Good

23
LEVY

800127

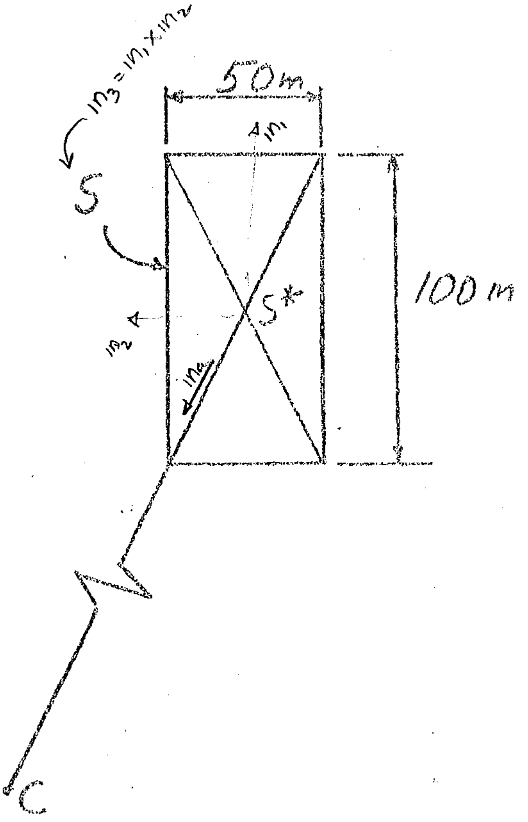
When the mass center S^* of an Earth satellite S moves on a circular orbit, the moment $\underline{M}$ about S^* of all gravitational forces exerted on S by the Earth is given approximately by

$$\underline{M} = 3\,\Omega^2\,\underline{n}_a \times \underline{I}_a$$

where Ω is the angular speed of the line joining S^* to the center C of the Earth, $\underline{n}_a$ is a unit vector parallel to this line, and $\underline{I}_a$ is the second moment of S relative to S^* for $\underline{n}_a$.

The sketch shows a satellite consisting of a uniform rectangular plate having a mass of 1000 kg and having sides of length 50 m and 100 m. This satellite has a period of 90 minutes.

Determine the magnitude of $\underline{M}$ and indicate the sense of $\underline{M}$ on a sketch showing C, S^*, and S.



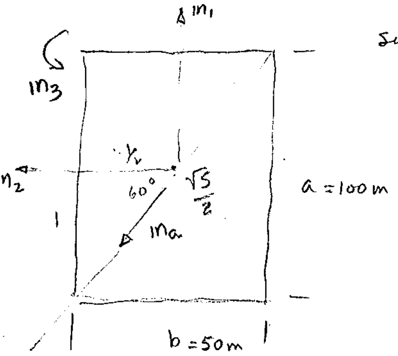
sense of M

$|M| = 2.37$ n-m.

LS
LEVY

Very good

1.



since $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are unit vectors in direction of principal axes

then $\mathbb{I} = \frac{mb^2}{12}\mathbf{n}_1\mathbf{n}_1 + \frac{ma^2}{12}\mathbf{n}_2\mathbf{n}_2 + \frac{m}{12}(a^2+b^2)\mathbf{n}_3\mathbf{n}_3$

$$\mathbf{n}_a = \frac{1}{\sqrt{5}}\mathbf{n}_2 - \frac{2}{\sqrt{5}}\mathbf{n}_1$$

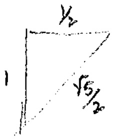
$$\mathbb{I}\cdot\mathbf{n}_a = \mathbb{I}_a = \frac{mb^2}{12}\left(-\frac{2}{\sqrt{5}}\mathbf{n}_1\right) + \frac{1}{\sqrt{5}}\frac{ma^2}{12}\mathbf{n}_2$$

$$\mathbf{n}_a \times \mathbb{I}_a = \left(\frac{1}{\sqrt{5}}\mathbf{n}_2 - \frac{2}{\sqrt{5}}\mathbf{n}_1\right) \times \left(\frac{-mb^2}{12}\cdot\frac{2}{\sqrt{5}}\mathbf{n}_1 + \frac{2}{\sqrt{5}}\frac{ma^2}{12}\mathbf{n}_2\right)$$

$$= \frac{mb^2}{30}\mathbf{n}_3 - \frac{1}{15}ma^2\mathbf{n}_3$$

$$= \frac{m}{30}(b^2 - 2a^2)\mathbf{n}_3 \quad = \frac{1000}{30}(-17500)\mathbf{n}_3 = -\frac{1.75}{30}\times 10^7\,\mathbf{n}_3 \text{ kg-m}$$

$$= \left(-\frac{1.75}{3}\times 10^6 \text{ kg-m}^2\right)\mathbf{n}_3$$



$\mathbf{n}_a = \frac{1}{\sqrt{5}}\mathbf{n}_2 - \frac{2}{\sqrt{5}}\mathbf{n}_1$

hence $\mathbb{M} = 3\Omega^2\left(-\frac{1.75\times 10^6}{3}\right)\mathbf{n}_3 \frac{\text{kg-m}^2}{\text{sec}^2}$ where Ω is in rads/sec.

$$= -1.75\times 10^6\,\Omega^2\,\mathbf{n}_3 \text{ kg-}\frac{\text{m}^2}{\text{sec}^2}$$

1 period = 90 min $\quad \Omega t = 2\pi$ rad. $\quad \Omega = \frac{2\pi}{90\times 60 \frac{\text{sec}}{\text{m}}} = \frac{2\pi}{5.4\times 10^3}$ ✓

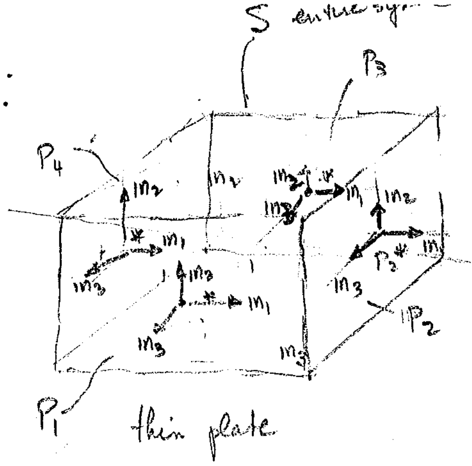
Ω-rad sec

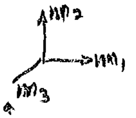
$$\mathbb{M} = \left(\frac{2\pi}{5.4\times 10^3}\right)^2\left(-1.75\times 10^6\right)\mathbf{n}_3 \text{ n-m} = -\frac{4\pi^2(1.75)}{(5.4)^2}\mathbf{n}_3 \text{ n-m}$$

$$|\mathbb{M}| = \underbrace{\frac{4\pi^2(1.75)}{(5.4)^2}}_{} \times \text{ n-m.} = \underbrace{2.369}_{} \times \text{ n-m}$$

Very good -
see p. 2

LEVY





let $\mathbf{n}_1\ \mathbf{n}_2\ \mathbf{n}_3$ be fixed in center of box

$$\mathbb{I}^{P_1/P_1^*} = \frac{ma^2}{12}(\mathbf{n}_1\mathbf{n}_1 + \mathbf{n}_2\mathbf{n}_2 + 2\mathbf{n}_3\mathbf{n}_3)$$

let each of these vectors be defined for each plate $\ni$ $\mathbf{n}_i \parallel \mathbf{m}_i$

$$\mathbb{I}^{P_2/P_2^*} = \frac{ma^2}{12}(2\mathbf{n}_1\mathbf{n}_1 + \mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3)$$

$$\mathbb{I}^{P_3/P_3^*} = \frac{ma^2}{12}(\mathbf{n}_1\mathbf{n}_1 + \mathbf{n}_2\mathbf{n}_2 + 2\mathbf{n}_3\mathbf{n}_3)$$

$$\mathbb{I}^{P_4/P_4^*} = \checkmark \frac{ma^2}{12}(2\mathbf{n}_1\mathbf{n}_1 + \mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3)$$

now $\mathbf{n}_a = \frac{1}{\sqrt{3}}(\mathbf{m}_2 + \mathbf{m}_3 - \mathbf{m}_1)$

then $\mathbb{I}^{P_1/P_1^*} \cdot \mathbf{n}_a = \frac{ma^2}{12\sqrt{3}}(-\mathbf{n}_1 + \mathbf{n}_2 + 2\mathbf{n}_3)$ $\qquad I_{aa} = \frac{ma^2}{12 \cdot 3} \cdot 4 = \frac{ma^2}{9}$

$\mathbb{I}^{P_2/P_2^*} \cdot \mathbf{n}_a = \frac{ma^2}{12 \cdot \sqrt{3}}(-2\mathbf{m}_1 + \mathbf{m}_2 + \mathbf{m}_3)$ $\qquad I_{aa} = \frac{ma^2}{12 \cdot 3}(\cdot 4) = \frac{ma^2}{9}$

$\mathbb{I}^{P_3/P_3^*} \cdot \mathbf{n}_a = \frac{ma^2}{12 \cdot \sqrt{3}}(-\mathbf{n}_1 + \mathbf{n}_2 + 2\mathbf{n}_3)$ $\qquad I_{aa} = +\frac{ma^2}{12 \cdot 3} \cdot 4 = \frac{ma^2}{9}$

$$\mathbf{n}_a \cdot \mathbb{I}^{P_4/P_4^*} \cdot \mathbf{n}_a = \checkmark \frac{ma^2}{9}$$

let S^* = center of box

now $\rightarrow I_{aa}^{P_1/S^*} = I_{aa}^{P/P_1^*} + I_{aa}^{P_1^*/S^*}$ w/. $\mathbf{p}^{P_1^*/S^*} = \frac{1}{2}a\mathbf{n}_3$

But $\rightarrow I_{aa}^{P_1^*/S^*} = m\left[\frac{1}{4}a^2 - \frac{a}{2\sqrt{3}} \cdot \frac{a}{2\sqrt{3}}\right] = \frac{a^2m}{6} = m(\mathbf{p}\cdot\mathbf{p} - (\mathbf{p}\cdot\mathbf{n}_a)(\mathbf{p}\cdot\mathbf{n}_a))$

Also: $I_{aa}^{P_1/S^*} = \frac{ma^2}{9} + \frac{ma^2}{6} = \checkmark \frac{10ma^2}{36}$

$I_{aa}^{P_2/S^*} = I_{aa}^{P_2/P_2^*} + I_{aa}^{P_2^*/S^*}$ w/: $I_{aa}^{P_2^*/S^*} = m(\mathbf{p}^{P_2^*/S^*} \cdot \mathbf{p} - (\mathbf{p}\cdot\mathbf{n}_a)(\mathbf{p}\cdot\mathbf{n}_a))$

thus $\mathbf{p}^{P_2^*/S^*} = \frac{1}{2}a\mathbf{n}_1 \Rightarrow$

$I_{aa}^{P_2/S^*} = m\left(\frac{a^2}{4} - \left(\frac{-a}{2\sqrt{3}}\right)\left(\frac{-a}{2\sqrt{3}}\right)\right) = \frac{ma^2}{6}$

$= \checkmark \frac{10a^2m}{36}$

thus we can similarly show the same results for

$I_{aa}^{P_3/S^*} = \checkmark \frac{10ma^2}{36}$ $\qquad I_{aa}^{P_4/S^*} = \checkmark \frac{10ma^2}{36}$

~~S = box thus~~ $\cancel{I_{aa}^{S/S^*} = \sum I_{aa}^{P_i/S^*}} = \frac{40ma^2}{36}$

but $I^{S/S^*} = Mk^2$ ~~$= 4mk^2 = \frac{40ma^2}{36}$ $\therefore k^2 = \frac{10a^2}{36}$ $k = \frac{a}{6}\sqrt{10}$~~

$M = 4m$

~~[illegible]~~ OVER please

on bottom $\mathbb{I}^{P_5/P_5^*} = \frac{ma^2}{12}(\mathbf{n}_1\mathbf{n}_1 + 2\mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3)$

$$I_{aa} = \frac{ma^2}{9}$$

& we can do same to show $I^{P/S^*} \overset{=}{\checkmark} \frac{10ma^2}{36}$

let S = box hence $I^{S/S^*} = \sum I = \frac{50ma^2}{36} = Mk^2$

$$5\cancel{4}mk^2 = \frac{50ma^2}{36}$$

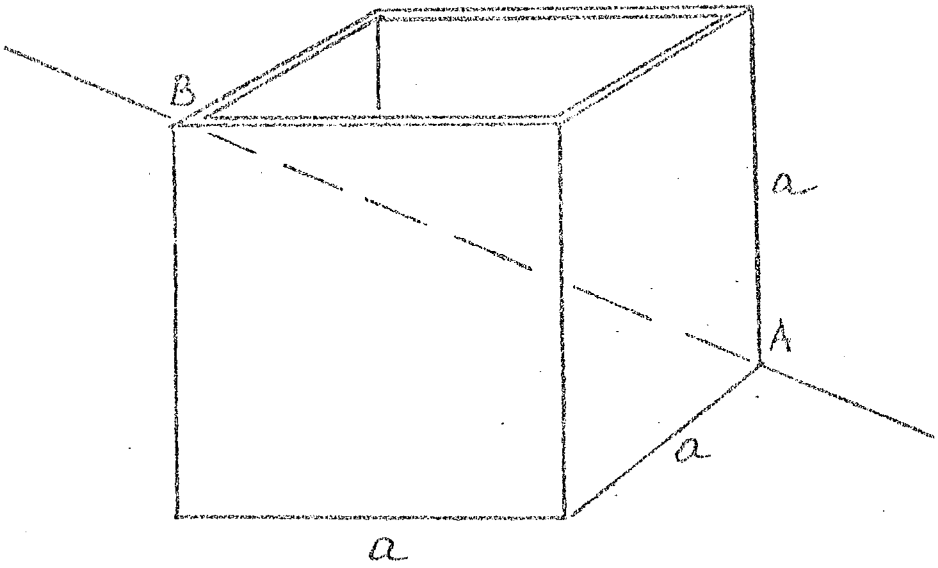
$$k^2 = \frac{50a^2}{144} \qquad k = \frac{a}{12}\sqrt{50} \cancel{=} \frac{a\cdot 5}{12}\sqrt{2}$$

$$\frac{\sqrt{10}}{6}a$$

23
LEVY

800128

The sketch shows an open box whose base and sides consist of identical, thin, uniform plates. Determine the radius of gyration of the box with respect to line AB.



23
LEVY

Poor

800129

A deformable body α consists of two rigid bodies β and γ which are connected to each other at their respective mass centers but are otherwise free to move relative to each other.

At a certain instant t^*, the central principal axes of inertia of β, called Y_1, Y_2, Y_3, coincide respectively with Z_1, Z_2, Z_3, the central principal axes of γ; and the angular velocity of γ relative to β at this instant is given by

$$^{\beta}\underline{\omega}^{\gamma} = \omega_1 \underline{n}_1 + \omega_2 \underline{n}_2 + \omega_3 \underline{n}_3$$

where $\underline{n}_1$, $\underline{n}_2$, $\underline{n}_3$ are unit vectors respectively parallel to Y_1, Y_2, Y_3 (and Z_1, Z_2, Z_3).

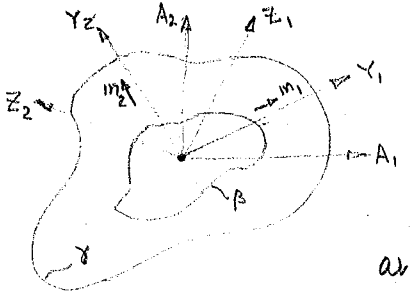
Letting A be a reference frame in which the central principal axes of α are fixed, determine the angular velocity of A relative to β for the instant t^*. Express the result in the form

$$^{\beta}\underline{\omega}^{A} = f_1 \underline{n}_1 + f_2 \underline{n}_2 + f_3 \underline{n}_3$$

where f_1, f_2, f_3 are functions of ω_1, ω_2, ω_3 and of the central principal moments of inertia β_1, β_2, β_3 and γ_1, γ_2, γ_3 of β and γ, respectively; that is, determine these functions.

#23

C. LEVY



Let Y_1, Y_2, Y_3 be the principal axes of inertia of β

Let Z_1, Z_2, Z_3 be the principal axes of inertia of γ

Let A_1, A_2, A_3 be the principal axes of inertia of α in a reference frame A which is fixed

At time $t = t^*$ the Z_i's coincide with the Y_i's. Furthermore Let them coincide with the A_i's. Let the $\mathbb{m}_i$'s be unit vectors // to Y_i's,

If @ $t = t^*$ ${}^{\beta}\omega^{\gamma} = \omega_1\mathbb{m}_1 + \omega_2\mathbb{m}_2 + \omega_3\mathbb{m}_3$ & β_i's and γ_i's are prine moments of inertia of β, γ

Find: ${}^{\beta}\omega^{A} = f_1\mathbb{m}_1 + f_2\mathbb{m}_2 + f_3\mathbb{m}_3$

In the reference frame A

$$ {}^{A}\mathbb{H}^{\alpha/\alpha^*} = {}^{A}\mathbb{H}^{\delta/\delta^*} + {}^{\gamma}\mathbb{H}^{\beta/\alpha^*} \qquad (1)$$

where δ is a fictitious rigid body having the mass distribution of α but the instantaneous motion of γ. Hence ${}^{A}\mathbb{H}^{\delta/\delta^*} = \mathbb{I}^{\alpha/\alpha^*} \cdot {}^{A}\omega^{\gamma}$ (2)

Yet we may also write

$$ {}^{\gamma}\mathbb{H}^{\beta/\alpha^*} = {}^{\gamma}\mathbb{H}^{\beta/\beta^*} + {}^{\gamma}\mathbb{H}^{\beta^*/\alpha^*} \qquad (3)$$

But ${}^{\gamma}\mathbb{H}^{\beta^*/\alpha^*} = m_\beta \mathbb{R}^{\beta^*/\alpha^*} \times {}^{\gamma}\mathbb{V}^{\beta^*} = 0$ since β^* and α^* coincide (4)

Thus $${}^{\gamma}\mathbb{H}^{\beta/\alpha^*} \underset{(4,3)}{=} {}^{\gamma}\mathbb{H}^{\beta/\beta^*} = \mathbb{I}^{\beta/\beta^*} \cdot {}^{\gamma}\omega^{\beta} \qquad (5)$$

Now at the instant of coincidence ${}^{A}\mathbb{H}^{\alpha/\alpha^*} = 0$ because ${}^{A}\omega^{\alpha} = 0$ (6)

α is not a rigid body. ${}^{A}\omega^{\alpha}$ is not meaning ful.

Also at that instant from (2)

$$ {}^{A}\mathbb{H}^{\delta/\delta^*} = \mathbb{I}^{\alpha/\alpha^*} \cdot {}^{A}\omega^{\gamma} = (\mathbb{I}^{\beta/\beta^*} + \mathbb{I}^{\gamma/\gamma^*}) \cdot {}^{A}\omega^{\gamma} \qquad (7)$$

due to all axes being principal axes and coinciding

Now we also know that at that instant

$$ \mathbb{I}^{\beta/\beta^*} = \beta_1\mathbb{m}_1\mathbb{m}_1 + \beta_2\mathbb{m}_2\mathbb{m}_2 + \beta_3\mathbb{m}_3\mathbb{m}_3 \qquad (8)$$

and $$\mathbb{I}^{\alpha/\alpha^*} = (\beta_1+\gamma_1)\mathbb{m}_1\mathbb{m}_1 + (\beta_2+\gamma_2)\mathbb{m}_2\mathbb{m}_2 + (\beta_3+\gamma_3)\mathbb{m}_3\mathbb{m}_3 \qquad (9)$$

But also

$$ {}^{A}\omega^{\gamma} = {}^{A}\omega^{\beta} + {}^{\beta}\omega^{\gamma} = (-f_1+\omega_1)\mathbb{m}_1 + (-f_2+\omega_2)\mathbb{m}_2 + (-f_3+\omega_3)\mathbb{m}_3 \qquad (10)$$

Thus using (5-10) in (3) we obtain

$$ \begin{aligned} 0 &= (\beta_1+\gamma_1)(-f_1+\omega_1)\mathbb{m}_1 + (\beta_2+\gamma_2)(-f_2+\omega_2)\mathbb{m}_2 + (\beta_3+\gamma_3)(-f_3+\omega_3)\mathbb{m}_3 - [\beta_1\omega_1\mathbb{m}_1 + \beta_2\omega_2\mathbb{m}_2 + \beta_3\omega_3\mathbb{m}_3 \\ &= (\omega_1\gamma_1 - f_1[\beta_1+\gamma_1])\mathbb{m}_1 + (\omega_2\gamma_2 - f_2[\beta_2+\gamma_2])\mathbb{m}_2 + (\omega_3\gamma_3 - f_3[\beta_3+\gamma_3])) \end{aligned} \qquad (11)$$

Solving the 3 scalar equations obtained from (11) yields

#23
C. LEVY

$f_1 = \frac{\gamma_1 \omega_1}{\gamma_1 + \beta_1}$ $f_2 = \frac{\gamma_2 \omega_2}{\gamma_2 + \beta_2}$ $f_3 = \frac{\gamma_3 \omega_3}{\gamma_3 + \beta_3}$ (12, 13, 14)

#27

Christine Miller

$$\text{Body } \beta \begin{cases} Y_1 \quad Y_2 \quad Y_3 & \text{prin. axes} \\ \mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3 & \text{unit vectors along prin. axes} \\ \beta_1 \quad \beta_2 \quad \beta_3 & \text{prin. moments} \end{cases}$$

$$\text{Body } \gamma \begin{cases} Z_1 \quad Z_2 \quad Z_3 & \text{prin. axes} \\ \mathbf{c}_1 \quad \mathbf{c}_2 \quad \mathbf{c}_3 & \text{unit vectors along prin axes} \\ \gamma_1 \quad \gamma_2 \quad \gamma_3 & \text{prin moments} \end{cases}$$

Let A be a co-ord. system in which the central prin. moments of $\alpha = \beta + \gamma$ are fixed, with unit vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$

At t^*, X, Y, & A correspond and:

$$^{\beta}\boldsymbol{\omega}^{\gamma} = \omega_1 \mathbf{n}_1 + \omega_2 \mathbf{n}_2 + \omega_3 \mathbf{n}_3 = \omega_1 \mathbf{b}_1 + \omega_2 \mathbf{b}_2 + \omega_3 \mathbf{b}_3$$

$$^{\beta}\boldsymbol{\omega}^{A} = f_1 \mathbf{n}_1 + f_2 \mathbf{n}_2 + f_3 \mathbf{n}_3 = f_1 \mathbf{b}_1 + f_2 \mathbf{b}_2 + f_3 \mathbf{b}_3$$

Let "~" denote quantities at t^*. We will work in a reference frame β where where $\mathbf{b}_i^{\circ}$ are fixed. At a very small time t after t^*,

$$\mathbf{a}_i = \tilde{\mathbf{a}}_i + \left(\frac{d\mathbf{a}_i}{dt}\right)_{t^*} t + \cdots \cong \tilde{\mathbf{a}}_i + \left({}^{\beta}\boldsymbol{\omega}^{A} \times \mathbf{a}_i\right)_{t^*} \cdot t$$

where we will drop all terms of order two or higher in t.

$$\mathbf{a}_1 = \mathbf{b}_1 + [(f_1 \mathbf{b}_1 + f_2 \mathbf{b}_2 + f_3 \mathbf{b}_3) \times \mathbf{b}_1] t$$
$$= \mathbf{b}_1 + f_3 t\, \mathbf{b}_2 - f_2 t\, \mathbf{b}_3$$

$$\mathbf{a}_2 = \mathbf{b}_2 + [(f_1 \mathbf{b}_1 + f_2 \mathbf{b}_2 + f_3 \mathbf{b}_3) \times \mathbf{b}_2] t$$
$$= -f_3 t\, \mathbf{b}_1 + \mathbf{b}_2 + f_1 t\, \mathbf{b}_3$$

$$\mathbf{a}_3 = \mathbf{b}_3 + [(f_1 \mathbf{b}_1 + f_2 \mathbf{b}_2 + f_3 \mathbf{b}_3) \times \mathbf{b}_3] t$$
$$= f_2 t\, \mathbf{b}_1 - f_1 t\, \mathbf{b}_2 + \mathbf{b}_3$$

For the vectors $\mathbf{a}_i$, for the reference frame γ:

$$^{\gamma}\omega^{A} = {}^{\gamma}\omega^{\beta} + {}^{\beta}\omega^{A} = {}^{\beta}\omega^{A} - {}^{\beta}\omega^{\gamma}$$

$$^{A}\omega^{\gamma} = (f_1-\omega_1)\,\mathbf{a}_1 + (f_2-\omega_2)\,\mathbf{a}_2 + (f_3-\omega_3)\,\mathbf{a}_3$$

$$\mathbf{a}_i' = \tilde{\mathbf{a}}_i + \left(\frac{d\mathbf{a}}{dt}\right)_{t^*}\cdot t + \dots \equiv \tilde{\mathbf{a}}_i + ({}^{\gamma}\omega^{A}\times\mathbf{a}_i)_{t^*}\cdot t$$

$$\mathbf{a}_1 = \mathbf{c}_1 + (f_3-\omega_3)t\,\mathbf{c}_2 - (f_2-\omega_2)t\,\mathbf{c}_3$$

$$\mathbf{a}_2 = -(f_3-\omega_3)t\,\mathbf{c}_1 + \mathbf{c}_2 + (f_1-\omega_1)t\,\mathbf{c}_3$$

$$\mathbf{a}_3 = (f_2-\omega_2)t\,\mathbf{c}_1 - (f_1-\omega_1)t\,\mathbf{c}_2 + \mathbf{c}_3$$

Summarizing,

	$\mathbf{b}_1$	$\mathbf{b}_2$	$\mathbf{b}_3$	$\mathbf{c}_1$	$\mathbf{c}_2$	$\mathbf{c}_3$
$\mathbf{a}_1$	1	f_3t	$-f_2t$	1	$(f_3-\omega_3)t$	$-(f_2-\omega_2)t$
$\mathbf{a}_2$	$-f_3t$	1	f_1t	$-(f_3-\omega_3)t$	1	$(f_1-\omega_1)t$
$\mathbf{a}_3$	f_2t	$-f_1t$	1	$(f_2-\omega_2)t$	$-(f_1-\omega_1)t$	1

At time t we can calculate the products of inertia of bodies β & γ about their mass centers for axes $\mathbf{a}_i$:

$$I_{ab} = \sum_{j=1}^{3}\sum_{k=1}^{3} a_j I_{jk} b_k$$

$$I_{12}^{\beta/\beta^*} = -\beta_1 f_3 t + \beta_2 f_3 t + -\beta_3 \cancel{f_2 t^2} f_1$$

$$I_{12}^{\gamma/\gamma^*} = -\gamma_1(f_3-\omega_3)t + \gamma_2(f_3-\omega_3)t - \gamma_3\cancel{(f_2-\omega_2)}(f_1-\omega_1)t^2$$

$$I_{23}^{\beta/\beta^*} = -\beta_1 \cancel{f_3 t^2} f_2 + -\beta_2 f_1 t + \beta_3 f_1 t$$

$$I_{23}^{\gamma/\gamma^*} = -\gamma_1(f_3-\omega_3)\cancel{(f_2-\omega_2)}t^2 - \gamma_2(f_1-\omega_1)t + \gamma_3(f_1-\omega_1)t$$

$$I_{13}^{\beta/\beta^*} = \beta_1 f_2 t - \cancel{\beta_2 f_3 t^2 f_1} - \beta_3 f_2 t$$

$$I_{13}^{\gamma/\gamma^*} = \gamma_1 (f_2 - \omega_2) t - \gamma_2 (f_3 - \cancel{\omega_3)(f_1 - \omega_1) t^2} - \beta_3 (f_2 - \omega_2) t$$

Since the a_i are the principal axes, the products of inertia for $\beta + \gamma$ must disappear.

$$I_{12}^{\beta/\beta^*} + I_{12}^{\gamma/\gamma^*} = 0$$

$$-\beta_1 f_3 \cancel{t} + \beta_2 f_3 \cancel{t} - \gamma_1 (f_3 - \omega_3) \cancel{t} + \gamma_2 (f_3 - \omega_3) \cancel{t} = 0$$

$$f_3 (-\beta_1 + \beta_2 - \gamma_1 + \gamma_2) = \omega_3 (\gamma_2 - \gamma_1)$$

$$\boxed{f_3 = \frac{\omega_3 (\gamma_2 - \gamma_1)}{\beta_2 - \beta_1 + \gamma_2 - \gamma_1}}$$

$$I_{23}^{\beta/\beta^*} + I_{23}^{\gamma/\gamma^*} = 0$$

$$-\beta_2 f_1 \cancel{t} + \beta_3 f_1 \cancel{t} - \gamma_2 (f_1 - \omega_1) \cancel{t} + \gamma_3 (f_1 - \omega_1) \cancel{t} = 0$$

$$f_1 (-\beta_2 + \beta_3 - \gamma_2 + \gamma_3) = \omega_1 (\gamma_3 - \gamma_2)$$

$$\boxed{f_1 = \frac{\omega_1 (\gamma_3 - \gamma_2)}{\beta_3 - \beta_2 + \gamma_3 - \gamma_2}}$$

$$I_{13}^{\beta/\beta^*} + I_{13}^{\gamma/\gamma^*} = 0$$

$$\beta_1 f_2 \cancel{t} - \beta_3 f_2 \cancel{t} + \gamma_1 (f_2 - \omega_2) \cancel{t} - \gamma_3 (f_2 - \omega_2) \cancel{t} = 0$$

$$f_2 (\beta_1 - \beta_3 + \gamma_1 - \gamma_3) = \omega_2 (\gamma_1 - \gamma_3)$$

$$\boxed{f_2 = \frac{\omega_2 (\gamma_1 - \gamma_3)}{\beta_1 - \beta_3 + \gamma_1 - \gamma_3}}$$

Very nice

23
LEVY

800130

A rigid body A and a rigid axisymmetric rotor B whose mass center B^* and axis of symmetry are fixed in A form a "gyrostat" C. Letting C^* designate the mass center of C, express the inertia torque of C in a reference frame R solely in terms of the inertia dyadics $\underline{I}^{C/C^*}$ and $\underline{I}^{B/B^*}$, the angular velocities ${}^{R}\underline{\omega}^{A}$ and ${}^{A}\underline{\omega}^{B}$, and the angular accelerations ${}^{R}\underline{\alpha}^{A}$ and ${}^{A}\underline{\alpha}^{B}$.

C. LEVY

Problem #1

From Class we had been given the following:

$$ {}^{R}\mathbb{H}^{C/C^*} \underset{\checkmark}{=} {}^{R}\mathbb{H}^{D/D^*} + {}^{A}\mathbb{H}^{B/C^*} \qquad (1)$$

where C makes up the system of the rotor B and the rigid body A; D is a fictitious rigid body whose mass distribution is the same as C but whose motion at any instant is that of A. The asterisk (*) denotes the mass center

In problem 7(l) we had showed that for a system of particles S

$$ {}^{R}\mathbb{H}^{S/O} \underset{\checkmark}{=} {}^{R}\mathbb{H}^{S/S^*} + {}^{R}\mathbb{H}^{S^*/O} \qquad (2a)$$

where it was shown that ${}^{R}\mathbb{H}^{S^*/O} = m_S\, \mathbb{R}^{S^*/O} \times {}^{R}\mathbb{V}^{S^*}$ (2b)

where ${}^{R}\mathbb{H}^{S/O}$ is the angular momentum of S with respect to point O and $m_S = \sum_{i=1}^{N} m_i$

Thus $$ {}^{A}\mathbb{H}^{B/C^*} \underset{(2a,1)}{=} {}^{A}\mathbb{H}^{B/B^*} + {}^{A}\mathbb{H}^{B^*/C^*} \qquad (2c)$$

where $$ {}^{A}\mathbb{H}^{B^*/C^*} \underset{(2b)}{=} m_B\, \mathbb{R}^{B^*/C^*} \times {}^{A}\mathbb{V}^{B^*} \qquad (2d)$$

But because B^* is fixed in A

$$ {}^{A}\mathbb{V}^{B^*} = 0 \quad \Rightarrow \quad {}^{A}\mathbb{H}^{B^*/C^*} \underset{\checkmark}{\neq} 0 \qquad (2e,f)$$

Thus $\boxed{{}^{A}\mathbb{H}^{B/C^*} \underset{\checkmark}{=} {}^{A}\mathbb{H}^{B/B^*}}$ using (2f) & (2c). (2g)

By problem 8(b) $\quad {}^{R}\Pi^{*C} \,\cancel{=}\checkmark\, -\frac{{}^{R}d}{dt}\,{}^{R}\mathbb{H}^{C/C^*}$ ~~[illegible]~~ (3a)

Thus by differentiation of

$$ {}^{R}\mathbb{H}^{C/C^*} \underset{(1,2g)}{=} {}^{R}\mathbb{H}^{D/D^*} + {}^{A}\mathbb{H}^{B/B^*} \qquad (3b)$$

we will find

$$ \frac{{}^{R}d\,{}^{R}\mathbb{H}^{C/C^*}}{dt} = -{}^{R}\Pi^{*C} = \frac{{}^{R}d\,{}^{R}\mathbb{H}^{D/D^*}}{dt} + \frac{{}^{R}d\,{}^{A}\mathbb{H}^{B/B^*}}{dt} \qquad (4)$$

Now. $$ \frac{{}^{R}d}{dt}{}^{A}\mathbb{H}^{B/B^*} \underset{(2.13)}{=} \frac{{}^{A}d\,{}^{A}\mathbb{H}^{B/B^*}}{dt} + {}^{R}\omega^{A} \times {}^{A}\mathbb{H}^{B/B^*} \underset{(3a)}{=} -{}^{A}\Pi^{B} + {}^{R}\omega^{A} \times {}^{A}\mathbb{H}^{B/B^*} \qquad (5a)$$

$$ = -{}^{A}\Pi^{*B} - \left(\mathbb{I}^{B/B^*} \cdot {}^{A}\omega^{B}\right) \times {}^{R}\omega^{A} \quad \text{using results of problem (7i).} \qquad (6)$$

Thus $$\boxed{\frac{{}^{R}d\,{}^{A}\mathbb{H}^{B/B^*}}{dt} \underset{(3.43)}{=} -\left[\left(\mathbb{I}^{B/B^*}\cdot{}^{A}\omega^{B}\right)\times{}^{A}\omega^{B} - \mathbb{I}^{B/B^*}\cdot{}^{A}\alpha^{B} + \left(\mathbb{I}^{B/B^*}\cdot{}^{A}\omega^{B}\right)\times{}^{R}\omega^{A}\right]} \qquad (7)$$

Now we look at $$\frac{{}^{R}d\,{}^{R}\mathbb{H}^{D/D^*}}{dt} \underset{(3.43)}{=} -\left[\left(\mathbb{I}^{D/D^*}\cdot{}^{R}\omega^{D}\right)\times{}^{R}\omega^{D} - \mathbb{I}^{D/D^*}\cdot{}^{R}\alpha^{D}\right] \qquad (8)$$

but since D represents a rigid body (fictitious though) whose motion is that of A at any instant and whose mass distribution is that of C, then

$$ {}^{R}\alpha^{D} = {}^{R}\alpha^{A} \qquad {}^{R}\omega^{D} = {}^{R}\omega^{A} \qquad \text{and} \quad \mathbb{I}^{D/D^*} = \mathbb{I}^{C/C^*} \qquad (9a,b,c)$$

Placing (9a-c) into (8) yields

#23
LEVY

$$\boxed{\frac{{}^{R}d\,{}^{R}\mathbb{H}^{D/D^*}}{dt} = -\left[\left(\mathbb{I}^{C/C^*}\cdot{}^{R}\boldsymbol{\omega}^{A}\right)\times{}^{R}\boldsymbol{\omega}^{A} - \mathbb{I}^{C/C^*}\cdot{}^{R}\boldsymbol{\alpha}^{A}\right]} \qquad (10)$$

Now using (10) & (7) in (4). then

$$\boxed{{}^{R}\mathbb{T}^{*C} = \left(\mathbb{I}^{C/C^*}\cdot{}^{R}\boldsymbol{\omega}^{A}\right)\times{}^{R}\boldsymbol{\omega}^{A} - \mathbb{I}^{C/C^*}\cdot{}^{R}\boldsymbol{\alpha}^{A} + \left(\mathbb{I}^{B/B^*}\cdot{}^{A}\boldsymbol{\omega}^{B}\right)\times{}^{A}\boldsymbol{\omega}^{B} - \mathbb{I}^{B/B^*}\cdot{}^{A}\boldsymbol{\alpha}^{B} + \left(\mathbb{I}^{B/B^*}\cdot{}^{A}\boldsymbol{\omega}^{B}\right)\times{}^{R}\boldsymbol{\omega}^{A}} \quad (11$$

which was to be obtained ✓

To show (10)

$$\frac{{}^{R}d\,{}^{R}\mathbb{H}^{D/D^*}}{dt} = \frac{{}^{R}d}{dt}\sum m_i\,\mathbf{r}^{P_i/C^*}\times\left({}^{R}\boldsymbol{\omega}^{A}\times\mathbf{r}^{P_i/C^*}\right) = \frac{{}^{D}d}{dt}\sum m_i\,\mathbf{r}^{P_i/C^*}\times\left({}^{R}\boldsymbol{\omega}^{A}\times\mathbf{r}^{P_i/C^*}\right) + {}^{R}\boldsymbol{\omega}^{D}\times{}^{R}\mathbb{H}^{D/D^*}$$

but $\frac{{}^{D}d}{dt}\sum m_i\,\mathbf{r}^{P_i/C^*}\times\left({}^{R}\boldsymbol{\omega}^{A}\times\mathbf{r}^{P_i/C^*}\right) = \sum m_i\,\mathbf{r}^{P_i/C^*}\times\left(\frac{{}^{D}d}{dt}{}^{R}\boldsymbol{\omega}^{A}\right)\times\mathbf{r}^{P_i/C^*}$ since $\mathbf{r}^{P_i/C^*}$ is fixed wrt body $\underset{A}{\underline{C}}$ as well as body D.

but $\frac{{}^{D}d}{dt}{}^{R}\boldsymbol{\omega}^{A} = \frac{{}^{R}d}{dt}{}^{R}\boldsymbol{\omega}^{A} + {}^{D}\boldsymbol{\omega}^{R}\times{}^{R}\boldsymbol{\omega}^{A} = {}^{R}\boldsymbol{\alpha}^{A} + {}^{D}\boldsymbol{\omega}^{R}\times{}^{R}\boldsymbol{\omega}^{A}$

but ${}^{D}\boldsymbol{\omega}^{R} = -{}^{R}\boldsymbol{\omega}^{A}$ by definition of body D hence $\frac{{}^{D}d}{dt}{}^{R}\boldsymbol{\omega}^{A} = {}^{R}\boldsymbol{\alpha}^{A}$

thus $\frac{{}^{R}d\,{}^{R}\mathbb{H}^{D/D^*}}{dt} = \mathbb{I}^{C/C^*}\cdot{}^{R}\boldsymbol{\alpha}^{A} + {}^{R}\boldsymbol{\omega}^{A}\times\left(\mathbb{I}^{C/C^*}\cdot{}^{R}\boldsymbol{\omega}^{A}\right)$ as required

✓

$${}^{A}\mathbb{H}^{\alpha/\alpha^*} = \mathbb{I}^{\alpha/\alpha^*}\cdot{}^{A}\omega^{\gamma} + \mathbb{I}^{\beta/\beta^*}\cdot{}^{\gamma}\omega^{\beta}$$

$$\mathbb{I}^{\alpha/\alpha^*}\cdot{}^{A}\omega^{\alpha}$$

$${}^{A}\omega^{\gamma}$$

$${}^{R}\mathbb{H}^{C/C^*} = {}^{R}\mathbb{H}^{D/D^*} + {}^{A}\mathbb{H}^{B/C^*}$$; since $${}^{A}\mathbb{H}^{B/C^*} = {}^{A}\mathbb{H}^{B/B^*} + {}^{A}\mathbb{H}^{B^*/C^*} = 0$$

~~$m_B\,\mathbb{R}^{B^*/C^*}\times{}^{A}\mathbb{V}^{B^*}$~~

$${}^{A}\mathbb{H}^{B/B^*}$$

$${}^{R}\mathbb{H}^{C/C^*} = {}^{R}\mathbb{H}^{D/D^*} + \mathbb{I}^{B/B^*}\cdot{}^{A}\omega^{B}$$

$${}^{R}\mathbb{H}^{C/C^*} = \mathbb{I}^{C/C^*}\cdot{}^{R}\omega^{A} + \mathbb{I}^{B/B^*}\cdot{}^{A}\omega^{B}$$

give ${}^{B}\omega^{A} = \omega_1\mathbf{n}_1 + \omega_2\mathbf{n}_2 + \omega_3\mathbf{n}_3$
need to find ${}^{B}\omega^{C}$

if $R = C$

$${}^{C}\mathbb{H}^{C/C^*} = 0 = \mathbb{I}^{C/C^*}\cdot{}^{C}\omega^{A} + \mathbb{I}^{B/B^*}\cdot{}^{A}\omega^{B}$$

Now $\mathbb{I}^{C/C^*} = (\beta_1+\gamma_1)\mathbf{n}_1\mathbf{n}_1 + (\beta_2+\gamma_2)\mathbf{n}_2\mathbf{n}_2 + (\beta_3+\gamma_3)\mathbf{n}_3\mathbf{n}_3$

~~${}^{C}\omega^{A} = \Omega_1\mathbf{n}_1 + \Omega_2\mathbf{n}_2 + \Omega_3\mathbf{n}_3$~~ ~~${}^{C}\omega^{B} + {}^{B}\omega^{A} = {}^{C}\omega^{A}$~~

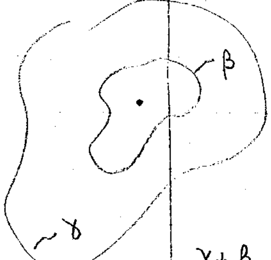
$$\mathbb{I}^{B/B^*} = (\beta_1)\mathbf{n}_1\mathbf{n}_1 + \beta_2\mathbf{n}_2\mathbf{n}_2 + \beta_3\mathbf{n}_3\mathbf{n}_3$$

$${}^{A}\omega^{B} = -(\omega_1\mathbf{n}_1 + \omega_2\mathbf{n}_2 + \omega_3\mathbf{n}_3)$$

hence.

$$(\beta_1+\gamma_1)\Omega_1\mathbf{n}_1 + (\beta_2+\gamma_2)\Omega_2\mathbf{n}_2 + (\beta_3+\gamma_3)\Omega_3\mathbf{n}_3 + \beta_1\omega_1\mathbf{n}_1 + \beta_2\omega_2\mathbf{n}_2 + \beta_3\omega_3\mathbf{n}_3 = 0$$

~~thus $(\beta_1+\gamma_1)\Omega_1 + \beta_1\omega_1 = 0$ thus $\Omega_1 = -\dfrac{\beta_1\omega_1}{\beta_1+\gamma_1}$ hence $\Omega_i = -\dfrac{\beta_i\omega_i}{\beta_i+\gamma_i}$~~ i not



$${}^{R}\mathbb{H}^{\alpha/\alpha^*} = {}^{R}\mathbb{H}^{\delta/\delta^*} + {}^{\gamma}\mathbb{H}^{\beta/\alpha^*}$$

$${}^{A}\mathbb{H}^{\alpha/\alpha^*} = {}^{A}\mathbb{H}^{\delta/\delta^*} + {}^{\gamma}\mathbb{H}^{\beta/\beta^*}$$

$$= \mathbb{I}^{\alpha/\alpha^*}\cdot{}^{A}\omega^{\gamma} + \mathbb{I}^{\beta/\beta^*}\cdot{}^{\gamma}\omega^{\beta}$$

now if $R = A$ & when θ's are zero

$${}^{\gamma}\mathbb{H}^{\beta/\alpha^*} = {}^{\gamma}\mathbb{H}^{\beta/\beta^*} + {}^{\gamma}\mathbb{H}^{\beta^*/\alpha^*}$$

$${}^{\gamma}\mathbb{H}^{\beta^*/C^*} = m_\beta\,\mathbb{R}^{\beta^*/\alpha^*}\times{}^{\gamma}\mathbb{V}^{\beta^*} = 0$$

since $\mathbb{R}^{\beta^*/\alpha^*} = 0$

$\gamma + \beta = \alpha$

δ is a fictitious body w/distrib of α but instantaneous motion of γ

now ${}^{A}\mathbb{H}^{\alpha/\alpha^*} = 0$ since at the instant the momentum of α wrt α^* in Reference frame $A = 0$

now $${}^{\gamma}\omega^{\beta} = -{}^{\beta}\omega^{\gamma} = -[\omega_1\mathbf{n}_1 + \omega_2\mathbf{n}_2 + \omega_3\mathbf{n}_3]$$

at that instant $\mathbb{I}^{\beta/\beta^*} = \beta_1\mathbf{n}_1\mathbf{n}_1 + \beta_2\mathbf{n}_2\mathbf{n}_2 + \beta_3\mathbf{n}_3\mathbf{n}_3$

$\mathbb{I}^{\alpha/\alpha^*} = (\beta_1+\gamma_1)\mathbf{n}_1\mathbf{n}_1 + (\beta_2+\gamma_2)\mathbf{n}_2\mathbf{n}_2 + (\beta_3+\gamma_3)\mathbf{n}_3\mathbf{n}_3$ since $\mathbb{I}^{\alpha/\alpha^*} = \mathbb{I}^{\beta+\gamma/\alpha^*}$

$= \mathbb{I}^{\beta/\beta^*} + \mathbb{I}^{\gamma/\gamma^*}$

and let ${}^{A}\varpi^{\gamma} = -{}^{\gamma}\varpi^{A} = -\left({}^{\gamma}\varpi^{\beta} + {}^{\beta}\varpi^{A}\right) = -{}^{\gamma}\varpi^{\beta} - {}^{\beta}\varpi^{A}$

$$= \left({}^{\beta}\varpi^{\gamma} - {}^{\beta}\varpi^{A}\right)$$

$$\Omega_1 m_1 + \Omega_2 m_2 + \Omega_3 m_3 = {}^{A}\varpi^{\gamma} = \omega_1 m_1 + \omega_2 m_2 + \omega_3 m_3 - \left[\eta_1 m_1 + \eta_2 m_2 + \eta_3 m_3\right]$$

$$\mathbb{I}^{\alpha/\alpha^*} \cdot {}^{A}\varpi^{\gamma} + \mathbb{I}^{\beta/\beta^*} \cdot {}^{\gamma}\varpi^{\beta} \qquad {}^{\gamma}\varpi^{\beta} = -\omega_1 m_1 - \omega_2 m_2 - \omega_3 m_3$$

$$(\gamma_1+\beta_1)\Omega_1 m_1 + (\gamma_2+\beta_2)\Omega_2 m_2 + (\gamma_3+\beta_3)\Omega_3 m_3 - \beta_1\omega_1 m_1 - \beta_2\omega_2 m_2 - \beta_3\omega_3 m_3$$

$$(\gamma_1+\beta_1)\left[\omega_1-\eta_1\right] m_1 + (\gamma_2+\beta_2)(\omega_2-\eta_2) m_2 + (\gamma_3+\beta_3)(\omega_3\mp\eta_3) m_3 - \beta_1\omega_1 m_1 - \beta_2\omega_2 m_2 - \beta_3\omega_3 m_3 = 0$$

$$\left[\omega_1\gamma_1 - (\gamma+\beta_1)\eta_1\right] m_1 + \left[\omega_2\gamma_2 - (\gamma_2+\beta_2)\eta_2\right] m_2 + \left[\omega_3\gamma_3 - (\gamma_3+\beta_3)\eta_3\right] m_3 = 0$$

$$\Rightarrow \qquad \frac{\gamma_i\omega_i}{\gamma_i+\beta_i} = \eta_i \qquad \therefore \ {}^{\beta}\varpi^{A} = \eta_i m_i$$

${}^{A}\varpi^{\alpha^*} : {}^{A}\varpi^{\beta} + {}^{\beta}\varpi^{\gamma} + {}^{\gamma}\varpi^{\alpha} + \omega$

$${}^{\beta}\varpi^{A} = \frac{\gamma_1\omega_1}{\gamma_1+\beta_1} m_1 + \frac{\gamma_2\omega_2}{\gamma_2+\beta_2} m_2 + \frac{\gamma_3\omega_3}{\gamma_3+\beta_3} m_3$$

$${}^{A}\mathbb{H}^{\alpha/\alpha^*} = \left(\mathbb{I}^{\beta/\beta^*} + \mathbb{I}^{\gamma/\gamma^*}\right) \cdot \left({}^{A}\varpi^{\beta} + {}^{\beta}\varpi^{\gamma}\right) = \mathbb{I}^{\alpha/\alpha^*} \cdot {}^{A}\varpi^{\gamma} + \mathbb{I}^{\beta/\beta^*} \cdot {}^{\gamma}\varpi^{\beta}$$

${}^{A}\varpi'$

${}^{R}\mathbb{H}^{C/C^*} = {}^{R}\mathbb{H}^{D/D^*} + {}^{A}\mathbb{H}^{B/C^*}$ (1) where C makes up the rotor B and the rigid body A
D is a fictitious rigid body whose mass distrib
is equal to C but whose motion at any instant
is that of A.

in 7(b) ${}^{R}\mathbb{H}^{S/O} = {}^{R}\mathbb{H}^{S/S^*} + {}^{R}\mathbb{H}^{S^*/O}$ was proven where ${}^{R}\mathbb{H}^{S^*/O} = m_S\, \mathbb{R}^{S^*/O} \times {}^{R}\mathbb{V}^{S^*}$

$$m_S = \sum_{i=1}^{m_S} m_i$$

Thus ${}^{A}\mathbb{H}^{B/C^*} = {}^{A}\mathbb{H}^{B/B^*} + {}^{A}\mathbb{H}^{B^*/C^*}$ where ${}^{A}\mathbb{H}^{B^*/C^*} = m_B\, \mathbb{R}^{B^*/C^*} \times {}^{A}\mathbb{V}^{B^*}$

now since the axis of symmetry of B is fixed in A and B* lies on the axis of symmetry and is also fixed in A then ${}^{A}\mathbb{V}^{B^*} \equiv 0$

Thus ${}^{A}\mathbb{H}^{B/C^*} = {}^{A}\mathbb{H}^{B/B^*}$ (2)

Placing (2) in (1) leads to

$${}^{R}\mathbb{H}^{C/C^*} = {}^{R}\mathbb{H}^{D/D^*} + {}^{A}\mathbb{H}^{B/B^*} \qquad (3)$$

but by problem 8(b) ${}^{R}\mathbb{T}^{*C} = -\frac{{}^{R}d}{dt}\,{}^{R}\mathbb{H}^{C/C^*}$

Thus we need to differentiate (3) wrt time in reference frame R.

thus
$$\frac{{}^{R}d\,\mathbb{H}^{C/C^*}}{dt} = -{}^{R}\mathbb{T}^{*C} = \frac{{}^{R}d}{dt}{}^{R}\mathbb{H}^{D/D^*} + \frac{{}^{R}d\,{}^{A}\mathbb{H}^{B/B^*}}{dt} \qquad (4)$$

but
$$\frac{{}^{R}d\,{}^{A}\mathbb{H}^{B/B^*}}{dt} \underset{(2.13)}{=} \frac{{}^{A}d\,{}^{A}\mathbb{H}^{B/B^*}}{dt} + {}^{R}\omega^{A} \times {}^{A}\mathbb{H}^{B/B^*} \qquad (5)$$

$$= -{}^{A}\mathbb{T}^{*B} + {}^{R}\omega^{A} \times \underset{\text{proble 7(i)}}{(\mathbb{I}^{B/B^*} \cdot {}^{A}\omega^{B})} \qquad (6)$$

$$\frac{{}^{R}d\,{}^{A}\mathbb{H}^{B/B^*}}{dt} \underset{(5,6)}{=} \underbrace{\mathbb{I}^{B/B^*} \cdot {}^{A}\alpha^{B} + {}^{A}\omega^{B} \times (\mathbb{I}^{B/B^*} \cdot {}^{A}\omega^{B})}_{(3.43)} + {}^{R}\omega^{A} \times (\mathbb{I}^{B/B^*} \cdot {}^{A}\omega^{B}) \qquad (7)$$

Now
$$\frac{{}^{R}d}{dt}\left({}^{R}\mathbb{H}^{D/D^*}\right) \underset{3.43}{=} \mathbb{I}^{D/D^*} \cdot {}^{R}\alpha^{D} + {}^{R}\omega^{D} \times (\mathbb{I}^{D/D^*} \cdot {}^{R}\omega^{D}) = -{}^{R}\mathbb{T}^{*D} \qquad (8)$$

but since D represents a fictitious rigid body whose mass distribution is that of C but whose motion at any instant is that of A, then

$${}^{R}\alpha^{D} = {}^{R}\alpha^{A} \;, \quad {}^{R}\omega^{D} = {}^{R}\omega^{A} \quad \text{and} \quad \mathbb{I}^{D/D^*} = \mathbb{I}^{C/C^*} \qquad (9,10,11)$$

Placing these into (8) give

$${}^{R}\dot{\mathbb{H}}^{D/D^*} = \mathbb{I}^{C/C^*} \cdot {}^{R}\alpha^{A} + {}^{R}\omega^{A} \times (\mathbb{I}^{C/C^*} \cdot {}^{R}\omega^{A}) \qquad (12)$$

Using (12), (7) into (4)

$${}^{R}\mathbb{T}^{*C} = {}^{A}\mathbb{T}^{*B} + {}^{R}\mathbb{T}^{*D} + (\mathbb{I}^{B/B^*} \cdot {}^{A}\omega^{B}) \times {}^{R}\omega^{A}$$
$$= (\mathbb{I}^{C/C^*} \cdot {}^{R}\omega^{A}) \times {}^{R}\omega^{A} + \mathbb{I}^{C/C^*} \cdot {}^{R}\alpha^{A} + (\mathbb{I}^{B/B^*} \cdot {}^{A}\omega^{B}) \times {}^{A}\omega^{B} - \mathbb{I}^{B/B^*} \cdot {}^{A}\alpha^{B} + (\mathbb{I}^{B/B^*} \cdot {}^{A}\omega^{B}) \times {}^{R}\omega$$

We can actually show that

$$ {}^{R}\frac{d}{dt}\mathbb{H}^{P/C^*} = {}^{R}\frac{d}{dt}\sum m_i \mathbf{r}_i^{P_i/C^*} \times {}^{R}\omega^{A} \times \mathbf{r}_i^{P_i/C^*} $$

$$ = \sum m_i \frac{{}^{R}d\mathbf{r}_i}{dt} \times {}^{R}\omega^{A} \times \mathbf{r}_i + \mathbb{I}^{C/C^*} \cdot {}^{R}\alpha^{A} + \sum m_i \mathbf{r}_i \times {}^{R}\omega^{A} \times \frac{{}^{R}d\mathbf{r}_i}{dt} $$

and that this becomes $\sum m_i \left\{ (\dot{\mathbf{r}}_i \cdot \mathbf{r}_i)\, {}^{R}\omega^{A} - (\dot{\mathbf{r}}_i \times \omega)\, \mathbf{r}_i + (\mathbf{r}_i \cdot \dot{\mathbf{r}}_i)\, \omega - (\mathbf{r}_i \cdot \omega)\, \dot{\mathbf{r}}_i \right\} \quad \mathbb{I}^{C/C^*} \cdot {}^{R}\alpha^{A}$

$$ \sum m_i \left\{ 2(\dot{\mathbf{r}}_i \cdot \mathbf{r}_i)\, {}^{R}\omega^{A} - \omega \cdot \left(\dot{\mathbf{r}}_i \mathbf{r}_i + \mathbf{r}_i \dot{\mathbf{r}}_i \right) \right\} $$

now $\frac{{}^{R}d\mathbf{r}_i}{dt} = \dot{\mathbf{r}}_i = \frac{{}^{C}d\mathbf{r}_i}{dt} + {}^{R}\omega^{C} \times \mathbf{r}_i$ but $\mathbf{r}_i = \mathbf{r}_i^{P/C^*}$ & are fixed in C thus

$$ \dot{\mathbf{r}}_i \cdot \mathbf{r}_i = ({}^{R}\omega^{C} \times \mathbf{r}_i) \cdot \mathbf{r}_i = (\mathbf{r}_i \times \mathbf{r}_i) \cdot {}^{R}\omega^{C} = 0 $$

thus we have $-\sum m_i\, {}^{R}\omega^{A} \cdot \frac{d}{dt}(\mathbf{r}_i \mathbf{r}_i) = + {}^{R}\omega^{A} \cdot {}^{R}\frac{d}{dt} \sum m_i \left(\mathbf{r}_i^2 \mathbb{U} - \mathbf{r}_i \mathbf{r}_i \right)$

$$ {}^{R}\omega^{A} \cdot {}^{R}\frac{d}{dt}\, \mathbb{I}^{C/C^*} = $$

Numerical Solution of Initial Value Problems in Dynamics

1. Introduction

The initial value problem in dynamics consists of determining certain quantities related to the motion of a system for time $t > t_1$ when the values of these quantities for time t_1 are known. Frequently, the solution of this problem involves the numerical integration of a set of ordinary differential equations. For example, Fig. 1 shows a system composed of a rigid body B and a particle P that moves in a tube T and is attached to B with a spring S and a damper D. For $t = 0$, the orientation of B, the angular velocity of B, the position of P in T, and the velocity of P in T are known, and it is desired to find θ as a function of t for $t > 0$. Here θ is the angle between Y_1 and

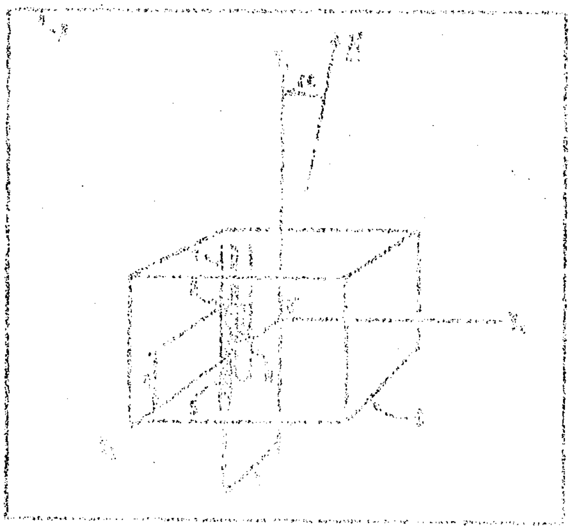


Fig. 1 [illegible]

of the central principal axes of inertia of B, and an inertially fixed vector, namely $\underline{H}$, the central angular momentum of the system, and q is the distance from P to Y_2, another central principal axis of inertia of B. It is the purpose of what follows to discuss the numerical solution of such problems.

The sequel is arranged as follows: Sec. 2 deals with the construction of a computational algorithm. In Sec. 3, a computer program based on the algorithm is discussed, and Sec. 4 contains a sample listing of the program.

2. Construction of an algorithm

Use of principles of mechanics leads to equations called dynamical equations (often differential equations). These govern certain scalar variables and involve certain system parameters. For example, use of $F_r + F_r^* = 0$ (see DYNAMICS, p.177) in connection with the system mentioned in Sec. 1 leads to four differential equations governing u_1, u_2, u_3, and u_4, the first three of which are the $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ measure numbers of the angular velocity of B, where b_1, b_2, b_3 are unit vectors parallel to Y_1, Y_2, Y_3, while u_4 is the time-rate of change of q, the distance between P and Y_2; the four differential equations involve the system parameters b, σ, δ, m_P, m_B, I_1, I_2, I_3 which characterize S, D, P, and B. They are

$$\begin{aligned}\dot{u}_1 = \Big\{ & -\Big(\frac{m_P m_B}{m_P + m_B} q^2 + I_2\Big) \Big[\frac{m_P m_B}{m_P + m_B} b \,(u_1 u_3 q + u_2 u_3 b \\ & - 2u_2 u_4) - u_2 u_3 (I_2 - I_3)\Big] + \frac{m_P m_B}{m_P + m_B} bq \Big[\frac{m_P m_B}{m_P + m_B} q \,(u_1 u_3 q \\ & + u_2 u_3 b - 2u_2 u_4) + u_3 u_1 (I_3 - I_1)\Big] \Big\} \Big[\frac{m_P m_B}{m_P + m_B} (q^2 I_1 \\ & + b^2 I_2) + I_1 I_2\Big]^{-1} \end{aligned} \qquad (1)$$

$$\dot{u}_2 = \Big\{ - \frac{m_P m_B}{m_P + m_B} bq \Big[\frac{m_P m_B}{m_P + m_B} b \, (u_1 u_3 q + u_2 u_3 b - 2 u_2 u_4)$$

$$- u_2 u_3 (I_2 - I_3) \Big] + \Big(\frac{m_P m_B}{m_P + m_B} b^2 + I_1\Big) \Big[\frac{m_P m_B}{m_P + m_B} q (u_1 u_3 q$$

$$+ u_2 u_3 b - 2 u_2 u_4) + u_3 u_1 (I_3 - I_1) \Big] \Big\} \Big[\frac{m_P m_B}{m_P + m_B} (q^2 I_1$$

$$+ b^2 I_2) + I_1 I_2 \Big]^{-1} \tag{2}$$

$$\dot{u}_3 = - \Big[(\sigma q + \delta u_4) b + \frac{m_P m_B}{m_P + m_B} q \, (2 u_3 u_4 - u_3^{\,2} b - u_1^{\,2} b + u_1 u_2 q)$$

$$- u_1 u_2 (I_1 - I_2) \Big] \Big(\frac{m_P m_B}{m_P + m_B} q^2 + I_3\Big)^{-1} \tag{3}$$

$$\dot{u}_4 = - (\sigma q + \delta u_4) \Big(\frac{m_P m_B}{m_P + m_B}\Big)^{-1} + \dot{u}_3 b - u_2 (u_1 b - u_2 q) + u_3^{\,2} q \tag{4}$$

Kinematical considerations furnish additional equations, called <u>kinematical equations</u>, some of which may be differential equations. For example,

$$\dot{q} = u_4 \tag{5}$$

is such an equation and, after introducing Euler parameters $\epsilon_1, \ldots, \epsilon_4$, one can write, in addition,

$$\dot{\epsilon}_1 = \frac{1}{2}(u_1\epsilon_4 - u_2\epsilon_3 + u_3\epsilon_2) \tag{6}$$

$$\dot{\epsilon}_2 = \frac{1}{2}(u_1\epsilon_3 + u_2\epsilon_4 - u_3\epsilon_1) \tag{7}$$

$$\dot{\epsilon}_3 = \frac{1}{2}(-u_1\epsilon_2 + u_2\epsilon_1 + u_3\epsilon_4) \tag{8}$$

$$\dot{\epsilon}_4 = \frac{1}{2}(-u_1\epsilon_1 - u_2\epsilon_2 - u_3\epsilon_3) \tag{9}$$

Furthermore, after defining c and s as

$$c \triangleq 1 - 2\,\epsilon_2{}^2 - 2\epsilon_3{}^2 \tag{10}$$

and

$$s \triangleq (1 - c^2)^{1/2} \tag{11}$$

one can express α as

$$\alpha = \tan^{-1}(s/c) \tag{12}$$

Simultaneous solution of (1)-(9) and subsequent use of (10)-(12) leads to the desired values of q and α. However, a program based on the equations as written would be inefficient because it would involve certain repetitious calculations. For example, since the expression $m_p m_R/(m_p + m_R)$ appears repeatedly, it is desirable to introduce a symbol to replace this expression, i.e., to let

$$\mu = m_p m_R/(m_p + m_R) \tag{13}$$

Similarly, one may introduce $z_1, \ldots, z_9$ as

$$z_1 = u_1 u_3 q + u_2 u_3 b - 2\, u_2 u_4 \tag{14}$$

$$z_2 = u_2 u_3 (I_2 - I_3) \tag{15}$$

$$z_3 = u_3 u_1 \, (I_3 - I_1) \tag{16}$$

$$z_4 = \mu b q \tag{17}$$

$$z_5 = [\mu (q^2 I_1 + b^2 \, I_2) + I_1 \, I_2]^{-1} \tag{18}$$

$$z_6 = \mu b z_1 - z_2 \tag{19}$$

$$z_7 = \mu q z_1 + z_3 \tag{20}$$

$$z_8 = \sigma q + \delta u_4 \tag{21}$$

$$z_9 = 2 u_3 u_4 - u_3^2 \, b - u_1^2 \, b + u_1 u_2 q \tag{22}$$

after which one can replace (1) - (4) with

$$\dot{u}_1 = [-(\mu q^2 + I_2) \; z_6 + z_4 z_7] z_5 \tag{23}$$

$$\dot{u}_2 = [- \, z_4 z_6 + (\mu b^2 + I_1) z_7] z_5 \tag{24}$$

$$\dot{u}_3 = -[z_8 b + \mu q z_9 - u_1 u_2 (I_1 - I_2)] (\mu q^2 + I_3)^{-1} \qquad (25)$$

$$\dot{u}_4 = -z_8/\mu + \dot{u}_3 b - u_2(u_1 b - u_2 q) + u_3^2 q \qquad (26)$$

9. Computer program

If a set of ordinary differential equations can be expressed as

$$\frac{dy_i}{dx} = F_i(z_1,\ldots,z_m;\ y_1,\ldots,y_n,x) \qquad (i=1,\ldots,n)$$

where $z_j (j = 1,\ldots, m)$ is a function of $y_1,\ldots, y_n$ and y_k $(k=1,\ldots,n)$ is a function of x, then the FORTRAN subroutine DFEQKM can be used to integrate the equations. (This subroutine is based on the KUTTA MERSON method as described in Fox, L., Numerical Solution of Ordinary and Partial Differential Equations, 1962, p. 24.) For example, (5)-(9) and (23)-(26) form such a set with

$$y_i = \begin{cases} u_i & i = 1,\ldots, 4 \\ q & i = 5 \\ \epsilon_{i-5} & i = 6,\ldots, 9 \end{cases} \qquad (27)$$

and

$$F_1 = [-(\mu y_5^2 + I_2)z_6 + z_4\ z_7]z_5 \qquad (28)$$

To use DFEQKM, one creates a FORTRAN program like the one in Sec. 4. An explanation of this program and a sample output can be obtained from LOTS by giving the TOPS-20 command

```
@ PRINT PS:<K.KANE>DOC..1
```

Execution of the program is initiated by giving the command

```
@ EXEC PS:<K. KANE>DYNEX.FOR, NAL:NALIB/LIB
```

After execution, printed output is obtained by giving the command

```
@ PRINT FOR20.DAT
```

To create a program intended for the solution of an initial value problem other than the one involving (5)-(9) and (23)-(26), one can modify DYNEX.FOR by giving the commands

```
@ EDIT PS:<K.KANE>DYNEX.FOR
```

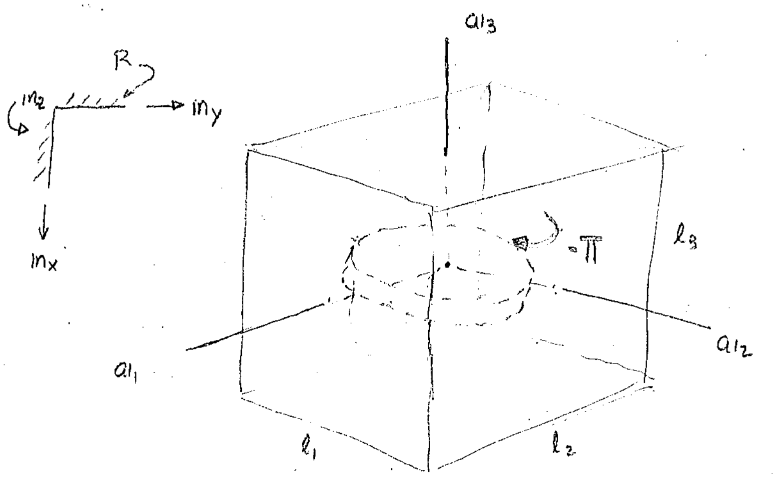
and then using editing commands to change or delete symbols underlined in the listing in Sec. 4.

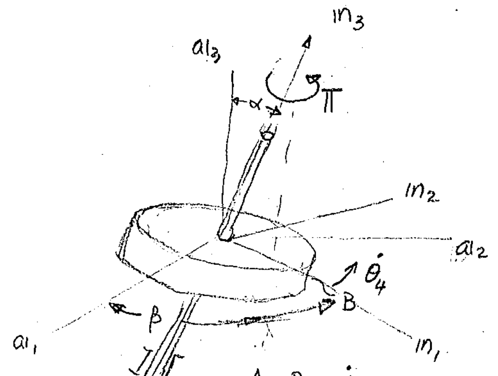
& FORTRAN program

```
C         THE NAME OF THIS FILE IS DYNEX.FOR
C
C         EXAMPLE PROBLEM
C
          IMPLICIT DOUBLE PRECISION (A-Z)
          INTEGER J,NCUTS,NEQ,NSTEPS
          LOGICAL STPSZ
          DIMENSION Y(9) —— 4
          EXTERNAL EQNS
          COMMON/DFOLST/X,STEP,RELERR,ABSERR,NCUTS,NEQ,STPSZ
          COMMON/PLIST/~~MP,MB,I1,I2,I3,SIG,DEL,B,MU~~
          DATA YES,NO/'YES','NO'/
C                                                    PLIST1/ CEE,BCE,BEE,C0,C1,B,C
          WRITE(20,600)                              PLIST2/ B1,B2,B3,B3C1,B2C1,B1C1
          WRITE(20,601)                              PLIST3/TI3,TI2,TI1,AJII3,AJII2,AJII1
C
    100   TYPE 602
          ACCEPT *,~~MP,MB~~ AMU,RM , T/JI
    ~~110~~   TYPE 603
          ACCEPT *,~~I1,I2,I3~~  ALP, BET
          ~~IF(I1+I2.LT.I3) TYPE 618~~
          ~~IF(I1+I2.LT.I3) GO TO 110~~
          ~~IF(I2+I3.LT.I1) TYPE 619~~
          ~~IF(I2+I3.LT.I1) GO TO 110~~
          ~~IF(I3+I1.LT.I2) TYPE 620~~
          ~~IF(I3+I1.LT.I2) GO TO 110~~
    110   TYPE 604
          ACCEPT *,~~SIG,DEL,B~~           ,B,C,D,ALI,
          TYPE 605                              IF (D.GT. AMIN (B,C)) TYPE 6041
          ACCEPT *,~~U1,U2,U3~~  W1,W2,W3,AWB     IF (D.GT. AMIN (B,C)) WRITE (20,6041)
          ~~TYPE 606~~                             IF (D.GT. AMIN (B,C)) D=AMIN (B,C)
          ~~ACCEPT *,Q,QDOT~~
          ~~TYPE 607~~
          ~~ACCEPT *,EPS1,EPS2,EPS3~~
          ~~EPS4=DSQRT(1.0-EPS1*EPS1-EPS2*EPS2-EPS3*EPS3)~~
          TYPE 608
          ACCEPT *,TINITL,TFINAL,STEP
C
          TYPE 600
          TYPE 609,~~MP,MB,I1,I2,I3,SIG,DEL,B~~  AMU,RM,T/JI, ALP,BET,B,C,D,ALI.
          WRITE(20,609) ~~MP,MB,I1,I2,I3,SIG,DEL,B~~  AMU,RM,T/JI,ALP,BET,B,C,D,ALI
W1,W2,W3,AWB TYPE 610,~~U1,U2,U3,Q,QDOT,EPS1,EPS2,EPS3,EPS4~~,TINITL,TFINAL,STEP
W1,W2,W3,AWB WRITE(20,610) ~~U1,U2,U3,Q,QDOT,EPS1,EPS2,EPS3,EPS4~~,TINITL,
                        TFINAL,STEP
C
          PI=4.0*DATAN(1.0D0)
          DEG=180.0/PI
          NSTEPS=IDINT((TFINAL-TINITL)/STEP+0.1)+1
          NEQ=~~9~~4
          RELERR=1.0D-4
          ABSERR=1.0D-4
          NCUTS=20
          STPSZ=.FALSE.
          X=TINITL
```

```
      Y(1)=U1  W1
      Y(2)=U2  W2
      Y(3)=U3  W3
      Y(4)=QDOT  AWB
      Y(5)=Q
      Y(6)=EPS1
      Y(7)=EPS2                           put in ** my notes
      Y(8)=EPS3
      Y(9)=EPS4
      ~~MU=MP*MB/(MP+MB)~~
      TYPE 611
      WRITE(20,611)
C
      DO 200 J=1,NSTEPS
      ~~C=1.0-2.0*(Y(7)*Y(7)+Y(8)*Y(8))~~
      ~~S=DSQRT(1.0-C*C)~~
      ~~A=DEG*DATAN2(S,C)~~
      TYPE 612, ~~X,Y(5),A~~   X,Y(1),Y(2),Y(3),Y(4)
      WRITE(20,612) ~~X,Y(5),A~~   X,Y(1),Y(2),Y(3),Y(4)
      IF(J.EQ.NSTEPS) GO TO 200
      CALL DFEQKM(EQNS,Y,&15)
  200 CONTINUE
C
  300 TYPE 613
      READ(5,614) CHOICE
      IF(CHOICE.NE.YES.AND.CHOICE.NE.NO) GO TO 300
      IF(CHOICE.EQ.YES) WRITE(20,617)
      IF(CHOICE.EQ.YES) GO TO 100
      TYPE 616
      STOP
   15 TYPE 615,NCUTS,X
      WRITE(20,615) NCUTS,X
C 6041 FORMAT(1X,'ROTOR DIAM EXCEEDS MIN. BODY LENGTH - WILL MAKE ROTOR DIAM = MIN BODY LENGTH')
  600 FORMAT(1X,'OUTPUT FROM PROGRAM DYNEX.FOR'//)
  601 FORMAT(1X,'(THIS IS FILE FOR20.DAT)'////)
  602 FORMAT(1X,'INPUT VALUES OF ~~MP,MB'~~) ROTOR/BODY MASS RATIO, RMASS, RTORQUE/J1'),
  603 FORMAT(1X,'INPUT VALUES OF ~~I1,I2,I3'~~) ROTOR AXIS ANGLES ALPHA,BETA (DEG)')
  604 FORMAT(1X,'INPUT VALUES OF ~~SIG,DEL,B~~') BODY LENGTH, ROTOR DIAM RATIOS-B,C,D AND LONGEST
                                                 BODY LENG L1 (M)')
  605 FORMAT(1X,'INPUT INITIAL VALUES OF U1,U2,U3 (RAD/SEC)')
  ~~606 FORMAT(1X,'INPUT INITIAL VALUES OF Q (M), QDOT (M/SEC)')~~
  ~~607 FORMAT(1X,'INPUT INITIAL VALUES OF EPSILON1,EPSILON2,EPSILON3~~
  608 FORMAT(1X,'INPUT TINITL,TFINAL,STEP (SEC)')
  609 FORMAT(1X,'SYSTEM PARAMETERS'//
     1         ~~2X,'MRH= ',F9.3,' KG'/~~
     2         2X,'~~MB~~RM= ',F9.3,' KG'/
     3   T/J1  2X,'~~I1~~ = ',F9.3,' ~~KG-M**2~~'/  'RAD/SEC**2'/
     4   ALP   2X,'~~I2~~ = ',F9.3,' ~~KG-M**2~~'/  'DEG'/
     5   BET   2X,'~~I3~~ = ',F9.3,' ~~KG-M**2~~'/  'DEG'/
     6   B     1X,'~~SIG~~B = ',F9.3,' ~~N/M~~/ 1X,'C = ', F9.3,/'1X,'D = ',F9.3,/
     7   L1    1X,'~~DEL~~ = ',F9.3,' ~~N-SEC/M~~'/  M'//)
     8         ~~9X,'B = ',F9.3,' M'//)~~
  610 FORMAT(1X,'INITIAL CONDITIONS'//
     1         7X,'~~U1~~W1 = ',1PD13.5,' RAD/SEC'/
     2         7X,'~~U2~~W1= ',1PD13.5,' RAD/SEC'/
     3         7X,'~~U3~~W3= ',1PD13.5,' RAD/SEC'/
     4         [illegible]
               'AWB'.
```

```
     4          ~~5X, 'X = ', 1PD13. 5, '~~
     5          ~~5X, 'QDOT = ', 1PD13. 5, ' M/SEC'/~~
     6          ~~1X, 'EPSILON1 = ', 1PD13. 5/~~
     7          ~~1X, 'EPSILON2 = ', 1PD13. 5/~~
     8          ~~1X, 'EPSILON3 = ', 1PD13. 5/~~
     9          ~~1X, 'EPSILON4 = ', 1PD13. 5//~~
     1          1X, 'TINITL = ', 0PF9. 3, ' SEC'/
     2          1X, 'TFINAL = ', F9. 3, ' SEC'/
     3          3X, 'STEP = ', F9. 3,  SEC'///)
   611    FORMAT(1X, 'SIMULATION RESULTS' 1X = TIME IN (SEC) , OMEGAS IN (RAD/SEC) '//
     1          ~~1X, 'T (SEC)', 7X, 'G (M)', 5X, 'ALPHA (DEG)'/)~~ 6X, 'T', 12X, 'W1', 12X, 'W2',
                12X, 'W3', 12X, 'AWB'/
   612    FORMAT(1X, ~~F7. 3, 4X, F10. 7, 3X, F10. 7)~~ 1X, 5(F10.7,4X))
   613    FORMAT(/1X, 'IF YOU WANT TO RUN ANOTHER CASE TYPE YES. '/
     1          2X, 'ELSE TYPE NO. ')
   614    FORMAT(A3)
   615    FORMAT(1X, 'STEPSIZE HAS BEEN HALVED THE ALLOTTED ', I2, ' TIMES.
     1          1X, 'SIMULATION TERMINATED AT X = ', 1PD13. 5/
     2          3X, 'AND DFEQKM IS UNABLE TO PROCEED. '/)
   616    FORMAT(//1X, '*** PRINTED OUTPUT IS ON FILE FOR20. DAT'//)
   617    FORMAT(//40('-')//)
   618    FORMAT(1X, 'I1 + I2 < I3.  THIS IS NOT PHYSICALLY POSSIBLE. '/)
   619    FORMAT(1X, 'I2 + I3 < I1.  THIS IS NOT PHYSICALLY POSSIBLE. '/)
   620    FORMAT(1X, 'I3 + I1 < I2.  THIS IS NOT PHYSICALLY POSSIBLE. '/)
          ~~2##~~
C  621    FORMAT (5X,'I1',12X,'I2',12X,'I3',12X,'I4',10X,'TORQUE')
          END
C
C
          SUBROUTINE EQNS(X, Y, DY)
          IMPLICIT DOUBLE PRECISION (A-Z)
          DIMENSION Y(9), DY(9)
          COMMON/~~PLIST/M2, M3, I1, I2, I3, SIG, DEL, B, MU~~   PLIST1/ CEE, BCE, BEE, C0, C1, B, C
                                                             PLIST2/ B1, B2, B3, B3C1, B2C1, B1C1
C                                                            PLIST3/TI3, TI2, TI1, AJ1I3, AJ1I2
          ~~Z1=Y(1)*Y(3)*Y(5)+Y(2)*Y(3)*B-2. 0*Y(2)*Y(4)~~               AJ1I1
          Z2=Y(2)*Y(3)*(I2-I3)
          Z3=Y(3)*Y(1)*(I3-I1)
          Z4=MU*B*Y(5)
          Z5=1. 0/(MU*(Y(5)*Y(5)*I1+B*B*I2)+I1*I2)
          Z6=MU*B*Z1-Z2
          Z7=MU*Y(5)*Z1+Z3
          Z8=SIG*Y(5)+DEL*Y(4)
          ~~Z9=2. 0*Y(2)*Y(4)-(Y(3)*Y(3)+Y(1)*Y(1))~~*B*Y(1)*Y(2)*Y(3)
C
          DY(1)=(-(MU*Y(5)*Y(5)+I2)*Z6+Z4*Z7)*Z5
          DY(2)=(-Z4*Z6+(MU*B*B+I1)*Z7)*Z5
          DY(3)=-(Z8*B+MU*Y(5)*Z9-Y(1)*Y(2)*(I1-I2))/(MU*Y(5)*Y(5)+I3)
          ~~DY(4)=-Z8/MU+DY(3)*B-Y(2)*(Y(1)*B-Y(2)*Y(5))+Y(3)*Y(3)*Y(5)~~
          DY(5)=Y(4)
          DY(6)=0. 5*(Y(1)*Y(9)-Y(2)*Y(8)+Y(3)*Y(7))
          DY(7)=0. 5*(Y(1)*Y(8)+Y(2)*Y(9)-Y(3)*Y(6))
          DY(8)=0. 5*(-Y(1)*Y(7)+Y(2)*Y(6)+Y(3)*Y(9))
          DY(9)=0. 5*(-Y(1)*Y(6)-Y(2)*Y(7)-Y(3)*Y(8))
C
          RETURN
          END
```





$$^A\omega^B = \dot\theta_4 n_3$$

Let x, y, z be the measure nos to the center of mass of the body A. Thus $r^{A^*} = x n_x + y n_y + z n_z$

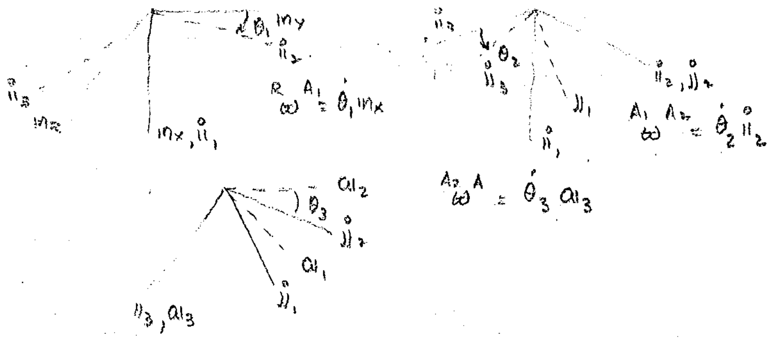
$$v^{A^*} = \dot x n_x + \dot y n_y + \dot z n_z$$
$$T^A = -T n_3$$

Let a_1, a_2, a_3 be the central axes of the body A fixed in A

Let n_x, n_y, n_z be a set of axes fixed in reference frame R thus we can define a set of intermediate reference planes A_1, A_2 in which we can define simple angular motions so that $^R\omega^A = {}^R\omega^{A_1} + {}^{A_1}\omega^{A_2} + {}^{A_2}\omega^A = \dot\theta_1 n_x + \dot\theta_2 \mathring{i}_2 + \dot\theta_3 a_3$

$$n_3 = a_3 \cos\alpha + a_1 \sin\alpha \cos\beta + a_2 \sin\alpha \sin\beta$$
$$T^B = T n_3$$
$$I^{B/B^*} = J_1 n_3 n_3 + J_2 (n_1 n_1 + n_2 n_2)$$

Let us also assume that the rotor is made of the same material as the body A and that its center of mass lies at the same point of body A, ie that a cut out is made in body A of same size as body B in which we may place body B.



We will assume a parallel gravitational field. Let body A have mass (M − m) and let body B have mass m.

thus $F_r = v^A_{\dot q_r} \cdot F^A + \omega^A_{\dot q_r} \cdot \overline{T}^A + v^B_{\dot q_r} \cdot F^B + \omega^B_{\dot q_r} \cdot \overline{T}^B$

$F_r^* = v^{A^*}_{\dot q_r} \cdot F^* + {}^R\omega^A_{\dot q_r} \cdot T^A + {}^R\omega^B_{\dot q_r} \cdot T^B$

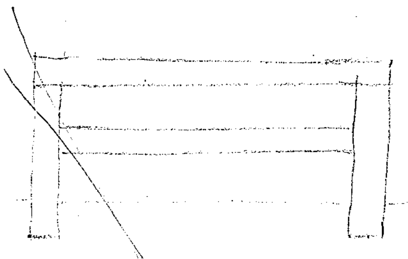
where $F^* = -M a^*$

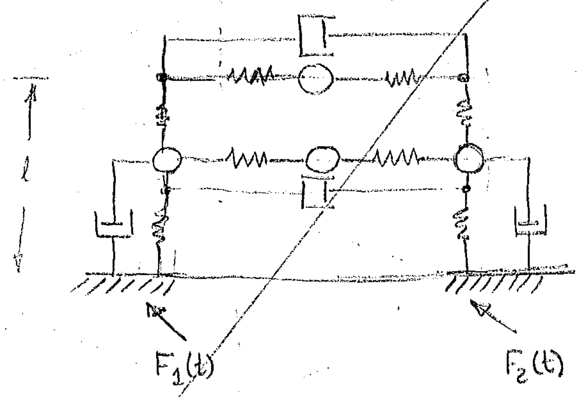
Let C be the composite of A & B.

Let D

$$^A\omega^B = {}^A\omega^B \beta = {}^A\omega^B \beta_i a_i$$

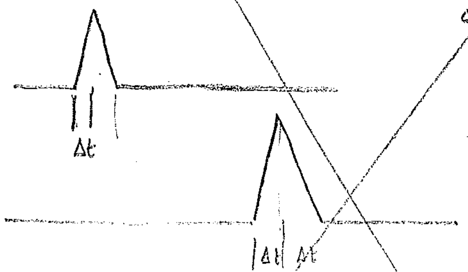
ME 431B Clear Lemy #23





A representation of a building by means of springs and masses & dampers to which forces at the bases are ~~imposed~~ applied (representation of an earthquake wave passing the building & setting the building in motion). The attempt here is to find what values of the spring constants for the lower & upper floors necessary to minimize the motion. What I'm attempting to do is model a "shake" table with intent on finding the "building code" for this model.

$\sum_{i=1}^{5} d(t) f_1(t-t_i) = F_2(t)$



This problem is quite practical in nature especially for an area such as california. The forces are modelled as as semi δ-functions which can be in fact periodic, yet of decreasing amplitude, to model reflections of P & S waves whose energy is dissipated as their travel paths through the earth's crust increase. $\left[\underset{\text{(dissipation fn)}}{d(t) =} e^{-\alpha t}\right]$

- My questions are: (1) should I model the floors by adding a light inextensible string if I want not only longitudinal but also transverse motion of the floors? As a first approximation I guess I can (to simplify model somewhat). (2) Is model too complicated for the purposes you intended
- This is my project This seems more like a linear vibrations problem than like an initial value problem to be solved numerically. I don't think it is suitable for our purposes.

P_i

$O \quad \mathbb{F} \quad \overset{d}{A^*}$

$$F_r = \sum_{i=1}^{N_1+N_2} \mathbb{V}_{\dot{q}_r}^{P_i} \cdot \mathbb{F}^{i} = \sum_{i=1}^{N_1} \mathbb{V}_{\dot{q}_r}^{P_i} \cdot \mathbb{F}^{i} + \sum_{i=1}^{N_2} \mathbb{V}_{\dot{q}_r}^{P_i} \cdot \mathbb{F}^{i}$$

for body A $\quad \mathbb{V}_{\dot{q}_r}^{P_i} = \mathbb{V}_{\dot{q}_r}^{A^*} + ({}^{R}\omega_{\dot{q}_r}^{A} \times \mathbb{p}_i)$ $\qquad$ for body B $\Rightarrow$ $\mathbb{V}_{\dot{q}_r}^{P_i} = \mathbb{V}_{\dot{q}_r}^{A^*} + {}^{R}\omega_{\dot{q}_r}^{B} \times \mathbb{p}_i$

$\mathbb{V}_{\dot{q}_r}^{A^*} \cdot \mathbb{F}^{i} \qquad \sum \mathbb{F}^{i} = (W - w)\, \mathbb{n}_y$

$\mathbb{V}_{\dot{q}_r}^{A^*} \cdot (M - w)\, \mathbb{n}_y$

$({}^{R}\omega_{\dot{q}_r}^{A} \times \mathbb{p}_i) \cdot \mathbb{F}^{i} = (\mathbb{p}_i \times \mathbb{F}^{i}) \cdot {}^{R}\omega_{\dot{q}_r}^{A} = {}^{R}\omega_{\dot{q}_r}^{A} \cdot \mathbb{T}$

$\mathbb{V}_{\dot{q}_r}^{A^*} \cdot \sum m_i g\, \mathbb{n}_y + \sum (m_i g\, {}^{R}\omega_{\dot{q}_r}^{B} \times \mathbb{p}_i) \cdot \mathbb{n}_y$

$\mathbb{V}_{\dot{q}_r}^{A^*} \cdot m g\, \mathbb{n}_y + \sum (\mathbb{p}_i \times \mathbb{F}_i) \cdot {}^{R}\omega_{\dot{q}_r}^{B}$

$\sum^{i=N_1+N_2} \mathbb{V}_{\dot{q}_r}^{P_i} \cdot \mathbb{F}_i^{*} \Rightarrow \sum^{N_1} \mathbb{V}_{\dot{q}_r}^{P_i} \cdot \mathbb{F}_i^{*}$ $\qquad$ where $\mathbb{F}_i^{*} = -m_i \mathbb{a}_i$

$= -m_i \left[\mathbb{a}^{*} + {}^{R}\alpha^{A} \times \mathbb{p}_i + {}^{R}\omega^{A} \times ({}^{R}\omega^{A} \times \mathbb{p}_i) \right.$

$(\mathbb{V}_{\dot{q}_r}^{A^*} + {}^{R}\omega_{\dot{q}_r}^{A} \times \mathbb{p}_i) \cdot \mathbb{F}_i^{*} = \mathbb{V}_{\dot{q}_r} \cdot \mathbb{F}_i^{*} + (\mathbb{p}_i \times \mathbb{F}_i^{*}) \cdot {}^{R}\omega_{\dot{q}_r}^{A}$

$= \mathbb{V}_{\dot{q}_r}^{A^*} \cdot \left[\frac{(M - w)}{g} \right] \mathbb{a}^{*} + {}^{R}\omega_{\dot{q}_r}^{A} \cdot {}^{R}\mathbb{T}^{*A}$ $\qquad$ where ${}^{R}\mathbb{T}^{*A} = -\sum m_i \mathbb{r}_i \times \mathbb{a}^{i}$

$\left[(\mathbb{I}^{A/A^*} \cdot {}^{R}\omega^{A}) \times {}^{R}\omega^{A} - \mathbb{I}^{A/A^*} \cdot {}^{R}\alpha^{A} \right]$

$$\sum^{N_2} \mathbb{V}_{\dot{q}_r} \cdot \mathbb{F}_i^{*} = -\mathbb{V}_{\dot{q}_r}^{A^*} \cdot \left[\frac{w}{g} \mathbb{a}^{*} \right] + {}^{R}\omega_{\dot{q}_r}^{B} \cdot \left[(\mathbb{I}^{B/B^*} \cdot {}^{R}\omega^{B}) \times {}^{R}\omega^{B} - \mathbb{I}^{B/B^*} \cdot {}^{R}\alpha^{B} \right]$$

Thus we get $-\mathbb{V}_{\dot{q}_r}^{A^*} \cdot M \mathbb{a}^{*} + \underbrace{{}^{R}\omega_{\dot{q}_r}^{A} \cdot {}^{R}\mathbb{T}^{A} + {}^{R}\omega_{\dot{q}_r}^{B} \cdot {}^{R}\mathbb{T}^{B}}$

but look at ${}^{R}\omega_{\dot{q}_r}^{A} \cdot ({}^{R}\mathbb{T}^{A} + {}^{R}\mathbb{T}^{B}) + {}^{A}\omega_{\dot{q}_r}^{B} \cdot {}^{R}\mathbb{T}^{B}$

$$-\frac{{}^{R}d\, {}^{R}\mathbb{H}^{A/A^*}}{dt} - \frac{d\, {}^{R}\mathbb{H}^{B/B^*}}{dt} = -\frac{{}^{R}d({}^{R}\mathbb{H}^{A/A^*} + {}^{R}\mathbb{H}^{B/B^*})}{dt} = {}^{R}\mathbb{T}^{A} + {}^{R}\mathbb{T}^{B}$$

but $\mathbb{H}^{A/A^*} = \mathbb{I}^{A/A^*} \cdot {}^{R}\omega^{A}$; $\mathbb{H}^{B/B^*} = \mathbb{I}^{B/B^*} \cdot {}^{R}\omega^{B} = \mathbb{I}^{B/B^*} \cdot {}^{R}\omega^{A} + \mathbb{I}^{B/B^*} \cdot {}^{A}\omega^{B}$

$$\mathbb{H}^{A/A^*} + \mathbb{H}^{B/B^*} = \left[\mathbb{I}^{C/C^*} \cdot {}^{R}\omega^{A} \right] + \mathbb{I}^{B/B^*} \cdot {}^{A}\omega^{B}$$

Now define a fictitious rigid body D whose mass distribution is that of C but whose motion is that of A. Thus $= {}^{R}\mathbb{H}^{D/D^*} + {}^{A}\mathbb{H}^{B/B^*}$

$$\frac{{}^{R}d({}^{R}\mathbb{H}^{A/A^*} + {}^{R}\mathbb{H}^{B/B^*})}{dt} = \frac{{}^{R}d}{dt} \mathbb{H}^{D/D^*} + \frac{{}^{R}d\, {}^{A}\mathbb{H}^{B/B^*}}{dt} = \frac{{}^{R}d\, \mathbb{H}^{D/D^*}}{dt} + \frac{{}^{A}d\, {}^{A}\mathbb{H}^{B/B^*}}{dt} + {}^{R}\omega^{A} \times {}^{A}\mathbb{H}^{B/B^*}$$

$$-\left[{}^{R}\mathbb{T}^{A} + {}^{R}\mathbb{T}^{B} \right] = \frac{{}^{R}d\, \mathbb{H}^{D/D^*}}{dt} - {}^{A}\mathbb{T}^{B} + {}^{R}\omega^{A} \times (\mathbb{I}^{B/B^*} \cdot {}^{A}\omega^{B})$$

but $\dfrac{{}^{R}d\, \mathbb{H}^{D/D^*}}{dt} = \mathbb{I}^{C/C^*} \cdot {}^{R}\alpha^{A} + {}^{R}\omega^{A} \times \mathbb{I}^{C/C^*} \cdot {}^{R}\omega^{A}$

thus ${}^{R}\mathbb{T}^{A} + {}^{R}\mathbb{T}^{B} = (\mathbb{I}^{C/C^*} \cdot {}^{R}\omega^{A}) \times {}^{R}\omega^{A} - \mathbb{I}^{C/C^*} \cdot {}^{R}\alpha^{A} + {}^{A}\mathbb{T}^{B} + {}^{A}\mathbb{H}^{B/B^*} \times {}^{R}\omega^{A}$

$$\mathbb{T}^{D/D^*}$$

$$\text{thus}\quad F_r^{*} = -\mathbb{V}_{\dot q_r}^{A^*}\cdot\frac{W}{g}\mathbf{a}^{*} + {}^{A}\boldsymbol{\omega}_{\dot q_r}^{B}\cdot{}^{R}\mathbb{T}^{B} + {}^{R}\boldsymbol{\omega}_{\dot q_r}^{A}\cdot\Big[\big(\mathbb{I}^{C/C^*}\cdot{}^{R}\boldsymbol{\omega}^{A}\big)\times{}^{R}\boldsymbol{\omega}^{A} - \mathbb{I}^{C/C^*}\cdot{}^{R}\boldsymbol{\alpha}^{A} + {}^{A}\mathbb{T}^{B}$$

$$+\,{}^{A}\mathbb{H}^{B/B^*}\times{}^{R}\boldsymbol{\omega}^{A}\Big]$$

Now back to $F_r = \mathbb{V}_{\dot q_r}^{A}\cdot\mathbb{F}^{A} + \mathbb{V}_{\dot q_r}^{B}\cdot\mathbb{F}^{B} + \boldsymbol{\omega}_{\dot q_r}^{A}\cdot\mathbb{T}^{A} + \boldsymbol{\omega}_{\dot q_r}^{B}\cdot\mathbb{T}^{B} = \mathbb{V}_{\dot q_r}^{C^*}\cdot Mg\,\mathbf{n}_y + {}^{R}\boldsymbol{\omega}_{\dot q_r}^{A}\cdot\mathbb{T}^{A} + {}^{R}\boldsymbol{\omega}_{\dot q_r}^{B}\cdot{}^{R}\mathbb{T}^{B}$

$${}^{R}\mathbb{V}_{\dot x}^{C^*} = \mathbf{n}_x \qquad {}^{R}\mathbb{V}_{\dot y}^{C^*} = \mathbf{n}_y \qquad {}^{R}\mathbb{V}_{\dot z}^{C^*} = \mathbf{n}_z \qquad {}^{R}\mathbb{V}_{\dot\theta_1}^{C^*} = {}^{R}\mathbb{V}_{\dot\theta_2}^{C^*} = {}^{R}\mathbb{V}_{\dot\theta_3}^{C^*} = {}^{R}\mathbb{V}_{\dot\theta_4}^{C^*} = 0 \qquad \mathbb{T}^{A} = -T\mathbf{n}_3\ \ \mathbb{T}^{B} = T\mathbf{n}_3$$

$${}^{R}\boldsymbol{\omega}_{\dot x}^{A} = {}^{R}\boldsymbol{\omega}_{\dot y}^{A} = {}^{R}\boldsymbol{\omega}_{\dot z}^{A} = {}^{R}\boldsymbol{\omega}_{\dot\theta_4}^{A} = 0 \qquad {}^{R}\boldsymbol{\omega}_{\dot\theta_1}^{A} = \mathbf{n}_x \qquad {}^{R}\boldsymbol{\omega}_{\dot\theta_2}^{A} = \mathring{\mathbf{i}}_2 \qquad {}^{R}\boldsymbol{\omega}_{\dot\theta_3}^{A} = \mathbf{a}_3$$

$${}^{R}\boldsymbol{\omega}^{B} = {}^{R}\boldsymbol{\omega}^{A} + {}^{A}\boldsymbol{\omega}^{B} = {}^{R}\boldsymbol{\omega}^{A} + \dot\theta_4\mathbf{n}_3 = \dot\theta_1\mathbf{n}_x + \dot\theta_2\mathring{\mathbf{i}}_2 + \dot\theta_3\mathbf{a}_3 + \dot\theta_4\mathbf{n}_3$$

$${}^{R}\boldsymbol{\omega}_{\dot x}^{B} = {}^{R}\boldsymbol{\omega}_{\dot y}^{B} = {}^{R}\boldsymbol{\omega}_{\dot z}^{B} = 0 \qquad {}^{R}\boldsymbol{\omega}_{\dot\theta_4}^{B} = \mathbf{n}_3 \qquad {}^{R}\boldsymbol{\omega}_{\dot\theta_1}^{B} = \mathbf{n}_y \qquad {}^{R}\boldsymbol{\omega}_{\dot\theta_2}^{B} = \mathring{\mathbf{i}}_2 \qquad {}^{R}\boldsymbol{\omega}_{\dot\theta_3}^{B} = \mathbf{a}_3$$

	$\mathbf{n}_x$	$\mathbf{n}_y$	$\mathbf{n}_z$
$\mathring{\mathbf{i}}_1$	1	0	0
$\mathring{\mathbf{i}}_2$	0	c_1	s_1
$\mathring{\mathbf{i}}_3$	0	$-s_1$	c_1

	$\mathring{\mathbf{i}}_1$	$\mathring{\mathbf{i}}_2$	$\mathring{\mathbf{i}}_3$
$\mathbf{j}_1$	c_2	0	$-s_2$
$\mathbf{j}_2$	0	1	0
$\mathbf{j}_3$	s_2	0	c_2

	$\mathbf{j}_1$	$\mathbf{j}_2$	$\mathbf{j}_3$
$\mathbf{a}_1$	c_3	s_3	0
$\mathbf{a}_2$	$-s_3$	c_3	0
$\mathbf{a}_3$	0	0	1

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{n}_1$	$c_\beta c_\alpha$	$s_\beta c_\alpha$	$-s_\alpha$
$\mathbf{n}_2$	$-s_\beta$	c_β	0
$\mathbf{n}_3$	$s_\alpha c_\beta$	$s_\alpha s_\beta$	c_α

$q_1 = x$ $\qquad F_1 = W + 0 + 0 = W$

$q_2 = y$ $\qquad F_2 = 0 + 0 + 0 = 0$

$q_3 = z$ $\qquad F_3 = 0 + 0 + 0 = 0$

$q_4 = \theta_1$ $\qquad F_4 = 0 + -T\mathbf{n}_x\cdot\mathbf{n}_3 + T\mathbf{n}_x\cdot\mathbf{n}_3 =$ ~~$T(c_2(c_3\mathbf{a}_1 - s_3\mathbf{a}_2) + s_2\mathbf{a}_3)$~~ ~~$T(c_1\mathring{\mathbf{j}}_2 - s_1(-s_2\mathbf{j}_1 + c_2\mathring{\mathbf{j}}_3))\cdot\mathbf{n}_3 = T(c_1\mathbf{j}_2 + s_1s_2\mathbf{j}_1 - s_1c_2)\mathbf{j}_3$~~ $\mathbf{n}$

$q_5 = \theta_2$ $\qquad F_5 = 0 + -T\mathring{\mathbf{i}}_2\cdot\mathbf{n}_3 + T\mathring{\mathbf{i}}_2\cdot\mathbf{n}_3 = T(s_3\mathbf{a}_1 + c_3\mathbf{a}_2)\cdot\mathbf{n}_3 = T($~~$s_3s_\alpha c_\beta + c_3$~~$s_\alpha s_\beta)$

$q_6 = \theta_3$ $\qquad F_6 = 0 + -T\mathbf{a}_3\cdot\mathbf{n}_3 + T\mathbf{a}_3\cdot\mathbf{n}_3 =$ ~~$T\cos\alpha$~~ $= 0$

$q_7 = \theta_4$ $\qquad F_7 = 0 + 0 + T\mathbf{n}_3\cdot\mathbf{n}_3 = T$

$F_4 =$ ~~$T(c_1\mathring{\mathbf{j}}_2 + s_1s_2\mathring{\mathbf{j}}_1 - s_1c_2\mathring{\mathbf{j}}_3)\cdot\mathbf{n}_3 = T(c_1[s_3\mathbf{a}_1 + c_3\mathbf{a}_2] + s_1s_2[c_3\mathbf{a}_1 - s_3\mathbf{a}_2] - s_1c_2\mathbf{a}_3)\cdot\mathbf{n}_3$~~ incorrect

~~$= T(c_1s_3s_\alpha c_\beta + c_1c_3s_\alpha s_\beta + s_1s_2c_3s_\alpha c_\beta - s_1s_2s_3s_\alpha s_\beta - s_1c_2c_\alpha)$~~

$$\mathbf{a}^{*} = \ddot x\,\mathbf{n}_x + \ddot y\,\mathbf{n}_y + \ddot z\,\mathbf{n}_y$$

$${}^{A}\boldsymbol{\omega}^{B} = \dot\theta_4\mathbf{n}_3 \quad\Rightarrow\quad {}^{A}\boldsymbol{\omega}_{\dot x}^{B} = {}^{A}\boldsymbol{\omega}_{\dot y}^{B} = {}^{A}\boldsymbol{\omega}_{\dot z}^{B} = {}^{A}\boldsymbol{\omega}_{\dot\theta_1}^{B} = {}^{A}\boldsymbol{\omega}_{\dot\theta_2}^{B} = {}^{A}\boldsymbol{\omega}_{\dot\theta_3}^{B} = 0 \qquad {}^{A}\boldsymbol{\omega}_{\dot\theta_4}^{B} = \mathbf{n}_3 \qquad {}^{A}\boldsymbol{\alpha}^{B} = \ddot\theta_4\mathbf{n}_3$$

$${}^{R}\boldsymbol{\omega}_{\dot\theta_1}^{A} = c_2(c_3\mathbf{a}_1 - s_3\mathbf{a}_2) + s_2\mathbf{a}_3 \qquad {}^{R}\boldsymbol{\omega}_{\dot\theta_2}^{A} = s_3\mathbf{a}_1 + c_3\mathbf{a}_2 \qquad {}^{R}\boldsymbol{\omega}_{\dot\theta_3}^{A} = \mathbf{a}_3$$

$$\mathbb{I}^{C/C^*} = I_1\mathbf{a}_1\mathbf{a}_1 + I_2\mathbf{a}_2\mathbf{a}_2 + I_3\mathbf{a}_3\mathbf{a}_3$$

$${}^{R}\boldsymbol{\omega}^{A} = \dot\theta_1\big(c_2(c_3\mathbf{a}_1 - s_3\mathbf{a}_2) + s_2\mathbf{a}_3\big) + \dot\theta_2(s_3\mathbf{a}_1 + c_3\mathbf{a}_2) + \dot\theta_3\mathbf{a}_3$$

$$= (\dot\theta_1c_2c_3 + \dot\theta_2s_3)\mathbf{a}_1 + (-\dot\theta_1c_2s_3 + \dot\theta_2c_3)\mathbf{a}_2 + (\dot\theta_1s_2 + \dot\theta_3)\mathbf{a}_3$$

$$= (\dot\theta_1c_2c_3 + \dot\theta_2s_3)\big[c_\beta c_\alpha\mathbf{n}_1 - s_\beta\mathbf{n}_2 + s_\alpha c_\beta\mathbf{n}_3\big] + (-\dot\theta_1c_2s_3 + \dot\theta_2c_3)\big[s_\beta c_\alpha\mathbf{n}_1 + c_\beta\mathbf{n}_2 + s_\alpha s_\beta\mathbf{n}_3\big]$$

$$+ (\dot\theta_1s_2 + \dot\theta_3)\big[-s_\alpha\mathbf{n}_1 + c_\alpha\mathbf{n}_3\big]$$

$$= \Big[(\dot\theta_1c_2s_3 + \dot\theta_2s_3)c_\beta c_\alpha + (-\dot\theta_1c_2s_3 + \dot\theta_2c_3)s_\beta c_\alpha - (\dot\theta_1s_2 + \dot\theta_3)s_\alpha\Big]\mathbf{n}_1 + \Big[-(\dot\theta_1c_2c_3 + \dot\theta_2s_3)s_\beta + (-\dot\theta_1c_2s_3 + \dot\theta_2c$$

$$F_r^* = -\mathbb{V}_{\dot q_r}^{A*} \cdot \frac{W}{g} a^* + {}^{A}\omega^{B}_{\dot q_r} \cdot {}^{R}\mathbb{T}^{B} + {}^{R}\omega^{A}_{\dot q_r} \cdot \left[(\mathbb{I}^{C/C*} \cdot {}^{R}\omega^{A}) \times {}^{R}\omega^{A} - \mathbb{I}^{C/C*} \cdot {}^{R}\alpha^{A} - \mathbb{I}^{B/B*} \cdot {}^{A}\alpha^{B} + {}^{A}\mathbb{H}^{B/B*} \times {}^{R}\omega^{A}\right.$$

$$F_1^* = -\frac{W}{g}\ddot x + 0 + 0 \qquad F_1 + F_1^* = -M\ddot x + Mg = 0 \quad \ddot x = g$$

$$F_2^* = -\frac{W}{g}\ddot y + 0 + 0 \qquad F_1 + F_2^* = -M\ddot y = 0 \quad \ddot y = 0$$

$$F_3^* = -\frac{W}{g}\ddot z + 0 + 0 \qquad F_3 + F_3^* = -M\ddot z = 0 \quad \ddot z = 0$$

$$F_4^* = 0 + 0 +$$

$$F_5^* = 0 + 0 +$$

$$F_6^* = 0 + 0 +$$

$$F_7^* = 0 + m_3 \cdot {}^{R}\mathbb{T}^{B} + 0 \qquad F_7 + F_7^* = T + J_1(\dot\omega_3 + \ddot\theta_4) = 0$$

$${}^{A}\mathbb{H}^{B/B*} = \mathbb{I}^{B/B*} \cdot {}^{A}\omega^{B} = J_1 \dot\theta_4 m_3$$

$$\text{Let } {}^{R}\omega^{A} = \tilde\omega_1 a_1 + \tilde\omega_2 a_2 + \tilde\omega_3 a_3 = \omega_1 m_1 + \omega_2 m_2 + \omega_3 m_3$$

$${}^{A}\mathbb{H}^{B/B*} \times {}^{R}\omega^{A} = J_1\dot\theta_4(\omega_1 m_2 - \omega_2 m_1) = J_1\dot\theta_4\omega_1(-s_\beta a_1 + c_\beta a_2) - J_1\dot\theta_4\omega_2(c_\beta c_\alpha a_1 + s_\beta c_\alpha a_2 - s_\alpha a_3) =$$

$${}^{R}\alpha^{A} = \dot{\tilde\omega}_1 a_1 + \dot{\tilde\omega}_2 a_2 + \dot{\tilde\omega}_3 a_3 = (\dot\omega_1 - \omega_2\dot\theta_4) m_1 + (\dot\omega_2 + \dot\theta_4\omega_1) m_2 + \dot\omega_3 m_3 \Big] \quad x_1 a_1 + x_2 a_2 + x_3 a_3$$

$$\mathbb{I}^{C/C*} \cdot {}^{R}\alpha^{A} = I_1 \dot{\tilde\omega}_1 a_1 + I_2 \dot{\tilde\omega}_2 a_2 + I_3 \dot{\tilde\omega}_3 a_3$$

$$(\mathbb{I}^{C/C*} \cdot {}^{R}\omega^{A}) = I_1 \tilde\omega_1 a_1 + I_2 \tilde\omega_2 a_2 + I_3 \tilde\omega_3 a_3$$

$$\mathbb{T}^{D/D*} = (\mathbb{I}^{C/C*} \cdot {}^{R}\omega^{A}) \times {}^{R}\omega^{A} - \mathbb{I}^{C/C*} \cdot {}^{R}\alpha^{A} = \left[(I_2 - I_3)\tilde\omega_2\tilde\omega_3 - I_1\dot{\tilde\omega}_1\right] a_1 + \left[(I_3 - I_1)\tilde\omega_3\tilde\omega_1 - I_2\dot{\tilde\omega}_2\right] a_2 +$$

$$\left[(I_1 - I_2)\tilde\omega_1\tilde\omega_2 - I_3\dot{\tilde\omega}_3\right] a_3 = T_1^D a_1 + T_2^D a_2 + T_3^D a_3$$

$${}^{A}\alpha^{B} = \frac{{}^{A}d}{dt}(\dot\theta_4 m_3) = \frac{{}^{B}d}{dt}\dot\theta_4 m_3 + {}^{A}\omega^{B} \times ({}^{A}\omega^{B}) = \ddot\theta_4 m_3 \qquad \therefore \quad \mathbb{I}^{B/B*} \cdot {}^{A}\alpha^{B} = J_1 \ddot\theta_4 m_3$$

$$\text{and } \mathbb{I}^{B/B*} \cdot {}^{A}\alpha^{B} = J_1\ddot\theta_4 c_\alpha a_3 + J_1\ddot\theta_4 s_\alpha s_\beta a_2 + J_1\ddot\theta_4 s_\alpha c_\beta a_1$$

$${}^{R}\omega^{B} = {}^{R}\omega^{A} + {}^{A}\omega^{B} = (\tilde\omega_1 + \dot\theta_4 s_\alpha c_\beta) a_1 + (\tilde\omega_2 + \dot\theta_4 s_\alpha s_\beta) a_2 + (\tilde\omega_3 + \dot\theta_4 c_\alpha) a_3 = \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3$$

$$= \omega_1 m_1 + \omega_2 m_2 + (\omega_3 + \dot\theta_4) m_3$$

$${}^{R}\alpha^{B} = \dot\omega_1 m_1 + \dot\omega_2 m_2 + (\dot\omega_3 + \ddot\theta_4) m_3 = \frac{{}^{B}d\,\omega^{B}}{dt} + {}^{R}\omega^{B} \times {}^{R}\omega^{B}$$

$$= (\dot{\tilde\omega}_1 + \ddot\theta_4 s_\alpha c_\beta) a_1 + (\dot{\tilde\omega}_2 + \ddot\theta_4 s_\alpha s_\beta) a_2 + (\dot{\tilde\omega}_3 + \ddot\theta_4 c_\alpha) a_3 + {}^{R}\omega^{A} \times {}^{A}\omega^{B}$$

$${}^{R}\omega^{A} \times {}^{A}\omega^{B} = {}^{R}\omega^{A} \times \dot\theta_4(s_\alpha c_\beta a_1 + s_\alpha s_\beta a_2 + c_\alpha a_3) = \dot\theta_4\left[(\gamma_2 c_\alpha - \gamma_3 s_\alpha s_\beta) a_1 + (\gamma_3 s_\alpha c_\beta - \gamma_1 c_\alpha) a_2 + (\gamma_1 s_\alpha s_\beta - \gamma_2 s_\alpha c_\beta\right.$$

$${}^{R}\alpha^{B} = \left[\dot{\tilde\omega}_1 + \ddot\theta_4 s_\alpha c_\beta + \dot\theta_4(\tilde\omega_2 c_\alpha - \tilde\omega_3 s_\alpha c_\beta)\right] a_1 + \left[\dot{\tilde\omega}_2 + \ddot\theta_4 s_\alpha s_\beta + \dot\theta_4(\tilde\omega_3 s_\alpha c_\beta - \tilde\omega_1 c_\alpha)\right] a_2 +$$

$$\left[\dot{\tilde\omega}_3 + \ddot\theta_4 c_\alpha + \dot\theta_4(\tilde\omega_1 s_\alpha s_\beta - \tilde\omega_2 s_\alpha c_\beta)\right] a_3$$

$${}^{R}\mathbb{T}^{B} = (\mathbb{I}^{B/B*} \cdot {}^{R}\omega^{B}) \times {}^{R}\omega^{B} - \mathbb{I}^{B/B*} \cdot {}^{R}\alpha^{B}$$

$$\mathbb{I}^{B/B*} \cdot {}^{R}\alpha^{B} = J_1(\dot\omega_3 + \ddot\theta_4) m_3 + J_2(\dot\omega_2 m_2 + \dot\omega_1 m_1)$$

$$(\mathbb{I}^{B/B*} \cdot {}^{R}\omega^{B}) \times {}^{R}\omega^{B} = \left[J_1(\omega_3 + \dot\theta_4) m_3 + J_2(\omega_2 m_2 + \omega_1 m_1)\right] \times \left[\omega_1 m_1 + \omega_2 m_2 + (\omega_3 + \dot\theta_4) m_3\right]$$

$$= J_1(\omega_3 + \dot\theta_4)(\omega_1 m_2 - \omega_2 m_1) + J_2\omega_2(-\omega_1 m_3 + (\omega_3 + \dot\theta_4) m_1) + J_2\omega_1(\omega_2 m_3 - (\omega_3 + \dot\theta_4) m_2)$$

$${}^{R}\mathbb{T}^{B} = \left[(J_2 - J_1)\omega_3\omega_2 + \dot\theta_4\omega_2(J_2 - J_1) - J_2\dot\omega_1\right] m_1 + \left[(J_1 - J_2)\omega_3\omega_1 + \dot\theta_4\omega_1(J_1 - J_2) - J_2\dot\omega_2\right] m_2 +$$

$[J_2\omega_1\omega_2 - J_2\omega_3\omega_1 - J_2(\dot{\omega}_3 + \dot{\theta}_4)]\, m_3$

${}^{A}\omega^{B}_{\dot{\theta}_1} = {}^{A}\omega^{B}_{\dot{\theta}_2} = {}^{A}\omega^{B}_{\dot{\theta}_3} = 0 \quad {}^{A}\omega^{B}_{\dot{\theta}_4} = m_3 \qquad {}^{R}\omega^{A}_{\dot{q}_r} : {}^{R}\omega^{A}_{\dot{\theta}_1} = m_x ,\ {}^{R}\omega^{A}_{\dot{\theta}_2} = \overset{o}{n}_2 ,\ {}^{R}\omega^{A}_{\dot{\theta}_3} = a_3 \quad {}^{R}\omega^{A}_{\dot{\theta}_4} = 0$

Only contrib to 2nd term is F_7 term of $J_1(\dot{\omega}_3 + \ddot{\theta}_4) = 0$

$m_x = c_2 c_3 a_1 - c_2 s_3 a_2 + s_2 a_3$
$\overset{o}{n}_2 = s_3 a_1 + c_3 a_2$
thus $m_x \cdot [\qquad] = c_2 c_3 T_1^D - c_2 s_3 T_2^D + s_2 T_3^D - J_1 \ddot{\theta}_4 (c_2 c_3 s_\alpha c_\beta - c_2 s_3 s_\alpha s_\beta + s_2 c_\alpha)$
$+ c_2 c_3 \chi_1 - c_2 s_3 \chi_2 + s_2 \chi_3$
thus $\overset{o}{n}_2 \cdot [\qquad] = s_3 T_1^D + c_3 T_2^D - J_1 \ddot{\theta}_4 (s_3 s_\alpha c_\beta + c_3 s_\alpha s_\beta) + s_3 \chi_1 + c_3 \chi_2$
thus $a_3 \cdot [\qquad] = T_3^D - J \ddot{\theta}_4 c_\alpha + \chi_3$

$F_4 + F_4^* = 0 + c_2 c_3 (T_1^D + \chi_1) - c_2 s_3 (T_2^D + \chi_2) + s_2 (T_3^D + \chi_3) - J_1 \ddot{\theta}_4 (c_2 s_\alpha (c_3 c_\beta - s_3 s_\beta) + s_2 c_\alpha) = 0$
$F_5 + F_5^* = 0 + s_3 (T_1^D + \chi_1) + c_3 (T_2^D + \chi_2) - J_1 \ddot{\theta}_4 s_\alpha (s_3 c_\beta + c_3 s_\beta) = 0$
$F_6 + F_6^* = 0 + (T_3^D + \chi_3) - J_1 \ddot{\theta}_4 c_\alpha = 0$
$F_7 + F_7^* = T^* J_1 (\dot{\omega}_3 + \ddot{\theta}_4) = 0$

$T_1^D = (I_2 - I_3)\tilde{\omega}_2 \tilde{\omega}_3 - I_1 \dot{\tilde{\omega}}_1 \qquad T_2^D = (I_3 - I_1)\tilde{\omega}_3 \tilde{\omega}_1 - I_2 \dot{\tilde{\omega}}_2$
$T_3^D = (I_1 - I_2)\tilde{\omega}_1 \tilde{\omega}_2 - I_3 \dot{\tilde{\omega}}_3$
$\chi_1 = -J_1 \dot{\theta}_4 (\omega_1 s_\beta + \omega_2 c_\beta c_\alpha) \qquad \chi_2 = J_1 \dot{\theta}_4 (\omega_1 c_\beta - \omega_2 s_\beta c_\alpha)$
$\chi_3 = J_1 \dot{\theta}_4 \omega_2 s_\alpha$

$\gamma_1 s_\alpha s_\beta a_3 \mp \gamma_1 c_\alpha a_2 - \gamma_2 s_\alpha c_\beta a_3 + \gamma_2 c_\alpha a_1 + \gamma_3 s_\alpha c_\beta a_2 - \gamma_3 s_\alpha s_\beta a_1$

$(\tilde{\omega}_2 + \dot{\theta}_4 s_\alpha s_\beta) c_\alpha - (\tilde{\omega}_3 + \dot{\theta}_4 c_\alpha) s_\alpha s_\beta$

$(\tilde{\omega}_2 c_\alpha - \tilde{\omega}_3 s_\alpha c_\beta) a_1$

$(\tilde{\omega}_3 + \cancel{\dot{\theta}_4 c_\alpha}) s_\alpha c_\beta - (\tilde{\omega}_1 + \cancel{\dot{\theta}_4 s_\alpha c_\beta}) c_\alpha$

$\tilde{\omega}_3 s_\alpha c_\beta - \tilde{\omega}_1 c_\alpha$

$\cancel{(\dot{\tilde{\omega}}_1 + \ddot{\theta}_4 s_\alpha c_\beta) s_\alpha s_\beta - (\dot{\tilde{\omega}}_2}$

$(\tilde{\omega}_1 + \cancel{\dot{\theta}_4 s_\alpha c_\beta}) s_\alpha s_\beta - (\tilde{\omega}_2 + \cancel{\dot{\theta}_4 s_\alpha s_\beta}) s_\alpha c_\beta$

$-J_1(\omega_3 + \dot{\theta}_4)\omega_2 + J_2 \omega_2(\omega_3 + \dot{\theta}_4) + J_2 \dot{\omega}_1$

$J_1(\omega_3 + \dot{\theta}_4)\omega_1 - J_2 \omega_1(\omega_3 + \dot{\theta}_4) - J_2 \dot{\omega}_2$

$\cancel{-J_2 \omega_2 \omega_1} \cancel{+ J_2 \omega_1 \omega_2}$

$c_2 c_3 (c_\alpha c_\beta m_1 - s_\beta m_2 + s_\alpha c_\beta m_3) - c_2 s_3 (s_\beta c_\alpha m_1 + c_\beta m_2 + s_\alpha s_\beta m_3) + s_2(-s_\alpha m_1 + c_\alpha m_3)$

$m_x = (c_2 c_3 s_\alpha c_\beta - c_2 s_3 s_\alpha s_\beta + s_2 c_\alpha) m_3 + (c_2 c_3 c_\alpha c_\beta - c_2 s_3 s_\beta c_\alpha - s_2 s_\alpha) m_1 + (-c_2 c_3 s_\beta - c_2 s_3 c_\beta) m_2$

$\hat{n}_2 = s_3(c_\beta c_\alpha m_1 - s_\beta m_2 + s_\alpha c_\beta m_3) + c_3(s_\beta c_\alpha m_1 + c_\beta m_2 + s_\alpha s_\beta m_3) = (s_3 c_\beta c_\alpha + c_3 s_\beta c_\alpha) m_1 \quad (s_3 s_\alpha c_\beta + c_3 s_\alpha s_\beta) m_3$

$-J_1 \dot{\theta}_4 (\omega_1 s_\beta + \omega_2 c_\beta c_\alpha) a_1 + J_1 \dot{\theta}_4 (\omega_1 c_\beta - \omega_2 s_\beta c_\alpha) a_2 + J_1 \dot{\theta}_4 \omega_2 s_\alpha a_3 = {}^{A}H^{B/A} \times {}^{A}\omega^{B}$

$-J_1 \dot{\theta}_4 (\omega_1 s_\beta + \omega_2 c_\beta c_\alpha) c_2 c_3 \mp J_1 \dot{\theta}_4 (\omega_1 c_\beta - \omega_2 s_\beta c_\alpha) c_2 s_3 + J_1 \dot{\theta}_4 \omega_2 s_2 s_\alpha$

now $T_3^D + \mathcal{N}_3 = J_1 \ddot{\theta}_4 c_\alpha$ put into $F_4 + F_4^* = 0$

$$c_2 c_3 (T_1^D + \mathcal{N}_1) - c_2 s_3 (T_2^D + \mathcal{N}_2) + \cancel{s_2 c_\alpha J_1 \ddot{\theta}_4} - \cancel{J_1 \ddot{\theta}_4 s_2 c_\alpha} - J_1 \ddot{\theta}_4 (c_2 s_\alpha (c_3 c_\beta - s_3 s_\beta)) = 0$$

$\Rightarrow$ if $c_2 \neq 0$

$$c_3^2 (T_1^D + \mathcal{N}_1) - s_3 c_3 (T_2^D + \mathcal{N}_2) - J_1 \ddot{\theta}_4 s_\alpha (c_3 c_\beta - s_3 s_\beta) c_3 = 0$$

$$s_3^2 (T_1^D + \mathcal{N}_1) + c_3 s_3 (T_2^D + \mathcal{N}_2) + J_1 \ddot{\theta}_4 s_\alpha (s_3 c_\beta + c_3 s_\beta) s_3 = 0$$

$$-(T_2^D + \mathcal{N}_2) + J_1 \ddot{\theta}_4 s_\alpha [s_3 c_3 c_\beta + c_3^2 s_\beta - s_3 c_3 c_\beta + s_3^2 s_\beta] = 0$$

$$\boxed{(T_2^D + \mathcal{N}_2) - J_1 \ddot{\theta}_4 s_\alpha s_\beta = 0}$$

$$(T_1^D + \mathcal{N}_1) - J_1 \ddot{\theta}_4 s_\alpha (c_3^2 c_\beta - s_3 \cancel{c_3} s_\beta + s_3^2 c_\beta + s_3 \cancel{c_3} s_\beta)$$

$$\boxed{(T_1^D + \mathcal{N}_1) - J_1 \ddot{\theta}_4 s_\alpha c_\beta = 0}$$

$$\boxed{(T_3^D + \mathcal{N}_3) - J_1 \ddot{\theta}_4 c_\alpha = 0}$$

$$T \mp J_1 (\dot{\omega}_3 + \ddot{\theta}_4) = 0$$

$$(T_3^D + \mathcal{N}_3) - J_1 \ddot{\theta}_4 c_\alpha = [(I_1 - I_2)\tilde{\omega}_1 \tilde{\omega}_2 - I_3 \dot{\tilde{\omega}}_3] + J_1 \dot{\theta}_4 \omega_2 s_\alpha - J_1 \ddot{\theta}_4 c_\alpha = 0$$

$$[\qquad\qquad\qquad\qquad] - J_1 ({}^A\dot{\omega}^B \beta_3 - {}^A\omega^B (\omega_2 s_\alpha)) = 0$$

is $\omega_2 s_\alpha = \beta_1 \tilde{\omega}_2 - \beta_2 \tilde{\omega}_1$ $\beta_1 = s_\alpha c_\beta$ $\beta_2 = s_\alpha s_\beta$

$\tilde{\omega}_2 = -\dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3$ $\tilde{\omega}_1 = \dot{\theta}_1 c_2 c_3 + \dot{\theta}_2 s_3$ $\omega_2 = -(\dot{\theta}_1 c_2 c_3 + \dot{\theta}_2 s_3) s_\beta + (-\dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3) c$

$$\tilde{\omega}_2 \beta_1 - \tilde{\omega}_1 \beta_2 = s_\alpha [(-\dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3) c_\beta - (\dot{\theta}_1 c_2 c_3 + \dot{\theta}_2 s_3) s_\beta] = \omega_2 s_\alpha$$

$\rightarrow$ $[(I_1 - I_2)\tilde{\omega}_1 \tilde{\omega}_2 - I_1 \dot{\tilde{\omega}}_1] - J_1 ({}^A\dot{\omega}^B \beta_3 - {}^A\omega^B (\tilde{\omega}_2 \beta_1 - \tilde{\omega}_1 \beta_2)) = 0$

Now

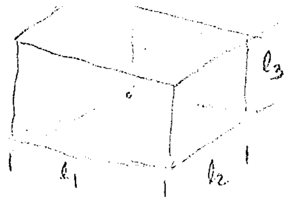
$$T_1^D + \mathcal{N}_1 - J_1 \ddot{\theta}_4 s_\alpha c_\beta = [(I_2 - I_3)\tilde{\omega}_2 \tilde{\omega}_3 - I_1 \dot{\tilde{\omega}}_1] - J_1 \dot{\theta}_4 (\omega_1 s_\beta + \omega_2 c_\beta c_\alpha) - J_1 \ddot{\theta}_4 s_\alpha c_\beta = 0$$

$\rightarrow$ $= [(I_2 - I_3)\tilde{\omega}_2 \tilde{\omega}_3 - I_1 \dot{\tilde{\omega}}_1] - J_1 ({}^A\dot{\omega}^B \beta_1 + {}^A\omega^B [\beta_2 \tilde{\omega}_3 - \beta_3 \tilde{\omega}_2]) = 0$

is $-\omega_1 s_\beta - \omega_2 c_\beta c_\alpha = \beta_2 \tilde{\omega}_3 - \beta_3 \tilde{\omega}_2 = s_\alpha s_\beta [\dot{\theta}_1 s_2 + \dot{\theta}_3] - c_\alpha [-\dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3]$ ✓

$-s_\beta [\cancel{(\dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 s_3)} c_\beta c_\alpha + (-\dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3) s_\beta c_\alpha - (\dot{\theta}_1 s_2 + \dot{\theta}_3) s_\alpha] - c_\beta c_\alpha [-\cancel{(\dot{\theta}_1 c_2 c_3 + \dot{\theta}_2 s_3)} s_\beta + (-\dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c$

$-c_\alpha (-\dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3) + (\dot{\theta}_1 s_2 + \dot{\theta}_3) s_\alpha s_\beta$ ✓



$\underset{\sim}{\beta} = (s\alpha c\beta\ ,\ s\alpha s\beta\ ,\ c_\alpha)$ $\qquad J_1 = \frac{mR^2}{2}$ $\qquad I_1 = \frac{M}{12}(l_1^2 + l_3^2)$ $\qquad I_2 = \frac{M}{12}(l_3^2 + l_2^2)$

$I_3 = (l_1^2 + l_2^2)\frac{M}{12}$ $\qquad {}^A\omega^B = \dot{\theta}_4$ $\qquad {}^A\dot{\omega}^B = \theta_4$; let $l_2 = b l_1$, $l_3 = c l_1$. $0 < b, c < 1$

$\tilde{\omega}_1 = \dot{\theta}_1 c_2 c_3 + \dot{\theta}_2 s_3$ $\qquad \tilde{\omega}_2 = -\dot{\theta}_1 c_2 s_3 + \dot{\theta}_2 c_3$ $\qquad \tilde{\omega}_3 = \dot{\theta}_1 s_2 + \dot{\theta}_3$

$\tilde{\omega}_1 s_3 + \tilde{\omega}_2 c_3 = \dot{\theta}_2$ and $(\tilde{\omega}_1 c_3 - \tilde{\omega}_2 s_3)/c_2 = \dot{\theta}_1$ $\qquad \dot{\theta}_3 = \tilde{\omega}_3 - (\tilde{\omega}_1 c_3 - \tilde{\omega}_2 s_3)\, s_2/c_2$

$I_1 = \frac{M l_1^2}{12}(1 + c^2)$ $\qquad I_2 = \frac{M}{12} l_1^2 (b^2 + c^2)$ $\qquad I_3 = \frac{M l_1^2}{12}(1 + b^2)$ $\qquad$ let $R = \frac{d}{2} l_1$ $\quad d \leq \min(b, c)$

$J_1 = \frac{m\, l_1^2}{4 \cdot 2} d^2$ $\qquad$ let $C_1 = 1 - J_1 \left(\frac{\beta_1^2}{I_1} + \frac{\beta_2^2}{I_2} + \frac{\beta_3^2}{I_3} \right)$ $\qquad 1 = \frac{1}{C_1} - \frac{J_1}{C_1}(\qquad)$

$${}^A\dot{\omega}^B = \frac{T}{J_1} \cdot \frac{1}{C_1} - \frac{1}{C_1}\left[\frac{\beta_1 F_1}{I_1} + \frac{\beta_2 F_2}{I_2} + \frac{\beta_3 F_3}{I_3} \right]$$

$$\dot{\tilde{\omega}}_3 = \frac{1}{I_3}\left[\frac{F_3}{I_3} - \frac{\beta_3}{C_1}\left\{ \frac{T}{I_3 C_1} - \frac{J_1}{I_3 C_1}\left[\frac{\beta_1 F_1}{I_1} + \frac{\beta_2 F_2}{I_2} + \frac{\beta_3 F_3}{I_3} \right] \right\}\right]$$ ok.

$$\dot{\tilde{\omega}}_2 = \frac{1}{I_2}\left[F_2 - \beta_2 \left\{ \frac{T}{C_1} - \frac{J_1}{C_1}\left[\frac{\beta_1 F_1}{I_1} + \frac{\beta_2 F_2}{I_2} + \frac{\beta_3 F_3}{I_3} \right] \right\}\right]$$ ok

$$\dot{\tilde{\omega}}_1 = \frac{1}{I_1}\left[F_1 - \beta_1 \left\{ \frac{T}{C_1} - \frac{J_1}{C_1}\left[\frac{\beta_1 F_1}{I_1} + \frac{\beta_2 F_2}{I_2} + \frac{\beta_3 F_3}{I_3} \right] \right\}\right]$$ ok.

$$F_1 = (I_2 - I_3)\tilde{\omega}_2 \tilde{\omega}_3 + J_1 {}^A\omega^B \left[\tilde{\omega}_3 \beta_2 - \tilde{\omega}_2 \beta_3 \right]$$
$$F_2 = (I_3 - I_1)\tilde{\omega}_3 \tilde{\omega}_1 + J_1 {}^A\omega^B \left[\tilde{\omega}_1 \beta_3 - \tilde{\omega}_3 \beta_1 \right]$$
$$F_3 = (I_1 - I_2)\tilde{\omega}_1 \tilde{\omega}_2 + J_1 {}^A\omega^B \left[\tilde{\omega}_2 \beta_1 - \tilde{\omega}_1 \beta_2 \right]$$
$$\dot{\theta}_1 = (\tilde{\omega}_1 c_3 - \tilde{\omega}_2 s_3)/c_2$$
$$\dot{\theta}_2 = \tilde{\omega}_1 s_3 + \tilde{\omega}_2 c_3$$
$$\dot{\theta}_3 = \tilde{\omega}_3 - s_2 (\tilde{\omega}_1 c_3 - \tilde{\omega}_2 s_3)/c_2$$

$$\text{Let}\quad y_i := \begin{cases} \tilde{\omega}_i & i=1,2,3 \quad , \quad \tilde{\omega}_1, \tilde{\omega}_2, \tilde{\omega}_3 \\ \theta_{i-3} & i=4,5,6,7 \quad \theta_1, \theta_2, \theta_3, \theta_4 \\ {}^A\omega^B = \dot{\theta}_4 & i=8 \quad \dot{\theta}_4 \end{cases}$$

define first

$\downarrow$

ALP, BET

```
Read L1, B, C, D, MB, MR
 A= AMIN (B,C)
IF (D.GT.A) D=A
 R= A*L1/2
 J1 = MR * R*R/2.
 A= MB* L1* L1/12.
 I1 = A*( 1+ C*C)
 I2 = A* (B*B + C*C)
 I3 = A* (1+B*B).
  PI04= ATAN(1.00)
  AL= ALP*PI04/45.
   BE= BET * PI04/45.
  B1 = SIN (AL) * COS (BE)
   B2 = SIN(AL)* SIN (BE)
   B3= COS(AL)
```

let $I_{23} = I_2 - I_3$ $\qquad$ $B11 = B1/I1$

$I_{31} = I_3 - I_1$ $\qquad$ $B22 = B2/I2$

$I_{12} = I_1 - I_2$ $\qquad$ $B33 = B3/I3$

$J_{11} = J_1 * B1$ $\qquad$ $C1 = 1 - J1*(B11*B1 + B22*B2 + B33*B3)$

$J_{12} = J_1 * B2$ $\qquad$ $JC = J1/C1$

$J_{13} = J_1 * B3$ $\qquad$ $TC = T/C1$

$Z_1 = I_{23} * Y(2)*Y(3) + Y(8)*[J12*Y(3) - J13*Y(2)]$ $\qquad$ $Z5 = \cos(Y(6))$

$Z_2 = I_{31} * Y(3)*Y(1) + Y(8)*[J13*Y(1) - J11*Y(3)]$ $\qquad$ $Z6 = \sin(Y(6))$

$Z_3 = I_{12} * Y(1)*Y(2) + Y(8)*[J11*Y(2) - J12*Y(1)]$ $\qquad$ $Z7 = \cos(Y(5))$

$Z4 = B11*Z1 + B22*Z2 + B33*Z3$ $\qquad$ $Z8 = \sin(Y(5))$

$Z9 = (Y(1)*Z5 - Y(2)*Z6)/Z7$

$\dot{\omega}_1$ $\quad DY(1) = [Z1 - B1*(TC - JC*Z4)]/I_1$

$\dot{\omega}_2$ $\quad DY(2) = [Z2 - B2*(TC - JC*Z4)]/I_2$

$\dot{\omega}_3$ $\quad DY(3) = [Z3 - B3*(TC - JC*Z4)]/I_3$

$\dot{\theta}_1$ $\quad DY(4) = Z9$

$\dot{\theta}_2$ $\quad DY(5) = Y(1)*Z6 + Y(2)*Z5$

$\dot{\theta}_3$ $\quad DY(6) = Y(3) - Z8*Z9$

$\dot{\theta}_4$ $\quad DY(7) = Y(8)$

${}^A\dot{\omega}^B$ $\quad DY(8) = TC/J1 - Z4/C1$

$\beta_3\omega_1 - \beta_1\omega_3)\Big]a_2 + \Big[\omega_1\omega_2(I_1-I_2) - \dot{\omega}_3 I_3 - J\,{}^{A}\dot{\omega}^{B}\beta_3 + J\,{}^{A}\omega^{B}(\beta_1\omega_2-\beta_2\omega_1)\Big]a_3$

$$F_4^{*} = \Big[\omega_2\omega_3(I_2-I_3) - I_1\dot{\omega}_1 - J\,{}^{A}\dot{\omega}^{B}\beta_1 + J\,{}^{A}\omega^{B}(\beta_2\omega_3-\beta_3\omega_2)\Big] = F^{*}_{\omega_1}$$

$$F_5^{*} = \Big[\omega_3\omega_1(I_3-I_1) - I_2\dot{\omega}_2 - J\,{}^{A}\dot{\omega}^{B}\beta_2 + J\,{}^{A}\omega^{B}(\beta_3\omega_1-\beta_1\omega_3)\Big] = F^{*}_{\omega_2}$$

$$F_6^{*} = \Big[\omega_1\omega_2(I_1-I_2) - I_3\dot{\omega}_3 - J\,{}^{A}\dot{\omega}^{B}\beta_3 + J\,{}^{A}\omega^{B}(\beta_1\omega_2-\beta_2\omega_1)\Big] = F^{*}_{\omega_3}$$

$$\mathbb{T}^{*B} = (\mathbb{I}^{B/B^*}\cdot{}^{R}\omega^{B})\times{}^{R}\omega^{B} - \mathbb{I}^{B/B^*}\cdot{}^{R}\alpha^{B}$$

$$ {}^{R}\alpha^{B} = \frac{{}^{R}d\,{}^{R}\omega^{B}}{dt} = \frac{{}^{A}d\,{}^{R}\omega^{A}}{dt} + \cancel{{}^{R}\omega^{A}\times{}^{R}\omega^{A}} + \frac{{}^{A}d\,{}^{A}\omega^{B}}{dt} + {}^{R}\omega^{A}\times{}^{A}\omega^{B}$$

$$= {}^{R}\dot{\omega}_i^{A}a_i + {}^{A}\dot{\omega}^{B}\beta_i a_i + {}^{R}\omega^{A}\times{}^{A}\omega^{B}$$

$${}^{R}\omega^{A}\times{}^{A}\omega^{B} = (\omega_i a_i)\times({}^{A}\omega^{B}\beta_j a_j) = {}^{A}\omega^{B}\Big[(\omega_2\beta_3-\omega_3\beta_2)a_1 + (\omega_3\beta_1-\omega_1\beta_3)a_2 + (\omega_1\beta_2-\omega_2\beta_1)$$

Now $\beta\cdot\mathbb{T}^{*B} = \beta\cdot(\mathbb{I}^{B/B^*}\cdot{}^{R}\omega^{B})\times{}^{R}\omega^{B} - \beta\cdot\mathbb{I}^{B/B^*}\cdot{}^{R}\alpha^{B}$

$$= (\mathbb{I}^{B/B^*}\cdot{}^{R}\omega^{B})\cdot({}^{R}\omega^{B}\times\beta) - \beta\cdot\mathbb{I}^{B}\cdot{}^{R}\alpha^{B}$$

$$({}^{R}\omega^{A}+{}^{A}\omega^{B})\times\beta - \beta\cdot\mathbb{I}^{B}\cdot{}^{R}\alpha^{A} - {}^{A}\dot{\omega}^{B}\beta\cdot\mathbb{I}^{B}\cdot\beta - \beta\cdot\mathbb{I}^{B}\cdot({}^{R}\omega^{A}\times{}^{A}\omega^{B})$$

$$= (\mathbb{I}^{B/B^*}\cdot{}^{R}\omega^{B})\cdot({}^{R}\omega^{A}\times\beta) - J\beta\cdot{}^{R}\alpha^{A} - {}^{A}\dot{\omega}^{B}J - \cancel{J\beta\cdot({}^{R}\omega^{A}\times{}^{A}\omega^{B})} \qquad {}^{A}\omega^{B} = {}^{A}\omega^{B}\beta$$

$$\Big[\mathbb{I}\cdot({}^{R}\omega^{A}+{}^{A}\omega^{B}\beta)\Big] \qquad - J\Big[\dot{\omega}_1\beta_1 + \dot{\omega}_2\beta_2 + \dot{\omega}_3\beta_3 + {}^{A}\dot{\omega}^{B}\Big]$$

$$= (\mathbb{I}^{B/B^*}\cdot{}^{R}\omega^{A}) + \cancel{J\,{}^{A}\omega^{B}\beta\cdot({}^{R}\omega^{A}\times\beta)} - J\Big[\dot{\omega}_1\beta_1 + \dot{\omega}_2\beta_2 + \dot{\omega}_3\beta_3 + {}^{A}\dot{\omega}^{B}\Big]$$

$$\uparrow \quad \cdot({}^{R}\omega^{A}\times\beta)$$

$$\beta\cdot\mathbb{T}^{*B} = \qquad - J\Big[\dot{\omega}_1\beta_1 + \dot{\omega}_2\beta_2 + \dot{\omega}_3\beta_3 + {}^{A}\dot{\omega}^{B}\Big]$$

$$-J[\quad] + T = 0 \qquad \frac{T}{J} = (\quad)$$

$$\frac{T}{J} - \Big[\dot{\omega}_1\beta_1 + \dot{\omega}_2\beta_2 + \dot{\omega}_3\beta_3\Big] = {}^{A}\dot{\omega}^{B}$$

$$\beta_1\dot{\omega}_1 = \frac{\beta_1}{I_1}\Big[\omega_2\omega_3(I_2-I_3) + J\,{}^{A}\omega^{B}(\beta_2\omega_3-\beta_3\omega_2)\Big] - \frac{J\beta_1^2}{I_1}{}^{A}\dot{\omega}^{B}$$

$$= \frac{\beta_1F_1}{I_1} - \frac{J\beta_1^2}{I_1}{}^{A}\dot{\omega}^{B} = \frac{\beta_1F_1}{I_1} - \frac{T}{I_1}\beta_1^2 + \frac{J}{I_1}\beta_1^2\Big[$$

$$\frac{T}{J} - \Big[\frac{\beta_1F_1}{I_1} + \frac{\beta_2F_2}{I_2} + \frac{\beta_3F_3}{I_3} + J\,{}^{A}\dot{\omega}^{B}\Big(\beta_1^2/I_1 + \beta_2^2/I_2 + \frac{\beta_3^2}{I_3}\Big)\Big] = {}^{A}\dot{\omega}^{B}$$

$$\frac{T}{J} - \Big[\frac{\beta_1F_1}{I_1} + \qquad\qquad\Big] = {}^{A}\dot{\omega}^{B}\Big[1 - J\Big(\beta_i^2/I_i \qquad\Big)\Big]$$

(3)

Derivation let $x, y, z,\ \omega_i,\ {}^{A}\omega^{B}$

$$F_r = \mathbb{V}^{A}_{\dot q_r}\cdot\mathbb{F}^{A} + \omega^{A}_{\dot q_r}\cdot\mathbb{T}^{A} + \mathbb{V}^{B}_{\dot q_r}\cdot\mathbb{F}^{B} + \omega^{B}_{\dot q_r}\cdot\mathbb{T}^{B} = \mathbb{V}_{\dot q_r}\left[\mathbb{F}^{A}+\mathbb{F}^{B}\right] + \omega^{A}_{\dot q_r}\cdot\left[\mathbb{T}^{A}+\mathbb{T}^{B}\right] + {}^{A}\omega^{B}_{\dot q_r}\cdot\mathbb{T}^{B}$$

$$F_r^{*} = \mathbb{V}^{A*}_{\dot q_r}\cdot\mathbb{F}^{*} + \omega^{A}_{\dot q_r}\cdot\mathbb{T}^{*A} + \omega^{B}_{\dot q_r}\cdot\mathbb{T}^{*B}$$ where $\mathbb{F}^{*} = -M\mathbf{a}^{*}$ $M = (\text{mass of rotor} + \square)$.

$\mathbb{T}^{A} = -T\beta = T\beta_i\mathbf{a}_i;\ \mathbb{T}^{B} = +T\beta_i\mathbf{a}_i;$ ${}^{R}\omega^{A} = \omega_i\mathbf{a}_i$ ${}^{R}\omega^{B} = {}^{R}\omega^{A} + {}^{A}\omega^{B} = (\omega_i + {}^{A}\omega^{B}\beta_i)\mathbf{a}_i$

$\mathbb{V}^{A} = \dot x\,\mathbf{n}_x + \dot y\,\mathbf{n}_y + \dot z\,\mathbf{n}_z$ $\mathbb{F}^{A} + \mathbb{F}^{B} = Mg\,\mathbf{n}_x$ ${}^{R}\omega^{B}_{\omega_i} = \mathbf{a}_i$ ${}^{R}\omega^{B}_{{}^{A}\omega^{B}} = \beta;$ ${}^{A}\omega^{B}_{\omega_i} = 0$

$\mathbf{a}^{*A} = \ddot x\,\mathbf{n}_x + \ddot y\,\mathbf{n}_y + \ddot z\,\mathbf{n}_z$ $\mathbb{T}_A + \mathbb{T}_B = 0\ ;$ ${}^{A}\omega^{B}_{{}^{A}\omega^{B}} = \beta$

$F_1 = Mg + 0 + 0$ $F_1^{*} = -M\ddot x$

$F_2 = 0 + 0 + 0$ $F_2^{*} = -M\ddot y$

$F_3 = 0 + 0 + 0$ $F_3^{*} = -M\ddot z$

ω_1 $F_4 = 0 + \mathbf{a}_1\cdot 0 + 0$ $F_4^{*} = \mathbb{V}^{A*}_{\omega_1}\cdot\mathbb{F}^{*} + \omega^{A}_{\omega_1}\cdot\mathbb{T}^{*A} + \omega^{B}_{\omega_1}\cdot\mathbb{T}^{*B}$

$0 + \mathbf{a}_1\cdot\mathbb{T}^{*A} + \mathbf{a}_1\cdot\mathbb{T}^{*B}$

ω_2 $F_5 = 0 + \mathbf{a}_2\cdot 0 + 0$ $F_5^{*} = 0 + \mathbf{a}_2\cdot(\mathbb{T}^{*A} + \mathbb{T}^{*B})$

ω_3 $F_6 = 0 + \mathbf{a}_3\cdot 0 + 0$ $F_6^{*} = 0 + \mathbf{a}_3\cdot(\mathbb{T}^{*A} + \mathbb{T}^{*B})$

${}^{A}\omega^{B}$ $F_7 = 0 + 0.0 + T$ $F_7^{*} = 0 + 0 + \beta\cdot\mathbb{T}^{*B}$

$$\mathbb{T}^{*A} + \mathbb{T}^{*B} = -\frac{{}^{R}d\,\mathbb{H}^{A/A*}}{dt} - \frac{{}^{R}d\,\mathbb{H}^{B/B*}}{dt} = -\frac{{}^{R}d}{dt}\left(\mathbb{H}^{A/A*} + \mathbb{H}^{B/B*}\right).$$

$$\mathbb{H}^{A/A*} = \mathbb{I}^{A/A*}\cdot{}^{R}\omega^{A} \qquad \mathbb{H}^{B/B*} = \mathbb{I}^{B/B*}\cdot{}^{R}\omega^{B} = \mathbb{I}^{B/B*}\cdot{}^{R}\omega^{A} + \mathbb{I}^{B/B*}\cdot{}^{A}\omega^{B}$$

$$\therefore\quad \mathbb{H}^{A/A*} + \mathbb{H}^{B/B*} = \underbrace{\left(\mathbb{I}^{A/A*} + \mathbb{I}^{B/B*}\right)}_{\mathbb{I}^{G/G*}}\cdot{}^{R}\omega^{A} + \mathbb{I}^{B/B*}\cdot{}^{A}\omega^{B}$$

Define a fictitious body (rigid D) whose mass distrib is that of G but whose motion is that of A.

Thus $\mathbb{H}^{A/A*} + \mathbb{H}^{B/B*} = \mathbb{H}^{D/D*} + {}^{A}\mathbb{H}^{B/B*}$

$$\frac{{}^{R}d}{dt}\left(\qquad\right) = \frac{{}^{R}d}{dt}\mathbb{H}^{D/D*} + \frac{{}^{R}d}{dt}{}^{A}\mathbb{H}^{B/B*} = \underbrace{\frac{{}^{R}d}{dt}\mathbb{H}^{D/D*}}_{-\mathbb{T}^{D/D*}} + \underbrace{\frac{{}^{A}d}{dt}{}^{A}\mathbb{H}^{B/B*}}_{-{}^{A}\mathbb{T}^{*B}} + \underbrace{{}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{B/B*}}_{+{}^{R}\omega^{A}\times\left(\mathbb{I}^{B/B*}\cdot{}^{A}\omega^{B}\right)}$$

$$\therefore\quad \mathbb{T}^{*A} + \mathbb{T}^{*B} = \mathbb{T}^{D/D*} + {}^{A}\mathbb{T}^{*B} - {}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{B/B*}$$

$$= \left[\left(\mathbb{I}^{C/C*}\cdot{}^{R}\omega^{A}\right)\times{}^{R}\omega^{A} - \mathbb{I}^{C/C*}\cdot{}^{R}\alpha^{A} + \left(\mathbb{I}^{B/B*}\cdot{}^{A}\omega^{B}\right)\times{}^{A}\omega^{B} - \mathbb{I}^{B/B*}\cdot{}^{A}\alpha^{B} + \left(\mathbb{I}^{B/B*}\cdot{}^{A}\omega^{B}\right)\times{}^{R}\omega^{A}\right.$$

$${}^{R}\alpha^{A} = \frac{{}^{R}d\,{}^{R}\omega^{A}}{dt} = \frac{{}^{A}d\,\omega^{A}}{dt} + \cancel{{}^{R}\omega^{A}\times{}^{R}\omega^{A}} = \dot\omega_i\,\mathbf{a}_i$$

$${}^{A}\alpha^{B} = \frac{{}^{A}d\,{}^{A}\omega^{B}}{dt} = {}^{A}\dot\omega^{B}\beta$$

$\mathbb{I}^{C/C*}\cdot{}^{R}\omega^{A} = I_i\,\omega_i\,\mathbf{a}_i$; $\mathbb{I}^{C/C*}\cdot{}^{R}\alpha^{A} = I_i\,\dot\omega_i\,\mathbf{a}_i$; $\mathbb{I}^{B/B*}\cdot{}^{A}\omega^{B} = J\,{}^{A}\omega^{B}\beta$ and $\left(\mathbb{I}^{B/B*}\cdot{}^{A}\omega^{B}\right)\times{}^{A}\omega^{B} = ($

$\mathbb{I}^{B/B*}\cdot{}^{A}\alpha^{B} = J\,{}^{A}\dot\omega^{B}\beta$; $\left(\mathbb{I}^{B/B*}\cdot{}^{A}\omega^{B}\right)\times{}^{R}\omega^{A} = \left(J\,{}^{A}\omega^{B}\beta_i\mathbf{a}_i\times\omega_j\mathbf{a}_j\right) = J\,{}^{A}\omega^{B}\left[(\beta_1\omega_2 - \beta_2\omega_1)\mathbf{a}_3 + (\beta_2\omega_3 - \beta_3\omega_2)\mathbf{a}_1 + (\beta_3\omega_1 - \beta_1\omega_3)\mathbf{a}_2\right]$

$$\mathbb{T}^{*A} + \mathbb{T}^{*B} = \Big[I_1\omega_1(\omega_2\mathbf{a}_3 - \omega_3\mathbf{a}_2) + I_2\omega_2(\omega_3\mathbf{a}_1 - \omega_1\mathbf{a}_3) + I_3\omega_3(\omega_1\mathbf{a}_2 - \omega_2\mathbf{a}_1) - I_1\dot\omega_1\mathbf{a}_1 - I_2\dot\omega_2\mathbf{a}_2 - I_3\dot\omega_3\mathbf{a}_3$$
$$- J\,{}^{A}\dot\omega^{B}\left[\beta_1\mathbf{a}_1 + \beta_2\mathbf{a}_2 + \beta_3\mathbf{a}_3\right] + J\,{}^{A}\omega^{B}\left(\beta_1\omega_2\mathbf{a}_3 - \beta_1\omega_3\mathbf{a}_2 + \beta_2\omega_3\mathbf{a}_1 - \beta_2\omega_1\mathbf{a}_3 + \beta_3\omega_1\mathbf{a}_2 - \beta_3\omega_2\mathbf{a}_1\right)\Big]$$

$$= \left[\omega_2\omega_3(I_2 - I_3) - I_1\dot\omega_1 - J\,{}^{A}\dot\omega^{B}\beta_1 + J\,{}^{A}\omega^{B}(\beta_2\omega_3 - \beta_3\omega_2)\right]\mathbf{a}_1 + \left[\omega_3\omega_1(I_3 - I_1) - I_2\dot\omega_2 - J\,{}^{A}\dot\omega^{B}\beta_2 + J\,{}^{A}\omega^{B}(\right.$$

Thus our 3 eqns reduce to

$\longrightarrow$ $(I_1-I_2)\tilde{\omega}_2\tilde{\omega}_1 - J_1\left({}^{A}\dot{\omega}^{B}\beta_3 - {}^{A}\omega^{B}\left[\tilde{\omega}_2\beta_1 - \tilde{\omega}_1\beta_2\right]\right) = I_3\dot{\tilde{\omega}}_3 \quad = F_3 - J_1\beta_3\,{}^{A}\dot{\omega}^{B}$

$\longrightarrow$ $(I_2-I_3)\tilde{\omega}_2\tilde{\omega}_3 - J_1\left({}^{A}\dot{\omega}^{B}\beta_1 - {}^{A}\omega^{B}\left[\tilde{\omega}_3\beta_2 - \tilde{\omega}_2\beta_3\right]\right) = I_1\dot{\tilde{\omega}}_1 \quad = F_1 - J_1\,{}^{A}\dot{\omega}^{B}\beta_1$

$\longrightarrow$ $(I_3-I_1)\tilde{\omega}_3\tilde{\omega}_1 - J_1\left({}^{A}\dot{\omega}^{B}\beta_2 - {}^{A}\omega^{B}\left[\tilde{\omega}_1\beta_3 - \tilde{\omega}_3\beta_1\right]\right) = I_2\dot{\tilde{\omega}}_2 \quad = F_2 - J_1\,{}^{A}\dot{\omega}^{B}\beta_2$

$\longrightarrow$ ${}^{A}\dot{\omega}^{B} = \ddot{\theta}_4 \qquad {}^{A}\omega^{B} = \dot{\theta}_4 \qquad \beta_3 = \cos\alpha \qquad \beta_1 = s_\alpha c_\beta \qquad \beta_2 = s_\alpha s_\beta$

$$T + J_1\left(\dot{\omega}_3 + {}^{A}\dot{\omega}^{B}\right) = 0 \qquad\qquad {}^{A}\dot{\omega}^{B} = -\frac{T}{J_1} - \dot{\omega}_3$$

Now $\dot{\tilde{\omega}}_1\left[c_\beta c_\alpha n_1 - s_\beta n_2 + s_\alpha c_\beta n_3\right] + \dot{\tilde{\omega}}_2\left[s_\beta c_\alpha n_1 + s_\beta n_2 + s_\alpha s_\beta n_3\right] + \dot{\tilde{\omega}}_3\left[-s_\alpha n_1 + c_\alpha n_3\right] = {}^{R}\tilde{\omega}^{A}$

$$\dot{\tilde{\omega}}_1 s_\alpha c_\beta + \dot{\tilde{\omega}}_2 s_\alpha s_\beta + \dot{\tilde{\omega}}_3 c_\alpha = \dot{\omega}_3$$

$$\dot{\tilde{\omega}}_1\beta_1 + \dot{\tilde{\omega}}_2\beta_2 + \dot{\tilde{\omega}}_3\beta_3 = \dot{\omega}_3$$

Solve the above for the derivatives $\dot{\tilde{\omega}}_1, \dot{\tilde{\omega}}_2, \dot{\tilde{\omega}}_3$ & put into eqn. for $\dot{\omega}_3$

$$\frac{\beta_1}{I_1}\left(F_1 - J_1\,{}^{A}\dot{\omega}^{B}\beta_1\right) + \frac{\beta_2}{I_2}\left(F_2 - J_1\,{}^{A}\dot{\omega}^{B}\beta_2\right) + \frac{\beta_3}{I_3}\left(F_3 - J_1\,{}^{A}\dot{\omega}^{B}\beta_3\right) = \dot{\omega}_3$$

$$\frac{\beta_1F_1}{I_1} + \frac{\beta_2F_2}{I_2} + \frac{\beta_3F_3}{I_3} = \dot{\omega}_3 + J_1\,{}^{A}\dot{\omega}^{B}\left(\frac{\beta_1^2}{I_1} + \frac{\beta_2^2}{I_2} + \frac{\beta_3^2}{I_3}\right)$$

Now ${}^{A}\dot{\omega}^{B} + \dot{\omega}_3 = \dfrac{T}{J_1}$; ${}^{A}\dot{\omega}^{B} + \dfrac{\beta_1F_1}{I_1} + \dfrac{\beta_2F_2}{I_2} + \dfrac{\beta_3F_3}{I_3} - J_1\,{}^{A}\dot{\omega}^{B}\left(\dfrac{\beta_1^2}{I_1} + \dfrac{\beta_2^2}{I_2} + \dfrac{\beta_3^2}{I_3}\right) = \dot{\omega}_3 + {}^{A}\dot{\omega}^{B} = +\dfrac{T}{J}$

$${}^{A}\dot{\omega}^{B}\left[1 - J_1\left(\frac{\beta_1^2}{I_1} + \frac{\beta_2^2}{I_2} + \frac{\beta_3^2}{I_3}\right)\right] = \frac{T}{J_1} - \left(\frac{\beta_1F_1}{I_1} + \frac{\beta_2F_2}{I_2} + \frac{\beta_3F_3}{I_3}\right)$$

$$-F_1 = (I_3-I_2)\hat{\omega}_2\hat{\omega}_3 - J_1(\tilde{\omega}_3\beta_2 - \tilde{\omega}_2\beta_3)$$

$$-F_2 = (I_1-I_3)\tilde{\omega}_3\tilde{\omega}_1 - J_1(\tilde{\omega}_1\beta_3 - \tilde{\omega}_3\beta_1)$$

$$-F_3 = (I_2-I_1)\tilde{\omega}_2\tilde{\omega}_1 - J_1(\tilde{\omega}_2\beta_1 - \tilde{\omega}_1\beta_2)$$

thus

$\longrightarrow$ $${}^{A}\dot{\omega}^{B}\left[1 - J_1\left(\frac{\beta_1^2}{I_1} + \frac{\beta_2^2}{I_2} + \frac{\beta_3^2}{I_3}\right)\right] = \frac{T}{J_1} + \frac{\beta_1}{I_1}\left[(I_3-I_2)\tilde{\omega}_2\tilde{\omega}_3 - J_1(\tilde{\omega}_3\beta_2 - \tilde{\omega}_2\beta_3)\right]$$

$$+ \beta_2/I_2\left[(I_1-I_3)\tilde{\omega}_3\tilde{\omega}_1 - J_1(\tilde{\omega}_1\beta_3 - \tilde{\omega}_1\beta_1)\right]$$

$$+ \beta_3/I_3\left[(I_2-I_1)\tilde{\omega}_2\tilde{\omega}_1 - J_1(\tilde{\omega}_2\beta_1 - \tilde{\omega}_1\beta_2)\right]$$

with the following definitions

4300

4400

5960 CO = 1.5 * r * amu

6220

(1. ± 6*6)

$$\left.\begin{array}{l}8750\\8775\end{array}\right\}\text{cont.}$$

$$\left.\begin{array}{l}9200\\9250\end{array}\right\}\text{cont.}$$

$$\left.\begin{array}{l}9700\\|\\10400\end{array}\right\}\text{contin}$$

11900

11950

12000

W1 W2 W3 AWB

1 10 4 4 4 |10| 4 , 4 4 |10 | 4 4 , 4 10 | 4 , 4 | 10 | 4

I1 I2 I3 J1 TORQUE

T/J1

ROTOR MASS

Input AMU, RM, B, C, D, AL1, , TJ1, ALP, BET.

Input AWB, W1, W2, W3

CEE = (C*C - 1.)/(C*C+1.)

BCE = (B*B - C*C)/(B*B + C*C)

BEE = (1.- B*B)/(1. + B*B).

~~IF (D. GT. AMIN (B,C)) D = AMIN (B,C)~~

R= D*AL1/2. radius of rotor

AL3 = C*AL1 l_3

AL2 = B*AL1 l_2

AJ1 = RM*R*R/2. $J1 = mR^2/2$ polar mom of inertia of rotor. $J = \frac{m}{2}\frac{d^2 l_1^2}{4}$

CO = 1.5*AMU*D*D $\frac{3}{2}\mu d^2$

AJ1I1 = CO/(1.+C*C) $J1/I1 = \frac{\frac{3}{2}\mu d^2}{1+c^2}$

AJ1I2 = Co/(B*B+C*C) $J1/I2$

AJ1I3 = Co/(1. +B*B) $J1/I3$

AI1 = AJ1/AJ1I1 $\frac{mR^2}{2}\cdot\frac{2}{3}\frac{1+c^2}{d^2}\frac{M}{m} = \frac{\cancel{m}}{\cancel{2}}\frac{d^2 l_1^2}{4}\cdot\frac{\cancel{2}}{3}\frac{(1+c^2)}{\cancel{d^2}}\frac{M}{\cancel{m}} = \frac{M l_1^2(1+c^2)}{12} = I_1$ ✓

AI2 = AJ1/AJ1I2 I_2

AI3 = AJ1/AJ1I3 I_3

IF (AI1+AI2. LT. AI3) TYPE 618
GO TO 110
IF (AI2+AI3. LT. AI1) TYPE 619
GO TO 110
IF (AI3+AI1 . LT. AI2) TYPE 620
GO TO 110

T = TJ1 * AJ1 $T = T/J_1 \cdot J1$

AL = ALP * ATAN (1.00)/45.0 α

BE = BET * ATAN (1.00)/45.0 β

B1 = SIN (AL) * COS (BE) β_1

B2 = SIN (AL) * SIN (BE) β_2

B3 = COS (AL) β_3

C1 = 1 - CO*(B1*B1/(1.+C*C) + B2*B2/(B*B+C*C) + B3*B3/(1+B*B))

B1C1 = B1/C1 β_1/c_1 B3C1 = B3/C1 β_3/c_3

B2C1 = B2/C1 β_2/c_2

TI3 = TJ1 * J1I3   T/I3

** TI2 = TJ1 * J1I2   T/I2

TI1 = TJ1 * J1I1   T/I1   TYPE 621 WRITE (20,621)

~~TYPE 612,~~ AI1, AI2, AI3, AJ1, T

WRITE (20,612) AI1, AI2, AI3, AJ1, T

G1 = CEE * ~~W2~~ Y(2) * ~~W3~~ Y(3) + Co * (~~W3~~ Y(3) * B2 - ~~W2~~ Y(2) * B3) * ~~AWB~~ Y(4) / (1+C*C)

G2 = BCE * ~~W3~~ Y(3) * ~~W1~~ Y(1) + Co * (~~W1~~ Y(1) * B3 - ~~W3~~ Y(3) * B1) * ~~AWB~~ Y(4) / (B*B+C*C)

G3 = BEE * ~~W1~~ Y(1) * ~~W2~~ Y(2) + Co * (~~W2~~ Y(2) * B1 - ~~W1~~ Y(1) * B2) * ~~AWB~~ Y(4) / (1+B*B)

H = B1*G1 + B2*G2 + B3*G3

DY(1) ~~W3DOT~~ = G3 - [~~B2~~ B3C1 * (TI3 - AJ1I3 * H)]

DY(2) ~~W2DOT~~ = G2 - [B2C1 * (TI2 - AJ1I2 * H)],

DY(3) ~~W1DOT~~ = G1 - [B1C1 * (TI1 - AJ1I1 * H)]

DY(4) ~~AWBDOT~~ = [TJ1 - H] / C1

~~DY(1) = W1DOT~~

~~DY(2) = W2DOT~~

~~DY(3) = W3DOT~~

~~DY(4) = AWBDOT~~

need to pass CEE, CO, C, B1, B2, B3, BCE, B, C, BEE

BBC1, B2C1, B1C1, TI3, TI2, TI1, AJ1I3, AJ1I3, AJ1I1, TJ1, C1.

~~or T, AJ1, AI1, AI2, AI3 ⇒ Bethe. & B1, B2, B3, C1~~

~~G1 = (AI2 - AI3) * Y(2) * Y(3) + AJ1 * [Y(3) * B2 - Y(2) * B3] * Y(4)~~

~~G2 = (AI3 - AI1) * Y(3) - Y(1) + AJ1 * [Y(1) * B3 - Y(3) * B1] * Y(4)~~

~~G2 = (AI1 - AI2) Y(1).~~

3.7 Dynamical equations for a simple gyrostat

In Fig. 3.7.1, G designates a simple gyrostat consisting of a rigid body A and an axisymmetric rotor B whose axis and mass center, B^*, are fixed in A. $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ form a dextral set of mutually perpendicular unit vectors fixed in A, each parallel to a central principal axis of inertia of G. The axis of B is parallel to $\underline{a}_3$, unit vector $\underline{\beta}$, and G has central principal moments of inertia I_1, I_2, I_3 while B has an axial moment of inertia J.

After defining ${}^A\omega^B$, ω_i, β_i, and M_i as

$$ {}^A\omega^B \triangleq {}^A\underline{\omega}^B \cdot \underline{\beta} \qquad (1)$$

$$ \omega_i \triangleq {}^N\underline{\omega}^A \cdot \underline{a}_i \quad (i=1,2,3) \qquad (2)$$

$$ \beta_i \triangleq \underline{\beta} \cdot \underline{a}_i \quad (i=1,2,3) \qquad (3)$$

and

$$ M_i \triangleq \underline{M} \cdot \underline{a}_i \quad (i=1,2,3) \qquad \cancel{(3)}\ (4)$$

where ${}^A\underline{\omega}^B$ is the angular velocity of B in A, ${}^N\underline{\omega}^A$ is the angular velocity of A in a Newtonian reference frame N, and $\underline{M}$ is the moment about the mass center G^* of G of all forces acting on G, one can write the following set of coupled differential equations governing ${}^A\omega^B$ and ω_i $(i=1,2,3)$:

~~$(I_3-J_3)\,{}^A\dot{\omega}^B = -(I_1-I_2)\omega_1\omega_2 - M_3 + (I_3/J_3)\,\underline{a}_3\cdot\underline{M}^{A/B}$ (4)~~

~~where $\underline{M}^{A/B}$ is the moment about B^* of all forces exerted by A on B, and~~

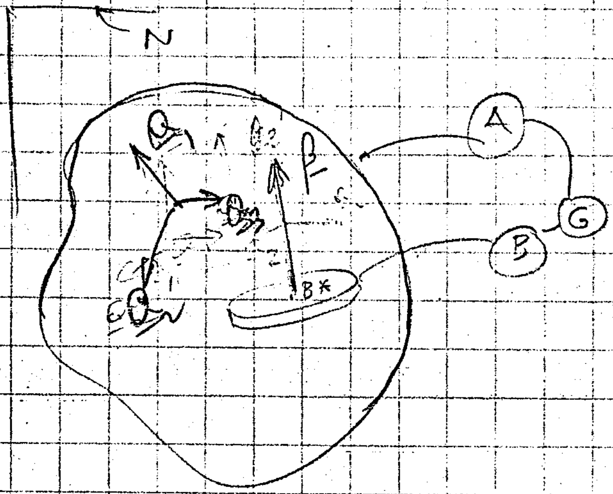


Fig. 3.7.1

$$ {}^{A}\dot{\omega}^{B}\left[1-J\left(\frac{\beta_1^2}{I_1}+\frac{\beta_2^2}{I_2}+\frac{\beta_3^2}{I_3}\right)\right] = \frac{\underline{\beta}\cdot\overset{T}{\underline{M}}{}^{A/B}}{J} + $$

$$ +\frac{\beta_1}{I_1}\left[(\overset{I_2-I_3}{I_3-I_2})\omega_2\omega_3 + J\,{}^{A}\omega^{B}(\beta_2\omega_3-\beta_3\omega_2)-\cancel{M_1}\right]+ $$

$$ +\frac{\beta_2}{I_2}\left[(\overset{I_3-I_1}{I_1-I_3})\omega_3\omega_1 + J\,{}^{A}\omega^{B}(\beta_3\omega_1-\beta_1\omega_3)-\cancel{M_2}\right]+ $$

$$ +\frac{\beta_3}{I_3}\left[(\overset{I_1-I_2}{I_2-I_1})\omega_1\omega_2 + J\,{}^{A}\omega^{B}(\beta_1\omega_2-\beta_2\omega_1)-\cancel{M_3}\right] \qquad (5) $$

where $\underline{M}^{A/B}$ is the moment about B^* of all forces exerted by A on B; and

$$ I_1\dot{\omega}_1 = (I_2-I_3)\omega_2\omega_3 - J\left[{}^{A}\dot{\omega}^{B}\beta_1 - {}^{A}\omega^{B}(\beta_2\omega_3-\beta_3\omega_2)\right]+M_1 \qquad (6) $$

$$ I_2\dot{\omega}_2 = (I_3-I_1)\omega_3\omega_1 - J\left[{}^{A}\dot{\omega}^{B}\beta_2 - {}^{A}\omega^{B}(\beta_3\omega_1-\beta_1\omega_3)\right]+M_2 \qquad (7) $$

$$ I_3\dot{\omega}_3 = (I_1-I_2)\omega_1\omega_2 - J\left[{}^{A}\dot{\omega}^{B}\beta_3 - {}^{A}\omega^{B}(\beta_1\omega_2-\beta_2\omega_1)\right]+M_3 \qquad (8) $$

If the axis of B is parallel to a central principal axis of inertia of G, the complexity of these equations is reduced considerably; and further simplifications can be made if B is completely free to rotate relative to A or if B is made to rotate relative to A with a constant angular speed, say by means of a motor each of whose parts belongs either to A or to B. Under these circumstances, and with $\underline{\beta} = \underline{a}_3$, the differential equations governing ω_1, ω_2, and ω_3 can be written

$$I_1\dot{\omega}_1 = (I_2 - \bar{I}_3)\omega_2\omega_3 - J\sigma\omega_2 + M_1 \qquad (9)$$

$$I_2\dot{\omega}_2 = (\bar{I}_3 - I_1)\omega_3\omega_1 + J\sigma\omega_1 + M_2 \qquad (10)$$

$$\bar{I}_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + M_3 \qquad (11)$$

where $\bar{I}_3$ and σ are given in Table 3.7.1, with ${}^{A}\hat{\omega}^{B}$ and $\hat{\omega}_3$ denoting the initial values of ${}^{A}\omega^{B}$ and ω_3, respectively.

Rotor motion	$\bar{I}_3$	σ
free	$I_3 - J$	$\hat{\omega}_3 + {}^{A}\hat{\omega}^{B}$
${}^{A}\omega^{B} = {}^{A}\hat{\omega}^{B}$	I_3	${}^{A}\hat{\omega}^{B}$

Table 3.7.1

Derivations: The inertia torque $\underline{T}_B$ of B in N is given by

$$\underline{T}_B = -{}^N\underline{\alpha}^B \cdot \underline{I}_B - {}^N\underline{\omega}^B \times \underline{I}_B \cdot {}^N\underline{\omega}^B \qquad (12)\ \cancel{(11)}$$

where ${}^N\underline{\alpha}^B$ and ${}^N\underline{\omega}^B$ are the angular acceleration and the angular velocity of B in N, and $\underline{I}_B$, the central inertia dyadic of B, can be expressed as

$$\underline{I}_B = K(\underline{\gamma}\,\underline{\gamma} + \underline{\delta}\,\underline{\delta}) + J\underline{\beta}\,\underline{\beta} \qquad \cancel{(12)}$$

$$= K\underline{U} + (J-K)\underline{\beta}\,\underline{\beta} \qquad (13)$$

Here K is the transverse central moment of inertia of B, $\underline{\gamma}$ and $\underline{\delta}$ are unit vectors such that β, $\underline{\gamma}$, and $\underline{\delta}$ are mutually perpendicular, and $\underline{U}$ is the unit dyadic. Hence

$${}^N\underline{\alpha}^B \cdot \underline{I}_B \underset{(13)}{=} K\,{}^N\underline{\alpha}^B + (J-K)\,{}^N\underline{\alpha}^B \cdot \underline{\beta}\,\underline{\beta} \qquad (14)\ \cancel{(13)}$$

and

$${}^N\underline{\omega}^B \times \underline{I}_B \cdot {}^N\underline{\omega}^B \underset{(13)}{=} (J-K)\,{}^N\underline{\omega}^B \times \underline{\beta}\,\underline{\beta} \cdot {}^N\underline{\omega}^B \qquad (15)\ \cancel{(14)}$$

from which it follows that

$$\underline{T}_B \cdot \overset{\omega_{\dot{q}_4}\cdot \Pi_B^A}{\underline{\beta}} \underset{(12)}{=} -J\,{}^N\underline{\alpha}^B \cdot \underline{\beta} = {}^A\omega^B_{u_4} \cdot \Pi^{*B} \qquad (16)\ \cancel{(15)}$$

Since by D'Alembert's Principle $\qquad \Pi$ (active torque) $\cdot\ {}^A\omega^B_{u_4}$

$$\underline{T}_B \cdot \underline{\beta} + \underline{M}^{A/B} \cdot \underline{\beta} = 0 \qquad (17)\ \cancel{(16)}$$

on thus has

$$J\,{}^N\underline{\alpha}^B \cdot \underline{\beta} \underset{(15,16)}{=} \underline{M}^{A/B} \cdot \underline{\beta} \qquad (18)\ \cancel{(17)}$$

Now,
$${}^N\underline{\alpha}^B \underset{(1.20.1)}{=} \frac{{}^N d\,{}^N\underline{\omega}^B}{dt} \underset{(1.16.1)}{=} \frac{{}^N d}{dt}({}^N\underline{\omega}^A + {}^A\omega^B\underline{\beta})$$

$$= \underset{(1.20.1)}{{}^{N}\underline{\alpha}^{A}} + {}^{A}\dot{\omega}^{B}\underline{\beta} + \underset{(1.11.9)}{{}^{A}\omega^{B}\,{}^{N}\underline{\omega}^{A} \times \underline{\beta}} \qquad (19)\ \cancel{(18)}$$

Consequently,

$$J\left({}^{N}\underline{\alpha}^{A}\cdot \underset{(19)}{\underline{\beta}} + {}^{A}\dot{\omega}^{B}\right) \underset{(18)}{=} \underline{\beta}\cdot\underline{M}^{A/B} \qquad (20)\ \cancel{(19)}$$

and, since

$${}^{N}\underline{\alpha}^{A}\cdot\underline{\beta} = \dot{\omega}_1\beta_1 + \dot{\omega}_2\beta_2 + \dot{\omega}_3\beta_3 \qquad (21)\ \cancel{(20)}$$

$${}^{N}\alpha^{A} = \frac{d\,{}^{N}\omega^{A}}{dt} = \frac{d}{dt}w_i a_{1i} = \dot{w}_i a_{1i} + \overset{(2)}{\left(w_i \frac{{}^{N}d a_{1i}}{dt} = w_i \cancel{\frac{{}^{A}d a_{1i}}{dt}} + w_i\,{}^{N}\omega^{A}\times a_{1i} = w_i w_j a_{1j}\times a_{1i}{}^{=0}\right)} = \dot{\omega} a_{1i}$$

$$w_i w_j e_{ikj} a_{1k} = (w_i w_j - w_j w_i) a_k$$

one can write

$$J\left(\dot{w}_1\beta_1 + \dot{w}_2\beta_2 + \dot{w}_3\beta_3 + {}^{A}\dot{\omega}^{B}\right) \underset{(20)}{=} \underline{\beta}\cdot\underline{M}^{A/B} \qquad (22)\ \cancel{(21)}$$

$\underline{T}_R$, the inertia torque of a rigid body R that has the same mass distribution as G but moves like A, is given by

$$\overset{(3.6.1)}{\underline{T}_R} = -{}^{N}\underline{\alpha}^{A}\cdot\underline{I}_G - {}^{N}\underline{\omega}^{A}\times\underline{I}_G\cdot{}^{N}\underline{\omega}^{A} \qquad (23)\ \cancel{(22)}$$

Hence, $\underline{T}_G$, the inertia torque of G in N, can be expressed with the aid of Eqs. (3.6.2) and (3.6.5) as

$$\underline{T}_G \underset{(3.6.2,\,3.6.5)}{=} -{}^{N}\underline{\alpha}^{A}\cdot\underline{I}_G - {}^{N}\underline{\omega}^{A}\times\underline{I}_G\cdot{}^{N}\underline{\omega}^{A} - J\left({}^{A}\dot{\underline{\omega}}^{B}\underline{\beta} + {}^{A}\omega^{B}\,{}^{N}\underline{\omega}^{A}\times\underline{\beta}\right) \qquad (24)\ \cancel{(23)}$$

Carrying out the operations indicated in Eq. (24) after noting that

$$\underline{I}_G = I_1\underline{a}_1\underline{a}_1 + I_2\underline{a}_2\underline{a}_2 + I_3\underline{a}_3\underline{a}_3 \qquad (25)\ \cancel{(24)}$$

and then appealing to D'Alembert's principle to write

$$\underline{T}_G + \underline{M} = 0 \qquad (26)\ \cancel{(25)}$$

one arrives at Eqs. (6) – (8); and solution of these equations for $\dot{\omega}_1, \dot{\omega}_2, \dot{\omega}_3$, followed by substitution into Eq. (22), produces Eq. (5).

Suppose now that $\underline{\beta} = \underline{a}_3$, so that [see Eq. (3)]

$$\beta_1 = \beta_2 = 0, \quad \beta_3 = 1 \qquad (27)$$

Then

$${}^{A}\dot{\omega}^{B}\left(1 - \frac{J}{I_3}\right) \underset{(5)}{=} \frac{\underline{\beta} \cdot \underline{M}^{A/B}}{J} + \frac{I_2 - I_1}{I_3}\omega_1\omega_2 - \frac{M_3}{I_3} \qquad (28)$$

and

$$I_1\dot{\omega}_1 \underset{(6)}{=} (I_2 - I_3)\omega_2\omega_3 - J\,{}^{A}\omega^{B}\omega_2 + M_1 \qquad (29)$$

$$I_2\dot{\omega}_2 \underset{(7)}{=} (I_3 - I_1)\omega_3\omega_1 + J\,{}^{A}\omega^{B}\omega_1 + M_2 \qquad (30)$$

$$I_3\dot{\omega}_3 \underset{(8)}{=} (I_1 - I_2)\omega_1\omega_2 - J\,{}^{A}\dot{\omega}^{B} + M_3 \qquad (31)$$

~~If B is completely free to rotate relative to A, that is, if $\underline{\beta} \cdot \underline{M}^{A/B} = 0$, then~~

$${}^{A}\dot{\omega}^{B} \underset{(28)}{=} \frac{I_2 - I_1}{I_3 - J}\omega_1\omega_2 - \frac{M_3}{I_3 - J} \qquad (32)$$

~~so that~~

$$I_3\dot{\omega}_3 \underset{(31)}{=} (I_1 - I_2)\omega_1\omega_2 - \frac{J(I_2 - I_1)}{I_3 - J}\omega_1\omega_2 + \frac{J M_3}{I_3 - J} + M_3 \qquad (33)$$

~~or, equivalently,~~

$$(I_3 - J)\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + M_3 \qquad (34)$$

If B is completely free to rotate relative to A, that is, if $\underline{\beta}\cdot\underline{M}^{A/B} = 0$, then [see Eqs. (27)]

$$\dot{\omega}_3 + {}^A\dot{\omega}^B = 0 \qquad (32)$$

which implies that

$$\omega_3 + {}^A\omega^B = \hat{\omega}_3 + {}^A\hat{\omega}^B \qquad (33)$$

where $\hat{\omega}_3$ and ${}^A\hat{\omega}^B$ are the initial values of ω_3 and ${}^A\omega^B$, respectively. Hence

$$I_1\dot{\omega}_1 \underset{(29)}{=} (I_2 - I_3)\omega_2\omega_3 - J(\underset{(33)}{\hat{\omega}_3 + {}^A\hat{\omega}^B} - \omega_3)\omega_2 + M_1$$

$$= (I_2 - I_3 + J)\omega_2\omega_3 - J(\hat{\omega}_3 + {}^A\hat{\omega}^B) + M_1 \qquad (34)$$

Similarly, from Eqs. (30) and (33),

$$I_2\dot{\omega}_2 = (I_3 - J - I_1)\omega_3\omega_1 + J(\hat{\omega}_3 + {}^A\hat{\omega}^B) + M_2 \qquad (35)$$

Furthermore,

$${}^A\dot{\omega}^B \underset{(28)}{=} \frac{I_2 - I_1}{I_3 - J}\omega_1\omega_2 - \frac{M_3}{I_3 - J} \qquad (36)$$

and substitution into Eq. (31) yields

$$(I_3 - J)\dot{\omega}_3 = (I_1 - I_2)\omega_1\omega_2 + M_3 \qquad (37)$$

Replacing $I_3 - J$ with $\bar{I}_3$ and $\hat{\omega}_3 + {}^A\hat{\omega}^B$ with σ in Eqs. (34), (35), and (37), one arrives at Eqs. (9), (10), and (11) respectively. On the other hand, if ~~If~~ B is made to rotate relative to A with the constant angular speed ${}^A\hat{\omega}^B$, then Eqs. (29) – (31)

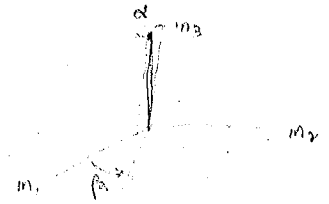
COPPE/UFRJ

DATA / /

NOME:

TRABALHO:

become Eqs. (9) – (11) when I_3 is replaced with $\overline{I}_3$, ${}^{A}\hat{\omega}^{B\prime}$ is replaced with σ, and ${}^{A}\mathring{\omega}^{B\prime\prime}$ is set equal to zero. Thus Eqs. (9) – (11) apply in both cases, provided $\overline{I}_3$ and σ be interpreted in accordance with Table 3.7.1



1. Trends if all else remains the same: as $\underline{\underline{\mu}} = \frac{rm}{rm+mb}$ ↓ angular velocity of body ↓ as well as that of the rotor. $rm\left(\frac{1-\mu}{\mu}\right) = mb$ for fixed rm as μ ↓ mb ↑ ∴ effect of rotor becomes smaller. Hence the induced $|\omega|$ of body ↓ as expected

2. for fixed μ $\frac{mb}{rm} = \frac{1-\mu}{\mu}$. Thus as the rotor mass changes for fixed μ the body mass changes also. Hence we get no effect here. Motion here is still unstable as ω increases as t increase

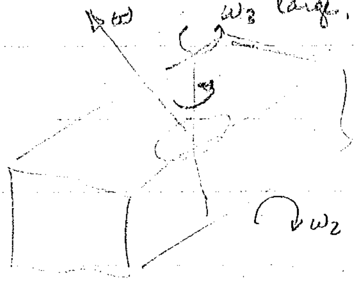
{ also the value of ω_3 remains negative for longer periods of time. Motion is quite unstable.

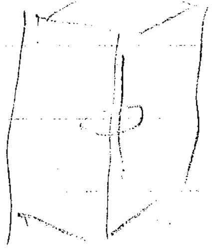
3. if we only change the T/J1 ratio and keep it very much below 1 then the value of $|\omega|$ ↓ as T/J1 ↓ and the order of magnitudes for $|\omega|$ ↓ drastically the ω_3 component decreases then starts to increase. ω_3 component is affect more than others because rotor axis is almost alligned with m_3 axis. For very low values of T/J1 ~~body~~ rotates very slowly and $|\omega|$ is very small as expected.

4&5 as α increases the ω's affected as ω_1 and ω_2 since the rotor axes approaches the m_1-m_2 plane this is noticed by the change in sign of ω_1 & ω_2. As α increases the largest effect is on ω_1, since large changes in α means that the rotor axis has a major component of itself ∥ to the m_1 axis. We note that the trend in the results show that the axis upon which the major component of the rotor axis lies will always have a positive component of ω associated with it. This is also true with β changes. Here the rotor's axis makes its major projection on the m_3 axis when $\beta = 90°$ hence sign of ω_3 = sign of ${}^{A}\omega^{B}$

6. If we keep everything else fixed and change the length of the side $\underline{\underline{l_2}}$ only we note that when l_2 ↑ to 1 Each value of ω_i slows down & speeds up in cycles with max. of speed up for each cycle increasing with time. ω_2 components changes direction cyclicly. When $l_2 > 1$ ω_1 slows down & speeds up with larger periods

as c gets larger that the motion is more stable & much less in magnitude than that of b getting larger. Reason. since c is generally in direction of axis it takes much less to move body B. In case where B is much larger, B is ⊥ generally to axis of rotor ∴ must increase ω in order to move body to stay in equilib.





Instabilities die out quicker in case where c is larger

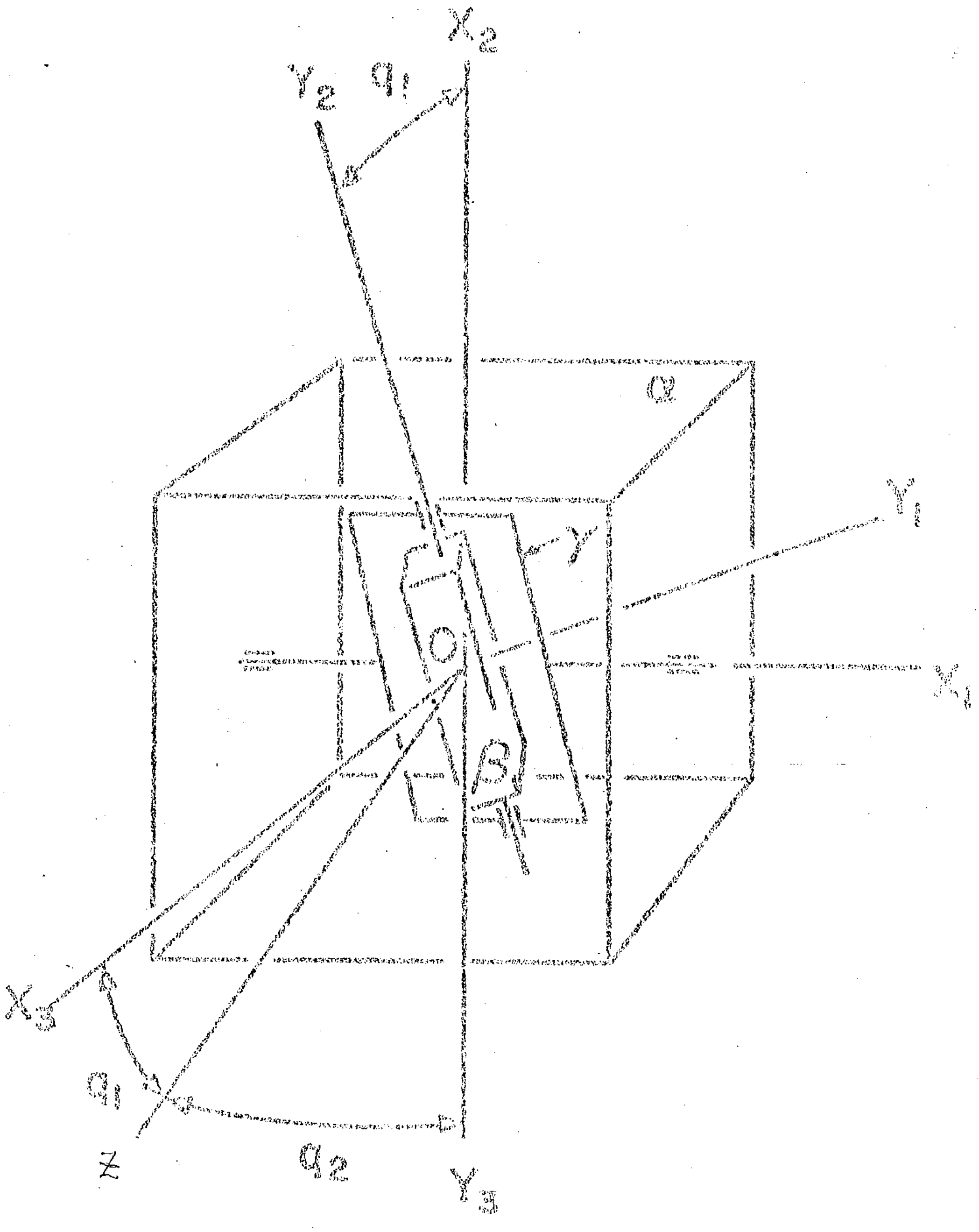
d. as d ↓, ω's decrease & effect of rotor decrease.

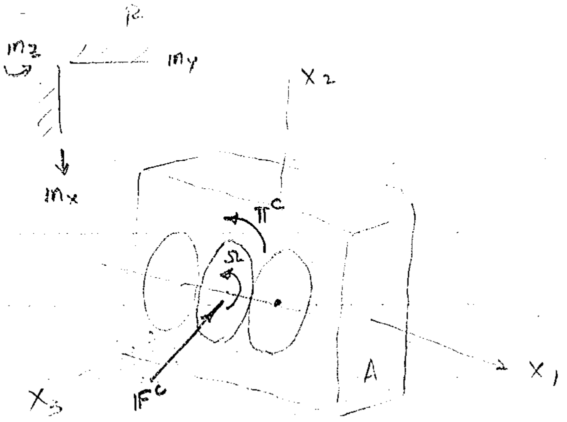
Thus for a stable system, make d small and elongate c_A over the other 2. (if direction of axis is kept fixed)
decrease μ and T/J1

The attached sketch shows a system S composed of two rigid bodies, α and β, which are connected to each other by means of a light gimbal γ. Point O is the mass center of α as well as β; X_1, X_2, X_3 are principal axes of α; Y_1, Y_2, Y_3 are principal axes of β; and γ can rotate relative to α and β only about X_1 and Y_2, respectively, so that the relative orientation of α and β depends solely on the angles q_1 and q_2. The central principal moments of inertia of α and β have the values A_1, A_2, A_3 and B_1, B_2, B_3, respectively.

When $q_1 = q_2 = 0$, the central principal axes of S coincide with those of α and β. Letting Z be the central principal axis of S that coincides with X_3 and Y_3 when $q_1 = q_2 = 0$, and taking $A_1 = 4$, $A_2 = 2$, $A_3 = 3$, $B_1 = 1.5$, $B_2 = 3$ kg cm^2, find a value of B_3 such that the moment of inertia of S about Z has a (local) minimum value when $q_1 = q_2 = 0$.

<u>Note:</u> $f(x,y)$ has a local minimum at $x = y = 0$ if $f_x = f_y = 0$, $f_{xx} > 0$, and $f_{xx}f_{yy} - (f_{xy})^2 > 0$, where subscripts denote partial differentiations and all derivatives are evaluated at $x = y = 0$.


X2
Y2
q1
a
Y1
γ
O
X1
β
X3
q1
Z
q2
Y3



this is a 7 degree of freedom system.

6 to describe motion of body A in R

1 to describe motion of body C in A

motions of B_1 and B_2 are known since their motion depend on motion of C

Assume That $\Pi^C = T_1\mathbf{n}_1 + T_2\mathbf{n}_2 + T_3\mathbf{n}_3$

~~$\mathbf{F}^C = \mathcal{F}_1\mathbf{n}_1 + \mathcal{F}_2\mathbf{n}_2 + \mathcal{F}_3\mathbf{n}_3$~~

$${}^A\alpha^C = \dot{\Omega}\mathbf{n}_3 = \dot{u}_7\mathbf{n}_3$$

$$\begin{aligned}{}^R\omega^A &= \omega_1\mathbf{n}_1 + \omega_2\mathbf{n}_2 + \omega_3\mathbf{n}_3\\ &= u_4\mathbf{n}_1 + u_5\mathbf{n}_2 + u_6\mathbf{n}_3\end{aligned} \qquad \begin{aligned}{}^A\omega^C &= \Omega\mathbf{n}_3\\ &= u_7\mathbf{n}_3\end{aligned} \qquad \begin{aligned}{}^A\omega^{B_1} = {}^A\omega^{B_2} &= -\Omega\mathbf{n}_3\\ &= -u_7\mathbf{n}_3\end{aligned}$$

Assume that C^* and A^* are same and rotors have radii R

$$u_1\mathbf{n}_1 + u_2\mathbf{n}_2 + u_3\mathbf{n}_3 = \mathbf{V}^{A^*} = v_1\mathbf{n}_1 + v_2\mathbf{n}_2 + v_3\mathbf{n}_3 = \mathbf{V}^{C^*}$$

$$u_1\mathbf{n}_1 + u_2\mathbf{n}_2 + u_3\mathbf{n}_3 = \mathbf{V}^{C^*} = v_1\mathbf{n}_1 + v_2\mathbf{n}_2 + v_3\mathbf{n}_3$$

$$\mathbf{V}^{C^*} + 2r[u_6\mathbf{n}_2 - u_5\mathbf{n}_3] = \mathbf{V}^{B_1^*} = \mathbf{V}^{C^*} + 2r[\omega_3\mathbf{n}_2 - \omega_2\mathbf{n}_3]$$

$$\mathbf{V}^{C^*} + 2r[-u_6\mathbf{n}_2 + u_5\mathbf{n}_3] = \mathbf{V}^{B_2^*} = \mathbf{V}^{C^*} + 2r[-\omega_3\mathbf{n}_2 + \omega_2\mathbf{n}_3]$$

$$\mathbf{r}^{B_1^*} = \mathbf{r}^{C^*} + 2r\,\mathbf{n}_1$$

$${}^R\mathbf{V}^{B_1^*} = {}^R\mathbf{V}^{C^*} + 2r\,\frac{{}^R d\mathbf{n}_1}{dt} \qquad \frac{{}^R d\mathbf{n}_1}{dt} = \overset{0}{\cancel{\frac{{}^A d\mathbf{n}_1}{dt}}} + {}^R\omega^A \times \mathbf{n}_1$$

$$= \mathbf{V}^{C^*} + 2r\,{}^R\omega^A \times \mathbf{n}_1 = \mathbf{V}^{C^*} + 2r[-\omega_2\mathbf{n}_3 + \omega_3\mathbf{n}_2]$$

$$\mathbf{V}^{B_2^*} = \mathbf{V}^{C^*} - 2r\,{}^R\omega^A \times \mathbf{n}_1 = \mathbf{V}^{C^*} + 2r[\omega_2\mathbf{n}_3 - \omega_3\mathbf{n}_2]$$

Now let the generalized coordinates of entire system be $v_i (i=1,2,3)$, ω_i $(i=1,2,3)$ and Ω

now u_1, u_2, u_3 equations of motion will decouple from rest.

$r=1,2,3$ $$F_r = \mathbf{V}^A_{u_r}\cdot\mathbf{F}^A + \mathbf{V}^C_{u_r}\cdot\mathbf{F}^C + \mathbf{V}^{B_1}_{u_r}\cdot\mathbf{F}^{B_1} + \mathbf{V}^{B_2}_{u_r}\cdot\mathbf{F}^{B_2} + {}^R\omega^A_{u_r}\cdot\Pi^A + {}^R\omega^C_{u_r}\cdot\Pi^C + {}^R\omega^{B_1}_{u_r}\cdot\Pi^{B_1} + {}^R\omega^{B_2}_{u_r}\cdot\Pi^{B_2}$$

since only body & contact forces acting are gravitational forces $\mathbf{F}_B^{B_2} = \mathbf{F}_B^{B_1} = \mathbf{F}_B^C = mg\mathbf{n}_x$ $\mathbf{F}_B^A = Mg\mathbf{n}_x$

the contact forces between the rotors contribute nothing

$$\mathbf{F}^A = Mg\mathbf{n}_x - \mathcal{F}_i\mathbf{n}_i \quad \mathbf{F}^{B_1} = \mathbf{F}^{B_2} = mg\mathbf{n}_x \quad \mathbf{F}^C = mg\mathbf{n}_x + \mathcal{F}_i\mathbf{n}_i$$

$$\Pi^C = -\Pi^A \quad \Pi^{B_1} = \Pi^{B_2} = 0$$

let
$u_i = v_i \qquad (i=1,2,3)$
$u_i = \omega_{i-3} \qquad (i=4,5,6)$
$u_i = \Omega \qquad i=7$

$$F_1 = \mathbf{n}_1\cdot\mathbf{F}^A + \mathbf{n}_1\cdot\mathbf{F}^C + \mathbf{n}_1\cdot\mathbf{F}^{B_1} + \mathbf{n}_1\cdot\mathbf{F}^{B_2} + 0 + 0 + 0 + 0$$ since $\omega^{(A,C,B_1,B_2)}_{\cdot\cdot} \neq fn(u_1, u_2, u_3)$

$$F_2 = \mathbf{n}_2\cdot\mathbf{F}^A + \mathbf{n}_2\cdot\mathbf{F}^C + \mathbf{n}_2\cdot\mathbf{F}^{B_1} + \mathbf{n}_2\cdot\mathbf{F}^{B_2} + 0 + 0 + 0 + 0$$

$$F_3 = \mathbf{n}_3\cdot\mathbf{F}^A + \mathbf{n}_3\cdot\mathbf{F}^C + \mathbf{n}_3\cdot\mathbf{F}^{B_1} + \mathbf{n}_3\cdot\mathbf{F}^{B_2} + 0 + 0 + 0 + 0$$

$$F_4 = 0 + 0 + 0 + 0 + \cancel{\mathbf{n}_1\cdot\Pi^A} + \cancel{\mathbf{n}_1\cdot\Pi^C} + \cancel{\mathbf{n}_1\cdot\Pi^{B_1}} + \cancel{\mathbf{n}_1\cdot\Pi^{B_2}}$$ since $\mathbf{V}^{(A,C,B_1,B_2)}_{\cdot\cdot} \neq fn\ of\ u_4$

$$F_5 = 0 + 0 - 2r\mathbf{n}_3\cdot\mathbf{F}^{B_1} + 2r\mathbf{n}_3\cdot\mathbf{F}^{B_2} + \cancel{\mathbf{n}_2\cdot\Pi^A} + \cancel{\mathbf{n}_2\cdot\Pi^C} + \cancel{\mathbf{n}_2\cdot\Pi^{B_1}} + \cancel{\mathbf{n}_2\cdot\Pi^{B_2}}$$

$$F_6 = 0 + 0 + 2r\mathbf{n}_2\cdot\mathbf{F}^{B_1} - 2r\mathbf{n}_3\cdot\mathbf{F}^{B_2} + \cancel{\mathbf{n}_3\cdot\Pi^A} + \cancel{\mathbf{n}_3\cdot\Pi^C} + \cancel{\mathbf{n}_3\cdot\Pi^{B_1}} + \cancel{\mathbf{n}_3\cdot\Pi^{B_2}}$$

$F_7 = 0 + 0 + 0 + 0 + 0 + m_3 \cdot \mathbb{T}^{C} - m_3 \cdot \mathbb{T}^{B_1} - m_3 \cdot \mathbb{T}^{B_2}$ since $\mathbb{V}_{u_7}^{(A,C,B_1,B_2)} \neq$ fn of u_7

Question : if motor exerts a force & a torque on C does it exert a reverse force & reverse torque to A? Assume yes $\therefore$ $\mathbb{T}^A = -\mathbb{T}^C$ and $\mathbb{F}^A = \mathbb{F}_B^A - \mathbb{F}_T^C$ $\mathbb{F}^C = \mathbb{F}_B^C + \mathbb{F}_T^C$

$\therefore \quad F_1 = m_1 \cdot [\mathbb{F}_B^A + \mathbb{F}_B^C + 2\mathbb{F}^{B_1}] = m_1 \cdot n_x [(M+3m)g]$

$F_2 = m_2 \cdot [\mathbb{F}_B^A + \mathbb{F}_B^C + 2\mathbb{F}^{B_1}] = m_2 \cdot n_x [(M+3m)g]$

$F_3 = m_3 \cdot [\mathbb{F}_B^A + \mathbb{F}_B^C + 2\mathbb{F}^{B_1}] = m_3 \cdot n_x [(M+3m)g]$

$F_4 = 0$

$F_5 = 0$

$F_6 = 0$

$F_7 = T_3$

To obtain $F_r^* = \mathbb{V}_{u_r}^{A^*} \cdot \mathbb{F}^{A^*} + \mathbb{V}_{u_r}^{C^*} \cdot \mathbb{F}^{C^*} + \mathbb{V}_{u_r}^{B_1^*} \cdot \mathbb{F}^{B_1^*} \cdot \mathbb{V}_{u_r}^{B_2^*} \cdot \mathbb{F}^{B_2^*} + \omega_{u_r}^{A} \cdot \mathbb{T}^{*A} + \omega_{u_r}^{C} \cdot \mathbb{T}^{*C} + \omega_{u_r}^{B_1} \cdot \mathbb{T}^{*B_1} + \omega_{u_r}^{B_2} \cdot \mathbb{T}$

$F_1^* = m_1 \cdot (-M\mathbf{a}^{A^*}) + m_1 \cdot (-m\mathbf{a}^{C^*}) + m_1 \cdot (-m\mathbf{a}^{B_1^*}) + m_1 \cdot (-m\mathbf{a}^{B_2^*}) + 0 + 0 + 0 + 0$ since $\omega_{u_1}^{(A,C,B_1,B_2)} \neq fn(u_1)$

$m_1 \cdot \mathbf{a}^{A^*} = m_1 \frac{{}^R d\mathbb{V}}{dt}\ m_1 \frac{{}^A d\mathbb{V}}{dt} + m_1 \cdot {}^R\omega^A \times \mathbb{V}^{A^*} = \dot{u}_1 + m_1 \cdot \overset{u_5u_3 - u_6u_2}{[u_4 m_1 + u_5 m_2 + u_6 m_3] \times [u_1 m_1 + u_2 m_2 + u_3 m_3]}$

$m_1 \cdot \mathbf{a}^{C^*} = m_1 \cdot \frac{d\mathbb{V}^{C^*}}{dt} = m_1 \cdot \frac{{}^A d\mathbb{V}^C}{dt} + m_1 \cdot {}^R\omega^A \times \mathbb{V}^{C^*} = \dot{u}_1 + u_5u_3 - u_6u_2$

$m_1 \cdot \mathbf{a}^{B_1^*} = m_1 \cdot \frac{d\mathbb{V}^{B_1^*}}{dt} = m_1 \frac{{}^A d\mathbb{V}^{B_1}}{dt} + m_1 \cdot {}^R\omega^A \times \mathbb{V}^{B_1} = \dot{u}_1 - 2r[u_5^2 + u_6^2]$ [illegible] $[u_5u_3 - u_6u_2]$

$m_1 \cdot \mathbf{a}^{B_2^*} = m_1 \cdot \frac{d\mathbb{V}^{B_2^*}}{dt} = m_1 \cdot \frac{{}^A d\mathbb{V}^{B_1}}{dt} + m_1 \cdot {}^R\omega^A \times \mathbb{V}^{B_2} = \dot{u}_1 + 2r[u_5^2 + u_6^2]$ [illegible] $[u_5u_3 - u_6u_2]$

$\boxed{F_1^* = -(M+3m)\dot{u}_1 - (M+m)(u_5u_3 - u_6u_2)} = -\cancel{(M+3m)[\dot{u}_1 + u_5u_3 - u_6u_2]}$ ✓

$F_2^* = m_2 \cdot (-M\mathbf{a}^{A^*} - m\mathbf{a}^{C^*} - m\mathbf{a}^{B_1^*} - m\mathbf{a}^{B_2^*})$ since $\omega_{u_2}^{(A,C,B,B_2)} \neq f_n(u_2)$

$= \boxed{-(M+3m)\dot{u}_2 - (M+m)(u_6u_1 - u_4u_3)} \neq -(M+3m)\cancel{[\dot{u}_2 + u_6u_1 - u_4u_3]}$ ✓

$F_3^* = m_3\ (-M\mathbf{a}^{A^*} - m\mathbf{a}^{C^*} - m\mathbf{a}^{B_1^*} - m\mathbf{a}^{B_2^*})$

$F_3^* = -(M+3m)\dot{u}_2 - (M+m)(u_4u_2 - u_5u_1) = -\cancel{(M+3m)[\dot{u}_3 + u_4u_2 - u_5u_1]}$

$F_4^* = 0 + 0 + 0 + 0 + m_1 \cdot \mathbb{T}^{*A} + m_1 \cdot \mathbb{T}^{*C} + m_1 \cdot \mathbb{T}^{*B_1} + m_1 \cdot \mathbb{T}^{*B_2}$ since $\mathbb{V}^{(A,B,C,D)} \neq f(u_4)$

$\boxed{F_4^* = m_1 \cdot [\mathbb{T}^{*A} + \mathbb{T}^{*C} + \mathbb{T}^{*B_1} + \mathbb{T}^{*B_2}]}$

$F_5^* = 0 + 0 - 2rm_3 \cdot [-m\mathbf{a}^{*B_1}] + 2rm_3[-m\mathbf{a}^{*B_2}] + m_2 \cdot [\mathbb{T}^{*A} + \mathbb{T}^{*C} + \mathbb{T}^{*B_1} + \mathbb{T}^{*B_2}]$

$= +2rm[m_3 \cdot (\mathbf{a}^{*B_1} - \mathbf{a}^{*B_2})] + m_2 \cdot [\qquad\qquad]$

$\boxed{F_5^* = 2rm[-4r\dot{u}_5 + 4ru_4u_6] + m_2 \cdot [\mathbb{T}^{*A} + \mathbb{T}^{*C} + \mathbb{T}^{*B_1} + \mathbb{T}^{*B_2}]}$

$$a^{B_1,B_2} = {}^{A}\frac{dv^{C}}{dt} + {}^{R}\omega^{A}\times v^{C} \pm 2r\,{}^{A}\frac{d}{dt}{}^{R}\omega^{A}\times n_1 \pm 2r\,{}^{R}\omega^{A}\times({}^{R}\omega^{A}\times n_1)$$

$({}^{R}\omega^{A}\cdot n_1)\,{}^{R}\omega^{A} - {}^{R}\omega^{A\,2}\,n_1 \leftarrow$ CHECK THIS

$$a_1^{B_1} - a_2^{B_2} = 4r\,\frac{{}^{A}d\,{}^{R}\omega^{A}}{dt}\times n_1 + 4r\left[({}^{R}\omega^{A}\cdot n_1)\,{}^{R}\omega^{A} - {}^{R}\omega^{A\,2}\,n_1\right]$$

$$F_6^* = 0 + 0 + 2r\,n_2\cdot[-m a^{B_1}] - 2r\,n_2[-m a^{B_2}] + n_3\cdot\left[\Pi^{*A} + \Pi^{*B} + \Pi^{*C} + \Pi^{*D}\right]$$

$$= -2rm\left[3(a^{B_1} - a^{B_2})\right] + n_3\cdot\left[\right.$$

$$\boxed{F_6^* = -2rm\left[4r\dot u_6 + 4r u_4 u_5\right] + n_3\cdot\left[\Pi^{*A} + \Pi^{*B} + \Pi^{*C} + \Pi^{*D}\right]}$$

$F_7^* = 0 + 0 + 0 + 0 + 0 + n_3\cdot\Pi^{C*} - n_3\cdot\Pi^{B_1^*} - n_3\cdot\Pi^{B_2^*}$ since $v^{(A,B_1,C,B_2)} \neq f(u_7)$ & $\omega^{A} \neq f(u_7)$

Since C, B_1, B_2 have simple angular velocities in reference frame A let us rewrite

~~$\Pi^{C*} = -{}^{R}\frac{d}{dt}\mathbb{H}^{C/C*} = \underbrace{-{}^{A}\frac{d}{dt}\mathbb{H}^{C/C*}}_{\text{not}} - {}^{R}\omega^{A}\times\mathbb{H}^{C/C*} \neq {}^{A}\Pi^{C*} + \mathbb{H}^{C/C*}\times{}^{R}\omega^{A} = {}^{A}\Pi^{C*} + (\mathbb{I}^{C/C*}\cdot{}^{R}\omega^{C})\times{}^{R}\omega^{A}$~~

~~$n_3\cdot\Pi^{C*} = n_3\cdot{}^{A}\Pi^{C*} + n_3\cdot(\mathbb{I}^{C/C*}\cdot{}^{R}\omega^{C})\times{}^{R}\omega^{A}$~~ ; $n_3\cdot(\mathbb{I}^{C/C*}\cdot{}^{A}\omega^{C})\times{}^{A}\omega^{C} - n_3\cdot\mathbb{I}^{C/C*}\cdot{}^{A}\alpha^{C} = -J\dot u_7$

$J u_7 n_3 \times u_7 n_3 \quad n_3 J n_3 n_3\cdot \dot u_7 n_3$

~~$= -J\dot\Omega + n_3\cdot(\mathbb{I}^{C/C*}\cdot{}^{R}\omega^{A})\times{}^{R}\omega^{A} + n_3\cdot(\mathbb{I}^{C/C*}\cdot{}^{A}\omega^{C})\times{}^{R}\omega^{A}$~~ $\quad \mathbb{I}^{C/C*} = \mathbb{I}^{B/B_1^*} = \mathbb{I}^{B/B_2^*}$

$n_3\cdot J\Omega n_3 \times {}^{R}\omega^{A} =$

~~$= -J\dot\Omega + 0 + 0$~~

~~$\Pi^{B_1^*} = {}^{A}\Pi^{B/B_1^*} + (\mathbb{I}^{B/B_1^*}\cdot{}^{R}\omega^{B_1})\times{}^{R}\omega^{A} = {}^{A}\Pi^{B/B_1^*} + (\mathbb{I}^{B/B_1^*}\cdot{}^{R}\omega^{A})\times{}^{R}\omega^{A} + (\mathbb{I}^{B/B_1^*}\cdot{}^{A}\omega^{B})\times{}^{R}\omega^{A}$~~

$-J\Omega n_3$

~~$n_3\cdot\Pi^{B/B_1^*} = +J\dot\Omega + 0 + 0$~~

~~$n_3\cdot\Pi^{B/B_2^*} = J\dot\Omega$ by same token.~~

~~$\therefore \boxed{F_7^* = -3J\dot\Omega}$~~

$\mathbb{I}^{C/C*}\cdot{}^{R}\omega^{C} = \left[J n_3 n_3 + K(n_1 n_1 + n_2 n_2)\right]\cdot\left[u_4 n_1 + u_5 n_2 + (u_6 + u_7) n_3\right]$

$(\mathbb{I}^{C/C*}\cdot{}^{R}\omega^{C})\times{}^{R}\omega^{A} = \left[J(u_6+u_7) n_3 + K(u_4 n_1 + u_5 n_2)\right]\times(u_4 n_1 + u_5 n_2 + u$

$= J(u_6+u_7)u_4 n_2 - J(u_6+u_7)u_5 n_1$

$+ K[-u_5 u_4 n_3 + u_4 u_5 n_3 - u_4 u_6 n_2$

$$\Pi^{C*} = -{}^{R}\frac{d}{dt}\mathbb{H}^{C/C*} = -{}^{R}\frac{d}{dt}(\mathbb{I}^{C/C*}\cdot{}^{R}\omega^{C}) = -{}^{R}\frac{d}{dt}(\mathbb{I}^{C/C*}\cdot{}^{R}\omega^{A}) - {}^{R}\frac{d}{dt}(\mathbb{I}\cdot{}^{A}\omega^{C})$$

$$-{}^{A}\frac{d}{dt}(\mathbb{I}\cdot{}^{A}\omega^{C}) - {}^{R}\omega^{A}\times(\mathbb{I}^{C/C*}\cdot{}^{A}\omega^{C})$$

$$= (\mathbb{I}^{C/C*}\cdot{}^{R}\omega^{A})\times{}^{R}\omega^{A} - \mathbb{I}^{C/C*}\cdot{}^{R}\alpha^{A} + {}^{A}\Pi^{C*} + {}^{A}\mathbb{H}^{C/C*}\times{}^{R}\omega^{A}$$

$$n_3\cdot\Pi^{C*} = -J n_3\cdot{}^{R}\alpha^{A} + n_3\cdot{}^{A}\Pi^{C*} = -J\dot u_6 - J\dot u_7 = -J(\dot u_6 + \dot u_7)$$

$n_3\cdot\Pi^{B_1^*} = -J(\dot u_6 - \dot u_7) = n_3\cdot\Pi^{B_2^*} \qquad n_3\cdot\Pi^{B_1^*} = \left(-\mathbb{I}^{B/B_1^*}\cdot{}^{R}\alpha^{A} + {}^{A}\Pi^{B_1^*}\right)\cdot n_3 = -J\dot u_6 + J\dot u_7$

$$F_7^* = -J(\dot u_6 + \dot u_7) + 2J(\dot u_6 - \dot u_7) = -3J\dot u_7 + J\dot u_6 \qquad \boxed{F_7^* = -3J\dot u_7 + J\dot u_6}$$

$\mathbb{V}^{A*} = \mathbb{V}^{C*} = u_1 \mathbf{m}_1 + u_2 \mathbf{m}_2 + u_3 \mathbf{m}_3$ $\qquad {}^{R}\omega^{A} = u_4 \mathbf{m}_1 + u_5 \mathbf{m}_2 + u_6 \mathbf{m}_3$

$\mathbb{V}^{B_1*} = \mathbb{V}^{C*} + \frac{d}{dt} 2r\mathbf{m}_1 = \mathbb{V}^{C*} + 2r\, {}^{R}\omega^{A} \times \mathbf{m}_1 = \mathbb{V}^{C*} + 2r\,(u_6 \mathbf{m}_2 - u_5 \mathbf{m}_3)$; $\mathbb{V}^{B_2} = \mathbb{V}^{C} - 2r\,(\underset{\omega \times \mathbb{V}}{u_6 \mathbf{m}_2 - u_5 \mathbf{m}_3})$

$\mathbf{a}^{B_1*} = \frac{{}^{R}d\mathbb{V}^{B_1*}}{dt} = \frac{{}^{A}d\mathbb{V}^{B_1*}}{dt} + {}^{R}\omega^{A} \times \mathbb{V}^{B_1*} = \dot{u}_1 \mathbf{m}_1 + \dot{u}_2 \mathbf{m}_2 + \dot{u}_3 \mathbf{m}_3 + 2r(\dot{u}_6 \mathbf{m}_2 - \dot{u}_5 \mathbf{m}_3) + {}^{R}\omega^{A} \times \mathbb{V}^{C} + {}^{R}\omega^{A} \times 2r\,({}^{R}\omega^{A} \times \mathbf{m}_1)$

$+2r\left[\underset{u_4}{(\omega^{A} \cdot \mathbf{m}_1)}\, \omega^{A} - \omega^{A2} \mathbf{m}_1\right]$

$=$

$\mathbf{a}^{B_2*} = \qquad = \dot{u}_1 \mathbf{m}_1 + \dot{u}_2 \mathbf{m}_2 + \dot{u}_3 \mathbf{m}_3 - 2r(\dot{u}_6 \mathbf{m}_2 - \dot{u}_5 \mathbf{m}_3) + {}^{R}\omega^{A} \times \mathbb{V}^{C} - 2r\, {}^{R}\omega^{A} \times ({}^{R}\omega^{A} \times \mathbf{m}_1)$

$\mathbf{a}^{A*} = \mathbf{a}^{C*} = \dot{u}_1 \mathbf{m}_1 + \dot{u}_2 \mathbf{m}_2 + \dot{u}_3 \mathbf{m}_3 + {}^{R}\omega^{A} \times \mathbb{V}^{C} = \left[(u_5 u_3 - u_6 u_2)\mathbf{m}_1 + (u_6 u_1 - u_4 u_3)\mathbf{m}_2 + (u_4 u_2 - u_5 u_1)\mathbf{m}_3\right]$

Now $\quad -M[\mathbf{m}_1 \cdot \mathbf{a}^{A}] - m[\mathbf{m}_1 \cdot \mathbf{a}^{C} + \mathbf{m}_1 \cdot \mathbf{a}^{B_1} + \mathbf{m}_1 \cdot \mathbf{a}^{B_2}]$

$-M\dot{u}_1 - M(u_5 u_3 - u_6 u_2) - m[\dot{u}_1 + (u_5 u_3 - u_6 u_2)] - m[\dot{u}_1 + (u_5 u_3 - u_6 u_2) + 2r(\cancel{u_4^2} - \cancel{u_4^2} - u_5^2 - u_6^2)]$

$-m[\dot{u}_1 + (u_5 u_3 - u_6 u_2) - 2r[\cancel{u_4^2} - \cancel{u_4^2} - u_5^2 - u_6^2]]$

$-(M+3m)(\dot{u}_1 + u_5 u_3 - u_6 u_2) - m[\cancel{2r(-u_5^2 - u_6^2)} - \cancel{2r(-u_5^2 - u_6^2)}]$

$-M[\mathbf{m}_2 \cdot \mathbf{a}^{A}] - m[\mathbf{m}_2 \cdot \mathbf{a}^{C} + \mathbf{m}_2 \cdot \mathbf{a}^{B_1} + \mathbf{m}_2 \cdot \mathbf{a}^{B_2}]$

$= -M[\dot{u}_2 + (u_6 u_1 - u_4 u_3)] - m[\dot{u}_2 + (u_6 u_1 - u_4 u_3) + \dot{u}_2 + \cancel{2r\dot{u}_6} + (u_6 u_1 - u_4 u_3) + \cancel{2r u_4 u_5} + \dot{u}_2 - \cancel{2r\dot{u}_6}$

$+ (u_6 u_1 - u_4 u_3) - \cancel{2r u_4 u_5}] = -(M+3m)[\dot{u}_2 + (u_6 u_1 - u_4 u_3)]$

$-2r\mathbf{m}_3 \cdot \mathbb{F}^{B_1*} + 2r\mathbf{m}_3 \cdot \mathbb{F}^{B_2*} = -2r\mathbf{m}_3 \cdot (-m\mathbf{a}^{B_1*}) + 2r\mathbf{m}_3 \cdot (-m\mathbf{a}^{B_2*})$

$= 2rm[\cancel{\dot{u}_3} - 2r\dot{u}_5 + (\cancel{u_4 u_2 - u_5 u_1}) + 2r u_4 u_6] - 2rm[\cancel{\dot{u}_3} + 2r\dot{u}_5$

$+ (u_4 u_2 - u_5 u_1) - 2r u_4 u_6] = 2rm[-4r\dot{u}_5 + 4r u_4 u_6]$

$-2rm[\mathbf{m}_2 \cdot \mathbf{a}^{B_1}] + 2rm[\mathbf{m}_2 \cdot \mathbf{a}^{B_2}] = -2rm[\cancel{\dot{u}_2} + 2r\dot{u}_6 + \cancel{(u_6 u_1 - u_4 u_3)} + 2r u_4 u_5] + 2rm[\cancel{\dot{u}_2} - 2r\dot{u}_6$

$+ \cancel{(u_6 u_1 - u_4 u_3)} - 2r u_4 u_5] = -2rm[4r\dot{u}_6 + 4r u_4 u_5]$

Now K_1, K_2, K_3 are the moments of inertia of entire system about x_1, x_2, x_3

$\mathbb{T}^{A} + \mathbb{T}^{B_1} + \mathbb{T}^{C} + \mathbb{T}^{B_2} = -\frac{{}^{R}d}{dt}\left[\mathbb{H}^{A/A*} + \mathbb{H}^{C/C*} + \mathbb{H}^{B_1/B_1*} + \mathbb{H}^{B_2/B_2*}\right]$

$= -\frac{d}{dt}\left[\mathbb{I}^{A/A*} \cdot {}^{R}\omega^{A} + \mathbb{I}^{C/C*} \cdot {}^{R}\omega^{C} + \mathbb{I}^{B/B_1} \cdot {}^{R}\omega^{B_1} + \mathbb{I}^{B/B_2} \cdot {}^{R}\omega^{B_2}\right]$

$\left[(\mathbb{I}^{A/A*} + \mathbb{I}^{C/C*} + \mathbb{I}^{B/B_1} + \mathbb{I}^{B/B_2}) \cdot {}^{R}\omega^{A} + \mathbb{I}^{C/C*} \cdot {}^{A}\omega^{C} + \mathbb{I}^{B/B_1} \cdot {}^{A}\omega^{B_1} + \mathbb{I}^{B/B_2} \cdot {}^{A}\omega^{B_2}\right]$

Now $\mathbb{I}^{B/A*} = \mathbb{I}^{B/B_1} + \mathbb{I}^{B_1*/A_1*}$ $\qquad$ Now ${}^{A}\omega^{C} = -{}^{A}\omega^{B_1} = -{}^{A}\omega^{B_2}$ & $\mathbb{I}^{C/C*} = \mathbb{I}^{B/B_1*} = \mathbb{I}^{B/B_2*}$

$\therefore \quad (\mathbb{I}^{A/A*} + \mathbb{I}^{C/A*} + \mathbb{I}^{B_1*/A*} + \mathbb{I}^{B/A*} + \mathbb{I}^{B_2*/A*} + \mathbb{I}^{B_2/A*}) = \mathbb{I}^{(A+C+B_1+B_2)/A*} - \mathbb{I}^{B_1*/A*} - \mathbb{I}^{B_2/A*}$

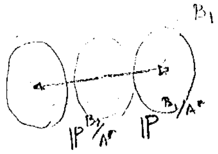
$I^{C/C}\cdot{}^{A}\omega^{C} + 2I^{B/B_1}\cdot - {}^{A}\omega^{C}$

$= {}^{A}\mathbb{H}^{C}$ ✓

$$\therefore\ \mathbb{T}^{A}+\mathbb{T}^{B}+\mathbb{T}^{C}+\mathbb{T}^{D} = -\frac{{}^{R}d}{dt}\left\{\mathbb{I}^{D/D^*}\cdot{}^{R}\omega^{A} - \left(\mathbb{I}^{B_1^*/A^*}+\mathbb{I}^{B_2^*/A^*}\right)\cdot{}^{R}\omega^{A} - \mathbb{I}^{C/C^*}\cdot{}^{A}\omega^{C}\right\}$$

$${}^{R}\mathbb{T}^{D} - {}^{A}\mathbb{T}^{C} + \frac{d}{dt}\left(8r^2m\left[\mathbf{n}_2\mathbf{n}_2+\mathbf{n}_3\mathbf{n}_3\right]\cdot{}^{R}\omega^{A}\right) + {}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C^*}$$

only nonzero terms are $\mathbf{n}_2\mathbf{n}_2$ $\mathbf{n}_3\mathbf{n}_3$



$\mathbb{P}^{B_1^*/A^*} = 2r\mathbf{n}_1 \quad \mathbb{P}^2 = 4r^2$

$$\mathbb{I}^{B_1^*/A^*} = m\left(\mathbb{P}^2\,\mathbb{U} - \mathbb{P}\mathbb{P}\right)$$

$$\mathbb{I}^{B_1^*/A^*} = m\left(4r^2\mathbb{U} - 4r^2\mathbf{n}_1\mathbf{n}_1\right) = m\cdot 4r^2\left(\mathbf{n}_2\mathbf{n}_2+\mathbf{n}_3\mathbf{n}_3\right)$$

$$\mathbb{I}^{B_2^*/A^*} = m\left(4r^2\mathbb{U} - 4r^2\mathbf{n}_1\mathbf{n}_1\right) \quad ; \quad \mathbb{P}^{B_2^*/A^*} = -2r\mathbf{n}_1 \quad \mathbb{P}^2 = 4r^2$$

$$= 4r^2m\left(\mathbf{n}_2\mathbf{n}_2+\mathbf{n}_3\mathbf{n}_3\right)$$

$+{}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C^*}$

$$= {}^{R}\mathbb{T}^{D} - {}^{A}\mathbb{T}^{C} + 8r^2m\frac{{}^{R}d}{dt}\left[\overset{\mathbf{w}}{u_5\mathbf{n}_2 + u_6\mathbf{n}_3}\right] = {}^{R}\mathbb{T}^{D} - {}^{A}\mathbb{T}^{C} + 8r^2m\left\{\left(\overset{\checkmark}{\dot u_5\mathbf{n}_2+\dot u_6\mathbf{n}_3}\right) + {}^{R}\omega^{A}\times\mathbf{w}\right\}$$

$+{}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C^*}$

$$\text{now}\quad \mathbf{w} = {}^{R}\omega^{A} - u_4\mathbf{n}_1 \qquad {}^{R}\omega^{A}\times\mathbf{w} = \left(\cancel{{}^{R}\omega^{A}\times{}^{R}\omega^{A}}^{\,0} - {}^{R}\omega^{A}\times u_4\mathbf{n}_1\right) = -\left(u_4\mathbf{n}_1+u_5\mathbf{n}_2+u_6\mathbf{n}_3\right)\times u_4\mathbf{n}_1$$

$$= +u_5u_4\mathbf{n}_3 - u_6u_4\mathbf{n}_2$$

$$\therefore\ \mathbb{T}^{A}+\mathbb{T}^{B}+\mathbb{T}^{C}+\mathbb{T}^{D} = {}^{R}\mathbb{T}^{D} - {}^{A}\mathbb{T}^{C} + 8r^2m\left[\left(\dot u_5 - u_6u_4\right)\mathbf{n}_2 + \left(\dot u_6+u_5u_4\right)\mathbf{n}_3\right] + {}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C^*}$$

${}^{A}\alpha^{C} = \dot u_7\mathbf{n}_3$

$$\therefore\ F_4^* = \mathbf{n}_1\cdot\left[\quad\right] = \mathbf{n}_1\cdot{}^{R}\mathbb{T}^{D} - \mathbf{n}_1\cdot{}^{A}\mathbb{T}^{C} + \mathbf{n}_1\cdot{}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C^*}$$

$$F_5^* = \cancel{-8r^2m\left[+\dot u_5 - u_4u_6\right]} + \mathbf{n}_2\cdot{}^{R}\mathbb{T}^{D} - \mathbf{n}_2\cdot{}^{A}\mathbb{T}^{C} \cancel{+ 8r^2m\left[\dot u_5 - u_6u_4\right]} + \mathbf{n}_2\cdot{}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C^*} = \mathbf{n}_2\cdot\left({}^{R}\mathbb{T}^{D} - {}^{A}\mathbb{T}^{C}\right) + \mathbf{n}_2\cdot{}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C^*}$$

$$F_6^* = \cancel{-8r^2m\left[\dot u_6 + u_4u_5\right]} + \mathbf{n}_3\cdot{}^{R}\mathbb{T}^{D} - \mathbf{n}_3\,{}^{A}\mathbb{T}^{C} \cancel{+ 8r^2m\left[\dot u_6 + u_5u_4\right]} + \mathbf{n}_3\cdot{}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C^*} = \mathbf{n}_3\cdot\left({}^{R}\mathbb{T}^{D} - {}^{A}\mathbb{T}^{C}\right) + \mathbf{n}_3\cdot{}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C^*}$$

$$\text{now}\quad {}^{R}\mathbb{T}^{D} = \left(\mathbb{I}^{D/D^*}\cdot{}^{R}\omega^{A}\right)\times{}^{R}\omega^{A} - \mathbb{I}^{D/D^*}\cdot{}^{R}\alpha^{A}$$

$$\mathbb{I}^{D/D^*} = K_1\mathbf{n}_1\mathbf{n}_1 + K_2\mathbf{n}_2\mathbf{n}_2 + K_3\mathbf{n}_3\mathbf{n}_3$$

$${}^{R}\alpha^{A} = \frac{{}^{R}d\,{}^{R}\omega^{A}}{dt} = \frac{{}^{A}d}{dt}{}^{R}\omega^{A} + \cancel{{}^{R}\omega^{A}\times{}^{R}\omega^{A}} = \dot u_4\mathbf{n}_1 + \dot u_5\mathbf{n}_2 + \dot u_6\mathbf{n}_3$$

$$\mathbb{I}^{D/D}\cdot{}^{R}\alpha^{A} = K_1\dot u_4\mathbf{n}_1 + K_2\dot u_5\mathbf{n}_2 + K_3\dot{}$$

$$\mathbb{I}^{D/D^*}\cdot{}^{R}\omega^{A} = K_1u_4\mathbf{n}_1 + K_2u_5\mathbf{n}_2 + K_3u_6\mathbf{n}_3$$

Now

$${}^{R}\mathbb{T}^{D} \qquad \cancel{\left(\mathbb{I}^{D/D^*}\cdot{}^{R}\omega^{A}\right)}\times{}^{R}\omega^{A} = \Big[\left(K_2-K_3\right)u_5u_6 - K_1\dot u_4\Big)\mathbf{n}_1 + \left(\left(K_3-K_1\right)u_4u_6 - K_2\dot u_5\right)\mathbf{n}_2 +$$

$$\left(\left(K_1-K_2\right)u_4u_5 \qquad - K_3\dot u_6\right)\mathbf{n}_3$$

$${}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C^*} = {}^{R}\omega^{A}\times\left[Ju_7\mathbf{n}_3\right] = Ju_7\left(u_5\mathbf{n}_1 -\right.$$

$${}^{A}\mathbb{T}^{C} = \left(\underset{u_7J\mathbf{n}_3}{\mathbb{I}^{C/C^*}\cdot{}^{A}\omega^{C}}\right)\times\underset{u_7\mathbf{n}_3}{{}^{A}\omega^{C}} - \underset{J\mathbf{n}_3\mathbf{n}_3\cdot\dot u_7\mathbf{n}_3}{\mathbb{I}^{C/C^*}\cdot{}^{A}\alpha^{C}} = -J\dot u_7\mathbf{n}_3$$

$$F_4^* = \left(K_2-K_3\right)u_5u_6 - K_1\dot u_4 + Ju_7u_5$$

$$F_5^* = \left(K_3-K_1\right)u_4u_6 - K_2\dot u_5 - Ju_7u_4$$

$$F_6^* = \left(K_1-K_2\right)u_4u_5 - K_3\dot u_6 + J\dot u_7$$

$$F_7^* = -3J\dot u_7 + J\dot u_6$$

$$-3J\dot u_7 - J\dot u_6 = T$$

$$\dot u_7 + \frac{\dot u_6}{3} = -\frac{T}{3J}$$

$$\dot u_7 = -\frac{T}{3J} - \frac{\dot u_6}{3}$$

$$\dot u_4 = \frac{J}{K_1}u_7u_5 + \frac{K_2-K_3}{K_1}u_5u_6 \Rightarrow \boxed{\frac{J}{K_1}\Omega\,\omega_2 + \frac{K_2-K_3}{K_1}\omega_2\omega_3 = \dot\omega_1}$$

$$\dot u_5 = -\frac{Ju_7u_4}{K_2} + \frac{K_3-K_1}{K_2}u_4u_6 = \boxed{-\frac{J}{K_2}\Omega\,\omega_1 + \frac{K_3-K_1}{K_2}\omega_1\omega_3 = \dot\omega_2}$$

$$\dot u_6 = \frac{J\dot u_7}{K_3} + \frac{K_1-K_2}{K_3}u_4u_5 \Rightarrow \frac{J\dot u_6}{3K_3} + \frac{T_3}{3K_3} + \frac{K_1-K_2}{K_3}u_4u_5 \Rightarrow \dot u_6 = \left[+\frac{T_3}{3K_3} + \frac{K_1-K_2}{K_3}u_4u_5\right]/1 - J/3K$$

$$\dot\omega_3 = \left[+\frac{T_3}{3K_3} + \frac{K_1-K_2}{K_3}\omega_1\omega_2\right]/(1 - J/3K_3)$$

$$\dot u_7 = \frac{\dot u_6}{3} + \frac{T_3}{3J} = \frac{J\dot u_7}{3K_3} + \frac{T_3}{3J} + \frac{K_1-K_2}{3K_3}u_4u_5 \Rightarrow$$

$$\dot u_6\ \frac{J\dot u_7}{K_3} = -\frac{T}{3K_3} - \frac{J\dot u_6}{3K_3} + \frac{K_1-K_2}{K_3}u_4u_5$$

$$\dot u_6\left(1 + \frac{J}{3K_3}\right).$$

$$\dot u_7 = \left[+\frac{T_3}{3J} + \cancel{\frac{K_1-K_2}{3K_3}} + \frac{K_1-K_2}{3K_3}u_4u_5\right] \Big/ \left(1 - \frac{J}{3K_3}\right)$$

$$\dot\Omega = \left[+T_3/3J + \cancel{\frac{K_1-K_2}{3K_3}} + \frac{K_1-K_2}{3K_3}\omega_1\omega_2\right]/(1 - J/3K_3)$$

$$\dot u_6 = \left[\frac{T_3}{3K_3} + \frac{K_1-K_2}{3K_3^2}Ju_4u_5\right]/\left(1 - \frac{J}{3K_3}\right) + \frac{K_1-K_2}{K_3}u_4u_5\ (1 - J/3K_3)/(1 - J/3K_3)$$

$$\boxed{\dot u_6 = \left[\frac{T_3}{3K_3} + \frac{K_1-K_2}{K_3}u_4u_5\right]/(1 - J/3K_3)}$$

$$\omega_2\omega_3(I_2 - I_3) - J\hat{\dot\omega}^B = I_1\dot\omega_1$$

$$\omega_3\omega_1(I_3 - I_1) + J\omega^B\omega_3 = I_2\dot\omega_2$$

$$\omega_1\omega_2(I_1 - I_2) + J\omega^B\omega_2 = I_3\dot\omega_3$$

$$\left(T/J - \frac{F_1}{I_1}\right)\Big/\left(1 - J/I_1\right) = \dot\omega^B = \left(T/J - \frac{\omega_2\omega_3\,I_2 - I_3}{I_1}\right)\Big/\left(1 - J/I_1\right)$$

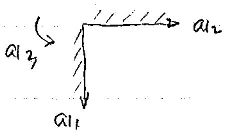
$$F_1 = \omega_2\omega_3(I_2 - I_3)$$

$$+\omega_1\omega_3(I_2 - I_3) + \left(-T/J + \frac{\omega_2\omega_3(I_2 - I_3)J}{I_1}\right) \Big/ \left(1 - J/I_1\right) = -J\dot\omega^B + \omega_2\omega_3(I_2 - I_3)$$

$$= I_1\dot\omega_1 = \frac{+\omega_2\omega_3(I_2 - I_3) - \omega_2\omega_3\cancel{(I_2 - I_3)\frac{J}{I_1}} - T + \cancel{\frac{\omega_2\omega_3(I_2 - I_3)J}{I_1}}}{1 - J/I_1}$$

$$\dot\omega_1 = \frac{\omega_2\omega_3\frac{(I_2 - I_3)}{I_1} - T/I_1}{1 - J/I_1}$$

From problem 2a we obtain the following if ${}^{A}\omega^{B} = {}^{A}\omega^{I_1} + {}^{I_1}\omega^{I_2} + {}^{I_2}\omega^{B} = \dot{q}_1 \mathbf{a}_1 + \dot{q}_2 \overset{\circ}{\mathbf{i}}_2 + \dot{q}_3 \mathbf{b}_3$

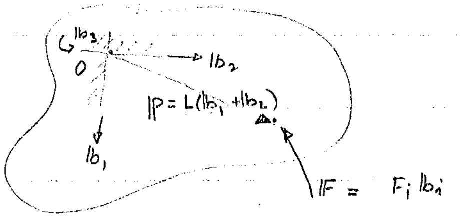


where initially $\mathbf{a}_i$ & $\mathbf{b}_i$ correspond before the successive rotations

	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\overset{\circ}{\mathbf{i}}_1$	1	0	0
$\overset{\circ}{\mathbf{i}}_2$	0	c_1	s_1
$\overset{\circ}{\mathbf{i}}_3$	0	$-s_1$	c_1

	$\overset{\circ}{\mathbf{i}}_1$	$\overset{\circ}{\mathbf{i}}_2$	$\overset{\circ}{\mathbf{i}}_3$
$\mathbf{j}_1$	c_2	0	$-s_2$
$\mathbf{j}_2$	0	1	0
$\mathbf{j}_3$	s_2	0	c_2

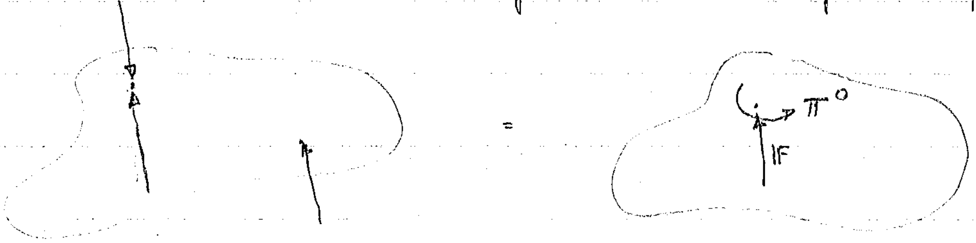
	$\mathbf{j}_1$	$\mathbf{j}_2$	$\mathbf{j}_3$
$\mathbf{b}_1$	c_3	s_3	0
$\mathbf{b}_2$	$-s_3$	c_3	0
$\mathbf{b}_3$	0	0	1



find a potential function for $\underline{\mathbf{F}}$. To find a potential fn. for $\mathbf{F}$ we must first find $\mathbf{F}$'s contribution to the generalized active forces

Since O is fixed let us find an equivalent Moment & force system about point O

$\therefore$ at O there will be a force $\mathbf{F}$ and a couple $= \mathbb{T} = \mathbf{P} \times \mathbf{F}$



$$\mathbb{T} = L(\mathbf{b}_1 + \mathbf{b}_2) \times F_1\mathbf{b}_1 + F_2\mathbf{b}_2 + F_3\mathbf{b}_3$$
$$= L\left(F_2\mathbf{b}_3 - F_3\mathbf{b}_2 - F_1\mathbf{b}_3 + F_3\mathbf{b}_1\right)$$
$$= L\left[(F_2 - F_1)\mathbf{b}_3 + F_3\mathbf{b}_1 - F_3\mathbf{b}_2\right]$$

$$\mathbb{T} = KL^2\left[c_2(s_3 + c_3)\mathbf{b}_3 - s_2\mathbf{b}_1 + s_2\mathbf{b}_2\right]$$

Now F_r' (being the contribution of $\mathbf{F}$ to the generalized active forces) $= \mathbf{V}^{O}_{\dot{q}_r} \cdot \mathbf{F}^{O} + {}^{A}\omega^{B}_{\dot{q}_r} \cdot \mathbb{T}^{O}$

but since O is fixed $\mathbf{V}^{O} = 0$ $\quad \therefore \quad F_r' = {}^{A}\omega^{B}_{\dot{q}_r} \cdot \mathbb{T}^{O}$ and $F_r' = -\dfrac{\partial P}{\partial q_r}$

${}^{A}\omega^{B}_{\dot{q}_1} = \mathbf{a}_1$, ${}^{A}\omega^{B}_{\dot{q}_2} = \overset{\circ}{\mathbf{i}}_2$, ${}^{A}\omega^{B}_{\dot{q}_3} = \mathbf{b}_3$

$$\therefore \quad F_1' = \mathbf{a}_1 \cdot \mathbb{T}^{O} = \overset{\circ}{\mathbf{i}}_1 \cdot \mathbb{T}^{O} = (c_2\mathbf{j}_1 + s_2\mathbf{j}_3) \cdot \mathbb{T}^{O} = \left(c_2[c_3\mathbf{b}_1 - s_3\mathbf{b}_2] + s_2\mathbf{b}_3\right) \cdot \mathbb{T}^{O}$$
$$= KL^2\left[(c_2c_3)\mathbf{b}_1 - c_2s_3\mathbf{b}_2 + s_2\mathbf{b}_3\right] \cdot \left[-s_2\mathbf{b}_1 + s_2\mathbf{b}_2 + c_2(s_3 + c_3)\right.$$
$$= KL^2\left[-c_2\cancel{c_3}s_2 - c_2\cancel{s_3}s_2 + s_2\cancel{c_2}s_3 + s_2\cancel{c_2}c_3\right] = 0 = -\frac{\partial P}{\partial q_1} \qquad (a)$$

$$F_2' = \overset{\circ}{\mathbf{i}}_2 \cdot \mathbb{T}^{O} = \mathbf{j}_2 \cdot \mathbb{T}^{O} = (s_3\mathbf{b}_1 + c_3\mathbf{b}_2) \cdot KL^2\left[-s_2\mathbf{b}_1 + s_2\mathbf{b}_2 + c_2(s_3 + c_3)\mathbf{b}_3\right]$$
$$= KL^2\left[-s_3s_2 + c_3s_2\right] = -\frac{\partial P}{\partial q_2} \qquad (b)$$

$$F_3' = \mathbf{b}_3 \cdot \mathbb{T}^{O} = \mathbf{b}_3 \cdot KL^2\left[-s_2\mathbf{b}_1 + s_2\mathbf{b}_2 + c_2(s_3 + c_3)\mathbf{b}_3\right] = KL^2\left[c_2s_3 + c_2c_3\right] = -\frac{\partial P}{\partial q_3} \qquad (c)$$

$\therefore$ integrating (a) $P = f(q_2, q_3, t)$

(b) $-P = KL^2(s_3c_2 - c_3c_2) + f(q_3, t) \Rightarrow P = KL^2(c_3 - s_3)c_2 + f(q_3, t)$

$$-\frac{\partial P}{\partial q_3} = KL^2 \left(c_3 c_2 + s_3 c_2 \right) - \frac{\partial f}{\partial q_3} = KL^2 \left(c_3 c_2 + c_2 s_3 \right) \quad \Rightarrow \quad \frac{\partial f}{\partial q_3} = 0 \quad \text{and} \quad f = f(t) \text{ only}$$

$$\therefore \quad P = KL^2 c_2 \left(c_3 - s_3 \right) + f(t) \qquad \text{for} \quad \mathbb{F}$$

Excellent

A

#23

LEVY

800311

A rigid body A carries three identical rotors B_1, B_2, and C on parallel axes fixed in A, as shown in the attached sketch. By means of a motor attached to A, C is made to rotate relative to A at a rate Ω (not necessarily constant), and B_1 and B_2 are driven by C at the same rate, but with opposite sense.

X_1, X_2, X_3 (see sketch) are principal axes of inertia of A for the mass center of A, and the moments of inertia of the entire system about these axes have the values K_1, K_2, K_3, respectively. The mass center of each of the three rotors lies on X_1, and each rotor has moment of inertia J about its axis of symmetry.

Let $\underline{n}_i$ be a unit vector pointing in the direction of the positive X_i-axis (i = 1 , 2 , 3); replace the system of forces exerted on C by A (by means of the motor) with a couple of torque

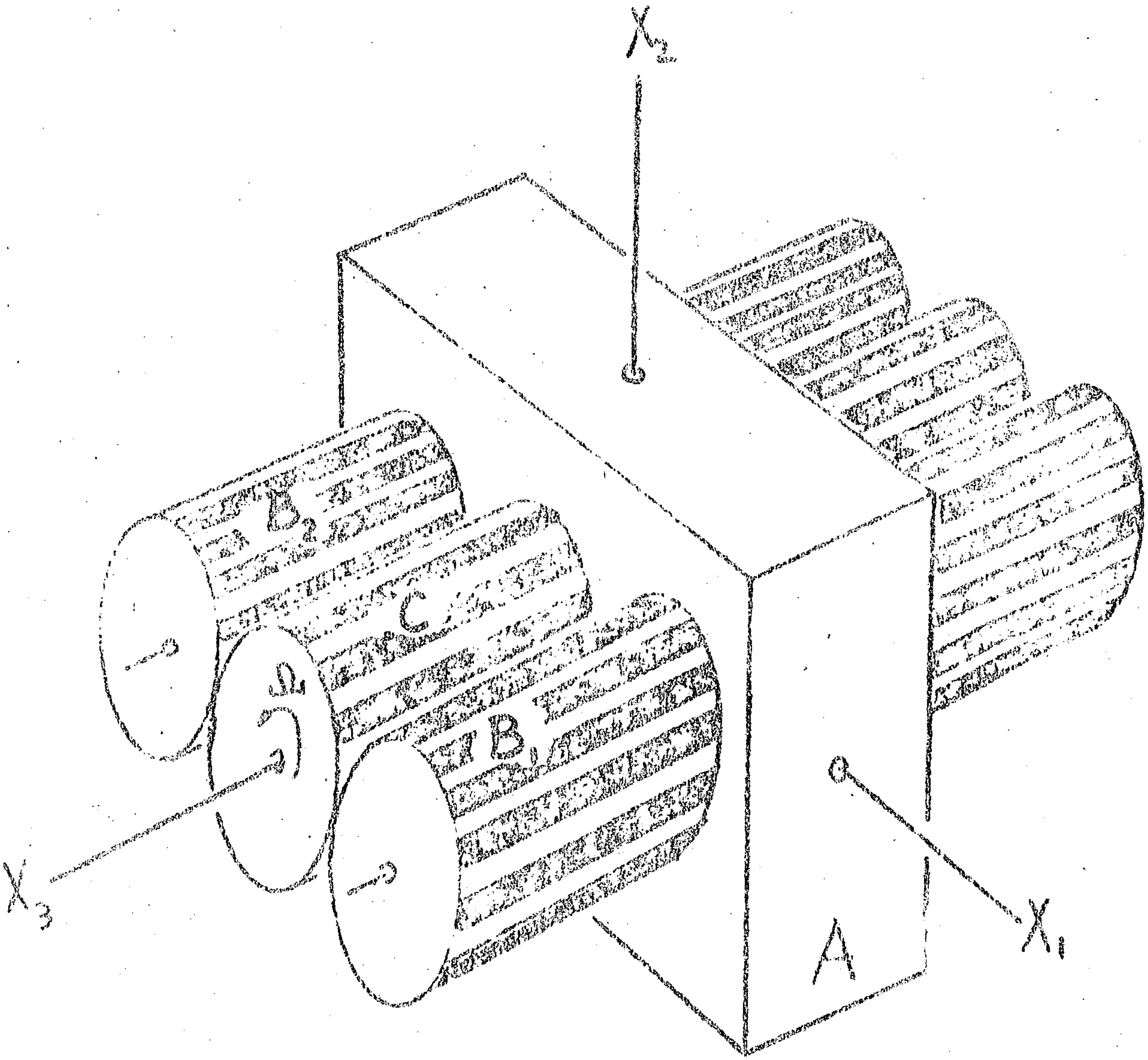
$$T_1\underline{n}_1 + T_2\underline{n}_2 + T_3\underline{n}_3$$

together with a force applied at the mass center of C; and assume that the only body and contact forces acting on the system are gravitational forces.

The differential equations of motion of this system can be expressed as

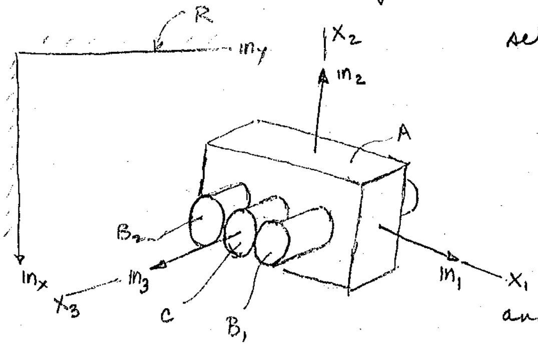
$$\dot{\Omega} = F ; \quad \dot{\omega}_i = G_i \quad (i = 1,2,3)$$

where F and G_i are functions of Ω, ω_1, ω_2, ω_3, J, K_1, K_2, K_3, T_1, T_2, T_3. Determine F and G_i (i=1,2,3).


X_2
B_2
C
Ω
B_1
X_3
A
X_1

#23
LEVY

We assume a reference frame R (with mutually perpendicular vectors $\mathbb{n}_x, \mathbb{n}_y, \mathbb{n}_z$) which is fixed and in which the system in question moves. The system in question has 7 degrees of freedom, 6 to describe the motion of body A and 1 to describe the motion of body C relative to body A (similarly this same degree of freedom describes the motion of bodies B_1 and B_2 relative to A). Since a motor causes a set of forces to act on body C and we replace them by a torque $\mathbb{T}^c$ and a resultant force $\mathbb{F}_T^c$ acting at C^*. Similarly by the law of action and reaction a torque $-\mathbb{T}^c$ and a resultant force $-\mathbb{F}_T^c$ acting at A^* exists. Since the only body and contact forces involved are $Mg\mathbb{n}_x$ on body A and $mg\mathbb{n}_x$ on each of the bodies C, B_1, B_2. Because the translational motion can be decoupled from the rotational motion we need only look at the rotational equations to obtain our results. However I will set up the general results.



If we assume $(\ )^*$ represent the values of the center of mass of the body and

$$\mathbb{V}^{A^*} = V_1\mathbb{n}_1 + V_2\mathbb{n}_2 + V_2\mathbb{n}_3 \tag{1}$$

then we can define the following generalized coordinates

$$u_i = \begin{cases} V_i & (i=1,2,3) \\ \omega_{i-3} & (i=4,5,6) \\ \Omega & i=7 \end{cases}$$

where

$${}^R\omega^A \cdot \mathbb{n}_i = \omega_i \tag{2}$$

$${}^A\omega^c = -{}^A\omega^{B_1} = -{}^A\omega^{B_2} = \Omega\,\mathbb{n}_3 \tag{3}$$

Also

$$\mathbb{F}^A = -\mathbb{F}_T^c + Mg\mathbb{n}_x \tag{4}$$

$$\mathbb{F}^c = mg\mathbb{n}_x + \mathbb{F}_T^c \tag{5}$$

$$\mathbb{F}^{B_1} = mg\mathbb{n}_x \tag{6}$$

$$\mathbb{F}^{B_2} = mg\mathbb{n}_x \tag{7}$$

Similarly

$$\mathbb{T}^c = T_1\mathbb{n}_1 + T_2\mathbb{n}_2 + T_3\mathbb{n}_3 \tag{8}$$

$$\mathbb{T}^A = -\mathbb{T}^c \tag{9}$$

$$\mathbb{T}^{B_1} = \mathbb{T}^{B_2} = 0 \tag{10}$$

#23

Now we can define

$$\mathbf{V}^{A^*} = \mathbf{V}^{C^*} = u_1\mathbf{n}_1 + u_2\mathbf{n}_2 + u_3\mathbf{n}_3 \qquad (11)$$

$$\mathbf{V}^{B_1} = \mathbf{V}^{C^*} + l\, {}^R\boldsymbol{\omega}^A \times \mathbf{n}_1 = \mathbf{V}^{C^*} + l\left[u_6\mathbf{n}_2 - u_5\mathbf{n}_3\right] \qquad (12)$$

$$\mathbf{V}^{B_2} = \mathbf{V}^{C^*} - l\, {}^R\boldsymbol{\omega}^A \times \mathbf{n}_1 = \mathbf{V}^{C^*} - l\left[-u_6\mathbf{n}_2 + u_5\mathbf{n}_3\right] \qquad (13)$$

Where l is the magnitude of the distance between A^* and B_1^* and between A^* and B_2^*. We now define the generalized active force F_r by

$$F_r = \mathbf{V}_{u_r}^{A^*}\cdot\mathbf{F}^A + \mathbf{V}_{u_r}^{C^*}\cdot\mathbf{F}^C + \mathbf{V}_{u_r}^{B_1^*}\cdot\mathbf{F}^{B_1} + \mathbf{V}_{u_r}^{B_2^*}\cdot\mathbf{F}^{B_2} + {}^R\boldsymbol{\omega}_{u_r}^{A}\cdot\mathbf{T}^A + {}^R\boldsymbol{\omega}_{u_r}^{C}\cdot\mathbf{T}^C + {}^R\boldsymbol{\omega}_{u_r}^{B_1}\cdot\mathbf{T}^{B_1} + \boldsymbol{\omega}_{u_r}\cdot\mathbf{T}^{B_2}$$

$$\text{for } r = 1,\dots,7 \qquad (14)$$

We note certain items: all the ω's are not functions of u_1, u_2, u_3; all the $\mathbf{V}$'s are not functions of u_4 and u_7 and $\mathbf{V}^{A^*}, \mathbf{V}^{C^*}$ are not functions of u_5 and u_6. Using these and (1-4) we obtain

$$F_1 = \mathbf{n}_1\cdot\mathbf{n}_x\left[(M+3m)g\right] \qquad (15)$$

$$F_2 = \mathbf{n}_2\cdot\mathbf{n}_x\left[(M+3m)g\right] \qquad (16)$$

$$F_3 = \mathbf{n}_3\cdot\mathbf{n}_x\left[(M+3m)g\right] \qquad (17)$$

$$F_4 = F_5 = F_6 = 0 \qquad (18\text{-}20)$$

$$F_7 = T_3 \qquad (21)$$

To obtain the generalized inertia forces F_r^*

$$F_r^* = \mathbf{V}_{u_r}^{A^*}\cdot\mathbf{F}^{A^*} + \mathbf{V}_{u_r}^{C^*}\cdot\mathbf{F}^{C^*} + \mathbf{V}_{u_r}^{B_1^*}\cdot\mathbf{F}^{B_1^*} + \mathbf{V}_{u_r}^{B_2^*}\cdot\mathbf{F}^{B_2^*} + {}^R\boldsymbol{\omega}_{u_r}^{A}\cdot\mathbf{T}^{*A} + {}^R\boldsymbol{\omega}_{u_r}^{C}\cdot\mathbf{T}^{*C} + {}^R\boldsymbol{\omega}_{u_r}^{B_1}\cdot\mathbf{T}^{B_1^*} + {}^R\boldsymbol{\omega}_{u_r}^{B_2}\cdot\mathbf{T}^{B_2} \qquad (22)$$

where

$$\mathbf{F}^{A^*} = -M\mathbf{a}^{A^*} \qquad \mathbf{T}^{A^*} = -\frac{{}^R d}{dt}\,{}^R\mathbf{H}^{A/A^*} \qquad (23\text{-}30).$$

$$\mathbf{F}^{C^*} = -m\mathbf{a}^{C^*} \qquad \mathbf{T}^{C^*} = -\frac{{}^R d}{dt}\,{}^R\mathbf{H}^{C/C^*}$$

$$\mathbf{F}^{B_1^*} = -m\mathbf{a}^{B_1^*} \qquad \mathbf{T}^{B_1^*} = -\frac{{}^R d}{dt}\,{}^R\mathbf{H}^{B_1/B_1^*}$$

$$\mathbf{F}^{B_2^*} = -m\mathbf{a}^{B_1^*} \qquad \mathbf{T}^{B_2^*} = -\frac{{}^R d}{dt}\,{}^R\mathbf{H}^{B_2/B_2^*}$$

Now

$$\mathbf{a}^{A^*} = \frac{{}^A d\mathbf{V}^{C^*}}{dt} + {}^R\boldsymbol{\omega}^A \times \mathbf{V}^C = \mathbf{a}^{C^*} \qquad (31, 32)$$

$$\mathbf{a}^{B_1^*} = \mathbf{a}^{C^*} + l\frac{{}^A d\,{}^R\boldsymbol{\omega}^A}{dt}\times\mathbf{n}_1 + l\left[({}^R\boldsymbol{\omega}^A\cdot\mathbf{n}_1)\,{}^R\boldsymbol{\omega}^A - {}^R\boldsymbol{\omega}^{A^2}\mathbf{n}_1\right] \qquad (33)$$

$$\mathbf{a}^{B_2^*} = \mathbf{a}^{C^*} - l\frac{{}^A d\,{}^R\boldsymbol{\omega}^A}{dt}\times\mathbf{n}_1 - l\left[({}^R\boldsymbol{\omega}^A\cdot\mathbf{n}_1)\,{}^R\boldsymbol{\omega}^A - {}^R\boldsymbol{\omega}^{A^2}\mathbf{n}_1\right] \qquad (34)$$

Using these results we find

$$F_1^* = -(M+3m)\left[\dot{u}_1 + u_5u_3 - u_6u_2\right] \qquad (35)$$

$$F_2^* = -(M+3m)\left[\dot{u}_2 + u_6u_1 - u_4u_3\right] \qquad (36)$$

$$F_3^* = -(M+3m)\left[\dot{u}_3 + u_4u_2 - u_5u_1\right] \qquad (37)$$

$$F_4^* = \mathbf{n}_1\cdot\left(\mathbf{T}^{A^*} + \mathbf{T}^{C^*} + \mathbf{T}^{B_1^*} + \mathbf{T}^{B_2^*}\right) \qquad (38)$$

$$F_5^* = -2l^2m\left[\dot{u}_5 - u_4u_6\right] + \mathbf{n}_2\cdot\left[\mathbb{T}^{*A} + \mathbb{T}^{*C} + \mathbb{T}^{*B_1} + \mathbb{T}^{*B_2}\right] \qquad (39)$$

$$F_6^* = -2l^2m\left[\dot{u}_6 + u_4u_5\right] + \mathbf{n}_3\cdot\left[\mathbb{T}^{*A} + \mathbb{T}^{*B} + \mathbb{T}^{*B_1} + \mathbb{T}^{*B_2}\right] \qquad (40)$$

$$F_7^* = \mathbf{n}_3\cdot\left[\mathbb{T}^{C*} - \mathbb{T}^{B_1*} - \mathbb{T}^{B_2*}\right] \qquad (41)$$

Now

$$\mathbb{T}^{C*} = -\frac{{}^{R}d}{dt}\mathbb{H}^{C/C*} = -\mathbb{I}^{C/C*}\cdot{}^{R}\alpha^{A} + {}^{A}\mathbb{T}^{C*} = -\mathbb{I}^{C/C*}\cdot\left({}^{R}\alpha^{A} + {}^{A}\alpha^{C}\right) \qquad (42)$$

$$\mathbb{T}^{B_1*} = -\frac{{}^{R}d}{dt}\mathbb{H}^{B_1/B_1*} = -\mathbb{I}^{C/C*}\cdot{}^{R}\alpha^{A} + {}^{A}\mathbb{T}^{B_1*} \quad \text{since} \quad \mathbb{I}^{C/C*} = \mathbb{I}^{B/B_1*} = \mathbb{I}^{B_2/B_2*} \qquad (43)$$

$$\mathbb{T}^{B_2*} = -\frac{{}^{R}d}{dt}\mathbb{H}^{B_2/B_2*} = -\mathbb{I}^{C/C*}\cdot{}^{R}\alpha^{A} + {}^{A}\mathbb{T}^{B_2*} = -\mathbb{I}^{C/C*}\cdot\left({}^{R}\alpha^{A} + {}^{A}\alpha^{B}\right) = \mathbb{T}^{B_1*} \qquad (44)$$

Using these in (41) leads to

$$F_7^* = -J(\dot{u}_6 + \dot{u}_7) + 2J(\dot{u}_6 - \dot{u}_7) = -3J\dot{u}_7 + J\dot{u}_6 \qquad (45)$$

Now

$$\mathbb{T}^{A*} + \mathbb{T}^{C*} + \mathbb{T}^{B_1*} + \mathbb{T}^{B_2*} = -\frac{{}^{R}d}{dt}\left[\mathbb{H}^{A/A*} + \mathbb{H}^{C/C*} + \mathbb{H}^{B_1/B_1*} + \mathbb{H}^{B_2/B_2*}\right]$$
$$= -\frac{{}^{R}d}{dt}\left[\left(\mathbb{I}^{A/A*} + \mathbb{I}^{C/C*} + \mathbb{I}^{B_1/B_1*} + \mathbb{I}^{B_2/B_2*}\right)\cdot{}^{R}\omega^{A} + \mathbb{I}^{C/C*}\cdot{}^{A}\omega^{C} + \mathbb{I}^{B_1/B_1*}\cdot{}^{A}\omega^{B_1} + \mathbb{I}^{B_2/B_2*}\cdot\right.$$
$$(46)$$

Using $\mathbb{I}^{B_1/A*} = \mathbb{I}^{B_1/B_1*} + \mathbb{I}^{B_1*/A*}$ and $\mathbb{I}^{B_2/A*} = \mathbb{I}^{B_2/B_2*} + \mathbb{I}^{B_2*/A*}$ (47 & 48)

then (46) becomes:

$$= -\frac{{}^{R}d}{dt}\left[\mathbb{I}^{(A+C+B_1+B_2)/A*}\cdot{}^{R}\omega^{A} - \left(\mathbb{I}^{B_1*/A*} + \mathbb{I}^{B_2*/A*}\right)\cdot{}^{R}\omega^{A} - \mathbb{I}^{C/C*}\cdot{}^{A}\omega^{C}\right]$$
$$= {}^{R}\mathbb{T}^{D} - {}^{A}\mathbb{T}^{C} + 2l^2m\frac{{}^{R}d}{dt}\left[(\mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3)\cdot{}^{R}\omega^{A}\right] + {}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C*} \qquad (49)$$

This is obtained since

$$\mathbb{I}^{B_1*/A*} = ml^2(\mathbb{U} - \mathbf{n}_1\mathbf{n}_1) = ml^2(\mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3) = \mathbb{I}^{B_2*/A*}$$ where $\mathbb{U}$ is the unit dyadic (50)

If we note that $\frac{{}^{R}d}{dt}\left[(\mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3)\cdot{}^{R}\omega^{A}\right] = (\dot{u}_5 - u_6u_4)\mathbf{n}_2 + (\dot{u}_6 + u_5u_4)\mathbf{n}_3$ then

$$\mathbb{T}^{A*} + \mathbb{T}^{C*} + \mathbb{T}^{B_1*} + \mathbb{T}^{B_2*} = {}^{R}\mathbb{T}^{D} - {}^{A}\mathbb{T}^{C} + 2l^2m\left\{(\dot{u}_5 - u_6u_4)\mathbf{n}_2 + (\dot{u}_6 + u_5u_4)\mathbf{n}_3\right\} + {}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C*} \qquad (51)$$

Here D is a fictitious rigid body whose mass distribution is the same as $A+B_1+C+B_2$ and whose motion is that of body A. (ie ${}^{R}\mathbb{T}^{D} = \left(\mathbb{I}^{D/D*}\cdot{}^{R}\omega^{A}\right)\times{}^{R}\omega^{A} - \mathbb{I}^{D/D*}\cdot{}^{R}\alpha^{A}\right)$). (52)

Using (51) in (38-40) and noting that $\mathbb{I}^{D/D*} = K_1\mathbf{n}_1\mathbf{n}_1 + K_2\mathbf{n}_2\mathbf{n}_2 + K_3\mathbf{n}_3\mathbf{n}_3$ (53)

$$F_4^* = (K_2 - K_3)u_5u_6 - K_1\dot{u}_4 + Ju_7u_5 \qquad (54)$$

$$F_5^* = (K_3 - K_1)u_4u_6 - K_2\dot{u}_5 - Ju_7u_4 \qquad (55)$$

$$F_6^* = (K_1 - K_2)u_4u_5 - K_3\dot{u}_6 + J\dot{u}_7 \qquad (56)$$

These are obtained by observing that ${}^{A}\mathbb{T}^{C} = -J\dot{u}_7\mathbf{n}_3$ and ${}^{R}\omega^{A}\times{}^{A}\mathbb{H}^{C/C*} = Ju_7(u_5\mathbf{n}_1 - u_4\mathbf{n}_2)$ (57, 58

Now using the equations of motion $F_r + F_r^* = 0$ for $r = 4, \ldots, 7$ ie eqns (18-21) & (45), (5

we obtain

$$K_1\dot{u}_4 = (K_2 - K_3)u_5u_6 + Ju_7u_5 \qquad (59)$$

$$K_2\dot{u}_5 = (K_3 - K_1)u_4u_6 - Ju_7u_4 \qquad (60)$$

$$K_3\dot{u}_6 = J\dot{u}_7 + (K_1 - K_2)u_4u_5 \qquad (61)$$

$$3J\dot{u}_7 = J\dot{u}_6 + T \qquad (62)$$

#23

$$K_3 \dot{u}_6 = \frac{J\dot{u}_6}{3} + \frac{T_3}{3} + (K_1-K_2)u_4u_5 \Rightarrow \boxed{\dot{u}_6 = \left(\frac{T_3}{3K_3} + \frac{K_1-K_2}{K_3}u_4u_5\right)\Big/\left(1-\frac{J}{3K_3}\right)} \tag{63}$$

$$\boxed{\dot{u}_7 = \frac{\dot{u}_6}{3} + \frac{T_3}{3J} = \left(\frac{T_3}{3J} + \frac{K_1-K_2}{3K_3}u_4u_5\right)\Big/\left(1-\frac{J}{3K_3}\right)} \tag{64}$$

$$\boxed{\dot{u}_4 = \frac{(K_2-K_3)}{K_1}u_5u_6 + \frac{J}{K_1}u_7u_5} \tag{65}$$

$$\boxed{\dot{u}_5 = \frac{(K_3-K_1)}{K_2}u_4u_6 - \frac{J}{K_2}u_7u_4} \tag{66}$$

Using the definitions for the u_i's to obtain

$$\dot{\omega}_1 = \left[\frac{K_2-K_3}{K_1}\omega_3 + \frac{J}{K_1}\Omega\right]\omega_2 \tag{67}$$

$$\dot{\omega}_2 = \left[\frac{K_3-K_1}{K_2}\omega_3 - \frac{J}{K_2}\Omega\right]\omega_1 \tag{68}$$

$$\dot{\omega}_3 = \left[\frac{T_3}{3K_3} + \frac{K_1-K_2}{K_3}\omega_1\omega_2\right]\Big/\left(1-J/3K_3\right) \tag{69}$$

$$\dot{\Omega} = \left[\frac{T_3}{3J} + \frac{K_1-K_2}{3K_3}\omega_1\omega_2\right]\Big/\left(1-J/3K_3\right) \tag{70}$$

23

Very nice

800310

A point 0 of a rigid body B is fixed. Mutually perpendicular axes B_1, B_2, B_3 are fixed in B and intersect at 0. Initially, B_1, B_2, B_3 are aligned with fixed axes A_1, A_2, A_3, respectively, and successive rotations of B of amounts q_1, q_2, q_3 are then performed about B_1, B_2, B_3, respectively, to bring B into a general position.

A force $\underline{F}$ is applied to B at a point P. $\underline{F}$ depends on the orientation of B as follows:

$$\underline{F} = kL[(1 - c_2c_3)\underline{b}_1 + (1 + c_2s_3)\underline{b}_2 - s_2\underline{b}_3]$$

where k and L are constants, $\underline{b}_i$ is a unit vector parallel to B_i, and $s_j = \sin q_j$, $c_j = \cos q_j$. The position vector $\underline{p}$ of P relative to 0 is given by

$$\underline{p} = L(\underline{b}_1 + \underline{b}_2)$$

Find a potential function for $\underline{F}$.

#23
LEVY

From problem 2a: If we denote fixed reference frame A having axes A_1, A_2, A_3 with unit vectors $\mathbb{a}_1, \mathbb{a}_2, \mathbb{a}_3$ along these axes respectively; and the body B having axes B_1, B_2, B_3 which are fixed in the body and mutually perpendicular with unit vectors $\mathbb{b}_1, \mathbb{b}_2, \mathbb{b}_3$ along these axes respectively, then by introducing intermediate reference frames I_1 and I_2 with unit vectors $(\mathbb{i}_1, \mathbb{i}_2, \mathbb{i}_3)$ and $(\mathbb{j}_1, \mathbb{j}_2, \mathbb{j}_3)$ which are fixed in each reference frame and mutually perpendicular we may write

$$ {}^A\omega^B \underset{(2.14)}{=} {}^A\omega^{I_1} + {}^{I_1}\omega^{I_2} + {}^{I_2}\omega^B = \dot{q}_1 \mathbb{a}_1 + \dot{q}_2 \mathbb{i}_2 + \dot{q}_3 \mathbb{b}_3 \qquad (1) $$

where
$$ \mathbb{a}_1 = c_2 c_3 \mathbb{b}_1 - c_2 s_3 \mathbb{b}_2 + s_2 \mathbb{b}_3 \qquad (2) $$

$$ \mathbb{i}_2 = s_3 \mathbb{b}_1 + c_3 \mathbb{b}_2 \qquad (3) $$

In order to find a potential function for $\mathbb{F}$ we must first form the contribution of $\mathbb{F}$ to the generalized active forces F_r' and then use

$$ F_r' \underset{(4.1)}{=} -\frac{\partial P}{\partial q_r} \qquad r = 1, 2, 3 \qquad (4) $$

It should be noted that because point O is fixed $\mathbb{v}^O = 0$ (5)

and it would be advantageous to replace $\mathbb{F}$ acting at P by an equivalent force at O and moment about point O. This leads to

$$ \mathbb{F}^O = KL\left[(1 - c_2 c_3)\mathbb{b}_1 + (1 + c_2 s_3)\mathbb{b}_2 - s_2 \mathbb{b}_3\right] \qquad (6) $$

and $\mathbb{T}^O = \mathbb{p} \times \mathbb{F}^P = L(\mathbb{b}_1 + \mathbb{b}_2) \times \mathbb{F}^O = KL^2\left[-s_2 \mathbb{b}_1 + s_2 \mathbb{b}_2 + c_2(s_3 + c_3)\mathbb{b}_3\right]$ (7)

To form F_r'

$$ F_r' = {}^A\mathbb{v}^O_{\dot{q}_r} \cdot \mathbb{F}^O + {}^A\omega^B_{\dot{q}_r} \cdot \mathbb{T}^O \underset{(5)}{=} {}^A\omega^B_{\dot{q}_r} \cdot \mathbb{T}^O \qquad r = 1, 2, 3 \qquad (8) $$

Thus $F_1' \underset{(1)}{=} \mathbb{a}_1 \cdot \mathbb{T}^O \underset{(2,7)}{=} 0 \underset{(4)}{=} -\frac{\partial P}{\partial q_1}$ (9)

$F_2' \underset{(1)}{=} \mathbb{i}_2 \cdot \mathbb{T}^O \underset{(3,7)}{=} KL^2\left[s_2(c_3 - s_3)\right] \underset{(4)}{=} -\frac{\partial P}{\partial q_2}$ (10)

$F_3' \underset{(1)}{=} \mathbb{b}_3 \cdot \mathbb{T}^O \underset{(7)}{=} KL^2\left[c_2(c_3 + s_3)\right] \underset{(4)}{=} -\frac{\partial P}{\partial q_3}$ (11)

from (9): $P = f(q_2, q_3, t)$ only (12)

from (10): $-P \underset{(12)}{=} KL^2\left[-c_2(c_3 - s_3)\right] - g(q_3, t)$ (13)

from (13): $-\frac{\partial P}{\partial q_3} = KL^2\left[c_2(s_3 + c_3)\right] - \frac{\partial g}{\partial q_3} \underset{(13)}{=} KL^2\left[c_2(c_3 + s_3)\right] \Rightarrow \frac{\partial g}{\partial q_3} = 0$ and $g = g(t)$ only

therefore using the results of (12), (13), (14) we obtain

$$P = KL^2 \left[c_2 (c_3 - s_3) \right] + g(t) \qquad \text{for force } \mathbb{F} \tag{15}$$

✓

23

DYNAMICS PROJECT:

STUDY
OF A
SIMPLE GYROSTAT

Cesar Levy
ME 231C
Winter 1980
Prof. Kane

Study of a Simple Gyrostat

The study of a simple gyrostat G consisting of an axisymmetric rotor B whose axis and mass center B^* are fixed in a rectangular body A is performed with the additional condition that the mass centers of both bodies are to coincide (se Figure 1). A set of unit vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ form a right-handed mutually perpendicula triad fixed in A, each vector parallel to a central principal axis of inertia of G. The axis of B is parallel to a unit vector $\boldsymbol{\beta}$ and the rotor-rectangular body system is such that the central principal moments of inertia of G are equal to those of the rectangular body. These central principal moments of inertia are denoted by I_1, I_2, I_3 and B has a polar moment of inertia about $\boldsymbol{\beta}$ equal to J.

If we define ${}^{A}\omega^{B}$, ω_i and β_i ~~by~~ as

$$ {}^{A}\omega^{B} \triangleq {}^{A}\boldsymbol{\omega}^{B} \cdot \boldsymbol{\beta} \qquad (1) $$

$$ \omega_i \triangleq {}^{R}\boldsymbol{\omega}^{A} \cdot \mathbf{a}_i \qquad (i=1,2,3) \qquad (2) $$

$$ \beta_i \triangleq \boldsymbol{\beta} \cdot \mathbf{a}_i \qquad (i=1,2,3) \qquad (3) $$

with ${}^{A}\boldsymbol{\omega}^{B}$ being the angular velocity of B in A and ${}^{R}\boldsymbol{\omega}^{A}$ being the angular velocity of A in a "fixed" reference frame R, we may then write the following set of differential equations that govern ${}^{A}\omega^{B}$ and ω_i $(i=1,2,3)$:

$$ {}^{A}\dot{\omega}^{B} = \left(\frac{I}{J} - H\right)/c_1 \; ; \qquad (4) $$

$$ \dot{\omega}_1 = \left[F_1 - \frac{\beta_1 J}{c_1}\left(T/J - H\right)\right]/I_1 \; ; \qquad (5) $$

$$ \dot{\omega}_2 = \left[F_2 - \frac{\beta_2 J}{c_1}\left(T/J - H\right)\right]/I_2 \; ; \qquad (6) $$

and

$$ \dot{\omega}_3 = \left[F_3 - \frac{\beta_3 J}{c_1}\left(T/J - H\right)\right]/I_3 \; . \qquad (7) $$

In the preceding equations c_1, F_i $(i=1,2,3)$, H and T are defined as follows:

$$ c_1 \triangleq 1 - J\left[\beta_1^2/I_1 + \beta_2^2/I_2 + \beta_3^2/I_3\right] \qquad (8) $$

$$H \triangleq \beta_1 F_1/I_1 + \frac{\beta_2 F_2}{I_2} + \beta_3 F_3/I_3 \qquad (9a)$$

$$F_1 \triangleq (I_2 - I_3)\,\omega_2\omega_3 + J\overset{A}{\omega}{}^{B}\left[\omega_3\beta_2 - \omega_2\beta_3\right] \qquad (9b)$$

$$F_2 \triangleq (I_3 - I_1)\,\omega_3\omega_1 + J\overset{A}{\omega}{}^{B}\left[\omega_1\beta_3 - \omega_3\beta_1\right] \qquad (9c)$$

$$F_3 \triangleq (I_1 - I_2)\,\omega_1\omega_2 + J\overset{A}{\omega}{}^{B}\left[\omega_2\beta_1 - \omega_1\beta_2\right] \qquad (9d)$$

and T is the induced torque on B by A when B is free to rotate relative to A. ?

With the equations (4-7) the significant parameters that affect the stability of motion of body A may be studied based upon the induced torque T. We note that β is defined in the derivation section.

What is "based upon ...T"?
Body A? Motion?

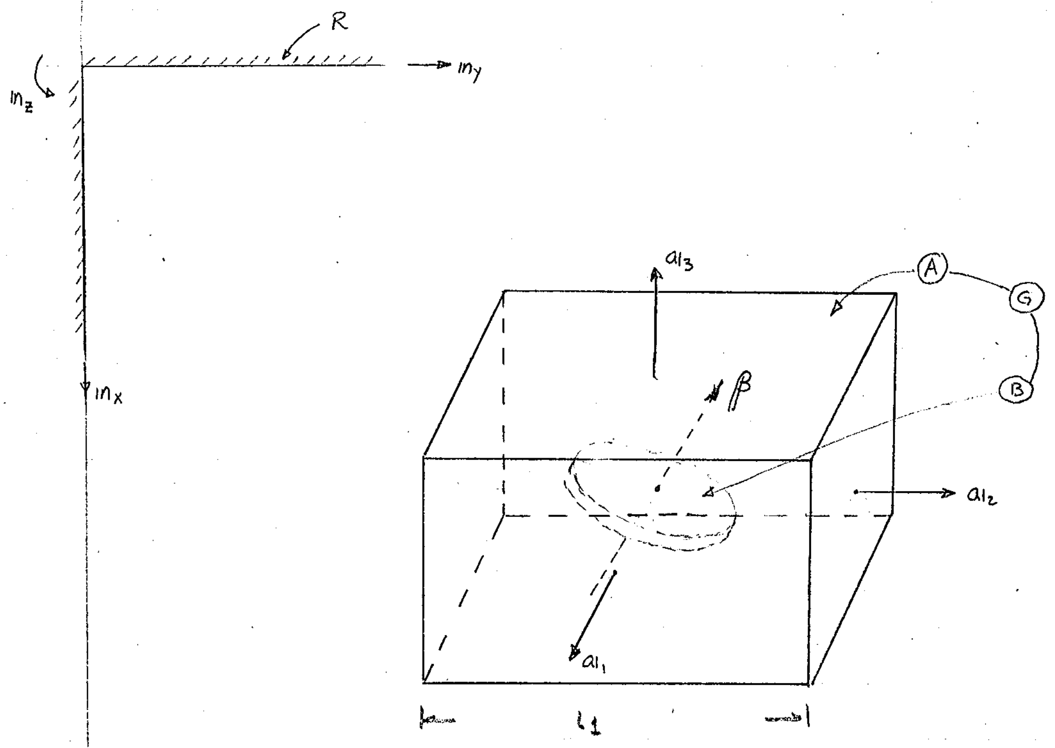


FIG. 1.

SIMPLE GYROSTAT SYSTEM

where is the program?

2. Results

An initial value problem was solved for starting values of ω_i $(i=1,2,3)$ and ${}^{A}\omega^{B}$ equal to zeros by varying the following parameters one at a time:

$\mu = \frac{m_b}{m_b+m_a}$, the ratio of the rotor mass to that of the rotor + body mass;

μ does not appear in the equations of motion. How can changing μ affect the motion?

T/J, the ratio of induced torque to rotor polar moment of inertia;

α, β the cant angle of the rotor axis; how defined?

$b = L_2/L_1$, $c = L_3/L_1$ where L_1, L_2, L_3 are the rectangular body's lengths;

$d = r/2L_1$ where r is the rotor radius; and

L_1 being the length of the side of the rectangular body whose normal is a_1,

Figures (2-6) present the results for the variables indicated on the figures for $|{}^{R}\omega^{A}|$ versus T (time) and ${}^{A}\omega^{B}$ versus T for times up to 40 seconds.

(same symbol as for torque!

Figures (2a,b) indicate the variation of μ with the remaining variables being unchanged. We note the linear character of the plots and that increase in μ causes corresponding increases in both $|{}^{R}\omega^{A}|$ and ${}^{A}\omega^{B}$. We also observe that ${}^{A}\omega^{B}$ is very much larger than $|{}^{R}\omega^{A}|$ but that the magnitude of the changes in $|{}^{R}\omega^{A}|$ between $\mu = 0.03$ and $\mu = 0.1$ is larger than the magnitude of ${}^{A}\omega^{B}$. The trend of the plots can be explained because the dominance of the rotor becomes significant.

We also find that by keeping μ fixed and varying the rotor mass m_b there is no change in the plots and the results for, let's say $\mu = .03$, are characterized by the $\mu = 0.03$ plots in Figures 2a and 2b. We see this when we note that $m_b/m_a = (1-\mu)/\mu$ and that for fixed μ, changes in m_b require changes in m_a such that the ratio of moments of inertia to rotor polar moment of inertia is unchanged.

Figure 3 indicates the variations of T/J, all else being fixed; again we note the linear character of the plots and that $\overset{A}{\omega}{}^{B}$ is still larger than $|^R\omega^A|$. As T/J increases tenfold we note that $\overset{A}{\omega}{}^{B}$ and $|^R\omega^A|$ does so, too. This is so because the dominant term in our differential equations is the T/J term.

Figure 4 provides a plot for variations in the cant angles α and β. We note that the same results occur no matter what the variation in α or no matter what the variations of β. The plots are such that each angle is varied separately while the other is kept fixed. This is due to the cube nature of body A. From the numerical results obtained when only varying α (not plotted) show that for $\alpha = 90°$ the ω_3 component is zero and that both ω_1 and ω_2 double when α changes from 30° to 90°. Similarly when β is the only parameter varied, from the numerical results (not plotted) the ω_1 component is zero for $\beta = 90°$, ω_2 doubles when β changes from 30° to 90°, but ω_3 remains unchanged. These phenomena can be explained due to the fact that the components of the rotor axis $(\beta_1, \beta_2, \beta_3)$ are close to being zero or one and therefore suppress terms in the differential equations. If the starting values of α and β were not 5° and 10° respectively, these effects would not be as apparent.

Figure 5 is a plot of the variations in both b and c. Again we note the linear character of all these plots and that as either b or c increases, $|^R\omega^A|$ decreases; this is due to conservation of momentum. As b, for instance, increases then correspondingly I_2 and I_3 increase. Since initially all the values of ω_i and $\overset{A}{\omega}{}^{B}$ were zero, the initial momentum was zero. Because of the increase in $\overset{A}{\omega}{}^{B}$ when the rotor is spun up, then the pertinent momentum equations

$$I_2\omega_2 + \beta_2 J \overset{A}{\omega}{}^{B} = 0$$

$$I_3\omega_3 + \beta_3 J \overset{A}{\omega}{}^{B} = 0$$

show that for fixed I_2, I_3, β_2, β_3 and J, that ω_2 and ω_3 must increase. Also that for fixed β_3, J and $\overset{A}{\omega}{}^{B}$, an increase in I_2 or I_3 (ie when b increases) requires a corresponding decrease in ω_2 and ω_3. Since $|^R\omega^A| = \sum_{i=1}^{3} (\omega_i\omega_i)^{1/2}$ and ω_1 being close the the same value (numerical results not shown) then $|^R\omega^A|$ must decrease.

Figure 6 provides us with the plots of the variations in d, the rotor diameter nondimensionalization factor. We note again the linear character of the plots and as in all cases that $|^R\omega^A|$ is less than its corresponding $^A\omega^B$ plot.

Here, as in the case of the μ variations, as d increases $|{}^R\omega^A|$ also increases. This is due to the rotor size being "felt" by the rectangular body (ie $J \propto d^2$).

It is interesting to note that even though there are large variations in all the plots of $|{}^R\omega^A|$, the variations of ${}^A\omega^B$ is still close even at time $T = 40$ sec. Of course we expect this to change over longer periods of time.

Based on the results obtained we may then recommend the following in order to keep the rectangular body's motion stable:

1. Keep μ as small as possible yet large enough for the rotor to do its job;
2. Keep T/J small or cycle the rotor so that T varies sinusoidally over a short period;
3. If the cant angle of the rotor axis is to lie along one of the central principle axes, increase the length of sides "perpendicular" to the rotor axis (ie if axis is parallel to L_3, increase L_1 and L_2);
4. Keep the rotor diameter as small as possible.

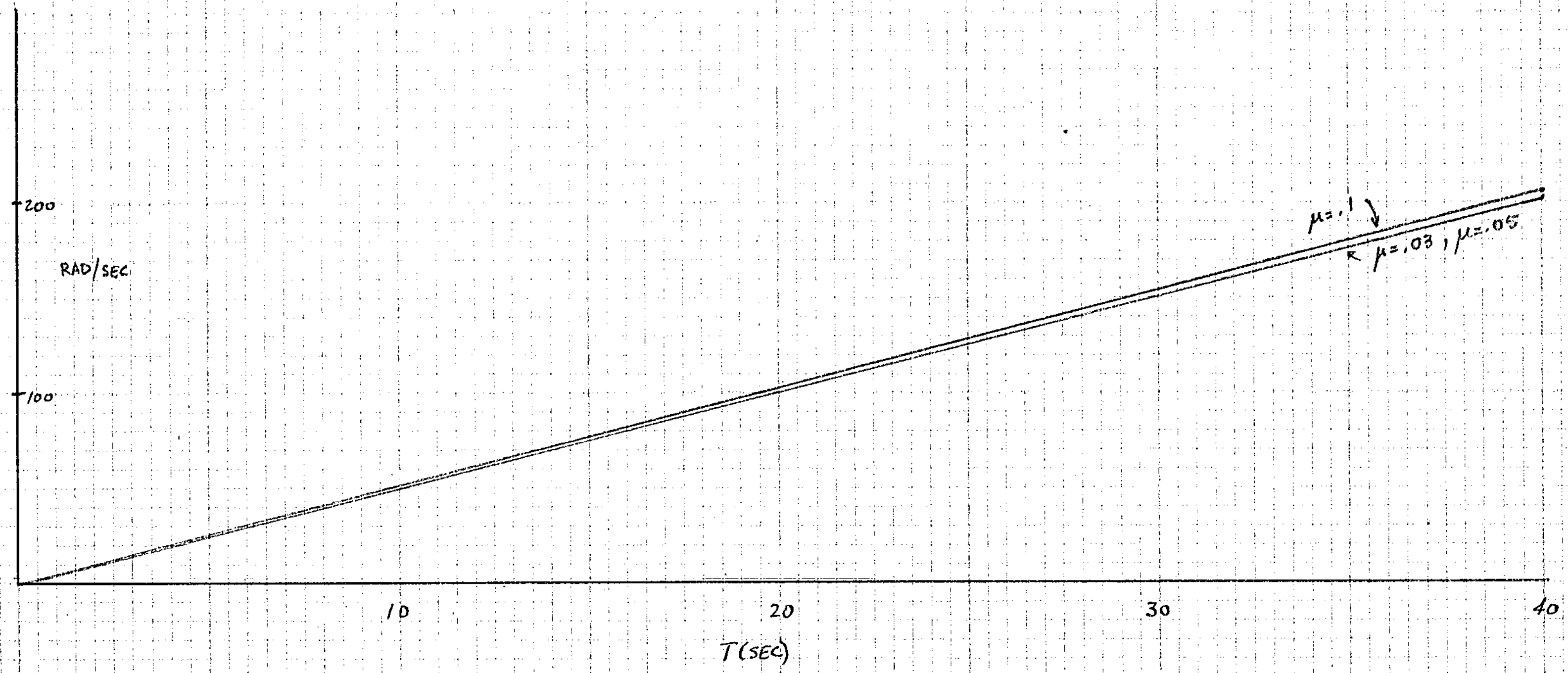
What does any of this have to do with stability? What do even mean by stability?

So far as I can see, the dynamics aspect is o.k. The use you made of the simulation is questionable.

FIGURES (2a, b)

$M_b = 10$ KG
$T/J = 5$ RAD/SEC^2
$\alpha = 5°$, $\beta = 10°$
$b = 1.0$, $c = 1.0$, $d = 0.5$
L1 = 10 M

$^A\omega^B$ vs. T



$|^R\omega^A|$ vs. T

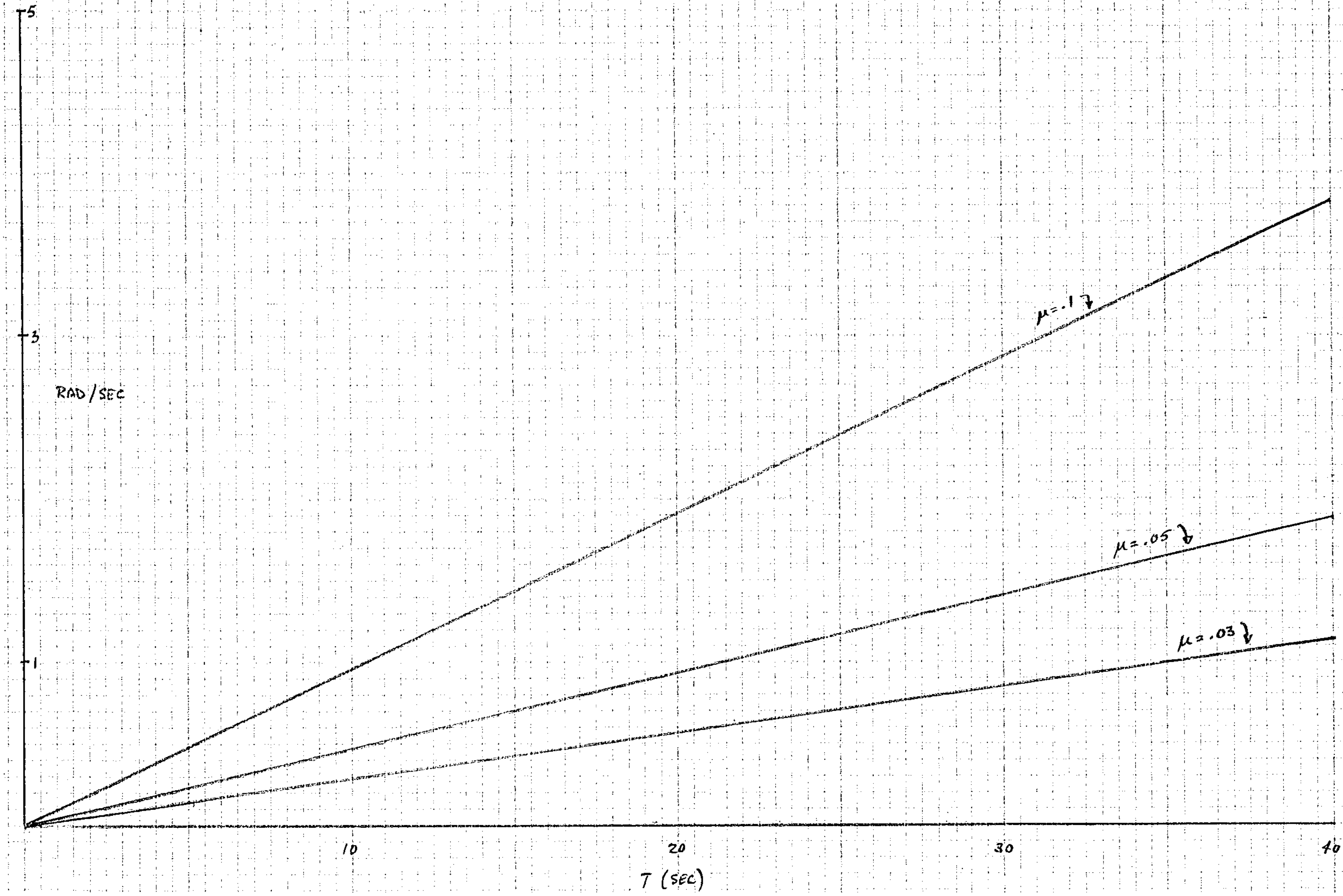
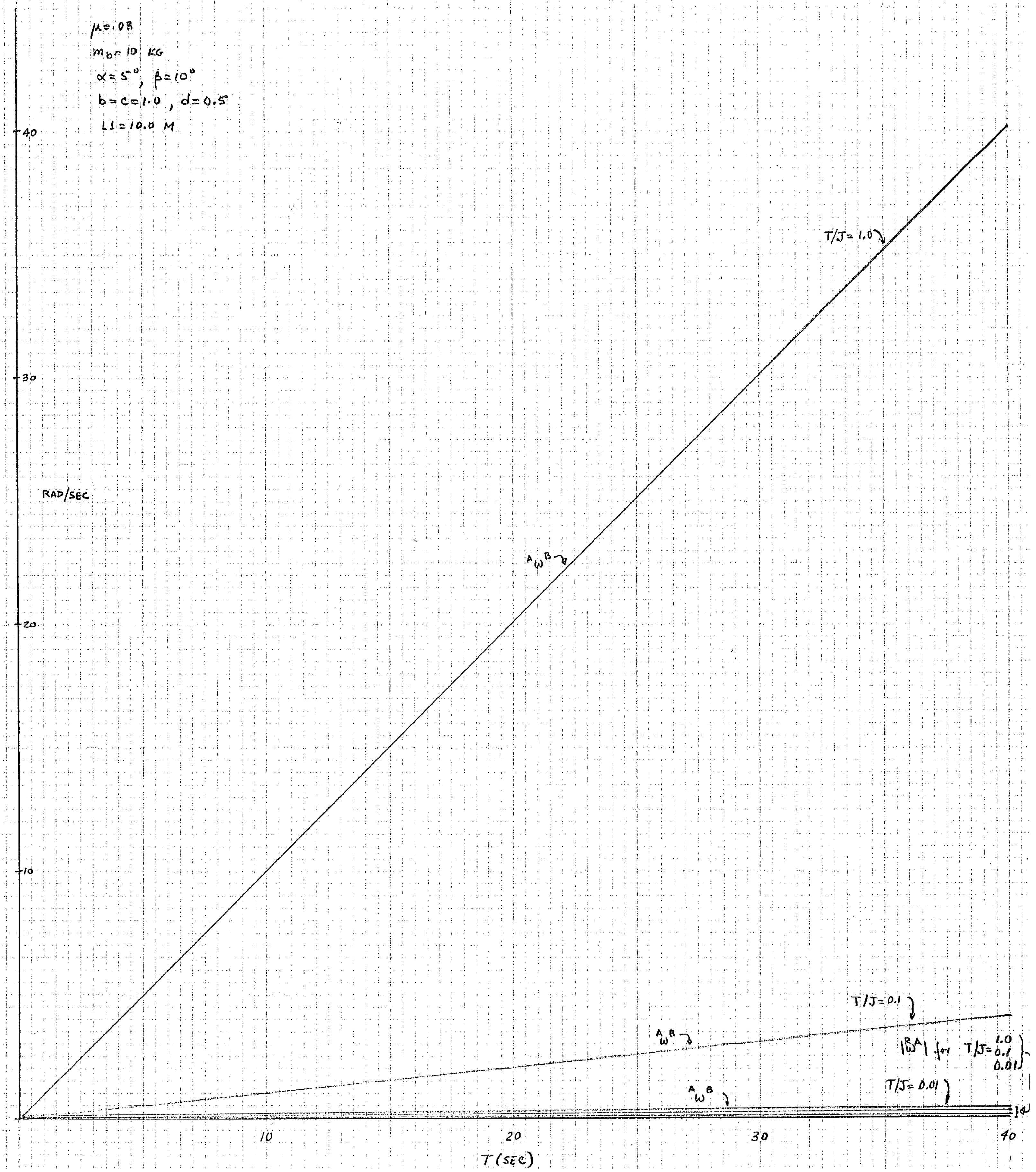
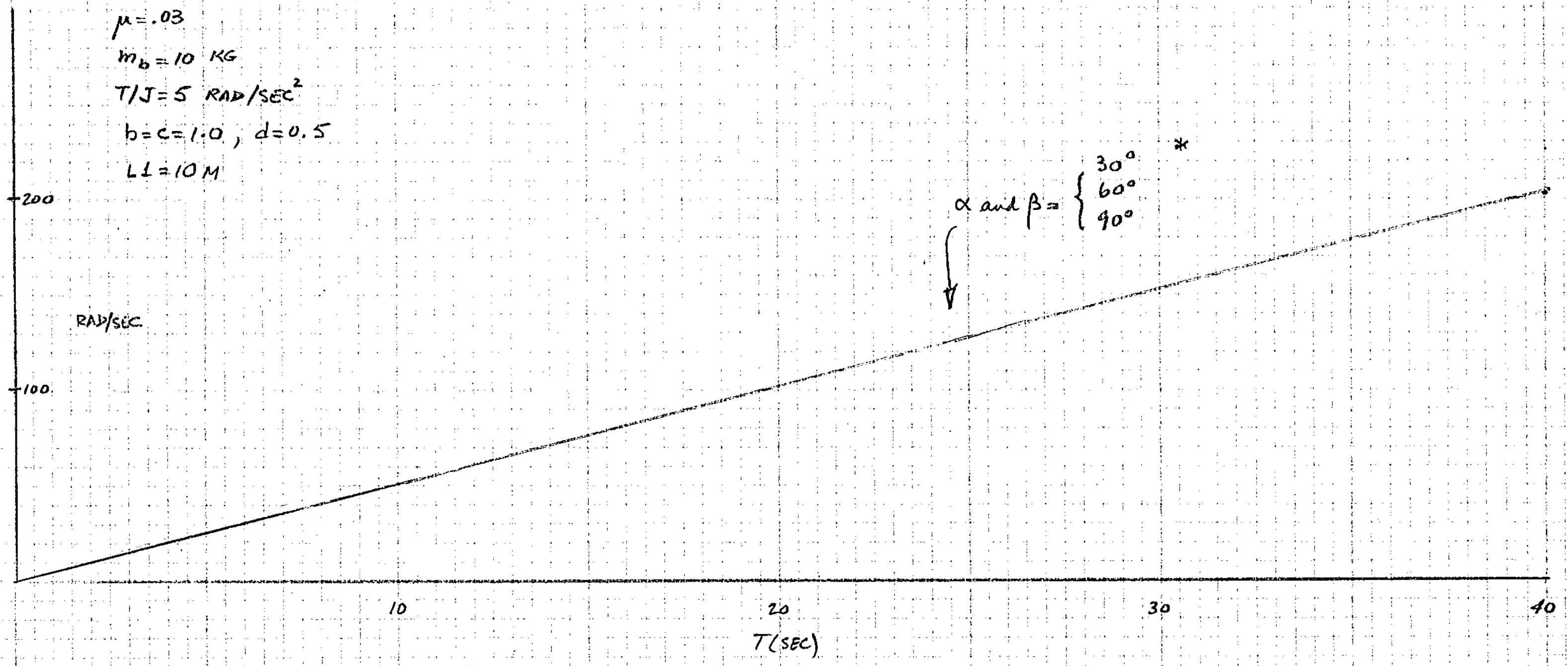


FIGURE 3

$^A\omega^B$ and $|^R\omega^A|$ vs. T

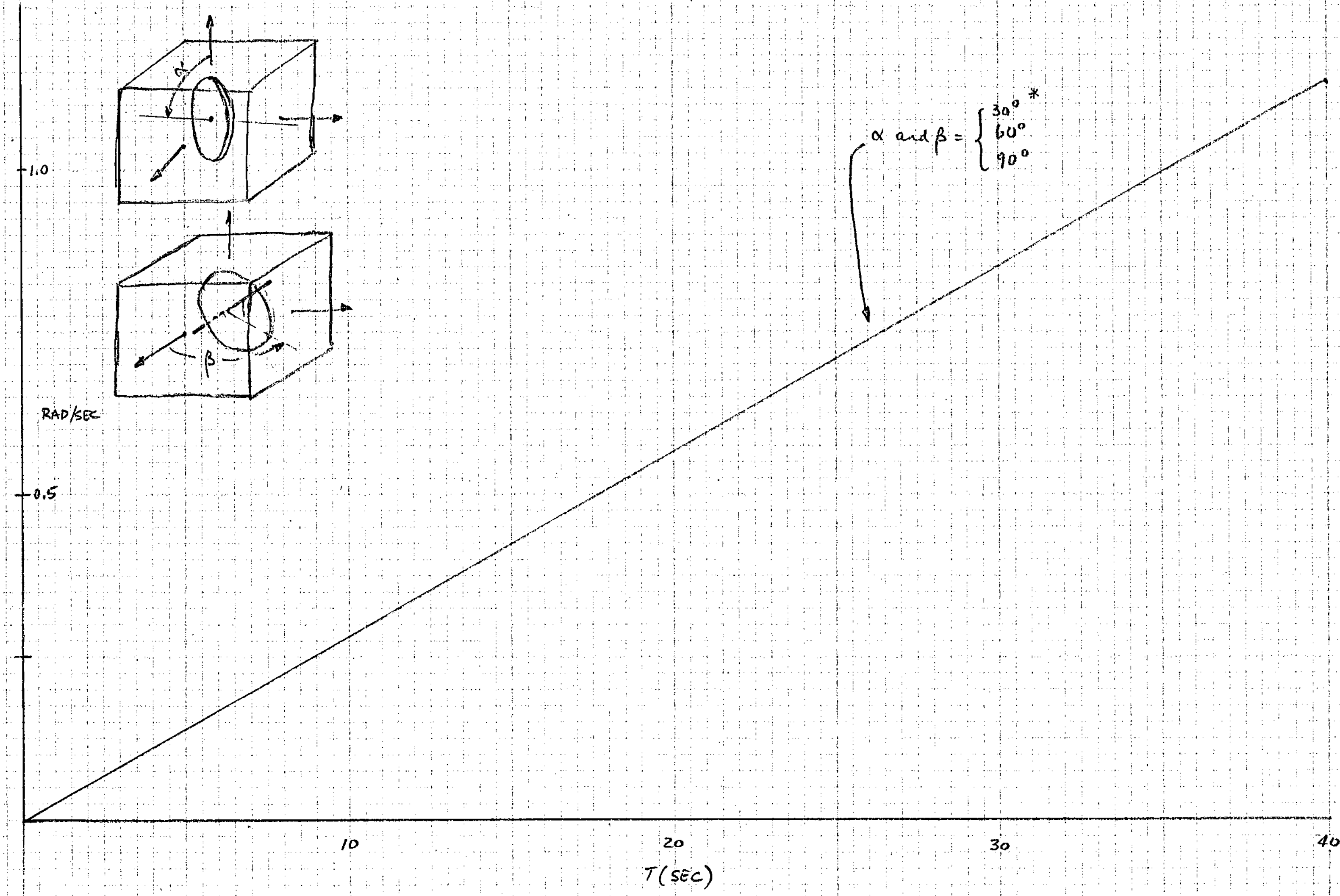


$^A\omega^B$ vs. T



$\mu = .03$
$m_b = 10$ KG
$T/J = 5$ RAD/SEC^2
$b = c = 1.0$, $d = 0.5$
$L1 = 10$ M

$|^B\omega^A|$ vs. T

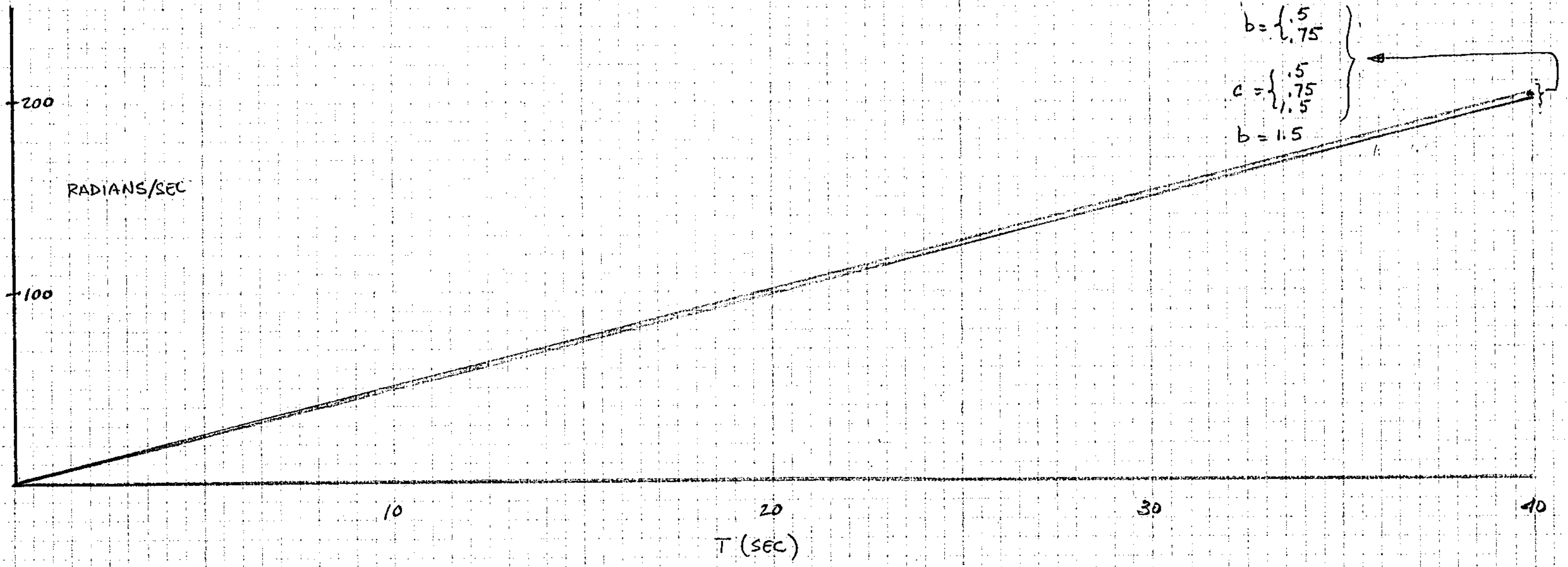


FIGURES (Aa, b)

* WHEN α IS VARIED β IS KEPT FIXED AT 10°
WHEN β IS VARIED α IS KEPT FIXED AT 5°

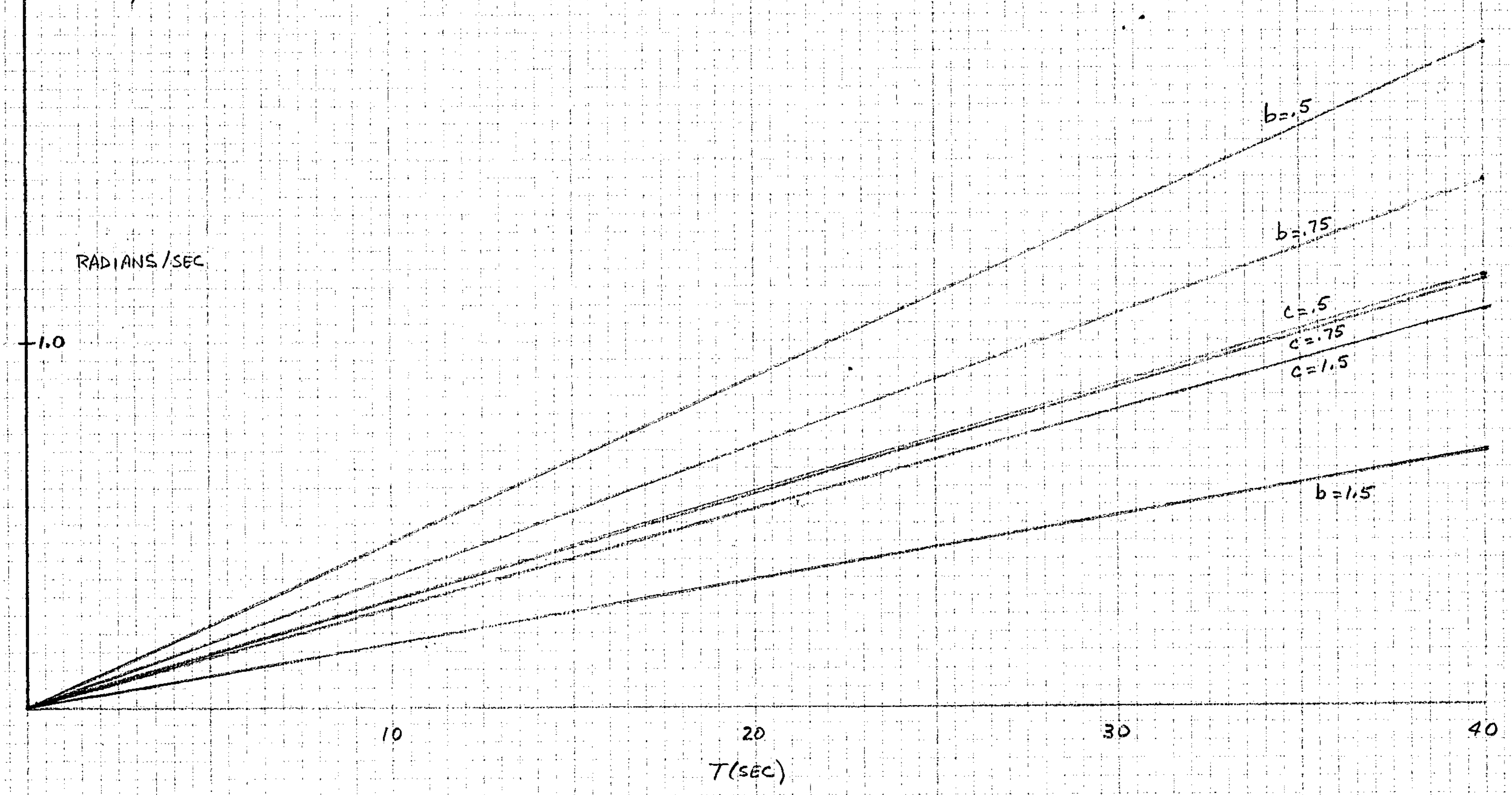
FIGURES (5a,b)

$^A\omega^B$ vs T



$\mu = .03$
$m_b = 10$ KG
$T/J = 5$ RADS/SEC^2
$\alpha = 5°$, $\beta = 10°$
$d = 0.5$
b, c SEE NOTE*

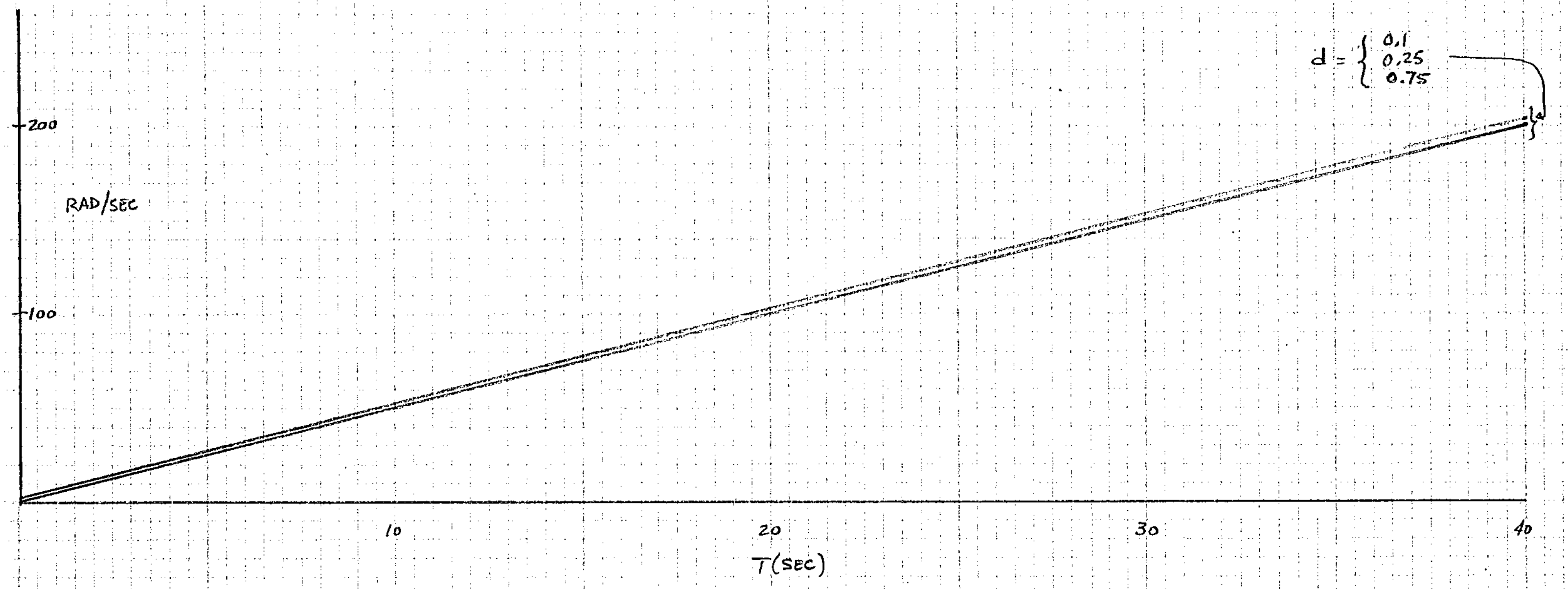
$|^R\omega^A|$ vs. T



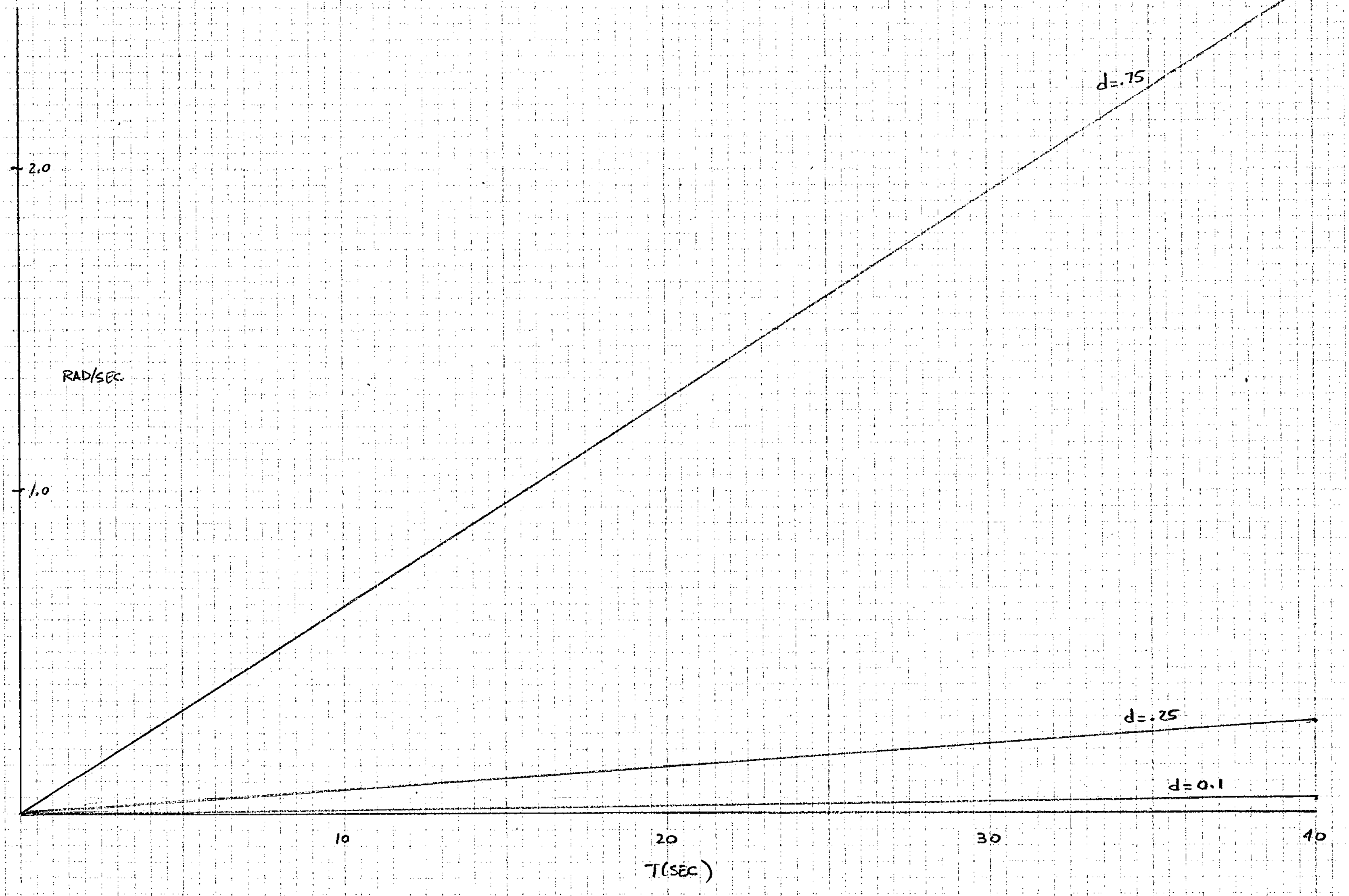
FOR VARIATIONS IN b, C=1.0 ALWAYS
FOR VARIATIONS IN C, b=1.0 ALWAYS

FIGURES (6a, b)

${}^A\omega^B$ vs. T



$|{}^R\omega^A|$ vs. T



3. Derivation

In deriving equations (4-7) we note that our system has seven degrees of freedom, namely: (x, y, z) the coordinates of the center of mass of G (also of A and B) ; ω_i $(i=1,2,3)$ and $\hat{\omega}^B$ as defined previously. We then use Lagrange's Form of D'Alembert's Principle

$$F_r + F_r^* = 0 \qquad r = 1, \ldots, 7 \qquad (1)$$

<u>For x, y, z</u> we may define $(x,y,z) = (q_1, q_2, q_3)$ coordinates and

$$F_r = \mathbb{V}^A_{\dot{q}_r} \cdot \mathbb{F}^A + \mathbb{V}^B_{\dot{q}_r} \cdot \mathbb{F}^B + \omega^A_{\dot{q}_r} \cdot \mathbb{T}^A + \omega^B_{\dot{q}_r} \cdot \mathbb{T}^B \qquad (2)$$

where $\mathbb{V}^A_{\dot{q}_r}$ and $\mathbb{V}^B_{\dot{q}_r}$ are the partial velocities of the points of application of resultant of all applied forces $\mathbb{F}^A$ and $\mathbb{F}^B$ on bodies A and B in R; $\omega_{\dot{q}_r}^A$ and $\omega_{\dot{q}_r}^B$ are the partial angular velocities of the points of application of the resultant of all applied torques $\mathbb{T}^A$ and $\mathbb{T}^B$ on bodies A and B in R.

We may also define

$$F_r^* = \mathbb{V}^{A^*}_{\dot{q}_r} \cdot \mathbb{F}^{*A} + \mathbb{V}^{B^*}_{\dot{q}_r} \cdot \mathbb{F}^{*B} + \omega^A_{\dot{q}_r} \cdot \mathbb{T}^{*A} + \omega^B_{\dot{q}_r} \cdot \mathbb{T}^{*B} \qquad (3)$$

where $\mathbb{V}_{\dot{q}_r}^{A^*}$ and $\mathbb{V}_{\dot{q}_r}^{B^*}$ are the partial velocities of the centers of mass of bodies A and B in R; $\mathbb{F}^{*A} = -m_A \hat{\mathbb{a}}^{*A}$ and $\mathbb{F}^{*B} = -m_B \hat{\mathbb{a}}^{*B}$, m_A and m_B are the masses of bodies A and B and $\hat{\mathbb{a}}^*$ is the acceleration of the center of mass ; $\mathbb{T}^{*A}$ and $\mathbb{T}^{*B}$ are the inertia torques of bodies A and B.

<u>For ω_i $(i=1,2,3)$ and $^A\omega^B$</u> we may define $(\omega_1, \omega_2, \omega_3, \hat{\omega}^B) = (u_1, u_2, u_3, u_4)$ generalized coordinates;

thus $$F_r = \mathbb{V}^A_{u_r} \cdot \mathbb{F}^A + \mathbb{V}^B_{u_r} \cdot \mathbb{F}^B + \omega^A_{u_r} \cdot \mathbb{T}^A + \omega^B_{u_r} \cdot \mathbb{T}^B \qquad (4)$$

and $$F_r^* = \mathbb{V}^{*A}_{u_r} \cdot \mathbb{F}^{*A} + \mathbb{V}^{*B}_{u_r} \cdot \mathbb{F}^{*B} + \omega^A_{u_r} \cdot \mathbb{T}^{*A} + \omega^B_{u_r} \cdot \mathbb{T}^{*B} \qquad (5)$$

where $(\ )_{u_r}$ are the partial rates with respect to the generalized coordinates.

Since $$\mathbb{V}^A = \mathbb{V}^B = \mathbb{V}^{*A} = \mathbb{V}^{*B} = \dot{x}\mathbb{n}_x + \dot{y}\mathbb{n}_y + \dot{z}\mathbb{n}_z \quad \text{and} \qquad (6)$$

$$\hat{\mathbb{a}}^{*A} = \hat{\mathbb{a}}^{*B} = \ddot{x}\,\mathbb{n}_x + \ddot{y}\,\mathbb{n}_y + \ddot{z}\,\mathbb{n}_z \quad \text{and because} \qquad (7)$$

$$^R\omega^A = \omega_1 \mathbb{a}_1 + \omega_2 \mathbb{a}_2 + \omega_3 \mathbb{a}_3 \quad \text{and} \quad ^R\omega^B = {}^R\omega^A + {}^A\omega^B = {}^R\omega^A + \hat{\omega}^B \beta \qquad (8,9)$$

we may then write, using $\mathbb{T}^A = -T\beta$ and $\mathbb{T}^B = T\beta$, $\mathbb{F}^A = m_A g\,\mathbb{n}_x$, $\mathbb{F}^B = m_B g\,\mathbb{n}_x$

$$\begin{array}{lll} F_1 = Mg & F_5 = 0 & F_2^* = -M\ddot{y} \\ F_2 = 0 & F_6 = 0 & F_3^* = -M\ddot{z} \qquad (10-18) \\ F_3 = 0 & F_7 = T & \quad \text{with } M = m_A + m_B \\ F_4 = 0 & F_1^* = -M\ddot{x} & \end{array}$$

Now to obtain $F_4 - F_7$ we may rewrite (6) as

$$F_r^* = \mathbb{V}_{u_r}^{*A} \cdot (\mathbb{F}^{*A} + \mathbb{F}^{*B}) + \omega_{u_r}^{A} \cdot (\mathbb{T}^{*A} + \mathbb{T}^{*B}) + {}^{A}\omega_{u_r}^{B} \cdot \mathbb{T}^{*B}$$

$$= \omega_{u_r}^{A} \cdot (\mathbb{T}^{*A} + \mathbb{T}^{*B}) + {}^{A}\omega_{u_r}^{B} \cdot \mathbb{T}^{*B} \qquad (19)$$

since $\mathbb{V}_{u_r}^{*A} = 0$

Now $\mathbb{T}^{*A} + \mathbb{T}^{*B} = -\frac{d}{dt}\left(\mathbb{H}^{A/A*} + \mathbb{H}^{B/B*}\right) = -\frac{d}{dt}\left(\mathbb{I}^{A/A*} \cdot \omega^{A} + \mathbb{I}^{B/B*} \cdot \omega^{B}\right)$

$$= -\frac{d}{dt}\left(\left[\mathbb{I}^{A/A*} + \mathbb{I}^{B/B*}\right] \cdot \omega^{A} + \mathbb{I}^{B/B*} \cdot {}^{A}\omega^{B}\right)$$

$$= (\mathbb{I}^{G/G*} \cdot \omega^{A}) \times \omega^{A} - \mathbb{I}^{G/G*} \cdot \alpha^{A} + (\mathbb{I}^{B/B*} \cdot {}^{A}\omega^{B}) \times {}^{A}\omega^{B} - \mathbb{I}^{B/B*} \cdot {}^{A}\alpha^{B} + (\mathbb{I}^{B/B*} \cdot {}^{A}\omega^{B}) \times \omega^{A} \qquad (20)$$

Using (8,9).

$$\alpha^{A} = \dot{\omega}_1 a_1 + \dot{\omega}_2 a_2 + \dot{\omega}_3 a_3 \qquad {}^{A}\alpha^{B} = {}^{A}\dot{\omega}^{B} \beta \qquad (21,22)$$

Remembering that $\mathbb{I}^{G/G*} = I_1 a_1 a_1 + I_2 a_2 a_2 + I_3 a_3 a_3$ and $\mathbb{I}^{B/B*} = J \beta\beta + \cdots$ (23,24)
and using these in (19) along with (20) and (8,9) we find

$$F_4^* = \left[\omega_2\omega_3(I_2 - I_3) - I_1\dot{\omega}_1 - J\,{}^{A}\dot{\omega}^{B}\beta_1 + J\,{}^{A}\omega^{B}(\beta_2\omega_3 - \beta_3\omega_2)\right] \qquad (25)$$

$$F_5^* = \left[\omega_3\omega_1(I_3 - I_1) - I_2\dot{\omega}_2 - J\,{}^{A}\dot{\omega}^{B}\beta_2 + J\,{}^{A}\omega^{B}(\beta_3\omega_1 - \beta_1\omega_3)\right] \qquad (26)$$

$$F_6^* = \left[\omega_1\omega_2(I_1 - I_2) - I_3\dot{\omega}_3 - J\,{}^{A}\dot{\omega}^{B}\beta_3 + J\,{}^{A}\omega^{B}(\beta_1\omega_2 - \beta_2\omega_1)\right] \qquad (27)$$

$$F_7^* = -J\left[\dot{\omega}_1\beta_1 + \dot{\omega}_2\beta_2 + \dot{\omega}_3\beta_3 + {}^{A}\dot{\omega}^{B}\right] \qquad (28)$$

Using (10-18, 25-28) and (1) we obtain after rearranging

$$\ddot{x} = g \qquad (29)$$

$$\ddot{y} = 0 \qquad (30)$$

$$\ddot{z} = 0 \qquad (31)$$

and the equations shown in section 1.

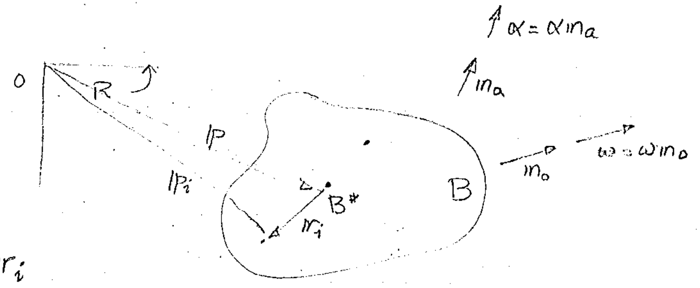
β is defined by

$$\beta = \cos\alpha\, a_3 + \sin\alpha \sin\beta\, a_2 + \sin\alpha\cos\beta\, a_1 \qquad (31)$$

For a set S of particles P_1 , ..., P_N , a quantity G , known as a Gibbs function, is defined as

$$G \triangleq \frac{1}{2} \sum_{i=1}^{N} m_i \underline{a}_i^2$$

where m_i and $\underline{a}_i$ are, respectively, the mass and the acceleration of P_i . Regarding a rigid body B as a set S of a large number of particles, express G as a function of the acceleration $\underline{a}$ of the mass center of B , the angular velocity $\underline{\omega}$ of B , the angular acceleration $\underline{\alpha}$ of B , the mass m of B , and the central inertia dyadic $\underline{I}$ of B .



$$G \triangleq \frac{1}{2} \sum_{i=1}^{N} m_i (\mathbf{a}_i \cdot \mathbf{a}_i)$$

Let $\mathbf{P}_i = \mathbf{P} + \mathbf{r}_i$

$$\mathbf{V}_i = \mathbf{V}^* + {}^{R}\boldsymbol{\omega}^{B} \times \mathbf{r}_i$$

$$\mathbf{a}_i = \mathbf{a}^* + \boldsymbol{\alpha} \times \mathbf{r}_i + {}^{R}\boldsymbol{\omega}^{B} \times ({}^{R}\boldsymbol{\omega}^{B} \times \mathbf{r}_i)$$

$$\mathbf{a}_i \cdot \mathbf{a}_i = \mathbf{a}^* \cdot \mathbf{a}^* + 2\mathbf{a}^* \cdot \boldsymbol{\alpha} \times \mathbf{r}_i + 2\mathbf{a}^* \cdot {}^{R}\boldsymbol{\omega}^{B} \times ({}^{R}\boldsymbol{\omega}^{B} \times \mathbf{r}_i) + \qquad + (\boldsymbol{\alpha} \times \mathbf{r}_i)^2$$
$$+ 2(\boldsymbol{\alpha} \times \mathbf{r}_i) \cdot [{}^{R}\boldsymbol{\omega}^{B} \times ({}^{R}\boldsymbol{\omega}^{B} \times \mathbf{r}_i)] + [{}^{R}\boldsymbol{\omega}^{B} \times ({}^{R}\boldsymbol{\omega}^{B} \times \mathbf{r}_i)]^2$$

now $\displaystyle\sum_{i=1}^{N} m_i (\mathbf{a}_i \cdot \mathbf{a}_i) = \left(\sum_{i=1}^{N} m_i\, \mathbf{a}^* \cdot \mathbf{a}^* = m\, \mathbf{a}^{*2}\right) + \left(\sum m_i \mathbf{a}^* \cdot \boldsymbol{\alpha} \times \mathbf{r}_i = \sum m_i \mathbf{r}_i \cdot (\mathbf{a}^* \times \boldsymbol{\alpha}) = 0\right)$

$$+ 2\left(\sum m_i\, \mathbf{a}^* \cdot {}^{R}\boldsymbol{\omega}^{B} \times ({}^{R}\boldsymbol{\omega}^{B} \times \mathbf{r}_i) = \mathbf{a}^* \cdot {}^{R}\boldsymbol{\omega}^{B} \times ({}^{R}\boldsymbol{\omega}^{B} \times \sum m_i \mathbf{r}_i) = 0\right)$$

$$+ \left(\sum m_i (\boldsymbol{\alpha} \times \mathbf{r}_i) \cdot (\boldsymbol{\alpha} \times \mathbf{r}_i) = \alpha^2 \mathbf{n}_a \cdot \sum m_i \mathbf{r}_i \times (\mathbf{n}_a \times \mathbf{r}_i) = \alpha^2 \mathbf{n}_a \cdot \mathbb{I}_a^{B/B^*}\right)$$

$$+ 2\left(\sum m_i (\boldsymbol{\alpha} \times \mathbf{r}_i) \cdot [{}^{R}\boldsymbol{\omega}^{B} \times ({}^{R}\boldsymbol{\omega}^{B} \times \mathbf{r}_i)] = \alpha \omega^2 \sum m_i (\mathbf{n}_a \times \mathbf{r}_i) \cdot [\mathbf{n}_o \times (\mathbf{n}_o \times \mathbf{r}_i)] = \right.$$

$$\alpha \omega^2 \sum m_i [\mathbf{r}_i \times (\mathbf{n}_o \times (\mathbf{n}_o \times \mathbf{r}_i))] \cdot \mathbf{n}_a = \alpha \omega^2\, \mathbf{n}_a \cdot \mathbf{n}_o \times \left[\sum m_i \mathbf{r}_i \times (\mathbf{n}_o \times \mathbf{r}_i)\right] =$$

$$\left. \omega \left[\boldsymbol{\alpha} \cdot \left\{{}^{R}\boldsymbol{\omega}^{B} \times \mathbb{I}_o^{B/B^*}\right\}\right]\right) + \left(\sum m_i [{}^{R}\boldsymbol{\omega}^{B} \times ({}^{R}\boldsymbol{\omega}^{B} \times \mathbf{r}_i)]^2 = \sum m_i \left\{{}^{R}\boldsymbol{\omega}^{B^2} [\boldsymbol{\omega} \times \mathbf{r}_i]^2 - [\boldsymbol{\omega} \cdot (\boldsymbol{\omega} \times \mathbf{r}_i\right.\right.$$

$$= \sum m_i\, {}^{R}\boldsymbol{\omega}^{B^2} [\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)] \cdot \boldsymbol{\omega} = ({}^{R}\boldsymbol{\omega}^{B})^2\, \boldsymbol{\omega} \cdot \sum m_i \mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i) = ({}^{R}\boldsymbol{\omega}^{B})^2\, \boldsymbol{\omega} \cdot \boldsymbol{\omega}\, \mathbb{I}_o^{B/B^*} \Big)$$

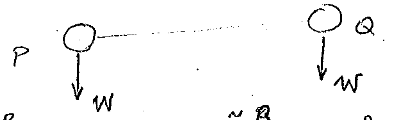
thus $\displaystyle G \triangleq \frac{1}{2}\left[m \mathbf{a}^2 + \boldsymbol{\alpha} \cdot \mathbb{I}^{B/B^*} \cdot \boldsymbol{\alpha} + 2\alpha \left\{{}^{R}\boldsymbol{\omega}^{B} \times [\boldsymbol{\omega} \cdot \mathbb{I}^{B/B^*}]\right\} + \omega^2 (\boldsymbol{\omega} \cdot \mathbb{I}^{B/B^*} \cdot \boldsymbol{\omega})\right]$

$$G \triangleq \frac{1}{2}\left[m \mathbf{a}^2 + \boldsymbol{\alpha} \cdot \mathbb{I}^{B/B^*} \cdot \boldsymbol{\alpha} + 2\, \boldsymbol{\omega} \cdot \left[(\mathbb{I}^{B/B^*} \cdot \boldsymbol{\omega}) \times \boldsymbol{\alpha}\right] + \omega^2\, \boldsymbol{\omega} \cdot \mathbb{I}^{B/B^*} \cdot \boldsymbol{\omega}\right]$$

6a. let P & Q in 3(j) has equal weights W find F_1^* assoc. with θ_1

$$F_{\theta_1}^* = \sum_{i=1}^{N} \tilde{\mathbb{V}}_{\dot\theta_1}^{P_i} \cdot \mathbb{F}_i^*$$

If we assume that the joints are smooth then



from previous $\tilde{\mathbb{V}}_{\dot\theta_1}^{P_1} = L\mathbb{j}_1$ $\qquad \tilde{\mathbb{V}}_{\dot\theta_1}^{R} = L\mathring{\mathbb{j}}_2 \dfrac{\sin(\theta_3-\theta_1)}{\sin(\theta_3-\theta_2)} = L\dfrac{\sin(\theta_3-\theta_1)}{\sin(\theta_3-\theta_2)}\left[\cos\theta_2\mathbb{n}_y - \sin\theta_2\mathbb{n}_x\right]$

Also $\mathbb{V}^P = \dot\theta_1 L\mathbb{j}_1$ and $\mathbb{V}^Q = \dot\theta_1 L\mathbb{j}_1 + \dot\theta_3\left[-3L\sin\theta_3\mathbb{n}_x + 3L\cos\theta_3\mathbb{n}_y\right] = L\mathring{\mathbb{j}}_2\dfrac{\sin(\theta_3-\theta_1)\dot\theta_1}{\sin(\theta_3-\theta_2)}$

$$\mathbb{a}^P = \frac{{}^R d\mathbb{V}^P}{dt} = \ddot\theta_1 L\mathbb{j}_1 + \dot\theta_1 L\left[-\sin\theta_1\dot\theta_1\mathbb{n}_y - \cos\theta_1\dot\theta_1\mathbb{n}_x\right] = \ddot\theta_1 L\mathring{\mathbb{j}}_1 - \dot\theta_1^2 L\mathring{\mathbb{i}}_1$$

thus $\tilde{\mathbb{V}}_{\theta_1}^{P} \cdot \mathbb{a}^P = L^2\ddot\theta_1$

$$\mathbb{a}^Q = \frac{{}^R d\mathbb{V}^Q}{dt} = \frac{d}{dt}\left(2\dot\theta_2 L\mathring{\mathbb{j}}_2\right) = 2\ddot\theta_2 L\mathring{\mathbb{j}}_2 + 2\dot\theta_2 L\frac{d\mathring{\mathbb{j}}_2}{dt}$$

$$+ 2\dot\theta_2 L * (\dot\theta_2\mathbb{k}\times\mathring{\mathbb{j}}_2) = 2\ddot\theta_2 L\mathring{\mathbb{j}}_2 - 2\dot\theta_2^2 L\mathring{\mathbb{i}}_2$$

$$\tilde{\mathbb{V}}_{\dot\theta_1}^{R} \cdot \mathbb{a}^Q = L\mathring{\mathbb{j}}_2\frac{\sin(\theta_3-\theta_1)}{\sin(\theta_3-\theta_2)} \cdot 2\ddot\theta_2 L\mathring{\mathbb{j}}_2 = 2\ddot\theta_2 L^2\frac{\sin(\theta_3-\theta_1)}{\sin(\theta_3-\theta_2)}$$

thus $F_{\theta_1}^* = -m\left[L^2\ddot\theta_1 + 2L^2\ddot\theta_2\dfrac{\sin(\theta_3-\theta_1)}{\sin(\theta_3-\theta_2)}\right] = -\dfrac{WL^2}{g}\left[\ddot\theta_1 + 2\ddot\theta_2\dfrac{\sin(\theta_3-\theta_1)}{\sin(\theta_3-\theta_2)}\right]$

6b. Example on pg 107 $\quad \mathbb{I}_2^{P/0}$ is not // $\mathbb{n}_2$

also in most cases $\mathbb{p}_i = a_i\mathbb{n}_a + b_i\mathbb{n}_b$ when $a_i = \mathbb{n}_a\cdot\mathbb{p}_i \quad b_i = \mathbb{n}_b\cdot\mathbb{p}_i \quad \mathbb{n}_a\cdot\mathbb{n}_b =$

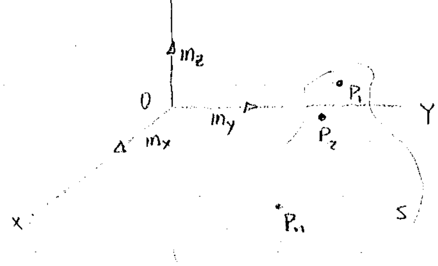
$\therefore \quad \mathbb{n}_a\times\mathbb{p}_i = b_i\,\mathbb{n}_a\times\mathbb{n}_b \qquad$ let $\mathbb{n}_a\times\mathbb{n}_b = \mathbb{n}_c$

now $\quad \mathbb{I}_a = \displaystyle\sum_{i=1}^{n} m_i\left[a_i\mathbb{n}_a + b_i\mathbb{n}_b\right]\times b_i\mathbb{n}_c$

$\qquad = \displaystyle\sum_{i=1}^{n} m_i a_i b_i(-\mathbb{n}_b) + \sum m_i b_i^2\mathbb{n}_a$

in most case $\displaystyle\sum_{i=1}^{n} m_i a_i b_i \neq 0 \quad \therefore \quad \mathbb{I}_a$ will not be parallel to $\mathbb{n}_a$

6c.



$$\mathbb{I}_x = \sum m_i\mathbb{p}_i\times(\mathbb{n}_x\times\mathbb{p}_i)$$

Let $\mathbb{p}_i = x_i\mathbb{n}_x + y_i\mathbb{n}_y + z_i\mathbb{n}_z$

$$\mathbb{n}_x\times\mathbb{p}_i = y_i\mathbb{n}_z - z_i\mathbb{n}_y$$

$$\mathbb{I}_x = \sum m_i(x_i\mathbb{n}_x + y_i\mathbb{n}_y + z_i\mathbb{n}_z)\times(y_i\mathbb{n}_z - z_i\mathbb{n}_y)$$

$$= \sum m_i\left[-x_i y_i\mathbb{n}_y - x_i z_i\mathbb{n}_z + y_i^2\mathbb{n}_x + z_i^2\mathbb{n}_x\right]$$

$$I_x = \mathbb{I}_x\cdot\mathbb{n}_x = \sum m_i(y_i^2 + z_i^2)$$

$$\mathbb{I}_y = \sum m_i \mathbb{p}_i \times (\mathbb{n}_y \times \mathbb{p}_i)$$

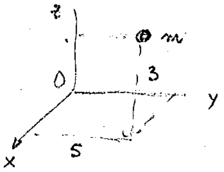
$$\mathbb{n}_y \times \mathbb{p}_i = -x_i \mathbb{n}_z + z_i \mathbb{n}_x$$

$$\mathbb{p}_i \times (-x_i \mathbb{n}_z + z_i \mathbb{n}_x) = +x_i^2 \mathbb{n}_y - y_i x_i \mathbb{n}_x - y_i z_i \mathbb{n}_z + z_i^2 \mathbb{n}_y$$

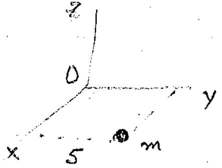
$$I_{yz} = \mathbb{I}_y \cdot \mathbb{n}_z = -\sum m_i y_i z_i$$

- RHS or LHS doesn't matter since your taking the double cross product. This produces two minus signs which cancel each other
- For non mutually ⊥ coordinate system we will find that results are not as simple as those shown, because we will get components that involve the sines of the angle between these vectors. In a ⊥ system the sines are = 0 or 1.
- The minus sign comes in from the definition of $I_{yz} = \mathbb{I}_y \cdot \mathbb{n}_z$

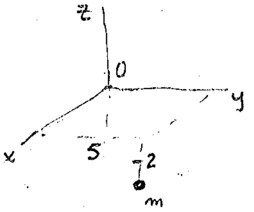
6(d)



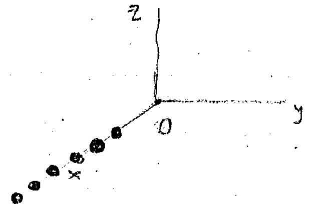
1 particle $I_{yz} = -m \cdot 15 = -15m < 0$



1 particle $I_{yz} = -m \cdot 5 \cdot 0 = 0$



$I_{yz} = -m\,(5)(-2) = 10m > 0$

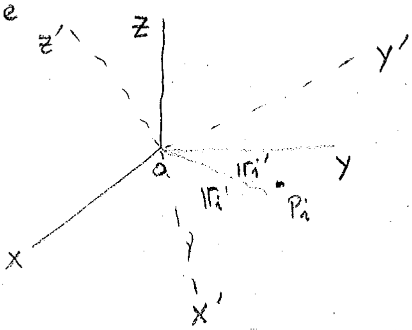


$$I_x = \sum_{i=1}^{6} m_i (z_i^2 + y_i^2) = 0$$

$$= M k^2 = 0$$

$$\therefore \; k = 0 \qquad M = \sum_{i=1}^{6} m$$

6e



$I_x = A \qquad I_y = B \qquad I_z = C$

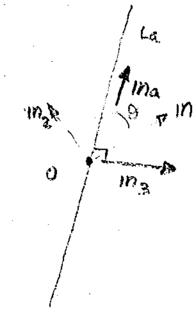
$I_x' = A' \qquad I_y' = B' \qquad I_z' = C'$

now $I_x + I_y + I_z = \sum_{i=1}^{N} 2m_i (x_i^2 + y_i^2 + z_i^2) = \sum 2m_i \mathbb{r}_i \cdot \mathbb{r}_i$

but $x_i^2 + y_i^2 + z_i^2$ = the distance squared from O to the point P_i. Distances are the same under transformation hence $I_x' + I_y' + I_z' = \sum 2m_i \mathbb{r}_i' \cdot \mathbb{r}_i' = \sum 2m_i \mathbb{r}_i \cdot \mathbb{r}_i$

thus $I_x + I_y + I_z = I_x' + I_y' + I_z'$

6f.



Since n_3 is ⊥ to La then La must lie in the plane of $n_1 \times n_2$ hence $n_a = (n_a \cdot n_1)n_1 + (n_a \cdot n_2)n_2$

$$\mathbb{I} = \sum_{j=1}^{3}\sum_{k=1}^{3} n_j I_{jk} n_k \qquad n_a \cdot n_3 = 0$$

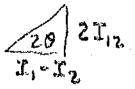
$$n_a \cdot \mathbb{I} \cdot n_a = I_a = \sum_{j=1}^{2}\sum_{k=1}^{2} n_a \cdot n_j I_{jk} n_k \cdot n_a$$

	n_1	n_2
n_a	$\cos\theta$	$\sin\theta$

now $I_a = I_1 \cos^2\theta + I_2 \sin^2\theta + I_{12}\sin 2\theta$

$$\frac{\partial I_a}{\partial\theta} = 0 \Rightarrow -I_1 \cdot 2\cos\theta\sin\theta + 2I_2\sin\theta\cos\theta + 2I_{12}\cos 2\theta = 0$$

$$-(-I_2 + I_1)\sin 2\theta + 2I_{12}\cos 2\theta = 0 \quad \text{or} \quad \tan 2\theta = \frac{2I_{12}}{I_1 - I_2}.$$



$$I_a = I_1\left(\frac{\cos 2\theta + 1}{2}\right) + I_2\left(\frac{1 - \cos 2\theta}{2}\right) + I_{12}\sin 2\theta$$

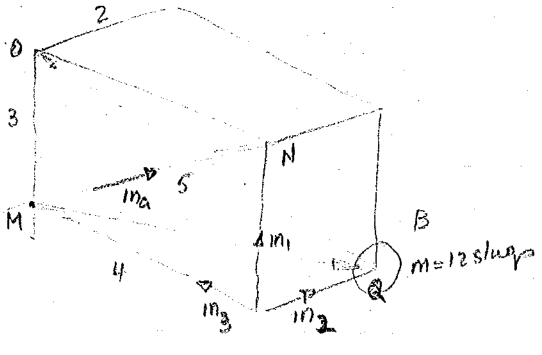
$$= \frac{I_1 + I_2}{2} + \frac{I_1 - I_2}{2} \cdot \frac{I_1 - I_2}{\sqrt{\quad}} + \frac{2I_{12}^2}{\sqrt{\quad}}$$

$$\frac{(I_1 - I_2)^2}{2\sqrt{\quad}} + \frac{4I_{12}^2}{2\sqrt{\quad}} = \frac{\left(\pm[\qquad]^{1/2}\right)^2}{2\sqrt{\quad}}$$

$$\sqrt{\quad} = \sqrt{(I_1 - I_2)^2 + 4I_{12}^2}$$

$$I_a = \frac{I_1 + I_2}{2} \pm \left[\left(\frac{I_1 - I_2}{2}\right)^2 + I_{12}^2\right]^{1/2} \quad \begin{array}{l} + \text{ gives max} \\ - \text{ gives min} \end{array}$$

6g.



$I_{jk}^{B/0}$	1	2	3
1	260	72	-144
2	72	325	96
3	-144	96	169

find $I_a^{B/M} = n_a \cdot \mathbb{I} \cdot n_a$

$n_a = (3n_1 - 4n_3)\frac{1}{5}$

$$\mathbb{I}^{B/0} = \sum_{j=1}^{3}\sum_{k=1}^{3} n_j I_{jk}^{B/0} n_k \qquad n_a \cdot n_* = \frac{3}{5} \quad n_a \cdot n_2 = 0 \quad n_a \cdot n_3 = -\frac{4}{5}$$

$$I_a^{B/0} = \frac{3}{5}\left[I_{11} \cdot \frac{3}{5} + I_{12} \cdot 0 + I_{13} \cdot -\frac{4}{5}\right] + 0[\qquad] - \frac{4}{5}\left[I_{31} \cdot \frac{3}{5} + I_{33}\left(-\frac{4}{5}\right)\right]$$

$$= \frac{9}{25} I_{11} - \frac{12}{25} I_{13} - \frac{12}{25} I_{31} + \frac{16}{25} I_{33}$$

$$I_a^{B/0} = \frac{1}{25}\left[9 \cdot 260 + 24 \cdot 144 + 169 \cdot 16\right] = \frac{8500}{25}$$

$$I_a^{B/M} = I_a^{B/Q} + I_a^{Q/M} \qquad I_a^{Q/M} = \mathbb{I}_a^{Q/M} \cdot n_a = n_a \cdot \left[\mathbb{p}^{Q/M} \times (n_a \times \mathbb{p}^{Q/M})\right] m$$

$$I_a^{B/Q} = I_a^{B/0} + I_a^{0/Q} \quad ; \quad I_a^{0/Q} = \mathbb{I}_a^{0/Q} \cdot n_a = n_a \cdot m\left[\mathbb{p}^{0/Q} \times (n_a \times \mathbb{p}^{0/Q})\right]$$

$$\begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ 3/5 & 0 & -4/5 \\ 3 & -2 & 4 \end{vmatrix}$$

$\mathbf{p}^{O/Q} = -2\mathbf{n}_2 + 3\mathbf{n}_1 + 4\mathbf{n}_3$ $\qquad \mathbf{n}_a \times \mathbf{p}^{O/Q} = -\frac{6}{5}\mathbf{n}_3 - \frac{12}{5}\mathbf{n}_2 - \frac{8}{5}\mathbf{n}_1 - \frac{12}{5}\mathbf{n}_2\Big] = \Big[+\frac{6}{5}\mathbf{n}_3 + \frac{24}{5}\mathbf{n}_2 + \frac{8}{5}\mathbf{n}_1\Big]$

$-\mathbf{p}^{O/Q} \times \left[+\frac{8}{5}\mathbf{n}_1 + \frac{24}{5}\mathbf{n}_2 + \frac{6}{5}\mathbf{n}_3\right]$ $\qquad \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ +3 & +2 & +4 \\ +8/5 & +24/5 & +6/5 \end{vmatrix} = +\frac{12}{5}\mathbf{n}_1 - \frac{32}{5}\mathbf{n}_2 - \frac{72}{5}\mathbf{n}_3 - \frac{16}{5}\mathbf{n}_3 + \frac{18}{5}\mathbf{n}_2 + \frac{96}{5}\mathbf{n}_1$

$= \frac{108}{5}\mathbf{n}_1 - \frac{14}{5}\mathbf{n}_2 - \frac{88}{5}\mathbf{n}_3$

$\therefore \mathbf{n}_a \cdot \left[\mathbf{p}^{O/Q} \times (\mathbf{n}_a \times \mathbf{p}^{O/Q})\right] = \frac{3\mathbf{n}_1 - 4\mathbf{n}_3}{5} \cdot \left(\frac{108}{5}\mathbf{n}_1 - \frac{14}{5}\mathbf{n}_2 - \frac{88}{5}\mathbf{n}_3\right)$

$\frac{324}{25} + \frac{352}{25} = \frac{676}{25}$ $\qquad \therefore \underline{I_a^{O/Q} = m \cdot \frac{676}{25}}$ $\qquad I_a^{Q/O} = -\frac{676\,m}{25}$

$\mathbf{p}^{Q/M} = (2\mathbf{n}_2 - 4\mathbf{n}_3)$ $\qquad \mathbf{n}_a \times \mathbf{p}^{Q/M} = \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ 3/5 & 0 & -4/5 \\ 0 & 2 & -4 \end{vmatrix} = 6/5\,\mathbf{n}_3 + \frac{12}{5}\mathbf{n}_2 + \frac{8}{5}\mathbf{n}_1$

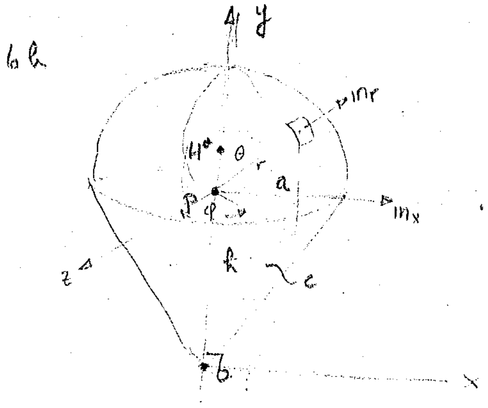
$\mathbf{p}^{Q/M} \times \left[\mathbf{n}_a \times \mathbf{p}^{Q/M}\right] = \begin{vmatrix} \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ 0 & 2 & -4 \\ +8/5 & +12/5 & +6/5 \end{vmatrix} = \frac{12}{5}\mathbf{n}_1 - \frac{32}{5}\mathbf{n}_2 - \frac{16}{5}\mathbf{n}_3 + \frac{48}{5}\mathbf{n}_1 = \frac{60}{5}\mathbf{n}_1 - \frac{32}{5}\mathbf{n}_2 - \frac{16}{5}\mathbf{n}_3$

$\mathbf{n}_a \cdot \left[\mathbf{p}^{Q/M} \times (\mathbf{n}_a \times \mathbf{p}^{Q/M})\right] = \frac{(3\mathbf{n}_1 - 4\mathbf{n}_3)}{5} \cdot \left(\frac{60\mathbf{n}_1}{5} - \frac{32}{5}\mathbf{n}_2 - \frac{16}{5}\mathbf{n}_3\right) = \frac{180 + 64}{25} = \frac{244}{25}$

$\underline{I_a^{Q/M} = m \cdot \frac{244}{25}}$ $\qquad$ now $\; I_a^{B/M} = I_a^{B/O} - I_a^{Q/O} + I_a^{B/Q} = \frac{8500 + m(244 - 676)}{25}$

$I_a^{B/M} = \frac{8500}{25} + \left(-\frac{676}{25} + \frac{244}{25}\right)m = 3316$

6 h



$I_x^{C/O} = \frac{3}{5} m\left(\frac{1}{4}a^2 + h^2\right)$ $\qquad$ if $r = h = a$

$= \frac{3}{5} m\left(\frac{5}{4} r^2\right) = \frac{3}{4} m r^2$

$I_x^{H/O} = I_x^{H/H^*} + I_x^{H^*/O}$

$= \frac{83}{320} \cdot 2mR^2 + 2m \cdot \left(\frac{3}{8}R\right)^2 = \frac{86}{20} m R^2$

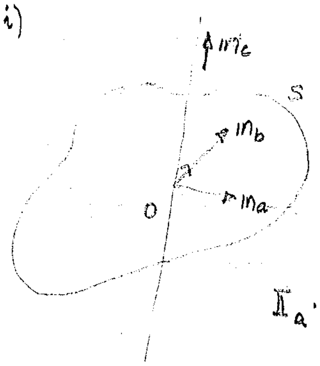
$I_x^{S/O} = I_x^{H/O} + I_x^{C/O} = \left(\frac{15}{20} + \frac{86}{20}\right) m R^2 = \frac{101}{20} m R^2$

$$\therefore \quad \mathbb{I}_x^{H/H^*} = -\mathbb{I}_x^{H^*/P} + \mathbb{I}_x^{H/P} = -\frac{16mR^2}{9\pi^2} + \frac{4mR^2}{5}$$

$$\mathbb{I}_x^{H/A} = \mathbb{I}_x^{H/H^*} + \mathbb{I}_x^{H^*/A} = -\frac{16mR^2}{9\pi^2} + \frac{4mR^2}{5} + 2m\left[\frac{4R}{3\pi} + R\right]^2 = \left(\frac{8R^2}{3\pi} + R^2\right)2m + \frac{4mR^2}{5}$$

$$\mathbb{I}_x^{\text{Cone}} = \frac{15}{20}mR^2 + \frac{16mR^2}{20} + \frac{40R^2}{20}$$

6i)



$$\mathbb{I}_c \cdot \mathbf{n}_c = I_c = \sum m_i \left[\mathbf{p}_i \times (\mathbf{n}_c \times \mathbf{p}_i)\right]$$

where $\mathbf{p}_i = a_i \mathbf{n}_a + b_i \mathbf{n}_b$

$$\mathbf{n}_c \times \mathbf{p}_i = a_i \mathbf{n}_b + b_i \mathbf{n}_a$$

$$(a_i \mathbf{n}_a + b_i \mathbf{n}_b) \times (a_i \mathbf{n}_b - b_i \mathbf{n}_a) = a_i a_i \mathbf{n}_c + b_i^2 \mathbf{n}_c = (a_i^2 + b_i^2)\mathbf{n}_c$$

$$\mathbb{I}_c \cdot \mathbf{n}_c = I_c = \sum m_i (a_i^2 + b_i^2)$$

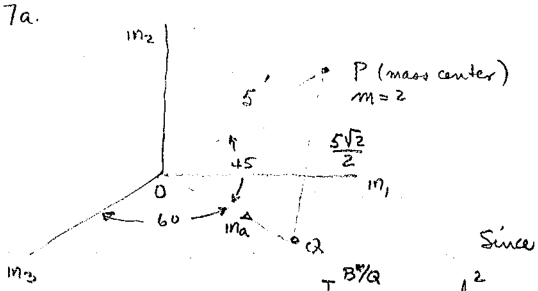
$$\mathbb{I}_a \cdot \mathbf{n}_a = I_a = \sum m_i \left[\mathbf{p}_i \times (\mathbf{n}_a \times \mathbf{p}_i) \cdot \mathbf{n}_a\right]$$

$$\mathbf{n}_a \times \mathbf{p}_i = b_i \mathbf{n}_c \qquad \mathbf{p}_i \times (\mathbf{n}_a \times \mathbf{p}_i) = -a_i b_i \mathbf{n}_b + b_i^2 \mathbf{n}_a$$

$I_a = \sum m_i b_i^2 \quad ; \quad I_b = \sum m_i a_i^2$ similarly

$$\therefore \quad I_c = I_a + I_b$$

7a.



$I^{B/B^*} = 20 \quad I_2 = 30 \quad I_3 = 40$

find $I_a^{B/Q}$

$\mathbf{n}_a = \frac{1}{2}\mathbf{n}_3 + \frac{\sqrt{3}}{2}\mathbf{n}_1$

Since we want $I_a^{B/Q} = I_a^{B/B^*} + I_a^{B^*/Q}$

$$I_a^{B^*/Q} = m \cdot l_{PQ}^2 = 2 \cdot \left(\frac{5\sqrt{2}}{2}\right)^2 = 2 \cdot \frac{25}{2} = 25 \text{ slug-ft}^2$$

$$I_a^{B/B^*} = \mathbf{n}_a \cdot \mathbb{I}^{B/B^*} \cdot \mathbf{n}_a$$

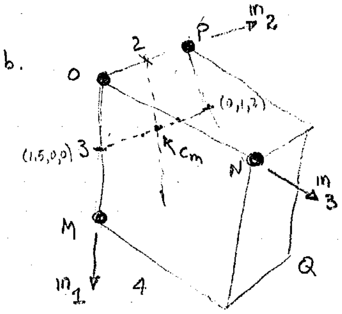
where $\mathbb{I}^{B/B^*} = I_1 \mathbf{n}_1\mathbf{n}_1 + I_2 \mathbf{n}_2\mathbf{n}_2 + I_3 \mathbf{n}_3\mathbf{n}_3$; since $\mathbf{n}_a$ has no $\mathbf{n}_2$ component

then $\mathbb{I}_a^{B/B^*} = (I_1 \mathbf{n}_1)\frac{\sqrt{3}}{2} + (I_3 \mathbf{n}_3) \cdot \frac{1}{2}$ $\qquad I_a = I_1\left(\frac{\sqrt{3}}{2}\right)^2 + I_3\left(\frac{1}{2}\right)^2 = I_1 \cdot \frac{3}{4} + I_3 \cdot \frac{1}{4}$

$$= 20 \cdot \frac{3}{4} + 40 \cdot \frac{1}{4} = 25$$

$$I_a^{B/Q} = I_a^{B/B^*} + I_a^{B^*/Q} = 25 + 25 = 50 \text{ slug-ft}^2$$

7b.



To find min radius of gyration we need to find the min moment of inertia, which will be the minimum principal moment of inertia. Thus if we call it I_{min} then $k = \sqrt{\frac{I_{min}}{4m}}$. Now the min principal moment of inertia will be at the mass center. By construction we can find the mass center. To this end, define a set of coordinates and let us find the moments of inertia about the given directions $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ for point O for the system.

Using (problem 6c) $I_1^{S/O} = \sum m_i \left((x_2)_i^2 + (x_3)_i^2\right) \qquad I_2^{S/O} = \sum m_i \left[(x_3)_i^2 + (x_1)_i^2\right]$

$I_3^{S/O} = \sum m_i \left[(x_1)_i^2 + (x_2)_i^2\right] \qquad I_{12}^{S/O} = -\sum m_i (x_1)_i (x_2)_i \qquad I_{23}^{S/O} = -\sum m_i (x_2)_i (x_3$

and $I_{31}^{S/O} = -\sum m_i (x_3)_i (x_1)_i$

P_1 located at $(0,0,0)$ $\quad P_2(3,0,0) \quad P_3(0,2,0) \quad P_4(0,0,4)$. Since all have same mass m

thus $\sum (x_2)_i^2 = 4 \quad \sum (x_1)_i^2 = 9 \quad \sum (x_3)_i^2 = 16. \qquad \sum (x_1 x_2)_i = 0 \qquad \sum \left[(x_2)(x_3)\right]_i = 0$

$I_1^{S/O} = m \cdot 20 \quad I_2^{S/O} = m \cdot 25 \quad I_3^{S/O} = m \cdot 13 \qquad \sum (x_3 x_1)_i = 0$

$I_{jk}^{S/O}$	1	2	3
1	20m	0	0
2	0	25m	0
3	0	0	13m

$\longrightarrow$ cm $= \left(\frac{3}{4}, \frac{1}{2}, 1\right)$ obtained by $\bar{x}_i = \sum_{j=1}^{4}(x_i m_i)_j / \sum m_i$

$\longrightarrow$ $I_{jk}^{S/S^*} = I_{jk}^{S/O} - I_{jk}^{S^*/O}$ w/. $I_{jk}^{S^*/O} = 4m\{\mathbf{p}^{S^*/O^2}\delta_{jk} - (\mathbf{n}_j \cdot \mathbf{p})(\mathbf{p}$

$\left(\mathbf{p}^{S^*/O}\right)^2 = \frac{9}{16} + \frac{4}{16} + \frac{16}{16} = \frac{29}{16}$

thus

$I_{jk}^{S^*/O}$	1	2	3
1	5m	$-\frac{3}{2}$m	-3m
2	$-\frac{3}{2}$m	$\frac{25}{4}$m	-2m
3	-3m	-2m	$\frac{13}{4}$m

NOTE :

$I_{jk}^{S^*/O} = 4m\left[\mathbf{p}^{S^*/O} \times (\mathbf{n}_j \times \mathbf{p}^{S^*/O})\right] \cdot \mathbf{n}_k$

$4m\left(\mathbf{p}^{S^*/O} \times (\mathbf{n}_j \times \mathbf{p}^{S^*/O}\right) \cdot \mathbf{n}_k$

but $\mathbf{p} \times (\mathbf{n}_j \times \mathbf{p}) = (\mathbf{p} \cdot \mathbf{p})\mathbf{n}_j - (\mathbf{p} \cdot \mathbf{n}_j)\mathbf{p}$

& dot w/ $\mathbf{n}_k = (\mathbf{p} \cdot \mathbf{p})\delta_{jk} - (\mathbf{p} \cdot \mathbf{n}_j)(\mathbf{p}$

thus

I_{jk}^{S/S^*}	1	2	3
1	15m	1.5m	3m
2	1.5m	18.75m	2m
3	3m	2m	9.75m

Aside: we note that when $j = k$ $\quad I_j^{S^*/O} = (\mathbf{p}^2 - (\mathbf{p}\cdot\mathbf{n}_j)^2)\cdot 4m$ but $(\mathbf{p}\cdot\mathbf{n}_j)$ is the proj onto the $\mathbf{n}_j$ axis

$$= 4m[\mathbf{p}^2 - \mathbf{p}^2 \cos(\mathbf{p},\mathbf{n}_j)] = [\mathbf{p}^2 \sin^2(\mathbf{p},\mathbf{n}_j)]4m = (4m)\,\ell^2$$ where $\ell = \perp$ distance from S^* to $\mathbf{n}_j$ axis.

thus to minimize I_{jk}^{S/S^*} $\Rightarrow$ det

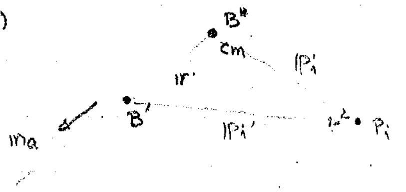
$$m^3\begin{bmatrix} 15-\tilde{I} & 1.5 & 3 \\ 1.5 & 18.75-\tilde{I} & 2 \\ 3 & 2 & 9.75-\tilde{I} \end{bmatrix} = 0$$

where $\tilde{I} = I/m$

thus $(15-\tilde{I})(18.75-\tilde{I})(9.75-\tilde{I}) + 18 - 9(18.75-\tilde{I}) - 2.25(9.75-\tilde{I}) - 4(15-\tilde{I}) = 0$ Solution to this for min I must be $< 9.75m$. $I_{min} = 8.243m$ is obtained after solution

now $k_{min} = \sqrt{\dfrac{I_{min}}{4m}} = \sqrt{2.061} = 1.436$ ft

7c.
(i)



$\mathbb{I}_a^{B/B^*} = \sum m_i \mathbf{p}_i \times (\mathbf{n}_a \times \mathbf{p}_i)$. for a principal axis $\mathbb{I}_a = \lambda\mathbf{n}_a$

thus $\mathbb{I}_a^{B/B^*} \times \mathbf{n}_a = 0$

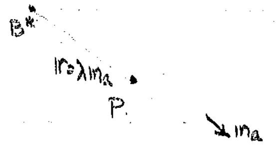
$\mathbb{I}_a^{B/B'} = \mathbb{I}_a^{B/B^*} + \mathbb{I}_a^{B^*/B'}$ if $\mathbf{p}^{B^*/B'} = \hat{\lambda}\mathbf{n}_a$

$= \mathbb{I}_a^{B/B^*} + M(\hat{\lambda}\mathbf{n}_a \times \underbrace{(\mathbf{n}_a \times \hat{\lambda}\mathbf{n}_a)})$

thus $\mathbb{I}_a^{B/B'} = \mathbb{I}_a^{B/B^*}$ but $\mathbb{I}_a^{B/B'} \times \mathbf{n}_a \overset{0}{=} \mathbb{I}_a^{B/B^*} \times \mathbf{n}_a = 0$

hence principal axis for B^* is a principal axis for any point on the axis.

7c (ii)

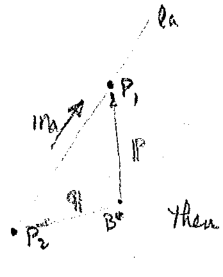


now $\mathbb{I}_a^{B/P} \times \mathbf{n}_a = 0$ since $\mathbf{n}_a$ is a principal axis for P. but it also passes through B^* hence we can form $\mathbb{I}_a^{B/B^*}$. Yet $\mathbb{I}_a^{B/B^*} = \mathbb{I}_a^{B/P} + \mathbb{I}_a^{P/B^*} = \mathbb{I}_a^{B/P} - \mathbb{I}_a^{B^*/P}$

now $\mathbb{I}_a^{B/B^*} \times \mathbf{n}_a = (\mathbb{I}_a^{B/P} \times \mathbf{n}_a = 0$ by hypothesis$) - \mathbb{I}_a^{B^*/P} \times \mathbf{n}_a$

but $\mathbb{I}_a^{B^*/P} \times \mathbf{n}_a = M[\lambda\mathbf{n}_a \times (\mathbf{n}_a \times \lambda\mathbf{n}_a)] \times \mathbf{n}_a = 0 \Rightarrow \mathbb{I}_a^{B/B^*} \times \mathbf{n}_a = 0$ which is only true if $\mathbb{I}_a^{B/B^*} = \mu\mathbf{n}_a \Rightarrow$ the axis is a principal axis for B^*

7c(iii)



if $\mathbb{I}_a^{B/P_1} \times \mathbf{n}_a = 0$ & $\mathbb{I}_a^{B/P_2} \times \mathbf{n}_a = 0 \Rightarrow \mathbb{I}_a^{B/B^*} \times \mathbf{n}_a = 0$

$\mathbb{I}_a^{B/P_1} = \mathbb{I}_a^{B/B^*} + \mathbb{I}_a^{B^*/P_1}$ and $-\mathbb{I}_a^{B/B^*} \times \mathbf{n}_a = \mathbb{I}_a^{B^*/P_1} \times \mathbf{n}_a$

$\mathbb{I}_a^{B/P_2} = \mathbb{I}_a^{B/B^*} + \mathbb{I}_a^{B^*/P_2}$ $\quad -\mathbb{I}_a^{B/B^*} \times \mathbf{n}_a = \mathbb{I}_a^{B^*/P_2} \times \mathbf{n}_a$

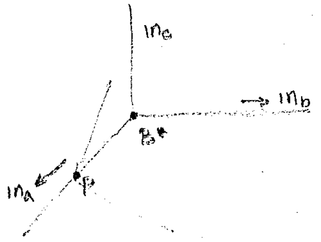
then $\mathbb{I}_a^{B/P_1} - \mathbb{I}_a^{B/P_2} = \mathbb{I}_a^{B^*/P_1} - \mathbb{I}_a^{B^*/P_2}$ and

$\mathbb{I}_a^{B^*/P_1} \times \mathbf{n}_a = \mathbb{I}_a^{B^*/P_2} \times \mathbf{n}_a$ but since P_2, P_1 are arbitrary

then $\Rightarrow$ either $\overset{m[-\mathbf{p}\cdot\mathbf{n}_a]\mathbf{p}\times\mathbf{n}_a \,=\, m[-\mathbf{q}\cdot\mathbf{n}_a]\mathbf{q}\times\mathbf{n}_a}{\mathbf{p} = \mathbf{q}}$ or $\mathbf{p} \neq \mathbf{q}$ must be $\parallel$ to $\mathbf{n}_a \Rightarrow \mathbb{I}_a^{B^*/P_1} \times \mathbf{n}_a = 0$ since $\mathbf{p} \neq \mathbf{q}$ in g

$\Rightarrow \mathbb{I}_a^{B/B^*} \times \mathbf{n}_a = 0 \Rightarrow \mathbb{I}_a^{B/B^*} \parallel \mathbf{n}_a \Rightarrow$ La is a principal axis for B^* by 7c ii

7c(iv)



since P lies on a principal axis of the mass center then $\mathbb{I}_a^{B/P} \times \mathbf{n}_a = 0$ thus $\mathbb{I}_a^{B/B^*} \cdot \mathbf{n}_b = 0$ and $\mathbb{I}_a^{B/B^*} \cdot \mathbf{n}_c = 0$

but $\mathbb{I}_a^{B/P} = \mathbb{I}_a^{B/B^*} + \mathbb{I}_a^{B^*/P}$ &

$\mathbb{I}_a^{B/P} \cdot \mathbf{n}_b = 0 + I_{ab}^{B^*/P}$ but $I_{ab}^{B^*/P} = m \cdot 0 = 0 \Rightarrow I_{ab}^{B/P} = 0$

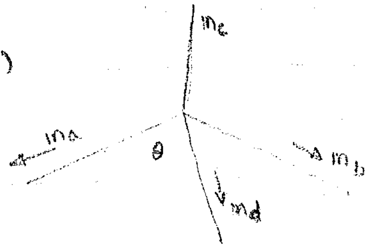
similarly we can show that $\mathbb{I}_a^{B/P} \cdot \mathbf{n}_c = I_{ac}^{B/P} = 0$

also $\mathbb{I}_b^{B/P} = \mathbb{I}_b^{B/B^*} + \mathbb{I}_b^{B^*/P}$ but $\mathbb{I}_b^{B^*/P} = m\left(\lambda\mathbf{n}_a \times (\mathbf{n}_b \times \lambda\mathbf{n}_a)\right) = +m\lambda^2\mathbf{n}_b$ where $\mathbf{r}^{B^*/P} = \lambda\mathbf{n}_a$

hence $\mathbb{I}_b^{B/P} \times \mathbf{n}_b = \mathbb{I}_b^{B/B^*} \times \mathbf{n}_b + m\lambda^2\mathbf{n}_b \times \mathbf{n}_b = 0 \Rightarrow \mathbb{I}_b^{B/P} \parallel \mathbf{n}_b$ and hence is a principal axis of $\mathbb{I}_b^{B/P}$. We can do this too for $\mathbb{I}_c^{B/P}$ & show that it is $\parallel$ to $\mathbf{n}_c$. Similarly $I_{ac}^{B/P} = I_{bc}^{B/P} = 0$

hence the principal axes for any point on a principal axis for the mass center are $\parallel$ to the principal axes for the mass center.

7c(v)



if $I_a = I_b \Rightarrow I_d = I_a = I_b$

$$\mathbb{I} = I(\mathbf{n}_a\mathbf{n}_a + \mathbf{n}_b\mathbf{n}_b) + J\mathbf{n}_c\mathbf{n}_c$$

$$\mathbf{n}_d \cdot \mathbb{I}\,\mathbf{n}_d = \mathbf{n}_d \cdot I(\mathbf{n}_a\mathbf{n}_a + \mathbf{n}_b\mathbf{n}_b) \cdot \mathbf{n}_d$$

$$I_d = I(\cos^2\theta + \sin^2\theta) = I$$

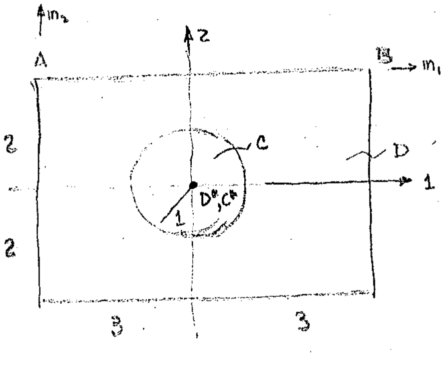
since $\mathbf{n}_a$ & $\mathbf{n}_b$ are principal axes $\therefore$ a direction $\mathbf{n}_c$ $\perp$ to $\mathbf{n}_a, \mathbf{n}_b$ also principal

$\mathbf{n}_d \cdot \mathbf{n}_a = \cos\theta$

$\mathbf{n}_d \cdot \mathbf{n}_b = \sin\theta$

thus $I_d = I_a = I_b$

7(d)



let $S = D - C$.

I_{jk}^{D/D^*}	1	2	3
1	$\frac{4m}{3}$	0	0
2	0	$3m$	0
3	0	0	$\frac{13m}{3}$

I_{jk}^{C/C^*}	1	2	3
1	$m/4$	0	0
2	0	$m/4$	0
3	0	0	$m/2$

thus

I_{jk}^{S/S^*}	1	2	3
1	$\frac{13m}{12}$	0	0
2	0	$11\frac{m}{4}$	0
3	0	0	$\frac{23}{6}m$

$$I_{jk}^{S/A} = I_{jk}^{S/S^*} + I_{jk}^{S^*/A}$$

$$I_{jk}^{S^*/A} = m\left[\mathbf{p}^{S^*/A} \times (\mathbf{n}_j \times \mathbf{p}^{S^*/A})\right] \cdot \mathbf{n}_k = (\mathbf{p}\cdot\mathbf{p})\delta_{jk} - (\mathbf{p}\cdot\mathbf{n}_j)(\mathbf{p}\cdot\mathbf{n}_k)$$

$\mathbf{p} = -2\mathbf{n}_2 + 3\mathbf{n}_1$, $\mathbf{p}\cdot\mathbf{p} = 13$

$I_{jk}^{S^*/A}$	1	2	3
1	$4m$	$6m$	0
2	$6m$	$9m$	0
3	0	0	$13m$

thus

$I_{jk}^{S/A}$	1	2	3
1	$\frac{61}{12}m$	$6m$	0
2	$6m$	$\frac{47m}{4}$	0
3	0	0	$\frac{101}{6}m$

now form the determinant $(I_{jk}-I\Lambda)=0$ $\quad \Lambda_{ii}=1$, $\ ij=0 \ \ i\neq j$

thus $I_3 = \frac{101}{6}m$ and $\left(\frac{61}{12}m-I\right)\left(\frac{47}{4}m-I\right)-36m^2=0$ for the other 3.

$$\frac{61\cdot 47}{48}m^2 - \frac{202}{12}mI + I^2 - 36m^2 = 0 \quad \text{or} \quad I^2 - 16.83\,mI + 23.73 = 0$$

$$I_{1,2} = \frac{16.83 \pm 13.73}{2} \Rightarrow \quad I_1 = 15.28\,m \qquad I_2 = 1.55\,m$$

$$\begin{bmatrix} 5.08m-I & 6m & 0 \\ 6m & 11.75m-I & 0 \\ 0 & 0 & 16.83m-I \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \underset{\sim}{0}$$

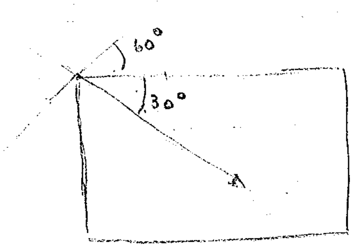
for $I_3 = 16.83\,m \Rightarrow \hat{\mathbb{n}}_3 = (0,0,1)$

for $I_1 = 15.28\,m \Rightarrow \hat{\mathbb{n}}_1 = (.5, .86, 0)$

for $I_2 = 1.55\,m \Rightarrow \hat{\mathbb{n}}_2 = (.86, -.5, 0)$

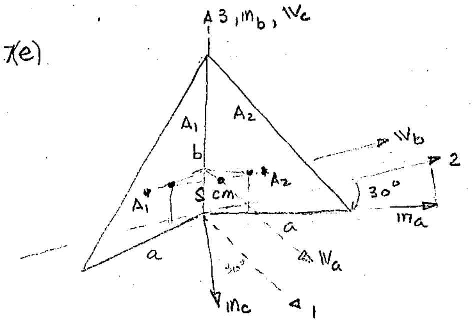
for I_1 $\quad -10.2m\cdot a + 6m\cdot b = 0$ $\quad$ let $a=1$ $\quad b = \frac{10.2}{6} = 1.7$ $\quad \therefore \sqrt{a^2+b^2} = 1.97$; $\quad \bar{a} = \frac{1}{1.97} \quad \bar{b} = \frac{1.7}{1.97}$

for I_2 $\quad 3.53m\cdot a + 6m\cdot b = 0$ $\quad$ let $a=1$ $\quad b = -\frac{3.53}{6} = -.588$ $\quad \therefore \sqrt{a^2+b^2} = 1.16$; $\quad \bar{a} = \frac{1}{1.16} \quad \bar{b} = \frac{-.588}{1.16}$



thus the $\overset{\text{min}}{\wedge}$angle is 30°

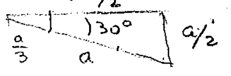
7(e)



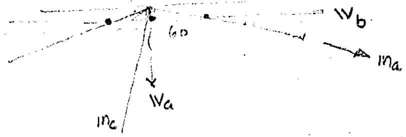
cm of $A_2 = \left(\frac{a}{6}, \frac{a}{2\sqrt{3}}, \frac{1}{3}b\right)$

cm of $A_1 = \left(\frac{a}{6}, -\frac{a}{2\sqrt{3}}, \frac{b}{3}\right)$

cm of $S\ (=A_1+A_2)$ is $\left(\frac{a}{6}, 0, \frac{1}{3}b\right)$



now by geometric construction I expect the principal axes to be the $\mathbb{w}$ axes show



To prove the above statement:

$I_{jk}^{A_2/A_2^*}$	$\mathbb{n}_a$	$\mathbb{n}_b$	$\mathbb{n}_c$
$\mathbb{n}_a$	$\frac{mb^2}{18}$	$\frac{mab}{36}$	0
$\mathbb{n}_b$	$\frac{mab}{36}$	$\frac{ma^2}{18}$	0
$\mathbb{n}_c$	0	0	$\frac{m(a^2+b^2)}{18}$

now $I_{jk}^{A_2/S^*} = I_{jk}^{A_2/A_2^*} + I_{jk}^{A_2^*/S^*} = I_{jk}^{A_2/A_2^*} - I_{jk}^{S^*/A_2^*}$

$\mathbb{p}^{S^*/A_2^*} = -\frac{a}{2\sqrt{3}}\mathbb{n}_2 \qquad \mathbb{n}_2 = -\frac{1}{2}\mathbb{n}_b + \frac{\sqrt{3}}{2}\mathbb{n}_a$

$\mathbb{p}^{S^*/A_2^*} = \frac{a}{2\sqrt{3}}\left(\frac{1}{2}\mathbb{n}_b - \frac{\sqrt{3}}{2}\mathbb{n}_a\right) = \frac{a}{4\sqrt{3}}\mathbb{n}_b - \frac{a}{4}\mathbb{n}_a$

now $I_{jk}^{S^*/A_2^*} = m\left[\mathbb{p}\cdot\mathbb{p}\ \delta_{jk} - (\mathbb{p}\cdot\mathbb{n}_j)(\mathbb{p}\cdot\mathbb{n}_k)\right]$

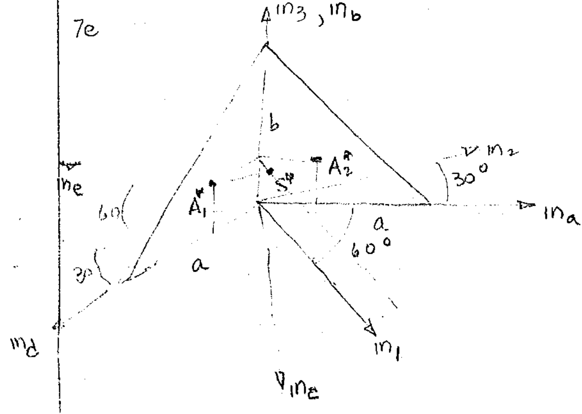
$\mathbb{p}\cdot\mathbb{p} = \frac{a^2}{48} + \frac{a^2}{16} = \frac{a^2}{12}$

$I_{jk}^{S^*/A^*}$	$\mathbb{n}_a$	$\mathbb{n}_b$	$\mathbb{n}_c$
$\mathbb{n}_a$	$ma^2/48$	$\frac{ma^2}{16\sqrt{3}}$	0
$\mathbb{n}_b$	$\frac{ma^2}{16\sqrt{3}}$	$ma^2/16$	0
$\mathbb{n}_c$	0	0	$ma^2/12$

thus

$I_{jk}^{A_2/S^*}$	$\mathbb{n}_a$	$\mathbb{n}_a$	$\mathbb{n}_c$
$\mathbb{n}_a$	$m\left(\frac{b^2}{18}-\frac{a^2}{48}\right)$	$m\left(\frac{ab}{36}-\frac{a^2}{16\sqrt{3}}\right)$	0
$\mathbb{n}_b$	$m\left(\frac{ab}{36}-\frac{a^2}{16\sqrt{3}}\right.$	$m\left(\frac{a^2}{18}-\frac{a^2}{16}\right)$	0
$\mathbb{n}_c$	0	0	$m\left(a^2/12-\frac{a^2+b^2}{18}\right)$

7e



cm of $A_2 = \left(\frac{a}{6}, \frac{a}{2\sqrt{3}}, \frac{1}{3}b\right)$

cm of $A_1 = \left(\frac{a}{6}, -\frac{a}{2\sqrt{3}}, \frac{1}{3}b\right)$

cm of $S = \left(\frac{a}{6}, 0, \frac{1}{3}b\right)$

$I_{jk}^{A_2/A_2^*}$	$\mathbf{n}_a$	$\mathbf{n}_b$	$\mathbf{n}_c$
$\mathbf{n}_a$	$\frac{mb^2}{18}$	$\frac{mab}{36}$	0
$\mathbf{n}_b$	$\frac{mab}{36}$	$\frac{ma^2}{18}$	0
$\mathbf{n}_c$	0	0	$m\frac{(a^2+b^2)}{18}$

now $I_{lm}^{A_2/A^*} = \sum_{j=a}^{c}\sum_{k=a}^{c} a_{lj}\, I_{jk}\, b_{km}$

where $a_{lj} \Rightarrow l = 1,2,3$

$j = a,b,c$

$b_{km} \Rightarrow k = a,b,c$

$m = 1,2,m$

	$\mathbf{n}_a$	$\mathbf{n}_b$	$\mathbf{n}_c$
$\mathbf{n}_1$	$\frac{1}{2}$	0	$\sqrt{3}/2$
$\mathbf{n}_2$	$\sqrt{3}/2$	0	$-1/2$
$\mathbf{n}_3$	0	1	0

a_{lj}

	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{n}_3$
$\mathbf{n}_a$	$1/2$	$\sqrt{3}/2$	0
$\mathbf{n}_b$	0	0	1
$\mathbf{n}_c$	$\sqrt{3}/2$	$-1/2$	0

b_{km}

thus

$$\begin{pmatrix} \frac{1}{2} & 0 & \sqrt{3}/2 \\ \sqrt{3}/2 & 0 & -1/2 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} mb^2/18 & \frac{mab}{36} & 0 \\ \frac{mab}{36} & ma^2/18 & 0 \\ 0 & 0 & m\frac{(a^2+b^2)}{18} \end{pmatrix}\begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \\ \sqrt{3}/2 & -1/2 & 0 \end{pmatrix} = \overset{I_{lm}^{A_2/A_2^*}}{\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}}$$

$$\begin{pmatrix} mb^2/36 & \frac{mab}{72} & \frac{m(a^2+b^2)\sqrt{3}}{36} \\ \frac{mb^2\sqrt{3}}{36} & \frac{mab\sqrt{3}}{72} & -\frac{m(a^2+b^2)}{36} \\ \frac{mab}{36} & \frac{ma^2}{18} & 0 \end{pmatrix} = \begin{pmatrix} \frac{mb^2}{72} + \frac{m(a^2+b^2)\cdot 3}{72} & \frac{mb^2\sqrt{3}}{72} - \frac{m(a^2+b^2)\sqrt{3}}{72} & \frac{mab}{72} \\ \frac{mb^2\sqrt{3}}{72} - \frac{m(a^2+b^2)\sqrt{3}}{72} & \frac{mb^2\cdot 3}{72} + \frac{m(a^2+b^2)}{72} & \frac{mab\sqrt{3}}{72} \\ \frac{mab}{72} & \frac{mab\sqrt{3}}{72} & \frac{ma^2}{18} \end{pmatrix}$$

now $\longrightarrow I_{lm}^{A_2/S^*} = I_{lm}^{A_2/A_2^*} + I_{lm}^{A_2^*/S^*}$

but first: for A_1

	$\mathbf{n}_d$	$\mathbf{n}_b$	$\mathbf{n}_e$
$\mathbf{n}_1$	$1/2$	0	$-\sqrt{3}/2$
$\mathbf{n}_2$	$-\sqrt{3}/2$	0	$-1/2$
$\mathbf{n}_3$	0	1	0

a_{lj}

	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{n}_3$
$\mathbf{n}_d$	$1/2$	$-\sqrt{3}/2$	0
$\mathbf{n}_b$	0	0	1
$\mathbf{n}_e$	$-\sqrt{3}/2$	$-1/2$	0

b_{km}

$$\begin{pmatrix} \frac{mb^2}{36} & \frac{mab}{72} & -\frac{m(a^2+b^2)\sqrt{3}}{36} \\ -\frac{mb^2\sqrt{3}}{36} & -\frac{mab\sqrt{3}}{72} & -\frac{m(a^2+b^2)}{36} \\ \frac{mab}{36} & \frac{ma^2}{18} & 0 \end{pmatrix} = \overset{I_{lm}^{A_1/A_1^*}}{\begin{pmatrix} \frac{mb^2}{72} + \frac{m(a^2+b^2)\cdot 3}{72} & -\frac{mb^2\sqrt{3}}{72} + \frac{m(a^2+b^2)\sqrt{3}}{72} & \frac{mab}{72} \\ -\frac{mb^2\sqrt{3}}{72} + \frac{m(a^2+b)\sqrt{3}}{72} & \frac{mb^2\cdot 3}{72} + \frac{m(a^2+b^2)}{72} & -\frac{mab\sqrt{3}}{72} \\ \frac{mab}{72} & -\frac{mab\sqrt{3}}{72} & \frac{ma^2}{18} \end{pmatrix}}$$

Next we must translate this to $I_{lm}^{A_1/S^*} = I_{lm}^{A_1/A_1^*} + I_{lm}^{A_1^*/S^*} =$

$\mathbb{P}^{A_2^*/S^*} = +\frac{a}{2\sqrt{3}}\mathbf{n}_2$ $\qquad$ $\mathbb{P}^{A_1^*/S^*} = \frac{-a}{2\sqrt{2}}\mathbf{n}_2$

$\mathbb{P}^2 = \frac{a^2}{12}$ $\qquad$ $\mathbb{P}^2 = a^2/12$

now $I_{lm}^{A_2^*/S^*} = m\left[\mathbb{P}^2\delta_{lm} - (\mathbb{P}\cdot\mathbf{n}_l)(\mathbb{P}\cdot\mathbf{n}_m)\right]$

$\mathbb{P}\cdot\mathbf{n}_2 = +\frac{a}{2\sqrt{3}}$ $\qquad$ $\mathbb{P}\cdot\mathbf{n}_1 = \mathbb{P}\cdot\mathbf{n}_3 = 0$

thus $I_{lm}^{A_2^{\#}/S^*}$

$$\begin{array}{c|ccc} & m_1 & m_2 & m_3 \\ \hline & \frac{ma^2}{12} & 0 & 0 \\ & 0 & 0 & 0 \\ & 0 & 0 & \frac{ma^2}{12} \end{array}$$

and the same is true for $I_{lm}^{A_1^{\#}/S^*}$

thus $I_{lm}^{A_1/S^*}$

$$\left(\begin{array}{c|c|c} \frac{mb^2}{72} + \frac{3m(a^2+b^2)}{72} + \frac{ma^2}{12} & -\frac{mb^2\sqrt{3}}{72} \; \frac{\sqrt{3}m(a^2+b^2)}{72} & \frac{mab}{72} \\ -\frac{mb^2\sqrt{3}}{72} + \frac{m(a^2+b^2)\sqrt{3}}{72} & \frac{3mb^2}{72} + \frac{m(a^2+b^2)}{72} & -\frac{mab\sqrt{3}}{72} \\ \frac{mab}{72} & -\frac{mab\sqrt{3}}{72} & \frac{ma^2}{18} + \frac{ma^2}{12} \end{array}\right)$$

and $I_{lm}^{A_2/S^*}$

$$\left(\begin{array}{c|c|c} \frac{mb^2}{72} + \frac{3m(a^2+b^2)}{72} + \frac{ma^2}{12} & \frac{mb^2\sqrt{3}}{72} - \frac{m(a^2+b^2)\sqrt{3}}{72} & \frac{mab}{72} \\ \frac{mb^2\sqrt{3}}{72} - \frac{m(a^2+b^2)\sqrt{3}}{72} & \frac{3mb^2}{72} + \frac{m(a^2+b^2)}{72} & \frac{mab\sqrt{3}}{72} \\ \frac{mab}{72} & \frac{mab\sqrt{3}}{72} & \frac{ma^2}{18} + \frac{ma^2}{12} \end{array}\right)$$

Now

$$I_{lm}^{S/S^{\#}} = I_{lm}^{A_1/S^*} + I_{lm}^{A_2/S^*}$$

$$\left(\begin{array}{c|c|c} \frac{mb^2}{36} + \frac{m(a^2+b^2)}{12} + \frac{ma^2}{6} & 0 & \frac{mab}{36} \\ \frac{ma^2}{4} + \frac{mb^2}{9} & & \\ 0 & \frac{m(a^2+b^2)}{36} + \frac{mb^2}{12} & 0 \\ & m\left(\frac{b^2}{9} + \frac{a^2}{36}\right) & \\ \frac{mab}{36} & 0 & \frac{ma^2}{9} + \frac{ma^2}{6} \\ & & \frac{10ma^2}{36} \end{array}\right)$$

for $a = 2b$ & $b = 1$ $a = 2$

$$\begin{pmatrix} m\left[1 + \tfrac{1}{9}\right] = m\frac{10}{9}b^2 & 0 & \frac{m}{18}b^2 \\ 0 & \frac{2m}{9}b^2 & 0 \\ \frac{m}{18}b^2 & 0 & m\cdot\frac{10}{9}b^2 \end{pmatrix}$$

13
18

thus $\left(\frac{10}{9}m - I\right)\left(\frac{2m}{9} - I\right)\left(\frac{10}{9}m - I\right) - \frac{m^2}{(18)^2}\left(\frac{2m}{9} - I\right) = 0 \quad \therefore \quad \left(\frac{2m}{9} - I\right)\left\{\left[\frac{10}{9}m - I\right]^2 - \frac{m^2}{324}\right\} = 0$

① $I = \frac{2m}{9}b^2$ ② $b^4\left(\frac{100}{81}m^2 - \frac{20m}{9}I + I^2 - \frac{m^2}{324}\right) = b^4\left(\frac{399}{324}m^2 - \frac{20m}{9}I + I^2\right) = 0$

$$I = b^2\left(\frac{20}{18}m \pm \frac{1}{2}\sqrt{\frac{400}{81}m^2 - 4\cdot\frac{399}{324}m^2}\right) = \frac{20}{18}m \pm \frac{1}{2}\sqrt{\frac{m^2}{81}} = \frac{20}{18}mb^2 \pm \frac{mb^2}{18} \Rightarrow \frac{21}{18}mb^2 \text{ or } \frac{19}{18}mb$$

$$I = 2mk'^2 = \frac{21mb^2}{18} \text{ or } \frac{19}{18}mb'^2 \qquad k^2 = \sqrt{\frac{21}{36}b^2} \text{ or } \sqrt{\frac{19}{36}b^2}$$

$$\frac{2m}{9}b^2 = I_{min} = 2mk^2 \Rightarrow k_{min} = \tfrac{1}{3}b$$

for $b = 2a$ let $a = 1$ $b = 2$ thus

$$I_{lm}^{S/S*} = \begin{pmatrix} m\left[\frac{1}{4}+\frac{4}{9}\right] = \frac{25}{36}ma^2 & 0 & \frac{m}{18}a^2 \\ 0 & m\left(\frac{4}{9}+\frac{1}{36}\right) = m\frac{17}{36}a^2 & 0 \\ \frac{m}{18}a^2 & 0 & \frac{10m}{36}a^2 \end{pmatrix}$$

thus $\left[\left(\frac{25m}{36}-I\right)\left(\frac{17m}{36}-I\right)\left(\frac{10m}{36}-I\right) - \frac{m^2}{324}\left(\frac{17m}{36}-I\right)\right]a^6 = 0$

$\therefore\ I = \frac{17m}{36}a^2$ [illegible] $= \left[\left(\frac{25m}{36}-I\right)\left(\frac{10m}{36}-I\right) - \frac{m^2}{324}\right]a^4 = 0$

$k = \sqrt{\frac{17}{72}}\,a = \frac{1}{12}\sqrt{\frac{17}{2}}\,b$

$a^4\left[\frac{250m^2}{36^2} - \frac{35}{36}mI + I^2 - \frac{m^2}{(18)^2}\right] = 0$

$a^4\left[\frac{246m^2}{(36)^2} - \frac{35}{36}mI + I^2\right] = 0$

$$I = a^2\left[\frac{35m}{72} \pm \frac{m}{2}\sqrt{\left(\frac{35}{36}\right)^2 - 4\cdot\frac{246}{(36)^2}}\right] = \left[\frac{35m}{72} \pm \frac{m}{72}\sqrt{(35)^2 - 4(246)}\right]a^2$$

$$= \left[\frac{35m}{72} \pm \frac{m}{72}\sqrt{241}\right]a^2$$

$\begin{array}{r} 35 \\ 35 \\ \hline 175 \\ 105\ \ \\ \hline 1225 \\ -984 \\ \hline 241 \end{array}$

984

$$k = \sqrt{35 \pm \sqrt{241}}\ \frac{1}{12}a = \frac{1}{24}\left[35 \pm (241)^{1/2}\right]^{1/2} b$$

$35 - \sim 16 \sim (18) \sim 19$ $\sqrt{18 \sim 19} = \frac{4.5}{24} = .16 \sim .2$

$35 + 16 \sim 51$ $\sqrt{51} \sim 7$ $\frac{7}{24} \cong .30$

$\frac{1}{12}\sqrt{\frac{17}{2}} = \frac{4.3}{12 \cdot 1.4} = \frac{1}{4} \cdot .25$

thus $k_{min} = \frac{1}{24}\left[35 - (241)^{1/2}\right]^{1/2} b$

now $\mathbf{n}_a = \frac{\sqrt{3}}{2}\mathbf{n}_2 + \frac{1}{2}\mathbf{n}_1$ $\mathbf{n}_b = \frac{\sqrt{3}}{2}\mathbf{n}_1 - \frac{1}{2}\mathbf{n}_2$ $\mathbf{n}_c = \mathbf{n}_2$

thus $I_{\ell m} = \sum_{\ell=1}^{3}\sum_{m=1}^{3} a_{\ell j}\, I_{jk}\, a_{km}$

7f $I_{jk}^{B/N} = I_{jk}^{B/Q} + I_{jk}^{Q/N}$ $I_{jk}^{B/O} = I_{jk}^{B/Q} + I_{jk}^{Q/O}$

$\mathbf{p}^{Q/N} = -3\mathbf{n}_1 + 2\mathbf{n}_2$

$\mathbf{p}\cdot\mathbf{p} = 13$ $I_{jk}^{Q/N} = m\left[13\delta_{jk} - (\mathbf{p}\cdot\mathbf{n}_j)(\mathbf{p}\cdot\mathbf{n}_k)\right]$

now $I_{jk}^{B/N} = I_{jk}^{B/O} - I_{jk}^{Q/O} + I_{jk}^{Q/N}$

$\mathbf{p}^{Q/O} = -3\mathbf{n}_1 - 4\mathbf{n}_3 + 2\mathbf{n}_2$

$\mathbf{p}\cdot\mathbf{p} = 29$ $I_{jk}^{Q/O} = m\left[29\delta_{jk} - (\mathbf{p}\cdot\mathbf{n}_j)(\mathbf{p}\cdot\mathbf{n}_k)\right]$

$I_{jk}^{Q/N}$	1	2	3
1	48	72	0
2	72	108	0
3	0	0	156

$I_{jk}^{Q/O}$	1	2	3
1	240	72	-144
2	72	300	96
3	-144	96	156

$\therefore$

$I_{jk}^{B/N}$	1	2	3
1	68	72	0
2	72	133	0
3	0	0	169

$\mathbb{I} = 68\,\mathbf{n}_1\mathbf{n}_1 + 72(\mathbf{n}_1\mathbf{n}_2 + \mathbf{n}_2\mathbf{n}_1) + 133\,\mathbf{n}_2\mathbf{n}_2 + 169\,\mathbf{n}_3\mathbf{n}_3$

7g $\mathbf{n} = \frac{3}{5}\mathbf{n}_1 - \frac{4}{5}\mathbf{n}_3$ $I_n = \mathbf{n}\cdot\mathbb{I}\cdot\mathbf{n} = 68\cdot\frac{9}{25} + 169\left(\frac{16}{25}\right) = \frac{3316}{25}$ Same result

hence $I_n = \mathbf{n}\cdot\mathbb{I}\cdot\mathbf{n}$

7h. Since $\mathbb{I}_a = \mathbb{I}\cdot\mathbf{n}_a = \sum m_i\,\mathbf{p}_i \times (\mathbf{n}_a \times \mathbf{p}_i)$

$$= \sum m_i\left[(\mathbf{p}_i\cdot\mathbf{p}_i)\,\mathbf{n}_a - (\mathbf{n}_a\cdot\mathbf{p}_i)\,\mathbf{p}_i\right] = \sum m_i\left[\mathbf{p}_i^2\{(\mathbf{n}_a\cdot\mathbf{n}_1)\mathbf{n}_1 + (\mathbf{n}_a\cdot\mathbf{n}_2)\mathbf{n}_2\right.$$

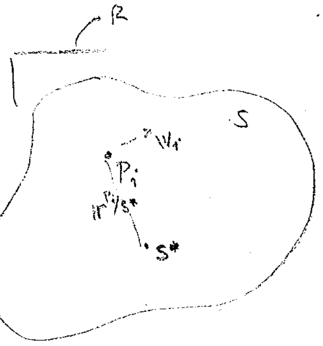
$$\left. + (\mathbf{n}_a\cdot\mathbf{n}_3)\mathbf{n}_3\} - (\mathbf{n}_a\cdot\mathbf{p}_i)\mathbf{p}_i\right]$$

$$= \sum m_i\left\{\mathbf{n}_a\cdot\left[\mathbf{n}_1\mathbf{n}_1 + \mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3\right]\mathbf{p}_i^2 - \mathbf{n}_a\cdot(\mathbf{p}_i\mathbf{p}_i)\right\} = \mathbf{n}_a\cdot\sum m_i\left(\mathbb{U}\,\mathbf{p}_i^2 - \mathbf{p}_i\mathbf{p}_i\right)$$

thus since $\mathbb{I}\cdot\mathbf{n}_a = \mathbf{n}_a\cdot\mathbb{I} = \mathbf{n}_a\cdot\sum m_i\left(\mathbb{U}\,\mathbf{p}_i^2 - \mathbf{p}_i\mathbf{p}_i\right) \Rightarrow$

$$\mathbb{I} = \sum_i m_i\left(\mathbb{U}\,\mathbf{p}_i\cdot\mathbf{p}_i - \mathbf{p}_i\mathbf{p}_i\right)$$

7i



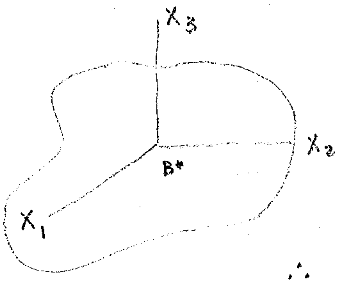
$${}^{R}\mathbb{H}^{S/S^*} = \sum_{i=1}^{N} m_i \mathbb{r}_i \times \mathbb{v}_i$$

now $\mathbb{v}_i = \mathbb{v}^{S^*} + {}^{R}\omega^{B} \times \mathbb{r}_i^{P_i/S^*}$

$$\therefore \quad {}^{R}\mathbb{H}^{S/S^*} = \underbrace{\sum m_i \mathbb{r}_i}_{=0} \times \mathbb{v}^{S^*} + \sum m_i \mathbb{r}_i \times ({}^{R}\omega^{B} \times \mathbb{r}_i) \quad \text{since } S^* \text{ is mass center}$$

$${}^{R}\mathbb{H}^{S/S^*} = \sum m_i \mathbb{r}_i \times (\omega \times \mathbb{r}_i) = \mathbb{I} \cdot {}^{R}\omega^{B} = \mathbb{I}_\omega$$

7j



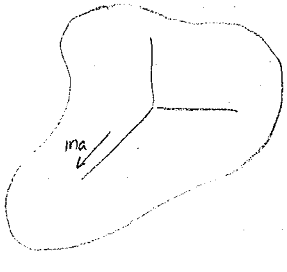
${}^{R}\omega^{B} = \omega_i \mathbb{n}_i$ where $\omega_i = {}^{R}\omega^{B} \cdot \mathbb{n}_i$

$${}^{R}\mathbb{H}^{B/B^*} = \mathbb{I} \cdot \omega$$

since X_i's are principal axes $\mathbb{I} = I_1 \mathbb{n}_1\mathbb{n}_1 + I_2 \mathbb{n}_2\mathbb{n}_2 + I_3 \mathbb{n}_3\mathbb{n}_3$

$$\therefore \quad \mathbb{I} \cdot \omega = I_1 \omega_1 \mathbb{n}_1 + I_2 \omega_2 \mathbb{n}_2 + I_3 \omega_3 \mathbb{n}_3 = {}^{R}\mathbb{H}^{B/B^*}$$

7k.



Show that if ${}^{R}\mathbb{H}^{B/B^*}$ is // to ${}^{R}\omega^{B} \Rightarrow {}^{R}\omega^{B}$ is $\parallel$ to $\mathbb{I}_a$

and if ${}^{R}\omega^{B}$ is // to $\mathbb{I}_a \Rightarrow \mathbb{H}^{B/B^*}$ is // to ${}^{R}\omega^{B}$

lets $\omega = \omega \mathbb{n}_a$ then lets $\mathbb{n}_a \cdot \mathbb{n}_i = a_i$

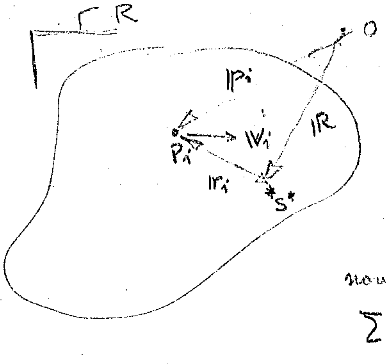
if ω is // $\mathbb{n}_1$ then $\mathbb{H} = \mathbb{I} \cdot \omega = \omega I_1 \mathbb{n}_1 \Rightarrow \mathbb{H}$ is $/\!/ \mathbb{n}_1 \Rightarrow \mathbb{H}$ is // ω

If ${}^{R}\mathbb{H}^{B/B^*}$ is // to ${}^{R}\omega^{B}$ then $\mathbb{H} = H \mathbb{n}_a = \mathbb{I} \cdot \omega = \omega \mathbb{I} \cdot \mathbb{n}_a = \omega \mathbb{I}_a \Rightarrow \mathbb{I}_a = \frac{H}{\omega} \mathbb{n}_a \Rightarrow$

$\mathbb{I}_a \parallel \mathbb{n}_a \Rightarrow \mathbb{n}_a$ must be a vector in principal direction $\Rightarrow \omega$ is $\parallel$

to a principal axis

7l)



$${}^{R}\mathbb{H}^{S/O} = \sum m_i \mathbb{p}_i^{P_i/O} \times \mathbb{v}_i^{P} \qquad \text{let } \mathbb{p}^{P_i/O} = \mathbb{r}^{P_i/S^*} + \mathbb{R}^{S^*/O}$$

$$= \sum m_i (\mathbb{r}_i + \mathbb{R}) \times (\mathbb{v}^{S^*} + {}^{R}\omega^{S} \times \mathbb{r}_i)$$

$$= \sum m_i \mathbb{r}_i \times \mathbb{v}^{S^*} + \sum m_i \mathbb{r}_i \times ({}^{R}\omega^{S} \times \mathbb{r}_i) + \sum m_i R \times \mathbb{v}^{S^*}$$

$$+ \sum m_i R \times ({}^{R}\omega^{S} \times \mathbb{r}_i)$$

now $\sum m_i \mathbb{r}_i = 0$ & $\sum m_i \mathbb{r}_i \times (\omega \times \mathbb{r}_i) = {}^{R}\mathbb{H}^{S/S^*}$

$$\sum m_i \mathbb{R} \times \mathbb{v}^{S^*} + \sum m_i \mathbb{R} \times ({}^{R}\omega^{S} \times \mathbb{r}_i) = m \mathbb{R} \times \mathbb{v}^{S^*} + \sum m_i \{(\mathbb{R} \cdot \mathbb{r}_i)\omega - (\omega \cdot \mathbb{R})$$

$$\sum m_i (\mathbb{R} \cdot \mathbb{r}_i)\omega = [\mathbb{R} \cdot (\sum m_i \mathbb{r}_i)]\omega = 0 \; ; \quad \sum m_i (\omega \cdot \mathbb{R})\mathbb{r}_i = (\omega \cdot \mathbb{R}) \sum m_i \mathbb{r}_i = 0$$

Thus $${}^{R}\mathbb{H}^{S/O} = {}^{R}\mathbb{H}^{S/S^*} + \underline{m \mathbb{R} \times \mathbb{v}^{S^*}} \qquad m = \sum m_i \qquad \underline{m \mathbb{R} \times \mathbb{v}^{S^*} = {}^{R}\mathbb{R} \times (m \mathbb{v}^{S^*})}^{S^*/O}$$

angular momentum of a

8b) From problem 7ii) we showed that ${}^{R}\mathbb{H}^{B/B*} = \mathbb{I}^{B/B*} \cdot {}^{R}\omega^{B}$

Now $\frac{{}^{R}d\,\mathbb{I}\cdot\omega}{dt} = \frac{{}^{B}d(\mathbb{I}\cdot\omega)}{dt} + {}^{R}\omega^{B} \times (\mathbb{I}\cdot\omega)$ but what is $\frac{{}^{B}d(\mathbb{I}\cdot\omega)}{dt}$

$$\mathbb{I}\cdot\omega \doteq \mathbb{I}_{\omega} = \sum m_i \mathbb{r}_i \times ({}^{R}\omega^{B} \times \mathbb{r}_i) \quad \text{thus} \quad \frac{{}^{B}d}{dt}(\ \ ) = \sum m_i \dot{\mathbb{r}}_i \times ({}^{R}\omega^{B} \times \mathbb{r}_i) +$$
$$\sum m_i \mathbb{r}_i \times \left(\frac{{}^{B}d\,{}^{R}\omega^{B}}{dt} \times \mathbb{r}_i\right) + \sum m_i \mathbb{r}_i \times ({}^{R}\omega^{B} \times \dot{\mathbb{r}}_i)$$

but since $\mathbb{r}_i$ are fixed wrt body B hence $\dot{\mathbb{r}}_i = \frac{{}^{B}d\,\mathbb{r}_i}{dt} = 0$. Thus 2nd & 3rd sums are zero. $\frac{{}^{B}d\,{}^{R}\omega^{B}}{dt} = {}^{R}\alpha^{B} + {}^{B}\omega^{R} \times {}^{R}\omega^{B} = {}^{R}\alpha^{B}$ since ${}^{B}\omega^{R} \times {}^{R}\omega^{B} \equiv 0$

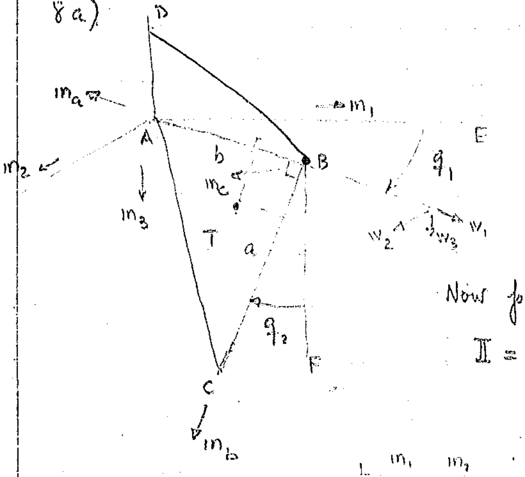
hence,

$$\frac{{}^{B}d}{dt}(\ \ ) = \sum m_i \mathbb{r}_i \times ({}^{R}\alpha^{B} \times \mathbb{r}_i) = \mathbb{I}\cdot{}^{R}\alpha^{B} \quad \text{thus}$$

$${}^{R}\dot{\mathbb{H}}^{B/B*} = \mathbb{I}\cdot{}^{R}\alpha^{B} + \omega \times (\mathbb{I}\cdot\omega) = \mathbb{I}\cdot{}^{R}\alpha^{B} - (\mathbb{I}\cdot\omega)\times\omega = -\mathbb{T}^{*}$$

hence $\mathbb{T}^{*} = -{}^{R}\dot{\mathbb{H}}^{B/B*}$ as required

8a)



$${}^{R}\omega^{T} = \dot{q}_1 \mathbb{n}_3 + \dot{q}_2 \mathbb{n}_a$$
$$F_2^{*} = \mathbb{v}^{*}_{\dot{q}_2}\cdot\mathbb{F}^{*} + \omega_{\dot{q}_2}\cdot\mathbb{T}^{*}$$

where $\mathbb{F}^{*} = -m\,\mathbb{a}^{*}$ & $\mathbb{T}^{*} = (\mathbb{I}\cdot\omega)\times{}^{R}\omega^{T} - \mathbb{I}\cdot{}^{R}\alpha^{T}$

and $\mathbb{v}^{*}$ is the velocity of the mass center

Now for the plate

$$\mathbb{I} = \frac{ma^2}{18}\mathbb{n}_a\mathbb{n}_a + \frac{mab}{36}(\mathbb{n}_a\mathbb{n}_b + \mathbb{n}_b\mathbb{n}_a) + \frac{mb^2}{18}\mathbb{n}_b\mathbb{n}_b + \frac{m(a^2+b^2)}{18}\mathbb{n}_c\mathbb{n}_c$$

$$\begin{array}{c|ccc} & \mathbb{n}_1 & \mathbb{n}_2 & \mathbb{n}_3 \\ \hline \mathbb{w}_1 & c_1 & s_1 & 0 \\ \mathbb{w}_2 & -s_1 & c_1 & 0 \\ \mathbb{w}_3 & 0 & 0 & 1 \end{array} \qquad \begin{array}{c|ccc} & \mathbb{w}_1 & \mathbb{w}_2 & \mathbb{w}_3 \\ \hline \mathbb{n}_a & -1 & 0 & 0 \\ \mathbb{n}_b & 0 & s_2 & c_2 \\ \mathbb{n}_c & 0 & c_2 & -s_2 \end{array}$$

$$\mathbb{n}_3 = \mathbb{w}_3 = c_2\mathbb{n}_b - s_2\mathbb{n}_c$$
$$\rightarrow {}^{R}\omega^{T} = \dot{q}_1(c_2\mathbb{n}_b - s_2\mathbb{n}_c) + \dot{q}_2\mathbb{n}_a$$
$${}^{R}\omega^{T}_{\dot{q}_2} = \mathbb{n}_a$$
$${}^{R}\alpha^{T} = {}^{T}\alpha^{T} + {}^{R}\omega^{T}\times{}^{R}\omega^{T} = \frac{d\omega}{dt}$$

$$\mathbb{I}_{\omega} = \mathbb{I}\cdot\omega = \frac{ma^2}{18}\dot{q}_2\mathbb{n}_a + \frac{mab}{36}\dot{q}_2\mathbb{n}_b + \frac{mab}{36}\dot{q}_1 c_2\mathbb{n}_a + \frac{mb^2}{18}\dot{q}_1 c_2\mathbb{n}_b + \frac{m(a^2+b^2)}{18}(-\dot{q}_1 s_2\mathbb{n}_c)$$
$$= \left(\frac{ma^2}{18}\dot{q}_2 + \frac{mab}{36}\dot{q}_1 c_2\right)\mathbb{n}_a + \left(\frac{mab}{34}\dot{q}_2 + \frac{mb^2}{18}\dot{q}_1 c_2\right)\mathbb{n}_b - \frac{m(a^2+b^2)}{18}\dot{q}_1 s_2\mathbb{n}_c$$

$$\mathbb{I}_{\omega}\times\omega = -\dot{q}_2\left[\frac{mab}{36}\dot{q}_2 + \frac{mb^2}{18}\dot{q}_1 c_2\right]\mathbb{n}_c - \frac{m(a^2+b^2)}{18}\dot{q}_1\dot{q}_2 s_2\mathbb{n}_b + \dot{q}_1 c_2\left(\frac{ma^2}{18}\dot{q}_2 + \frac{mab}{36}\dot{q}_1 c_2\right)\mathbb{n}_c +$$
$$\dot{q}_1 s_2\left(\frac{ma^2}{18}\dot{q}_2 + \frac{mab}{36}\dot{q}_1 c_2\right)\mathbb{n}_b - \dot{q}_1 s_2\left(\frac{mab}{36}\dot{q}_2 + \frac{mb^2}{18}\dot{q}_1 c_2\right)\mathbb{n}_a + \dot{q}_1 c_2\left[\frac{m(a^2+b^2)}{18}\dot{q}_1 s_2\right]\mathbb{n}_a$$

$${}^{R}\alpha^{T} = \frac{{}^{T}d\omega}{dt} = \ddot{q}_1(c_2\mathbb{n}_b - s_2\mathbb{n}_c) + \dot{q}_1(-s_2\dot{q}_2\mathbb{n}_b - c_2\dot{q}_2\mathbb{n}_c) + \ddot{q}_2\mathbb{n}_a = \ddot{q}_2\mathbb{n}_a + (\ddot{q}_1 c_2 - \dot{q}_1\dot{q}_2 s_2)\mathbb{n}_b - (\ddot{q}_1 s_2 + \dot{q}_1\dot{q}_2 c_2)$$

$$\longrightarrow \quad \mathbb{I}\cdot\alpha = \frac{ma^2}{18}\ddot{q}_2\mathbb{n}_a + \frac{mab}{36}\ddot{q}_2\mathbb{n}_b + \frac{mab}{36}(\ddot{q}_1 c_2 - \dot{q}_1\dot{q}_2 s_2)\mathbb{n}_a + \frac{mb^2}{18}(\ddot{q}_1 c_2 - \dot{q}_1\dot{q}_2 s_2)\mathbb{n}_b - \frac{m(a^2+b^2)}{18}(\ddot{q}_1 s_2 + \dot{q}_1\dot{q}_2 c_2)\mathbb{n}_c$$

$$\therefore \quad \omega_{\dot q_2}\cdot \mathbb{T}^* = \mathbf{n}_a\cdot\mathbb{T}^* = -\dot q_1 s_2\left(\frac{mab}{36}\dot q_2 + \frac{mb^2}{18}\dot q_1 c_2\right) + \dot q_1 c_2\left[\frac{m(a^2+b^2)}{18}\dot q_1 s_2\right]$$
$$-\frac{ma^2}{18}\ddot q_2 - \frac{mab}{36}\left(\ddot q_1 c_2 - \dot q_1\dot q_2 s_2\right)$$

now $\mathbf{r}^{T^*} = -\frac{2}{3}b\,\mathbf{n}_a + \frac{a}{3}\mathbf{n}_b$

$${}^R\mathbf{v}^{T^*} = \underset{\to 0}{{}^T\mathbf{v}^{T^*}} + {}^R\omega^T\times\mathbf{r}^T = \left[\dot q_1(c_2\mathbf{n}_b - s_2\mathbf{n}_c) + \dot q_2\mathbf{n}_a\right]\times\left(-\frac{2b}{3}\mathbf{n}_a + \frac{a}{3}\mathbf{n}_b\right)$$
$$= \frac{2b}{3}\dot q_1 c_2\mathbf{n}_c + \dot q_1 s_2\cdot\frac{2b}{3}\mathbf{n}_b + \dot q_1 s_2\frac{a}{3}\mathbf{n}_a + \dot q_2\frac{a}{3}\mathbf{n}_c$$
$$= \dot q_1 s_2 a\frac{\mathbf{n}_a}{3} + \dot q_1 s_2\cdot\frac{2b}{3}\mathbf{n}_b + \left(\frac{2b}{3}\dot q_1 c_2 + \dot q_2\frac{a}{3}\right)\mathbf{n}_c$$

$$\boxed{{}^R\mathbf{v}^{T^*}_{\dot q_2} = a\frac{\mathbf{n}_c}{3}}$$

$${}^R\mathbf{a}^{T^*} = {}^T\mathbf{a}^{T^*} + {}^R\omega^T\times\mathbf{v}^{T^*}$$

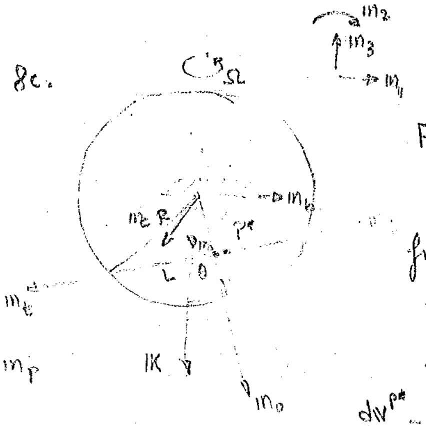
$${}^T\mathbf{a}^{T^*} = \frac{2b}{3}\mathbf{n}_c\left(\ddot q_1 c_2 + \dot q_1 s_2\dot q_2\right) + \frac{2b}{3}\mathbf{n}_b\left(\ddot q_1 s_2 + \dot q_1\dot q_2 c_2\right) + \left(\ddot q_1 s_2 a + \dot q_1\dot q_2 c a\right)\frac{\mathbf{n}_a}{3} + \ddot q_2\frac{a}{3}\mathbf{n}_c$$

$$\omega\times\mathbf{v} = \dot q_2\dot q_1 s_2\frac{2b}{3}\mathbf{n}_c - \dot q_2\left(\frac{2b}{3}\dot q_1 c_2 + \dot q_2\frac{a}{3}\right)\mathbf{n}_b - \dot q_1 c_2 a\left(\dot q_1\frac{s_2}{3}\right)\mathbf{n}_c + \dot q_1 c_2\left(\frac{2b}{3}\dot q_1 c_2 + \dot q_2\frac{a}{3}\right)\mathbf{n}_a$$
$$-\dot q_1 s_2 a\left(\dot q_1\frac{s_2}{3}\right)\mathbf{n}_b + \dot q_1 s_2\left(\dot q_1 s_2\cdot\frac{2b}{3}\right)\mathbf{n}_a$$

Thus $\mathbf{v}_{\dot q_2}\cdot\mathbf{a}^{T^*} = \frac{2ba}{9}\left(\ddot q_1 c_2 - \dot q_1 s_2\dot q_2\right) + \ddot q_2\frac{a^2}{9} + \dot q_2\dot q_1 s_2\frac{2ba}{9} - \dot q_1\dot q_1 c_2 s_2\frac{a^2}{9}$

hence
$$F_2^* = -\frac{2bma}{9}\left(\ddot q_1 c_2 + \dot q_1\dot q_2 s_2\right) - \frac{m\ddot q_2 a^2}{9} - m\,\dot q_2\dot q_1 s_2\cdot\frac{2ba}{9} + m\,\dot q_1^2\frac{c_2 s_2}{9}a^2$$
$$- \dot q_1\dot q_2\frac{mab}{36}s_2 - \dot q_1^2\frac{mb^2}{18}s_2c_2 + \dot q_1^2 c_2 s_2\frac{m(a^2+b^2)}{18} - \frac{ma^2}{18}\ddot q_2 - \frac{mab}{36}\ddot q_1 c_2 + \frac{mab}{36}\dot q_1\dot q_2 s_2$$
$$= \ddot q_1\left[-\frac{2bma}{9}c_2 - \frac{mab}{36}c_2\right] + \ddot q_2\left[-\frac{ma^2}{9} - \frac{ma^2}{18}\right] + \dot q_1^2\left[\frac{ma^2}{9}c_2s_2 - \cancel{\frac{mb^2}{18}s_2c_2} + \frac{m(a^2+\cancel{b^2})}{18}c_2s_2\right]$$
$$+ \dot q_1\dot q_2\left[\cancel{\frac{2mba}{9}s_2} - \cancel{\frac{2mba}{9}s_2} - \cancel{\frac{mabs_2}{36}} + \cancel{\frac{mabs_2}{36}}\right]$$
$$= \ddot q_1\left[-\frac{1}{4}mba\,c_2\right] + \ddot q_2\left[-\frac{ma^2}{6}\right] + \dot q_1^2\left[\frac{ma^2}{6}c_2s_2\right]$$
$$= \frac{ma}{12}\left[2a\left(c_2s_2\dot q_1^2 - \ddot q_2\right) - 3\ddot q_1 b c_2\right] \quad ✓✓$$

8c.



$\mathbf{n}_t\times\mathbf{n}_b = \mathbf{n}_p$

$\mathbf{n}_c, \mathbf{n}_b, \mathbf{n}_a$ are fixed in plane of ring

$${}^R\omega^{RING} = -\Omega\mathbb{K}$$

$$F_\theta^* = \mathbf{v}_{\dot\theta}\cdot\mathbb{F}^* + \omega_{\dot\theta}\cdot\mathbb{T}^*$$

where $\mathbb{F}^* = -m\,\mathbf{a}^*$ $\qquad {}^R\omega^{ROD} = \dot\theta\mathbf{n}_c - \Omega\mathbb{K}$ $\qquad \mathbf{r} = (R^2-L^2)^{1/2}\mathbf{n}_a$

from 5a we found $\mathbf{v}^{P^*} = (R^2-L^2)^{1/2}\left[\dot\theta\cos\theta\,\mathbf{n}_b - \dot\theta\sin\theta\,\mathbb{K} - \Omega s\,\mathbf{n}_c\right]$

thus $\mathbf{v}^{P^*}_{\dot\theta} = (R^2-L^2)^{1/2}\left[\cos\theta\,\mathbf{n}_b - \sin\theta\,\mathbb{K}\right] = (R^2-L^2)^{1/2}\left[\cos\theta\,\mathbf{n}_b - s\theta\,\mathbf{n}_a\right]$

now $\mathbf{a}^* = \frac{{}^R d\mathbf{v}^{P^*}}{dt} = \frac{^{ring}d\mathbf{v}^{P^*}}{dt} + \omega\times\mathbf{v}^{P^*}$ $\qquad {}^R\omega^{ring} = -\Omega\mathbb{K}$

$$\frac{d\mathbf{v}^{P^*}}{dt} = (R^2-L^2)^{1/2}\left[\ddot\theta c\,\mathbf{n}_b - \dot\theta^2 s\,\mathbf{n}_b - \ddot\theta s\,\mathbb{K} - \dot\theta^2 c\,\mathbb{K} - \Omega\dot\theta c\,\mathbf{n}_c\right]$$
$$\omega\times\mathbf{W} = (R^2-L^2)^{1/2}\left[\quad \mathbf{n}_b - \Omega\dot\theta c\,\mathbf{n}_c - \Omega^2 s\,\mathbf{n}_b\right]$$
$$\mathbf{a}^* = (R^2-L^2)^{1/2}\left[\left(\cancel{-\dot\theta^2 c} - \dot\theta^2 c - \ddot\theta s\right)\mathbb{K} + \left(\cancel{-\dot\theta s} - \Omega s^2 - \dot\theta^2 s + \ddot\theta c\right)\mathbf{n}_b - 2\Omega\dot\theta c\,\mathbf{n}_c\right]$$
$= (R^2-L^2)^{1/2}$ [illegible]

Thus $\mathbb{V}_{\dot\theta}\cdot\mathbb{F}^* = +m(R^2-L^2)\left[-s(\dot\theta^2 c+\ddot\theta s)+(\dot\theta^2 s+\Omega^2 s-\ddot\theta c)c\right] = m(R^2-L^2)\left[\quad +\Omega^2 cs-\ddot\theta(s^2+c^2)\right]$

$\omega_{\dot\theta} = \mathbb{n}_c$ $\quad = m(R^2-L^2)\left[\Omega^2 sc-\ddot\theta\right]$

Now
$$\begin{array}{c|ccc} \mathbb{I} & \mathbb{n}_t & \mathbb{n}_o & \mathbb{n}_c \\ \hline \mathbb{n}_t & 0 & 0 & 0 \\ \mathbb{n}_o & 0 & \frac{mL^2}{3} & 0 \\ \mathbb{n}_c & 0 & 0 & \frac{mL^2}{3} \end{array}$$

$\mathbb{I} = \frac{mL^2}{3}(\mathbb{n}_o\mathbb{n}_o+\mathbb{n}_c\mathbb{n}_c)$ $\quad \mathbb{n}_o = \cos\theta\,\mathbb{k}+\sin\theta\,\mathbb{n}_b$

$\mathbb{n}_o\mathbb{n}_o = \cos^2\theta\,\mathbb{k}\mathbb{k}+\frac{\sin 2\theta}{2}(\mathbb{k}\mathbb{n}_b+\mathbb{n}_b\mathbb{k}) + \sin^2\theta\,\mathbb{n}_b\mathbb{n}_b$

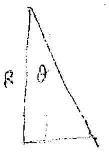
$\mathbb{I}\times{}^R\omega^{rod} = \frac{mL^2}{3}\left(c^2\mathbb{k}\mathbb{k}+sc[\mathbb{k}\mathbb{n}_b+\mathbb{n}_b\mathbb{k}]+s^2\mathbb{n}_b\mathbb{n}_b+\mathbb{n}_c\mathbb{n}_c\right)\cdot(\dot\theta\mathbb{n}_c-\Omega\mathbb{k})$

$= \frac{mL^2}{3}\left[\dot\theta\mathbb{n}_c-c^2\Omega\mathbb{k}-\Omega sc\,\mathbb{n}_b\right]$

$(\mathbb{I}\cdot\omega)\times\omega = \frac{mL^2}{3}\left[\dot\theta\mathbb{n}_c-c^2\Omega\mathbb{k}-\Omega sc\,\mathbb{n}_b\right]\times\left[\dot\theta\mathbb{n}_c-\Omega\mathbb{k}\right] = \frac{mL^2}{3}\left[-\dot\theta\Omega\mathbb{n}_b+c^2\Omega\dot\theta\mathbb{n}_b-\Omega sc\dot\theta\,\mathbb{n}_a-\Omega^2 sc\,\mathbb{n}_c\right]$

${}^R\alpha^{ROD} = \frac{{}^Rd\,{}^R\omega^{ROD}}{dt} = \frac{{}^{RING}d\,{}^R\omega^{ROD}}{dt} + {}^R\omega^{RING}\times{}^R\omega^{ROD} = \ddot\theta\mathbb{n}_c+(-\Omega\mathbb{k})\times(\dot\theta\mathbb{n}_c-\Omega\mathbb{k}) = \ddot\theta\mathbb{n}_c+\Omega\dot\theta\mathbb{n}_b$

since $\mathbb{n}_b, \mathbb{n}_a, \mathbb{n}_c$ are fixed in ring



now $\mathbb{I}\cdot\alpha = \frac{mL^2}{3}\left\{c^2\mathbb{k}\mathbb{k}+sc[\mathbb{k}\mathbb{n}_b+\mathbb{n}_b\mathbb{k}]+s^2\mathbb{n}_b\mathbb{n}_b+\mathbb{n}_c\mathbb{n}_c\right\}\cdot(\ddot\theta\mathbb{n}_c+\Omega\dot\theta\mathbb{n}_b)$.

$= \frac{mL^2}{3}\left\{\ddot\theta\mathbb{n}_c+\Omega\dot\theta sc\,\mathbb{k}+\Omega\dot\theta s^2\mathbb{n}_b\right\}$

$\mathbb{T}^* = (\mathbb{I}\cdot\omega)\times\omega-\mathbb{I}\cdot\alpha = \frac{mL^2}{3}\left\{(-2\Omega sc\dot\theta)\mathbb{n}_a+(-\dot\theta\Omega\mathbb{n}_b+c^2\Omega\dot\theta-\Omega\dot\theta s^2)\mathbb{n}_b+(-\Omega^2 sc-\ddot\theta)\mathbb{n}_c\right\}$

now $\omega_{\dot\theta}\cdot\mathbb{T}^* = -(\Omega^2 sc+\ddot\theta)\frac{mL^2}{3}$

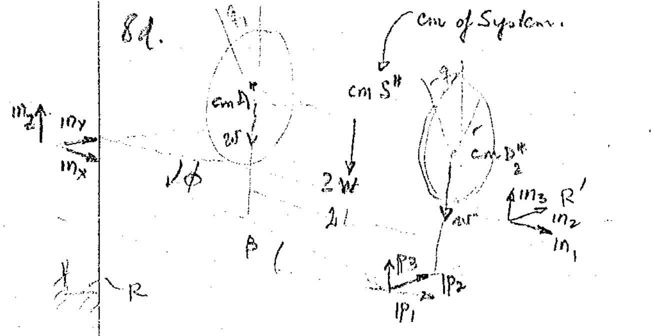
Hence

$F_\theta^* = \mathbb{V}_{\dot\theta}\mathbb{F}^*+\omega_{\dot\theta}\cdot\mathbb{T}^* = \frac{3m(R^2-L^2)}{3}\left[\Omega^2 sc-\ddot\theta\right]-\frac{mL^2}{3}(\Omega^2 sc+\ddot\theta)$

$= \frac{-W}{3g}\left[\underset{(3R^2-2L^2)\ddot\theta}{(3R^2-3L^2+L^2)\ddot\theta}+\Omega^2 sc\underset{(4L^2-3R^2)}{(3L^2-3R^2+L^2)}\right]$

$= -\frac{W}{3g}\left[(3R^2-2L^2)\ddot\theta+\Omega^2 sc(4L^2-3R^2)\right]$ ✓✓ whew!!

8d.



${}^{R'}\mathbb{V}^{D_2^*} = r\dot q_2(\sin\phi\cos\beta\,\mathbb{p}_1-\sin\phi\sin\beta\,\mathbb{p}_3-\cos\phi\,\mathbb{p}_2) = r\dot q_2(\sin\phi\,\mathbb{n}_x-\cos\phi\,\mathbb{n}_y)$

$\mathbb{V}^{D_1^*} = r\dot q_1(\sin\phi\,\mathbb{n}_x-\cos\phi\,\mathbb{n}_y) = r\dot q_1(\sin\phi\cos\beta\,\mathbb{p}_1-\sin\phi\sin\beta\,\mathbb{p}_3-\cos\phi\,\mathbb{p}$

$\mathbb{V}^{S^*} = \frac{r}{2}(\dot q_1+\dot q_2)(\sin\phi\,\mathbb{n}_x-\cos\phi\,\mathbb{n}_y) = -\frac{r}{2}(\dot q_1+\dot q_2)\mathbb{n}_2$

${}^{R'}\omega^{D_2} = \frac{r}{2l}[\dot q_1-\dot q_2]\mathbb{n}_3+\dot q_2\mathbb{n}$ [illegible] $\quad {}^{R'}\omega^{S} = \dot\phi\,\mathbb{n}_z = \dot\phi\,\mathbb{n}_3 = \frac{r}{2L}\left[\dot q_1-\dot q_2\right]\mathbb{n}_3$

${}^{R'}\omega^{D_1} = \dot\phi\,\mathbb{n}_z+\dot q_1\mathbb{n}$ [illegible] $= \frac{r}{2L}[\dot q_1-\dot q_2]\mathbb{n}_3+\dot q_1\mathbb{n}_1$

now $(F_r^*)_B = \mathbb{V}_{\dot q_r}^*\cdot\mathbb{F}^*+\omega_{\dot q_r}\cdot\mathbb{T}^*$ where $\mathbb{F}^* = -m\,\mathbb{a}^*$

now $\mathbb{V}_{\dot q_1}^{D_2^*} = 0$ $\quad \mathbb{V}_{\dot q_2}^{D_2^*} = r(\sin\phi\cos\beta\,\mathbb{p}_1-\sin\phi\sin\beta\,\mathbb{p}_3-\cos\phi\,\mathbb{p}_2)$.

$\mathbb{V}_{\dot q_1}^{D_1^*} = r(\sin\phi\cos\beta\,\mathbb{p}_1-\sin\phi\sin\beta\,\mathbb{p}_3-\cos\phi\,\mathbb{p}_2)$ $\quad \mathbb{V}_{\dot q_2}^{D_1^*} = 0$

$\mathbb{V}_{\dot q_1}^{S^*} = \frac{r}{2}(\sin\phi\cos\beta\,\mathbb{p}_1-\sin\phi\sin\beta\,\mathbb{p}_3-\cos\phi\,\mathbb{p}_2) = \mathbb{V}_{\dot q_2}^{S^*}$

$\mathbb{a}^{D_2^*} = r\ddot q_2(\sin\phi\cos\beta\,\mathbb{p}_1-\sin\phi\sin\beta\,\mathbb{p}_3-\cos\phi\,\mathbb{p}_2)+r\dot q_2\dot\phi(\cos\phi\cos\beta\,\mathbb{p}_1-\cos\phi\sin\beta\,\mathbb{p}_3+\sin\phi\,\mathbb{p}_2)$

$\mathbb{a}^{D_1^*} = r$ (

$$ {}^{R'}a^{D_2*} = -r\ddot{q}_2 n_2 - r\dot{q}_2(-\dot{\phi} n_1 + \dot{q}_2 n_3) \qquad v^{D_2*} = -r\dot{q}_2 n_2 \qquad \ddot{\phi} = \frac{r}{2L}[\ddot{q}_1 - \ddot{q}_2] $$

$$ {}^{R'}a^{D_1*} = -r\ddot{q}_1 n_2 - r\dot{q}_1(-\dot{\phi} n_1 + \dot{q}_1 n_3) \qquad v^{D_1*} = -r\dot{q}_1 n_2 $$

$$ {}^{R'}a^{S*} = -\tfrac{r}{2}(\ddot{q}_1 + \ddot{q}_2) n_2 - \tfrac{r}{2}(\dot{q}_1 + \dot{q}_2)(-\dot{\phi} n_1) \qquad v^{S*} = -\tfrac{r}{2}(\dot{q}_1 + \dot{q}_2) n_2 $$

$$ \mathbb{I}^{S*} = \frac{2ML^2}{3}(n_2n_2 + n_3n_3) = \frac{2WL^2}{3g}(n_2n_2 + n_3n_3) $$

$$ \mathbb{I}^{D_2*} = \mathbb{I}^{D_1*} = \frac{Wr^2}{4g}[2n_1n_1 + n_2n_2 + n_3n_3] $$

$$ {}^{R'}\omega^{D_2} = \dot{\phi} n_3 + \dot{q}_2 n_1 \qquad {}^{R'}\omega^{D_1} = \dot{\phi} n_3 + \dot{q}_1 n_1 $$

$$ {}^{R'}\omega^{S} = \dot{\phi} n_3 \; ; \; {}^{R'}\alpha^{D_2} = \ddot{\phi} n_3 + \ddot{q}_2 n_1 + \dot{q}_2\dot{\phi} n_2 $$

$$ {}^{R'}\alpha^{D_1} = \ddot{\phi} n_3 + \ddot{q}_1 n_1 + \dot{q}_1\dot{\phi} n_2 \; ; \; {}^{R'}\alpha^{S*} = \ddot{\phi} n_3 $$

$$ (\mathbb{I}^{D_1}\cdot\omega) = \frac{Wr^2}{4g}[2\dot{q}_1 n_1 + \dot{\phi} n_3] \qquad (\mathbb{I}^{D_1}\cdot\omega)\times\omega = \frac{Wr^2}{4g}[-2\dot{q}_1\dot{\phi} n_2 + \dot{\phi}\dot{q}_1 n_2] = -\frac{Wr^2}{4g}\dot{q}_1\dot{\phi} n_2 $$

$$ \mathbb{I}^{D_2}\cdot\omega = '' \; [2\dot{q}_2 \qquad] \qquad (\mathbb{I}^{D_2}\cdot\omega)\times\omega = -\frac{Wr^2}{4g}[\dot{q}_2\dot{\phi} n_2] $$

$$ \mathbb{I}^{D_1}\cdot\alpha = \frac{Wr^2}{4g}[2\ddot{q}_1 n_1 + \dot{q}_1\dot{\phi} n_2 + \ddot{\phi} n_3] \qquad \mathbb{I}^{D_2}\cdot\alpha = \frac{Wr^2}{4g}[2\ddot{q}_2 n_1 + \dot{q}_2\dot{\phi} n_2 + \ddot{\phi} n_3] $$

$$ {}^{R'}\omega^{D_1}_{\dot{q}_1} = \left(\frac{r}{2L} n_3 + n_1\right) \qquad \omega^{D_1}_{\dot{q}_2} = -\frac{r}{2L} n_3 \qquad {}^{R'}\omega^{D_2}_{\dot{q}_1} = \frac{r}{2L} n_3 \qquad \omega^{D_2}_{\dot{q}_2} = -\frac{r}{2L} n_3 + n_1 $$

$$ {}^{R'}\omega^{S}_{\dot{q}_1} = \frac{r}{2L} n_3 \qquad {}^{R'}\omega^{S}_{\dot{q}_2} = \frac{r}{2L} n_3 \; ; \; (\mathbb{I}^{S}\cdot\omega) = \frac{2WL^2}{3g}(\dot{\phi} n_3) \; ; \; (\mathbb{I}^{S}\cdot\omega)\times\omega = 0 \qquad \mathbb{I}^{S}\cdot\alpha = \frac{2WL^2}{3g}\ddot{\phi} n_3 $$

Thus $\underline{\omega^{D_1}_{\dot{q}_1}\cdot T^{*D_1} = \frac{r}{2L}\left(-\ddot{\phi}\frac{Wr^2}{4g}\right) - \frac{Wr^2}{4g}(2\ddot{q}_1)}$ $\qquad \underline{\omega^{D_1}_{\dot{q}_2}\cdot T^{D_1} = -\frac{r}{2L}\left[-\frac{Wr^2}{4g}\ddot{\phi}\right]}$

$$ \underline{\omega^{D_2}_{\dot{q}_1}\cdot T^{*D_2} = +\frac{r}{2L}\left[-\frac{Wr^2}{4g}\ddot{\phi}\right]} \qquad \omega^{D_2}_{\dot{q}_2}\cdot T^{*D_2} = -\frac{r}{2L}\left[-\frac{Wr^2}{4g}\ddot{\phi}\right] - \frac{Wr^2}{4g}\cdot 2\ddot{q}_2] $$

$$ \underline{\omega^{S}_{\dot{q}_1}\cdot T^{S} = -\frac{r}{2L}\cdot\frac{2WL^2}{3g}\ddot{\phi}} \qquad \omega^{S}_{\dot{q}_2}\cdot T^{S} = \underline{+\frac{r}{2L}\cdot\frac{2WL^2}{3g}\ddot{\phi}} $$

Now $v^{D_2*}_{\dot{q}_1} = 0 \quad v^{D_2*}_{\dot{q}_2} = -r n_2 \quad v^{D_1*}_{\dot{q}_1} = -r n_2 \quad v^{D_1*}_{\dot{q}_2} = 0 \quad v^{S*}_{\dot{q}_1} = -\tfrac{r}{2} n_2 \quad v^{S*}_{\dot{q}_2} = -\tfrac{r}{2} n_2$

$-mv^{D_2*}_{\dot{q}_1}\cdot a^{D_2*} = 0 \; ; \; -mv^{D_2*}_{\dot{q}_2}\cdot a^{D_2*} = -mr^2\ddot{q}_2 \; ; \; -mv^{D_1}_{\dot{q}_1}\cdot a^{D_1} = -mr^2\ddot{q}_1 \; ; \; -mv_{\dot{q}_2}\cdot a^{D_1} = 0 \; ;$

$-Mv^{S*}_{\dot{q}_1}\cdot a^{S} = \underline{-\tfrac{Mr^2}{4}(\ddot{q}_1 + \ddot{q}_2)} \; ; \; -Mv^{S*}_{\dot{q}_2}\cdot a^{S} \; \underline{= -\tfrac{Mr^2}{4}(\ddot{q}_1 + \ddot{q}_2)}$

Now

$$ F_1^* = \Big[-\frac{2Wr^2}{4g}(\ddot{q}_1 + \ddot{q}_2) - \frac{Wr^2}{g}\ddot{q}_1 - \frac{r}{2L}\cdot\frac{2W\cancel{L^2}}{3g}\cdot\frac{r}{2L}(\ddot{q}_1 - \ddot{q}_2) - \frac{r}{2L}\left[\frac{Wr^2}{4g}\cdot\frac{r}{2L}(\ddot{q}_1 - \ddot{q}_2)\right] - \frac{Wr^2}{4g}\cdot 2\ddot{q}_1 $$

$$ - \frac{r}{2L}\frac{Wr^2}{4g}\cdot\frac{r}{2L}(\ddot{q}_1 - \ddot{q}_2)\Big] = -\frac{3Wr^2}{2g}\ddot{q}_1 - \frac{Wr^4}{8gL^2}(\ddot{q}_1 - \ddot{q}_2) - \frac{Wr^2}{6g}(\ddot{q}_1 - \ddot{q}_2) - \frac{Wr^2}{2g}(\ddot{q}_1 + \ddot{q}_2) $$

$$ \frac{Wr^2}{g}\left(-\frac{2}{3}\ddot{q}_1 - \frac{1}{3}\ddot{q}_2\right) $$

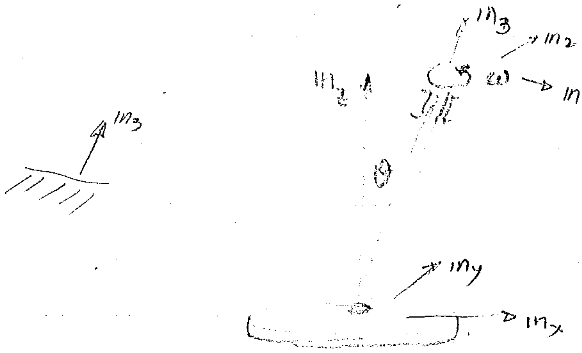
$$ = -\ddot{q}_1\left(+\frac{3W}{2g}r^2 + \frac{Wr^4}{8gL^2} + \frac{2Wr^2}{3g}\right) - \ddot{q}_2\left(\frac{1}{3}\frac{Wr^2}{g} - \frac{Wr^4}{8gL^2}\right) $$

$$F_2^* = \Big[-\frac{2Wr^2}{4g}(\ddot{q}_1+\ddot{q}_2) - \frac{Wr^2}{g}\ddot{q}_2 + \frac{r^2}{24L^2}\cdot\frac{2WL^2}{3g}(\ddot{q}_1-\ddot{q}_2) + \frac{r^4}{4L^2}\frac{W}{4g}(\ddot{q}_1-\ddot{q}_2) + \frac{r^4 W}{4L^2\cdot 4g}(\ddot{q}_1-\ddot{q}_2)$$

$$-\frac{Wr^2}{2g}\ddot{q}_2\Big] = -\frac{3r^2W}{2g}\ddot{q}_2 + \frac{r^4 W}{8L^2 g}(\ddot{q}_1-\ddot{q}_2) + \frac{Wr^2}{g}\left(\frac{1}{3}\ddot{q}_1 - \frac{2}{3}\ddot{q}_2\right)$$

$$= -\ddot{q}_1\left(-\frac{r^4 W}{8gL^2} + \frac{1Wr^2}{3g}\right) - \ddot{q}_2\left(\frac{r^4W}{8L^2 g} + \frac{3r^2W}{2g} + \frac{2}{3}\frac{Wr^2}{g}\right)$$

8e.



$${}^R\omega^{ROD} = \omega\,\mathbf{n}_3$$

$${}^R\omega^{DISK} = {}^R\omega^{ROD} + \overset{\to 0}{{}^{ROD}\omega^{DISK}}$$

$$= \omega\cos\theta\,\mathbf{n}_z + \omega\sin\theta\,\mathbf{n}_x$$

what is T^* disk $\mathbf{n}_z = \cos\theta\,\mathbf{n}_1$

$$\mathbb{I} = \frac{mR^2}{2}\mathbf{n}_z\mathbf{n}_z + \frac{mR^2}{4}(\mathbf{n}_x\mathbf{n}_x + \mathbf{n}_y\mathbf{n}_y)$$

$$\mathbb{I}\cdot\omega = \frac{mR^2}{2}\omega\cos\theta\,\mathbf{n}_z + \frac{mR^2}{4}\omega\sin\theta\,\mathbf{n}_x$$

$$(\mathbb{I}\cdot\omega)\times\omega = \frac{mR^2}{2}\omega^2\sin\theta\cos\theta\,\mathbf{n}_y - \frac{mR^2}{4}\omega^2\sin\theta\cos\theta\,\mathbf{n}_y$$

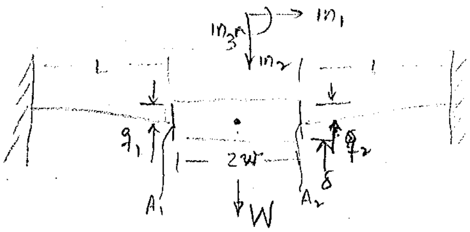
$$= \frac{mR^2\omega^2}{4}\sin\theta\cos\theta\,\mathbf{n}_y = \frac{mR^2\omega^2\sin 2\theta}{8}\,\mathbf{n}_y$$

$$\alpha^{ROD} = \dot{\omega}\,\mathbf{n}_3 \quad \text{but} \quad \omega = \text{const} \ \therefore\ \alpha = 0 \Rightarrow \mathbb{I}\cdot\alpha = 0$$

$$T^* = (\mathbb{I}\cdot\omega)\times\omega - \mathbb{I}\cdot\alpha = \frac{mR^2\omega^2\sin 2\theta}{8}\,\mathbf{n}_y$$

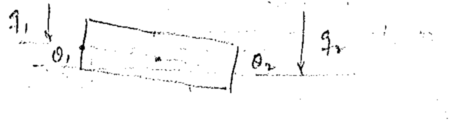
Dalembert Principal $T + T^* = 0$ $-T^* = +T$ = torque applied by bearings

8f.



Assume $\delta \ll 2w$ thus block is like a thin rod.

We had formed $\omega^B = (-\dot{q}_1+\dot{q}_2)/2w\ \mathbf{n}_3$ and that $\mathbf{v}^{A_1} = \dot{q}_1\mathbf{n}_2$ $\mathbf{v}^{A_2} = \dot{q}_2\mathbf{n}_2$



$$\theta_2 = \frac{q_2 - q_1}{2w}$$

$$\theta_1 = \frac{q_2 - q_1}{2w}$$

$$\omega = \dot{\theta}\,\mathbf{n}_3 \qquad \omega_{\dot{q}_1} = -\frac{1}{2w}\mathbf{n}_3 \quad \omega_{\dot{q}_2} = \frac{1}{2w}\mathbf{n}_3$$

$$\mathbf{v}^{B^*} = \mathbf{v}^{A_1} + \omega\times w\,\mathbf{n}_1 = \frac{\dot{q}_2+\dot{q}_1}{2}\mathbf{n}_2$$

$$\mathbf{v}^{B^*}_{\dot{q}_1} = \frac{1}{2}\mathbf{n}_2 \quad \mathbf{v}^{B^*}_{\dot{q}_2} = \frac{\mathbf{n}_2}{2}$$

$$\mathbf{a}^{B^*} = \frac{\ddot{q}_2+\ddot{q}_1}{2}\mathbf{n}_2$$

$$\therefore\ -m\,\mathbf{v}_{\dot{q}_1}\cdot\mathbf{a}^{B^*} = -\frac{m}{4}(\ddot{q}_2+\ddot{q}_1) \qquad -m\,\mathbf{v}_{\dot{q}_2}\cdot\mathbf{a}^{B^*} = -\frac{m}{4}(\ddot{q}_2+\ddot{q}_1)$$

$\mathbb{I}^{B*} = \frac{m l^2}{12}(\mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3)$ $\quad m = \frac{W}{g} \quad l = 2w$

$= \frac{W}{3g} w^2 (\mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3)$

$\mathbb{I}\cdot\omega = \frac{Ww^2}{3g}\left(\frac{\ddot q_2 - \ddot q_1}{2w}\right)\mathbf{n}_3 \qquad (\mathbb{I}\cdot\omega)\times\omega = 0 \qquad \alpha = \frac{\ddot q_2 - \ddot q_1}{2w}\,\mathbf{n}_3$

$\mathbb{I}\cdot\alpha = \frac{W}{3g} w^2 \frac{\ddot q_2 - \ddot q_1}{2w}\,\mathbf{n}_3 \qquad T^* = -\frac{W}{6g} w (\ddot q_2 - \ddot q_1)\,\mathbf{n}_3$

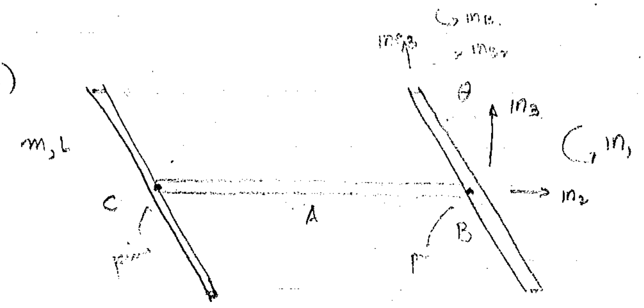
$\omega_{\dot q_1}\cdot T^* = \frac{W}{12g}(\ddot q_2 - \ddot q_1) \qquad \omega_{\dot q_2}\cdot T^* = -\frac{W}{12g}(\ddot q_2 - \ddot q_1)$

$F_1^* = -\frac{W}{4g}(\ddot q_2 + \ddot q_1) + \frac{W}{12g}(\ddot q_2 - \ddot q_1) = -\ddot q_1\left(\frac{1}{3}\frac{W}{g}\right) - \ddot q_2\left(\frac{1}{6}\frac{W}{g}\right)$

$= -\frac{W}{6g}\left[2\ddot q_1 + \ddot q_2\right]$

$F_2^* = -\frac{W}{4g}(\ddot q_2 + \ddot q_1) - \frac{W}{12} g(\ddot q_2 - \ddot q_1) = -\ddot q_1\left(\frac{1}{6}\frac{W}{g}\right) - \ddot q_2\left(\frac{1}{3}\frac{W}{g}\right) = -\frac{1}{6}\frac{W}{g}\left[\ddot q_1 + 2\ddot q_2\right]$

8g)



let $u_i = \begin{cases} \omega\cdot\mathbf{n}_i & i = 1,2,3 \\ \dot\theta & i = 4 \\ \mathbf{v}\cdot\mathbf{n}_{i-4} & i = 5,6,7; \end{cases}$

where $\mathbf{v}^A$, ω^A

drop all 2nd & higher degrees of θ

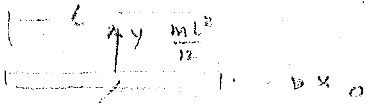
& 2nd & higher degrees of $\dot\theta$, $\ddot\theta$ etc

find F_1^*, F_4^*, F_7^*

now ${}^R\mathbf{v}^{A*} = (\mathbf{v}\cdot\mathbf{n}_{i-4})\,\mathbf{n}_{i-4} = u_5\mathbf{n}_1 + u_6\mathbf{n}_2 + u_7\mathbf{n}_3$ $\quad \left| \mathbf{v}^{A*}_{u_1} = 0 \quad \mathbf{v}^{A*}_{u_4} = 0 \quad \mathbf{v}^{A*}_{u_7} = \mathbf{n}_3 \right|$

$\omega^A = u_1\mathbf{n}_1 + u_2\mathbf{n}_2 + u_3\mathbf{n}_3 \; ;$ $\quad \left| \omega^A_{u_7} = 0 \quad \omega^A_{u_1} = \mathbf{n}_1 \;\&\; \omega^A_{u_4} = 0 \right|$

$\mathbb{I}^{A/A*} = \frac{mL^2}{12}(\mathbf{n}_3\mathbf{n}_3 + \mathbf{n}_1\mathbf{n}_1)$



for $\mathbf{n}_i$ fixed in A only

$\mathbf{a}^{A*} = \dot u_5\mathbf{n}_1 + \dot u_6\mathbf{n}_2 + \dot u_7\mathbf{n}_3 + (-u_5u_2 + u_6u_1)\mathbf{n}_3 + (u_5u_3 - u_7u_1)\mathbf{n}_2 + (-u_6u_3 + u_7u_2)\mathbf{n}$

$-m\,\mathbf{v}^{A*}_{u_7}\cdot\mathbf{a}^{A*} = -m\,\mathbf{n}_3\cdot\mathbf{a}^* = -m\dot u_7 - m(u_6u_1 - u_5u_2)$

$T^{A*} = (\mathbb{I}\cdot\omega)\times\omega - \mathbb{I}\cdot\alpha$

$\alpha^A = \dot u_1\mathbf{n}_1 + \dot u_2\mathbf{n}_2 + \dot u_3\mathbf{n}_3 + u_1(-u_2\mathbf{n}_3 + u_3\mathbf{n}_2) + u_2(u_1\mathbf{n}_3 - u_3\mathbf{n}_1) + u_3(-u_1\mathbf{n}_2 + u_2\mathbf{n}_1)$

$(\mathbb{I}^A\cdot\omega^A) = \frac{mL^2}{12}(u_3\mathbf{n}_3 + u_1\mathbf{n}_1) \qquad (\mathbb{I}^A\cdot\omega^A)\times\omega^A = \frac{mL^2}{12}(u_3u_1\mathbf{n}_2 - \dot u_3u_2\mathbf{n}_1 + u_1u_2\mathbf{n}_3 - u_1u_3\mathbf{n}_2)$

$\mathbb{I}\cdot\alpha^A = \frac{mL^2}{12}\left[(\dot u_1 - u_2u_3 + u_2u_3)\mathbf{n}_1 + (\dot u_3 - u_2u_1 + u_2u_1)\mathbf{n}_3\right]$

$\therefore \; T^{*A} = \frac{mL^2}{12}\left[(-u_3u_2 - \dot u_1)\mathbf{n}_1 + (u_1u_2 - \dot u_3)\mathbf{n}_3\right] \qquad \omega^A_{u_7}\cdot T^{*A} = 0$

${}^A\omega^B = \dot\theta\,\mathbf{n}_1 = u_4\mathbf{n}_1 \qquad \omega^B = u_4\mathbf{n}_1 + \omega^A \; ; \qquad \omega^B_{u_7} = 0 \quad \omega^B_{u_4} = \mathbf{n}_1 \quad \omega^B_{u_1} = \mathbf{n}_1 \;|$

$\mathbf{v}^{B*} = \mathbf{v}^{A*} + \omega^A \times \frac{l}{2}\mathbf{n}_2 = \mathbf{v}^A + \left[(u_4 + u_1)\mathbf{n}_1 + u_2\mathbf{n}_2 + u_3\mathbf{n}_3\right]\times \frac{l}{2}\mathbf{n}_2 = u_5\mathbf{n}_1 + (u_6 - u_3\tfrac{l}{2})\mathbf{n}_2 + (u_7 + (u_1 + u_4$ $\mathbf{n}_3$

$\mathbf{a}^{B*} = \dot u_5\mathbf{n}_1 + (\dot u_6 - \dot u_3\tfrac{l}{2})\mathbf{n}_2 + (\dot u_7 + (\dot u_4 + \dot u_1)\tfrac{l}{2})\mathbf{n}_3 + u_5\big(\overset{\omega^B\times\mathbf{n}_1}{\cancel{-u_2\mathbf{n}_3 + u_3\mathbf{n}_2}}\big) + (u_6 - u_3\tfrac{l}{2})\big(\overset{\omega^B\times\mathbf{n}_2}{\cancel{u_1\mathbf{n}_3 - u_3\mathbf{n}_1}}\big)$

$+ (u_7 + (u_1 + u_4)\tfrac{l}{2})\big(\overset{\omega^B\times\mathbf{n}_3}{\cancel{-u_1\mathbf{n}_2 + u_2\mathbf{n}_1}}\big)$

$\mathbf{v}_{u_7} = \mathbf{n}_3 \; ; \; \mathbf{v}_{u_4} = \tfrac{l}{2}\mathbf{n}_3 \quad \mathbf{v}_{u_1} = \tfrac{l}{2}\mathbf{n}_2$

$${}^{A}\omega^{C} = \dot{\theta}\,\mathbf{n}_1 = u_4\,\mathbf{n}_1 \qquad \omega^{C} = \omega^{A} + u_4\mathbf{n}_1 \qquad \boxed{\omega^{C}_{u_7} = 0 \quad \omega^{C}_{u_4} = \mathbf{n}_1 \quad \omega^{C}_{u_1} = \mathbf{n}_1}$$

$$\omega^{C} = (u_4+u_1)\mathbf{n}_1 + u_2\mathbf{n}_2 + u_3\mathbf{n}_3$$

$$\mathbf{v}^{C*} = \mathbf{v}^{A*} + \omega^{A} \times -\frac{l}{2}\mathbf{n}_2 \qquad u_5\mathbf{n}_1 + (u_6 + u_3 l/2)\mathbf{n}_2 + (u_7 + (u_1+u_4)l/2)\mathbf{n}_3$$

$$\boxed{\mathbf{v}^{C*}_{u_7} = \mathbf{n}_3 \quad \mathbf{v}^{C*}_{u_4} = -l/2\,\mathbf{n}_3 \quad \mathbf{v}^{C*}_{u_1} = -\frac{l}{2}\mathbf{n}_3}$$

$$\mathbf{a}^{C*} = \dot{u}_5\mathbf{n}_1 + (\dot{u}_6 + \dot{u}_3 l/2)\mathbf{n}_2 + [\dot{u}_7 - (\dot{u}_1+\dot{u}_4)l/2]\mathbf{n}_3 + u_5(\overset{B}{\omega}{}^{C}\times\mathbf{n}_1) + (u_6+u_3l/2)(\omega^{C}\times\mathbf{n}_2) + [u_7-(u_1+u_4l/2)](\omega^{C}\times\mathbf{n}_3) ; \; \mathbb{I}^{B/B*} = \frac{mL^2}{12}(\mathbf{n}_{B_2}\mathbf{n}_{B_2} + \mathbf{n}_{B_1}\mathbf{n}_{B_1})$$

$$\begin{array}{c|ccc} & \mathbf{n}_1 & \mathbf{n}_2 & \mathbf{n}_3 \\ \hline \mathbf{n}_{B_1} & 1 & 0 & 0 \\ \mathbf{n}_{B_2} & 0 & c_\theta & +s_\theta \\ \mathbf{n}_{B_3} & 0 & -s_\theta & c_\theta \end{array}$$

$$= \frac{mL^2}{12}\left[(c_\theta\mathbf{n}_2 + s_\theta\mathbf{n}_3)(c_\theta\mathbf{n}_2 + s_\theta\mathbf{n}_3) + \mathbf{n}_1\mathbf{n}_1\right]$$

$$= \frac{mL^2}{12}\left[c_\theta^2\mathbf{n}_2\mathbf{n}_2 + c_\theta s_\theta(\mathbf{n}_2\mathbf{n}_3 + \mathbf{n}_3\mathbf{n}_2) + s_\theta^2\mathbf{n}_3\mathbf{n}_3 + \mathbf{n}_1\mathbf{n}_1\right]$$

$$\omega^{B} = (u_4+u_1)\mathbf{n}_1 + u_2\mathbf{n}_2 + u_3\mathbf{n}_3$$

$$(\mathbb{I}^{B/B*}\cdot\omega^{B}) = \left[(u_4+u_1)\mathbf{n}_1 + u_2(c_\theta^2\mathbf{n}_2 + c_\theta s_\theta\mathbf{n}_3) + u_3(s_\theta^2\mathbf{n}_2 + c_\theta s_\theta\mathbf{n}_2)\right]\frac{mL^2}{12}$$

$$(\mathbb{I}\cdot\omega^{B})\times\omega^{B} = \Big[(u_4+u_1)u_2\mathbf{n}_3 - (u_4+u_1)u_3\mathbf{n}_2 - u_2c_\theta^2(u_4+u_1)\mathbf{n}_3 + u_2u_3c_\theta^2\mathbf{n}_1 + u_2(u_4+u_1)c_\theta s_\theta\mathbf{n}_2$$
$$- u_2c_\theta s_\theta u_2\mathbf{n}_1 + u_3s_\theta^2(u_4+u_1)\mathbf{n}_2 - u_2u_3s_\theta^2\mathbf{n}_1 - u_3c_\theta s_\theta(u_4+u_1)\mathbf{n}_3 + u_3^2c_\theta s_\theta\mathbf{n}_1\Big]\frac{mL^2}{12}$$
$$= \Big[(u_4+u_1)u_2s_\theta^2\mathbf{n}_3 - u_3(u_4+u_1)c_\theta^2\mathbf{n}_2 + u_2u_3c_{2\theta}\mathbf{n}_1 + u_2(u_4+u_1)c_\theta s_\theta\mathbf{n}_2$$
$$- u_2^2c_\theta s_\theta\mathbf{n}_1 - u_3(u_4+u_1)c_\theta s_\theta\mathbf{n}_3 + u_3^2c_\theta s_\theta\mathbf{n}_1\Big]\frac{mL^2}{12}$$

~~$\alpha^{B} = (\dot{u}_4+\dot{u}_1)\mathbf{n}_1 + \dot{u}_2\mathbf{n}_2 + \dot{u}_3\mathbf{n}_3 + (u_4+u_1)(-u_2\mathbf{n}_3 + u_3\mathbf{n}_2) + u_2(u_1\mathbf{n}_3 - u_3\mathbf{n}_1) + u_3(-u_1\mathbf{n}_2 + u_2\mathbf{n}_1)]$~~

~~$\mathbb{I}\cdot\alpha^{B} = \frac{mL^2}{12}\left[(\dot{u}_4+\dot{u}_1 - u_2u_3 + u_2u_3)\mathbf{n}_1 + (\dot{u}_2 + (u_4+u_1)u_3 - u_3u_1)\mathbf{n}_2 + (\dot{u}_3 - u_2(u_4+u_1) + u_2u_1)\mathbf{n}_3\right]$~~

$$\omega^{B}_{u_7} = 0 \qquad \omega^{B}_{u_7}\cdot\mathbf{T}^{*} = \omega^{B}_{u_7}\cdot\left[(\mathbb{I}\cdot\omega)\times\omega^{B} - \mathbb{I}\cdot\alpha^{B}\right] = 0$$

Since bar C has same $\mathbb{I}^{C/C*}$ as $\mathbb{I}^{B/B*}$ $\quad \omega^{C}_{u_7} = \omega^{B}_{u_7} = 0$

then $(\mathbb{I}\cdot\omega^{C})\times\omega^{C} - \mathbb{I}\cdot\alpha^{C} = \mathbf{T}^{*C} = \mathbf{T}^{*B}$ hence $\omega^{C}_{u_7}\cdot\mathbf{T}^{*} = 0$ also

thus only contribution

to: $F_7^{*} = -m\mathbf{v}^{A}_{u_7}\cdot\mathbf{a}^{A} - m\mathbf{v}^{B}_{u_7}\cdot\mathbf{a}^{B} - m\mathbf{v}^{C}_{u_7}\cdot\mathbf{a}^{C}$

$$= -m\dot{u}_7 - m(u_6u_1 - u_5u_2) - m(\dot{u}_7 + \dot{u}_4 + \dot{u}_1)l/2) - m(u_1u_6 - u_2u_5)$$
$$-m(\dot{u}_7 - (\dot{u}_1+\dot{u}_4)l/2) - m(u_1u_6 - u_2u_5) = -3m\left[\dot{u}_7 + (u_6u_1 - u_5u_2)\right]$$

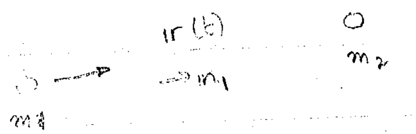
to $F_4^{*} = +\omega^{C}_{u_4}\cdot\mathbf{T}^{C} + \mathbf{v}^{C}_{u_4}\cdot\mathbf{F}^{*C} + \omega^{B}_{u_4}\cdot\mathbf{T}^{B} + \mathbf{v}^{B}_{u_4}\cdot\mathbf{F}^{*B}$ since $\mathbf{v}^{A*}_{u_4} = \omega^{A}_{u_4} = 0$

to $F_1^{*} = \left(\omega^{B}_{u_1}\cdot\mathbf{T}^{*B} + \mathbf{v}^{B*}_{u_1}\cdot\mathbf{F}^{B}\right) + \left(\omega^{C}_{u_1}\cdot\mathbf{T}^{*C} + \mathbf{v}^{C}_{u_1}\cdot\mathbf{F}^{C}\right) + \omega^{A}_{u_1}\cdot\mathbf{T}^{*A}$

in obtaining α & $\mathbf{a}$ we must use $\dot{\mathbf{n}}_1 = \omega\times\mathbf{n}_1 \quad \dot{\mathbf{n}}_2 = \omega\times\mathbf{n}_1 \quad \dot{\mathbf{n}}_3 = \omega\times\mathbf{n}$

Set #9

9a.



$\mathbf{F}_1 = \frac{Gm_1m_2}{r^2}\mathbf{n}_1$ $\mathbf{F}_2 = -\frac{Gm_1m_2}{r^2}\mathbf{n}_1$ $\mathbf{r}^{P_2} = \mathbf{r}^{P_1} + \mathbf{r}(t)$

$\mathbf{v}^{P_1} = u_{1i}\mathbf{n}_i$ $\mathbf{v}^{P_2} = u_{2i}\mathbf{n}_i$; $\mathbf{v}^{P_2} = \mathbf{v}^{P_1} + \dot{\mathbf{r}}$

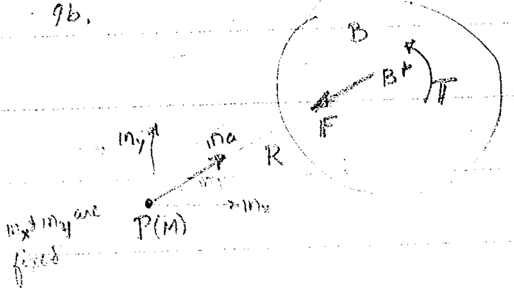
∴ $\mathbf{v}^{P_1}_{u_{11}} = \mathbf{n}_1$ $\mathbf{v}^{P_2}_{u_1} = \mathbf{n}_1$ since $\dot{\mathbf{r}}$ is not a fn of u_i

∴ $F_i' = \mathbf{v}^{P_1}_{u_{1i}} \cdot \mathbf{F}_1 + \mathbf{v}^{P_2}_{u_{1i}} \cdot \mathbf{F}_2 = 0$ $i=1,2,3$; $F_4' = \mathbf{v}^{P_2}_{\dot{r}} \cdot \mathbf{F}_2 = -\frac{Gm_1m_2}{r^2}\mathbf{n}_1 \cdot \mathbf{n}_1 = -\frac{Gm_1m_2}{r^2}$

but $F_4' = -\frac{Gm_1m_2}{r^2} = -\frac{\partial P}{\partial r}$ ∴ $\frac{Gm_1m_2}{r^2} = \frac{\partial P}{\partial r} \Rightarrow P = -\frac{Gm_1m_2}{r} + f(u_1,u_2,u_3,t$

but $\frac{\partial P}{\partial u_1} = 0$; $\frac{\partial P}{\partial u_2} = 0$, $\frac{\partial P}{\partial u_3} = 0 \Rightarrow f(u_1,u_2,u_3,t) = w(t)$ ∴ $P = -\frac{Gm_1m_2}{r} + w(t)$

9b.



${}^R\omega^B = {}^R\omega^I + {}^I\omega^B$ ${}^R\omega^I = \Omega\mathbf{n}_z + {}^I\omega^B$ let ${}^I\omega^B = \omega_i\mathbf{n}_i$ where

$\mathbf{n}_i$'s are ∥ to principal axes of B

$\mathbf{F} = -F\mathbf{n}_a$ and $\mathbf{T} = \frac{3GM}{R^3}\mathbf{n}_a \times \mathbb{I}_a$

$\mathbf{v}^{B^*} = \omega^B \times R\mathbf{n}_a = R(\omega^B \times \mathbf{n}_a)$ $\mathbf{v}^{B^*}_{\dot{q}_r} = \omega_{\dot{q}_r} \times \mathbf{n}_a$

now $F_r = \mathbf{v}_{\dot{q}_r} \cdot \mathbf{F} + \omega_{\dot{q}_r} \cdot \mathbf{T} = -F\,\overset{\to 0}{\underline{\mathbf{n}_a \cdot (\omega_{\dot{q}_r} \times \mathbf{n}_a)}} + \omega_{\dot{q}_r} \cdot \frac{3GM}{R^3}\mathbf{n}_a \times \mathbb{I}_a$

$= \frac{3GM}{R^3}\omega_{\dot{q}_r} \cdot (\mathbf{n}_a \times \mathbb{I}_a) = \frac{3GM}{R^3}\mathbb{I}_a \cdot (\omega_{\dot{q}_r} \times \mathbf{n}_a)$

but $\frac{{}^R\partial \mathbf{n}_a}{\partial q_r} = \frac{{}^B\partial \mathbf{n}_a}{\partial q_r} + \omega^B_{q_r} \times \mathbf{n}_a$ and $\frac{{}^R\partial \mathbf{n}_a}{\partial q_r} = 0$ since $\mathbf{n}_a$ is not a fn of q_r in R

∴

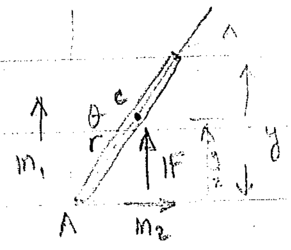
$$F_r = \frac{3GM}{R^3}\mathbb{I}_a \cdot \left(-\frac{{}^B\partial \mathbf{n}_a}{\partial q_r}\right) = -\frac{3GM}{R^3}\mathbb{I}_a \cdot \frac{{}^R\partial \mathbf{n}_a}{\partial q_r}$$

if $\mathbf{n}_a = a_i\mathbf{n}_i$ then $\mathbb{I}_a = I_1a_1\mathbf{n}_1 + \dots$ then $\mathbb{I}_a \cdot \frac{{}^B\partial \mathbf{n}_a}{\partial q_r} = I_i\left(a_i\frac{\partial a_i}{\partial q_r}\right) = I_i\frac{1}{2}\frac{\partial}{\partial q_r}a_i^2$

∴ $F_r = -\frac{3GM}{2R^3}\frac{\partial}{\partial q_r}(a_i^2 I_i)$ but $a_i^2 I_i = I = \mathbf{n}_a \cdot \mathbb{I} \cdot \mathbf{n}_a$

$F_r = -\frac{\partial}{\partial q_r}\left(\frac{3GM}{2R^3}I\right)$ ∴ $P = \frac{3GMI}{2R^3} + w(t)$

9c.



$w = \rho g$ displaced volume is $\frac{Ay}{\cos\theta}$ ∴ by Archimedes's princip

The buoyant force = displaced volume × weight density

$\mathbf{F} = \frac{Ay}{\cos\theta} \cdot w\mathbf{n}_1$ acting upward at mass center of submerged portion

now $\mathbf{v}^C = \mathbf{v}^A + \dot{\theta}\mathbf{n}_3 \times \mathbf{n}_a$ now $\mathbf{v}^C_{\dot{\theta}} = r\mathbf{n}_3 \times \mathbf{n}_a + \mathbf{v}^A_{\dot{\theta}}$ $\mathbf{v}^A_{\dot{\theta}} = 0$

∴ $(F_\theta)_{\text{buoyant force}} = \frac{Ay}{\cos\theta}w\,r\mathbf{n}_3 \times \mathbf{n}_a \cdot \mathbf{n}_1$

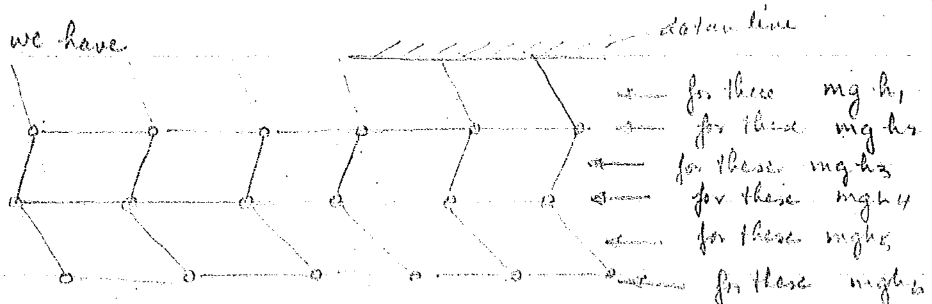
but $m_a = m_1 \cos\theta + m_2 \sin\theta$ $\quad m_3 \times m_a = \cos\theta\, m_2 - m_1 \sin\theta$ $\quad$ and $\quad m_3 \times m_a \cdot m_1 = -\sin\theta$

$\therefore \quad F_\theta = -rAy\omega \tan\theta$ $\quad$ but $r = \dfrac{y}{2\cos\theta}$ $\quad \therefore \quad F_\theta = -\dfrac{Ay^2}{2}\omega \dfrac{\sin\theta}{\cos^2\theta}$

but $F_\theta = -\dfrac{\partial P}{\partial\theta} = -\dfrac{Ay^2}{2}\omega \dfrac{\sin\theta}{\cos^2\theta}$ $\qquad P = +\dfrac{Ay^2\omega}{2\cos\theta} + f(t))$

or $\underline{P = \dfrac{Ay^2\omega}{2} \sec\theta + f(t)}$

9d Now we have $\qquad P = \sum mg\,h_i$



$h_1 = -\frac{l}{2}\cos q_1 \quad h_2 = -l\sin q_1 \quad h_3 = -l(\cos q_1 + \frac{1}{2}\cos q_2) \quad h_4 = -l(\cos q_1 + \cos q_2)$

$h_5 = -l(\cos q_1 + \cos q_2 + \frac{1}{2}\cos q_3) \quad h_6 = -l(\cos q_1 + \cos q_2 + \cos q_3)$

$\therefore \quad 6mgh_1 + 5mgh_2 + 6mgh_3 + 5mgh_4 + 6mgh_5 + 5mgh_6$

$-lmg\left[+\overset{/}{3}\cos q_1 + \overset{/}{5}\cos q_1 + \overset{/}{6}\cos q_1 + \overset{\times}{3}\cos q_2 + \overset{/}{5}\cos q_1 + \overset{\times}{5}\cos q_2 + \overset{/}{6}\cos q_1 + \overset{\times}{6}\cos q_2 + \overset{//}{3}\cos q_3 + \overset{/}{5}\cos q_1 + \overset{\times}{5}\cos q_2 + \overset{//}{5}\cos q_3\right)$

$P = -Wl\left[30\cos q_1 + 19\cos q_2 + 8\cos q_3\right)$

9e Since we found in 5(a) $\quad F_\theta = -W(R^2 - L^2)^{1/2}\sin\theta = -\dfrac{\partial P}{\partial\theta}$

$\therefore \dfrac{\partial P}{\partial\theta} = W(R^2 - L^2)^{1/2}\sin\theta \qquad \therefore \quad P = -W(R^2 - L^2)^{1/2}\cos\theta + f(t)$

9f) From 5(d) we found

$F_1 = a_1 q_1 + b_1 q_2 + \dfrac{W}{2} = -\dfrac{\partial P}{\partial q_1} \qquad \therefore \quad P = -a_1 q_1^2/2 + b_1 q_2 q_1 + \dfrac{W}{2} q_1 + f(q_2, t)$

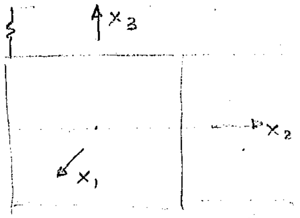
$F_2 = b_1 q_1 + a_1 q_2 + \dfrac{W}{2} = -\dfrac{\partial P}{\partial q_2} \qquad \dfrac{\partial P}{\partial q_2} = b_1 q_1 + \dfrac{\partial f}{\partial q_2} = b_1 q_1 + a_1 q_2 + \dfrac{W}{2}$

$\dfrac{\partial f}{\partial q_2} = a_1 q_2 + \dfrac{W}{2} \qquad \therefore \quad f = a_1 q_2^2/2 + \dfrac{W}{2} q_2 + g(t)$

$\therefore \quad P = -a_1 q_1^2/2 + b_1 q_2 q_1 + \dfrac{W}{2}(q_1 + q_2) + a_1 q_2^2/2 + g(t)$

$= \left(\dfrac{q_1^2 - q_2^2}{2}\right)\left[\dfrac{12EI}{L^3}\left(1 + \dfrac{L}{2w} + \dfrac{1}{6}\dfrac{L^2}{w^2}\right)\right] + \dfrac{6EI}{L^2 w}\left(1 + \dfrac{L}{3w}\right) q_2 q_1 + \dfrac{W}{2}(q_1 + q_2) + g(t)$

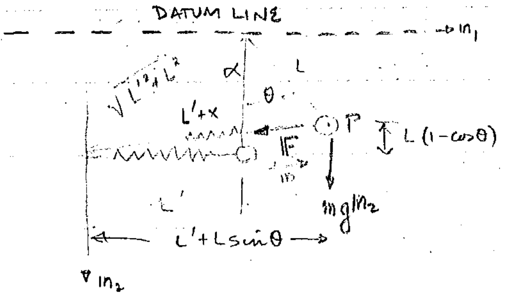
9(g) look at one corner



Movement of the corner of the cube: By looking the movement of the cube corner as the cube center undergoes displacement to a new point; The cube corner undergoes an extension in the $-x_3$ direction. Now if the cube is rotated about the x_1 axis will give rise to a displacement $a\theta_1$. Now if the cube is rotated about the x_2 direction gives rise to an $a\theta_2$ displacement. Rotation about the x_3 axis gives rise to displacements in the x_1 & x_2 direction thus the overall displacement of the spring is $-x_3+a\theta_1+a\theta_2=x$

Now P for a spring is $\frac{1}{2}kx^2$ $\therefore$ $P_1=\frac{1}{2}ka^2\left(\theta_1+\theta_2-x_3/a\right)^2$. Similarly we can do the same for the other springs to get P_2 & P_3. Thus $P=\Sigma P_i$. What we've done here is good for small displacements ie displacements for terms of 0 or 1st degree in θ_i and x_i.

9(h)



$$r^P = L\cos\theta\, n_2 + L\sin\theta\, n_1$$

$$v^P = (-L\sin\theta\, n_2 + L\cos\theta\, n_1)\dot{\theta}$$

now $(L'+x)^2 = L^2+(L'^2+L^2) - 2L\sqrt{L'^2+L^2}\cos(\theta+\alpha)$

$$L'^2+2L'x+x^2 = 2L^2+L'^2 - 2L\sqrt{L'^2+L^2}\cos(\theta+\alpha)$$

$$= 2L^2 - 2L\sqrt{L'^2+L^2}\left[\cos\theta\frac{L}{\sqrt{L'^2+L^2}} - \sin\theta\frac{L'}{\sqrt{L'^2+L^2}}\right]$$

$$2L'x+x^2+L'^2 = 2L^2 + 2L^2\cos\theta + 2LL'\sin\theta + L'^2$$

$$x^2+2L'x - 2L^2(1-\cos\theta) - 2LL'\sin\theta = 0$$

$\mathbb{F} = -kx n$ $\quad n = \dfrac{(L'+L\sin\theta)n_1 - L(1-\cos\theta)n_2}{\sqrt{(L'+L\sin\theta)^2+L^2(1-\cos\theta)^2}}$

$$F_\theta = -kx\left\{\frac{L\cos\theta(L'+L\sin\theta)+L^2\sin\theta(1-\cos\theta)}{\sqrt{\ }}\right\} - mgL\sin\theta$$

the pot. fn due to gravity is $P=mgh$ $\quad h=-L\cos\theta$ $\quad\therefore\quad P=-mgL\cos\theta$ from the datum line

the potential fn due to the spring $=\frac{1}{2}kx^2$

$$\text{Now}\quad (x+L')^2 = 2L^2(1-\cos\theta) + L'(L'+2L\sin\theta)$$

$$(x+L') = L'\sqrt{2\frac{L^2}{L'^2}(1-\cos\theta) + \left(1+\frac{2L}{L'}\sin\theta\right)}$$

Use expansion $(1+p)^{1/2} = 1+\frac{1}{2}p - \frac{1}{8}p^2 + \frac{1}{16}p^3$ $\quad$ where $p = \frac{2L}{L'}\sin\theta + \frac{2L^2}{L'^2}(1-\cos\theta)$

$$\therefore\quad p \cong \frac{2L}{L'}\left(\theta - \theta^3/3! + \dots\right) + \frac{2L^2}{L'^2}\left(\theta^2/2 - \theta^4/4!\right) = \frac{2L}{L'}\left(\theta + \frac{L}{2L'}\theta^2 - \theta^3/3! - \frac{L}{4!L'}\theta^4\right.$$

$$= 1 + \frac{L}{L'}\left(\theta - \theta^3/3!\right) + \frac{L^2}{L'^2}\left(\theta^2/2 - \theta^4/4!\right) - \frac{1}{8}\cdot\frac{4L^2}{L'^2}\left[\theta^2 + \frac{L^2}{4L'^2}\theta^4 + \frac{2L}{2L'}\theta^3 - \frac{2\theta^4}{3!}\right] + \dots$$

$$+\frac{1}{16}\left[\frac{8L^3}{L'^3}\left(\theta^3 + \frac{3L}{2L'}\theta^4\right) + \dots\right.$$

$$\therefore\quad x = L(\theta - \theta^3/3!) + 0(\theta^4)$$

$$x^2 = L^2\left(\theta^2 - 2\theta^4/6 + h.o.t.\right)$$

$$P_{TOT} = -mgL\left(1 - \theta^2/2! + \theta^4/4! \ldots\right) + \frac{1}{2}\cdot\frac{5mg}{k}\cdot L^2\left[\theta^2 - \frac{\theta^4}{3} + \ldots\right]$$

$$= -mgL + mgL\theta^2/2! - mgL\theta^4/4! + \frac{5}{2}mgL\theta^2 - \frac{5mg}{6}L\theta^4 + \ldots$$

$$P_{TOT} - mgL = mgL\left[\theta^2\cdot 3\right] - mgL\theta^4\left[\frac{1}{24} + \frac{5}{6}\right] = mgL\left[3\theta^2 - \frac{7}{8}\theta^4\right]$$

$$a_1 = 0\ ,\ a_2 = 3,\ a_3 = 0\ ,\ a_4 = -7/8$$

Now $F_\theta = -\frac{Kx}{x+L'}\left\{LL'\cos\theta + L^2\sin\theta\right\} - mgL\sin\theta = -\frac{KL^2(\theta - \theta^3/3!)}{}\left\{\left(1 - \theta^2/2! + \theta^4/4! + \ldots\right) + \frac{L}{L'}\left(\theta - \theta^3/3! + \ldots\right)\right\}\left\{1 - \frac{L}{L'}\left(\theta - \theta^3/3!\right) + \frac{L^2}{L'^2}\left(\theta - \theta^3/3!\right)^2 + \ldots\right\}$

$- mgL\left\{\theta - \theta^3/3!\right\} = -KL^2(\theta - \theta^3/3!)\left\{1 + L/L'\,\theta - \theta^2/2! - \frac{L}{L'}\theta^3/3!\right\}\left\{1 - \frac{L}{L'}\theta + L^2/L'^2\,\theta^2 + \frac{L}{L'}\theta^3/3! \ldots\right\} = -KL^2\left(\theta - \theta^3/3!\right)\left\{1 + 0 + 2L^2/L'^2\,\theta^2 - \theta^2/2!\right\} - mgL\left\{\theta - \theta^3/3!\right\}$

$= -6mgL\theta + 5/6\,mgL\theta^3 + \frac{5mg}{2}\theta^3 + mgL\frac{\theta^3}{6}$

q_i since $P = mgL\sum a_n\theta^n$ then $\frac{\partial P}{\partial\theta} = mgL\sum_{n=1}^{\infty} a_n n\theta^{n-1} = mgL[-6\theta + 3.5\theta^3 + \ldots]$

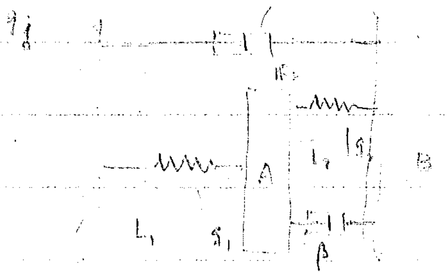
$$F_\theta = -\frac{\partial P}{\partial\theta} = -mgL\left[a_1 + a_2\cdot 2\theta + a_3\cdot 3\theta^2 + \ldots\right.$$

$$mgL\left[b_1\theta + b_2\theta^2 + \ldots\right. = mgL\left[-a_1 - a_2\cdot 2\theta - a_3\cdot 3\theta^2 \ldots\right.$$

$a_1 = 0\quad b_1 = -2a_2 = -6\ ,\ b_2 = -3a_3 = 0\ ,\ b_3 = -4a_4 = +4\cdot\frac{7}{8} = +3.5$

easier to find the potential energy then differentiate. Both methods are cumbersome.

$\mathbb{F}_\alpha = -\alpha\mathbb{W}^B$



$\mathbb{F}_\beta = -\beta(\mathbb{W}^B - \mathbb{W}^A)$

$\rightarrow$

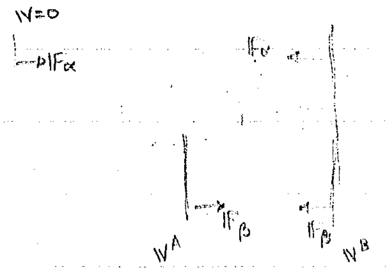
$\mathbb{r}^B = (L_1 + q_1 + L_2 + q_2)\,\mathbb{m}$ $\qquad \mathbb{r}^A = (L_1 + q_1)\mathbb{m}$

$\mathbb{W}^B = (\dot q_1 + \dot q_2)\ \mathbb{m}$ $\qquad \mathbb{W}^B - \mathbb{W}^A = \dot q_2\mathbb{m}$

$\mathbb{W}^A = \dot q_1\mathbb{m}$

$\mathbb{F}_\beta = -\beta\dot q_2\mathbb{m}$

$\mathbb{F}_\alpha = -\alpha[\dot q_1 + \dot q_2]\mathbb{m}_1$



$$\therefore\ F_1' = \mathbb{F}_\alpha\cdot\mathbb{W}^B_{\dot q_1} + \mathbb{F}_\beta\cdot\mathbb{W}^B_{\dot q_1} + \mathbb{F}_\alpha\cdot 0 - \mathbb{F}_\beta\cdot\mathbb{W}^A_{\dot q_1}$$

$$= -\alpha[\dot q_1 + \dot q_2] - \beta\dot q_2 + \beta\dot q_2 = -\alpha[\dot q_1 + \dot q_2]$$

$$F_2' = \mathbb{F}_\alpha\cdot\mathbb{W}^B_{\dot q_2} + \mathbb{F}_\beta\cdot\mathbb{W}^B_{\dot q_2} + \mathbb{F}_\alpha\cdot 0 - \mathbb{F}_\beta\cdot\mathbb{W}^A_{\dot q_2}$$

$$= -\alpha[\dot q_1 + \dot q_2] - \beta\dot q_2 = -(\alpha+\beta)\dot q_2 - \alpha\dot q_1$$

$$\frac{\partial\mathcal{F}}{\partial\dot q_1} = -F_1' = \alpha[\dot q_1 + \dot q_2] \qquad \mathcal{F} = \alpha\left[\frac{\dot q_1^2}{2} + \dot q_2\dot q_1\right] + f(\dot q_2, t)$$

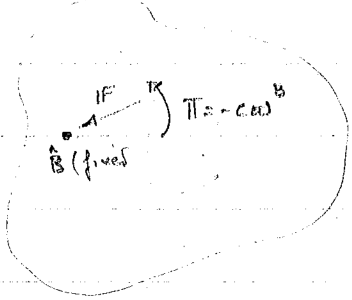
$$\frac{\partial\mathcal{F}}{\partial\dot q_2} = \alpha\dot q_1 + f' = -F_2' = (\alpha+\beta)\dot q_2 + \alpha\dot q_1$$

$$f' = (\alpha+\beta)\dot q_2$$

$$f = (\alpha+\beta)\dot q_2^2/2 + g(t)$$

$$\therefore\ \mathcal{F} = \alpha\dot q_1^2/2 + \alpha\dot q_2\dot q_1 + (\alpha+\beta)\dot q_2^2/2$$

9k



$$F_r = \overset{\hat{B}}{W}_{\dot q_r} \cdot \mathbb{F} + \omega_{\dot q_r} \cdot \overline{\Pi} \qquad \Pi = -c\,\omega$$

since $W^{\hat{B}} = 0$ ($\hat{B}$ is fixed)

$$F_r = \omega_{\dot q_r} \cdot \overline{\Pi} = \omega_{\dot q_r} \cdot (-c\,\omega)$$

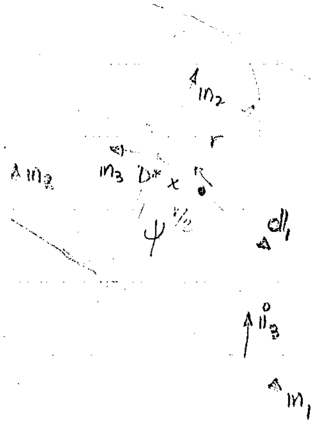
now if $\omega_{\dot q_r} = \dfrac{\partial \omega}{\partial \dot q_r}$ then $\omega = \omega_{\dot q_r}\,\dot q_r + \omega_t$

hence $F_r = -c\,(\omega \cdot \omega_{\dot q_r}') - c\,(\omega_{\dot q_r} \cdot \omega_t)$

$$F_r = -c\left(\frac{\partial}{\partial \dot q_r}\frac{\omega^2}{2}\right) - c\,\frac{\partial}{\partial \dot q_r}\left[\frac{d}{dt}\frac{\omega^2}{2}\right]$$

Now $F_r = -\dfrac{\partial \mathcal{F}}{\partial \dot q_r} = -c\,\dfrac{\partial}{\partial \dot q_r}\dfrac{\omega^2}{2} \Rightarrow \mathcal{F} = \dfrac{c\omega^2}{2} +$ fn of time.

10a



$$\mathbf{v}^{D^*} = (\dot\phi\,\mathbf{n}_2 - \dot\theta\,\mathbf{n}_1 + \dot\psi\,\mathbf{n}_3)\times r\,\mathbf{n}_2 = \cancel{r\dot\phi\cdot 0} - r\dot\theta\,\mathbf{n}_3 - r\dot\psi\,\mathbf{n}_1$$

$$\mathbf{r}^P = \mathbf{r}^{D^*} + x\,\mathbf{d}_1 \qquad \mathbf{v}^P = \mathbf{v}^{D^*} - v\,\mathbf{d}_1 + x\,\boldsymbol{\omega}^D\times\mathbf{d}_1$$

$${}^R\boldsymbol{\omega}^D = \dot\phi\,\mathbf{i}^0_3 - \dot\theta\,\mathbf{n}_1 + \dot\psi\,\mathbf{n}_3\,;\quad {}^R\boldsymbol{\omega}^D\times\mathbf{d}_1 = -\dot\phi\,\mathbf{n}_3 + \dot\psi\,\mathbf{n}_2$$

when $\theta = 0,\ \psi = \pi/2$: $\mathbf{n}_3\times\mathbf{d}_1 = \mathbf{n}_2$; $\mathbf{n}_1\times\mathbf{d}_1 = 0$; $\mathbf{i}^0_3\times\mathbf{d}_1 = -\mathbf{n}_3$ $\quad \mathbf{d}_1 = \mathbf{n}_1$

$$\mathbf{v}^{P^2} = \mathbf{v}^{D^{*2}} - 2v\,\mathbf{v}^{D^*}\cdot\mathbf{d}_1 + r\,\mathbf{v}^{D^*}\cdot\underline{(\boldsymbol{\omega}^D\times\mathbf{d}_1)} + v^2 + \cancel{2\frac{vr}{2}\,\mathbf{d}_1\cdot(\boldsymbol{\omega}^D\times\mathbf{d}_1)} + \frac{r^2}{4}(\boldsymbol{\omega}\times\mathbf{d}_1)^2$$

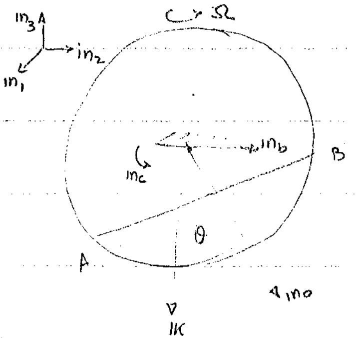
$$= r^2\dot\theta^2 + r^2\dot\psi^2 - 2v(-r\dot\psi) + r\left[+r\dot\theta\dot\phi\right] + v^2 + \frac{r^2}{4}\left[\dot\phi^2 + \dot\psi^2\right]$$

$$= r^2\left[\dot\theta^2 + \frac{5}{4}\dot\psi^2 + \frac{\dot\phi^2}{4} + 2\frac{v}{r}\dot\psi + \dot\theta\dot\phi + \frac{v^2}{r^2}\right]$$

$$\mathbf{v}^{P^2} = \left[r^2\left(\dot\theta^2 + \tfrac{5}{4}\dot\psi^2 + \tfrac{\dot\phi^2}{4} + \dot\theta\dot\phi\right) + v(v + 2r\dot\psi)\right]$$

$$K = \frac{1}{2}m\mathbf{v}^{P^2} = \frac{1}{2}m\left[r^2\left(\dot\theta^2 + \tfrac{5}{4}\dot\psi^2 + \tfrac{\dot\phi^2}{4} + \dot\theta\dot\phi\right) + v(v + 2r\dot\psi)\right]$$

10b $\quad K = K_v + K_\omega \qquad K_v = \frac{1}{2}m v^{*2} \qquad K_\omega = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbb{I}\cdot\boldsymbol{\omega} \qquad \boldsymbol{\omega}^{ROD} = \dot\theta\,\mathbf{n}_c - \Omega\,\mathbf{k}$



$$v^{*2} = (R^2 - L^2)\left[\dot\theta^2 + \Omega^2\sin^2\theta\right]$$

$$\mathbb{I} = \frac{mL^2}{3}\mathbf{n}_o\mathbf{n}_o + \frac{mL^2}{3}\mathbf{n}_c\mathbf{n}_c \qquad \mathbf{n}_o = c\,\mathbf{k} + s\,\mathbf{n}_b$$

$$\mathbf{n}_o\mathbf{n}_o = c^2\mathbf{kk} + s^2\mathbf{n}_b\mathbf{n}_b + sc(\mathbf{k}\mathbf{n}_b + \mathbf{n}_b\mathbf{k})$$

$$\mathbb{I}\cdot\boldsymbol{\omega} = \frac{mL^2}{3}\left\{-\Omega c^2\mathbf{k} - \Omega sc\,\mathbf{n}_b + \dot\theta\,\mathbf{n}_c\right\}$$

$$\frac{1}{2}\boldsymbol{\omega}\cdot\mathbb{I}\cdot\boldsymbol{\omega} = \Omega^2c^2\frac{mL^2}{6} + \frac{mL^2}{6}\dot\theta^2$$

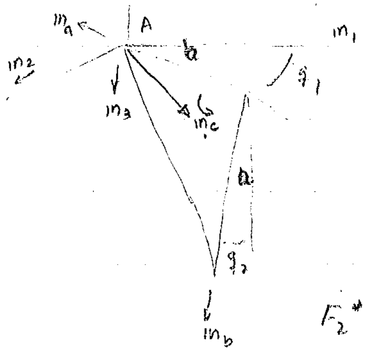
$$K = K_v + K_\omega = \frac{1}{2}m\left[(R^2 - L^2)\left\{\dot\theta^2 + \Omega^2\sin^2\theta\right\} + \Omega^2\cos^2\theta\frac{L^2}{3} + \frac{L^2}{3}\dot\theta^2\right]$$

$$F_\theta^* = \frac{\partial K}{\partial\theta} - \frac{d}{dt}\frac{\partial K}{\partial\dot\theta} = \frac{1}{\cancel{2}}m\left[(R^2 - L^2)\Omega^2\cdot\cancel{2}\sin\theta\cos\theta - \cancel{2}\Omega^2\cos\theta\sin\theta\,L^2/3\right] - \frac{1}{\cancel{2}}m\left[(R^2 - L^2)\cdot\cancel{2}\ddot\theta + \cancel{2}\frac{L^2}{3}\ddot\theta\right]$$

$$= -\frac{m}{3}\left[\Omega^2\sin\theta\cos\theta\,(4L^2 - 3R^2) + (3R^2 - 2L^2)\ddot\theta\right]$$

Same result. Much easier tho.

10c $\quad \mathbf{v}^{T^*} = \dot q_1 s_2\frac{a}{3}\mathbf{n}_a + \dot q_1 s_2\cdot\frac{2b}{3}\mathbf{n}_b + \left(\frac{2b}{3}\dot q_1 c_2 + \dot q_2\, a/3\right)\mathbf{n}_a \qquad {}^R\boldsymbol{\omega}^T = \dot q_1\left[\mathbf{n}_b\cos\theta_2 - \mathbf{n}_c\sin\theta_2\right] + \dot q_2\mathbf{n}_a$



$$\mathbb{I}^{T/T^*} = \frac{ma^2}{18}\mathbf{n}_a\mathbf{n}_a + \frac{mab}{36}(\mathbf{n}_a\mathbf{n}_b + \mathbf{n}_b\mathbf{n}_a) + \frac{mb^2}{18}\mathbf{n}_b\mathbf{n}_b + \frac{m(a^2+b^2)}{18}\mathbf{n}_c\mathbf{n}_c$$

$$\mathbf{p} = -\tfrac{2}{3}b\,\mathbf{n}_a + \tfrac{1}{3}a\,\mathbf{n}_b \qquad \mathbf{p}\cdot\mathbf{n}_a = -\tfrac{2}{3}b \qquad \mathbf{p}\cdot\mathbf{n}_b = +\tfrac{1}{3}a \qquad \mathbf{p}\cdot\mathbf{p} = \frac{4}{9}b^2 + \frac{a^2}{9}$$

$$\mathbb{I}^{T/A} = \begin{bmatrix} \frac{ma^2}{18} + \frac{ma^2}{9} = \frac{ma^2}{6} & \frac{mab}{4} & 0 \\ \frac{mab}{4} & \frac{mb^2}{18} + \frac{8mb^2}{18} = \frac{mb^2}{2} & 0 \\ 0 & 0 & \frac{m(a^2+b^2)}{18} + m\frac{(8b^2+2a^2)}{18} = m\left(\frac{a^2}{6} + \frac{b^2}{2}\right) \end{bmatrix}$$

$$F_2^* = \frac{\partial K}{\partial q_2} - \frac{d}{dt}\frac{\partial K}{\partial\dot q_2}$$

$$I^{A/T^*}_{\ell m} = m\left[\mathbf{p}^2\delta_{\ell m} - (\mathbf{p}\cdot\mathbf{n}_\ell)(\mathbf{p}\cdot\mathbf{n}_m)\right]$$

$$\mathbb{I}\cdot\omega = \frac{ma^2}{6}\dot q_2 \mathbf{n}_a + \frac{mab}{4}\left[\dot q_1 c_2 \mathbf{n}_a + \dot q_2 \mathbf{n}_b\right] + \frac{mb^2}{2}\dot q_1 c_2 \mathbf{n}_b - \dot q_1 s_2\, m\left(\frac{3a^2+9b^2}{18}\right)\mathbf{n}_c$$

$$\frac{1}{2}\omega\cdot\mathbb{I}\cdot\omega = \frac{ma^2}{12}\dot q_2^2 + \frac{mab}{8}\left[2\dot q_1 \dot q_2 c_2\right] + \frac{mb^2}{4}\dot q_1^2 c_2^2 + \dot q_1^2 s_2^2\, m\left(\frac{3a^2+9b^2}{36}\right)$$

$$\frac{\partial K}{\partial q_2} = -\frac{mab}{4}\dot q_1 \dot q_2 s_2 - \frac{9mb^2}{18}\dot q_1^2 c_2 s_2 + \dot q_1^2\, m\left(\frac{3a^2+9b^2}{18}\right)s_2 c_2 = \dot q_1^2\,\frac{ma^2}{6}s_2 c_2$$

$$\frac{\partial K}{\partial \dot q_2} = \frac{ma^2}{6}\dot q_2 + \frac{mab}{4}\dot q_1 c_2\, ; -\frac{d}{dt}\left(\frac{\partial K}{\partial \dot q_2}\right) = -2\frac{ma^2}{12}\ddot q_2 + \frac{mab}{4}\dot q_1 s_2 \dot q_2 - \frac{3mab}{12}\ddot q_1 c_2$$

$$F_2^* = \frac{ma^2}{12}\left[2a\left(s_2 c_2 \dot q_2^2 - \ddot q_2\right) - 3b\, c_2 \ddot q_1\right]$$

Method of Solution

10 d. Find the velocity of the mass centers ; $\mathbb{I}\cdot\omega = I_3\omega_3$ where I_3, ω_3 are components in direction out of paper. Thus for each rod form $\sum \frac{1}{2} I_3 \omega_{3_i}^2 = K_\omega$ $\quad I_3 = m\ell^2/12$ for rods length ℓ.

square velocity & form $\sum \frac{1}{2} m_i V_i^{*2} = K_V$ now that $\frac{\partial K_T}{\partial q_1} - \frac{d}{dt}\frac{\partial K_T}{\partial \dot q_1} = F_1^*$

where $K_T = K_\omega + K_V$

Better to use $\mathbb{F}^*\cdot V_{\dot q_1} + \mathbb{T}^*\cdot\omega_{\dot q_1} = F_1^*$

10e. (8d) is a non holonomic system.

(8e) Because of generalized speeds we have problem

10f. must form $KE_\omega + KE_V$ $\qquad KE_V = \frac{1}{2} m V_{sp}^{*2}\cdot 4 + \frac{1}{2} m V_c^{*2} \nearrow 0$

$$KE_\omega = \frac{1}{2}\omega^c\cdot\mathbb{I}^c\cdot\omega^c + \frac{4}{2}\omega^s\cdot\mathbb{I}^s\cdot\omega^s = \frac{1}{2}J\omega^2 +$$

$\mathbb{I}^S = \frac{2}{5}mr^2\left(\mathbf{n}_1\mathbf{n}_1 + \mathbf{n}_2\mathbf{n}_2 + \mathbf{n}_3\mathbf{n}_3\right)$. $\qquad \mathbb{I}^c = J\mathbf{n}_2\mathbf{n}_2 + \dots \qquad {}^R\omega^c = \omega\mathbf{n}_2$

$V_s^*\cdot\mathbf{n}_3 = {}^RV^S = \dfrac{\omega\sin\theta\, r}{\cos\theta - \sin\theta}$ from problem (3d) $\quad$ at 30° $\quad \dfrac{\omega\cdot\frac{1}{2}r}{\frac{\sqrt3}{2} - \frac{1}{2}} = \dfrac{\omega r}{\sqrt3 - 1}$

$$\omega^s = \omega^c + {}^c\omega^s = \cancel{\omega\mathbf{n}_2} + \left(-\frac{{}^RV^S}{r} - \cancel{\omega}\right)\mathbf{n}_2 + \left(\frac{{}^RV^S}{r}\right)\mathbf{n}_1 = (\mathbf{n}_1 - \mathbf{n}_2)\frac{{}^RV^S}{r}$$

$$\omega^s = (\mathbf{n}_1 - \mathbf{n}_2)\frac{\omega}{\sqrt3 - 1} \qquad \omega^c = \omega\mathbf{n}_2 \qquad V^{*2} = \frac{\omega^2 r^2}{4 - 2\sqrt3}$$

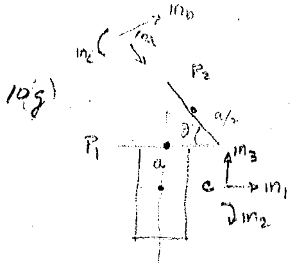
$$\therefore\; K = \frac{4}{2}m\frac{\omega^2 r^2}{4-2\sqrt3} + \frac{1}{2}J\omega^2 + \frac{4}{2}\frac{\omega^2}{4-2\sqrt3}(\mathbf{n}_1 - \mathbf{n}_2)\cdot\underbrace{\left[\frac{2}{5}mr^2\left(\mathbf{n}_1\mathbf{n}_1 + \mathbf{n}_2\mathbf{n}_2 + \dots\right)\right](\mathbf{n}_1 - \mathbf{n}_2)}_{\frac{2}{5}mr^2(\mathbf{n}_1 - \mathbf{n}_2)}$$

$$= \frac{4}{2}\frac{m\omega^2 r^2}{4-2\sqrt3} + \frac{1}{2}J\omega^2 + \frac{4}{2}\frac{\omega^2}{4-2\sqrt3}\cdot\frac{4}{5}mr^2 = \frac{2m\omega^2 r^2}{4-2\sqrt3} + \frac{8}{5}\frac{\omega^2 mr^2}{4-2\sqrt3} + \frac{1}{2}J\omega^2$$

$$= \frac{m\omega^2 r^2}{2-\sqrt3} + \frac{4}{5}\frac{m\omega^2 r^2}{2-\sqrt3} + \frac{1}{2}J\omega^2$$

$$= \frac{9}{5}m\omega^2 r^2\left(2+\sqrt3\right) + \frac{1}{2}J\omega^2$$

10g)



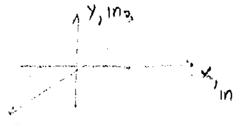
$\omega^c = \omega n_3$ $\omega^{P_1} = \omega n_3$ $\omega^{P_2} = \omega n_3$

$\mathbb{V}^{c^*} = 0$ $\mathbb{V}^{P_1^*} = 0$ $\mathbb{r}^{P_2} = (h + a/2 \sin\theta) n_3 + \frac{a}{2}(1-\cos\theta) n_1$

$\mathbb{V}^{P_2^*} = \omega^{P_2} \times \mathbb{r} = \omega \frac{a}{2}(1-\cos\theta) n_2$

$$K_{TOT} = \frac{1}{2} m \mathbb{V}^{P_2^2} + \frac{1}{2}\omega^{P_1}\cdot\mathbb{I}\cdot\omega^{P_1} + \frac{1}{2}\omega^{P_2}\cdot\mathbb{I}^{P_2}\cdot\omega^{P_2}$$

$$= \frac{1}{2} m\omega^2\frac{a^2}{4}(1-\cos\theta)^2 + \frac{1}{2}\omega^2\cdot\frac{ma^2}{12} + \frac{1}{2}\omega^{P_2}\cdot\mathbb{I}^{P_2}\cdot\omega^{P_2}$$



$I_{xx} = \frac{ma^2}{12}$, $I_{yy} = \frac{m}{6}a^2$, $I_{zz} = \frac{ma^2}{12}$

$\mathbb{I}^{P_2}$	n_a	n_b	n_c
n_a	$\frac{ma^2}{12}$	0	0
n_b	0	$\frac{ma^2}{6}$	0
n_c	0	0	$\frac{ma^2}{12}$

	n_a	n_b	n_c
n_1	c	s	0
n_2	0	0	~~0~~ -1
n_3	-s	c	0

only need I_{33}.

$$\mathbb{I}^{P_2} = \frac{ma^2}{12}(n_a n_a + 2 n_b n_b + n_c n_c)$$

$$\mathbb{I} = \frac{ma^2}{12}\cdot s^2 n_3 n_3 + 0 + \frac{ma^2}{6} c^2 n_3 n_3 = \frac{ma^2}{12} n_3 n_3 + \cdots \qquad \therefore \frac{1}{2}\frac{ma^2}{12}(s^2 + 2c^2) = \frac{1}{2}\omega^{P_2}\cdot\mathbb{I}^{P_2}\cdot\omega^{P_2}$$

$$\therefore K_{TOT} = \cancel{\frac{\omega^2 ma^2}{12}} + \frac{1}{2} m \frac{\omega^2 a^2}{12} 3(1-\cos\theta)^2 + \frac{1}{2}\omega^2\frac{ma^2}{12}(s^2+2c^2) + \frac{1}{2}\omega^2\frac{ma^2}{12}$$

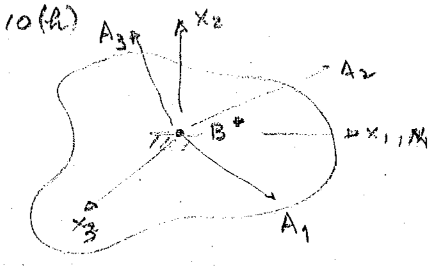
$$\frac{\partial K}{\partial\theta} = 0 = 6(1-\cos\theta)s + 2sc + 4cs$$

$$= 6s - 6cs + 2sc - 4cs = 6s - 8cs = 0 \qquad \therefore \frac{3}{4} = \cos\theta \qquad \theta = \arccos\left(\frac{3}{4}\right)$$

$$\frac{\partial^2 K}{\partial\theta^2} = 6c - 8c^2 + 8s^2 = 6c - 8(c^2 - s^2) = 6c - 8(\cos 2\theta)$$

since $\theta < 90°$ then $\cos 2\theta < 0$ $\therefore \frac{\partial^2 K}{\partial\theta^2} > 0$ $\therefore$ min

10(h)



for the fixed point of a rigid body $\Rightarrow (m_{rs})_B = \omega_{\dot{q}_r}^B \cdot \mathbb{I} \cdot \omega_{\dot{q}_s}^B$

let I_1, I_2, I_3 be the value of the principal moments of inertia $\omega = \dot{q}_1 x_1 + \dot{q}_2 n_2 + \dot{q}_3 a_3$

$$= (\dot{q}_1 c_2 c_3 + s_2\dot{q}_2) a_1 + (-\dot{q}_1 s_3 c_2 + c_3\dot{q}_2) a_2 + (s_2\dot{q}_1 + \dot{q}$$

from the form of $\omega_{\dot{q}_1} = (c_2 c_3 a_1 - s_3 c_2 a_2 + s_2 a_3)$

$\omega_{\dot{q}_2} = (s_2 a_1 + c_3 a_2)$

$\omega_{\dot{q}_3} = a_3$

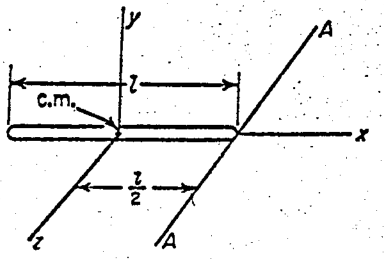
$\therefore m_{12} = c_2 c_3 s_2 I_1 - s_3 c_2 c_3 I_2$ $\qquad m_{13} = s_2 I_3$ $\qquad m_{23} = 0$

$\therefore$ q_1 & q_2 and q_1 and q_3 are coupled dynamically

Note that i, j, k, l are virtual work problems

APPENDIX A

INERTIAL PROPERTIES OF HOMOGENEOUS BODIES

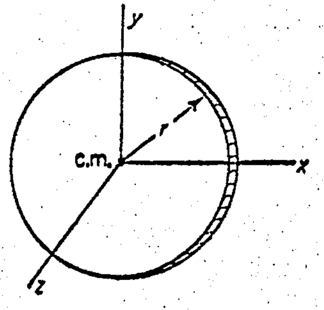


Thin rod

$$I_{xx} = 0$$

$$I_{yy} = I_{zz} = \frac{ml^2}{12}$$

$$I_{AA} = \frac{ml^2}{3}$$

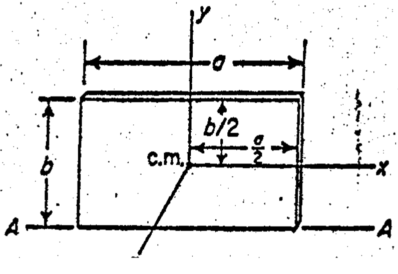


Thin circular disk

$$A = \pi r^2$$

$$I_{xx} = I_{yy} = \frac{mr^2}{4}$$

$$I_{zz} = \frac{mr^2}{2}$$



$$A = ab$$

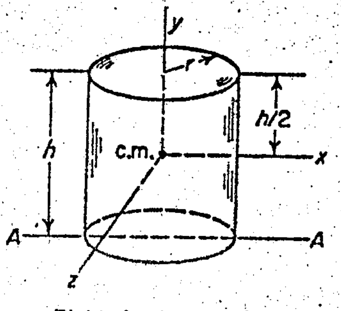
see pg 108 for more

$$I_{xx} = \frac{mb^2}{12}$$

$$I_{yy} = \frac{ma^2}{12}$$

$$I_{zz} = \frac{m}{12}(a^2 + b^2)$$

$I_{AA} = m\underline{b}^2$



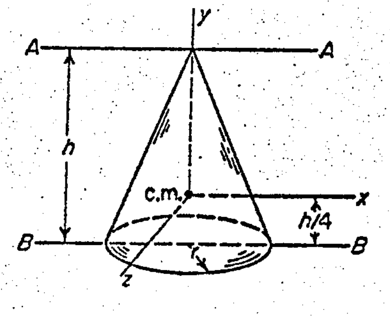
Right circular cylinder

$$V = \pi r^2 h$$

$$I_{xx} = I_{zz} = \frac{m}{12}(3r^2 + h^2)$$

$$I_{yy} = \frac{1}{2} mr^2$$

$$I_{AA} = \frac{m}{12}(3r^2 + 4h^2)$$



Right circular cone

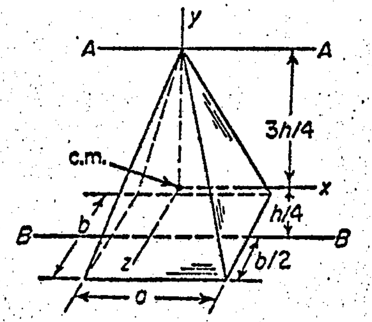
$$V = \frac{1}{3}\pi r^2 h$$

$$I_{xx} = I_{zz} = \frac{3m}{80}(4r^2 + h^2)$$

$$I_{yy} = \frac{3}{10} mr^2$$

$$I_{AA} = \frac{3m}{20}(r^2 + 4h^2)$$

$$I_{BB} = \frac{m}{20}(3r^2 + 2h^2)$$



Right rectangular pyramid

$$V = \frac{abh}{3}$$

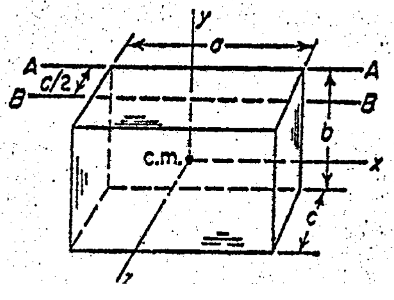
$$I_{xx} = \frac{m}{80}(4b^2 + 3h^2)$$

$$I_{yy} = \frac{m}{20}(a^2 + b^2)$$

$$I_{zz} = \frac{m}{80}(4a^2 + 3h^2)$$

$$I_{AA} = \frac{m}{20}(b^2 + 12h^2)$$

$$I_{BB} = \frac{m}{20}(b^2 + 2h^2)$$



$$V = abc$$

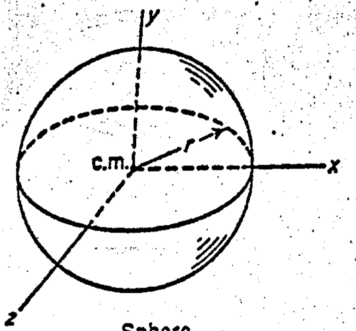
$$I_{xx} = \frac{m}{12}(b^2 + c^2)$$

$$I_{yy} = \frac{m}{12}(a^2 + c^2)$$

$$I_{zz} = \frac{m}{12}(a^2 + b^2)$$

$$I_{AA} = \frac{m}{3}(b^2 + c^2)$$

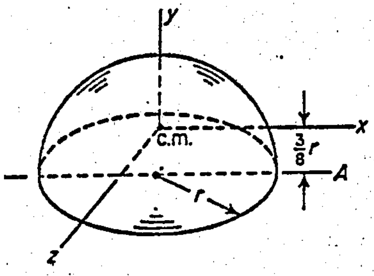
$I_{BB} = \frac{m}{12}(4b^2 + c^2)$



Sphere

$$V = \frac{4}{3}\pi r^3$$

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{5} mr^2$$



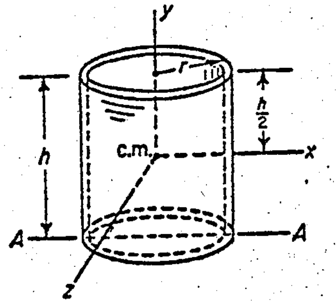
Hemisphere

$$V = \frac{2}{3}\pi r^3$$

$$I_{xx} = I_{zz} = \frac{83}{320} mr^2$$

$$I_{yy} = \frac{2}{5} mr^2$$

$$I_{AA} = \cancel{\frac{2}{5} mr^2} \quad \frac{5}{2} mr^2$$



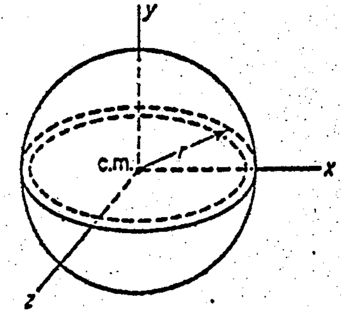
Thin circular cylindrical shell

$$A = 2\pi rh$$

$$I_{xx} = I_{zz} = \frac{m}{12}(6r^2 + h^2)$$

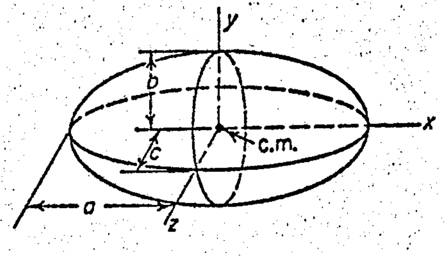
$$I_{yy} = mr^2$$

$$I_{AA} = \frac{m}{6}(3r^2 + 2h^2)$$



$$A = 4\pi r^2$$

$$I_{xx} = I_{yy} = I_{zz} = \frac{2}{3} mr^2$$



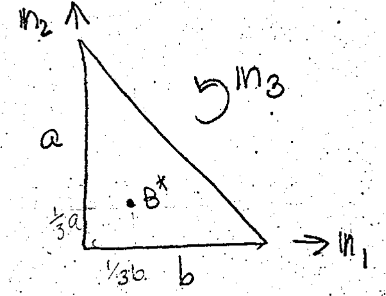
Ellipsoid

$$V = \frac{4}{3}\pi abc$$

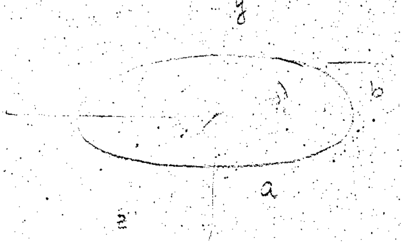
$$I_{xx} = \frac{m}{5}(b^2 + c^2)$$

$$I_{yy} = \frac{m}{5}(a^2 + c^2)$$

$$I_{zz} = \frac{m}{5}(a^2 + b^2)$$



$I_{ij}^{B^*}$	1	2	3
1	$\frac{ma^2}{18}$	$\frac{mab}{36}$	0
2	$\frac{mab}{36}$	$\frac{mb^2}{18}$	0
3	0	0	$\frac{m(a^2+b^2)}{18}$



$I_x = m b^2/4$

$I_y = m a^2/4$

$I_z = \frac{m}{4}(a^2 + b^2)$