frac{\Delta\theta}{2}}{\Delta\theta} = 1$

$= \dfrac{2(\Delta\theta/2)}{\Delta\theta} = 1$

$\dfrac{d\bar{i}_r}{d\theta}$ is $\perp$ to $\bar{i}_r$: $\bar{i}_\theta$

$$\bar{a} = \ddot{r}\bar{i}_r + \dot{r}\dot{\theta}\bar{i}_\theta + \dot{r}\dot{\theta}\bar{i}_\theta + r\ddot{\theta}\bar{i}_\theta - r\dot{\theta}^2\bar{i}_r$$
$$= (\ddot{r} - r\dot{\theta}^2)\bar{i}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\bar{i}_\theta$$

3. Tangent-Normal Coordinates (Path)



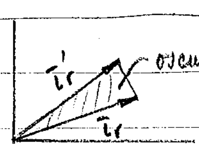
$\lim\limits_{\Delta t\to 0} \dfrac{|\Delta\bar{r}|}{\Delta s} = 1 = |\bar{i}_T|$

$\bar{v} = v\,\bar{i}_T$ (unit tangent vector)

$\bar{a} = \dot{v}\,\bar{i}_T + \dfrac{v^2}{\rho}\,\bar{i}_N$ (principal normal)

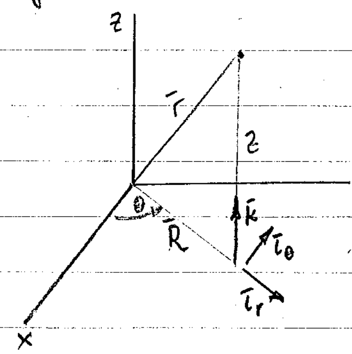
$\bar{i}_T$ & $\bar{i}_N$ lie in the osculating plane

Note: positive $\bar{i}_N$ points toward center of curvature.



$\bar{i}_T \times \bar{i}_N = \bar{i}_b$ — binormal.

4. Cylindrical Coordinates



$$\bar{r} = R\bar{\iota}_r + z\bar{k}$$

$$\bar{v} = \dot{R}\bar{\iota}_r + R\dot{\theta}\bar{\iota}_\theta + \dot{z}\bar{k}$$

$$\bar{a} = (\ddot{R} - R\dot{\theta}^2)\bar{\iota}_r + (2\dot{R}\dot{\theta} + R\ddot{\theta})\bar{\iota}_\theta + \ddot{z}\bar{k}$$

5. Frenet - Serret Formulas

$$\frac{d\bar{\iota}_T}{ds} = \frac{1}{\rho}\bar{\iota}_N \quad ; \quad \frac{d\bar{\iota}_N}{ds} = -\frac{1}{\rho}\bar{\iota}_T + \frac{1}{\tau}\bar{\iota}_b$$

$$\frac{d\bar{\iota}_b}{ds} = -\frac{1}{\tau}\bar{\iota}_N$$ τ - radius of torsion

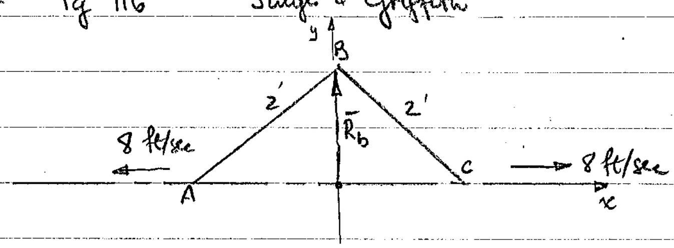
$$\bar{\iota}_T \cdot \bar{\iota}_T = 1 \rightarrow 2\frac{d\bar{\iota}_T}{ds} \cdot \bar{\iota}_T = 0$$

$$\frac{d\bar{\iota}_N}{ds} = a\bar{\iota}_T + b\bar{\iota}_b \qquad \frac{d\bar{\iota}_b}{ds} = \alpha\bar{\iota}_T + \beta\bar{\iota}_N$$

and $\bar{\iota}_T \cdot \bar{\iota}_N = 0 \qquad \bar{\iota}_T \cdot \bar{\iota}_b = 0 \qquad \bar{\iota}_N \cdot \bar{\iota}_b = 0$

gives $a = -\frac{1}{\rho} \qquad b = \frac{1}{\tau} \qquad \alpha = 0 \qquad \beta = -\frac{1}{\tau}$

Problem 4 Pg 116 Singe & Griffith



find $\bar{v}_B$ & $\bar{a}_B$ when $\overline{AB}$ is $\perp$ to $\overline{BC}$

$\bar{r}_B = y_B\bar{j}$

$\bar{v}_B = \dot{y}_B\bar{j} \qquad y_B = \sqrt{4 - x_c^2}$

$\bar{a}_B = \ddot{y}_B\bar{j} \qquad \dot{y}_B = \frac{-1}{(4 - x_c^2)^{1/2}} x_c \dot{x}_c = -\frac{8x_c}{(4 - x_c^2)^{1/2}}$

$\dot{x}_c$ = 8 ft/sec

$$\ddot{y} = \frac{(4 - x_c^2)^{1/2} \cdot -8\dot{x}_c + 8x_c(4 - x_c^2)^{-3/2}\left(-\frac{1}{2} \cdot -2x_c\dot{x}_c\right)}{(4 - x_c^2)}$$

$$\ddot{y} = -8x_c^2(4-x_c^2)^{-3/2} - 64(4-x_c^2)^{-1/2}$$

when $\overline{AB} \perp \overline{BC}$ $\quad x_c = \sqrt{2}$

$\therefore \quad \dot{y}_B = -8 \text{ ft/sec} \qquad \ddot{y}_B = -64\sqrt{2} \text{ ft/sec}^2$

Problem 2.2 from Fowles

Given $\bar{r} = (\cos\omega t)\bar{i} + 2\sin\omega t\,\bar{j}$

Find $\bar{v}$ & $\bar{a}$ when $t = \pi/4\omega$ sec

$$\bar{v} = -\omega\sin\omega t\,\bar{i} + 2\omega\cos\omega t\,\bar{j}$$

$$\bar{a} = -\omega^2\cos\omega t\,\bar{i} - 2\omega^2\sin\omega t\,\bar{j}$$

$\bar{v}\cdot\bar{a} = |\bar{v}||\bar{a}|\cos\varphi_{va}$

@ $t = \pi/4\omega$ $\qquad \bar{v} = -\frac{\omega\sqrt{2}}{2}\bar{i} + 2\omega\frac{\sqrt{2}}{2}\bar{j}$

$$\bar{a} = -\omega^2\frac{\sqrt{2}}{2}\bar{i} - \omega^2\sqrt{2}\,\bar{j}$$

$\frac{\bar{v}\cdot\bar{a}}{|\bar{v}||\bar{a}|} = \cos\varphi$

$\cos\varphi; \quad \omega^3/2 - 2\omega^3 = -\frac{3\omega^3}{2}$

$|\bar{v}| = \sqrt{\frac{\omega^2}{2} + 2\omega^2} = \omega\sqrt{\frac{5}{2}}\;; \quad |\bar{a}| = \sqrt{\frac{\omega^4}{2} + 2\omega^4} = \omega^2\sqrt{5/2}$

$\cos\varphi = \frac{-\frac{3\omega^2}{2}}{5\omega^3/2} = -3/5 \qquad \varphi = \cos^{-1}(-3/5)$

6. Equations of Motion

Rectangular Coord.

$$\bar{F} = m\bar{a} = m[\ddot{x}\,\bar{i} + \ddot{y}\,\bar{j} + \ddot{z}\,\bar{k}]$$

$F_x = m\ddot{x}\;,\; F_y = m\ddot{y}\;,\; F_z = m\ddot{z}$

Polar Coordinates (Plane)

$F_r = (\ddot{r} - r\dot{\theta}^2)m \qquad F_\theta = (2\dot{r}\dot{\theta} + r\ddot{\theta})m$

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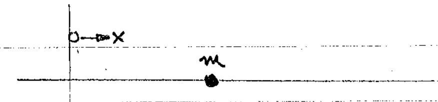
Eq of Motion

1. Rect: $F_x = m\ddot{x}$ $\quad F_y = m\ddot{y}$ $\quad F_z = m\ddot{z}$
2. Path: $F_T = m\dot{v}$ $\quad F_N = \dfrac{mv^2}{\rho}$ $\quad F_b = 0$
3. Cylind: $F_r = m\left[\ddot{R} - R\dot{\varphi}^2\right]$ $\quad F_\varphi = m\dfrac{1}{R}\dfrac{d}{dt}\left[R^2\dot{\varphi}\right]$ $\quad F_z = m\ddot{z}$

Rectilinear Motion of a particle

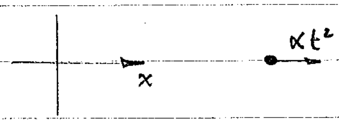
I.



$x(0) = x_0$

$\dot{x}(0) = v_0$

$F = F(t) = \alpha t^2$



$$F_x = m a_x$$
$$= m\frac{dv}{dt} = \alpha t^2 \qquad dv = \frac{\alpha}{m}t^2 dt \longrightarrow \quad v + c_1 = \frac{\alpha t^3}{3m}$$

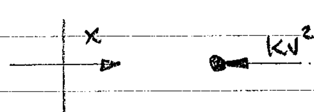
at $t = 0$ $\quad v = v_0$ $\quad c_1 = -v_0$ $\quad \therefore$ $\boxed{v - v_0 = \dfrac{\alpha t^3}{3m}}$

Now $\quad v = \dfrac{dx}{dt} = v_0 + \dfrac{\alpha t^3}{3m} \longrightarrow \quad x + c_2 = \dfrac{\alpha t^4}{12m} + v_0 t$

at $t = 0$ $\quad x = x_0$ $\quad \therefore$ $\quad c_2 = -x_0$ $\qquad \boxed{x - x_0 = v_0 t + \dfrac{\alpha t^4}{12m}}$

II $\quad F = F(v)$ $\quad F = -kv^2$ $\quad k > 0$ $\qquad x(0) = x_0$

$\dot{x}(0) = v_0$



$$F_x = m a_x = -kv^2$$
$$= mv\frac{dv}{dx} = -kv^2$$

finally $\quad -\dfrac{k}{m}x = \ln v + \ln c_1 \qquad \ln c_1 = -\dfrac{k}{m}x_0 - \ln v_0$

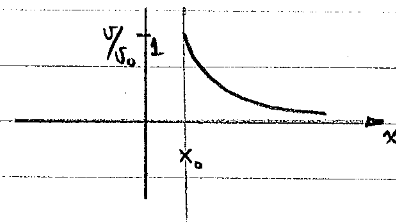
$\therefore$ $\quad \boxed{-\dfrac{k}{m}\left[x - x_0\right] = \ln v/v_0}$ $\qquad$ call $\dfrac{k}{m} = \lambda$

or $\quad e^{\lambda(x_0 - x)} = v/v_0$

$$\frac{dx}{dt} = v_0\, e^{\lambda(x_0 - x)} \quad ; \qquad t = \frac{1}{\lambda v_0} e^{\lambda(x - x_0)} + C_2$$

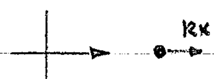
at $t=0 \quad x = x_0 \qquad \therefore \qquad C_2 = -\frac{1}{v_0 \lambda}$

or $$\boxed{t = \frac{1}{v_0 \lambda}\left[e^{\lambda(x - x_0)} - 1\right]}$$



III $\qquad F = F(x) \qquad F = kx \qquad k > 0 \qquad\qquad x(0) = x_0$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dot{x}(0) = v_0$



$$kx = mv\frac{dv}{dx}$$

$$kx\,dx = mv\,dv \qquad \text{or} \qquad \frac{kx^2}{2} = \frac{mv^2}{2} + C_1 \qquad C_1 = \frac{k}{2}x_0^2 - \frac{mv_0^2}{2}$$

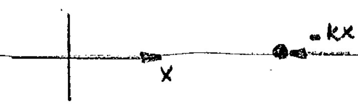
or $$\boxed{\frac{k}{m}\left[x^2 - x_0^2\right] + v_0^2 = v^2}$$

$$\frac{dx}{dt} = \sqrt{\lambda\left[x^2 - x_0^2\right] + v_0^2} \quad \longrightarrow \quad dt = \frac{dx}{\sqrt{\lambda x^2 + (v_0^2 - \lambda x_0^2)}}$$

$$\boxed{t = \frac{1}{\sqrt{\lambda}} \ln\left[x + \sqrt{\frac{v_0^2}{\lambda} - (x_0^2 - x^2)}\right]}$$

IV $\qquad F = F(x) \qquad F = -kx \qquad\qquad x(0) = x_0$

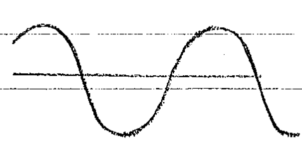
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad \dot{x}(0) = v_0$



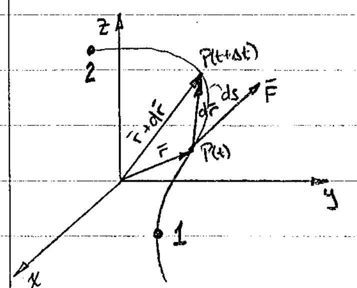
$$-kx = m\ddot{x}$$

$$m\ddot{x} + kx = 0$$

$$x = A \sin \omega t + B \cos \omega t \qquad \omega^2 = \frac{k}{m}$$



Energy Principles



$dU = \bar{F} \cdot d\bar{r}$ definition of work.

Principle of work & Energy

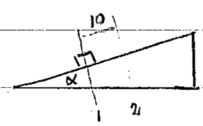
$$dU = m\frac{d\bar{v}}{dt} \cdot \bar{v}\, dt$$

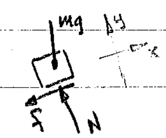
$$= m\bar{v} \cdot d\bar{v} = d\left(\frac{m}{2}\, \bar{v} \cdot \bar{v}\right) = dT$$

Call $T = \frac{m}{2} \bar{v} \cdot \bar{v} = \frac{mv^2}{2}$ (kinetic Energy)

$$\int_1^2 dU = T_2 - T_1$$

work done on a particle is equal to the change in kinetic energy





$$T_2 - T_1 = -\left[mg \sin\alpha + \mu N\right] S$$

Consider a scalar point fn. $V(x, y, z)$

$$\nabla V = \frac{\partial V}{\partial x}\bar{i} + \frac{\partial V}{\partial y}\bar{j} + \frac{\partial V}{\partial z}\bar{k}$$ gradient vector.

$$\frac{dV}{ds} = \nabla V \cdot \frac{d\bar{r}}{ds} = \frac{\partial V}{\partial x}\frac{dx}{ds} + \frac{\partial V}{\partial y}\frac{dy}{ds} + \frac{\partial V}{\partial z}\frac{dz}{ds}$$ directional derivative

or $$\frac{dV}{ds} = \nabla V \cdot \bar{i}_T$$

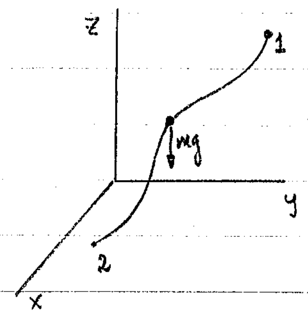
Definition of a conservative force field: the work done by a system of forces independent of the path. Work done over a closed path is 0.

Definition of Potential Energy – for a cons. system of forces define $V(x, y, z)$

$$dU = \bar{F} \cdot d\bar{r} = -dV$$

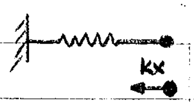
Recall $dV = \nabla V \cdot d\bar{r} = -\bar{F} \cdot d\bar{r}$ $\rightarrow$ $\boxed{\bar{F} = -\nabla V}$

$$F_x = -\frac{\partial V}{\partial x} \quad F_y = -\frac{\partial V}{\partial y} \quad F_z = -\frac{\partial V}{\partial z} \quad \text{if} \quad \bar{F} = F_x\bar{i} + F_y\bar{j} + F_z\bar{k}$$



x,y plane is in plane of earth surface.

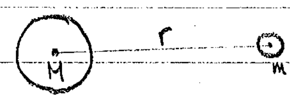
$V = (mg)z$



$V = \frac{1}{2}kx^2$

Inverse square law.

$V = -\frac{GMm}{r}$



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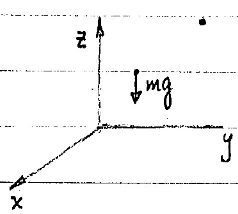
Prob 3.3,5,12,20 Mimeo 9,10

$$\int_1^2 dU = T_2 - T_1$$

In a conservative force field $\nabla \times \bar{F} = 0$.

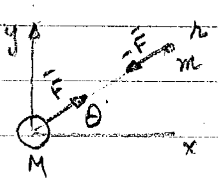
Know derivation of following:



$V = mgz$

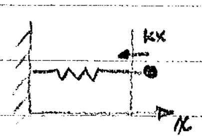
$\bar{F}_z = -mg = -\nabla V = -\frac{\partial V}{\partial z}$

$\therefore V_2 - V_1 = mg(z_2 - z_1)$



$\bar{F} = -\nabla V = -\frac{\partial V}{\partial r}$

$\frac{GMm}{r^2} = -\frac{\partial V}{\partial r}$ $\therefore V_2 - V_1 = \left[\frac{GMm}{r_1} - \frac{GMm}{r_2}\right]$



$\bar{F} = -kx = -\frac{\partial V}{\partial x} = -\nabla V$ $\therefore V_2 - V_1 = \frac{k}{2}\left[x_2^2 - x_1^2\right]$

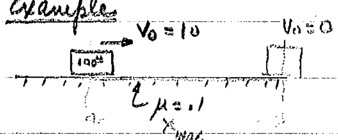
Principle of Conservation of Energy - Conservative force field

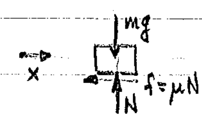
$dU = \bar{F} \cdot d\bar{r} = -dV$ Def.

$dU = \bar{F} \cdot d\bar{r} = dT$ Principle of work.

$$dT = -dV \quad \rightarrow \quad T_2 - T_1 = V_1 - V_2 \qquad \boxed{T_2 + V_2 = T_1 + V_1}$$

Example





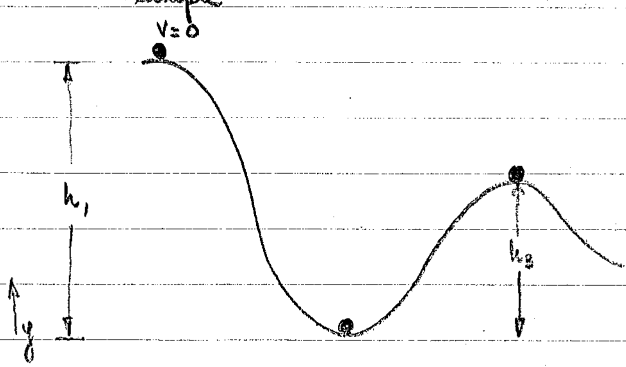
1. $$\int F_x dx = T_2 - T_1$$

$$\int_1^2 -\mu N dx = -\mu \int_1^2 mg dx = -\mu mg \Delta x = 0 - \frac{1}{2} m V_0^2$$

$$x_m = \frac{V_0^2}{2\mu g} = \frac{100}{2(.1)(32.2)} \cong 15 \text{ ft}$$

2. $$F_x = m a_x = -\mu N = m v \frac{dv}{dx}; \quad -\mu g (x_2 - x_1) = \frac{V_2^2}{2} - \frac{V_1^2}{2}; \qquad \frac{V_1^2}{2\mu g} = x_2 - x_1 \cong 15 \text{ ft}.$$

Example

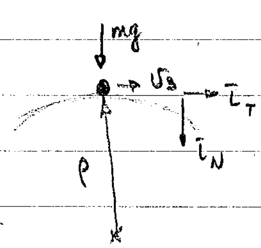


$$T_1 + V_1 = T_3 + V_3$$

$$0 + mgh_1 = \frac{mV_3^2}{2} + mgh_3$$

$$V_3 = \sqrt{2g(h_1 - h_3)}$$

$$v_2 = \sqrt{2gh_1}$$



$$F = \frac{mV_3^2}{\rho_{min}} = mg \qquad \rho_{min} = 2(h_1 - h_3)$$

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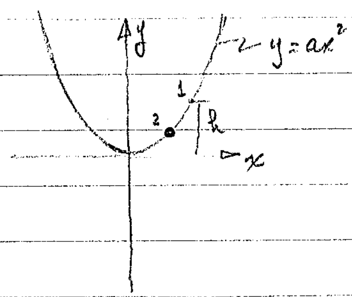
3 - 1 hr. Quizzes.

Example in Constrained Motion.



$$T + V = \text{constant}.$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 + mgh = \tfrac{1}{2}mv^2 + mgy$$

$$\tfrac{1}{2}mv^2 = m(gh - gy)$$

$$V^2 = 2g(h-y) \qquad (1)$$

$$\dot{x}^2 + \dot{y}^2 = V^2 \qquad (2) \qquad \text{also} \qquad \dot{y} = 2ax\dot{x}$$

$$\dot{x}^2 + \dot{y}^2 = 2g(h - ax^2)$$

$$\dot{x}^2\left[1 + 4a^2x^2\right] = 2g(h - ax^2) \qquad (3)$$

$$\frac{dx}{dt} = \sqrt{\frac{2g(h - ax^2)}{1 + 4a^2x^2}} \quad ; \quad dt = dx\sqrt{\frac{1 + 4a^2x^2}{2g(h - ax^2)}} \qquad (4)$$

If $h \ll 1 \quad y \ll 1$ go back to (3)

$$\dot{x}^2\left[1 + 4ay\right] = 2g(h - y) \qquad \text{neglect } 4ay$$

$$\therefore \quad \dot{x}^2 = 2g(h - ax^2)$$

$$2\dot{x}\ddot{x} = -4gax\dot{x} \quad \rightarrow \quad \ddot{x} + 2gax = 0$$

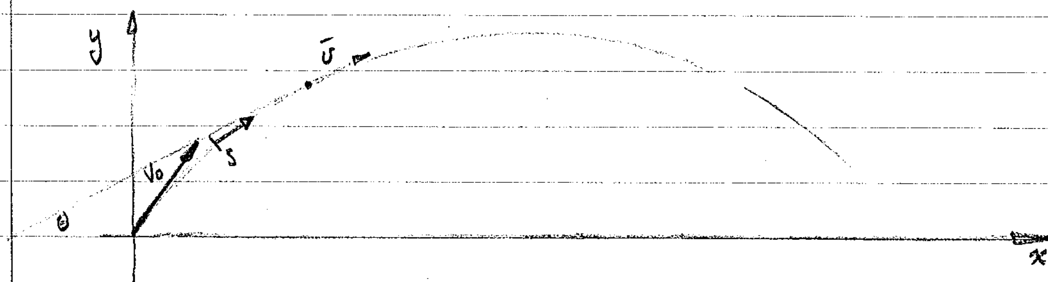
$$x = C_1 \sin\omega t + C_2 \cos\omega t \qquad \omega^2 = 2ga$$

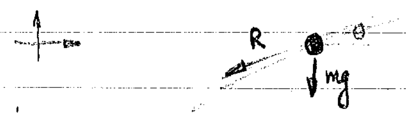
$$\dot{x} = C_1\omega \sin\omega t - C_2\omega\cos\omega t \qquad \text{if } \dot{x} = 0 \quad C_1 = 0 \quad \text{if } v = 0 \text{ at } y = h.$$

$$\therefore \quad x = C_2 \cos\omega t$$

Projectile Motion (with resistance)

Planar Motion - assume uniform grav. field.





1 Cartesian

(1) $F_x = ma_x = m\ddot{x} = -R\cos\theta$ $\qquad V\cos\theta = \dot{x}\ ;\ V\operatorname{sen}\theta = \dot{y} \ \rightarrow\ \dot{x}^2+\dot{y}^2 = V^2$

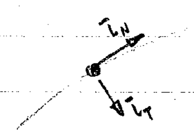
(2) $F_y = ma_y = m\ddot{y} = -R\operatorname{sen}\theta - mg$

from (1) $-R\dfrac{\dot{x}}{V} = m\ddot{x}$ (1a) $\qquad$ let $\phi(v,y) = \dfrac{R}{mV}$

" (2) $-R\dfrac{\dot{y}}{V} - mg = m\ddot{y}$ (2a)

then $\ddot{x} + \Phi\dot{x} = 0 \qquad \ddot{y} + \Phi\dot{y} = -g$

2 Tangent-Normal Coordinate.



(1) $F_T = ma_T = m\dot{v} = -R - mg\sin\theta$

(2) $F_N = ma_N = mg\cos\theta = m\dfrac{v^2}{\rho}$ $\qquad \rho d\theta = -ds$

(1) can be written as $\quad -\dfrac{R}{mv} - g\dfrac{\sin\theta}{v} = -\dfrac{dv}{\rho\, d\theta}$ (3a)

(2) can be written as $\quad mg\cos\theta = -\dfrac{m v^2 d\theta}{ds}$ (4)

using (3) $\dfrac{R}{mv} + g\dfrac{\sin\theta}{v} = -\dfrac{dv}{ds}$ $\qquad$ we can get

$$\frac{dv}{d\theta} = \left[\frac{R}{mg} + \sin\theta\right]\frac{v}{\cos\theta}$$

3 Equation of the hodograph.

4a $\quad \frac{1}{v}\frac{dv}{d\theta} = \Phi \sec\theta + \tan\theta \qquad$ if $R/mg = \Phi$ & $\Phi = cv^2$

substitute $\tanh\psi = \sin\theta$

$$\operatorname{sech}^2\psi\, d\psi = \cos\theta\, d\theta \qquad 1 - \tanh^2\psi = \cos^2\theta$$

$$\operatorname{sech}^2\psi = \cos^2\theta$$

$$\frac{d}{d\psi}\left(\frac{1}{v^2}\right) = -2c - 2\left(\frac{1}{v^2}\right)\tanh\psi$$

when $\frac{1}{v^2} = \frac{1}{v^2}(\psi)$

$$\frac{1}{v^2} = \cos^2\theta\left[A - c\tanh^{-1}(\sin\theta)\right] - c\sin\theta$$

$$\frac{dx}{d\theta} = \frac{dx}{ds}\frac{ds}{d\theta} = -\rho\cos\theta \qquad x = \int -\rho\cos\theta\, d\theta + C_1 = \int -\frac{v^2}{g}\, d\theta + C_1$$

$$\frac{dy}{d\theta} = -\rho\tan\theta\cos\theta \qquad y = \int -\rho\cos\theta\tan\theta\, d\theta + C_2 = \int -\frac{v^2}{g}\tan\theta\, d\theta + C_2$$

4b $\quad$ If $R = mgcv \qquad \Phi = cg$

Use rectangular coord.

$$\ddot{x} + gc\dot{x} = 0 \qquad \text{let } \varphi = \dot{x} \quad \dot{\varphi} = \ddot{x}$$

$$\ddot{y} + gc\dot{y} = -g$$

$$\dot{\varphi} + gc\varphi = 0 \qquad \text{let } \varphi = Ae^{+\lambda t} \qquad +A\lambda e^{\lambda t} + gcAe^{\lambda t} = 0$$

$$\lambda = -gc$$

$$\therefore \varphi = Ae^{-gct} \qquad \dot{x}(0) = \dot{x}_0 = A$$

$$x = \frac{A}{-gc}e^{-gct} = \frac{\dot{x}_0}{-gc}e^{-gct} + B \qquad B = \frac{\dot{x}_0}{gc} + x_0$$

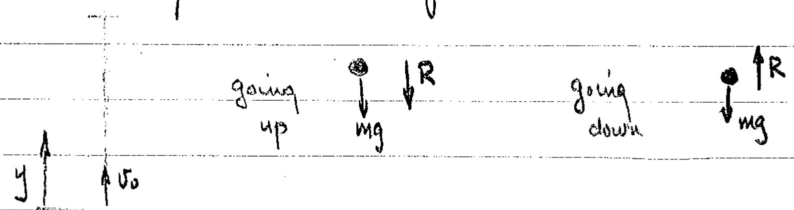
$$\boxed{x = \frac{\dot{x}_0}{gc}\left[1 - e^{-gct}\right] + x_0}$$

y turns out to be $\quad \boxed{y = y_0 - \frac{t}{c} + \frac{1}{gc}\left[\dot{y}_0 + \frac{1}{c}\right]\left[1 - e^{-gct}\right]}$

Vertical Motion of Projectiles in $R = kv^2$ Medium.

Eq of Hodograph is no good.

Initially $v(0) = v_0$, $y = 0$



$$-R - mg = mv\frac{dv}{dy} \qquad \text{if } R = mgcv^2$$

$$-g[1 + cv^2] = v\frac{dv}{dy} \quad ; \quad -g\,dy = \frac{v\,dv}{1 + cv^2}$$

$$-g\int dy = \frac{1}{2c}\ln(1 + cv^2) \rightarrow -2gcy = \ln(1 + cv^2) + A$$

$$A = -\ln(1 + cv_0^2) \quad \therefore \quad +2gcy = \ln\left(\frac{1 + cv_0^2}{1 + cv^2}\right)$$

$y\downarrow$ $$mg - mgcv^2 = mv\frac{dv}{dy}$$

$$g(1 - cv^2) = v\frac{dv}{dy}$$

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UP $\qquad\qquad\qquad\qquad$ DOWN.

$$y = -\frac{1}{2gc}\ln(1 + cv^2) + C_1 \qquad y = \frac{1}{2gc}\ln(1 - cv^2) + C_2$$

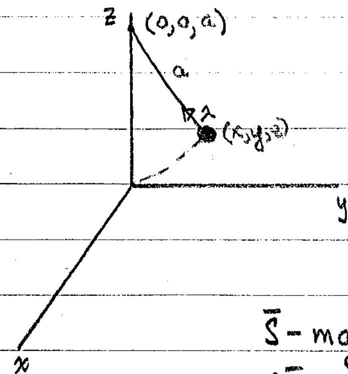
$$\Delta KE = \Delta Work \qquad -\int_0^H (mg + R)\,dy = -\frac{1}{2}mv_0^2$$

H.W. Chapter 4. 4.1 abc 4.3 4.5 $\bar{F} = y\bar{i} + x\bar{j}$ $\bar{F} = y\bar{i} - x\bar{j}$ both conserv.

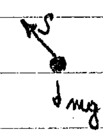
Energy principle needed

Quiz #1   2/25/71   Kinematics, Kinetics (Rectilinear), Work & Energy.

Pendulum Motion.



Small Oscillations of a Spherical Pendulum.



$$\bar F = m\bar a$$

$$\bar F = F_x \bar i + F_y \bar j + F_z \bar k = m(\ddot x \bar i + \ddot y \bar j + \ddot z \bar k)$$

$$\bar S - mg\bar k = m(\ddot x \bar i + \ddot y \bar j + \ddot z \bar k)$$

$$S\bar\lambda - mg\bar k = m\ddot x \bar i + m\ddot y \bar j + m\ddot z \bar k$$

$$\bar\lambda = \frac{-x\bar i - y\bar j + (a-z)\bar k}{\sqrt{x^2+y^2+(a-z)^2}}$$

$$S\left[\frac{-x\bar i - y\bar j + (a-z)\bar k}{a}\right] - mg\bar k = m\ddot x \bar i + m\ddot y \bar j + m\ddot z \bar k$$

(1) $$-\frac{Sx}{a} = m\ddot x \;;\quad -\frac{Sy}{a} = m\ddot y \;;\quad \frac{S(a-z)}{a} - mg = m\ddot z$$

for small excursions $a^2 = x^2 + y^2 + (a-z)^2$

$$a^2 = x^2 + y^2 + \cancel{a^2} - 2az + z^2$$

$$z = \frac{1}{2a}\left[x^2 + y^2 + z^2\right] \qquad \text{if } x, y \ll 1 \quad z \text{ is second order small}$$

$\therefore \ddot z$ " very small.

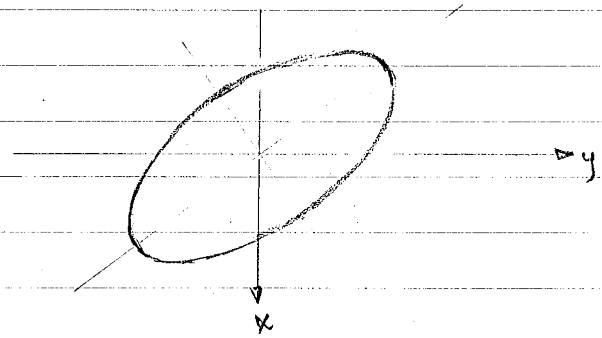
$$\therefore \quad \frac{S(a)}{a} - mg = 0 \qquad \boxed{S \cong mg}$$

$$-\frac{mgx}{a} = m\ddot x \;;\quad -\frac{mgy}{a} = m\ddot y$$

$$\ddot x + \frac{g}{a}x = 0 \;;\quad \ddot y + \frac{g}{a}y = 0$$

$$x = C_1\cos\omega t + C_2 \sin\omega t \;;\quad y = C_3\cos\omega t + C_4\sin\omega t$$

$$\omega^2 = g/a \qquad\qquad \omega^2 = g/a$$



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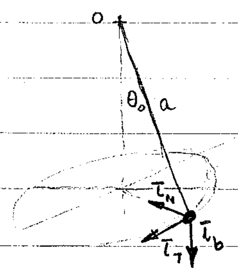
<u>Conical Pendulum.</u>

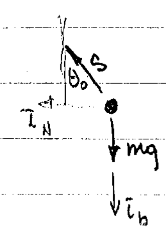
Is motion at constant θ_0 possible



$$\bar{F} = m\bar{a}$$

$$F_T\,\bar{\iota}_T + F_N\,\bar{\iota}_N + F_b\,\bar{\iota}_b = m\left[\dot{v}\,\bar{\iota}_T + \frac{v^2}{\rho}\,\bar{\iota}_N\right]$$

$$F_T = m\frac{dv}{dt} \qquad F_N = \frac{mv^2}{\rho} \qquad F_b = 0 \qquad (1,2,3)$$



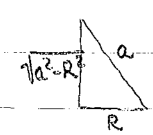
$$\bar{F}_T = 0 \qquad \bar{F}_N = S\sin\theta_0 \qquad F_b = mg - S\cos\theta_0$$

(4)

$$0 = m\frac{dv}{dt} \rightarrow v = c \qquad F_T \quad (5)$$

$$S\sin\theta_0 = \frac{mv^2}{R} \qquad F_N \quad (6)$$

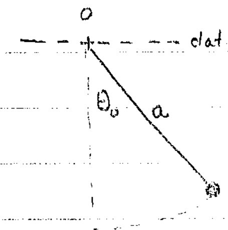
$$mg - S\cos\theta_0 = 0 \qquad F_b \quad (7)$$



$$mg\tan\theta_0 = \frac{mv^2}{R} \qquad \boxed{g\frac{R^2}{\sqrt{a^2-R^2}} = v^2} \quad (8)$$

$$S\cdot\frac{R}{a} = \frac{mg}{R}\frac{R^2}{\sqrt{a^2-R^2}} \qquad \boxed{S = \frac{mga}{\sqrt{a^2-R^2}}} \quad (9)$$

<u>Simple Pendulum. (Motion is in a plane)</u>



Linear approximation.

$$T + V = E \quad ; \quad v = a\ddot{\theta}$$

$$\tfrac{1}{2} m a^2 \dot{\theta}^2 - mga\cos\theta = E$$

diff the expres.

$$\tfrac{1}{2} m a^2 2\dot{\theta}\ddot{\theta} + mga \sin\theta\, \dot{\theta} = 0$$

$$\dot{\theta}\left[\ddot{\theta} + g/a \sin\theta\right] = 0$$

either $\dot{\theta} = 0$ trivial, or $\ddot{\theta} + g/a \sin\theta = 0$ if $\theta < 10^\circ$ $\sin\theta \sim \theta$

$$\ddot{\theta} + g/a\, \theta = 0$$

let $\theta = A \sin\omega t + B\cos\omega t$ $\qquad \omega = \sqrt{g/a}$

period $\tau = \dfrac{2\pi}{\omega} = 2\pi\sqrt{a/g}$

Exact Solution

$T + V = \text{const.}$ $\qquad \theta(0) = \alpha \qquad \dot{\theta}(0) = 0$

$$\tfrac{1}{2} m a^2 \dot{\theta}^2 - mga\cos\theta = -mga\cos\alpha$$

$$\dot{\theta}^2 = \frac{2g}{a}(\cos\theta - \cos\alpha) = 2\omega^2(\cos\theta - \cos\alpha)$$

$$\dot{\theta}^2 = 4\omega^2\left(\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right)$$

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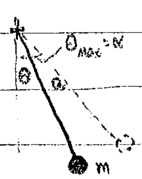
H.W. #4.9, 4.10



$$\dot{\theta}^2 = 2\omega^2\left[\cos\theta - \cos\alpha\right] \qquad \omega^2 = g/a$$

$$\dot{\theta}^2 = 4\omega^2\left[\sin^2\frac{\alpha}{2} - \sin^2\frac{\theta}{2}\right]$$

let $\sin\frac{\theta}{2} = \sin\frac{\alpha}{2}\sin\varphi$ $\qquad$ let $k = \sin\frac{\alpha}{2}$

$$\frac{\dot{\theta}}{2}\cos\frac{\theta}{2} = k\cos\varphi\,\dot{\varphi}$$

$$\dot{\theta}^2 = 4k^2\frac{\cos^2\varphi\,\dot{\varphi}^2}{\cos^2\theta/2} = 4\omega^2 k^2\left[1 - \sin^2\varphi\right]$$

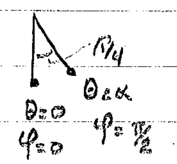
$$\dot{\varphi}^2 = \omega^2 \left[1 - \sin^2 \frac{\theta}{2}\right] = \omega^2 \left[1 - k^2 \sin^2 \varphi\right]$$

$$\dot{\varphi} = \omega \sqrt{1 - k^2 \sin^2 \varphi}$$

$$dt = \frac{1}{\omega} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

$$t = \int \frac{1}{\omega} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} + C \quad \text{second.}$$

$$\text{Period} - P = 4 \frac{1}{\omega} \int \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$



$$T = 4 \frac{1}{\omega} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}}$$

$$\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - k^2 \sin^2 \varphi}} = \frac{2\pi}{\omega} \left[1 + \left(\frac{1}{2}\right)^2 k^2 + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 k^4 + \left(\frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\right)^2 k^6 + \dots\right]$$

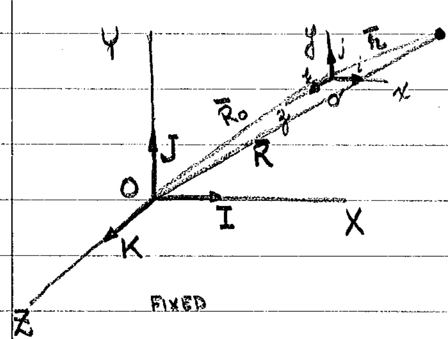
$k \sim \alpha/2$

$$T \cong \frac{2\pi}{\omega} \left[1 + \frac{\alpha^2}{16}\right] \cong 2\pi \sqrt{\frac{a}{g}} \left[1 + \frac{\alpha^2}{16}\right] \quad \text{soft spring.}$$

as $\alpha \uparrow$ $T \uparrow$ causing the pendulum to slow down.

Moving Frames of Reference

A. Frames in Translation Only



OXYZ - fixed

O'xyz - translating frame

$$\bar{R} = \bar{R}_0 + \bar{r}$$

$$\bar{R} = X I + Y J + Z K + x i + y j + z k$$

$$\bar{V}_{ABS} = \frac{d\bar{R}}{dt} = \dot{X} I + \dot{Y} J + \dot{Z} K + \dot{x} i + \dot{y} j + \dot{z} k$$

$$\bar{V}_{abs} = \frac{d\bar{R}_0}{dt} + \frac{d\bar{r}}{dt} = \bar{V}_0 + \bar{v} \qquad \bar{a}_{abs} = \bar{A}_0 + \bar{a}$$

In translating frame $I, i; J, j; K, k$ are the same we can then write

$$\bar{R} = (X + x, Y + y, Z + z)$$

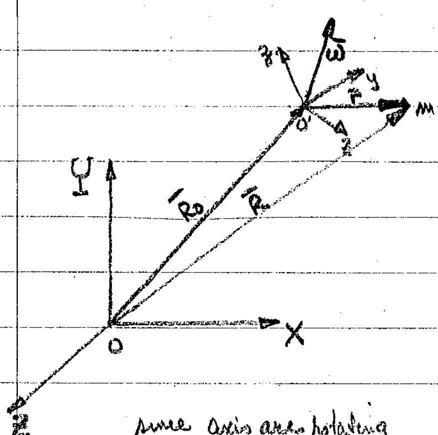
$$\bar{F} = m\bar{a} = m\left[\ddot{\bar{R}}_o + \ddot{\bar{r}}\right]$$

$$\underbrace{\left[\bar{F} - m\ddot{\bar{R}}_o\right]}_{\text{inertia force}} = m\ddot{\bar{r}}$$

A. General Frame of Motion

OXYZ - fixed

O'xyz - moving $\quad \bar{\omega}$ - axis of rotation



$$\bar{R} = \bar{R}_o + \bar{r}$$

$$\bar{V}_{abs} = \frac{d\bar{R}}{dt} = \underbrace{\frac{d\bar{R}_o}{dt}}_{\text{vel of } o'} + \underbrace{\frac{d\bar{r}}{dt}}_{\text{vel of m wrt } o'}$$

$$\frac{d\bar{R}_o}{dt} = \dot{X} I + \dot{Y} J + \dot{Z} K$$

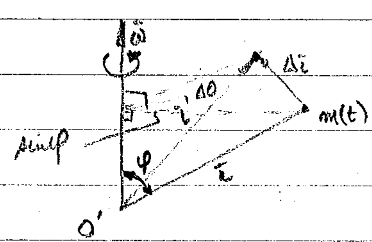
since axis are rotating

$$\frac{d\bar{r}}{dt} = \dot{x}\bar{i} + x\frac{d\bar{i}}{dt} + \dot{y}\bar{j} + y\frac{d\bar{j}}{dt} + \dot{z}\bar{k} + z\frac{d\bar{k}}{dt}$$

$$= \underbrace{(\dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k})}_{\text{v of translation; relative velocity}} + \underbrace{\left(x\frac{d\bar{i}}{dt} + y\frac{d\bar{j}}{dt} + z\frac{d\bar{k}}{dt}\right)}_{\text{v of rotation of axes}}$$

$$= \left(\frac{\delta \bar{r}}{\delta t}\right) \qquad (\bar{\omega} \times \bar{r})$$

$$\bar{V}_{abs} = \frac{d\bar{R}}{dt} = \bar{V}_o + \frac{\delta \bar{r}}{\delta t} + (\bar{\omega} \times \bar{r})$$



$$\left|\frac{d\bar{i}}{dt}\right| = \lim_{\Delta t \to 0} \left|\frac{\Delta \bar{i}}{\Delta t}\right|$$

$$= \lim_{\Delta t \to 0} \left|\frac{\sin\varphi \, \Delta\theta}{\Delta t}\right| = \sin\varphi \, \omega = \bar{\omega} \times \bar{i}$$

$$x\frac{d\bar{i}}{dt} + y\frac{d\bar{j}}{dt} + z\frac{d\bar{k}}{dt} = x(\bar{\omega} \times \bar{i}) + y(\bar{\omega} \times \bar{j}) + z(\bar{\omega} \times \bar{k})$$

$$= \bar{\omega} \times (\bar{i}x + \bar{j}y + \bar{k}z) = \bar{\omega} \times \bar{r}$$

$\bar{A}$ – moving frame of reference.

$$\frac{d}{dt}[\bar{A}] = \frac{\delta[\bar{A}]}{\delta t} + \bar{\omega} \times [\bar{A}]$$

In Summary

$$\bar{R} = R_0 + \bar{r}$$

$$\bar{V}_{abs} = \dot{\bar{R}}_0 + \frac{\delta \bar{r}}{\delta t} + \bar{\omega} \times \bar{r}$$

$$\bar{a}_{abs} = \ddot{\bar{R}}_0 + \frac{\delta}{\delta t}\left[\frac{\delta \bar{r}}{\delta t} + \bar{\omega}\times\bar{r}\right] + \bar{\omega}\times\left\{\frac{\delta \bar{r}}{\delta t} + \bar{\omega}\times\bar{r}\right\}$$

$$= \ddot{\bar{R}}_0 + \underbrace{\frac{\delta^2 \bar{r}}{\delta t^2}}_{\text{relative acc.}} + \underbrace{\frac{\delta\bar{\omega}}{\delta t}\times\bar{r}}_{\text{transverse}} + \underbrace{2\bar{\omega}\times\frac{\delta \bar{r}}{\delta t}}_{\text{Coriolis}} + \underbrace{\bar{\omega}\times(\bar{\omega}\times\bar{r})}_{\text{Centripetal accel.}}$$

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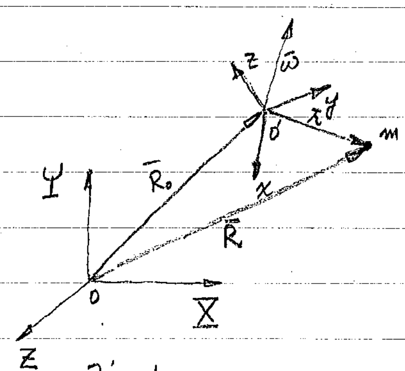


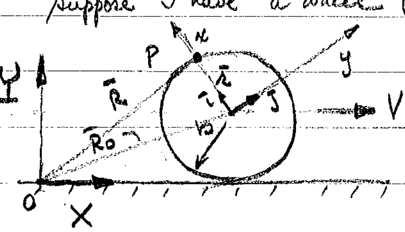
$$\bar{R} = \bar{R}_0 + \bar{r}$$

$$\bar{v}_{abs} = \frac{d\bar{R}}{dt} = \frac{d\bar{R}_0}{dt} + \frac{\delta \bar{r}}{\delta t} + \bar{\omega}\times\bar{r}$$

$$\bar{a}_{abs} = \frac{d^2\bar{R}}{dt^2} = \underbrace{\ddot{\bar{R}}_0}_{\text{accel of } o'} + \underbrace{\frac{\delta^2 \bar{r}}{\delta t^2}}_{\text{relative acc.}} + \underbrace{\frac{\delta\bar{\omega}}{\delta t}\times\bar{r}}_{\text{transverse}} + \underbrace{2\bar{\omega}\times\frac{\delta \bar{r}}{\delta t}}_{\text{Coriolis}} + \underbrace{\bar{\omega}\times(\bar{\omega}\times\bar{r})}_{\text{Centripet}}$$

Example:

Suppose I have a wheel (rad. b) rolling in a straight line at V (const)



$$\bar{r} = b\bar{i}$$

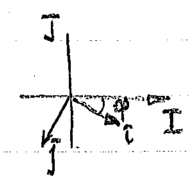
$$\frac{\delta \bar{r}}{\delta t} = 0 = \frac{\delta^2 \bar{r}}{\delta t} \qquad \bar{\omega} = \frac{V}{b}\bar{k}$$

$$\frac{d\bar{R}_0}{dt} = V \qquad \frac{\delta\bar{\omega}}{\delta t} = 0$$

$$\bar{a}_{abs} = 0+0+0+0+\frac{V}{b}\bar{k}\times\left(\frac{V}{b}\bar{k}\times b\bar{i}\right) = \frac{V}{b}\bar{k}\times\left(V\bar{j}\right) = -\frac{V^2}{b}\bar{i}$$

What is $\bar{V}_{abs}$ of point P?

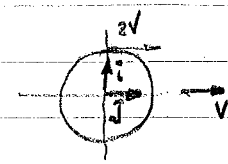
$$\bar{V}_{abs} = \frac{d\bar{R}_0}{dt} + \frac{\delta \bar{r}}{\delta t} + \bar{\omega} \times \bar{r}$$



$$= V\bar{I} + 0 + V\bar{j}$$

$$I = \bar{i}\cos\varphi - \bar{j}\sin\varphi$$

$$\bar{V}_{abs} = V\cos\varphi\,\bar{i} + V(1-\sin\varphi)\,\bar{j}$$



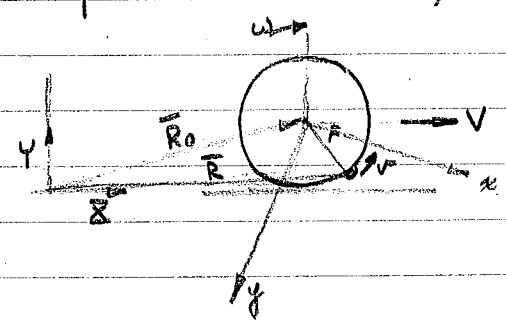
for $\varphi = 270°$ $\quad \bar{v} = V(0 + (1+1)\bar{j}) = 2V\bar{j}$

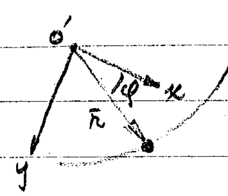
for $\varphi = 90°$ $\quad \bar{v} = V(0 + (1-1)\bar{j}) = 0$

Example: wheel, rad. b, uniform speed V.

calculate acceleration on bug

bug moving at unif speed v.





$$\bar{r} = r\cos\varphi\,\bar{i} + r\sin\varphi\,\bar{j}$$

$$\frac{\delta \bar{r}}{\delta t} = -r\dot{\varphi}\sin\varphi\,\bar{i} + r\dot{\varphi}\cos\varphi\,\bar{j} \qquad \text{but } -r\dot{\varphi} = v$$

$$\frac{\delta^2 \bar{r}}{\delta t} = -r\dot{\varphi}^2\cos\varphi\,\bar{i} - r\dot{\varphi}^2\sin\varphi\,\bar{j} = -\frac{v^2}{r}(\cos\varphi\,\bar{i} + \sin\varphi\,\bar{j})$$

$$\bar{\omega} = \frac{V}{b}\bar{k} \qquad \bar{\omega}\times\bar{r} = \frac{V}{b}\bar{k}\times(r\cos\varphi\,\bar{i} + r\sin\varphi\,\bar{j})$$

$$\bar{\omega}\times\bar{r} = \frac{Vr}{b}\cos\varphi\,\bar{j} - \frac{Vr}{b}\sin\varphi\,\bar{i}$$

$$\bar{\omega}\times(\bar{\omega}\times\bar{r}) = \frac{V}{b}\bar{k}\times(\bar{\omega}\times\bar{r}) = -\frac{V^2 r}{b^2}\cos\varphi\,\bar{i} - \frac{V^2 r}{b^2}\sin\varphi\,\bar{j}$$

but $b = r$ $\quad \therefore \quad \bar{\omega}\times(\bar{\omega}\times\bar{r}) = -\frac{V^2}{b}\cos\varphi\,\bar{i} - \frac{V^2}{b}\sin\varphi\,\bar{j}$

$$\bar{a}_{abs} = \frac{d\bar{R}_0}{dt^2} + \frac{\delta^2\bar{r}}{\delta t^2} + \frac{\delta\bar{\omega}}{\delta t}\times\bar{r} + 2\bar{\omega}\times\frac{\delta\bar{r}}{\delta t} + \bar{\omega}\times(\bar{\omega}\times\bar{r})$$

$$= 0 - \frac{v^2}{b}(\cos\varphi\,\bar{i} + \sin\varphi\,\bar{j}) + 0 + 2\frac{V}{b}\bar{k}\times(v\sin\varphi\,\bar{i} - v\cos\varphi\,\bar{j})$$

$$- \frac{V^2}{b}(\cos\varphi\,\bar{i} + \sin\varphi\,\bar{j})$$

$$= -\frac{v^2}{b}(\cos\varphi\,\bar{i} + \sin\varphi\,\bar{j}) + \frac{2Vv}{b}(\sin\varphi\,\bar{j} + \cos\varphi\,\bar{i}) - \frac{V^2}{b}(\cos\varphi\,\bar{i} + \sin\varphi\,\bar{j})$$

$$= -\frac{\cos\varphi\,\bar{i}}{b}\left[(v-V)^2\right] - \frac{\sin\varphi}{b}\bar{j}\left[(v-V)^2\right]$$

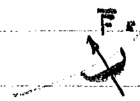
$$= -\frac{1}{b}\left[(v-V)^2\right]\left\{\cos\varphi\,\bar{i} + \sin\varphi\,\bar{j}\right\}$$

if $\varphi = 0$ $\qquad \bar{a}_{abs} = -\frac{1}{b}\left[(v-V)^2\right]\bar{i}$

Force on Bug.

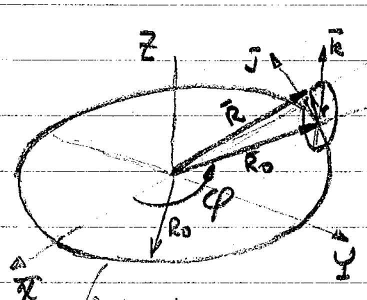
$$\bar{F} = m\bar{a}_{abs.}$$

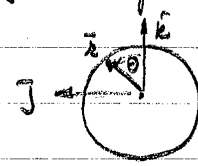
$$\therefore\ \bar{F} = -\frac{m}{b}(v-V)^2\,\bar{i}$$



Example: wheel moves in a circle at uniform speed V (rad b)
circle rad R_0

$\bar{k}$ is always vertical $\quad\bar{j}$ is tangent to traj of axis of wheel. $\quad\bar{i}$ in $\bar{R}_0$ direct.
$\bar{j}$ is horiz.





$$\bar{v} = \frac{d\bar{R}_0}{dt} + \frac{\delta\bar{r}}{\delta t} + \bar{\omega}\times\bar{r}$$

note $V = b\dot{\theta} = R_0\dot{\varphi}$

$$\frac{d\bar{R}_0}{dt} = V\bar{j}$$

$$\bar{r} = b\cos\theta\,\bar{k} + b\sin\theta\,\bar{j}$$

$$\frac{\delta\bar{r}}{\delta t} = -V\sin\theta\,\bar{k} + V\cos\theta\,\bar{j}$$

$\bar{\omega} = \frac{V}{R_0}\bar{k}$ $\qquad$ $\bar{\omega}\times\bar{r} = \frac{V}{R_0}\bar{k}\times b\left(\cos\theta\,\bar{k} + \sin\theta\,\bar{j}\right)$

$$= \frac{Vb}{R_0}\left(-\sin\theta\,\bar{i}\right)$$

$$\bar{v} = V\bar{j} + V\left(-\sin\theta\,\bar{k} + \cos\theta\,\bar{j}\right) - \frac{Vb}{R_0}\left(\sin\theta\,\bar{i}\right)$$

$$= V\left[(1+\cos\theta)\bar{j} - \frac{b}{R_0}\sin\theta\,\bar{i} - \sin\theta\,\bar{k}\right]$$

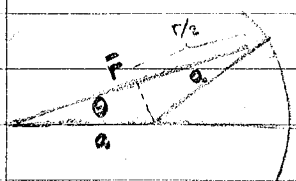
when $\theta = 0$ $\qquad$ $\bar{V} = V\left[2\bar{j}\right]$

when $\theta = 90^\circ$ $\qquad$ $\bar{v} = V\left[\bar{j} - \frac{b}{R_0}\bar{i} - \bar{k}\right]$

2/4/71

Ch. 5-1-5-7

Test.



$\cos\theta = \frac{r}{2a}$ $\qquad$ $r = 2a\cos\theta$

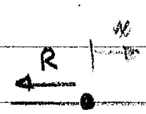
$\dot{r} = -2a\dot{\theta}\sin\theta$

$\ddot{r} = -2a\dot{\theta}^2\cos\theta$

$a_r = -2a\dot{\theta}^2\cos\theta - 2a\cos\theta\,\dot{\theta}^2 = -4a\dot{\theta}^2\cos\theta$

$a_\theta = 2\left(-2a\dot{\theta}\sin\theta\right)\dot{\theta} = -4a\dot{\theta}^2\sin\theta$

$$|a| = \sqrt{a_r^2 + a_\theta^2} = \sqrt{16a^2\dot{\theta}^4} = 4a\dot{\theta}^2$$



$-mcv = mv\frac{dv}{ds}$ $\qquad$ $-c\,ds = dv$

$-cs = v + B$

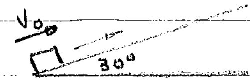
$-cx = v - v_0$

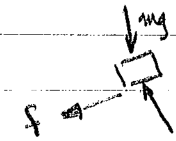
$x_{max} = \frac{v_0}{c}$

$-mcv = m\frac{dv}{dt}$ $\qquad$ $-c\,dt = \frac{dv}{v}$

$-ct = \ln\left(v/v_0\right)$

when $v = 0$ $\quad$ $\underbrace{\ln 0}_{-\infty} = -ct$ $\qquad$ $t = \infty$



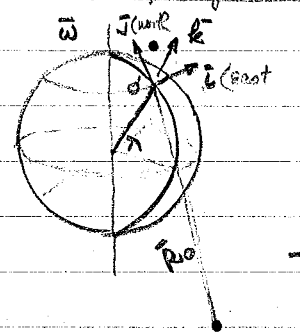


$$-f - mg\sin\theta = \bar{F}$$

$$(-\mu mg\cos\theta - mg\sin\theta)\,dx = -\frac{mv_0^2}{2}$$

$$x_{max} = \frac{v_0^2}{2g(\mu\cos\theta + \sin\theta)} = 16.5\text{ ft.}$$

<u>Motion of Projectile near surface of Earth.</u>



$$x_0 = y_0 = z_0 = 0$$

$$\dot{x}(0) = \dot{x}_0 \;;\; \dot{y}_0 \;;\; \dot{z}_0$$

$$\bar{F} = m\bar{a}$$

$$-mg\bar{k} = m\left\{\frac{d^2\bar{R}_0}{dt^2} + \frac{\delta^2\bar{r}}{\delta t^2} + \cancel{\frac{\delta\bar{\omega}}{\delta t}\times\bar{r}} + 2\bar{\omega}\times\frac{\delta\bar{r}}{\delta t} + \bar{\omega}\times(\bar{\omega}\times\bar{r})\right\}$$

$\frac{\delta\bar{\omega}}{\delta t}\times\bar{r}$; $\frac{d^2R_0}{dt^2}$ is small & can be neglected. so can $\bar{\omega}(\bar{\omega}\times\bar{r})$

$$m\frac{\delta^2\bar{r}}{\delta t^2} = -mg\bar{k} - m\,2\bar{\omega}\times\frac{\delta\bar{r}}{\delta t}$$

$$\frac{\delta^2\bar{r}}{\delta t^2} = -g\bar{k} - 2\bar{\omega}\times\frac{\delta\bar{r}}{\delta t} \qquad (1)$$

$$\ddot{x}\,\bar{i} + \ddot{y}\,\bar{j} + \ddot{z}\,\bar{k} = -g\bar{k}$$

$$\bar{\omega} = \omega\cos\lambda\,\bar{j} + \omega\sin\lambda\,\bar{k}$$

$$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \qquad (2)$$

$$\frac{\delta\bar{r}}{\delta t} = \dot{x}\bar{i} + \dot{y}\bar{j} + \dot{z}\bar{k}$$

3/8/71

Exam on 3/25   Start reading chapter 6.

└ Pendulum, Moving Frames, Central Force.

$$\frac{\delta^2 \bar{r}}{\delta t^2} = \ddot{x}\bar{i} + \ddot{y}\bar{j} + \ddot{z}\bar{k}$$

$$-2\bar{\omega} \times \frac{\delta \bar{r}}{\delta t} = -2\omega \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & \cos\lambda & \sin\lambda \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix}$$

$$-2\omega \left[(\dot{z}\cos\lambda - \dot{y}\sin\lambda)\bar{i} - (0 - \dot{x}\sin\lambda)\bar{j} + (0 - \dot{x}\cos\lambda)\bar{k} \right] \quad (3)$$

$$\ddot{x}\bar{i} + \ddot{y}\bar{j} + \ddot{z}\bar{k} = -g\bar{k} - 2\omega \left[(\dot{z}\cos\lambda - \dot{y}\sin\lambda)\bar{i} + (\dot{x}\sin\lambda)\bar{j} - (\dot{x}\cos\lambda)\bar{k} \right]$$

$$\ddot{x} = -2\omega(\dot{z}\cos\lambda - \sin\lambda\, \dot{y}) \quad (4)$$

$$\ddot{y} = -2\omega(\dot{x}\sin\lambda) \quad (5)$$

$$\ddot{z} = -g + 2\omega\dot{x}\cos\lambda \quad (6)$$

Integrate 5 & 6

$$\dot{y} = -2\omega\sin\lambda\, x + \dot{y}_0 \quad (7)$$

$$\dot{z} = -gt + 2\omega\cos\lambda\, x + \dot{z}_0 \quad (8)$$

$$\ddot{x} = -2\omega\left(-gt\cos\lambda + \underline{2\omega\cos^2\lambda\, x} + \dot{z}_0\cos\lambda + \underline{2\omega\sin^2\lambda\, x} - \dot{y}_0\sin\lambda\right)$$

but 2ω terms is small so we can neglect it

$$\ddot{x} = -2\omega\left[\cos\lambda(-gt + \dot{z}_0) - \sin\lambda(\dot{y}_0)\right]$$

$$\dot{x} = -2\omega\left[\cos\lambda\left(-\frac{gt^2}{2} + \dot{z}_0 t\right) - \sin\lambda(\dot{y}_0 t)\right] + \dot{x}_0$$

$$x = -2\omega\left[\cos\lambda\left(-\frac{gt^3}{6} + \frac{\dot{z}_0 t^2}{2}\right) - \sin\lambda\left(\frac{\dot{y}_0 t^2}{2}\right)\right] + \dot{x}_0 t + \cancel{x_0}^{\,0} \quad (10)$$

10 into 7 and 8

$$\left.\begin{aligned} y &= \dot{y}_0 t - \omega\sin\lambda\, \dot{x}_0 t \\ z &= -\tfrac{1}{2}gt^2 + \dot{z}_0 t + \omega\cos\lambda\, \dot{x}_0 t^2 \end{aligned}\right\} \text{neglecting } \omega^2 \text{ terms.}$$

Fire from origin at V_0 vertically.

$x_0 = y_0 = z_0 = 0$

$\dot{x}_0 = \dot{y}_0 = \dot{z}_0 = v_0$

$x = -2\omega\left[\cos\lambda\left(-\frac{gt^3}{6} + \frac{v_0 t^2}{2}\right)\right]$ ① $\quad y = 0 \quad z = -\frac{1}{2}gt^2 + v_0 t$ ③

$\dot{z} = -gt_1 + v_0 = 0$; $t_1 = \frac{v_0}{g}$ ④ ; $x_1 = \frac{\omega\cos\lambda}{3}\left[g\frac{v_0^3}{g^3}\right] - \omega v_0 \frac{v_0^2}{g^2}\cos\lambda$

$= -\frac{2}{3}\omega\cos\lambda\frac{v_0^3}{g^2}$ ⑤

$y_1 = 0$; $z_1 = -\frac{1}{2}\frac{v_0^2}{g} + \frac{v_0^2}{g} = \frac{1}{2}\frac{v_0^2}{g}$ ⑦

Down assume start at origin $\dot{x}_0 = \dot{y}_0 = \dot{z}_0 = 0$

$x = \frac{1}{3}\omega g t^3 \cos\lambda \quad y = 0 \quad z = -\frac{1}{2}gt^2$

when $z = -\frac{1}{2}\frac{v_0^2}{g}$ t_2 is time of fall.

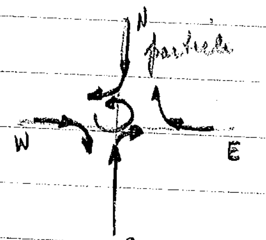
t_2 turns out to be $\frac{v_0}{g}$

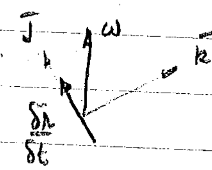
$x_2 = \frac{1}{3}\omega g\frac{v_0^3}{g^3}\cos\lambda = \frac{1}{3}\omega\frac{v_0^3}{g^2}\cos\lambda$ $\qquad x = x_1 + x_2 = -\frac{1}{3}\omega\cos\lambda\frac{v_0^3}{g^2}$

$v_2 = 0 \qquad z_2 = -\frac{1}{2}v_0^2/g$

$$\boxed{\begin{array}{l} X = -\frac{1}{3}\omega\cos\lambda\frac{v_0^3}{g^2} \\ z_{TOT} = 0 \qquad y_{TOT} = 0 \end{array}}$$

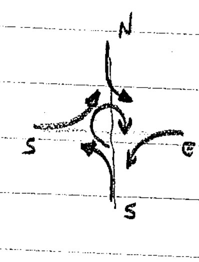
Suppose a particle moves north.

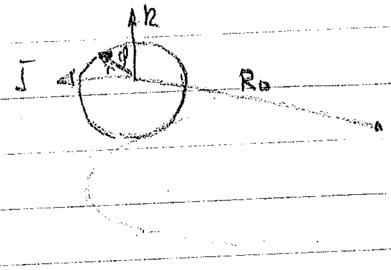




tends to curve around to east in northern hemisphere.

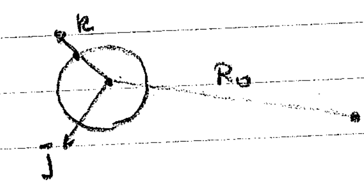
in southern hemisphere





$2\bar{\omega} \times \frac{\delta \bar{r}}{\delta t}$ exists due to rotation

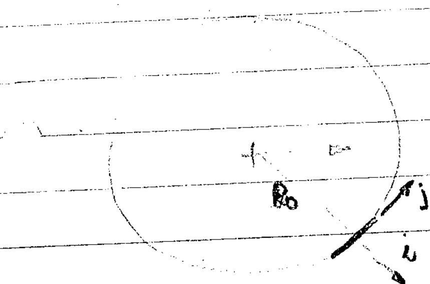
if



$\bar{r} = b\bar{k}$ $\bar{\omega} \times \frac{\delta \bar{k}}{\delta t}$ does not exist here

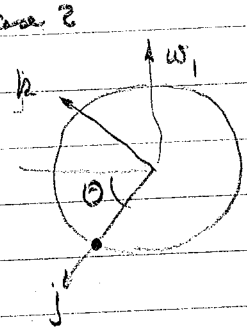
due

in case 1



$\bar{\omega} = \frac{V}{R_0}\bar{k}$

in case 2



$\bar{\omega} = \overbrace{\omega_1}^{V/R_0}(-\sin\theta j + \cos\theta i) + \underbrace{\omega_2}_{V/b}\bar{i}$

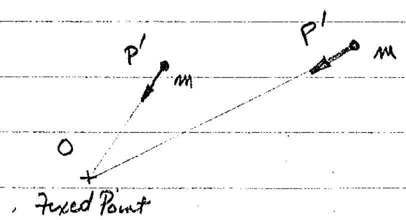
b = rad of wheel

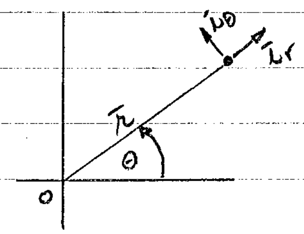
$\bar{r} = b\bar{j}$

$\frac{\delta \bar{r}}{\delta t} = \frac{\delta^2 \bar{r}}{\delta t^2} = 0$ $\frac{\delta \bar{\omega}}{\delta t} \times \bar{r} = 0$

3/11/70

Central Force Systems.





Angular Momentum $\bar{h}_o = \bar{r} \times m\bar{v}$

$$\frac{d\bar{h}_o}{dt} = \underbrace{\bar{v} \times m\bar{v}}_{0} + \bar{r} \times m\frac{d\bar{v}}{dt} = \bar{r} \times \bar{F} = 0 \quad \text{in a central force system.}$$

therefore $\bar{h}_o$ = constant. Motion is planar since $\bar{h}_o \perp \bar{r}$ & $\perp m\bar{v}$

from this $m r_1 v_1 \sin\phi_1 = m r_2 v_2 \sin\phi_2$ $\qquad v_1 \sin\phi_1 = v_{\theta_1} \quad v_2 \sin\phi_2 = v_{\theta_2}$

$\bar{F} = m\bar{a} = \frac{d}{dt}(m\bar{v})$ $\qquad$ also $\bar{M} = \frac{d}{dt}(\bar{h})$

$F = m\bar{a}$

$$F_r \bar{i}_r + F_\theta \bar{i}_\theta = m\left[\ddot{r} - r\dot{\theta}^2\right]\bar{i}_r + m\left[\frac{1}{r}\frac{d}{dt}\left(r^2\dot{\theta}\right)\right]\bar{i}_\theta$$

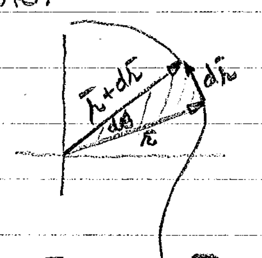
$$F_r = m\left(\ddot{r} - r\dot{\theta}^2\right) \quad (1)$$

$$F_\theta = m\left(\frac{1}{r}\frac{d}{dt}\left[r^2\dot{\theta}\right]\right) \quad (2)$$

$F_\theta = 0 \Rightarrow \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0 \qquad r^2\dot{\theta} = h$ = constant. (3)

but $r^2\dot{\theta} = r v_\theta$

note:



$dA = \frac{1}{2} r^2 d\theta \qquad \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta}$ = const.

$F_r = -mP \qquad P' = mP$

$= \left(\ddot{r} - r\dot{\theta}^2\right) m$

$$P = \ddot{r} - r\dot{\theta}^2$$

let $r = \frac{1}{u}$ $\qquad \dot{\theta} = \frac{h}{r^2} = hu^2$

$$\frac{dr}{dt} = \frac{d}{d\theta}(1/u)\frac{d\theta}{dt} = hu^2\left(-\frac{du}{d\theta}\right)\frac{1}{u^2} = -h\frac{du}{d\theta}$$

$$\frac{d^2r}{dt^2} = \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right)\frac{d\theta}{dt} = \dot{\theta}\left(-h\frac{d^2u}{d\theta^2}\right) = -h^2u^2\frac{d^2u}{d\theta^2}$$

$$-P = -h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}h^2u^4 \; ; \; \boxed{\frac{P}{h^2u^2} = \frac{d^2u}{d\theta^2} + u} \quad (4)$$

Energy form of equation of motion

$T + V = E$

$\frac{1}{2}v^2 + V = E$ $\qquad$ for unit mass

$$\frac{1}{2}\left[\dot{r}^2 + (r\dot{\theta})^2\right] = E - V$$

$$h^2\left(\frac{du}{d\theta}\right)^2 + \left(\frac{1}{u}hu^2\right)^2 = \frac{2(E-V)}{1}$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-V)}{h^2} \quad (5)$$

$\frac{d}{dt}$ of eq 5 = eq 4

Try Problems 4-17, 19, 21

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$$\frac{d^2u}{d\theta^2} + u = \frac{P(u)}{h^2u^2}$$

or

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \frac{2(E-V)}{h^2}$$

Orbits in Inverse Square Force Fields

$$P = \frac{\mu}{r^2} = \mu u^2$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}$$

$$u_h = A\cos\theta + B\sin\theta \qquad u_p = \mu/h^2$$

$$u = A\cos\theta + B\sin\theta + \mu/h^2$$

or

$$u = C\cos(\theta - \theta_0) + \frac{\mu}{h^2}$$

Arbitrary Constants from initial conditions

$\bar{r}(t=0)$; $\bar{v}(t=0) \to r, \theta, v_r, v_\theta$

set up coordinate system so that $\theta_0 = 0$

Rotate axis to make $\theta_0 = 0$ $\qquad u = C\cos\theta + \mu/h^2$

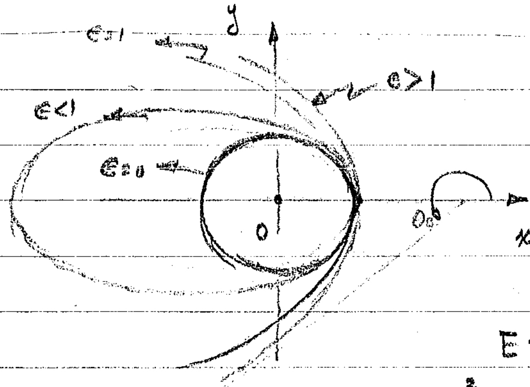
$$u = \mu/h^2\left[1 + \epsilon\cos\theta\right] \qquad \epsilon = \frac{Ch^2}{\mu}$$

$$u = \frac{1}{l}\left[1 + \epsilon\cos\theta\right] \qquad l = h^2/\mu$$

$1+\epsilon\cos\theta$ equation of conic section with one of foci at origin in terms of E & h

$$\epsilon = \sqrt{1 + \frac{2Eh^2}{\mu^2}}$$

for earth orbits $P = \mu/r^2 = \frac{GM}{r^2} = \frac{gR^2}{r^2}$ where $\mu = 1.41\times10^{16}\ \frac{ft^3}{sec^2}$



$\frac{du}{d\theta} = 0$

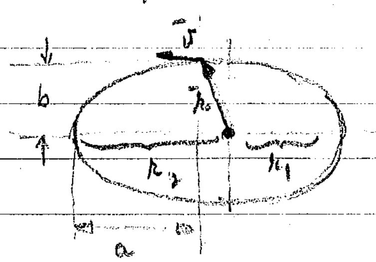
$\epsilon > 1$	hyperbola	$E > 0$
$\epsilon = 1$	parabola	$E = 0$
$\epsilon < 1$	ellipse	$E < 0$
$\epsilon = 0$	circle	"

$$E = T + V = \frac{v^2}{2} - \frac{\mu}{r}$$ grav. potential

as $r \to \infty$ $\quad E \to \frac{v^2}{2} > 0$; $v = \sqrt{2E}$

Escape velocity as $r \to \infty$ $\quad v \to 0$ $\quad E \to 0$

<u>Elliptic Orbits</u>



$$a = \frac{1}{2}(r_1 + r_2) = \frac{1}{2}\left(\frac{l}{1+\epsilon} + \frac{l}{1-\epsilon}\right)$$

$$\longrightarrow a = \frac{l}{1-\epsilon^2} = \frac{h^2}{\mu\left[1 - \left(1 + \frac{2Eh^2}{\mu^2}\right)\right]} = -\frac{\mu}{2E}$$

$b = a\sqrt{1-\epsilon^2}$ from $a = \frac{l}{1-\epsilon^2}$ $\qquad b = \sqrt{\frac{l^2}{1-\epsilon^2}} = \sqrt{\frac{-h^2}{2E}}$ $\longleftarrow$

$$v^2 = 2\left(E + \mu/r\right) = -\frac{\mu}{a} + \frac{2\mu}{r}$$

$$\boxed{v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]}$$

$\frac{dA}{dt} = \frac{h}{2}$ $\qquad \int_0^{\tau} dt = \int \frac{2}{h}\, dA$

$$\tau = \frac{2}{h}\pi ab$$

$$\tau^2 = \frac{4}{h^2}\pi^2 a^2 b^2 = \frac{4\pi^2}{h^2} a^2\left(\frac{-h^2}{2E}\right) = \frac{-4\pi^2 a^2}{-\mu/a} = \frac{4\pi^2}{\mu}a^3$$

$$\tau = 2\pi\sqrt{a^3/\mu}$$

Elementary Problems in Space Mechanics
(Use Conservation of Energy & Momentum Theorem)

Escape Velocity

Def: v_e $\quad r \to \infty \quad v \to 0$

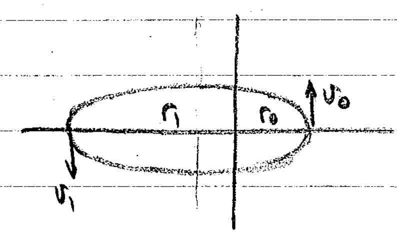
$$E = 0 = \frac{v_e^2}{2} - \frac{gR^2}{r}$$

$$\text{at } r = r_0 \qquad v_{e_0}^2 = \frac{2gR^2}{r_0} \qquad v_{e_0} = \sqrt{\frac{2gR^2}{r_0}}$$

at $r_0 = R$ $\quad v_e = \sqrt{2gR}$ at earth's surface, $\approx$ 25000 mph.

1.467 mph = 1 ft/sec

$$v_{e_0} = v_e \sqrt{\frac{R}{r_0}} \qquad R = 3960 \text{ miles}$$



$$\frac{v_1^2}{2} - \frac{gR^2}{r_1} = \frac{v_0^2}{2} - \frac{gR^2}{r_0}$$

$$r_0 v_0 = r_1 v_1$$

$$\frac{v_0^2}{2} - \frac{gR^2}{r_0} = \frac{1}{2}\left(\frac{r_0}{r_1} v_0\right)^2 - \frac{gR^2}{r_1}$$

$$v_0^2\left(1 - \frac{r_0^2}{r_1^2}\right) = \frac{2gR^2}{r_0}\left(1 - \frac{r_0}{r_1}\right)$$

$$v_0^2\left(1 + \frac{r_0}{r_1}\right) = \frac{2gR^2}{r_0}$$

$$\boxed{\frac{r_0}{r_1} = \frac{2gR^2}{r_0 v_0^2} - 1}$$

$$\frac{v_1}{v_0} = \frac{r_0}{r_1}$$

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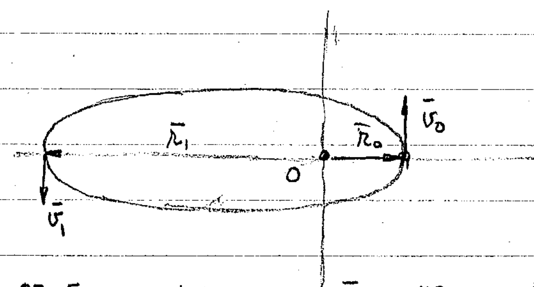
Do problems 1, 2, 3

Example:



$v_0 = 22,500$ mph $\qquad \bar{r}_0 = 4300$ miles

① $\dfrac{v_0^2}{2} - \dfrac{\mu}{r_0} = \dfrac{v_1^2}{2} - \dfrac{\mu}{r_1}$

② $v_0 r_0 = v_1 r_1 \qquad (h = r^2\dot{\theta} = r v_\theta)$

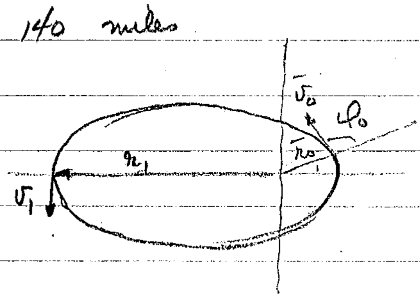
$$\frac{r_0}{r_1} = \frac{2(gR^2)}{r_0 v_0^2} - 1 = \frac{v_e^2 R}{r_0 v_0^2} - 1 = \left(\frac{25000}{22500}\right)^2 \frac{3960}{4300} - 1$$

$$\frac{r_0}{r_1} = 1.139 - 1 = .139$$

$$r_1 = \frac{4300}{.139} = 30,950 \qquad h_1 = 26,990 \text{ miles}$$

$$v_1 = \frac{r_0}{r_1} v_0 = .139(22500) =$$

How much error in φ_0 of satellite is not to come closer than 140 miles



$$\frac{v_1^2}{2} - \frac{\mu}{r_1} = \frac{v_0^2}{2} - \frac{\mu}{r_0}$$

$$r_1 v_1 = r_0 v_0 \sin\varphi_0$$

$$r_1 = 4100 \qquad r_0 = 4300$$

$$\frac{1}{2}\left(\frac{r_0 v_0 \sin\varphi_0}{r_1}\right)^2 - \frac{gR^2}{r_1} = \frac{v_0^2}{2} - \frac{gR^2}{r_0}$$

$$\left(\frac{r_0}{r_1}\right)^2 v_0^2 \sin^2\varphi_0 = v_0^2 + \frac{2gR^2}{r_1}\left(1 - \frac{r_1}{r_0}\right)$$

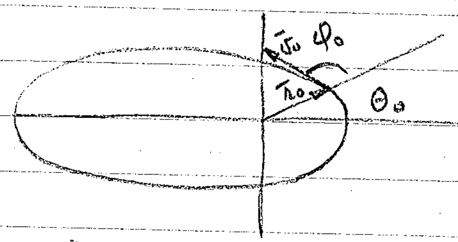
$$\frac{r_0}{r_1} = 1.0488$$

$v_0 = 33{,}000$ ft/sec

$$(33\times10^3)^2\left[(1.0488)^2 \sin^2\varphi_0 - 1\right] = \frac{2(1.408)\times10^{16}}{4100(5280)}\left[1 - \frac{1}{1.0488}\right]$$

$$\sin\varphi_0 = 0.9796$$

$\varphi_0 = 78.4^\circ$ true $\varphi_0 = 90^\circ \pm 11.6^\circ$



Given $\bar{r}_0$ & $\bar{v}_0$ what is equation of trajectory

$$u = \frac{\mu}{h^2}\left[1 + \epsilon\cos(\theta - \theta_0)\right]$$

$h = r_0 v_0 \sin\varphi_0$ $\mu = gR^2$

$$E = \frac{v_0^2}{2} - \frac{\mu}{r_0}$$

$$\epsilon = \sqrt{1 + \frac{2Eh^2}{\mu^2}} = \sqrt{1 + \frac{2\left(\frac{v_0^2}{2} - \frac{\mu}{r_0}\right) r_0^2 v_0^2 \sin^2\varphi_0}{(1.408\times10^{16})^2}}$$

at $\theta = 0$ $u_0 = \frac{\mu}{h^2}\left(1 + \epsilon\cos\theta_0\right)$

$$\cos\theta_0 = \frac{1}{\epsilon}\left[\frac{h^2}{r_0\mu} - 1\right]$$

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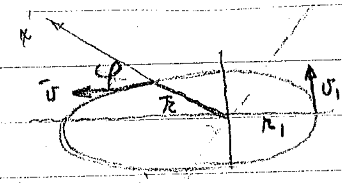


Given $\vec{r}(0)$ $v(0)$

$$u = \mu/h^2 \left[1 + \epsilon \cos\theta\right]$$

or $$u = C\cos(\theta - \theta_0) + \frac{\mu}{h^2}$$

$$= \frac{\mu}{h^2}\left(1 + \epsilon\cos(\theta - \theta_0)\right)$$

$$h = r^2\dot{\theta} = r v_\theta = |\bar{r}||\bar{v}|\sin\varphi$$

apsidal angle lies between 2 pts where $\frac{du}{d\theta} = 0$.

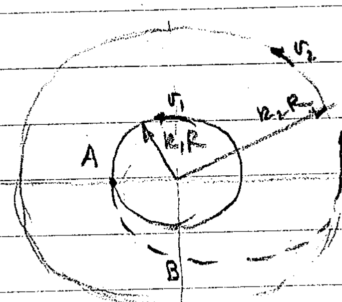
$$h = r(0)\,v(0)\sin\varphi = r_1 v_1$$

$$\epsilon = \sqrt{1 + \frac{2Eh^2}{\mu^2}} \qquad E = \frac{v(0)^2}{2} - \frac{\mu}{r(0)} \qquad F_r = -\frac{\partial V}{\partial r}$$

at $t=0$ $$\frac{1}{r(0)} = \frac{\mu}{h^2}\left[1 + \epsilon\cos(-\theta_0)\right]$$

after θ_0 is obtained

$$u = \frac{\mu}{h^2}\left[1 + \epsilon\cos(\theta - \theta_0)\right]$$



Instantaneous veloc. changes $K_1R \to K_2R$

$$v_{@C} + \Delta v = v_2$$

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

$$v_1^2 = gR^2\left[\frac{2}{k_1R} - \frac{1}{k_1R}\right] = \frac{gR}{k_1} \quad ①$$

$$v_A^2 = gR^2\left[\frac{2}{k_1R} - \frac{2}{(k_1+k_2)R}\right] = 2gR\left[\frac{k_2}{k_1(k_1+k_2)}\right] \quad ②$$

$$v_C^2 = 2gR\left[\frac{k_1}{k_2(k_1+k_2)}\right] \quad ③$$

Algazi
Farhi

$$v_2^2 = \frac{gR}{k_2} \quad (4)$$

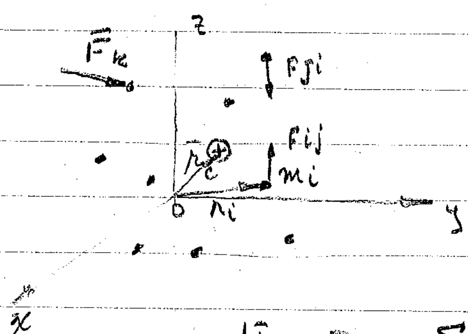
$$\Delta V_A = V_A - V_1 = \sqrt{2gR\left[\frac{k_2}{k_1(k_1+k_2)}\right]} - \sqrt{\frac{gR}{k_1}}$$

$$\Delta V_c = V_2 - V_c = \sqrt{\frac{gR}{k_2}} - \sqrt{2gR\left[\frac{k_1}{k_2(k_1+k_2)}\right]}$$

$$\text{total Energy} = \frac{1}{2}m(v_A^2 - v_1^2) + \frac{1}{2}m(v_2^2 - v_c^2)$$

$$= \frac{mgR}{2}\left[\frac{k_2 - k_1}{k_1 k_2}\right]$$

System of Particles



Linear Momentum Theorem

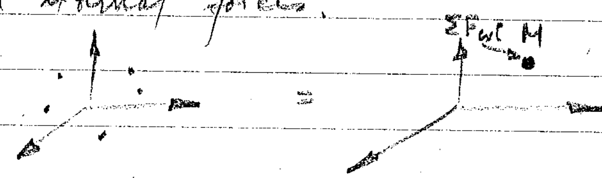
$$M\bar{v}_c = \sum_i m_i \bar{v}_i \; ; \; M = \sum_i m_i$$

$$\text{Define } \bar{r}_c = \frac{\sum_i m_i \bar{r}_i}{\sum m_i}$$

$$\frac{d\bar{r}_c}{dt} = \bar{v}_c = \frac{\sum_i m_i \bar{v}_i}{M}$$

$$\sum_i m_i \bar{v}_i = M\bar{v}_c \quad \text{QED}$$

The center of mass of system of particles moves as if it were a single particle of mass M acted upon by the resultant of all external forces.



Theorem: $\bar{F}_i = m_i \frac{d}{dt}\bar{v}_i$

$$\bar{F}_i^{(e)} + \sum_j \bar{F}_{ij} = m_i \frac{d}{dt}\bar{v}_i \qquad \text{for } i^{th} \text{ particle}$$

the total sum: $\sum_i \bar{F}_i^{(e)} + \sum_i \sum_j \bar{F}_{ij} = \sum m_i \frac{d}{dt}\bar{v}_i$

according to newton's third law $F_{ji} = -F_{ij} \Rightarrow \sum_i \sum_j \bar{F}_{ij} = 0$
the forces do not have to be colinear.

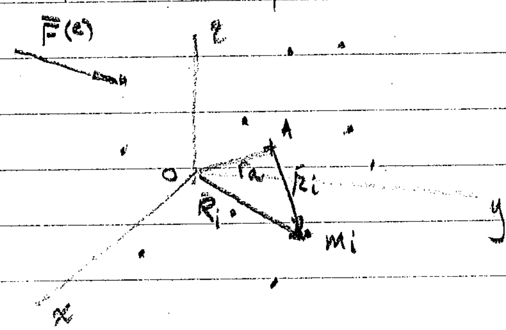
$$\sum_i \bar{F}_i^{(e)} = \sum m_i \frac{d}{dt}\bar{v}_i = M\frac{d}{dt}\bar{v}_c$$

If $\sum_i \bar{F}_i^{(e)} = 0$ then $M\frac{d}{dt}\bar{v}_c = \frac{d}{dt}(M\bar{v}_c) = 0$

$\therefore$ $M\bar{v}_c$ is a constant and is the principle of Constant Linear Momentum

<u>Angular Momentum</u> total

The time rate of change of angular momentum of a system of particles about a pt a is equal to the total External moment about pt A if a is a fixed pt or if a is the center of mass.



$\bar{h}_A = \sum_i \bar{r}_i \times m_i \bar{v}_i$

where $\bar{v}_i = \frac{d\bar{R}_i}{dt} = \frac{d}{dt}(\bar{r}_a + \bar{r}_i)$

$$\frac{d\bar{h}_A}{dt} = \sum_i \dot{\bar{r}}_i \times m_i(\dot{\bar{r}}_a + \dot{\bar{r}}_i) + \sum_i \bar{r}_i \times m_i \frac{d\bar{v}_i}{dt}$$

$$= \sum_i \dot{\bar{r}}_i \times m_i \dot{\bar{r}}_a + \sum_i \bar{r}_i \times \bar{F}_i$$

if pt A is fixed $\dot{\bar{r}}_A = 0$ $\quad \therefore \frac{d\bar{h}_a}{dt} = \sum r_i \times \bar{F}_i$

if pt A is at center of mass $\sum m_i \dot{\bar{r}}_i \times \dot{\bar{r}}_A = 0$

$$\bar{r}_c = \frac{\sum_i m_i \bar{r}_i}{\sum_i m_i} = 0 \qquad \sum m_i \bar{r}_i = 0 \rightarrow \sum m_i \frac{d\bar{r}_i}{dt} = 0$$

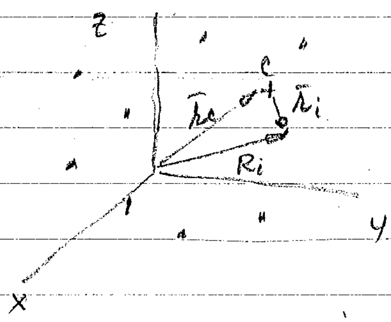
$$\therefore \quad (m_1 \bar{r}_1 + \dots m_i \bar{r}_i + \dots) \times \ddot{\bar{r}}_a = 0$$

$$\therefore \quad \dot{\bar{h}}_A = \sum_i \bar{G}_i$$

if forces are colinear $\sum_i \bar{G}_i^{(e)} + \sum_i \bar{G}_i^{(int)} = \sum \bar{G}_i$

and $\sum_i \bar{G}_i^{(int)} = 0 \quad \therefore \quad \sum_i \bar{G}_i^{(e)} = \sum \bar{G}_i = \dot{\bar{h}}_A$

Theorem: the total angular momentum of a system of particles about the mass center may be calculated from the absolute velocities or the velocities relative to mass center.



$$\bar{h}_c = \sum_i \bar{r}_i \times m_i \bar{v}_i = \sum_i \bar{r}_i \times m_i (\dot{\bar{r}}_c + \dot{\bar{r}}_i)$$

if pt c is a center of mass $\sum_i \bar{r}_i \times m_i \dot{\bar{r}}_c = 0$

$$\therefore \quad \bar{h}_c = \sum \bar{r}_i \times m_i \dot{\bar{r}}_i$$

find: $\dot{\bar{h}}_c = \sum_i \dot{\bar{r}}_i \times m_i \dot{\bar{r}}_i + \sum_i \bar{r}_i \times m_i \frac{d\bar{v}_i}{dt} = \sum_i \bar{r}_i \times \bar{F}_i$

since $\dot{\bar{r}}_i \times \dot{\bar{r}}_i = 0$; since $\bar{r}_i = R_i - \bar{r}_c$ then

$$\sum_i \bar{r}_i \times F = \sum \bar{r}_i \times m_i [\ddot{\bar{R}}_i - \ddot{\bar{r}}_c] = \sum \bar{r}_i \times m_i \ddot{\bar{R}}_i$$

but $\bar{F}_i = m_i \ddot{\bar{R}}_i$ then $\sum \bar{r}_i \times \bar{F}_i$

Ch. 7 1, 3.

3/30/71

Work & Energy Principle

$$\bar{F}_i = m_i \frac{d^2 \bar{r}_i}{dt^2}$$

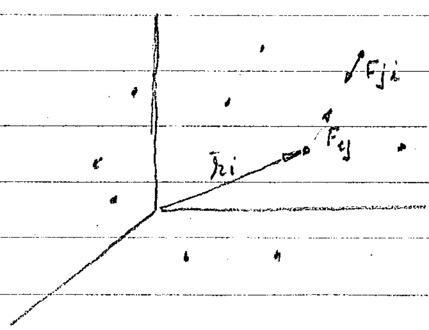
$$dW = \bar{F}_i \cdot d\bar{r}_i$$

$$= \sum m_i \frac{d\bar{v}_i}{dt} \cdot \bar{v}_i \, dt$$

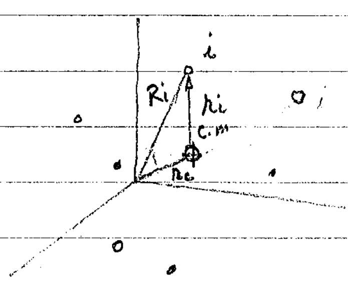
$$= d\sum \frac{m_i}{2} \bar{v}_i \cdot \bar{v}_i = d\sum \frac{m_i}{2} v_i^2$$

$$dW = dT$$

Exception: you cannot exclude the internal forces which produce work



Kinetic Energy



$$T = \sum_i \frac{1}{2} m_i \bar{v}_i \cdot \bar{v}_i$$

$$= \sum_i \frac{1}{2} m_i (\dot{\bar{R}}_c + \dot{\bar{r}}_i) \cdot (\dot{\bar{R}}_c + \dot{\bar{r}}_i)$$

$$= \sum_i \frac{1}{2} m_i \dot{\bar{R}}_c^2 + \overset{0}{\cancel{\sum_i m_i \dot{\bar{r}}_i \dot{\bar{R}}_c}} + \sum \frac{1}{2} m_i \dot{\bar{r}}_i^2$$

$$T = \underbrace{\frac{1}{2} M v_c^2}_{\text{K.E. of c.m.}} + \sum \frac{1}{2} m_i v_i'^2$$

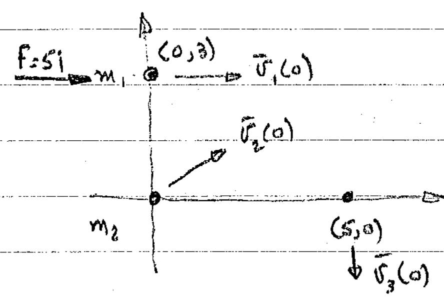
v_i' - relative velocity.

D'alembert Forces.

$$\sum_i \bar{F}_i^{(e)} = M \frac{d\bar{v}_c}{dt}$$

$$\sum_i \bar{F}_i^{(e)} - M \frac{d\bar{v}_c}{dt} = 0$$

Example



@ $t=0$ $\bar{r}_1 = 3j$ $\bar{v}_1 = 10i$

$\bar{r}_2 = 0$ $\bar{v}_2 = 5i + 5j$

$\bar{r}_3 = 5i$ $\bar{v}_3 = -10j$

$m_1 = 1$

$m_2 = 3$

$m_3 = 6$

at $t=0$ what is $\bar{r}_c = \frac{\sum_i m_i \bar{r}_i}{\sum_i m_i} = \frac{3j + 6(5i)}{10} = .3j + 3i$

at $t=0$ $\bar{p} = \sum_i m_i \bar{v}_i = 10i + 15i + 15j - 60j = 25i - 45j$

at $t=0$ $\bar{v}_c = \frac{\bar{p}}{\sum m_i} = 2.5i - 4.5j$

$\bar{h}_o(0) = \sum \bar{r}_i \times m_i \bar{v}_i = 3j \times (10i) + 0 + 5i \times (-60j)$
$= -30\bar{k} - 300\bar{k} = -330\bar{k}$

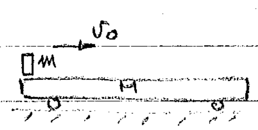
$$\sum_i \bar{F}_i^{(e)} = M\frac{d\bar{v}_c}{dt} = 5i = M\left[\ddot{x}_c i + \ddot{y}_c j\right]$$

$5 = M\ddot{x}_c$ $\quad 0 = M\ddot{y}_c$

$\boxed{\frac{5}{M}t = \dot{x}_c - 2.5}$ $\quad \boxed{\dot{y}_c = -4.5}$

$\boxed{x_c = \frac{5}{2M}t^2 - 2.5t + 3}$ $\quad \boxed{y_c = -4.5t + .3}$

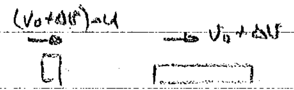
Examples



$$\sum_i \bar{F}_i^{(e)} = M \frac{d\bar{v}_c}{dt} = 0 \quad \rightarrow \quad M\bar{v}_c = \text{constant}.$$

initial momentum = final momentum.

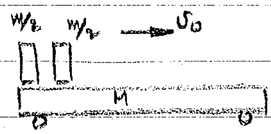
$$(M+m)v_0 = M(v_0+\Delta v) + m(v_0+\Delta v - u)$$



u is relative of man

$$0 = \Delta v\,(M+m) - mu$$

$$\Delta v = \frac{m}{M+m}\,u$$



1 jumps off first relative speed u

$$(M+m)v_0 = (M+m/2)(v_0+\Delta v) + m/2\,(v_0+\Delta v - u)$$

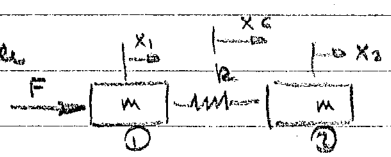
$$0 = \Delta v\,(M+m) - m/2\; u \qquad \Delta v_1 = \frac{m}{2(M+m)}\,u$$

$$(M+m/2)(v_0+\Delta v_1) = M(v_0+\Delta v_1+\Delta v_2) + m/2\,(v_0+\Delta v_1+\Delta v_2-u)$$

$$0 = M\Delta v_2 + \frac{m}{2}\Delta v_2 - u\,m/2 \qquad \Delta v_2 = \frac{um}{2(M+m/2)}$$

$$\Delta V_T = \Delta V_1 + \Delta V_2$$

Example



$m_1 = m_2 = m$

$x(0) = 0 \qquad \dot{x}(0) = 0$

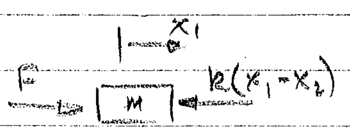
$x_1 = x_1(t)$

$x_2 = x_2(t)$

$$\frac{x_1+x_2}{2} = x_c$$

$$F = 2m\frac{dv_c}{dt} \qquad \boxed{v_c = \frac{F}{2m}t}$$

$$\boxed{x_c = \frac{F}{4m}t^2}$$



$$F - k(x_1 - x_2) = m\ddot{x}_1$$

$$F - k(x_1 - 2x_c + x_1) = m\ddot{x}_1$$

$$F - 2k(x_1 - x_c) = m\ddot{x}_1 \qquad F/2 - 2k(x_1-x_c) = m(\ddot{x}_1 - \ddot{x}_c)$$

$$F/2 = m\ddot{x}_c \qquad (\ddot{x}_1 - \ddot{x}_c) + \frac{2k}{m}(x_1 - x_c) = \frac{F}{2m}$$

$$(x_1 - x_c) = A\sin\sqrt{\frac{2k}{m}}\,t + B\cos\sqrt{\frac{2k}{m}}\,t + \frac{F}{4k}$$

$$x_1 = \frac{Ft^2}{4m} + \frac{F}{4k}\left(1 - \cos\sqrt{\frac{2k}{m}}\,t\right)$$

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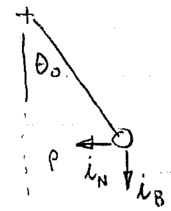
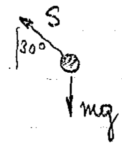
Test Review

1.

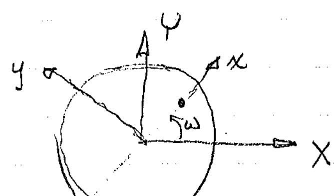




$$S\cos\theta_0 = mg$$
$$S\sin\theta_0 = \frac{mv^2}{\rho}$$

$v \cong 6.1$ ft/sec

2.

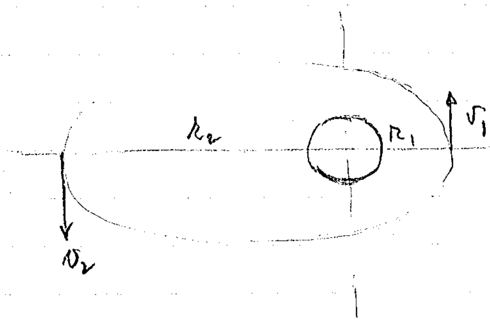


$$\bar{a}_{abs} = \cancel{\frac{d^2\bar{R}_0}{dt^2}} + \cancel{\frac{\delta^2\bar{r}}{\delta t^2}} + \cancel{\frac{\delta\bar{\omega}}{\delta t}\times\bar{r}} + 2\bar{\omega}\times\frac{\delta\bar{r}}{\delta t} + \bar{\omega}\times(\bar{\omega}\times\bar{r})$$

$$\bar{r} = xi$$
$$\frac{\delta\bar{r}}{\delta t} = vi$$
$$\bar{\omega} = \omega k$$

$$\bar{a}_{abs} = 2\omega\bar{k}\times v\bar{i} + (\omega\bar{k})\times(\omega\bar{k}\times xi)$$

3.



$$r_1 v_1 = r_2 v_2$$

Impulsive Forces.

<u>Particle</u>

$$\bar{F} = \frac{d}{dt}(m\bar{v}) \qquad \underbrace{\int_{t_0}^{t_1}\bar{F}\,dt}_{\text{Impulse}} = \int\frac{d}{dt}(m\bar{v})\,dt = d(m\bar{v})$$

$$\boxed{(m\bar{v})_i + \int_{t_0}^{t_1}\bar{F}\,dt = (m\bar{v})_f}$$

$$\bar{G} = \frac{d}{dt}\bar{h} \qquad \text{when} \qquad \bar{h} = \bar{r}\times m\bar{v}$$

$$\int_{t_0}^{t_1} G\,dt = d\bar{h} \rightarrow \boxed{\bar{h}_i + \int_{t_0}^{t_1} G\,dt = \bar{h}_f}$$

angular impulse.

Collision $F \rightarrow \infty$

$$\lim_{t_1\to t_0}\int_{t_0}^{t_1}\bar{F}\,dt = \hat{F} \quad \text{impulsive Force}$$

$\therefore$ we get $\hat{F} = \Delta m\bar{v}$ $\quad \boxed{(m\bar{v})_{init} + \hat{F} = (m\bar{v})_{final}}$

$$\lim_{t_1 \to t_0} \int_{t_0}^{t_1} \bar{G}\,dt = \hat{G} \text{ - impulsive moment } = \lim_{t_1 \to t_0} \int (\bar{r} \times \bar{F})\,dt \quad \text{if } \bar{r} \text{ is a constant}$$

$$\hat{G} = \bar{r} \times \lim_{t_1 \to t_0} \int_{t_0}^{t_1} \bar{F}\,dt = \bar{r} \times \hat{F} \qquad \boxed{\hat{G} = \bar{r} \times \hat{F}}$$

$$\boxed{\bar{h}_{init} + \hat{G} = \bar{h}_f}$$

<u>Summary</u>

$\hat{F} = \Delta m\bar{v}$; $\hat{G} = \Delta \bar{h}$ $\qquad$ particle

$\hat{F} = \Delta m\bar{v}_c$; $\hat{G} = \Delta \bar{h}_c$ $\qquad$ system.

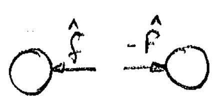
<u>Collision of two particles</u>

line of motion.

$u_1 \quad u_2$ are speeds before collision

$u_1' \quad u_2'$ after collision

$(m\bar{v})_i + \hat{F} = (m\bar{v})_f$ $\qquad$ Consider whole system.



$$m_1 u_1 + m_2 u_2 + 0 = m_1 u_1' + m_2 u_2' \quad (1)$$

$$m_1 u_1 + \hat{F} = m_1 u_1'$$

coefficient of restitution $\quad \epsilon = \dfrac{v_{separat}}{v_{approach}} = \dfrac{u_2' - u_1'}{u_1 - u_2} \quad (2)$

given

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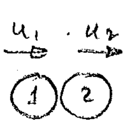

$u_1' \quad u_2'$

$$m_1 u_1 + m_2 u_2 = m_1 u_1' + m_2 u_2' \quad (1)$$

$$e = \frac{u_1' - u_2'}{u_2 - u_1} \quad (2)$$

$$u_1' = \frac{\left[(m_1 - m_2 e) u_1 + m_2(1+\epsilon) u_2\right]}{m_1 + m_2}$$

$$u_2' = \left[m_1(1+\epsilon) u_1 + (m_2 - m_1 \epsilon) u_2\right] / (m_1 + m_2)$$

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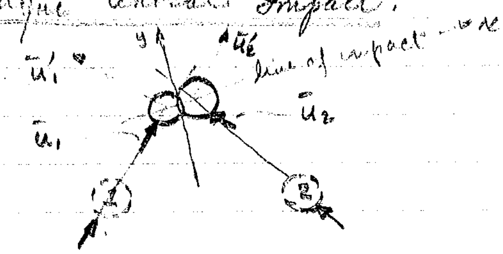
if $e = 0$ $\quad u_2' = u_1'$ (plastic collision)

if $e = 1$ also $m_1 = m_2$ $\quad \begin{matrix} u_1 = u_2' \\ u_2 = u_1' \end{matrix}$ (elastic collision)

Oblique Central Impact.



known $\bar u_1, \bar u_2$ line of impact
find $\bar u_1', \bar u_2'$

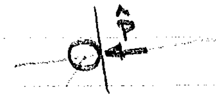
Consider Entire System.

$$m_1 \bar u_1 + m_2 \bar u_2 = m_1 \bar u_1' + m_2 \bar u_2'$$

$$m_1 u_{1x} + m_2 u_{2x} = m_1 u_{1x}' + m_2 u_{2x}' \quad ①$$

$$m_1 u_{1y} + m_2 u_{2y} = m_1 u_{1y}' + m_2 u_{2y}' \quad ②$$ not indep. of 3 & 4

Particle 1 alone



in y direct. impulse

$$m_1 u_{1y} + 0 = m_1 u_{1y}'$$

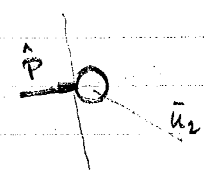
$$\boxed{u_{1y} = u_{1y}'} \quad ③$$

$\bar u_1$

in x direct.

$$m_1 u_{1x} - \hat P = m_1 u_{1x}' \quad ⑥$$

Particle 2 alone



in y direct

$$m_2 u_{2y} + 0 = m_2 u_{2y}'$$

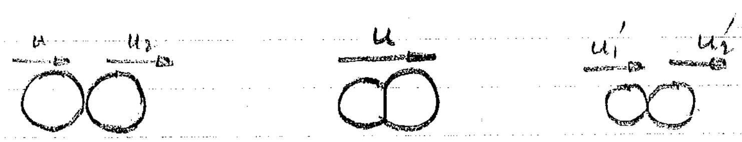
$$\boxed{u_{2y} = u_{2y}'} \quad ④$$

$$m_2 u_{2x} + \hat P = m_2 u_{2x}' \quad ⑦$$

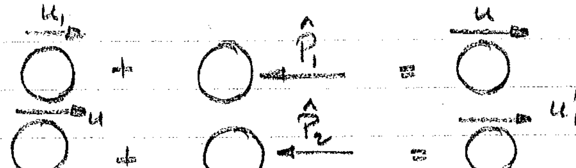
$$e = \frac{(u_2')_x - (u_1')_x}{(u_1)_x - (u_2)_x} \quad ⑤$$ adding 6 & 7 gives equ. 1.

Coefficient of Restitution may be defined also as

$$e = \frac{\hat P_2}{\hat P_1}$$ impulsive force of restitution
" " " compression



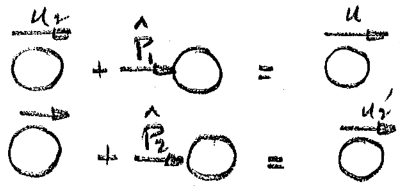
Sphere 1



$$m_1 u_1 - \hat P_1 = m_1 u$$

$$m_1 u - \hat P_2 = m_1 u_1'$$

Sphere 2



$$m_2 u_2 + \hat{P}_1 = m_2 u$$

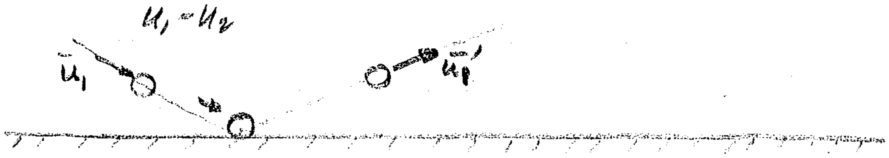
$$m_2 u + \hat{P}_2 = m_2 u_2'$$

Consider

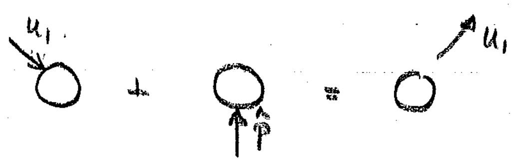
$$e = \frac{\hat{P}_2}{\hat{P}_1} = \frac{u - u_1'}{u_1 - u} \qquad e = \frac{\hat{P}_2}{\hat{P}_1} = \frac{u_2' - u}{u - u_2}$$

$$\frac{u - u_1'}{u_1 - u} = \frac{u_2' - u}{u - u_2} \quad \longrightarrow \text{find } u \text{ and put into } e$$

$$\therefore \quad e = \frac{u_2' - u_1'}{u_1 - u_2}$$



$\cancel{m_1 \bar{u}_1 = m_1 \bar{u}_1'}$ $\qquad$ $\cancel{m_1 u_{1x} = m_1 u_{1x}'}$ $\qquad$ $\cancel{u_{1x} = u_{1x}'}$



$$m_1 u_{1x} = m_1 u_{1x}'$$

$$e = \frac{(u_1')_y - (u_2')_y}{(u_2)_y - (u_1)_y} = \frac{(u_1')_y - 0}{0 - (u_1)_y}$$

$$-e(u_1)_y = (u_1')_y$$

$\bar{u}_1 = 10$ ft/sec

$\bar{u}_2 = -20$ ft/sec

$m_1 = 1$ unit $\qquad m_2 = 10$ units

$$m_1 \bar{u}_1 + m_2 \bar{u}_2 = m_1 \bar{u}_1' + m_2 \bar{u}_2'$$

$$1 \cdot 10 + 10(-20) = 1 \cdot \bar{u}_1' + 10\,\hat{u}_2'$$

$$-190 = u_1' + 10 u_2' \longrightarrow (1)$$

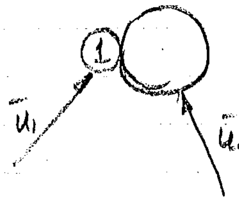
$$e = \frac{u_2' - u_1'}{u_1 - u_2} \qquad \text{let} \quad e = .5 \qquad .5 = \frac{u_2' - u_1'}{30}$$

$$u_2' - u_1' = 15 \qquad u_1' + 10(15 + u_1') = -190$$

$$11 u_1' = -340$$

$$u_1' = \frac{-340}{11} = -30.9 \longleftarrow$$

$$u_2' = 15 + u_1' = 15 - 30.9 = -15.9 \longleftarrow$$



H.W.

$e = .5 \quad m_1 = 1 \quad m_2 = 10$

$\bar{u}_1 = 10i + 10j \qquad \bar{u}_2 = -20i + 15j$

find $\bar{u}_1'$ & $\bar{u}_2'$

$$T - T' = (1 - e^2)\frac{1}{2}\left[m_1 v_1^2 + m_2 v_2^2 \right]$$

$u_1 = v_c + v_1 \qquad\qquad u_1' = v_c + v_1'$

v_c = center of mass $\quad v_1$ speeds relative to center of mass

$u_2 = v_c + v_2 \qquad\qquad u_2' = v_c + v_2'$

$$T = \frac{1}{2} m_1 (v_c + v_1)^2 + \frac{1}{2} m_2 (v_c + v_2)^2$$

$$T' = \frac{1}{2} m_1 (v_c + v_1')^2 + \frac{1}{2} m_2 (v_c + v_2')^2$$

H.W. Problem 7.13

before impact

$$m_1(v_c + v_1) + m_2(v_c + v_2) = (m_1 + m_2) v_c$$

after impact

$$m_1(v_c + v_1') + m_2(v_c + v_2') = (m_1 + m_2) v_c$$

and $e =$

$$e = \frac{u_2' - u_1'}{u_1 - u_2} = \frac{v_c + v_2' - v_c - v_1'}{v_c + v_1 - v_c - v_2} = \frac{v_2' - v_1'}{v_1 - v_2}$$

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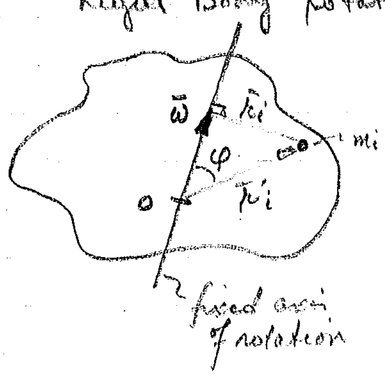
<u>Rigid Bodies</u>

Collection of particles where relative position of particles remains unchanged.

Eh!

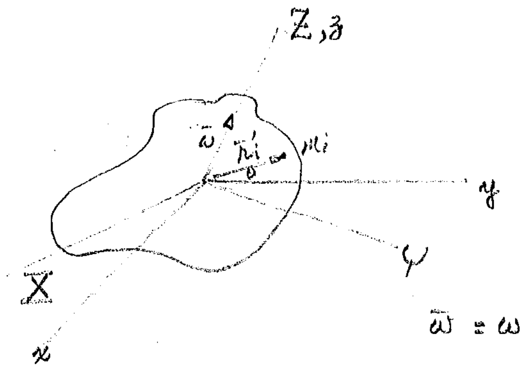
<u>Plane Motions of Rigid Bodies.</u>

Rigid Body rotating about a fixed axis



$$\boxed{\bar{v}_i = \bar{\omega} \times \bar{r}_i'}$$

$$|\bar{v}_i| = \omega r_i' \sin\varphi = \omega r_i$$



rotation with 3 fixed

$$\bar r_i = r\cos\theta\,\bar j + r\sin\theta\,\bar k$$

$$\bar v_i = \frac{\delta \bar r_i'}{\delta t} + \bar\omega\times\bar r_i' = \bar\omega\times\bar r_i'$$

$$\bar\omega = \omega k \qquad \omega\times\bar r_i = -i\omega r\cos\theta.$$

Kinetic Energy

$$T = \sum_i \tfrac{1}{2} m_i v_i^2 = \sum_i \tfrac{1}{2} m_i \omega^2 r_i^2 = \boxed{\tfrac{1}{2} I\omega^2} \qquad \text{since } \sum m_i r_i^2 = I$$

Angular Momentum

$$\bar h_o = \sum_i \bar r_i \times m_i \bar v_i \qquad |h_o| = \sum_i m_i\, \bar r_i \times \bar v_i = \sum_i m_i r_i' \omega r_i' \sin\varphi$$

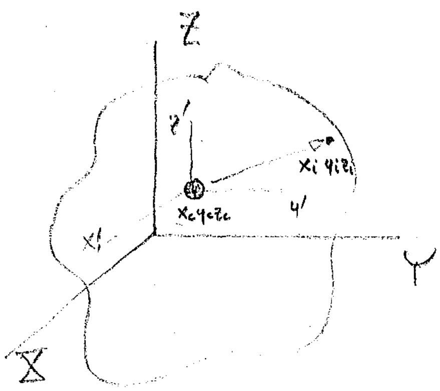
$$h_o \text{ along } \omega \text{ axis} = \left\{\sum_i m_i r_i^2 \omega \sin\varphi\right\} \sin\varphi = \sum_i \omega\, [m_i](r_i' \sin\varphi)^2$$

$$(h_o)_\omega = I\omega$$

If rigid body rotates and translates.

$$T = \underset{\text{translation}}{\tfrac{1}{2} M v_c^2} + \underset{\text{rotation}}{\tfrac{1}{2} I\omega^2}$$

rotation. I is through center of mass



$$x_i = x_c + x_i'$$
$$y_i = y_c + y_i'$$
$$z_i = z_c + z_i'$$

$$T = \sum_i \tfrac{1}{2} m_i v_i^2$$

$$= \sum_i \tfrac{1}{2} m_i [\dot x_i^2 + \dot y_i^2 + \dot z_i^2]$$

since $\sum m_i \dot x_c \dot x_i' = \sum m_i \dot y_c \dot y_i' = \sum m_i \dot z_c \dot z_i' = 0$

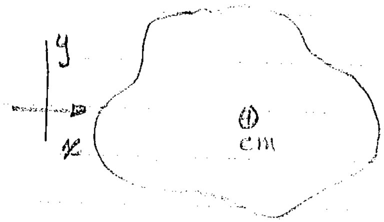
$$= \sum_i \tfrac{1}{2} m_i [(\dot x_c + \dot x_i')^2 + (\dot y_c + \dot y_i')^2 + (\dot z_c + \dot z_i')^2]$$

$$\sum \tfrac{1}{2} m_i (\dot x_i'^2 + \dot y_i'^2 + \dot z_i'^2) = \sum \tfrac{1}{2} m_i v_i'^2 \qquad = \tfrac{1}{2} M v_c^2 + \tfrac{1}{2} I\omega^2$$

but $v_i' = \omega r_i$

also $\underbrace{\sum \tfrac{1}{2} m_i (\dot x_i^2 + \dot y_i^2 + \dot z_i^2)}_{v_c^2}$

Eqs of Motion for Rigid Body
Plane Motion



$$F_x = M\ddot{x}_c \qquad F_y = M\ddot{y}_c$$

with respect to fixed scale.

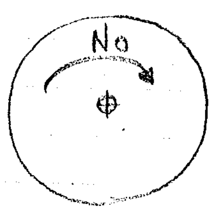
$$\sum \overline{G}_i^{(e)} = \frac{d\bar{h}}{dt}$$

$\sum \overline{G}_i^{(e)} \to N$, $\quad \frac{dh}{dt} \to \frac{d}{dt}(I\omega) = I\dot{\omega}$

$$N = I_{f,\,cm}\,\dot{\omega}$$

I is about a fixed pt or the center of mass.

Example:



$N = N_0 =$ constant.

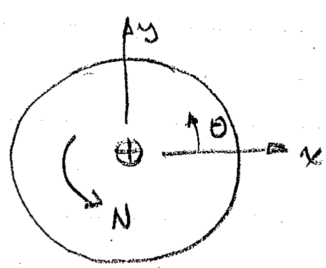
$N = I_0\dot{\omega}$

$$\omega = \frac{N}{I_0}t + C_1 \qquad \text{at } t=0 \;\; \omega = \omega_0$$

$$\theta = \frac{N}{2I_0}t^2 + \omega_0 t + C_2 \qquad \text{at } t=0 \;\; \theta = \theta_0$$

$$\theta = \frac{N}{2I_0}t^2 + \omega_0 t + \theta_0$$

Example 4/20/71



$\omega = \omega(t) \quad [\dot{\theta}(t) = ?$

$$N = N_0 + N_1\cos ct - L\dot{\theta}$$

$$N = I_0\ddot{\theta} = I_0\dot{\omega}$$

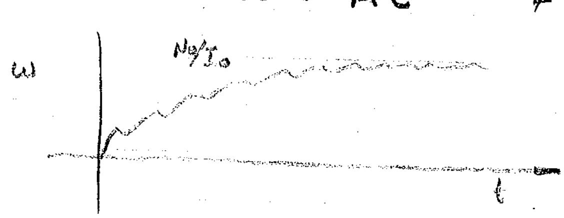
$I_0 = mk_0^2. \qquad mk_0^2\dot{\omega} = N_0 + N_1\cos ct - L\omega$

$$\dot{\omega} = \frac{N}{I_0} + \frac{N_1}{I_0}\cos ct - \frac{L}{I_0}\omega$$

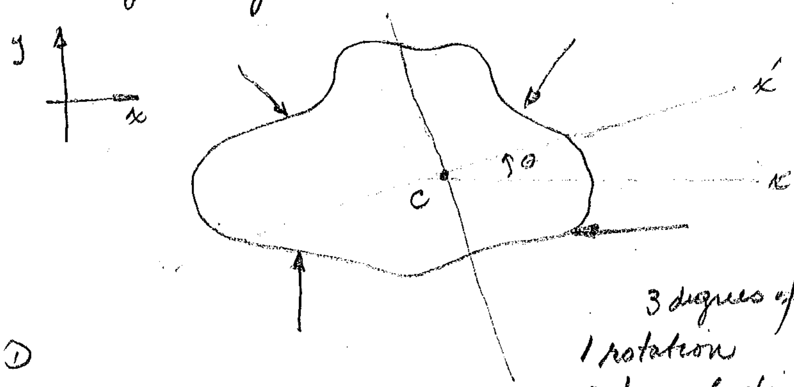
$$\dot{\omega} + \frac{L}{I_0}\omega = \frac{N_0 + N_1\cos ct}{I_0}$$

complem. sol. $\qquad$ partic. Solution

$$\omega = Ae^{-L/I_0\, t} + \frac{N_0}{I_0} + \frac{N_1}{I_0\sqrt{(L/I_0)^2 + c^2}}\cos(ct + \epsilon)$$



Equations of Motion - Rigid Body in Plane Motion



3 degrees of freedom.
1 rotation
2 translation.

$\underline{X} = \sum F_x = m\ddot{x}_c$ ①

$Y = \sum F_y = m\ddot{y}_c$ ②

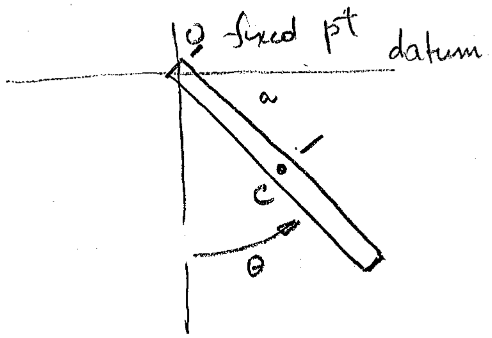
$N = I_c \cdot \ddot{\theta}$ ③

If conservative forcefield

$T + V = E$

$\frac{1}{2} m v_c^2 + \frac{1}{2} I_c \dot{\theta}^2 + V = E$ ④

Compound Pendulum.



$|\theta - |0|| \leq 0$

$T + V = E$

$\frac{1}{2} I_o \dot{\theta}^2 - mga\cos\theta = E$

$\frac{1}{2} I_o \cdot 2\dot{\theta}\ddot{\theta} + mga\sin\theta\,\dot{\theta} = 0$

$\dot{\theta}\left(I_o\ddot{\theta} + mg\,a\sin\theta\right) = 0$

$I_o = m k_o^2$

$m k_o^2 \ddot{\theta} + mga\sin\theta = 0$

$k_o^2\ddot{\theta} + ga\sin\theta = 0$ for $-10° \leq \theta \leq 10°$ $\sin\theta \sim \theta$

$\ddot{\theta} + \underbrace{ga/k_o^2}_{\omega_1^2}\,\theta = 0$ for simple pendulum $\ddot{\theta} + \underbrace{g/l}_{\omega_2^2}\,\dot{\theta} = 0$

$l_{eff} = \frac{k_o^2}{a}$ $k_o^2 = k_c^2 + a^2$

$\omega_1 = \sqrt{ga/k_o^2} = \sqrt{\frac{ga}{k_c^2 + a^2}}$; $\tau = \frac{2\pi}{\sqrt{\frac{ga}{k_c^2 + a^2}}}$

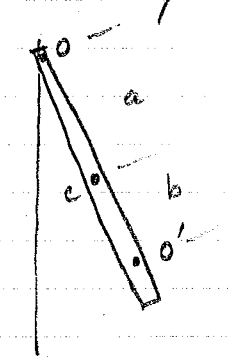
$$T^2 = \frac{4\pi^2}{g}\left[\frac{k_c^2}{a} + a\right] \qquad \text{as } a \to 0 \quad T \to \infty, \qquad a \to \infty \quad T \to \infty.$$

find $\frac{dT}{da} = 0$ $\qquad 2T\frac{dT}{da} = C\left[-\frac{k_c^2}{a^2} + 1\right] = 0$

$k_c = a \quad \therefore \quad T$ is min at this pt.

$$\therefore \quad T_{min}^2 = \frac{4\pi^2}{g}\left[2k_c\right] = \frac{8k_c\pi^2}{g}$$

Is the another pt O' such that $T_{min} = T_{O'}$ $\qquad O'$ is center of oscillation



$$\frac{k_c^2}{a} + a = \frac{k_c^2}{b} + b$$

$$\frac{k_c^2 b}{a} + ab = k_c^2 + b^2$$

$$b^2 - b\left(\frac{k_c^2}{a} + a\right) + k_c^2 = 0$$

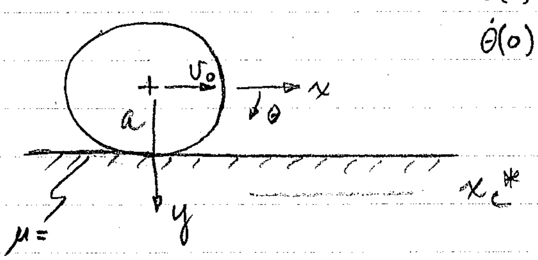
$$\left(b - \frac{k_c^2}{a}\right)\left(b - a\right) = 0 \qquad b = \frac{k_c^2}{a}$$

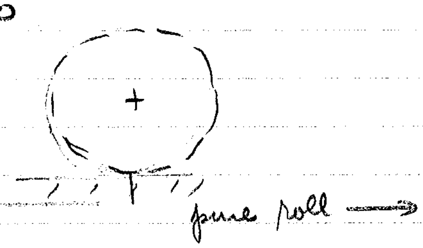
$$x_c = \frac{\int x\,dm}{\int dm} \qquad \text{from } x = x_c \text{ axis} \qquad I_c = \int (x - x_c)^2\,dm$$

Example:

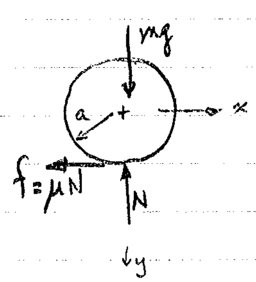
$v(0) = v_0 \qquad x_c(0) = 0$

$\dot\theta(0) = 0 \qquad \theta(0) = 0$





roll and slip



$$Y = mg - N = m\ddot y_c = 0 \qquad N = mg \qquad ①$$

$$X = m\ddot x_c = -\mu N \qquad ②$$

$$N = \mu N a = I_c \ddot\theta \qquad ③$$

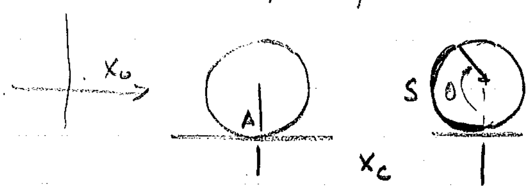
integrate two $-\mu m g = m\ddot{x}_c$ $\qquad \dot{x}_c = -\mu g t + C_1, \quad C_1 = v_0$ (4)

$$x_c = -\frac{\mu g t^2}{2} + v_0 t + \cancel{C_2}^{\,0} \quad (5)$$

integrate three $\mu m g a = m k_c^2 \ddot{\theta}$ $\qquad \dot{\theta} = \frac{\mu g a}{k_c^2} t + \cancel{C_3}^{\,0}$ (6)

$$\theta = \frac{\mu g a}{2 k_c^2} t^2 + \cancel{C_4}^{\,0} \quad (7)$$

check speed of bottom most pt if equal to 0 pure roll if $\neq 0$ slip



$$s = r\theta = x_c = a\theta + x_0$$
$$v_c = \dot{x}_c = a\dot{\theta} \quad \text{pure roll.} \quad (8)$$
$$\ddot{x}_c = a\ddot{\theta}$$

To calculate x_c^*, t^*. $\dot{x}_c = a\dot{\theta}$

$$\dot{x}_c = -\mu g t^* + v_0 = a\left[\frac{\mu g a}{k_c^2} t^*\right]$$

$$t^* = \frac{+v_0}{\mu g\left[\frac{a^2}{k_c^2} + 1\right]} \quad (9)$$

put (9) into (5)

$$x_c^* = -\frac{v_0^2}{2\mu g\left[\frac{a^2}{k_c^2}+1\right]^2} + \frac{2 v_0^2\left[\frac{a^2}{k_c^2}+1\right]}{2\mu g\left[\frac{a^2}{k_c^2}+1\right]^2} = \frac{v_0^2\left(1 + 2\frac{a^2}{k^2}\right)}{2\mu g\left[\frac{a^2}{k_c^2}+1\right]^2}$$

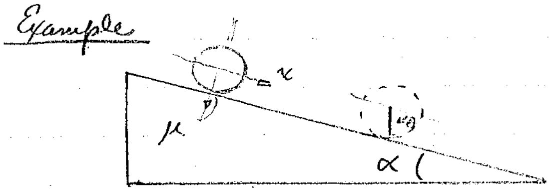
$$\bar{r} = a\bar{j} \qquad \bar{\omega} = \dot{\theta}\bar{k} \qquad \bar{v}_A = \bar{\omega}\times\bar{r} + \bar{v}_c$$

8.1 abc

4/22/71

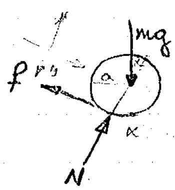
3rd quiz – May 6   2 questions
H.W. Problems   8.13, 15a, 17, 19

Example



Ball starts from rest on rough plane.

What is μ_{min} for pure roll of ball – sphere.



$$f \le \mu N$$

$$-N + mg\cos\alpha = m\ddot{y}_c = 0$$
$$N = mg\cos\alpha \quad (1)$$
$$-f + mg\sin\alpha = m\ddot{x}_c \quad (2)$$
$$T = fa = I_c\ddot{\theta} \quad (3)$$

Kinematic condition pure roll
$$\left.\begin{aligned} x_c &= a\theta \\ \dot{x}_c &= a\dot{\theta} \\ \ddot{x}_c &= a\ddot{\theta} \end{aligned}\right\} (4)$$

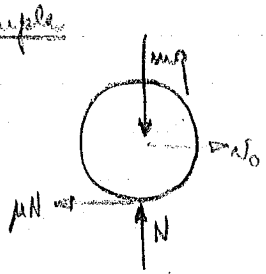
$$f = \frac{mk_c^2}{a^2}\ddot{x}_c \quad (5) \qquad -f + mg\sin\alpha = \frac{fa^2}{k_c^2}$$

$$f = \frac{mg\sin\alpha}{(1 + a^2/k_c^2)} \le \mu\, mg\cos\alpha$$

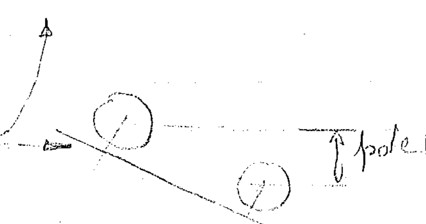
$$\mu \ge \frac{\tan\alpha}{(1 + a^2/k_c^2)} = \frac{\tan\alpha\,(k_c^2)}{k_c^2 + a^2}$$

$$\mu_{min} = \tan\alpha\,\frac{k_c^2}{k_c^2 + a^2} = \frac{I_c\tan\alpha}{I_0} \qquad I_0 = mk_c^2 + ma^2$$

Example



$$V + T = \tfrac{1}{2}mv_c^2 + \tfrac{1}{2}I_c\dot{\theta}^2 + \text{potential}$$

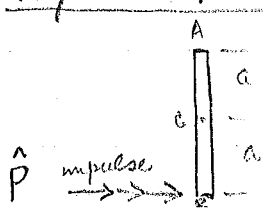


$\dot{x}_c = a\dot{\theta}$ previous example.

Impulse Momentum $mv_0 - \mu N\Delta t = mv_c^*$ $\Delta t \cdot t^*$
Angular $0 + \mu N a\Delta t = I_c\dot{\theta}^*$

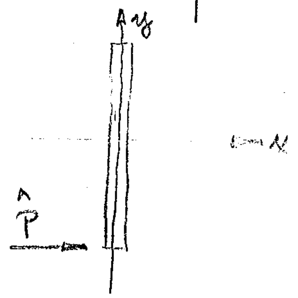
Impulsive Forces on Rigid Bodies

$m, 2a$



impulse given

Bar initially at rest, free. Calculate the velocity at A.



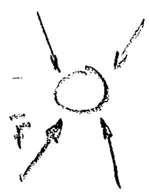
$$0 + \hat{P} = m v_{c_x} \quad (1)$$

$$0 + \hat{P} a = I_c \omega \quad (2)$$

$$v_{c_x} = \frac{\hat{P}}{m} \qquad \omega = \frac{\hat{P} a}{m k_c^2} \qquad v_A = v_c - a\dot{\theta}$$

$$v_a = \frac{\hat{P}}{m} - \frac{\hat{P} a^2}{m k_c^2}$$

Review 4/29/91



$$\sum \bar{F}_i = \bar{F}$$

$$\bar{F} = \frac{d}{dt}(m\bar{v}) \qquad \int_0^t \bar{F}\,dt = \int_{m\bar{v})_1}^{m\bar{v})_2} d(m\bar{v})$$

linear impulse $= \Delta (m\bar{v})_1^2$

$$m\bar{v})_1 + \int_0^t \bar{F}\,dt = (m\bar{v})_2$$

$$\bar{h}_1 + \int_0^t \bar{N}\,dt = \bar{h}_2 \quad \text{— angular momentum} \qquad \bar{N} = \frac{dh}{dt}$$

Impulsive Force $- \hat{F} = \lim_{\Delta t \to 0} \int_0^t \bar{F}\,dt$

Impulsive Moment $- \hat{N} = \lim_{\Delta t \to 0} \int_0^t \bar{N}\,dt = \lim_{\Delta t \to 0} \int_0^t \bar{r} \times \bar{F}\,dt$

$$= \bar{r} \times \lim_{\Delta t \to 0} \int_0^t \bar{F}\,dt = \bar{r} \times \hat{F}$$

$\bar{r}$ doesn't change instantly however $\bar{v}$ does.

Particle

$$(m\bar{v})_1 + \hat{F} = (m\bar{v})_2$$

$$\bar{h}_1 + \hat{N} = \bar{h}_2$$

System of Particles

$$(m\bar{v}_c)_1 + \hat{F} = (m\bar{v}_c)_2$$

$$\bar{h}_1 + \hat{N} = \bar{h}_2$$

about cm or fixed pt.

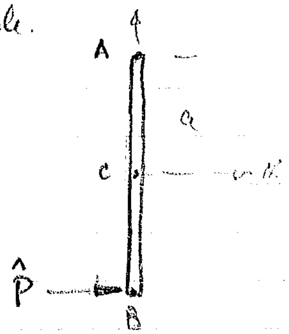
Rigid Body (Plane Motion)

$$(m\bar{v}_c)_1 + \hat{F} = (m\bar{v}_c)_2$$

relative posit of part in rigid bodies remain unchanged

$$(I\omega)_1 + \hat{N} = (I\omega)_2$$

Example.



$\hat P$ given. Initially at rest.
$v_A = ?$

$v_{A_x} = v_{c_x} - a\omega$

$(m\bar v)_1 + \hat F = (m\bar v)_2$

$0 + |\hat P| = m v_c$ $\qquad v_c = \frac{|\hat P|}{m}$ (1)

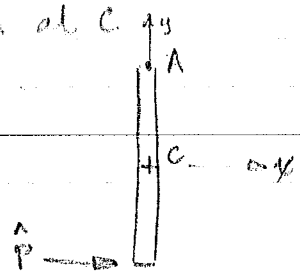
$\bar h_1 + \hat N = \bar h_2$

$\boxed{I_c = \frac{ma^2}{3}}$ $\qquad 0 + a|\hat P| = I_c\omega$ $\qquad \omega = \frac{a|\hat P|}{I_c}$

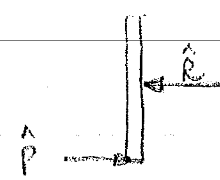
$\omega = \frac{3|\hat P|}{ma}$ (2)

$v_A = \frac{|\hat P|}{m}\left[1 - 3\right] = -\frac{2|\hat P|}{m}$

Pin fixed at C



no translation



$\Sigma m\bar v_i \qquad 0 + |\hat P| - |\hat R| = (m\bar v_c)_2 = 0$

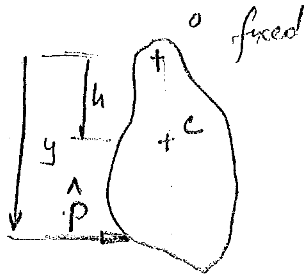
$\boxed{\hat P = \hat R}$ (1)

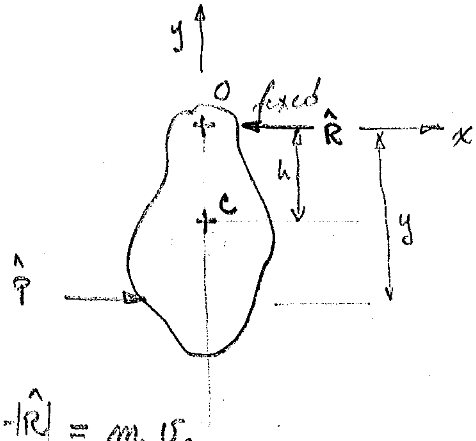
$\Sigma \bar h_i \qquad 0 + |\hat P| a = I_c\,\omega$

$\hat P a = \frac{ma^2}{3}\omega \qquad \omega = \frac{3\hat P}{ma}$

$v_{A_x} = v_{c_x} - a\omega$
$= 0 - \frac{3\hat P}{m}$

Center of Percussion – pt along axis of pt fixed & cm ∋ $\hat P$ gives 0 motion at fixed pt.





lin mom. $\quad 0 + |\hat{P}| - |\hat{R}| = m\, v_c$

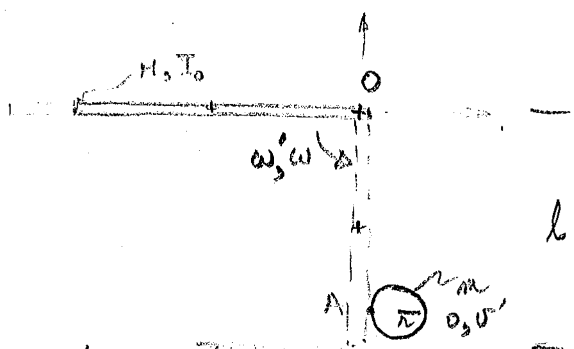
ang mom $\quad 0 + |\hat{P}|y = I_o\, \omega$

C.P. $\quad y$ for $|\hat{R}| = 0$

$$\hat{P} = m v_c = m\, h\, \omega \quad (1a)$$
$$|\hat{P}|y = m k_o^2\, \omega \quad (2a)$$

$$m h \omega\, y = m k_o^2\, \omega$$
$$\boxed{y = k_o^2/h}$$

Problems :

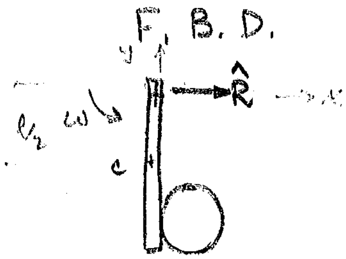


To calc. ω use principle of energy

$$T + V = E$$
$$0 + 0 = \tfrac{1}{2} I_o \omega^2 - Mg\,l/2 \qquad \frac{Ml^2}{3} = I_o$$
$$0 = \tfrac{1}{2}\frac{Ml^2}{3}\omega^2 - Mg\,l/2$$
$$\omega = \sqrt{\frac{3g}{l}}$$

F. B. D.

$\omega', v', \hat{R}$



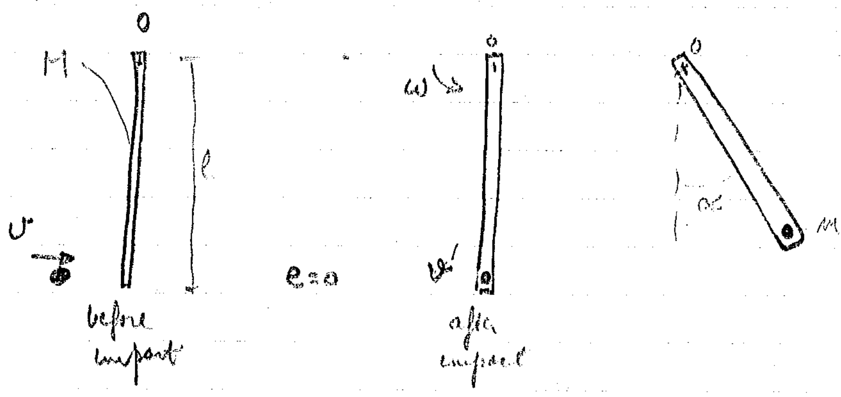
$$v_c = l/2\, \omega$$
$$M\,l/2\,\omega + \hat{R} = M\frac{l}{2}\omega' + m v'$$
$$I_o \omega + 0 = I_o \omega' + m v'(l - r)$$

$$e = \frac{v_{sep}}{v_{app}} = \frac{\omega'(l-r) - v'}{0 - \omega(l-r)}$$

$$v' = \frac{(1+e)(l-r)\sqrt{\frac{3g}{l}}}{\left[1 + 3\frac{m}{M}(1-r/l)^2\right]} \qquad |\hat{R}| = \left[1 - \tfrac{3}{2}(1 - r/l)\right] m v'$$

$$\omega' = \omega\left[\frac{1 - 3e\frac{m}{M}(1-r/l)^2}{1+3m/M\,(1-r/l)^2}\right]$$

Ballistic Pendulum $\qquad m \ll M$



$$mvl + 0 = mv'l + I_0\omega \qquad v', \omega, v$$

$$e = \frac{v_{sep}}{v_{app}} = \frac{\omega l - v'}{v - 0} = 0 \qquad \boxed{v' = \omega l}$$

$$\omega = \frac{mvl}{ml^2 + I_0}$$

measure alpha + apply conserv of energy

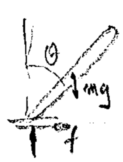
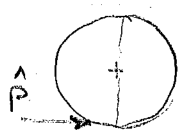
$$\tfrac{1}{2}I_0\omega^2 + \tfrac{1}{2}mv'^2 - mgl - Mg\,l/2 = 0 - Mg\,l/2\cos\alpha - mgl\cos\alpha$$

$$\left[\tfrac{1}{2}I_0 + \tfrac{1}{2}ml^2\right]\omega^2 = \left[mgl + Mg\,l/2\right](1-\cos\alpha)$$

Assume $m l \ll \frac{Ml}{2} \qquad ml^2 \ll I_0$

$$v = \frac{\sqrt{gI_0 Ml(1-\cos\alpha)}}{ml}$$

4/29/71

8.21  8.23  8.27 8.25 - not needed

General Motion of a Rigid Body

Kinematics

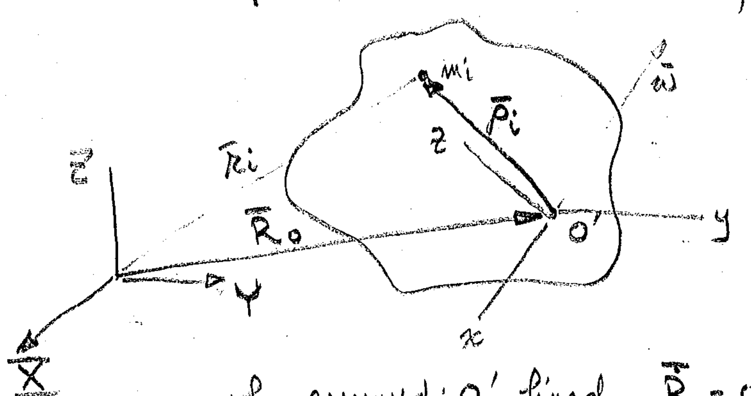
Axiom:

The displ of rigid body is completely descr. by displ. of 3 non-colinear pts.

Euler's theorem:

The most gen. displ of rigid body about fixed pt is equiv to a rotation about a line through pt.

Velocity of a Pt in a Rigid Body



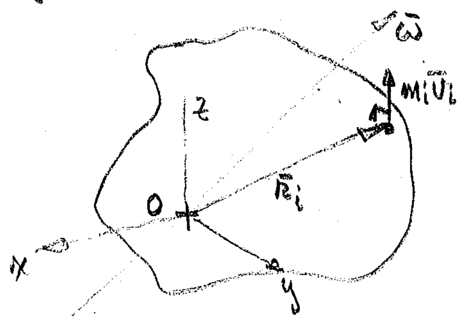
$$\bar{r}_i = \bar{R}_o + \bar{\rho}_i$$

$$\dot{\bar{r}}_i = \bar{v}_{ab} = \frac{d\bar{R}_o}{dt} + \frac{\delta \bar{\rho}_i}{\delta t} + \bar{\omega} \times \bar{\rho}_i$$

if assumed: O' fixed $\bar{R}_o = 0$ + if x, y, z axis fixed in body then $\frac{\delta \bar{\rho}_i}{\delta t} = 0$

$$\therefore \quad \bar{v}_{abs} = \bar{\omega} \times \bar{\rho}_i$$

Angular Momentum of a Rigid Body Rotating About fixed pt.



$$\bar{h}_o = \sum_i \bar{r}_i \times m_i \bar{v}_i$$

$$= \sum_i m_i \bar{r}_i \times (\bar{\omega} \times \bar{r}_i)$$

$$\bar{\omega} \times \bar{r}_i = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ x_i & y_i & z_i \end{vmatrix}$$

$$= \bar{i}\,(\overbrace{\omega_y z_i - \omega_z y_i}^{\lambda_1}) + \bar{j}\,(\overbrace{\omega_z x_i - \omega_x z_i}^{\lambda_2}) + \bar{k}\,(\overbrace{\omega_x y_i - \omega_y x_i}^{\lambda_3})$$

$$\bar{r}_i \times (\bar{\omega} \times \bar{r}_i) = \begin{vmatrix} i & j & k \\ x_i & y_i & z_i \\ \lambda_1 & \lambda_2 & \lambda_3 \end{vmatrix}$$

$$= i\,(y_i\lambda_3 - z_i\lambda_2) + j\,(z_i\lambda_1 - x_i\lambda_3) + k\,(x_i\lambda_2 - y_i\lambda_1)$$

$$\bar{h}_o = \bar{i} \sum m_i \left[\omega_x (y_i^2 + z_i^2) - \omega_y (x_i y_i) - \omega_z (x_i z_i)\right] + \bar{j} \sum m_i \left[\omega_y (x_i^2 + z_i^2) \ldots\right.$$

let $I_{xx} = \sum_i m_i (y_i^2 + z_i^2) = \int (y^2 + z^2)\, dm$

$I_{yy} = \sum_i m_i (z_i^2 + x_i^2) = \int (z^2 + x^2)\, dm$

$I_{zz} = \sum_i m_i (x_i^2 + y_i^2) = \int (x^2 + y^2)\, dm$

Mass Moments of inertia

define $I_{xy} = -\sum_i m_i x_i y_i = -\int xy\, dm$

$$\bar{h}_o = (I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z)\bar{i} + (I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z)\bar{j} + (I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z)\bar{k}$$

If at O we find principle axes $I_{12} \to 0$

if x, y, z principal axes then

$$\bar{h}_o = I_{xx}\omega_x \bar{i} + I_{yy}\omega_y \bar{j} + I_{zz}\omega_z \bar{k}$$

ang moment vector doesn't lie on $\bar{\omega}$ vector.

if x_p, y_p, z_p and $\bar{\omega}$ lies on one of the axes then $h_o = I_{ii}\omega_i$

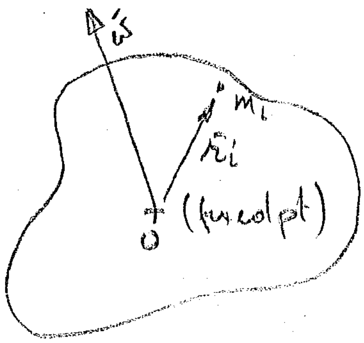
$$\bar{N} = \frac{d}{dt}\bar{h}$$

5/4/71

Exam Motion of Rigid Bodies in Plane, Collisions - 2 quest. 2:00 PM exam 5th HR 215

$$\bar{h} = I_{xx}\omega_x \bar{i} + I_{yy}\omega_y \bar{j} + I_{zz}\omega_z \bar{k}$$ principal axes only

Kinetic Energy for Rigid Body Rotating about Fixed Point



$$T = \sum_i \tfrac{1}{2} m_i \bar{v}_i \cdot \bar{v}_i$$

$$= \sum_i \tfrac{1}{2} (\bar{\omega} \times \bar{r}_i) \cdot (m \bar{v}_i)$$

$$= \sum_i \tfrac{1}{2} \, \bar{\omega} \cdot \underbrace{(\bar{r}_i \times m_i \bar{v}_i)}_{h_i}$$

$$T = \sum_i \tfrac{1}{2} \bar{\omega} \cdot \bar{h}_i = \tfrac{1}{2} \bar{\omega} \cdot \bar{h}$$

$$\bar{\omega} = \omega_x \bar{i} + \omega_y \bar{j} + \omega_z \bar{k}$$

$$\bar{h} = I_{xx} \omega_x \bar{i} + I_{yy} \omega_y \bar{j} + I_{zz} \omega_z \bar{k}$$

$$T = \tfrac{1}{2} \left[I_{xx} \omega_x^2 + I_{yy} \omega_y^2 + I_{zz} \omega_z^2 \right]$$

for general motion

$$T = \tfrac{1}{2} \left[I_{xx} \omega_x^2 + \ldots \right] + \tfrac{1}{2} M \left[V_{c_x}^2 + V_{c_y}^2 + V_{c_z}^2 \right]$$

Mass Moment of Inertia - $I_{xx} = \int_M (y^2 + z^2)\, dm$

Principal Axes - $I_{xy} = I_{yz} = I_{zx} = 0$

<u>Euler's Equation of Motion</u>

$$\bar{N} = \frac{d\bar{h}}{dt} \qquad \bar{h} \text{ for cm or fixed pt.}$$

$\bar{h} = I_{xx} \omega_x \bar{i} + I_{yy} \omega_y \bar{j} + I_{zz} \omega_z \bar{k}$ principal axes only

$\dfrac{d\bar{h}}{dt} = \dfrac{\delta \bar{h}}{\delta t} + \bar{\omega} \times \bar{h}$ for axes fixed in body Euler's Equation will be developed.

$$\frac{d\bar{h}}{dt} = \bar{N} = I_{xx}\dot{\omega}_x\,\bar{i} + I_{yy}\dot{\omega}_y\,\bar{j} + I_{zz}\dot{\omega}_z\,\bar{k} + \begin{vmatrix} i & j & \bar{k} \\ \omega_x & \omega_y & \omega_z \\ I_{xx}\omega_x & I_{yy}\omega_y & I_{zz}\omega_z \end{vmatrix}$$

$$\bar{N} = \bar{i}\left[I_{xx}\dot{\omega}_x + I_{zz}\omega_z\omega_y - I_{yy}\omega_y\omega_z\right] + \bar{j}\left[I_{yy}\dot{\omega}_y + I_{xx}\omega_x\omega_z - I_{zz}\omega_z\omega_x\right]$$

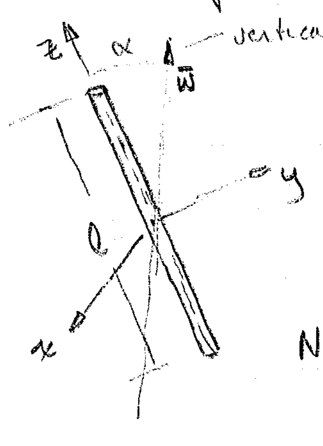
$$\bar{k}\left[I_{zz}\dot{\omega}_z + I_{yy}\omega_y\omega_x - I_{xx}\omega_x\omega_y\right]$$

$$\boxed{\begin{aligned} N_x &= I_{xx}\dot{\omega}_x + (I_{zz} - I_{yy})\omega_y\omega_z \\ N_y &= I_{yy}\dot{\omega}_y + (I_{xx} - I_{zz})\omega_z\omega_x \\ N_z &= I_{zz}\dot{\omega}_z + (I_{yy} - I_{xx})\omega_x\omega_y \end{aligned}}$$

Example Rigid Body – Constrained to move about fixed axis $\bar{\omega}$ const.

$$\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0 \qquad \begin{aligned} N_x &= (I_{zz} - I_{yy})\omega_y\omega_z \\ N_y &= (I_{xx} - I_{zz})\omega_z\omega_x \\ N_z &= (I_{yy} - I_{xx})\omega_x\omega_y \end{aligned}$$

For a thin uniform rod.



x, y, z are principal axes fixed to body

$$I_{zz} = 0$$

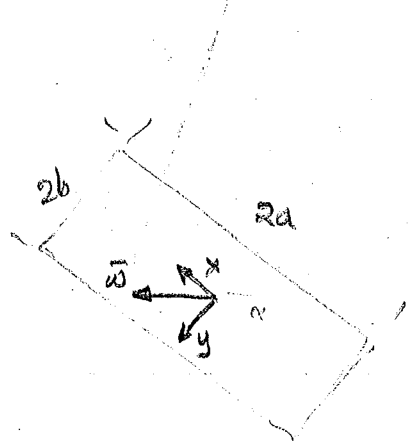
$$I_{xx} = I_{yy} = \frac{ml^2}{12} \qquad N_z = 0 \quad I_{xx} = I_{yy}$$

$$N_y = 0 \quad \omega_x = 0$$

$$N_x = (I_{zz} - I_{yy})\omega_y\omega_z = -\frac{ml^2}{12}(\omega\sin\alpha)(\omega\cos\alpha)$$

$$= -\frac{ml^2}{12}\omega^2\sin\alpha\cos\alpha$$

<u>Rectangular Plate</u> rotates with constant speed about diagonal $\bar{N} = ??$

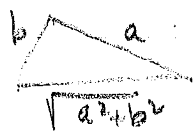


$$N_x = (I_{zz} - I_{yy})\,\omega_y \omega_z$$
$$N_y = (I_{xx} - I_{zz})\,\omega_z \omega_x$$
$$N_z = (I_{yy} - I_{xx})\,\omega_x \omega_y$$

$$\bar{\omega} = \omega \cos\alpha\,\hat{i} + \omega \sin\alpha\,\hat{j}$$

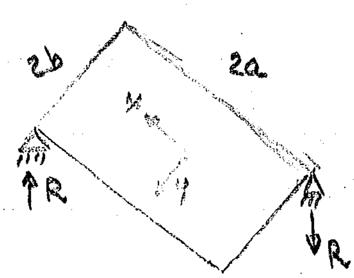
$$I_{xx} = \tfrac{1}{3} m b^2, \quad I_{yy} = \tfrac{1}{3} m a^2$$
$$I_{zz} = \tfrac{1}{3} m (b^2 + a^2)$$

$$N_x, N_y = 0$$
$$N_z = \left(\tfrac{1}{3} m a^2 - \tfrac{1}{3} m b^2\right) \omega^2 \cos\alpha \sin\alpha$$



$$= \frac{1}{3} m \frac{(a^2 - b^2)}{a^2 + b^2}\, a b\, \omega^2$$

Calculate Reaction



if $a > b$

$$R\left[\sqrt{(2b)^2 + (2a)^2}\right] - N_z = 0$$

$$R = \frac{N_z}{2\sqrt{a^2 + b^2}} = \frac{N_z}{2\sqrt{a^2+b^2}} = \frac{1}{6} \frac{(a^2 - b^2)}{\left(\sqrt{a^2+b^2}\right)^3}\, a b\, \omega^2$$

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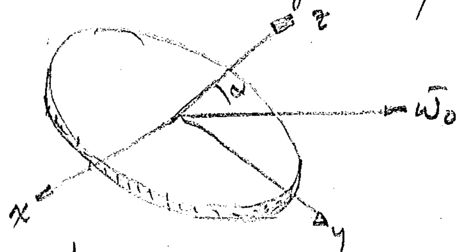
$$N_x = I_{xx}\dot{\omega}_x + \omega_y\omega_z(I_{zz} - I_{yy})$$

$$N_y = I_{yy}\dot{\omega}_y + \omega_z\omega_x(I_{xx} - I_{zz})$$

$$N_z = I_{xx}\dot{\omega}_z + \omega_x\omega_y(I_{yy} - I_{xx})$$

Free Rotation, Rigid Body Axis of Symmetry,



Disc mass m, Rad a

Set spinning $\bar{\omega}_0$ in yz plane.

$$\bar{\omega} = \omega_x i + \omega_y j + \omega_z k$$

$\bar{N} = 0$ no torque

$$\bar{\omega}_0 = 0 + \omega_0 \sin\alpha\, \bar{j} + \omega_0\cos\alpha\, \bar{k}$$

Let $I = I_{xx} = I_{yy}$ $\quad I_{ss} = I_{zz}$

$$0 = I\dot{\omega}_x + \omega_y\omega_z(I_{ss} - I)$$

$$0 = I\dot{\omega}_y + \omega_z\omega_x(I - I_{ss})$$

$$0 = I_{ss}\dot{\omega}_z + \omega_x\omega_y\cancel{(I - I)} \rightarrow \quad \dot{\omega}_z = 0 \quad \omega_z = \text{const}$$

$\boxed{\omega_z = \omega_0\cos\alpha}$ Call $\Omega = \left(\frac{I_s}{I} - 1\right)\cos\alpha$

$$\dot{\omega}_x + \omega_y\Omega = 0$$

$$\dot{\omega}_y - \omega_x\Omega = 0 \qquad \ddot{\omega}_y - \dot{\omega}_x\Omega = 0$$

$$\dot{\omega}_x = \frac{\ddot{\omega}_y}{\Omega} \longrightarrow \ddot{\omega}_y + \omega_y\Omega^2 = 0$$

$$\omega_y = A\cos\Omega t + B\sin\Omega t = \omega_1\cos\Omega t$$

$\omega_y(0) = \omega_1 = \omega_0\sin\alpha$ $\quad \boxed{\omega_y = \omega_0\sin\alpha\cos\Omega t}$

$$\dot{\omega}_y = -\omega_0\sin\alpha(\Omega\sin\omega t) \qquad \therefore \boxed{\omega_x = -\omega_0\sin\alpha(\sin\omega t)}$$

$$\omega_x^2 + \omega_y^2 = \omega_1^2 = \text{const.} = \omega_0^2\sin^2\alpha$$

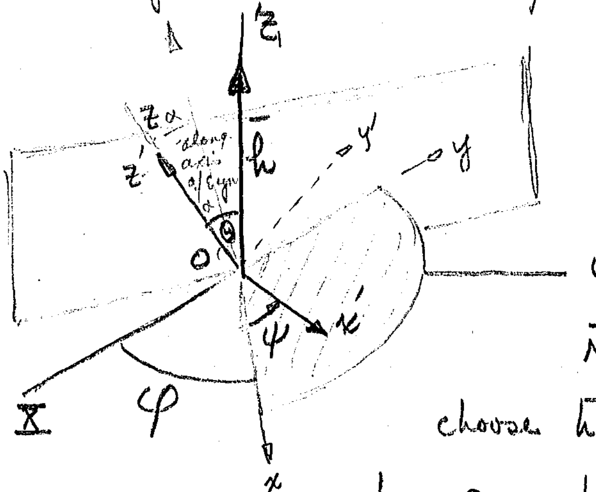
the precession of $\bar{\omega} = \Omega$ as seen from the moving frame of reference

$$\frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right) = T = \text{const.} \quad C_1$$

$\bar{N} = \frac{dh}{dt}$ = constant in space

$\bar{h}\cdot\bar{h} = h^2 = I_{xx}^2\omega_x^2 + I_{yy}^2\omega_y^2 + I_{zz}^2\omega_z^2 = C_2$

Eulerian Angles $\rightarrow$ Orientation of Rotation Relative to a fixed Coord. System.



Eulerian Angles φ, ψ, Θ

$\bar{N} = 0 \qquad \bar{h} = \text{const.}$

choose $\bar{h}$ to coincide with Z_1

$h_x = 0 \qquad h_y = h\sin\Theta \qquad h_z = h\cos\Theta$

$h_x = I_{xx}\omega_x = 0 \qquad \omega_x = 0$

$h_y = I_{yy}\omega_y = h\sin\Theta = \omega\sin\alpha\, I \qquad \dfrac{h_y}{h_z} = \tan\Theta = \dfrac{I}{I_s}\tan\alpha$

$h_z = I_{zz}\omega_z = I_s\,\omega\cos\alpha = h\cos\Theta$

In terms of Eulerian Angles

$\omega_x = 0 \qquad \omega_y = \dot{\varphi}\sin\Theta \qquad \omega_z = \dot{\varphi}\cos\Theta + \dot{\psi}$

$\omega_y = \omega\sin\alpha = \dot{\varphi}\sin\Theta \qquad \dot{\varphi} = \dfrac{\omega\sin\alpha}{\sin\alpha}$

$\dot{\varphi} = \omega\sqrt{1 + \left(\dfrac{I_s^2}{I^2} - 1\right)\cos^2\alpha}$

for a disk $\dfrac{I_s}{I} = 2 \qquad \Omega = \omega\cos\alpha$

$\dot{\varphi} = \omega\sqrt{1 + 3\cos^2\alpha}$

if α is small $\quad \Omega \approx \omega$

$\dot{\varphi} \approx 2\omega$

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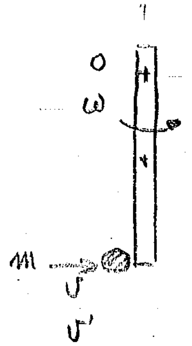
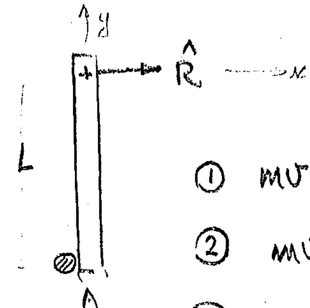




① $mv + \hat{R} = mv' + Mv_c$ $\qquad v_c = \omega \frac{L}{2}$

② $mvL = mv'L + I_o \omega$ $\qquad \frac{ML^2}{3}$

③ $e = \dfrac{\omega L - v'}{v - 0}$

find v', ω, $\hat{R}$

$$(.01)(100) + \hat{R} = (.01)v' + .5\omega \frac{3}{2} \quad (1A)$$

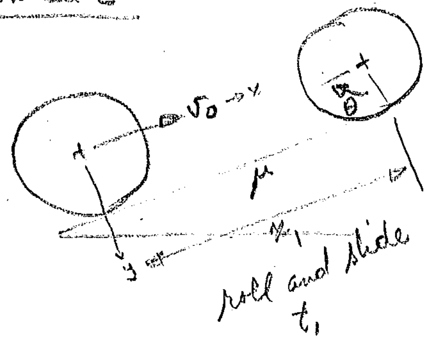
$$(.01)(100)3 = (.01)(3)v' + \left(\frac{.5}{3}\right)(9)\omega \quad (2A)$$

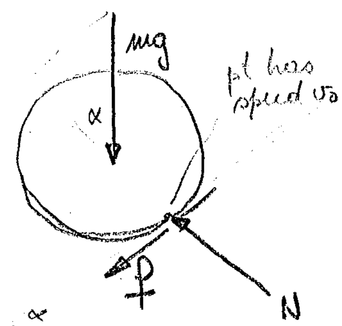
$$.5 = \frac{3\omega - v'}{100} \quad (3A)$$

$v' = -41.5$ ft/sec
$\omega = 2.83$ rad/sec
$\hat{R} = +.705$ lbsec

or $\quad mv - \hat{P} = mv'$
$\hat{P} = m(v - v') = 1.415$ lbsec

Problem 2





$$mg\cos\alpha - N = m\ddot{y}_c = 0 \quad ①$$

$$N = mg\cos\alpha$$

$$-mg\sin\alpha - \mu\, mg\cos\alpha = m\ddot{x}_c \quad ②$$

$$(\mu mg\cos\alpha)\, r = I_c \ddot{\theta} \quad ③$$

$$\ddot{x}_c = -g(\sin\alpha + \mu\cos\alpha)$$

$$\dot{x}_c = -gt(\sin\alpha + \mu\cos\alpha) + v_0 \quad ④$$

$$x_c = -\frac{gt^2}{2}(\sin\alpha + \mu\cos\alpha) + v_0 t \quad ⑤$$

$$\ddot{\Theta} = \frac{(\mu m g \cos\alpha) r}{\frac{2}{5} m r^2} = \frac{5 \mu g \cos\alpha}{2 r}$$

$$\dot{\Theta} = \frac{5}{2} \frac{\mu g \cos\alpha}{r} t + 0 \quad (6)$$

$$\Theta = \left(\frac{5}{2} \frac{\mu g \cos\alpha}{r}\right) \frac{t^2}{2} \quad (7)$$

at pure roll $\dot{x}_c = r \dot{\Theta}$

$$\therefore \quad -g(\sin\alpha + \mu\cos\alpha) t^* + v_0 = \frac{5}{2} (\mu g \cos\alpha) t^*$$

$$t^* = \frac{v_0}{g(\sin\alpha + \mu \frac{7}{2} \cos\alpha)}$$

$$x^* = -\frac{g}{2}(\sin\alpha + \mu\cos\alpha)\left[\frac{v_0}{g(\sin\alpha + \frac{7}{2}\mu\cos\alpha)}\right]^2 + v_0 \left[\frac{v_0}{g(\sin\alpha + \mu \frac{7}{2} \cos\alpha)}\right]$$

$$= \frac{v_0^2}{2g} \left[\frac{\sin\alpha + 6\mu\cos\alpha}{(\sin\alpha + \frac{7}{2}\mu\cos\alpha)^2}\right]$$

$$2v_0^2 \left[(\sin\alpha + \mu \tfrac{7}{2}\cos\alpha\right] - v_0^2 \left[\sin\alpha + \mu\cos\alpha\right] =$$

$$v_0^2 \left[\sin\alpha + 6\mu\cos\alpha\right] \checkmark\checkmark$$

1. A motorist enters a curve of 500-ft. radius at a speed of 45 mph. As he applies his brakes, he decreases his speed at a constant rate of 5 ft/sec^2. Determine the magnitude of the total acceleration of the automobile when its speed is 40 mph.
2. A particle moves in a plane with constant speed. Prove that its acceleration is perpendicular to its velocity.
3. A particle moves in an elliptical path with constant speed. At what points is the magnitude of the acceleration (1) a maximum, (2) a minimum? If the axes of the elliptical path are 4 and 2 ft. respectively, and if the speed of the particle is 8 ft/sec, determine the magnitudes of these accelerations.
4. A particle moves along a curve

$$y = c \sin px$$

where c and p are constants. The component of velocity in the x-direction is a constant (u). Find the acceleration and describe the hodograph.

5. Starting from

$$x = r \cos \theta \qquad\qquad y = r \sin \theta$$

calculate $\ddot{x}$ and $\ddot{y}$. Hence, by resolving the acceleration vector along and perpendicular to the radius vector, establish the acceleration components for plane polar coordinate system derived in class.

6. A motorboat experiences a resistance proportional to the square of the speed. The engine is switched off when the speed is 50 ft/sec. When the boat has moved through a distance of 60 feet, its speed has been reduced to 20 ft/sec. Find (to the nearest foot) the total distance traversed when the speed has been reduced to 10 ft/sec.
7. The two-dimensional motion of a particle is defined by the relations

$$r = k\left(1 + \cos \frac{\pi t}{2}\right) \qquad\qquad \theta = \frac{\pi t}{2}$$

Determine the velocity and acceleration of the particle when (a) $t = 1$ and (b) $t = 2$.

8. A particle moves along the hyperbolic spiral $r\theta = b$. (a) Determine the magnitude of the velocity of the particle in terms of b, θ, and $d\theta/dt$. (b) Knowing that $d\theta/dt$ is constant and denoting this by ω, determine the magnitude of the acceleration in terms of b, θ, and ω.

9. If $V = x^2 + y^2 + z^2 + xy + x$, at what points in space is the vector grad V parallel to the z=axis?

10. A scalar field is given over a plane by

$$V = \frac{x^2 + y^2}{2x}$$

What are the level curves? (Curves of constant V). Show that, at the point with polar coordinates (r, θ), grad V is inclined to the x-axis at an angle 2θ and its magnitude is $(\sec^2\theta)/2$.

2

H.W. #1

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1. $a_T = -5$ ft/sec^2 $a_N = \frac{v^2}{\rho}$

$\rho = 500$ ft $v = 40$ mph $= 58.7$ ft/sec

$a_N = \frac{(58.7)^2}{500} = 6.88$ ft/sec^2

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$a = \sqrt{(6.88)^2 + (-5)^2} = \sqrt{25 + 47.4} = 8.51$ ft/sec^2

2.

Given $|\bar{v}| = c$ Prove: $\bar{a} \perp \bar{v}$

$\bar{v} = v\,\bar{i}_T$, $\bar{a} = \dot{v}\,\bar{i}_T + \frac{v^2}{\rho}\,\bar{i}_N$

then

$\bar{v} \cdot \bar{a} = v\dot{v} = 0$ (since $v = c$ $\frac{dv}{dt} = 0$)

this gives two cases,

1. $\bar{v}, \bar{a} = 0$ 2. $\bar{v} \perp \bar{a}$

in first case $\bar{v} \neq 0$ since v is finite; $\bar{a} \neq 0$ but $a = a_N = \frac{v^2}{\rho} \neq 0$

this leaves case 2.

(The case where $\rho = \infty$ is not considered.)

Therefore $\bar{a} \perp \bar{v}$.

QED

Excellent

10

AE 211 Dr. Nardo

Cesar Levy

1. $x = r\cos\theta$

$\dot{x} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$

$\ddot{x} = (\ddot{r} - r\dot{\theta}^2)\cos\theta - (2\dot{r}\dot{\theta} + r\ddot{\theta})\sin\theta$

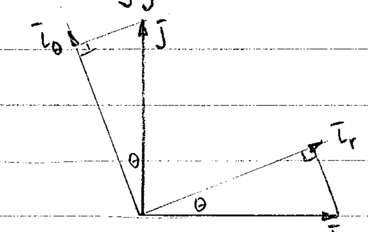
$y = r\sin\theta$

$\dot{y} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$

$\ddot{y} = (\ddot{r} - r\dot{\theta}^2)\sin\theta + (2\dot{r}\dot{\theta} + r\ddot{\theta})\cos\theta$

$\bar{r} = r\cos\theta\,\bar{i} + r\sin\theta\,\bar{j}$

$\bar{a} = \ddot{x}\,\bar{i} + \ddot{y}\,\bar{j}$



$\bar{i}_r = \bar{i}\cos\theta + \bar{j}\sin\theta$

$\bar{i}_\theta = \bar{j}\cos\theta - \bar{i}\sin\theta$

$\bar{i} = \bar{i}_r\cos\theta - \bar{i}_\theta\sin\theta$

$\bar{j} = \bar{i}_r\sin\theta + \bar{i}_\theta\cos\theta$

$$\bar{a} = \ddot{x}\left[\bar{i}_r\cos\theta - \bar{i}_\theta\sin\theta\right] + \ddot{y}\left[\bar{i}_r\sin\theta + \bar{i}_\theta\cos\theta\right]$$

$$= \bar{i}_r\left[\ddot{r} - r\dot{\theta}^2\right]\cos^2\theta - \bar{i}_r\left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\cos\theta\sin\theta$$

$$- \bar{i}_\theta\left[\ddot{r} - r\dot{\theta}^2\right]\sin\theta\cos\theta + \bar{i}_\theta\left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\sin^2\theta$$

$$+ \bar{i}_r\left[\ddot{r} - r\dot{\theta}^2\right]\sin^2\theta + \bar{i}_r\left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\sin\theta\cos\theta$$

$$+ \bar{i}_\theta\left[\ddot{r} - r\dot{\theta}^2\right]\sin\theta\cos\theta + \bar{i}_\theta\left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]\cos^2\theta$$

$$\bar{a} = \bar{i}_r\left[\ddot{r} - r\dot{\theta}^2\right] + \bar{i}_\theta\left[2\dot{r}\dot{\theta} + r\ddot{\theta}\right]$$

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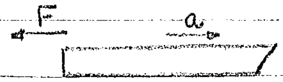
2.



$$\Sigma F_x = m\ddot{x} + kV^2 = 0$$

$$\ddot{x} = -\frac{kV^2}{m}$$

@ $t=0 \quad V_0 = 50 \quad S_0 = 0$

$t = t^* \quad V = 20 \quad S = 60$

$$a = V\frac{dV}{dS} = -\frac{kV^2}{m} \; ; \quad -\frac{dV}{V} = \frac{k}{m}dS$$

$$\ln\left[V_0/V\right] = \frac{k}{m}\left[S - S_0\right] \; ; \quad \ln\left[50/V\right] = \frac{k}{m}S$$

@ $t = t^*$

$$\ln\left[5/2\right] = \frac{k}{m}60 \qquad \frac{k}{m} = \frac{1}{60}\ln\left[5/2\right]$$

@ $V = 10$ ft/sec

$$\ln\left[50/10\right] = \frac{1}{60}\ln\left[5/2\right]S$$

10

$$S = 60\,\frac{\ln(5)}{\ln(5/2)} \approx 105 \text{ ft}$$

3. $r = k\left(1 + \cos\frac{\pi t}{2}\right) \qquad \theta = \frac{\pi t}{2}$

$$\bar{v} = \dot{r}\,\bar{i}_r + r\dot{\theta}\,\bar{i}_\theta = \frac{k\pi}{2}\left[\left(1+\cos\frac{\pi t}{2}\right)\bar{i}_\theta - \sin\frac{\pi t}{2}\,\bar{i}_r\right]$$

@ $t=1 \qquad \bar{V} = \frac{k\pi}{2}\left[\bar{i}_\theta - \bar{i}_r\right]$

@ $t=2 \qquad \bar{V} = 0$

$$\bar{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\bar{i}_r + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\bar{i}_\theta$$

$$= -\frac{k\pi^2}{4}\left[\left(1 + 2\cos\frac{\pi t}{2}\right)\bar{i}_r + 2\sin\frac{\pi t}{2}\,\bar{i}_\theta\right]$$

@ $t=1 \qquad \bar{a} = -\frac{k\pi^2}{4}\left[\bar{i}_r + 2\,\bar{i}_\theta\right]$

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@ $t=2 \qquad \bar{a} = \frac{k\pi^2}{4}\,\bar{i}_r$

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9. $\nabla V = (2x+y+1)\bar{i} + (2y+x)\bar{j} + 2z\bar{k}$

if $\nabla V = \epsilon \bar{k}$ then $\nabla V = 2z\bar{k}$ only where $\epsilon = 2k$

$\therefore$ $2x+y+1=0$ & $2y+x=0$ simultaneously.

from these two equations: $x = -2/3$ $y = 1/3$ $-\infty \le z \le \infty$

10

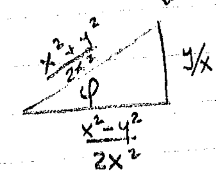
10a. $V = \dfrac{x^2+y^2}{2x} = C$

$x^2+y^2 = 2xC \rightarrow (x-C)^2 + y^2 = C^2$ or a circle having a radius of C and having origin at $(C,0)$.

10b. $\nabla V = \left\{ \dfrac{2x[2x] - 2[x^2+y^2]}{4x^2} \right\} \bar{i} + \left\{ \dfrac{2x[2y]}{4x^2} \right\} \bar{j}$

$= \dfrac{x^2-y^2}{2x^2}\bar{i} + y/x\,\bar{j}$

$|\nabla V| = \sqrt{\left(\dfrac{x^2-y^2}{2x^2}\right)^2 + \left(y/x\right)^2} = \dfrac{x^2+y^2}{2x^2}$



but $x = r\cos\theta$ $y = r\sin\theta$

$\therefore$ $|\nabla V| = r^2/[2r^2\cos^2\theta] = \sec^2\theta/2$

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$\cos\varphi = \dfrac{x^2-y^2}{2x^2} \Big/ \dfrac{x^2+y^2}{2x^2} = r^2\cos 2\theta / r^2 = \cos 2\theta$

$\therefore$ $\varphi = 2\theta$

3.3. $F = ma_x = ct \Rightarrow mv = \dfrac{ct^2}{2} + C_1$ at $t=0$ $v=0$ $\therefore C_1 = 0$

$mx = \dfrac{ct^3}{6} + C_2$ at $t=0$ $v=0$ $x=0$ $\therefore C_2 = 0$

$\therefore$ $\boxed{x = \dfrac{ct^3}{6m} \quad \text{for} \quad 0 \le t \le t_0}$

in region where $t_0 \le t \le 2t_0$

$F = 2ct_0 - ct = ma_x$

$$mV = 2ct_0t - \frac{ct^2}{2} + C_3$$

$$mV_0 = \frac{ct_0^2}{2} = 2ct_0^2 - \frac{ct_0^2}{2} + C_3 \quad \text{at } t = t_0 \quad V = V_0$$

$$C_3 = -ct_0^2$$

$$mV = 2ct_0t - \frac{ct^2}{2} - ct_0^2$$

$$mX = ct_0t^2 - \frac{ct^3}{6} - ct_0^2t + C_4 \quad \text{at} \quad t = t_0 \quad X = X_0$$

$$mX_0 = \frac{ct_0^3}{6} = ct_0^3 - \frac{ct_0^3}{6} - ct_0^3 + C_4 \quad \therefore C_4 = \frac{ct_0^3}{3}$$

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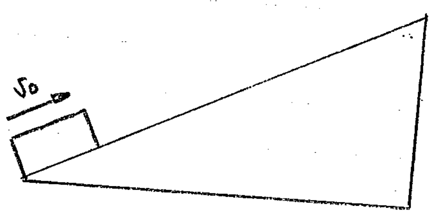
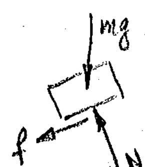
$$mX = ct_0t^2 - \frac{ct^3}{6} - ct_0^2t + \frac{ct_0^3}{3}$$

$$\Delta X = \frac{1}{m}\left[\left(ct_0t^2 - \frac{ct^3}{6} - ct_0^2t + \frac{ct_0^3}{3}\right)\Big|_{t_0}^{2t_0} + \left(\frac{ct^3}{6}\right)\Big|_0^{t_0}\right] = \frac{ct_0^3}{m}$$

3.5.





$$-mg\cos\theta + N = ma_y = 0$$

$$N = mg\cos\theta$$

$$F_x = ma_x = -mg\sin\theta - \mu N$$

$$a_x = -g\sin\theta - \mu g\cos\theta$$

$$V = -gt\sin\theta - \mu tg\cos\theta + C_1 \quad \text{at } t = 0 \quad V = V_0 \quad \therefore C_1 = V_0$$

$$\therefore \quad V_{up} = -gt\sin\theta - \mu gt\cos\theta + V_0$$

$$X_{up} = -g\frac{t^2}{2}\sin\theta - \frac{\mu}{2}gt^2\cos\theta + V_0t + C_2; \quad t = 0 \quad x = 0 \quad \therefore \quad C_2 = 0$$

$$\therefore \quad X_{up} = -g\frac{t^2}{2}\sin\theta - \mu g\frac{t^2}{2}\cos\theta + V_0t$$

at $S = S_0$ $\quad V = 0 \quad \therefore \quad t^*_{up} = \dfrac{V_0}{g[\sin\theta + \mu\cos\theta]}$

the distance traveled in $t^*_{up} = \dfrac{V_0^2}{2g[\sin\theta + \mu\cos\theta]}$

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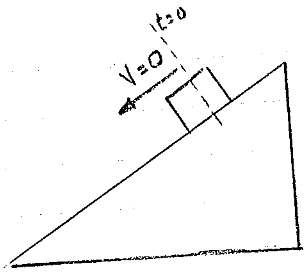

$$F_y = ma_y = -mg\cos\theta + N = 0$$
$$N = mg\cos\theta$$
$$F_x = ma_x = mg\sin\theta - \mu N$$

$$a_x = g\sin\theta - \mu g\cos\theta$$

$$V = gt\sin\theta - \mu gt\cos\theta + C_3 \quad \text{at } t=0 \quad V=0 \quad C_3 = 0$$

$$\therefore \quad V = gt\sin\theta - \mu gt\cos\theta$$

$$x = g\frac{t^2}{2}\sin\theta - \mu g\frac{t^2}{2}\cos\theta + x^* \quad \text{at } t=0 \quad V=0 \quad x^*=0$$

$$\therefore \quad x = g\frac{t^2}{2}\sin\theta - \mu g\frac{t^2}{2}\cos\theta$$

$$@\ t = t^*_{down} \qquad x = \frac{V_0^2}{2g[\sin\theta + \mu\cos\theta]} = \frac{g}{2}t^{*2}_{down}\sin\theta - \frac{\mu}{2}g\,t^{*2}_{down}$$

$$t^*_{down} = \sqrt{\frac{V_0^2}{g^2[\sin\theta+\mu\cos\theta][\sin\theta-\mu\cos\theta]}} = \frac{V_0}{g}\sqrt{\frac{1}{\sin^2\theta - \mu^2\cos^2\theta}}$$

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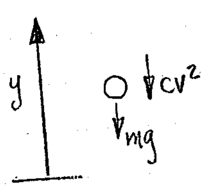
$$t_{TOT} = t^*_{up} + t^*_{down} = \frac{V_0}{g}\left[(\sin\theta + \mu\cos\theta)^{-1} + (\sin^2\theta - \mu^2\cos^2\theta)^{-1/2}\right]$$

3.12

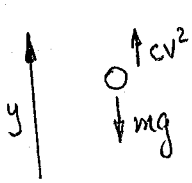


$$m\ddot{y} = -mg - cv^2 = -c\left(\frac{mg}{c} + v^2\right) = mv\frac{dv}{dy}$$

$$dy = -\frac{m}{c}\left(\frac{v\,dv}{V_t^2 + V^2}\right) \qquad \text{using the cond. } \underline{V = V_0 \ @\ y=0}$$

we get
$$y = \frac{m}{2c}\ln\left(\frac{V_t^2 + V_0^2}{V_t^2 + V^2}\right)$$

when $V=0$:
$$\boxed{y^* = \frac{m}{2c}\ln\left(\frac{V_t^2 + V_0^2}{V_t^2}\right)}$$



$$mv\frac{dv}{dy} = cv^2 - mg$$

$$\frac{m}{c} \frac{v\,dv}{v^2 - V_t^2} = dy \quad \Rightarrow \quad \frac{m}{2c} \ln\left(v^2 - V_t^2\right) = y - y_2$$

when $v = 0$ $\quad y = y^*$ which leads to

$$y = \frac{m}{2c} \ln\left[\frac{(v^2 - V_t^2)(V_t^2 + V_0^2)}{-V_t^4}\right] \checkmark$$

when $y = 0$

either $\frac{m}{2c} = 0$ or $\left[\frac{(v^2 - V_t^2)(V_t^2 + V_0^2)}{-V_t^4}\right] = 1$

if $\frac{m}{2c} = 0$ we have a trivial solution

$\therefore$

$$\left[\frac{(v^2 - V_t^2)(V_t^2 + V_0^2)}{-V_t^4}\right] = 1$$

$$v^2 - V_t^2 = \frac{-V_t^4}{V_t^2 + V_0^2}$$

$$v^2 = \frac{-V_t^4}{V_t^2 + V_0^2} + V_t^2$$

$$= \frac{-V_t^4 + V_t^2(V_t^2 + V_0^2)}{V_t^2 + V_0^2}$$

$$= \frac{V_t^2 V_0^2}{V_t^2 + V_0^2}$$

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finally:

$$\boxed{V = \frac{V_t V_0}{(V_t^2 + V_0^2)^{1/2}}} \checkmark$$

QED

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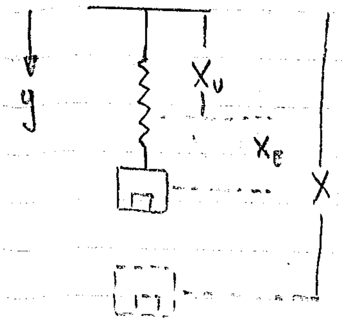
3.20



$$(M+m)g - k(X - X_U) = F$$

let $y_r = X - X_U - X_E$

$$(M+m)g = +k(X_E) \qquad X_E = \frac{(M+m)g}{k}$$

then $y_r = X - X_U - \frac{(M+m)g}{k}$

$$\therefore \quad -ky_r = (m+M)\ddot{y}_r$$

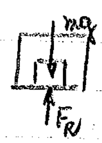
from this we get $\quad y_r = a \sin\sqrt{\frac{k}{M+m}}\,t + b \cos\sqrt{\frac{k}{M+m}}\,t$

at $y_r = d,\ t = 0 \quad v = 0 \qquad \dot{y}_r = a\omega\cos\omega t - b\omega\sin\omega t \qquad$ where $\omega = \sqrt{\frac{k}{M+m}}$

$\therefore \quad 0 = a\omega \quad \therefore \quad a = 0 \implies \dot{y}_r = -b\omega\sin\omega t$

and $y_r = b\cos\omega t \implies y_r = d\cos\omega t$ ✓

at $y_r = -d \qquad \dot{y}_r = 0 \qquad t = \frac{\pi}{\omega}$



$$mg - F_R = m\ddot{y}_r = \cancel{-ky_r}$$

solv. for $\ddot{y}_r$

$\therefore \quad F_R = mg + ky_r \implies mg - F_R = -\omega^2 d\,m\cos\omega t$

$$F_R = m\left[g - \frac{k}{M+m}d\cos\omega t\right]$$

$$F_R = mg + kd\cos\sqrt{\frac{k}{M+m}}\,t$$

$F_R = 0 \qquad d_{min} = d_{crit} = \frac{g(M+m)}{k}$

at $y_r = -d \qquad F_R = 0$

$$0 = mg + kd\cos\sqrt{\frac{k}{M+m}}\left(\pi\sqrt{\frac{M+m}{k}}\right) = mg - kd$$

$$\boxed{d = mg/k}$$

8

$$X = X_U + \frac{mg}{k} + \frac{(m+M)g}{k} = X_U + \frac{(2m+M)g}{k}$$

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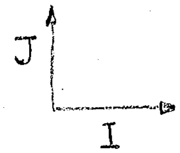
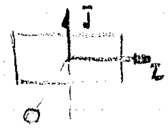
5.1

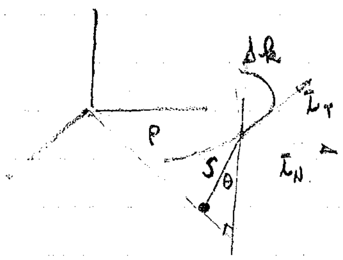




$m a_0 \bar{i} = S\sin\theta\,\bar{i} + S\cos\theta\,\bar{j} - mg\bar{j}$

since the total force on the body is equal to the acceleration of the system x mass.

from this $\left.\begin{array}{l} S\sin\theta = ma_0 \\ S\cos\theta = mg \end{array}\right\}$ $S = m(a_0^2+g^2)^{1/2}$ $\tan\theta = \frac{a_0}{g}$



$\ddot{\bar r} = \frac{v_0^2}{\rho}\bar{i}_N$ the only acceleration due to the turning about the center of the circle

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$m\ddot{\bar r} = \bar S - mg\bar k = +S\sin\theta\,\bar i_N + S\cos\theta\,\bar k - mg\bar k$

and $\left.\begin{array}{l} \frac{mv_0^2}{\rho} = S\sin\theta \\ mg = S\cos\theta \end{array}\right\}$ $S = m\left(g^2+\frac{v_0^4}{\rho^2}\right)^{1/2}$; $\tan\theta = \frac{v_0^2}{\rho g}$

5.5

$b\bar i_r = \bar r$ $\bar\omega = \frac{v}{b}\bar k$

$\frac{d\bar r}{dt} = \dot r\bar i_r + r\dot\theta\bar i_\theta$ $\frac{d^2\bar r}{dt^2} = (\ddot r - r\dot\theta^2)\bar i_r + (2\dot r\dot\theta + r\ddot\theta)\bar i_\theta$

from $\frac{d^2\bar r}{dt^2}$ we get $-b\omega^2\bar i_r$ from $\frac{d\bar r}{dt}$ we get $b\omega\bar i_\theta$

$\bar\omega\times\bar r$ gives $v\bar i_\theta$; $\bar\omega\times(\bar\omega\times\bar r)$ gives $-\frac{v^2}{b}\bar i_r$; $2\bar\omega\times\frac{d\bar r}{dt}$ gives $-2\omega V\bar i_r$

∴ $\bar a = -\bar i_r\left[b\omega^2 + 2\omega V + \frac{v^2}{b}\right]$ and $\bar F = m\bar a$

if $V = b\omega$ $\bar a = -\bar i_r[b\omega^2+2\omega^2 b + b\omega^2] = -\bar i_r[4\omega^2 b]$

$V=-b\omega$ $\bar a = -\bar i_r[b\omega^2 - 2\omega^2 b + b\omega^2] = \bar 0$

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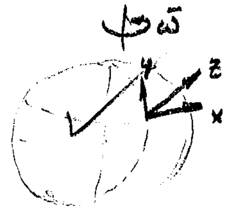
5.7. $\bar{a}_{cor} = -2\bar{\omega}\times\bar{v} = -2 \begin{vmatrix} i & j & k \\ 0 & \omega\cos\lambda & \omega\sin\lambda \\ 0 & 350\text{mph} & 0 \end{vmatrix}$



$$= -2\left(-\bar{i}\left[350\times\frac{88\text{ fps}}{60\text{ mph}}\right]\omega\sin\lambda\right) = \bar{i}\left\{1027\times\frac{\sqrt{2}}{2}\,(7.3\times10^{-5}\right\}$$

$$\bar{F} = ma\,\bar{i} = \bar{i}\left\{\frac{4000}{32.2}(1.027)(.707)(7.3)\right\} = 6.57 \text{ lb east.}$$

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Cuarleroy
AE 311

4.1 $\bar{F} = cyz\,\bar{i} + czx\,\bar{j} + cxy\,\bar{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ cyz & czx & cxy \end{vmatrix}$$

$$= \bar{i}\left[cx - cx\right] + \bar{j}\left[cy - cy\right] + \bar{k}\left[cz - cz\right] = \bar{0}$$

Conservative

$\bar{F} = cyz\,\bar{i} + cxy\,\bar{j} + czx\,\bar{k}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ cyz & cxy & czx \end{vmatrix}$$

$$= \bar{i}\left[0 - 0\right] + \bar{j}\left[cy - cz\right] + \bar{k}\left[cy - cz\right] \neq \bar{0}$$

non-conservative.

$\bar{F} = cy/z\,\bar{i} + cx/z\,\bar{j} - \frac{cxy}{z^2}$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ cy/z & cx/z & -\frac{cxy}{z^2} \end{vmatrix}$$

$$= \bar{i}\left[-\frac{cx}{z^2} - \left[-\frac{cx}{z^2}\right]\right] + \bar{j}\left[-\frac{cy}{z^2} - \left[-\frac{cy}{z^2}\right]\right] + \bar{k}\left[c/z - c/z\right] = \bar{0}$$

Conservative.

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4.3 Given:

$V = ax + by^2 + cz^3$, particle m mass, at $(x,y,z) = (0,0,0)$ $v = v_0$

Find:

v when $(x,y,z) = (1,1,1)$ if it passes through it.

$$\bar{F} = -\nabla V = -a\bar{i} - 2by\,\bar{j} - 3cz^2\bar{k}$$

to pass through 1,1,1 $\bar{F} \neq \bar{F}(t)$ or $\bar{F} = \bar{F}(x,y,z)$ or $\nabla \times \bar{F} = 0$

$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -a & -2by & -3cz^2 \end{vmatrix} = \bar{0}$$

$\therefore$ $\bar{F}$ is conservative $\therefore$ $\bar{F} = \bar{F}(x,y,z)$

we can then use $T + V = E \Rightarrow \frac{1}{2}mv_0^2 = \frac{1}{2}mv_*^2 + (a+b+c)$

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or $$v_* = \sqrt{v_0^2 - \frac{2}{m}(a+b+c)}$$

4.5 $\bar{F}_1 = y\bar{i} + x\bar{j}$; $\bar{F}_2 = y\bar{i} - x\bar{j}$

$$\nabla \times \bar{F}_1 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x & 0 \end{vmatrix} = \bar{k}(1-1) = \bar{0}$$ Conservative

$$\nabla \times \bar{F}_2 = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix} = \bar{k}(-1-1) = -2\bar{k} \neq \bar{0}$$ non conservative,

ⓐ $$\int \bar{F}_1 \cdot d\bar{r} = \int (F_{x_1}\, dx + F_{y_1}\, dy) = \int (y\, dx + x\, dy)$$

if $x = y$

$$\int_0^1 x\, dx + \int_0^1 y\, dy = 1$$

if $x = C = 0$ to $(0,1)$
then $y = C = 1$ to $(1,1)$

$$\int_0^1 dx = 1$$ (the first integral from 0,0 to 0,1 is 0)

ⓑ $$\int \bar{F}_2 \cdot dr = \int (y\, dx - x\, dy)$$

if $x = y$

$$\int_0^1 x\, dx - \int_0^1 y\, dy = 0$$

if $x = C = 0$ $(0,0)$ to $(0,1)$
$y = C = 1$ $(0,1)$ to $(1,1)$

$$\int_0^1 dx = 1$$

OK

As you can see $\int \bar{F}_1 \cdot d\bar{r} = 1$ both ways but $\int \bar{F}_2 \cdot dr$ is not.

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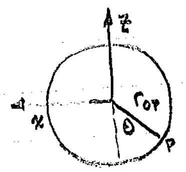
AE 211

4.9.



$$\bar{r}_{op} = -b\sin\theta\,\bar{i} - b\cos\theta\,\bar{j}$$
$$\bar{V}_{rel} = -V_o\cos\theta\,\bar{i} + V_o\sin\theta\,\bar{j}$$

$$b\frac{d\theta}{dt} = b\omega = V_o$$

$$\bar{V} = V_o(1-\cos\theta)\bar{i} + \bar{j}\,V_o\sin\theta$$

$$v^2 = 2v_o^2(1-\cos\theta)$$ v at any pt p with respect to the ground

for this velocity and taking datum line at center of wheel:

$$z = -b\cos\theta$$

$$\therefore \quad mv_o^2(1-\cos\theta) - mgb\cos\theta = \text{Constant.}$$

at highest point in the trajectory:

$$\bar{v} = v_o(1-\cos\theta)\bar{i} \text{ only} \quad \text{and } z = z^*_{rel}$$ z^*_{rel} = relative height wrt datum line

thus,

$$mv_o^2(1-\cos\theta) - mgb\cos\theta = \frac{mv_o^2}{2}(1-\cos\theta)^2 + mgz^*_{rel} \qquad (1)$$

take $\frac{dz^*_{rel}}{d\theta} = 0 \;=\; v_o^2\sin\theta + gb\sin\theta = v_o^2(\sin\theta - \sin\theta\cos\theta)$

$$gb\sin\theta = -v_o^2\sin\theta\cos\theta$$

$$\boxed{\cos\theta = -\frac{gb}{v_o^2}}$$

Substitute the following into the equat (1) we finally get:

$$\therefore \quad z^*_{rel} = \frac{v_o^2}{2g} + \frac{gb^2}{2v_o^2} \quad ; \qquad z = z^*_{rel} + b$$

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$$\boxed{z = b + \frac{v_o^2}{2g} + \frac{gb^2}{2v_o^2}}$$

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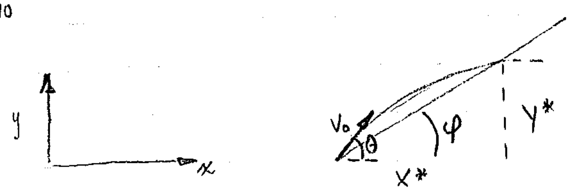
4.10



$$x = v_0 \cos\theta \, t \qquad y = v_0 \sin\theta \, t - \tfrac{1}{2} g t^2$$

$$\text{at} \quad t = t^* \quad x = x^* \quad y = y^* \qquad R = \sqrt{x^{*2} + y^{*2}}$$

$$\tan\varphi = y^*/x^* \qquad x^* = v_0 \cos\theta \, t^* ; \quad y^* = v_0 \sin\theta \, t^* - \tfrac{1}{2} g t^{*2}$$

$$y^* = x^* \tan\varphi = v_0 \cos\theta \tan\varphi \, t^* = v_0 \sin\theta \, t^* - \tfrac{1}{2} g t^{*2}$$

$$\underline{\text{or}} \quad t^* = \frac{2v_0}{g} \left(\sin\theta - \cos\theta \tan\varphi \right)$$

$$x^* = v_0 \cos\theta \left[\frac{2v_0}{g} \left(\sin\theta - \cos\theta \tan\varphi \right) \right]$$

$$y^* = v_0 \sin\theta \left[\frac{2v_0}{g} \left(\sin\theta - \cos\theta \tan\varphi \right) \right] - \frac{g}{2} \left[\frac{2v_0}{g} \left(\sin\theta - \cos\theta \tan\varphi \right) \right]^2$$

$$y^* = \frac{2v_0^2}{g} \left[\sin\theta \cos\theta \tan\varphi - \cos^2\theta \tan^2\theta \right.$$

$$R^2 = x^{*2} + y^{*2} = \frac{4v_0^4}{g^2} \cos^2\theta \left[\sin^2\theta - \sin 2\theta \tan\varphi + \tan^2\varphi + \cos^2\theta \tan^4\varphi - \sin 2\theta \tan^3\varphi \right]$$

$$= \frac{4v_0^4}{g^2} \frac{\cos^2\theta}{\cos^4\varphi} \left[\sin^2\theta \cos^4\varphi - \sin 2\theta \sin\varphi \cos\varphi + \sin^2\varphi \cos^2\varphi + \cos^2\theta \sin^4\varphi \right]$$

$$= \frac{4v_0^4}{g^2} \frac{\cos^2\theta}{\cos^4\varphi} \left[\sin^4\varphi - \sin^4\varphi \sin^2\theta + \sin^2\varphi \cos^2\varphi + \sin^2\theta \cos^4\varphi - \sin 2\theta \sin\varphi \cos\varphi \right.$$

$$= \frac{4v_0^4}{g^2} \frac{\cos^2\theta}{\cos^4\varphi} \left[\sin^2\varphi \cos^2\theta + \sin^2\theta \cos^2\varphi - 2 \sin\theta \cos\theta \sin\varphi \cos\varphi \right]$$

$$R = \frac{2v_0^2}{g} \frac{\cos\theta}{\cos^2\varphi} \left[\sin(\theta - \varphi) \right]$$

QED.

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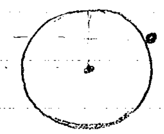
4.17



$$\frac{1}{2} m v^2 + mgz = E$$

when $z = b/2$ $\quad v = 0 \quad \Rightarrow \quad mg\,b/2 = \frac{1}{2} m v^2 + mgz$

or $\quad v^2 = g\,(b - 2z) \quad ①$

but $mg\cos\theta = \dfrac{mv^2}{b}$ $\qquad$ but $\cos\theta = \dfrac{z}{b} \quad ②$

$\therefore$ ① into ② gives $\quad g\,z/b = g(b - 2z)\cdot 1/b$

or $\quad z = b - 2z \quad \Rightarrow \quad z = b/3$

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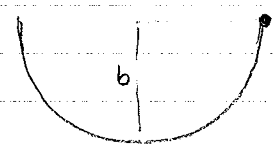
4.19



$$E = \tfrac{1}{2} m v^2 + mgz$$

when $v = 0 \quad z = b$

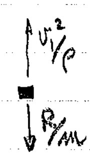
$$E = mgb$$

when $z = 0 \quad v = v_1$

$$\therefore \quad mgb = \frac{1}{2} m v_1^2$$

$$v_1 = \sqrt{2gb}$$

$$-\frac{m v_1^2}{b} = +mg + R \qquad R = -3mg$$



10

4.21 $\quad T_{ELEM} = 2\pi\sqrt{l/g}$

$$T_{NEW} = 2\pi\sqrt{l/g}\left(1 + \frac{\theta_0^2}{16} + \ldots\right) = T_{ELEM}\left(1 + \frac{\theta_0^2}{16} + \ldots\right)$$

if $\theta_0 = 30^\circ = \dfrac{\pi}{6}$ rad. $\qquad 1 + \dfrac{\theta_0^2}{16} = 1 + \dfrac{\pi^2}{36\cdot 16} = 1.01715$

Error incurred is 1.715% ; T_{ELEM} is too small by 1.715%

10

Cesar Levy
AE 211

1. $r_L = 29960$ miles $r_S = 4360$ miles

$$\frac{V_L^2}{2} - \frac{\mu}{r_L} = \frac{V_S^2}{2} - \frac{\mu}{r_S} \quad ; \quad r_L V_L = r_S V_S$$

$$\frac{1}{2}\left(\frac{r_S V_S}{r_L}\right)^2 - \frac{V_S^2}{2} = \mu\left[\frac{1}{r_L} - \frac{1}{r_S}\right]$$

$$V_S = \sqrt{\frac{2\mu\left[\frac{1}{r_L} - \frac{1}{r_S}\right]}{\left[\left(\frac{r_S}{r_L}\right)^2 - 1\right]}} \quad ; \quad \frac{r_S}{r_L} = .1455$$

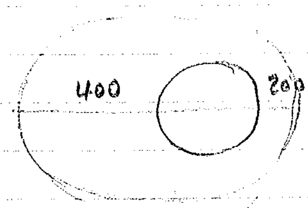
a) $V_S = 32650$ ft/sec

10

b) $V_L = \frac{r_S V_S}{r_L} = .1455\,(32650) = 4750$ ft/sec

2. $Ve_0 = Ve\sqrt{\frac{R}{r_0}}$ $Ve_0 = \sqrt{2}\,Vc_0$

$$\therefore \quad Vc_0 = Ve\sqrt{\frac{R}{2r_0}} = 36700\sqrt{\frac{3960}{8720}} = .674\,(36700) = 24700 \text{ ft/sec}$$



$a = 4260$ miles

$$v = \sqrt{\mu\left[\frac{2}{r_0} - \frac{1}{a}\right]} \quad \text{where } r_0 = 4360 \text{ miles}$$

$$v = \sqrt{(1.408\times10^6)\left[\frac{2}{4360\times5280} - \frac{1}{4260\times5280}\right]}$$

$v = 24400$ ft/sec

10

$$\epsilon = \frac{24700 - 24400}{24700} \times 100 = 1.215\,\%$$

3. $h = r_0 V_0 \sin \varphi_0$ where $r_0 = 5460$ miles, $V_0 = 26700$ ft/sec, $\varphi_0 = 76°$

$$h = 7.47 \times 10^{11} \text{ ft}^2/\text{sec}$$

$$E = \frac{v_0^2}{2} - \frac{\mu}{r_0} = \frac{(26700)^2}{2} - \frac{1.408 \times 10^{16}}{5460 \times 5280} = -1.315 \times 10^8$$

$$\epsilon = \sqrt{1 + \frac{2Eh^2}{\mu^2}} = .511$$

at $\theta = 0$ $\qquad \cos\theta_0 = \frac{1}{\epsilon}\left[\frac{h^2}{r\mu} - 1\right]$

$$\cos\theta_0 = .73 \qquad \theta_0 = 43.0°$$

$$u = \frac{\mu}{h^2}\left[1 + \epsilon\cos(\theta - \theta_0)\right] \qquad \theta - \theta_0 = 0, 180°$$

$\therefore \quad \frac{1}{r_{LARGE}} = \frac{\mu}{h^2}\left[1 - \epsilon\right] \quad r_{LARGE} = 15350$ miles $\quad ht = 11390$ miles

$\therefore \quad \frac{1}{r_{SMALL}} = \frac{\mu}{h^2}\left[1 + \epsilon\right] \quad r_{SMALL} = 4980$ miles $\quad ht = 1020$ miles

10

Cesar Leavy
AE211

1. $\bar{r} = \sum \frac{(i+j)+(j+k)+k}{3} = \frac{i}{3} + \frac{2j}{3} + \frac{2k}{3}$

$\bar{v} = \sum \frac{(2i)+j+(i+j+k)}{3} = i + \frac{2j}{3} + \frac{k}{3}$

10

$p = \sum m_i \bar{v}_i = 3i + 2j + k$

2. $v_0 = v_B - v_g$

$-v_g = v_0 - v_B$

$m v_B = -M v_g$

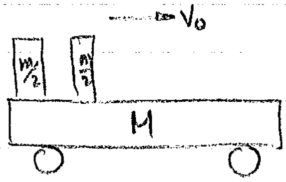
$-M v_g = M v_0 - M v_B$

$m v_B + M v_B = M v_0$

$$v_B = \frac{v_0}{\left(1 + \frac{m}{M}\right)}$$

10

Cesar Levy
AE211



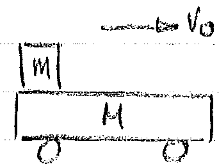
$$(M+m)v_0 = (M+m/2)(v_0+\Delta v_1) + m/2\,(v_0+\Delta v_1-u)$$

$$\Delta v_1 = \frac{mu}{2[M+m]}$$

$$(M+m/2)(v_0+\Delta v_1) = M(v_0+\Delta v_1+\Delta v_2) + m/2\,(v_0+\Delta v_1+\Delta v_2-u)$$

$$\Delta v_2 = \frac{mu}{2[M+m/2]}$$

$$\Delta v_T = \Delta v_1+\Delta v_2 = \frac{mu}{2M+2m} + \frac{mu}{2M+m}$$



$$(M+m)v_0 = M(v_0+\Delta v) + m(v_0+\Delta v-u)$$

$$\Delta v = \frac{mu}{M+m}$$

Assume

$$\Delta v \geq \Delta v_T$$

$$\frac{2mu}{2m+2M} \geq \frac{mu}{2M+2m} + \frac{mu}{2M+m}$$

$$\frac{mu}{2m+2M} \geq \frac{mu}{2M+m}$$ but this is not true since $2M+m \leq 2m+2M$

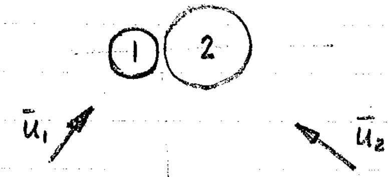
$$\therefore \quad \Delta v_T \geq \Delta v$$

10

Cesar Levy
AE 211

1. $e = .5$ $\quad m_1 = 1$ $\quad m_2 = 10$

$\bar{u}_1 = 10i + 10j$ $\qquad \bar{u}_2 = -20i + 15j$



In y direction it was proven that

$u_{1y} = u_{1y}' = 10$

and $u_{2y} = u_{2y}' = 15$

since $e = \dfrac{u_{2x}' - u_{1x}'}{u_{1x} - u_{2x}} = .5$ we obtain $u_{2x}' - u_{1x}' = 15$

solving for u_{1x}' we obtain $u_{1x}' = u_{2x}' - 15$

Substituting into the equation obtained from entire system in x direction

$m_1 u_{1x} + m_2 u_{2x} = m_1 \bar{u}_{1x}' + m_2 \bar{u}_{2x}'$ we obtain

$m_1 u_{1x} + m_2 u_{2x} = m_1 (u_{2x}' - 15) + m_2 u_{2x}'$; solving for u_{2x}'

$$\frac{m_1 u_{1x} + m_2 u_{2x} + m_1 \cdot 15}{m_1 + m_2} = \frac{1(10) + 10(-20) + 15}{11} = \frac{-175}{11} = -15.91$$

since $\hat{P}$ is the impulsive force for particle 2, we can write

$m_2 u_{2x} + \hat{P} = m_2 u_{2x}'$; solving for $\hat{P}$ we obtain

$$\hat{P} = m_2 (u_{2x}' - u_{2x}) = 10(-15.91 + 20) = 40.9$$

since we can also write an equation for u_{1x}' as a function of $m_1, u_{1x}, \hat{P}$

$$u_{1x}' = u_{1x} - \frac{\hat{P}}{m_1} = 10 - 40.9 = -30.9$$

10

$\therefore$

$$\bar{u}_1' = -30.9\, i + 10 j \qquad \bar{u}_2' = -15.91\, i + 15 j$$

2. $$T = \tfrac{1}{2} m_1 (v_c + v_1)^2 + \tfrac{1}{2} m_2 (v_c + v_2)^2 \quad (1)$$

$$T' = \tfrac{1}{2} m_1 (v_c + v_1')^2 + \tfrac{1}{2} m_2 (v_c + v_2')^2 \quad (2)$$

$$T - T' = \tfrac{1}{2} m_1 (2v_c v_1 + v_1^2 - 2v_c v_1' - v_1'^2) + \tfrac{1}{2} m_2 (2v_c v_2 + v_2^2 - 2v_c v_2' - v_2'^2)$$

$$T - T' = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 + \hat{Q}$$

from momentum $\quad m_1 u_1 + m_2 u_2 = (m_1 + m_2) v_c \quad$ ③

$\qquad\qquad\qquad\quad m_1 u_1' + m_2 u_2' = (m_1 + m_2) v_c \quad$ ④

where $\quad u_1 = v_1 + v_c \quad u_2 = v_2 + v_c \quad u_1' = v_1' + v_c \quad u_2' = v_2' + v_c$

$\hat{Q}$ can be evaluated as $\; -\tfrac{1}{2} m_1 v_1'^2 - \tfrac{1}{2} m_2 v_2'^2$

$$\therefore \quad T - T' = \tfrac{1}{2} m_1 (v_1^2 - v_1'^2) + \tfrac{1}{2} m_2 (v_2^2 - v_2'^2) \quad ⑤$$

from eq ③ and ④

$$m_1 v_1 = -m_2 v_2$$
$$m_1 v_1' = -m_2 v_2' \quad ⑥$$

from these we can get the following

$$\frac{v_1}{v_1'} = \frac{v_2}{v_2'} \quad ⑦ \quad \Longrightarrow \quad v_1 = \frac{v_1' v_2}{v_2'}$$

putting them into the equation for coefficient of Restitution

$$\epsilon = \frac{v_1' - v_2'}{v_2 - v_1} \quad ⑧ \quad = \frac{v_1' - v_2'}{\frac{v_2}{v_2'}(v_2' - v_1')} = -\frac{v_2'}{v_2}$$

$$\therefore \quad \epsilon = -\frac{v_2'}{v_2} \longrightarrow \boxed{-\frac{1}{\epsilon} = \frac{v_2}{v_2'} = \frac{v_1}{v_1'}}$$

$\therefore \; -v_2 \epsilon = v_2' \;$ & $\; -v_1 \epsilon = v_1' \quad$ squaring & putting into ⑤ gives

$$T - T' = \tfrac{1}{2} m_1 v_1^2 (1 - \epsilon^2) + \tfrac{1}{2} m_2 v_2^2 (1 - \epsilon^2)$$

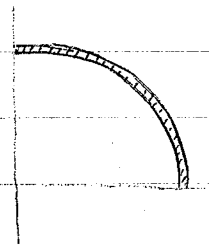
Q.E.D.

Cesar Levy

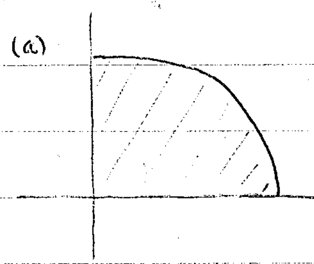
AE 211

8.1

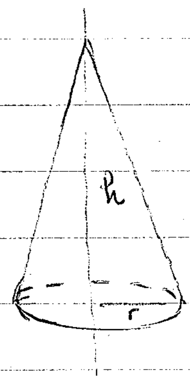
(b)



(a)



(c)



for (a)

$$\bar{x} = \frac{\int \rho x \, ds}{\int \rho \, ds}$$ ✓ since $x = r\cos\theta$ and $ds = r\,dr\,d\theta$ and ρ = constant

$$\bar{x} = \frac{\iint r^2\cos\theta \, dr\,d\theta}{\iint r\,dr\,d\theta} = \frac{\int_0^a r^2 dr \int_0^{\pi/2} \cos\theta \, d\theta}{\int_0^a r\,dr \int_0^{\pi/2} d\theta} = \frac{a^3/3}{\frac{\pi a^2}{4}}$$

10

$$\bar{x} = \frac{4a}{3\pi}$$ ✓

since $y = r\sin\theta$ & $ds = r\,dr\,d\theta$

$$\bar{y} = \frac{\int \rho y \, dS}{\int \rho \, dS} = \frac{\int_0^a r^2 dr \int_0^{\pi/2} \sin\theta \, d\theta}{\int_0^a r\,dr \int_0^{\pi/2} \cancel{\frac{\pi a^2}{4}} d\theta} = \frac{a^3/3}{\frac{\pi a^2}{4}}$$

$$\bar{y} = \frac{4a}{3\pi}$$ ✓

(b) $\bar{x} = \dfrac{\int x\rho\, dl}{\int \rho\, dl}$ since ρ = constant and $dl = a\,d\theta$

$$\bar{x} = \frac{\int_0^{\pi/2} a^2 \cos\theta\, d\theta}{\int_0^{\pi/2} a\, d\theta} = \frac{a^2 \cdot 1}{\frac{a\pi}{2}} = \frac{2a}{\pi}$$

$$\bar{y} = \frac{\int_0^{\pi/2} a^2 \sin\theta\, d\theta}{\int_0^{\pi/2} a\, d\theta} = \frac{\int \rho y\, dl}{\int \rho\, dl}$$

20

$$= \frac{-a^2 \cos\theta \Big|_0^{\pi/2}}{a\theta \Big|_0^{\pi/2}} = \frac{a^2}{a\pi/2} = \frac{2a}{\pi}$$

(c) $\bar{x} = \dfrac{\int \rho x\, dV}{\int \rho\, dV}$ $\quad dV = \pi r^2 dx \quad r = a\left[-\frac{x}{h} + 1\right]$

ρ = const.

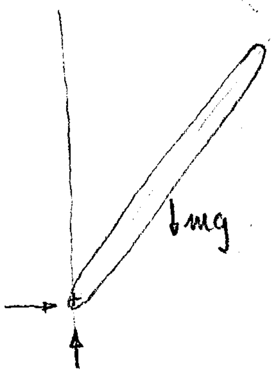
$$\bar{x} = \frac{a^2\pi \int \left[\frac{x}{h} - 1\right]^2 x\, dx}{a^2\pi \int \left[\frac{x}{h} - 1\right]^2 dx} = \frac{\frac{x^4}{4h^2} - \frac{2x^3}{3h} + \frac{x^2}{2}\Big|_0^h}{\frac{x^3}{3h^2} - \frac{x^2}{h} + x\Big|_0^h} = \frac{\frac{h^2}{12}}{h/3}$$

$$\bar{x} = h/4$$

10

1.

Cesar Levy
AE211



$$f = m\ddot{x}_{cm}$$

$$N - mg = m\ddot{y}_{cm}$$

$$I_o \ddot{\theta} = mg\frac{l}{2}\sin\theta$$

$$mg\frac{l}{2} = \frac{1}{2}I_o\dot{\theta}^2 + mg\tfrac{l}{2}\cos\theta \Rightarrow \dot{\theta}^2 = \frac{mgl(1-\cos\theta)}{I_o} = \frac{3g}{l}(1-\cos\theta)$$

since $y_{cm} = \frac{l}{2}\cos\theta$; $\dot{y}_{cm} = \frac{l}{2}(-\sin\theta\,\dot{\theta})$

$x_{cm} = \tfrac{l}{2}\sin\theta$ $\quad \dot{x}_{cm} = \tfrac{l}{2}(\cos\theta\,\dot{\theta})$

this leads to $\ddot{y}_{cm} = \frac{l}{2}(-\sin\theta\,\ddot{\theta} - \cos\theta\,\dot{\theta}^2)$

$\ddot{x}_{cm} = \frac{l}{2}[\cos\theta\,\ddot{\theta} - \dot{\theta}^2\sin\theta]$

thus

$$\ddot{y}_{cm} = -\frac{l}{2}\left[\sin\theta\, \frac{3\cancel{m}g\cancel{l}\sin\theta}{2\cancel{m}l^{\cancel{2}}} + 2\cos\theta\,\frac{3g}{2\cancel{l}}(1-\cos\theta)\right]$$

$$= -\frac{3}{4}\left[g\sin^2\theta - g(2\cos^2\theta - 2\cos\theta)\right] = +\tfrac{3}{4}g\left[(3\cos^2\theta - 1) - 2\cos\theta\right]$$

$$N = mg + m\ddot{y}_{cm} = \frac{1}{4}mg(3\cos\theta - 1)^2$$

$$f = m\frac{l}{2}\left[\cos\theta \cdot \frac{3\cancel{m}g\cancel{l}\sin\theta}{2\cancel{m}l^{\cancel{2}}} - 2\frac{3g}{2\cancel{l}}(1-\cos\theta)\sin\theta\right]$$

$$= \frac{3}{4}mg\sin\theta\left[\cos\theta - 2 + 2\cos\theta\right] = \frac{3}{4}mg\sin\theta\left[3\cos\theta - 2\right]$$

for slipping $f = \mu N$ $\therefore$

$$\frac{3}{4}mg\sin\theta\left[3\cos\theta - 2\right] = \mu\,\frac{1}{4}mg(3\cos\theta - 1)^2$$

10

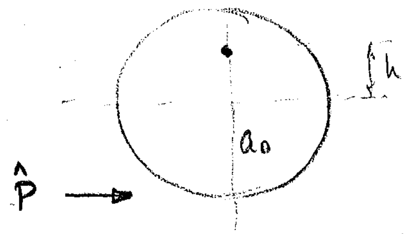
$$3\sin\theta\left[3\cos\theta - 2\right] = \mu(3\cos\theta - 1)^2$$

$$\mu = \frac{3\sin\theta\left[3\cos\theta - 2\right]}{(3\cos\theta - 1)^2}$$

for slipping to left since denom. > 0 and since $0 \le \theta \le \frac{\pi}{2}$

$3\cos\theta - 2 > 0$ or $\theta > \arccos\frac{2}{3}$

2.



$$0 + \hat{P} = m v_c = m\omega h$$

center of percussion

$$I_0 \omega = \hat{P}\cdot(a_0 + h)$$

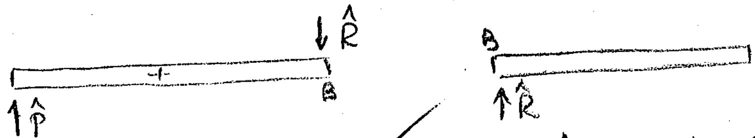
$$I_0 = \frac{m a_0^2}{2} + m h^2 \quad \therefore$$

$$m\left(\frac{a_0^2}{2} + h^2\right)\omega = \hat{P}\,(a_0 + h) = m\omega h\,(a_0 + h)$$

$$\frac{a_0^2}{2} = a_0 h \qquad h = \frac{a_0}{2}$$

10

3.



$$(\hat{P} + \hat{R})\frac{l}{2} = I_c \dot{\theta}_1 \qquad \frac{l}{2}\hat{R} = I_c \dot{\theta}_2$$

$$\hat{P} - \hat{R} = m v_{c_1} \qquad \hat{R} = m v_{c_2}$$

$$v_B = v_{c_1} - \frac{l}{2}\dot{\theta}_1 = v_{c_2} + \tfrac{l}{2}\dot{\theta}_2$$

$$\hat{P} = m(v_{c_1} + v_{c_2})$$

$$\hat{P} = \frac{2 I_c}{l}(\dot{\theta}_1 - \dot{\theta}_2) = \frac{ml}{6}(\dot{\theta}_1 - \dot{\theta}_2)$$

9

$$v_{c_2} + v_{c_1} = 2 v_{c_2} + \frac{l}{2}(\dot{\theta}_2 + \dot{\theta}_1) = \frac{2\,ml^2}{12\,l\,m}\dot{\theta}_2 + \frac{l}{2}(\dot{\theta}_2 + \dot{\theta}_1)$$

$$= \frac{5}{6} l \dot{\theta}_2 + \frac{l}{2}\dot{\theta}_1$$

$$\hat{P} = \frac{ml}{6}\left[5\dot{\theta}_2 + 3\dot{\theta}_1\right]$$

$$\hat{P} = \frac{ml}{6}\left(\dot{\theta}_1 - \dot{\theta}_2\right)$$

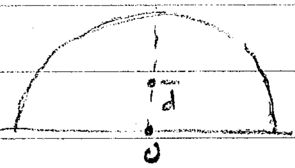
$$\Longrightarrow \dot{\theta}_1 = \cancel{\tfrac{3}{4}\frac{\hat{P}}{ml}}\ \frac{9}{2}\frac{\hat{P}}{ml} \qquad \dot{\theta}_2 = -\frac{3}{2}\frac{\hat{P}}{ml}$$

$$v_{c_2} = \frac{l}{6}\dot{\theta}_2 = -\frac{\hat{P}}{4m} \qquad v_{c_1} = \frac{5}{4}\frac{\hat{P}}{m}$$

$$v_B = v_{c_2} + \frac{l}{2}\dot{\theta}_2 = -\frac{\hat{P}}{4m} - \frac{3\hat{P}}{4m} = -\frac{\hat{P}}{m}$$

AE 211

8.13.



$$d = \rho\int_0^a r^2 dr \int_0^\pi \sin\theta\, d\theta \Big/ \rho\int_0^a r\,dr \int_0^\pi d\theta$$

$$= \frac{4a}{3\pi}$$

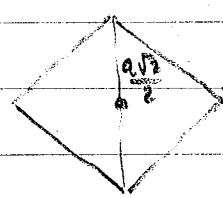
$$I_{cm} = I_0 - Ad^2 \qquad \rho A d^2 = \rho\frac{\pi a^2}{2}\cdot\frac{16a^2}{9\pi^2} = \rho\frac{16a^4}{18\pi} = \rho\frac{8a^4}{9\pi}$$

$$I_0 = \rho\int r^2 \cdot r\,dr\,d\theta = \rho\cdot\frac{\pi a^4}{4}$$

$$I_{cm} = \rho\frac{\pi a^2}{2}\left[\frac{a^2}{2} - \frac{16a^2}{9\pi^2}\right] = ma^2\left[\frac{1}{2} - \left(\frac{4}{3\pi}\right)^2\right]$$

where $m = \rho A$

8.15a



$$-mg\frac{a\sqrt{2}}{2}\sin\theta = mk^2\ddot{\theta} \Rightarrow$$

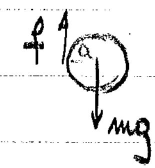
$$T = 2\pi\sqrt{\frac{2k^2}{ga\sqrt{2}}}$$

$$k^2 = k_{cm}^2 + l^2 = a^2/6 + a^2/2 = 2a^2/3$$

$$T = 2\pi\sqrt{\frac{2a}{3g}}\sqrt[4]{2}$$

$$\frac{a\sqrt{2}}{2}\cdot l = k_{cm}^2 = a^2/6 \qquad l = \frac{a\sqrt{2}}{6}$$

8.17



$$mg - f = m\ddot{y}_c$$

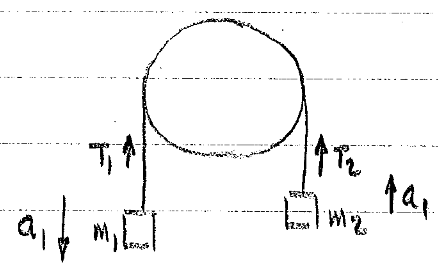
$$f \cdot a = I_c \ddot{\theta} = \frac{2}{5} m a^2 \ddot{\theta}$$

$$f = \frac{2}{5} a m \ddot{\theta} \qquad mg - \frac{2}{5} m a \ddot{\theta} = m\ddot{y}_c$$

$$\ddot{y}_{cm} = a\ddot{\theta} \qquad mg - \frac{2}{5} m \ddot{y}_c = m\ddot{y}_c$$

$$\ddot{y}_c = \frac{5}{7} g \leftarrow$$

8.19



$$m_1 g - T_1 = m_1 a_1$$

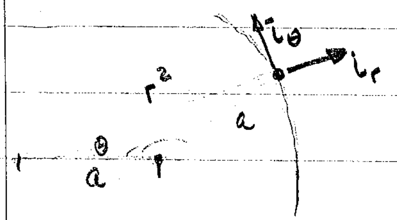
$$T_2 - m_2 g = m_2 a_1$$

$$(T_1 - T_2) a = I\ddot{\theta} = I a_1 / a$$

$$T_2 - T_1 = (m_1 + m_2) a_1 - g(m_1 - m_2)$$

$$-\frac{I}{a^2} a_1 - (m_1 + m_2) a_1 = -g(m_1 - m_2)$$

$$a_1 = g(m_1 - m_2) \Big/ \left(m_1 + m_2 + I/a^2\right)$$



$$\bar v = \dot r\,\bar\imath_r + r\dot\theta\,\bar\imath_\theta$$

$$\bar a = (\ddot r - r\dot\theta^2)\,\bar\imath_r + (r\ddot\theta + 2\dot r\dot\theta)\,\bar\imath_\theta$$

$$\dot\theta = c$$

And you would have had much easier time.

$$r^2 = 2a^2 + 2a^2\cos(2\theta) = 2a^2[1+\cos2\theta] = 4a^2\cos^2\theta$$

$$2r\dot r = -2a^2\sin2\theta\,(2\dot\theta) = -4a^2\dot\theta\sin2\theta$$

$$r\dot r = -2a^2\dot\theta\sin2\theta \qquad \dot r = -\frac{2a^2\dot\theta\sin2\theta}{r}\,; \quad r = -\frac{2a^2\dot\theta\sin2\theta}{\dot r}$$

since $\dot\theta = c$ $\quad\ddot\theta = 0$ $\therefore$

$$\bar a = (\ddot r - r\dot\theta^2)\,\bar\imath_r + (2\dot r\dot\theta)\,\bar\imath_\theta$$

since $r\dot r = -2a^2\dot\theta\sin2\theta$

$$\dot r\dot r + r\ddot r = -4a^2\dot\theta^2\cos2\theta$$

$$\ddot r = -\frac{4a^2\dot\theta^2\cos2\theta}{r} - \dot r^2 = -\left[\frac{4a^2\dot\theta^2\cos2\theta}{r} + \frac{4a^4\dot\theta^2\sin^2 2\theta}{r^3}\right]$$

$$\bar a = \left(-\frac{[4a^2\dot\theta^2\cos2\theta + \dot r^2]}{r} + \frac{2a^2\dot\theta\sin2\theta}{\dot r}\,\dot\theta^2\right)\bar\imath_r - 2\left(\frac{a^2\dot\theta^2\sin2\theta}{r}\right)\bar\imath_\theta$$

$$\frac{4a^4\dot\theta^2\sin^2 2\theta}{r}$$

$$\cancel{a = |\bar a| = \sqrt{}}$$

$$\cancel{a_r = -\frac{[4a^2\dot\theta^2\cos2\theta + \dot r^2 + 2a^2\dot\theta^2 + 2a^2\dot\theta^2\cos2\theta]}{r}}$$

Good try

$$\ddot r = -\frac{4a^2}{r}\dot\theta^2\left[\cos2\theta + \frac{a^2\sin^2 2\theta}{r^2}\right] = -\frac{4a^2\dot\theta^2}{r^3}\left[\frac{2a^2\cos2\theta + a^2 + a^2\cos^2 2\theta}{a^2(1+\cos2\theta)^2}\right]$$

$$r\dot\theta^2 = \frac{r^2\dot\theta^2}{r} = \frac{2a^2\dot\theta^2 + 2a^2\dot\theta^2\cos2\theta}{r}$$

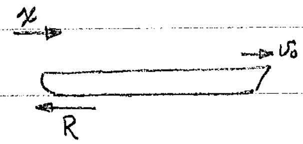
$$\dot r\dot\theta = -\frac{2a^2\dot\theta^2\sin2\theta}{r}$$

$$\ddot r = -\frac{4a^2\dot\theta^2}{r}\,\frac{[1+\cos2\theta]}{2}$$

$$\ddot r = -\frac{2a^2\dot\theta^2[1+\cos2\theta]}{r}$$

B

II



$\sum F_x = m\ddot{x}$ $\qquad -mcv = m\ddot{x}$

$$-\cancel{m}c\cancel{v} = \cancel{m}\cancel{v}\frac{dv}{d\cancel{x}}$$

$-c\,dt = dv$ $\qquad -c\int_0^t dt = \int_{v_0}^{v} dv$

$-ct = -v_0 + v$ $\qquad v = v_0 - ct \longleftarrow (1)$

when $v=0$ $\quad t = t^*$ $\quad \therefore \boxed{t^* = \frac{v_0}{c}}$!!

(1) $\dfrac{ds}{dt} = v_0 - ct$ should be x or s → obtain $-ct = \ln(v_0 - cx) + C$

etc

$$\int_0^s ds = \int_0^t (v_0 - ct)\,dt$$

$s = v_0 t - \dfrac{ct^2}{2}$ $\qquad$ when $t = t^* = \dfrac{v_0}{c}$ $\qquad s = s^*$

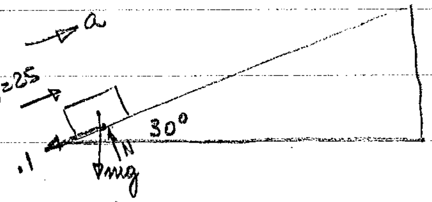
$$s^* = v_0 t^* - \frac{ct^{*2}}{2} = \frac{v_0^2}{c} - \frac{c}{2}\frac{v_0^2}{c^2} = \frac{v_0^2}{2c}$$

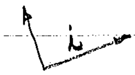
$$\boxed{s^* = \frac{v_0^2}{2c}}$$

B

≈90%

III





$$\int_1^2 \bar{F}\cdot d\bar{r} = T_2 - T_1$$

$$F_y = N - mg\cos\theta = 0$$
$$N = mg\cos\theta$$

Good

$$\bar{F} = (-\mu N - mg\sin\theta)\,\bar{i}$$
$$d\bar{r} = dx\,\bar{i}$$
$$\int \bar{F}\cdot d\bar{r} = -\int_0^x (\mu N + mg\sin\theta)\,dx = T_2 - T_1$$

$$\cancel{+\mu mg x + mg\sin 30° x = +\frac{mv_0^2}{2}}$$

$$\cancel{x = \frac{v_0^2}{2\left[\mu g + \frac{g}{2}\right]} = \frac{v_0^2}{g\left[2\mu + 1\right]}}$$

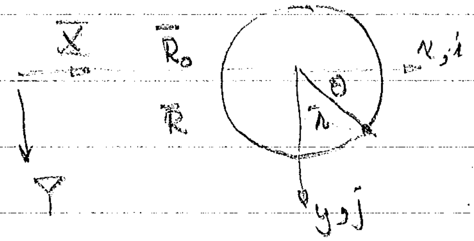
$$-\left(\mu mg\cos\theta + mg\sin\theta\right)x = -\frac{mv_0^2}{2}$$

$$x = \frac{v_0^2}{2g\left[\mu\cos\theta + \sin\theta\right]} = \frac{625}{2(32)\left[.0866 + .5\right]}$$

$$x = \frac{625}{64\left[.5866\right]} \approx 16.5 \text{ ft}$$

30

Problem seminar



$a_0,\ v_0$ fixed x, y, axis

$$\bar{a} = \frac{d^2\bar{R}_0}{dt^2} + \frac{\delta^2 \bar{r}}{\delta t^2} + \cancel{\frac{\delta \bar{\omega}}{\delta t} \times \bar{r}} + \cancel{2\bar{\omega} \times \frac{\delta \bar{r}}{\delta t}} + \cancel{\bar{\omega} \times (\bar{\omega} \times \bar{r})}$$

$\bar{\omega} = 0$

$$\bar{r} = r\cos\theta\, i + r\sin\theta\, j$$

$$\frac{\delta \bar{r}}{\delta t} = -r\sin\theta\,\dot{\theta}\, i + r\dot{\theta}\cos\theta\, j$$

$$\frac{\delta^2 \bar{r}}{\delta t^2} = r\left[-\left[\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right] i + \left[\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right] j\right]$$

$$r\theta = X \qquad \bar{a} = a_0\, I - \left[+a_0\sin\theta + \frac{v_0^2}{r}\cos\theta\right] i + \left[a_0\cos\theta - \frac{v_0^2}{r}\sin\theta\right] j$$

$$r\dot{\theta} = \dot{X} = v_0$$

$$r\ddot{\theta} = \ddot{X} = a_0$$

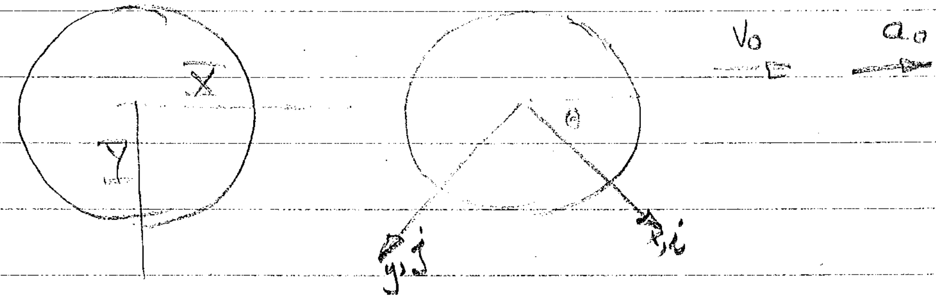
$$a^2 = \left[a_0(1 - \sin\theta) - \frac{v_0^2}{r}\cos\theta\right]^2 + \left[a_0\cos\theta - \frac{v_0^2}{r}\sin\theta\right]^2$$

$$2a\frac{da}{d\theta} = 2\left[a_0(1-\sin\theta) - \frac{v_0^2}{r}\cos\theta\right]\left[-a_0\cos\theta + \frac{v_0^2}{r}\sin\theta\right]$$

$$+ 2\left[a_0\cos\theta - \frac{v_0^2}{r}\sin\theta\right]\left[-a_0\sin\theta - \frac{v_0^2}{r}\cos\theta\right] = 0$$

if x, y imbedded in wheel.

the ω is the rotation of relative frame seen from fixed syst.



$$\frac{d^2\bar{R}_0}{dt^2} = \bar{a}_0 \qquad \bar{r} = r\bar{i}$$

$$\frac{\delta \bar{r}}{\delta t} = 0 \qquad \frac{\delta^2 r}{\delta t^2} = 0$$

$$\bar{a} = \frac{d^2R_0}{dt^2} + \frac{\delta\bar{\omega}}{\delta t}\times\bar{r} + \bar{\omega}\times(\bar{\omega}\times\bar{r})$$

$$\bar{\omega} = \frac{v_0}{r}\bar{k} \qquad \frac{\delta\omega}{\delta t} = \frac{a_0}{r}\bar{k} \quad \text{since} \quad r\ddot{\theta} = r\frac{\delta\omega}{\delta t} = a_0$$

$$\bar{a} = a_0 I + \frac{a_0}{r}\bar{k}\times r\bar{i} + \frac{v_0}{r}\bar{k}\times\left(\frac{v_0}{r}\bar{k}\times r\bar{i}\right)$$

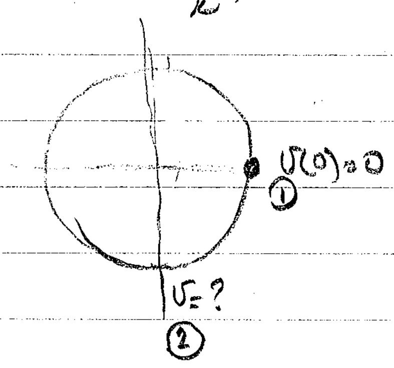
$$= a_0 I + a_0\bar{j} - \frac{v_0}{r}\bar{i}$$

$$\bar{i} = I\cos\theta + J\sin\theta \qquad I = \bar{i}\cos\theta - \sin\theta\,\bar{j}$$

$$\bar{j} = J\cos\theta - I\sin\theta \qquad J = \bar{i}\sin\theta + \bar{j}\cos\theta$$

$$\bar{a} = a_0\cos\theta\,\bar{i} - a_0\sin\theta\,\bar{j} - \frac{v_0}{r}\bar{i} + a_0\bar{j}$$

$$= \left(a_0\cos\theta - \frac{v_0}{r}\right)\bar{i} + a_0(1-\sin\theta)\bar{j}$$



$$T + V = E$$

$$T_1 + V_1 = T_2 + V_2$$

$$0 = \tfrac{1}{2} m V_2^2 - mgr$$

$$v_2 = \sqrt{2gr}$$

3. $h = 340$ miles $\qquad r_1 = 3960 + 340 = 4300$ miles

$v_1 = 27000$ ft/sec

$$v_1^2 = \mu\left(\frac{2}{r_1} - \frac{1}{a}\right) \qquad \frac{1}{a} = \frac{2}{r_1 + r_2}$$

$$v_1^2 = \mu\left(\frac{2(r_1+r_2) - 2r_1}{r_1(r_1+r_2)}\right) = \mu\left[\frac{2r_2}{r_1(r_1+r_2)}\right]$$

$$v_1^2 r_1 (r_1 + r_2) = 2\mu r_2$$

$$v_1^2 r_1^2 + v_1^2 r_1 r_2 = 2\mu r_2$$

$$v_1^2 r_1^2 = r_2(2\mu - v_1^2 r_1)$$

$$r_2 = \frac{v_1^2 r_1^2}{2\mu - v_1^2 r_1}$$

$3.\times 10^4 \quad 9\times10^8 \quad 4\times10^3 \cdot 5\times10^3 \quad (20\times10^7)^2 \quad 4\times10^{14}\cdot 9\times10^8 = 36\times10^{22}$

$$r_2 = \frac{(27000)^2(4300\times5280)^2}{2.82\times10^{16} - (27000)^2(4300\times5280)} = \frac{37.6\times10^{22}}{1.165\times10^{16}}$$

$9\times10^8 \qquad 1.655\times10^{16}$

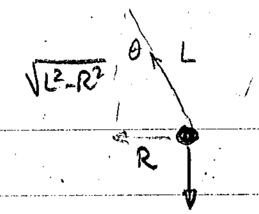
$9\times10^8\,(20\times10^7) = 18\times10^{16}$

$$r_2 = \cancel{227\times10^6 \text{ ft}}\ 32.3\times10^6 \text{ ft} = \cancel{4300 \text{ miles}}\ 6120 \text{ miles}$$

~~h = 340 miles~~ $\qquad \boxed{h = 2160 \text{ mile}}$

~~max height~~ $\qquad$ max height

1.



$$S\cos\theta = mg$$

$$S\sin\theta = \frac{mv^2}{R}$$

$$\cos\theta = \frac{\sqrt{L^2-R^2}}{L}$$

$$\tan\theta = \frac{R}{\sqrt{L^2-R^2}}$$

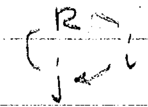
$$\sin\theta = \frac{R}{L}$$

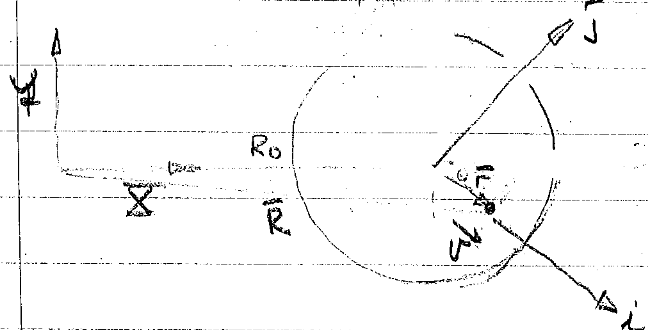
$$\frac{gR^2}{\sqrt{L^2-R^2}} = v^2$$

$$v^2 = \frac{(32.2)(4)}{\sqrt{16-4}} = \frac{32.2(4)}{2\sqrt{3}} = \frac{(32.2)(2)}{1.732}$$

$$\frac{64.4}{1.732} = v^2 \qquad v = \sqrt{37.2} = \underline{6.1 \text{ ft/sec}}$$

2.





$\cancel{r\cos\theta\,\bar{i} - r\sin\theta\,\bar{j}}$

$\omega = V/R$

$\frac{dR_0}{dt} = 0$ since the wheel is not moving in translation with respect to fixed axes. $\therefore$ $\frac{d^2R_0}{dt^2} = 0$

$\cancel{\bar{r} = (r\cos\theta\,\bar{i} - r\sin\theta\,\bar{j})}$ $\quad \bar{r} = r\bar{i}$ $\quad \frac{\delta \bar{r}}{\delta \theta} \cdot \frac{\delta\theta}{\delta t}$

$\frac{\delta \bar{r}}{\delta t} = V\bar{i}$ $\qquad \frac{\delta^2 \bar{r}}{\delta t^2} = 0$

if ~~wheel has radius b.~~

$\bar{\omega} = \omega\bar{k}$ since it is const angular velocity $\frac{\delta\bar{\omega}}{\delta t} = 0$

$$\bar{a} = 0 + 0 + 0 + 2\omega\bar{k} \times v\bar{i} + \omega\bar{k} \times (\bar{\omega}k \times r\bar{i})$$

$$\bar{a} = 2\omega v\bar{j} + \omega\bar{k} \times \omega r\bar{j} = 2\omega V\bar{j} - \omega^2 r\,\bar{i}$$

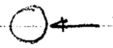
$\bar{a} = 2\omega v\bar{j} - \omega^2 r\,\bar{i}$ where v = bugs velocity

if $\omega = bv$ $\quad \frac{V^2}{b^2} r$ $\quad$ is $r = b$

$\frac{V^2}{b}$

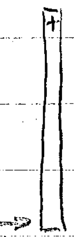
VD

1.



p'

mv $\qquad mv - \hat{P} = mv' \qquad v' = v - \dfrac{\hat{P}}{m}$ $\qquad v_{cm},\ v',\ \hat{P},\ \omega$



$\hat{P}$

$$-\hat{R} + \hat{P} = M v_{cm}$$

$$\hat{P} l = I_o \omega \qquad I_o = \frac{Ml^2}{3} = \frac{.5(9)}{3} = 1.5$$

$$\omega = \frac{\hat{P}\,3}{1.5} = 2\hat{P}$$

$$e = .5 = \frac{v_{cm} + \omega l/2 - v'}{v - 0} = \frac{v_{cm} + 1.5\omega - v'}{v}$$

$$.5v = v_{cm} + 1.5\omega - v' = \frac{\hat{P}}{.5} + \frac{1.5\,\hat{P}3}{1.5} - v + \frac{\hat{P}}{m}$$

$$.5v = \frac{\hat{P}}{M} + 1.5\frac{\hat{P}l}{I_o} - v + \frac{\hat{P}}{m}$$

$$1.5v = \frac{\hat{P}}{.5} + 3\frac{1.5}{1.5}\hat{P} + \frac{\hat{P}}{.01} = 2\hat{P} + 3\hat{P} + 100\hat{P} = 105\hat{P}$$

$$\hat{P} = \frac{1.5}{105}v = \frac{150}{105} = 1.43 \text{ lb-sec}$$

$$v_{cm} = \frac{.5}{1.43} = .35 \text{ ft/sec}$$

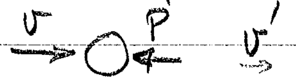
② $$\omega = \frac{\hat{P}l}{I_o} = \frac{1.43(3)}{1.5} = 2.86 \text{ rad/sec}$$

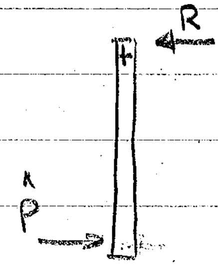
① $$v' = 100 - \frac{1.43}{.01} = 100 - 143 = -43 \text{ ft/sec}$$

$$e = .5 = \frac{v_{cm} + \omega l/2 - v'}{v} = .5v = v_{cm} + 1.5\omega - v'$$

$$= 3\hat{P}$$

$v', \hat{P}, v_{cm}, \hat{R}, \omega$





$m v - \hat{P} = m v'$

$\hat{P} - \hat{R} = M v_{cm}$ $\qquad e = .5 = \dfrac{v_{cm} + \frac{1}{2}\omega - v'}{v - 0}$

$\hat{P} l = I_o \omega$

$\cancel{1(-\hat{P}) = .01 v'}$

$\cancel{\hat{P} + \hat{R}}$

$v_{cm} = \frac{1}{2}\omega \qquad \omega = \dfrac{3\hat{P}}{1.5}$

$e = .5$

$.5v = \dfrac{l\omega - v'}{v} = .5v = 3(2\hat{P}) + 100\hat{P} - v$

$v' = v - \dfrac{\hat{P}}{m}$

$\dfrac{1.5v}{106} = \hat{P}$

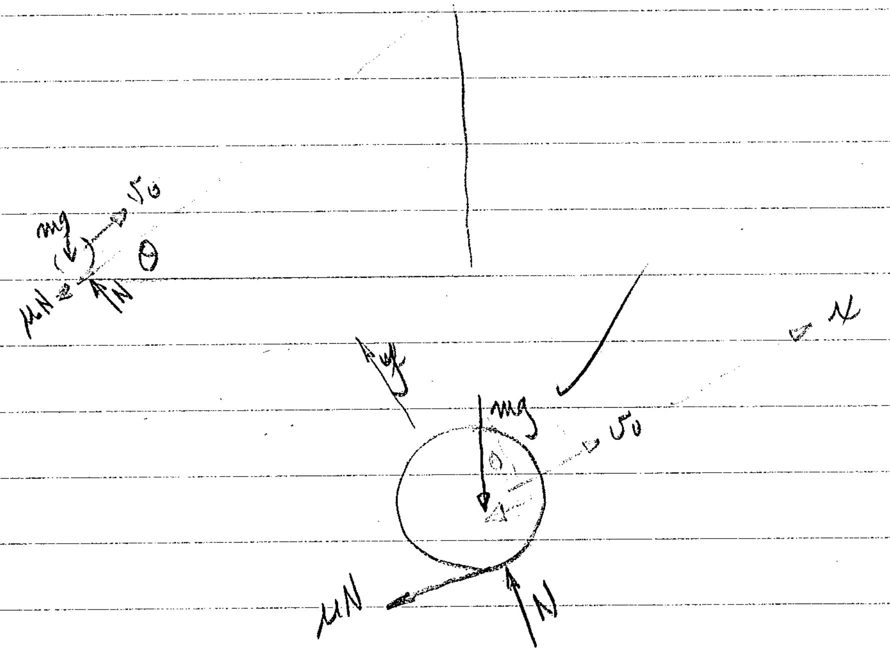
$\hat{P} = 1.415$ lb-sec

$\omega = 2\hat{P} = 2.83$ rad/sec ✓

$v_{cm} = 1.5(2.83) = 4.24$ ft/sec

$\hat{R} = \hat{P} - M v_{cm} = 1.415 - 2.12 = -.705$ lb/sec ✓

$v' = 100 - 141.5 = -41.5$ ft/sec ✓



$$N - mg\cos\theta = m\ddot{y}_{cm} = 0 \qquad N = mg\cos\theta$$

I forgot a $-mg\sin\theta$ in here!

$$-\mu N = m\ddot{x}_{cm} = -\mu mg\cos\theta = m\ddot{x}_{cm}$$

should be $-\mu mg\cos\theta - mg\sin\theta = m\ddot{x}_{cm}$

$$-\mu g\cos\theta\, t = v_{cm} - v_0$$

$$v_{cm} = -\mu g\cos\theta\, t + v_0 \quad \leftarrow \quad \dot{x}_{cm} = -\mu g(\cos\theta + \sin\theta)t + $$

a) $x_{cm} = -\mu g\cos\theta \dfrac{t^2}{2} + v_0 t$

$$\mu N a = I_{cm}\dot{\omega}$$

b) when pure rolling $v_{cm} = a\omega$

$$t^* = \frac{a\omega - v_0}{-\mu g\cos\theta} = \frac{v_0 - a\omega}{\mu g\cos\theta}$$

$$x^* = -\frac{\mu g\cos\theta}{2}\left[\frac{v_0 - a\omega}{\mu g\cos\theta}\right]^2 + v_0\left[\frac{v_0 - a\omega}{\mu g\cos\theta}\right]$$

$\mu N a = I_{cm} \ddot{\Theta}$

$\mu mg \cos\Theta a_y = I_{cm} \ddot{\theta}$ $\qquad \overset{60}{\ddot{\Theta}} = \text{Cc}$

~~Integrate~~ to obtain $\dot{\theta}$ or ω

$\int K$

It just took longer to roll without the ($mg \sin\theta$) F_x component of

R