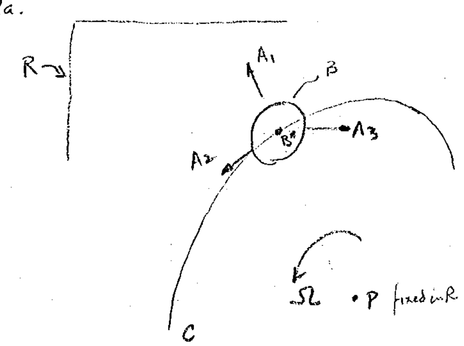
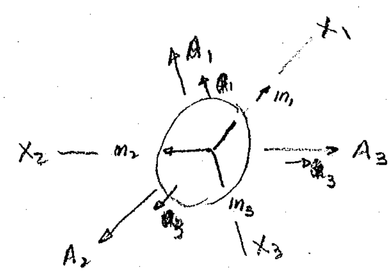
2a.

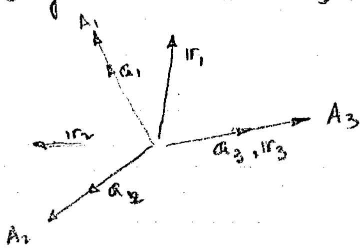




$${}^R\omega^B = \omega_1 m_1 + \omega_2 m_2 + \omega_3 m_3$$

find $\omega_i = f_i(\theta_1, \theta_2, \theta_3, \dot\theta_1, \dot\theta_2, \dot\theta_3, \Omega)$

reference frame A be the plane of the circular orbit & ~~normal outward to plane in direction of A_3~~ aligned with A_1, A_2, A_3. let r_1, r_2, r_3 be unit vectors fixed in the reference R but in the plane of the orbit and r_3 is aligned with a_3

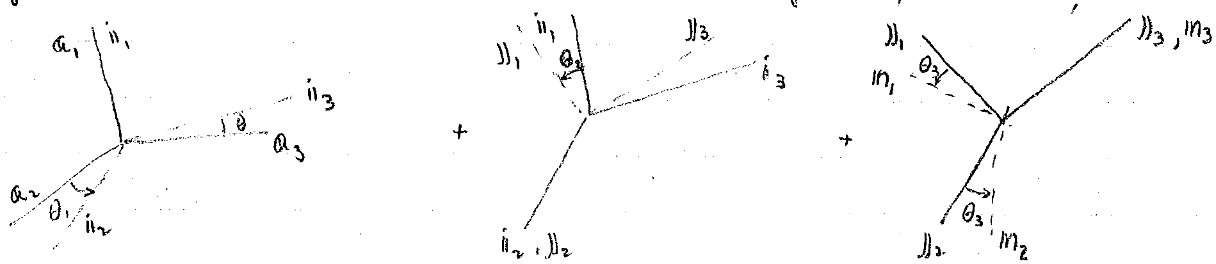


thus

	r_1	r_2	r_3
a_1	$\cos\Omega t$	$\sin\Omega t$	0
a_2	$-\sin\Omega t$	$\cos\Omega t$	0
a_3	0	0	1

Using the chain rule $\quad {}^R\omega^B = {}^R\omega^A + {}^A\omega^B = \Omega a_3 + {}^A\omega^B \qquad (1)$

Now since m_1, m_2, m_3 are the unit vectors fixed in B, and we can obtain the by starting w/ A and rotate about the 3 axes separately,



Then $\quad {}^A\omega^B = {}^A\omega^I + {}^I\omega^J + {}^J\omega^B = \dot\theta_1 i_1 + \dot\theta_2 j_2 + \dot\theta_3 m_3$

but

	a_1	a_2	a_3
i_1	1	0	0
i_2	0	c_1	s_1
i_3	0	$-s_1$	c_1

	i_1	i_2	i_3
j_1	c_2	0	$-s_2$
j_2	0	1	0
j_3	s_2	0	c_2

	j_1	j_2	j_3
m_1	c_3	s_3	0
m_2	$-s_3$	c_3	0
m_3	0	0	1

$$i_1 = c_2 j_1 + s_2 j_3 = c_2(c_3 m_1 - s_3 m_2) + s_2 m_3 = c_2 c_3 m_1 - c_2 s_3 m_2 + s_2 m_3$$

$$j_2 = s_3 m_1 + c_3 m_2$$

$$\therefore \quad {}^A\omega^B = (\dot\theta_1 c_2 c_3 + s_3 \dot\theta_2) m_1 + (-\dot\theta_1 c_2 s_3 + c_3 \dot\theta_2) m_2 + (s_2 \dot\theta_1 + \dot\theta_3) m_3 \qquad (2)$$

but $a_3 = s_1 i_2 + c_1 i_3 = s_1 j_2 + c_1(-s_2 j_1 + c_2 j_3) = s_1(s_3 m_1 + c_3 m_2) - s_2 c_1(c_3 m_1 - s_3 m_2) + c_1 c_2 m_3$

thus $\quad {}^R\omega^A \underset{(1)}{=} \Omega\left[(s_1 s_3 - s_2 c_1 c_3) m_1 + (s_1 c_3 + s_2 c_1 s_3) m_2 + c_1 c_2 m_3\right] \qquad (3)$

finally ${}^{R}\omega^{B}_{(2,3)} = \omega_1 m_1 + \omega_2 m_2 + \omega_3 m_3$

$$\Rightarrow \quad \omega_1 = \Omega(s_1 s_3 - s_2 c_1 c_3) + \dot\theta_1 c_2 c_3 + s_3 \dot\theta_2$$

$$\omega_2 = \Omega(s_1 c_3 + s_2 c_1 s_3) + (c_3 \dot\theta_2 - \dot\theta_1 c_2 s_3)$$

$$\omega_3 = \Omega(c_1 c_2) + s_2 \dot\theta_1 + \dot\theta_3$$

$\therefore$ we can write

$$\begin{bmatrix} c_2 c_3 & s_3 & 0 \\ -c_2 s_3 & c_3 & 0 \\ s_2 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot\theta_1 \\ \dot\theta_2 \\ \dot\theta_3 \end{pmatrix} = \begin{bmatrix} \omega_1 - \Omega(s_1 s_3 - s_2 c_1 c_3) \\ \omega_2 - \Omega(c_3 s_1 + s_2 c_1 s_3) \\ \omega_3 - c_1 c_2 \Omega \end{bmatrix}$$

$$\dot\theta_1 = \begin{bmatrix} \omega_1 - \Omega(s_1 s_3 - s_2 c_1 c_3) & s_3 & 0 \\ \omega_2 - \Omega(c_3 s_1 + s_2 c_1 s_3) & c_3 & 0 \\ \omega_3 - c_1 c_2 \Omega & 0 & 1 \end{bmatrix} \Big/ c_2 = \frac{\omega_1 c_3 - \Omega(c_3 s_1 s_3 - s_2 c_1 c_3^2) - \omega_2 s_3 + \Omega(c_3 s_1 s_3 + s_2 c_1 s_3^2)}{c_2}$$

$$\dot\theta_1 = (\omega_1 c_3 - \omega_2 s_3 + \Omega s_2 c_1)/c_2$$

$$\dot\theta_2 = \begin{bmatrix} c_2 c_3 & \omega_1 - \Omega(\quad) & 0 \\ -c_2 s_3 & \omega_2 - \Omega(\quad) & 0 \\ s_2 & \omega_3 - c_1 c_2 \Omega & 1 \end{bmatrix} \Big/ c_2 = \frac{\omega_2 c_2 c_3 + \omega_1 c_2 s_3 - \Omega(c_2 s_1 c_3^2 + s_2 c_1 s_3 c_2 c_3) - \Omega(c_2 s_1 s_3^2 - c_2 s_3 s_2 c_1 c_3)}{c_2}$$

$$\dot\theta_1 = \omega_2 c_3 + \omega_1 s_3 - \Omega s_1$$

The third follows by simple algebra. $\dot\theta_3 = [(\omega_2 s_3 - \omega_1 c_3)s_2 + \omega_3 c_2 - \Omega c_1]/c_2$

2b. Find $\omega_{\dot\theta_1}, \omega_{\dot\theta_2}, \omega_{\dot\theta_3}$ for ${}^{R}\omega^{B}$ we know that

${}^{R}\omega^{B} = \omega_1 m_1 + \omega_2 m_2 + \omega_3 m_3 \Leftrightarrow {}^{R}\omega^{B} = \sum \omega_{\dot q_r} \dot q_r + \omega_t$

$\omega_t = {}^{R}\omega^{A} \qquad \sum \omega_{\dot q_r} \dot q_r = {}^{A}\omega^{B}$

thus $\omega_{\dot\theta_1 (2)} = c_2 c_3 m_1 - c_2 s_3 m_2 + s_2 m_3 \quad ; \quad \omega_{\dot\theta_2} = s_3 m_1 + c_3 m_2 \quad ; \quad \omega_{\dot\theta_3} = m_3$

2c. if ${}^{B}\omega^{A}$ prove ${}^{A}\omega^{B} = -{}^{B}\omega^{A}$ look at ${}^{A}\omega^{A}$ if we fix vectors a_1, a_2, a_3 in A then ${}^{A}\omega^{A} = \sum \omega_{\dot q_r} \dot q_r + \omega_t$ and $\omega_t = e_{ijk} \frac{\partial a_i}{\partial t} \cdot a_j a_k$

since a_i are fixed $\frac{{}^{A}d a_i}{dt}$ and $\frac{{}^{A}d a_i}{d q_r} = 0$ but ${}^{A}\omega^{A} = \omega_t = 0$

now ${}^{A}\omega^{A} = {}^{A}\omega^{B} + {}^{B}\omega^{A} = 0 \quad \rightarrow \quad {}^{A}\omega^{B} = -{}^{B}\omega^{A}$

prove $\frac{{}^{A}d\,{}^{A}\omega^{B}}{dt} = \frac{{}^{B}d\,{}^{A}\omega^{B}}{dt}$ now let $v = {}^{A}\omega^{B}$ then $\frac{{}^{A}dv}{dt} = \frac{{}^{B}dv}{dt} + {}^{A}\omega^{B} \times {}^{A}\omega^{B}$

but ${}^{A}\omega^{B} \times {}^{A}\omega^{B} = |{}^{A}\omega^{B}|^2 \sin({}^{A}\omega^{B}, {}^{A}\omega^{B}) = 0 \quad \therefore \quad \frac{{}^{A}d\,{}^{A}\omega^{B}}{dt} = \frac{{}^{B}d\,{}^{A}\omega^{B}}{dt}$

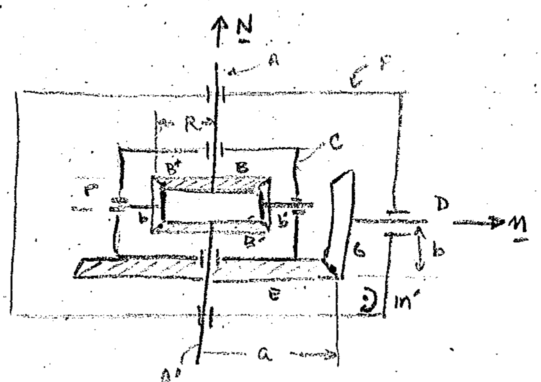
part 1 of 2e: can also be proved by

$\forall\ \mathbb{V} \quad \frac{{}^{A}d\mathbb{V}}{dt} = \frac{{}^{B}d\mathbb{V}}{dt} + {}^{A}\omega^{B} \times \mathbb{V} \qquad$ and $\quad \frac{{}^{B}d\mathbb{V}}{dt} = \frac{{}^{A}d\mathbb{V}}{dt} + {}^{B}\omega^{A} \times \mathbb{V}$

$\Rightarrow \frac{{}^{B}d\mathbb{V}}{dt} = \frac{{}^{B}d\mathbb{V}}{dt} + {}^{A}\omega^{B} \times \mathbb{V} + {}^{B}\omega^{A} \times \mathbb{V} \Rightarrow ({}^{A}\omega^{B} + {}^{B}\omega^{A}) \times \mathbb{V} = 0$. But since this is true $\forall\ \mathbb{V}$

then $\Rightarrow {}^{A}\omega^{B} = -{}^{B}\omega^{A}$

2(d)



given ${}^{F}\omega^{A} = \Omega \mathbb{N}$

${}^{F}\omega^{A'} = \Omega' \mathbb{N}$

${}^{F}\omega^{D} = \omega \mathbb{n}$

find $\omega = \omega(\Omega, \Omega')$

${}^{F}\mathbb{V}^{G^*} = {}^{F}\omega^{G^*} \times \mathbb{r}^{Q/G^*} = \omega b\, \mathbb{n}' = {}^{F}\mathbb{V}^{E^*} = {}^{F}\omega^{E^*} \times \mathbb{r}^{Q/E^*} = {}^{F}\omega^{E} \cdot a\, \mathbb{N} = {}^{F}\omega^{C} a\, \mathbb{n}' \qquad {}^{F}\omega^{C} = \omega b / a$

Now $\ {}^{F}\omega^{B'} = {}^{F}\omega^{C} + {}^{C}\omega^{B'} = {}^{F}\omega^{C} \mathbb{N} + {}^{C}\omega^{B'} = \Omega \mathbb{N} \qquad \therefore\ {}^{C}\omega^{B'} = \frac{\omega b}{a} \mathbb{N} - \Omega \mathbb{N} \qquad (*)$

${}^{F}\omega^{B} = {}^{F}\omega^{C} + {}^{C}\omega^{B} = \omega \frac{b}{a} \mathbb{N} + {}^{C}\omega^{B} = \Omega' \mathbb{N} \qquad \therefore\ {}^{C}\omega^{B} = \left(\frac{\omega b}{a} - \Omega'\right) \mathbb{N} \qquad (**)$

Now to relate ${}^{C}\omega^{B'}$ to ${}^{C}\omega^{B}$: $\quad {}^{C}\mathbb{V}^{B^*} = {}^{C}\omega^{B^*} \times \mathbb{r}^{Q/B^*} = {}^{C}\omega^{B} \mathbb{N} \times (-R\mathbb{n}) = -{}^{C}\omega^{B} R\, \mathbb{n}'$

but $\ {}^{C}\mathbb{V}^{B^*} = {}^{C}\mathbb{V}^{b^*} = {}^{C}\omega^{b^*} \times \mathbb{r}^{Q/b^*} = {}^{C}\omega^{b} \mathbb{n} \times r\mathbb{N} = -{}^{C}\omega^{B} \mathbb{n}'$

also $\ {}^{C}\mathbb{V}^{b^{**}} = {}^{C}\omega^{b^{**}} \times \mathbb{r}^{Q/b^{**}} = -{}^{C}\mathbb{V}^{b^*} = {}^{C}\omega^{b} \mathbb{n} \times -r\mathbb{N} = {}^{C}\omega^{B} \mathbb{n}' = {}^{C}\mathbb{V}^{B'^*} = {}^{C}\omega^{B'^*} \times \mathbb{r}^{Q/B'^*} = {}^{C}\omega^{B'} \mathbb{N} \times -R\mathbb{n}$

$= {}^{C}\omega^{B'} R\mathbb{n}' = {}^{C}\mathbb{V}^{b^{**}} = -{}^{C}\mathbb{V}^{b^*}$

Now $\ {}^{C}\mathbb{V}^{b^*} = -{}^{C}\mathbb{V}^{b^{**}} \Rightarrow {}^{C}\omega^{B} = -{}^{C}\omega^{B'}$

Thus from $(*)$ and $(**)$ $\qquad \left(\frac{\omega b}{a} - \Omega\right) = -\left(\frac{\omega b}{a} - \Omega'\right) \Rightarrow \omega = \frac{a}{2b}(\Omega' + \Omega$

2e. $\ {}^{R}\omega^{B} = \omega_i \mathbb{n}_i \qquad$ where $\mathbb{n}_i$ are RHS on unit vectors fixed in B

$\mathbb{A} = I_1 \omega_1 \mathbb{n}_1 + I_2 \omega_2 \mathbb{n}_2 + I_3 \omega_3 \mathbb{n}_3$

when can $\ \frac{{}^{R}d\mathbb{A}}{dt} = M_1 \mathbb{n}_1 + M_2 \mathbb{n}_2 + M_3 \mathbb{n}_3 \qquad$ where $M_i = M_i(\omega_i \,\&\, I_i)$

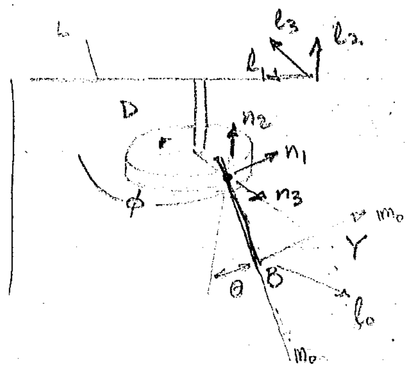
$$\frac{{}^{R}d\mathbb{A}}{dt} = \frac{{}^{B}d\mathbb{A}}{dt} + {}^{R}\omega^{B} \times \mathbb{A}$$

$M_1 \mathbb{n}_1 + M_2 \mathbb{n}_2 + M_3 \mathbb{n}_3 = I_1 \dot{\omega}_1 \mathbb{n}_1 + I_2 \dot{\omega}_2 \mathbb{n}_2 + I_3 \dot{\omega}_3 \mathbb{n}_3 + \omega_i \mathbb{n}_i \times I_j \omega_j \mathbb{n}_j$

$= \omega_i \omega_j I_j\, e_{ijk} \mathbb{n}_k + I_k \dot{\omega}_k \mathbb{n}_k$

$\therefore\ M_1 = I_1 \dot{\omega}_1 + \omega_2 I_3 \omega_3 - \omega_3 I_2 \omega_2 = I_1 \dot{\omega}_1 + \omega_2 \omega_3 (I_3 - I_2)$

2f.



	l_1	l_2	l_3
n_1	s_1	0	c_1
n_2	0	1	0
n_3	c_1	0	$-s_1$

	n_1	n_2	n_3
m_0	s_2	$-c_2$	0
m_0	c_2	s_2	0
l_0	0	0	1

find ${}^L\alpha^B$

first ${}^L\omega^B = {}^L\omega^D + {}^D\omega^B$

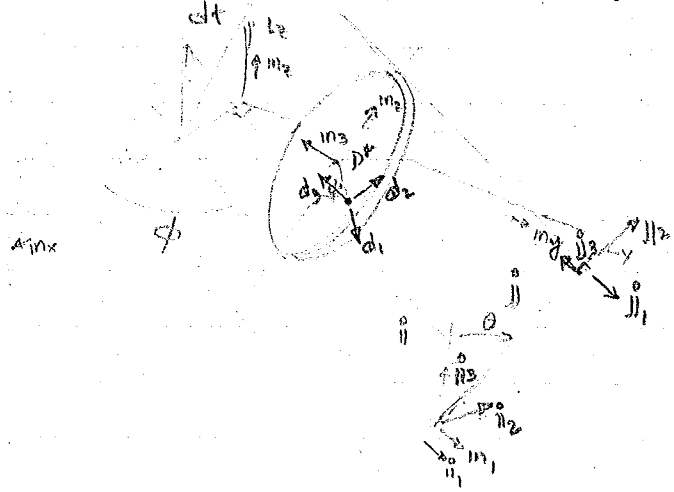
$= \dot\phi n_2 + \dot\theta n_3$

$$\frac{{}^L d\,{}^L\omega^B}{dt} = \frac{{}^D d\,{}^L\omega^B}{dt} + {}^L\omega^D \times {}^L\omega^B$$

$$= \ddot\phi n_2 + \ddot\theta n_3 + \underbrace{\left[\dot\phi n_2 \times (\dot\phi n_2 + \dot\theta n_3)\right]}_{\dot\phi\dot\theta n_1}$$

$$\therefore\ {}^L\alpha^B = \frac{{}^L d\,{}^L\omega^B}{dt} = \dot\theta\dot\phi n_1 + \ddot\phi n_2 + \ddot\theta n_3$$

2g.



	i_1	i_2	i_3
m_x	$c\phi$	$-s\phi$	0
m_y	$s\phi$	$+c\phi$	0
m_z	0	0	1

	i_1	i_2	i_3
$m_1 = j_1$	1	0	0
$m_2 = j_2$	0	$+s\theta$	$c\theta$
$m_3 = j_3$	0	$-c\theta$	$s\theta$

	j_1	j_2	j_3
d_1	$s\psi$	$-c\psi$	0
d_2	$c\psi$	$s\psi$	0
d_3	0	0	1

find ${}^R\alpha^D$ ${}^R\omega^D = {}^R\omega^I + {}^I\omega^J + {}^J\omega^D = \dot\phi i_3 - \dot\theta m_1 + \dot\psi m_3$

$= \dot\phi(\cos\theta\, m_2 + \sin\theta\, m_3) - \dot\theta m_1 + \dot\psi m_3$

$$\frac{{}^R d\,{}^R\omega^D}{dt} = -\ddot\theta m_1 - \dot\theta\frac{{}^R dm_1}{dt} + \dot f_2(\dot\phi\cos\theta) m_2 + f_2(\dot\phi\cos\theta)\frac{{}^R dm_2}{dt} + \dot f_3(\dot\psi, \sin\theta) m_3 + f_3(\dot\psi, \sin\theta)\frac{{}^R dm_3}{dt}$$

$$\frac{{}^R dm_1}{dt} = \underbrace{\frac{{}^I dm_1}{dt}}_{=0\ m_1 \text{ fixed in } I} + {}^R\omega^I \times m_1 = \dot\phi i_3 \times i_1 = \dot\phi i_2 = \dot\phi(\sin\theta\, m_2 - \cos\theta\, m_3)$$

$$\frac{{}^R dm_2}{dt} = \underbrace{\frac{{}^J dm_2}{dt}}_{=0 \text{ since } m_2 \text{ is fixed in } J} + {}^R\omega^J \times m_2 = (\dot\phi i_3 - \dot\theta m_1)\times m_2 = \left[\dot\phi(\cos\theta\, m_2 + \sin\theta\, m_3) - \dot\theta m_1\right]\times m_2$$

$$= -\dot\phi\sin\theta\, m_1 - \dot\theta m_3$$

$$\frac{{}^R dm_3}{dt} = \underbrace{\frac{{}^J dm_3}{dt}}_{=0} + \left[\dot\phi(\cos\theta\, m_2 + \sin\theta\, m_3) - \dot\theta m_1\right]\times m_3 = \dot\phi\cos\theta\, m_1 + \dot\theta m_2$$

$$\therefore\ -\ddot\theta m_1 - \dot\theta\left[\dot\phi\sin\theta\, m_2 - \cos\theta\,\dot\phi\, m_3\right] + (\ddot\phi\cos\theta - \dot\phi\dot\theta\sin\theta) m_2 + \dot\phi\cos\theta(-\dot\phi\sin\theta\, m_1 - \dot\theta m_3)$$
$$+ (\ddot\psi + \ddot\phi\sin\theta + \dot\phi\dot\theta\cos\theta) m_3 + (\dot\psi + \dot\phi\sin\theta)(\dot\phi\cos\theta\, m_1 + \dot\theta m_2) = {}^R\alpha^D$$

$$\alpha_1 = -\ddot\theta - \dot\phi^2\sin\phi\cos\phi + (\dot\phi\cos\phi)(\dot\psi + \dot\phi\sin\phi) = -\ddot\theta + \dot\psi\dot\phi\cos\theta$$

$$\alpha_2 = -\dot\theta\dot\phi\sin\theta + \ddot\phi\cos\theta - \dot\phi\dot\theta\sin\theta + \dot\theta(\dot\psi + \dot\phi\sin\theta) = \dot\theta\dot\psi + \ddot\phi\cos\theta - \dot\phi\dot\theta\sin\theta$$

$$\alpha_3 = \dot\psi\dot\theta\cos\theta - \dot\theta\dot\phi\cos\theta + \ddot\psi + \ddot\phi\sin\theta + \dot\phi\dot\theta\cos\theta$$

$$^R\omega^D = \dot\phi\, n_z - \dot\theta\left(\cos\phi\, n_x + \sin\phi\, n_y\right) + \dot\psi\left(-\cos\theta\left[-\sin\phi\, n_x + \cos\phi\, n_y\right] + \sin\theta\, n_z\right)$$

$$\frac{^R d\,{^R\omega^D}}{dt} = \left(-\ddot\theta\cos\phi + \dot\theta\dot\phi\sin\phi + \ddot\psi\cos\theta\sin\phi - \dot\psi\dot\theta\sin\theta\sin\phi + \dot\psi\dot\phi\cos\theta\cos\phi\right) n_x +$$
$$\left(-\ddot\theta\sin\phi - \dot\theta\dot\phi\cos\phi - \ddot\psi\cos\theta\cos\phi + \dot\psi\dot\theta\sin\theta\cos\phi + \dot\psi\dot\phi\cos\theta\sin\phi\right) n_y +$$
$$\left(\ddot\phi + \ddot\psi\sin\theta + \dot\psi\dot\theta\cos\theta\right) n_z = \alpha_x n_x + \alpha_y n_y + \alpha_z n_z$$

2h. $\quad {^R\omega^B} = \omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 \qquad n_1, n_2, n_3$ are fixed in N

$\quad {^R\alpha^B} = \alpha_1 n_1 + \alpha_2 n_2 + \alpha_3 n_3$

$\quad {^R\omega^N} = \Omega$

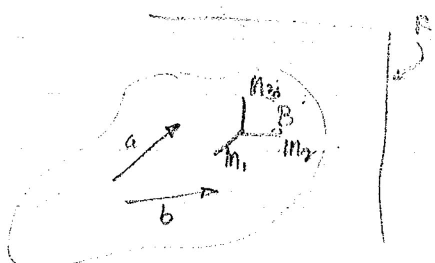
now $\quad \dfrac{^N d v}{dt} = \dfrac{^R d v}{dt} + {^N\omega^R} \times v \qquad$ if $\quad \underbrace{\dfrac{^N d\,{^R\omega^B}}{dt}}_{\frac{d\omega_i}{dt} n_i} = \dfrac{^R d\,{^R\omega^B}}{dt} + \underbrace{{^N\omega^R} \times {^R\omega^B}}_{-\Omega \times {^R\omega^B}}$

$$\frac{d\omega_i}{dt} n_i = \alpha_i n_i \quad \text{iff} \quad \Omega \times {^R\omega^B} = 0$$

now if $\quad N = R \implies {^R\omega^R} = 0$ as shown previously

$\quad N = B \implies {^B\omega^R} = -{^R\omega^B} \quad \& \quad {^B\omega^R} \times {^R\omega^B} = -\left({^R\omega^B} \times {^R\omega^B}\right) = 0$

2i.



let n_1, n_2, n_3 be mutually ⊥ vectors of unit length in B. now $a = a_i n_i$ and $b = b_i n_i$

Find ${^R\omega^B}$

$$\frac{^R d a}{dt} = \frac{^B d a}{dt} + {^R\omega^B} \times {^B a} \quad \therefore \quad \frac{^R d a}{dt} = {^R\omega^B} \times {^B a} \quad \text{since } a \text{ is fixed in } B$$

$$\frac{^R d b}{dt} = \frac{^B d b}{dt} + {^R\omega^B} \times {^B b} \quad \therefore \quad \frac{^R d b}{dt} = {^R\omega^B} \times b$$

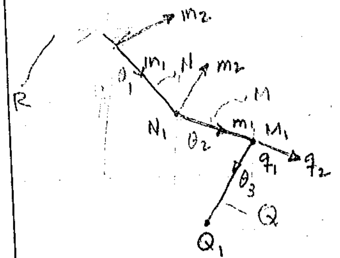
$$\frac{\dfrac{^R d a}{dt} \times \dfrac{^R d b}{dt}}{\dfrac{^R d a}{dt} \cdot b} = \frac{\left({^R\omega^B} \times a\right) \times \left({^R\omega^B} \times b\right)}{\left({^R\omega^B} \times a\right) \cdot b} = \frac{\left[\left({^R\omega^B} \times a\right) \cdot b\right] {^R\omega^B} - \left[\left({^R\omega^B} \times a\right) \cdot {^R\omega^B}\right] b}{\left({^R\omega^B} \times a\right) \cdot b}$$

$$= {^R\omega^B} - 0$$

since ${^R\omega^B} \times a$ is a vector ⊥ to both ${^R\omega^B}$ & a

Dot product of 2 vectors ⊥ to each other = 0.

3a.



$\mathbf{r}^{P_1} = \frac{L}{2}\mathbf{n}_1$ $\mathbf{r}^{P_2} = L\mathbf{n}_1 + \frac{L}{2}\mathbf{m}_1$ $\mathbf{r}^{P_3} = L\mathbf{n}_1 + L\mathbf{m}_1 + L/2\,\mathbf{q}_1$

$\frac{{}^R d\mathbf{r}^{P_1}}{dt} = \frac{L}{2}\frac{{}^R d\mathbf{n}_1}{dt}$ $\frac{{}^R d\mathbf{n}_1}{dt} = {}^R\boldsymbol{\omega}^N \times \mathbf{n}_1 = \dot{\theta}_1\mathbf{n}_3\times\mathbf{n}_1$; $\therefore \left.\frac{{}^R d\mathbf{r}^{P_1}}{dt}\right|_{t=t^*} = \mathbf{v}^{P_1} = \frac{L\dot{\theta}_1}{2}\mathbf{n}_2$

$\therefore \boxed{\mathbf{v}^{P_1}_{\dot{q}_1} = \frac{L}{2}\mathbf{n}_2 \quad \text{and} \quad \mathbf{v}^{P_1}_{\dot{q}_2} = \mathbf{v}^{P_1}_{\dot{q}_3} = 0}$ when $\theta_1 = \theta_2 = \theta_3 = \pi/4$

$\left.\frac{{}^R d\mathbf{r}^{P_2}}{dt}\right|_{t=t^*} = \left.\frac{{}^R d\mathbf{r}^{N_1}}{dt}\right|_{t=t^*} + {}^R\boldsymbol{\omega}^M \times {}^R\mathbf{r}^{P_2/M}$; ${}^R\boldsymbol{\omega}^M = \dot{\theta}_2\mathbf{m}_3 = \dot{\theta}_2\mathbf{n}_3$; ${}^R\mathbf{r}^{P_2/N_1} = {}^R\mathbf{r}^{P_2} - {}^R\mathbf{r}^{N_1} = \frac{3L}{2}\mathbf{n}_1 - \frac{L}{2}\mathbf{n}_1 = \frac{L}{2}\mathbf{n}_1$ when $\theta_1 = \theta_2 = \dots$

$\frac{{}^R d\mathbf{r}^{N_1}}{dt} = L\dot{\theta}_1\mathbf{n}_2$

$\frac{{}^R d\mathbf{r}^{P_2}}{dt} = L\dot{\theta}_1\mathbf{n}_2 + \left(\dot{\theta}_2\mathbf{n}_3 \times \frac{L}{2}\mathbf{n}_1\right) = L\dot{\theta}_1\mathbf{n}_2 + \frac{L\dot{\theta}_2}{2}\mathbf{n}_2$ $\therefore \boxed{\mathbf{v}^{P_2}_{\dot{q}_1} = L\mathbf{n}_2 ;\ \mathbf{v}^{P_2}_{\dot{q}_2} = \frac{L\mathbf{n}_2}{2} ;\ \mathbf{v}^{P_2}_{\dot{q}_3} = 0}$

$\left.\frac{{}^R d\mathbf{r}^{P_3}}{dt}\right|_{t^*} = \left.\frac{{}^R d\mathbf{r}^{M_1}}{dt}\right|_{t^*} + {}^R\boldsymbol{\omega}^Q \times {}^R\mathbf{r}^{M_1/Q} = L\dot{\theta}_1\mathbf{n}_2 + L\dot{\theta}_2\mathbf{n}_2 + \left(-\dot{\theta}_3\mathbf{q}_3\right)\times -\frac{L}{2}\mathbf{n}_2 = L\dot{\theta}_1\mathbf{n}_2 + L\dot{\theta}_2\mathbf{n}_2 - \frac{L\dot{\theta}_3}{2}\mathbf{n}_1$

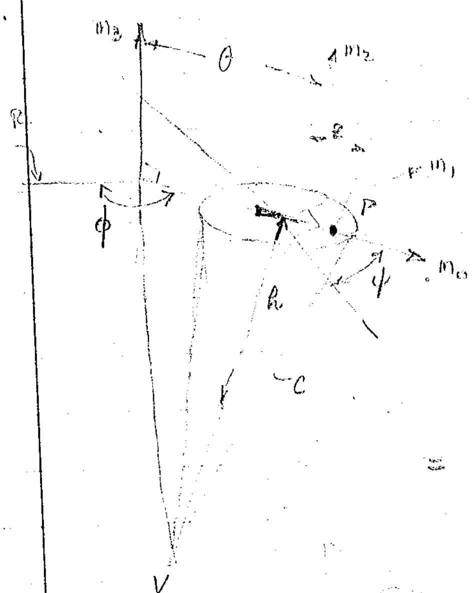
since ${}^R\mathbf{r}^{M_1/Q} = {}^R\mathbf{r}^{Q} - {}^R\mathbf{r}^{M_1} = (2L\mathbf{n}_1 - 1/2\,\mathbf{n}_2) - 2L\mathbf{n}_1 = -L/2\,\mathbf{n}_2$

and ${}^R\boldsymbol{\omega}^Q = -\dot{\theta}_3\mathbf{q}_3 = -\dot{\theta}_3\mathbf{n}_3$

$\therefore \boxed{\mathbf{v}^{P_3}_{\dot{q}_1} = L\mathbf{n}_2 \quad \mathbf{v}^{P_3}_{\dot{q}_2} = L\mathbf{n}_2 \quad \mathbf{v}^{P_3}_{\dot{q}_3} = -\frac{L}{2}\mathbf{n}_1}$ when $\theta_1 = \theta_2 = \theta_3 = \pi/4$

3b.

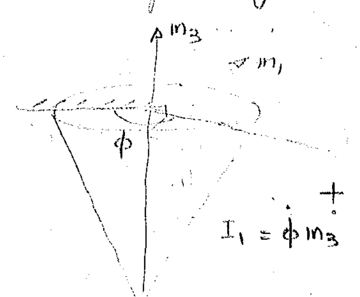
let N be the plane of $\mathbf{n}_0, \mathbf{n}_2$



we want to find $\frac{{}^R d\mathbf{r}^P}{dt}$

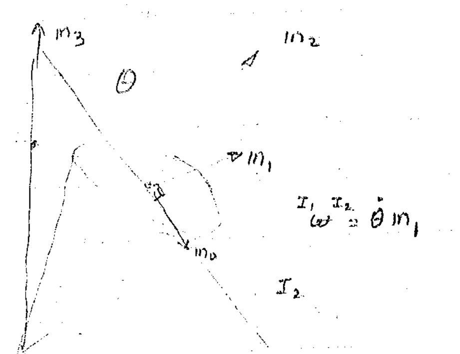
decompose as follows

=



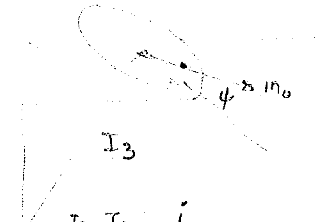
$I_1 = \dot{\phi}\mathbf{n}_3$

+



${}^{I_1}\boldsymbol{\omega}^{I_2} = \dot{\theta}\mathbf{n}_1$

+



I_3

${}^{I_2}\boldsymbol{\omega}^{I_3} = \dot{\psi}\mathbf{n}_2$

let V be fixed pt in R $\therefore {}^R\mathbf{r}^P = h\mathbf{n}_2 + z\mathbf{n}_0$

$\frac{{}^R d\mathbf{r}^P}{dt} = h\dot{\mathbf{n}}_2 + \dot{z}\mathbf{n}_0 + z\dot{\mathbf{n}}_0$

$\dot{\mathbf{n}}_2 = \cancel{\frac{{}^{I_3} d\mathbf{n}_2}{dt}}^{\,0} + {}^R\boldsymbol{\omega}^{I_3}\times\mathbf{n}_2$ since $\mathbf{n}_2, \mathbf{n}_0$ are fixed in I_3

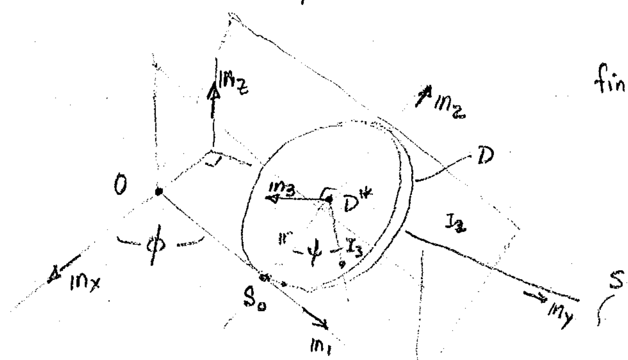
$\dot{\mathbf{n}}_0 = \cancel{\frac{{}^{I_3} d\mathbf{n}_0}{dt}}^{\,0} + {}^R\boldsymbol{\omega}^{I_3}\times\mathbf{n}_0$

$\therefore \frac{{}^R d\mathbf{r}^P}{dt} = {}^R\boldsymbol{\omega}^{I_3}\times\mathbf{r}^P + \dot{z}\mathbf{n}_0$

${}^R\boldsymbol{\omega}^{I_3} = {}^R\boldsymbol{\omega}^{I_1} + {}^{I_1}\boldsymbol{\omega}^{I_2} + {}^{I_2}\boldsymbol{\omega}^{I_3} = \dot{\phi}\mathbf{n}_3 + \dot{\theta}\mathbf{n}_1 + \dot{\psi}\mathbf{n}_2$

thus $\displaystyle \frac{{}^R d\mathbb{r}^P}{dt} = \underbrace{(\mathbb{n}_3 \times \mathbb{r}^P)}_{\mathbb{v}^P_{\dot\phi}} \dot\phi + \underbrace{(\mathbb{n}_1 \times \mathbb{r}^P)}_{\mathbb{v}^P_{\dot\theta}} \dot\theta + \underbrace{(\mathbb{n}_2 \times \mathbb{r}^P)}_{\mathbb{v}^P_{\dot\psi}} \dot\psi + \underbrace{\dot z\, \mathbb{n}_0}_{\mathbb{v}^P_{\dot z}}$

3c



find ${}^S\mathbb{a}^{S_0}$

find ${}^S\mathbb{r}^{D^*} = {}^S\mathbb{r}^{S_0} + {}^S\mathbb{r}^{D^*/S_0}$

$\displaystyle \frac{{}^S d\mathbb{r}^{D^*}}{dt} = \frac{{}^S d\mathbb{r}^{S_0}}{dt} + \frac{{}^S d}{dt}\mathbb{r}^{D^*/S_0}$

$\displaystyle {}^S\mathbb{v}^{D^*} = \underbrace{{}^S\mathbb{v}^{S_0}}_{=0 \text{ rolling condition}} + \frac{{}^S d\mathbb{r}^{D^*/S_0}}{dt}$

but $\displaystyle \frac{{}^S d\mathbb{r}^{D^*/S_0}}{dt} = \frac{{}^{I_3} d\mathbb{r}^{D^*/S_0}}{dt} + {}^S\omega^{I_3} \times {}^{I_3}\mathbb{r}^{D^*/S_0}$

$\displaystyle = \underbrace{\frac{{}^{I_3} d}{dt} r\mathbb{n}_2}_{=0} + {}^S\omega^{I_3} \times r\mathbb{n}_2$

since $\mathbb{r}^{D^*/S_0}$ is fixed in D but not in I_2 (it translates in I_2)

$\displaystyle \therefore \frac{{}^S d\mathbb{r}^{D^*/S_0}}{dt} = {}^S\omega^{I_3} \times r\mathbb{n}_2 \qquad I_3 = D$

${}^S\omega^{I_3} = {}^S\omega^{I_1} + {}^{I_1}\omega^{I_2} + {}^{I_2}\omega^{D} = \dot\phi\mathbb{n}_z - \dot\theta\mathbb{n}_1 + \dot\psi\mathbb{n}_3 \quad \therefore \displaystyle \frac{{}^S d\mathbb{r}^{D^*/S_0}}{dt} = {}^S\mathbb{v}^{D^*} = (\dot\phi\mathbb{n}_z - \dot\theta\mathbb{n}_1 + \dot\psi\mathbb{n}_3) \times r\mathbb{n}_2$

$\therefore {}^S\mathbb{v}^{D^*} = r\,\omega^{D} \times \mathbb{n}_2$

$\displaystyle \therefore \frac{{}^S d\mathbb{v}^{D^*}}{dt} = \mathbb{a}^{D^*} = r\frac{{}^S d\,{}^S\omega^{D}}{dt} \times \mathbb{n}_2 + r\,{}^S\omega^{D} \times \frac{{}^S d\mathbb{n}_2}{dt}$; $\displaystyle \frac{{}^S d\mathbb{n}_2}{dt} = \underbrace{\frac{{}^{I_2} d\mathbb{n}_2}{dt}}_{=0} + {}^S\omega^{I_2} \times \mathbb{n}_2$ since $\mathbb{n}_2$ is fixed in I_2

$= r\,{}^S\alpha^{D} \times \mathbb{n}_2 + r\,{}^S\omega^{D} \times ({}^S\omega^{I_2} \times \mathbb{n}_2)$

${}^S\omega^{I_2} = {}^S\omega^{D} - \dot\psi\mathbb{n}_3$

Now ${}^S\mathbb{a}^{D} = {}^S\mathbb{a}^{D^*} + {}^S\alpha^{D} \times \mathbb{r}^{S_0/D^*} + {}^S\omega^{D} \times ({}^S\omega^{D} \times \mathbb{r}^{S_0/D^*})$

$= r\,\alpha^{D} \times \mathbb{n}_2 + r\,{}^S\omega^{D} \times ({}^S\omega^{I_2} \times \mathbb{n}_2) + {}^S\alpha^{D} \times (-r\mathbb{n}_2) + {}^S\omega^{D} \times ({}^S\omega^{D} \times (-r\mathbb{n}_2))$

$= r\,{}^S\omega^{D} \times ({}^S\omega^{D} \times \mathbb{n}_2) - r\,{}^S\omega^{D} \times \dot\psi\mathbb{n}_3 \times \mathbb{n}_2 + {}^S\omega^{D} \times {}^S\omega^{D} \times (-r\mathbb{n}_2) = r\,{}^S\omega^{D} \times \dot\psi\mathbb{n}_1$

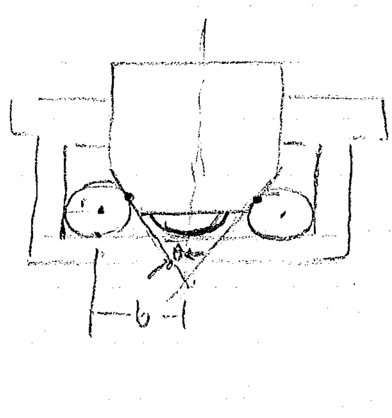
${}^S\mathbb{a}^{D} = r\dot\psi\,(\dot\phi\cos\theta\,\mathbb{n}_2 + [\dot\phi\sin\theta + \dot\psi]\mathbb{n}_3 - \dot\theta\mathbb{n}_1) \times \mathbb{n}_1$

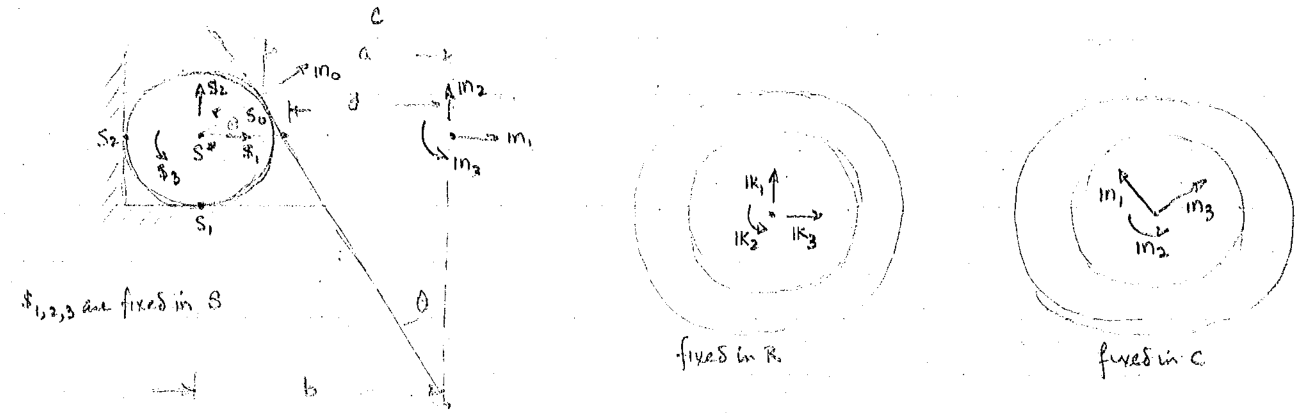
$= r\dot\psi\,(-\dot\phi\cos\theta\,\mathbb{n}_3 + (\dot\phi\sin\theta + \dot\psi)\mathbb{n}_2)$

$|{}^S\mathbb{a}^{D}| = |r\dot\psi|\,(\dot\phi^2\cos^2\theta + \dot\psi^2 + 2\dot\psi\dot\phi\sin\theta + \dot\phi\sin^2\theta)^{1/2}$

$= |r\dot\psi|\,(\dot\phi^2 + \dot\psi^2 + 2\dot\psi\dot\phi\sin\theta)^{1/2}$

3d.





${}^R\mathbf{r}^{s_2} = {}^R\mathbf{r}^{s^*} + {}^R\mathbf{r}^{s_2/s^*} \Rightarrow {}^R\mathbf{V}^{s_2} = {}^R\mathbf{V}^{s^*} + \frac{{}^R d\mathbf{r}^{s_2/s^*}}{dt}$ but now $\mathbf{r}^{s_2/s^*}$ is fixed in S

thus $\frac{{}^R d\mathbf{r}^{s_2/s^*}}{dt} = \underbrace{\frac{{}^S d\mathbf{r}^{s_2/s^*}}{dt}}_{\to 0} + {}^R\boldsymbol{\omega}^S \times \mathbf{r}^{s_2/s^*}$, $\quad \mathbf{r}^{s_2/s^*} = -r\,\mathbf{n}_1 \quad \therefore \frac{{}^R d\mathbf{r}^{s_2/s^*}}{dt} = -r\,{}^R\boldsymbol{\omega}^S \times \mathbf{n}_1$

but since s_2 is fixed in R it has no velocity (R is the fixed Ref frame)

$\therefore \quad \boxed{{}^R\mathbf{V}^{s^*} - r\,{}^R\boldsymbol{\omega}^S \times \mathbf{n}_1 = 0} \quad (1)$

Similarly ${}^R\mathbf{r}^{s_1} = {}^R\mathbf{r}^{s^*} + {}^R\mathbf{r}^{s_1/s^*} \Rightarrow {}^R\mathbf{V}^{s_1} = 0 \Rightarrow \boxed{{}^R\mathbf{V}^{s^*} + {}^R\boldsymbol{\omega}^S \times (-r\,\mathbf{n}_2) = 0} \quad (2)$

Now ${}^R\mathbf{r}^{s_0} = {}^R\mathbf{r}^{s^*} + {}^R\mathbf{r}^{s_0/s^*} \Rightarrow {}^R\mathbf{V}^{s_0} = {}^R\mathbf{V}^{s^*} + \frac{{}^R d\mathbf{r}^{s_0/s^*}}{dt} = {}^R\mathbf{V}^{s^*} + \underbrace{\frac{{}^S d\mathbf{r}^{s_0/s^*}}{dt}}_{\to 0} + {}^R\boldsymbol{\omega}^S \times (+r\,\mathbf{n}_0)$

thus $\boxed{{}^R\mathbf{V}^{s_0} = {}^R\mathbf{V}^{s^*} + {}^R\boldsymbol{\omega}^S \times r\,\mathbf{n}_0} \quad (3)$ since $\mathbf{r}^{s_0/s^*}$ is fixed in S

but ${}^R\mathbf{r}^{s_0} = {}^C\mathbf{r}^{s_0} \Rightarrow {}^R\mathbf{V}^{s_0} = {}^C\mathbf{V}^{s_0} + {}^R\boldsymbol{\omega}^C \times (-a\,\mathbf{n}_1)$ but ${}^C\mathbf{r}^{s_0}$ is fixed in C $\therefore {}^C\mathbf{V}^{s_0} = 0$

$\therefore \boxed{{}^R\mathbf{V}^{s_0} = {}^R\boldsymbol{\omega}^C \times (-a\,\mathbf{n}_1)} = {}^R\omega^C a\,\mathbf{n}_3 \quad (4)$

not correct here →

Now $\mathbf{r}^{s^*}$ is a fixed vector in C and $\frac{{}^R d\mathbf{r}^{s^*}}{dt} = {}^R\mathbf{V}^{s^*} = \underbrace{\frac{{}^C d\mathbf{r}^{s^*}}{dt}}_{\to 0} + {}^R\boldsymbol{\omega}^C \times -b\,\mathbf{n}_1$ where $b = d + \frac{r}{\cos\theta}$

but $\boxed{{}^R\boldsymbol{\omega}^C = {}^R\omega^C\,\mathbf{n}_2}\ (5) \quad \therefore \quad \boxed{{}^R\mathbf{V}^{s^*} = {}^R\omega^C b\,\mathbf{n}_3} \quad (6)$

Now ${}^R\boldsymbol{\omega}^S = {}^R\boldsymbol{\omega}^C + {}^C\boldsymbol{\omega}^S$ and for pure rolling ${}^C\boldsymbol{\omega}^S \cdot \mathbf{n}_0 = 0$ but ${}^C\boldsymbol{\omega}^S \cdot [\mathbf{n}_1\cos\theta + \mathbf{n}_2\sin\theta] = 0$

$\Rightarrow \boxed{{}^C\boldsymbol{\omega}^S = \tilde{C}(-\sin\theta\,\mathbf{n}_1 + \cos\theta\,\mathbf{n}_2)}\ (7)$ where $\tilde{C}$ is a pure constant

$\boxed{{}^R\boldsymbol{\omega}^S = -\tilde{C}\sin\theta\,\mathbf{n}_1 + (\tilde{C}\cos\theta + {}^R\omega^C)\,\mathbf{n}_2}\ (8)$

from (1),(6)&(8) ${}^R\mathbf{V}^{s^*} = {}^R\omega^C b\,\mathbf{n}_3 = r\left\{-\tilde{C}\sin\theta\,\mathbf{n}_1 + (\tilde{C}\cos\theta + {}^R\omega^C)\,\mathbf{n}_2\right\} \times \mathbf{n}_1 = r(\tilde{C}\cos\theta + {}^R\omega^C)(-\mathbf{n}_3)$

$= -(r\tilde{C}\cos\theta + r\,{}^R\omega^C)\,\mathbf{n}_3$ Solving for b

$\therefore \quad \boxed{b = -\left(r\cos\theta\ \tilde{C}/{}^R\omega^C + r\right)} \quad (9)$

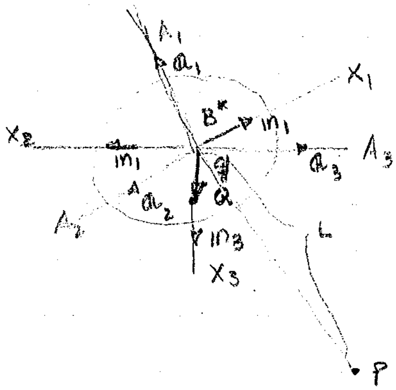
also (6)&(2) ${}^R\omega^C b\,\mathbf{n}_3 = r\left[-\tilde{C}\sin\theta\,\mathbf{n}_1 + (\tilde{C}\cos\theta + {}^R\omega^C)\,\mathbf{n}_2\right] \times \mathbf{n}_2$

$= -r\tilde{C}\sin\theta\,\mathbf{n}_3 \quad \therefore \quad \boxed{\tilde{C}/{}^R\omega^C = \frac{-b}{r\sin\theta}} \quad (10)$

$$\therefore \quad b = -r\left\{\cos\theta \cdot \frac{-b}{r\sin\theta} + 1\right\} = -\left(-\frac{b\cos\theta}{\sin\theta} + r\right) = \frac{b\cos\theta}{\sin\theta} - r$$

$$b\left[1 - \frac{\cos\theta}{\sin\theta}\right] = -r \quad \Rightarrow \quad b = \frac{r\sin\theta}{\cos\theta - \sin\theta}$$

3(e)



$q = r^{Q/B^*} = c(\Omega^2 t^2 - 1)\, n_3$

find ${}^R a^Q$ for the case where $t = 1/\Omega$ and $\theta_1, \theta_2, \theta_3 = 0$

$${}^R r^Q = {}^R r^{B^*} + {}^R r^{Q/B^*}$$

$$\frac{{}^R d r^Q}{dt} = \frac{{}^R d r^{B^*}}{dt} + \frac{{}^R d r^{Q/B^*}}{dt}$$

$$\frac{{}^R d r^{B^*}}{dt} = \underbrace{\cancel{\frac{{}^A d r^{B^*}}{dt}}}_{\to 0} + {}^R\omega^A \times r^{B^*}$$ now r^{B^*} is fixed in A and ${}^R\omega^A = \Omega\, a_3$ & $r^{B^*} = L a$

$\therefore \dfrac{{}^R d r^{B^*}}{dt} = \Omega L\, a_2$ (1) and $\dfrac{{}^R d r^{Q/B^*}}{dt} = \dfrac{{}^B d r^{Q/B^*}}{dt} + {}^R\omega^B \times {}^B r^{Q/B^*}$

but $\dfrac{{}^B d r^{Q/B^*}}{dt} = \dfrac{{}^B d}{dt}\left(c[\Omega^2 t^2 - 1]\, n_3\right) = 2c\Omega^2 t\, n_3$ (2) since n_3 is fixed in B

Now ${}^R v^Q = \dfrac{{}^R d r^Q}{dt} = \Omega L\, a_2 + 2c\Omega^2 t\, n_3 + {}^R\omega^B \times {}^B r^{Q/B^*}$

$$\frac{{}^R d v^Q}{dt} = {}^R a^Q = \Omega L \frac{{}^R d a_2}{dt} + 2c\Omega^2 n_3 + 2c\Omega^2 t \frac{{}^R d n_3}{dt} + \frac{d\, {}^R\omega^B}{dt} \times {}^B r^{Q/B^*} + {}^R\omega^B \times \frac{{}^R d\, {}^B r^{Q/B^*}}{dt}$$

$$= \Omega L \Big[\underbrace{\cancel{\frac{{}^A d a_2}{dt}}}_{=0 \text{ since } a_2 \text{ is fixed in A}} + {}^R\omega^A \times a_2 \Big] + 2c\Omega^2 n_3 + 2c\Omega^2 t \Big[\underbrace{\cancel{\frac{{}^B d n_3}{dt}}}_{=0 \text{ since } n_3 \text{ is fixed in B}} + {}^R\omega^B \times n_3 \Big]$$

$$+ {}^R\alpha^B \times {}^B r^{Q/B^*} + {}^R\omega^B \times \left[\frac{{}^B d\, {}^B r^{Q/B^*}}{dt} + {}^R\omega^B \times {}^B r^{Q/B^*} \right]$$

$$= -\Omega^2 L \underbrace{a_1}_{n_1} + 2c\Omega^2 n_3 + 2c\Omega^2 t \{ \dot\theta_1 \underbrace{i_1}_{n_1} + \dot\theta_2 \underbrace{j_2}_{n_2} + \dot\theta_3 n_3 \} \times n_3 + {}^R\alpha^B \times \overset{=0}{\cancel{q}}$$

$$+ [\Omega \underbrace{a_3}_{n_3} + \dot\theta_1 n_1 + \dot\theta_2 n_2 + \dot\theta_3 n_3] \times 2c\Omega^2 t\, n_3 + {}^R\omega^B \times {}^R\omega^B \times \overset{=0}{\cancel{{}^B r^{Q/B^*}}}$$

$${}^R a^Q = -\Omega^2 L\, n_1 + 2c\Omega^2 n_3 + 2c\Omega \{ -\dot\theta_1 n_2 + \dot\theta_2 n_1 \} + \{ -\dot\theta_1 n_2 + \dot\theta_2 n_1 \} \times 2c\Omega$$

$${}^R a^Q = (-\Omega^2 L + 4c\Omega\dot\theta_2)\, n_1 - 4c\Omega\dot\theta_1\, n_2 + 2c\Omega^2 n_3 \quad ✓$$

3(f) Since ${}^R r^P$ is a fn of $q_1(t), q_2(t), q_3(t)$ $\Rightarrow$ ${}^R v^P$ is then of $q_1(t), q_2(t), q_3$ and $\dot q_1(t), \dot q_2(t), \dot q_3(t)$ and we assume the system is holonomic then we can use (2.25) to get $v_{\dot q_r} \cdot a = \frac{1}{2}\left[\frac{d}{dt}\frac{\partial v^2}{\partial \dot q_r} - \frac{\partial v^2}{\partial q_r}\right]$

$\Rightarrow$ $v^2 = v \cdot v = \sum_{r,s} f_r \dot q_r n_r \cdot f_s \dot q_s n_s$ & $\dfrac{\partial v^2}{\partial \dot q_r} = \sum_r 2 f_r^2 \dot q_r$; $\dfrac{\partial}{\partial q_r} v^2 = 2 \dfrac{\partial v}{\partial q_r} \cdot v$

$$\frac{\partial v}{\partial q_r} \cdot v = \sum_s \left(\frac{\partial v_{\dot q_r}}{\partial q_s} \right) \dot q_s \cdot \sum_r f_r \dot q_r n_r = \sum_s \frac{\partial f_r n_r}{\partial q_s} \dot q_s \cdot \sum_r f_r \dot q_r n_r = \sum\sum_s \left(\frac{\partial f_s}{\partial q_r} f_s \dot q_s^2 \right)$$

over

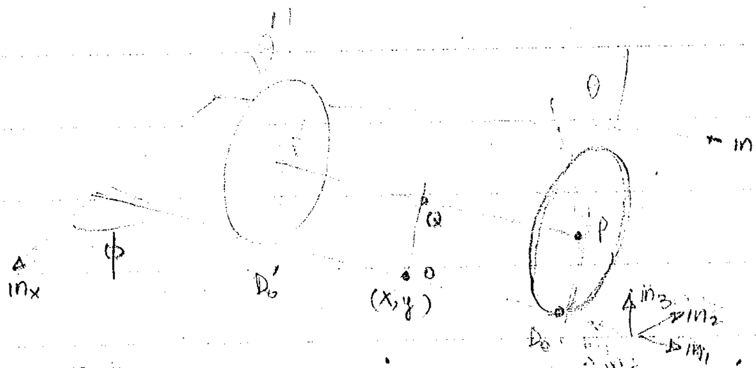
$$\mathbb{v}_{\dot q_r} \cdot \mathbb{a} = \frac{d}{dt}\left\{\sum_r \left(f_r^2 \dot q_r\right)\right\} - \sum_r \sum_s f_s \frac{\partial f_s}{\partial q_r} \dot q_s^2 = \sum_r \left\{\frac{d}{dt}\left(f_r^2 \dot q_r\right) - \sum_s f_s \frac{\partial f_s}{\partial q_r} \dot q_s^2\right\}$$

but $\mathbb{v}_{\dot q_r} \cdot \mathbb{a} = \sum f_r a_r = \sum_r \left\{ \qquad\qquad \right\} \Rightarrow \sum_{r=1} \left[f_r a_r - \left\{ \qquad \right\} \right] = 0$

since we can look at this one at a time $\Rightarrow a_r = \frac{1}{f_r}\left\{\frac{d}{dt}\left(f_r^2 \dot q_r\right) - \sum_s f_s \frac{\partial f_s}{\partial q_r} \dot q_s^2\right\}$ Q E D

a_r gives acceleration of rth q curvilinear coords.

3h)



$\mathbf{r}^O = x\mathbf{n}_x + y\mathbf{n}_y \qquad \mathbf{r}^Q = x\mathbf{n}_x + y\mathbf{n}_y + r\mathbf{n}_3$

$\mathbf{r}^P = x\mathbf{n}_x + y\mathbf{n}_y + r\mathbf{n}_3 + L\mathbf{n}_1$

$\mathbf{r}^{P'} = x\mathbf{n}_x + y\mathbf{n}_y + r\mathbf{n}_3 - L\mathbf{n}_1$

${}^R\omega^D = {}^R\omega^I + {}^I\omega^D = \dot\phi\mathbf{n}_3 + \dot\theta\mathbf{n}_1$

${}^R\omega^{D'} = {}^R\omega^I + {}^I\omega^{D'} = \dot\phi\mathbf{n}_3 + \dot\theta'\mathbf{n}_1$

${}^R\mathbf{v}^P = \dot x\mathbf{n}_x + \dot y\mathbf{n}_y + L(\dot\phi\mathbf{n}_3 \times \mathbf{n}_1) = \dot x\mathbf{n}_x + \dot y\mathbf{n}_y + L\dot\phi\mathbf{n}_2$

${}^R\mathbf{v}^{D_0} = 0 = {}^R\mathbf{v}^P + {}^R\omega^D \times \mathbf{r} = \dot x\mathbf{n}_x + \dot y\mathbf{n}_y + L\dot\phi\mathbf{n}_2 + [(\dot\phi\mathbf{n}_3 + \dot\theta\mathbf{n}_1) \times -r\mathbf{n}_3] = \dot x\mathbf{n}_x + \dot y\mathbf{n}_y + L\dot\phi\mathbf{n}_2 + r\dot\theta\mathbf{n}_2$

$$\begin{array}{c|ccc} & \mathbf{n}_x & \mathbf{n}_y & \mathbf{n}_z \\ \hline \mathbf{n}_1 & c & s & 0 \\ \mathbf{n}_2 & -s & c & 0 \\ \mathbf{n}_3 & 0 & 0 & 1 \end{array} \qquad 0 = [\dot x(c\mathbf{n}_1 - s\mathbf{n}_2) + \dot y(s\mathbf{n}_1 + c\mathbf{n}_2) + L\dot\phi\mathbf{n}_2 + r\dot\theta\mathbf{n}_2]$$

${}^R\mathbf{v}^{P'} = \dot x\mathbf{n}_x + \dot y\mathbf{n}_y - L\dot\phi\mathbf{n}_2 \qquad {}^R\mathbf{v}^{D_0'} = {}^R\mathbf{v}^P + {}^R\omega^{D'} \times \mathbf{r} = \dot x\mathbf{n}_x + \dot y\mathbf{n}_y - L\dot\phi\mathbf{n}_2 + [(\dot\phi\mathbf{n}_3 + \dot\theta'\mathbf{n}_1) \times (-r\mathbf{n}_3)]$

$0 = \dot x\mathbf{n}_x + \dot y\mathbf{n}_y - L\dot\phi\mathbf{n}_2 + r\dot\theta'\mathbf{n}_2$

$= \dot x(c\mathbf{n}_1 - s\mathbf{n}_2) + \dot y(c\mathbf{n}_2 + s\mathbf{n}_1) - L\dot\phi\mathbf{n}_2 + r\dot\theta'\mathbf{n}_2$

$\therefore \quad \dot x c + \dot y s = 0 \quad (1) \qquad -\dot x s + \dot y c + L\dot\phi + r\dot\theta = 0 \quad (2) \qquad (3)\ -\dot x s + \dot y c - L\dot\phi + r\dot\theta' = 0$

$$\begin{pmatrix} c & s & 0 \\ -s & c & 1 \\ -s & c & -1 \end{pmatrix}\begin{pmatrix} \dot x \\ \dot y \\ L\dot\phi \end{pmatrix} \begin{matrix} = 0 \\ = -r\dot\theta \\ = -r\dot\theta' \end{matrix}$$

$-c^2 - s^2 - s^2 - c^2 = -2 = \text{denom.}$

$\dot x = \dfrac{-s(r\dot\theta' + r\dot\theta)}{-2} = \dfrac{+r}{2}(\dot\theta' + \dot\theta)\sin\phi$

$\dot y = \dfrac{c}{-2}[r\dot\theta' + r\dot\theta] = \dfrac{-r(\dot\theta' + \dot\theta)}{2}\cos\phi \qquad L\dot\phi = \dfrac{1}{-2}[-r\dot\theta' c^2 + r\dot\theta s^2 - s^2 r\dot\theta' + r\dot\theta]$

$= \dfrac{1}{-2}[-r\dot\theta' + r\dot\theta] \Rightarrow \dot\phi = \dfrac{1}{2L}[r\dot\theta' - r\dot\theta]$

$\mathbf{v}^P = \dot x\mathbf{n}_x + \dot y\mathbf{n}_y + L\dot\phi\mathbf{n}_2 = \left\{\dfrac{r}{2}(\dot\theta' + \dot\theta)\sin\phi\right\}\underbrace{\{c\mathbf{n}_1 - s\mathbf{n}_2\}}_{sc\mathbf{n}_1 - s^2\mathbf{n}_2} + \left\{-\dfrac{r}{2}(\dot\theta' + \dot\theta)\cos\phi\right\}\underbrace{\{c\mathbf{n}_2 + s\mathbf{n}_1\}}_{-c^2\mathbf{n}_2 - cs\mathbf{n}_1} + \dfrac{r}{2}[\dot\theta' - \dot\theta]$

$\mathbf{v}^P = \sum \tilde{\mathbf{v}}^P_{\dot q_r}\dot q_r \Rightarrow \quad \tilde{\mathbf{v}}^P_{\dot\theta} = \dfrac{r}{2}s(\mathbf{n}_x) - \dfrac{r}{2}c\,\mathbf{n}_y - \dfrac{r}{2}\mathbf{n}_2 = \left(\dfrac{-r}{2}\mathbf{n}_2\right)2 = -r\mathbf{n}_2$

$\tilde{\mathbf{v}}^P_{\dot\theta'} = \dfrac{r}{2}(sc\mathbf{n}_1 - s^2\mathbf{n}_2) + \dfrac{r}{2}(-c^2\mathbf{n}_2 - cs\mathbf{n}_1) + \dfrac{r}{2}\mathbf{n}_2 = 0$

$\mathbf{v}^{P'} = \dot x\mathbf{n}_x + \dot y\mathbf{n}_y - L\dot\phi\mathbf{n}_2 = \mathbf{v}^P - 2L\dot\phi\mathbf{n}_2 = \mathbf{v}^P + (r\dot\theta - r\dot\theta')\mathbf{n}_2$

$\therefore \quad \tilde{\mathbf{v}}^{P'}_{\dot\theta} = \tilde{\mathbf{v}}^P_{\dot\theta} + r\mathbf{n}_2 = -r\mathbf{n}_2 + r\mathbf{n}_2 = 0$

$\tilde{\mathbf{v}}^{P'}_{\dot\theta'} = \tilde{\mathbf{v}}^{P}_{\dot\theta'} - r\mathbf{n}_2 = 0 - r\mathbf{n}_2 = -r\mathbf{n}_2$

${}^R\omega^{D'} = \dot\phi\mathbf{n}_3 + \dot\theta'\mathbf{n}_1 \qquad \tilde\omega^{D'}_{\dot\theta} = \dfrac{-r}{2L}\mathbf{n}_3 \qquad \tilde\omega^{D'}_{\dot\theta'} = \mathbf{n}_1 + \dfrac{r}{2L}\mathbf{n}_3$

6

3(g) $\mathbb{v} = \dot{r}\,\mathbb{n}_r + r\dot\theta\,\mathbb{n}_\theta + r\sin\theta\,\dot\phi\,\mathbb{n}_\phi$

$f_1 = 1 \quad f_2 = r \quad f_3 = r\sin\theta \qquad \dot q_1 = \dot r \quad \dot q_2 = \dot\theta \quad \dot q_3 = \dot\phi$

$q_1 = r \quad q_2 = \theta \quad q_3 = \phi$

$$a_1 = \frac{1}{f_1}\left\{\frac{d}{dt}\left(f_1^2 \dot q_1\right) - \left[f_1\cdot 0 + r\cdot 1\cdot \dot\theta^2 + r\sin\theta\cdot\sin\theta\cdot\dot\phi^2\right]\right\} = \ddot r - r\dot\theta^2 - r\sin^2\theta\,\dot\phi^2$$

$$a_2 = \frac{1}{r}\left\{\frac{d}{dt}\left(r^2\dot\theta\right) - \left[1\cdot 0 + r\cdot 0 + r\sin\theta\cdot r\cos\theta\cdot\dot\phi^2\right]\right\} = \frac{1}{r}\left\{2r\dot r\dot\theta + r^2\ddot\theta - r^2\sin\theta\cos\theta\,\dot\phi^2\right\}$$

$$= 2\dot r\dot\theta + r\ddot\theta - r\sin\theta\cos\theta\,\dot\phi^2 = \frac{1}{r}\frac{d}{dt}(r^2\dot\theta) - r\frac{\sin 2\theta}{2}\dot\phi^2$$

$$a_3 = \frac{1}{r\sin\theta}\left\{\frac{d}{dt}\left(r^2\sin^2\theta\,\dot\phi\right) - \left\{1\cdot 0 + r\cdot 0 + r\sin\theta\cdot 0\right\}\right\} = \frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\,\dot\phi\right)$$

3(i). if B is the same as body B in 2(a) then we showed that

$${}^R\omega^B = \left\{\dot\theta_1 c_2c_3 + s_3\dot\theta_2 + \Omega\left(s_1s_3 - s_2c_1c_3\right)\right\}\mathbb{n}_1 + \left\{-\dot\theta_1 c_2 s_3 + c_3\dot\theta_2 + \Omega(s_1c_3 + s_2c_1s_3)\right\}\mathbb{n}_2$$
$$+ \left\{s_2\dot\theta_1 + \dot\theta_3 + \Omega c_1c_2\right\}\mathbb{n}_3$$

$= \omega_1\mathbb{n}_1 + \omega_2\mathbb{n}_2 + \omega_3\mathbb{n}_3$ where $\omega_1 = u_1$, $\omega_2 = u_2$, $\omega_3 = u_3$

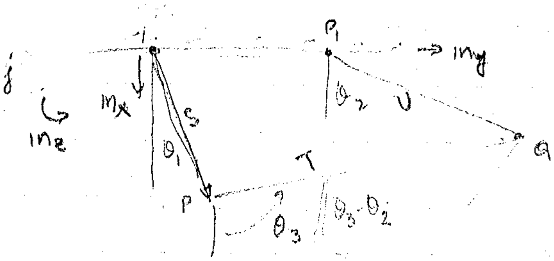
and since ${}^B\omega^G = -\mathbb{n}_1\dot\theta$ and ${}^G\omega^{Rot} = \frac{d}{dt}\Omega\cdot\left[\cos\phi\,\mathbb{n}_3 + \sin\theta\,\mathbb{n}_2\right]$

but since $\Omega = $ const $\frac{d\Omega}{dt} = 0 \Rightarrow {}^G\omega^{Rot} \equiv 0$

$\therefore\ {}^R\omega^{Rot} = u_1\mathbb{n}_1 + u_2\mathbb{n}_2 + u_3\mathbb{n}_3 - u_4\mathbb{n}_4$

$\therefore\ \omega_{u_i} = \mathbb{n}_1, \mathbb{n}_2, \mathbb{n}_3, -\mathbb{n}_4$ for $i = 1,\dots,4$

3 j



geom. const $\Rightarrow \mathbb{r}^{P_1} + \mathbb{r}^{Q/P} + \mathbb{r}^{P_1/Q} = \mathbb{r}^{P_1} = (3-\beta)L\,\mathbb{n}_y$

$\mathbb{r}^{P_1} = L\sin\theta_1\,\mathbb{n}_y + L\cos\theta_1\,\mathbb{n}_x$

$\mathbb{r}^{Q/P} = 3L\sin\theta_3\,\mathbb{n}_y + 3L\cos\theta_3\,\mathbb{n}_x$

$\mathbb{r}^{P_1/Q} = -\mathbb{r}^{Q/P_1} = -L\sin\theta_2\,\mathbb{n}_y - L\cos\theta_2\,\mathbb{n}_x$

$\cos\theta_1\,\dot\theta_1 - 2\cos\theta_2\,\dot\theta_2 + 3\cos\theta_3\,\dot\theta_3 = 0$

$-\sin\theta_1\,\dot\theta_1 + 2\sin\theta_2\,\dot\theta_2 + 3\sin\theta_3\,\dot\theta_3 = 0$

$\mathbb{v}^P = \frac{d}{dt}\mathbb{r}^P = \omega^S \times \mathbb{r}^P = \dot\theta_1\mathbb{n}_z \times L\,\mathbb{i}_1 = \dot\theta_1 L\,\mathbb{j}_1 \quad \therefore$ if $\mathbb{v}^P = \sum \tilde{\mathbb{v}}_{q_r}\dot q_r + \tilde{\mathbb{v}}_t$

then $\tilde{\mathbb{v}}^P_{\dot\theta_1} = L\,\mathbb{j}_1 \quad \tilde{\mathbb{v}}_{\dot\theta_2} = \tilde{\mathbb{v}}_{\dot\theta_3} = \tilde{\mathbb{v}}_t = 0$

also $\tilde{\mathbb{v}}^Q = \tilde{\mathbb{v}}^P + \omega^T \times \mathbb{r}^{P/Q} = \dot\theta_1 L\,\mathbb{j}_1 + \dot\theta_3\,\mathbb{n}_z \times \left[3L\sin\theta_3\,\mathbb{n}_y + 3L\cos\theta_3\,\mathbb{n}_x\right]$

$-3\dot\theta_3 L\sin\theta_3\,\mathbb{n}_x + \dot\theta_3 3L\cos\theta_3\,\mathbb{n}_y$

3j ...

now $-3\dot\theta_3 \sin\theta_3 = +\sin\theta_1\dot\theta_1 - 2\sin\theta_2\dot\theta_2$ $\qquad \mathbb{j}_1 = \cos\theta_1 \mathbb{m}_y - \sin\theta_1\mathbb{m}_x$

$3\cos\theta_3\dot\theta_3 = 2\cos\theta_2\dot\theta_2 - \cos\theta_1\dot\theta_1$ $\qquad \mathbb{j}_2 = \cos\theta_2\mathbb{m}_y - \sin\theta_2\mathbb{m}_x$

$\therefore \mathbb{V}^Q = \dot\theta_1 L\mathbb{j}_1 + L\left[\sin\theta_1\dot\theta_1 - 2\sin\theta_2\dot\theta_2\right]\mathbb{m}_x + L\left[2\cos\theta_2\dot\theta_2 - \cos\theta_1\dot\theta_1\right]\mathbb{m}_y$

$= \left[-2L\sin\theta_2 \mathbb{m}_x + 2\cos\theta_2 L\mathbb{m}_y\right]\dot\theta_2 = 2L\dot\theta_2\mathbb{j}_2$

but $2\cos\theta_2\dot\theta_2 = \cos\theta_1\dot\theta_1 + 3\cos\theta_3\dot\theta_3$

$2\sin\theta_2\dot\theta_2 = \sin\theta_1\dot\theta_1 + 3\sin\theta_3\dot\theta_3$ $\Rightarrow$ $2\left[\sin\theta_3\cos\theta_2 - \cos\theta_3\sin\theta_2\right]\dot\theta_2 = \left[\cos\theta_1\sin\theta_3 - \sin\theta_1\cos\theta_3\right]\dot\theta_1$

$2\sin(\theta_3-\theta_2)\dot\theta_2 = \sin(\theta_3-\theta_1)\dot\theta_1$

$\therefore 2\dot\theta_2 = \dfrac{\sin(\theta_3-\theta_1)\dot\theta_1}{\sin(\theta_3-\theta_2)}$ or $\mathbb{V}^Q = L\mathbb{j}_2\dfrac{\sin(\theta_3-\theta_1)\dot\theta_1}{\sin(\theta_3-\theta_2)} = \tilde{\mathbb{V}}_{\dot\theta_1}\dot\theta_1$

3k Given: $\hat{\mathbb{V}}^{P_1} = -3v\mathbb{m}_2$ $\quad \hat{\mathbb{V}}^{P_3} = 5v\mathbb{m}_1 + 4v\mathbb{m}_2$ $\quad$ Find: $\hat{\mathbb{V}}^{P_2} = ?$ $\quad$ if $\theta_1 = \theta_2 = \theta_3 = \pi/4$

$\hat{\mathbb{V}} = \sum_{r=1}^{n-m}\tilde{\mathbb{V}}_{\dot q_r}\hat{\dot q}_r \qquad \mathbb{V} = \sum_{r=1}^{n-m}\tilde{\mathbb{V}}_{u_r}u_r$

$\mathbb{V}^{P_1} = \frac{L\dot\theta_1}{2}\mathbb{m}_2 \qquad \mathbb{V}^{P_2} = L\dot\theta_1\mathbb{m}_2 + \frac{L\dot\theta_2}{2}\mathbb{m}_2 \qquad \mathbb{V}^{P_3} = L\dot\theta_1\mathbb{m}_2 + L\dot\theta_2\mathbb{m}_2 - \frac{L\dot\theta_3}{2}\mathbb{m}_1$

let $u_1 = \dot\theta_1 \quad u_2 = \dot\theta_2 \quad u_3 = \dot\theta_3$

$\tilde{\mathbb{V}}^{P_1}_{\dot\theta_1} = \frac{L\mathbb{m}_2}{2} \quad \therefore \hat{\mathbb{V}}^{P_1} = \frac{L\mathbb{m}_2}{2}\hat{\dot q}_1 = -3v\mathbb{m}_2 \Rightarrow \hat{\dot q}_1\frac{L}{2} = -3v \Rightarrow \boxed{\hat{\dot q}_1 = -\frac{6v}{L}}$

$\tilde{\mathbb{V}}^{P_3}_{\dot\theta_1} = L\mathbb{m}_2 \quad \tilde{\mathbb{V}}^{P_3}_{\dot\theta_2} = L\mathbb{m}_2 \quad \tilde{\mathbb{V}}^{P_3}_{\dot\theta_3} = -\frac{L\mathbb{m}_1}{2} \Rightarrow \hat{\mathbb{V}}^{P_3} = L\mathbb{m}_2\hat{\dot q}_1 + L\mathbb{m}_2\hat{\dot q}_2 - \frac{L\mathbb{m}_1}{2}\hat{\dot q}_3$

$= 5v\mathbb{m}_1 + 4v\mathbb{m}_2$

thus $L\hat{\dot q}_1 + L\hat{\dot q}_2 = 4v \quad -\frac{L}{2}\hat{\dot q}_3 = 5v \Rightarrow \boxed{\hat{\dot q}_3 = -\frac{10v}{L}} \quad \boxed{\hat{\dot q}_2 = \frac{10v}{L}}$

$\tilde{\mathbb{V}}^{P_2}_{\dot\theta_1} = L\mathbb{m}_2 \quad \tilde{\mathbb{V}}^{P_2}_{\dot\theta_2} = \frac{L\mathbb{m}_2}{2} \Rightarrow \hat{\mathbb{V}}^{P_3} = L\mathbb{m}_2\hat{\dot q}_1 + \frac{L\mathbb{m}_2}{2}\hat{\dot q}_2 = L\mathbb{m}_2\left(-\frac{6v}{L}\right) + \frac{L\mathbb{m}_2}{2}\left(\frac{10v}{L}\right)$

$= -6v\mathbb{m}_2 + 5v\mathbb{m}_2 = \boxed{-v\mathbb{m}_2 = \hat{\mathbb{V}}^{P_3}}$

3l. if $\delta\mathbb{q}^Q = 3L\mathbb{j}_2$ when $\theta_1 = 0$; find $|\delta\alpha|$

now $\delta\mathbb{q}^Q = \hat{\mathbb{V}}^Q\delta t$; $\quad$ but $\mathbb{V}^Q = L\mathbb{j}_2\dfrac{\sin(\theta_3-\theta_1)}{\sin(\theta_3-\theta_2)}\dot\theta_1 = \tilde{\mathbb{V}}^Q_{\dot\theta_1}\dot\theta_1$ or $\tilde{\mathbb{V}}^Q_{\dot\theta_1} = L\mathbb{j}_2\dfrac{\sin(\theta_3-\theta_1)}{\sin(\theta_3-\theta_2)}$

and $\mathbb{V}^P = L\mathbb{j}_1\dot\theta_1$ and $\tilde{\mathbb{V}}^P_{\dot\theta_1} = L\mathbb{j}_1$

$\hat{\mathbb{V}}^Q = \tilde{\mathbb{V}}^Q_{\dot\theta_1}\hat{\dot q}_1 \Rightarrow \delta\mathbb{q}^Q = \tilde{\mathbb{V}}^Q_{\dot\theta_1}\delta t\,\hat{\dot q}_1 = L\mathbb{j}_2\dfrac{\sin\theta_3}{\sin(\theta_3-\theta_2)}\hat{\dot q}_1\delta t = 3L\mathbb{j}_2$

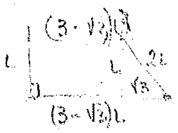
thus $\hat{\dot q}_1\delta t = \dfrac{3\sin(\theta_3-\theta_2)}{\sin\theta_3} = \delta q_1$; but $\delta\mathbb{p}^P = \tilde{\mathbb{V}}^P_{\dot\theta_1}\delta q_1 = L\mathbb{j}_1\dfrac{3\sin(\theta_3-\theta_2)}{\sin\theta_3}$

$\omega = \dot\theta_3\mathbb{m}_z = \dfrac{2\sin\theta_2}{3\sin\theta_3}\dot\theta_2\mathbb{m}_z$ from 2nd constraint $\therefore \omega = \dfrac{\sin\theta_2}{3\sin\theta_3}\mathbb{m}_z \cdot \dfrac{\sin\theta_3\,\dot\theta_1}{\sin(\theta_3-\theta_2)} = \dfrac{\sin\theta_2\,\dot\theta_1\,\mathbb{m}_z}{3\sin(\theta_3-\theta_2)}$

but $\tilde{\omega}_{\dot{\theta}_1} = \dfrac{\sin\theta_2\, n_z}{3\sin(\theta_3-\theta_2)}$ now $\delta\alpha = \tilde{\omega}_{\dot{\theta}_1}\,\delta q_1$

$$= \frac{\sin\theta_2\; n_z}{3\sin(\theta_3-\theta_2)} \cdot \frac{3\sin(\theta_3-\theta_2)}{\sin\theta_3}$$

$$\delta\alpha = \frac{\sin\theta_2}{\sin\theta_3}\, n_z$$



$\dfrac{\sin\theta_2}{\sin\theta_3} = \dfrac{\frac{\sqrt{3}}{2}}{1}$ $\quad \sin\theta_3 = \sin 90^\circ = 1$ $\quad \therefore\ \delta\alpha = \dfrac{\sqrt{3}}{2}\, n_z$ $\quad \therefore \boxed{|\delta\alpha| = \sqrt{3}/2}$

4a. Fix a set of vectors $\ni$. m_1 is // to line AB m_2 is // to line AL_2 m_3 is // to line AL3

thus.

	m_1	m_2	m_3
$\mathbb{F}$	0	-30	0
$\mathbb{G}$	32.77	+16.22	+16.22
$r^{G/A}$	0	$-\frac{7\sqrt{2}}{2}$	$\frac{7\sqrt{2}}{2}$
$r^{F/A}$	10	0	-9

triangle that G lies in has unit vector $\left[10m_1 - \frac{7\sqrt{2}}{2}m_2 + \frac{7\sqrt{2}}{2}m_3\right]/\sqrt{149}$

$-\mathbb{G} \times r^{G/A} = M^{G/A}$ 0 +162.2 +162.2 $|M^{G/A}| = |m_i m_i| = (m_i m_i)^{1/2} = 229.4$ in/lb

$-\mathbb{F} \times r^{F/A} = M^{F/A}$ +270 0 -300 $|M^{F/A}| = (m_i m_i) = (m_i m_i)^{1/2} = 403.6$ in/lb

the angle between them can be found by $\mathbb{A}\cdot\mathbb{B} = |\mathbb{A}||\mathbb{B}|\cos(\mathbb{A},\mathbb{B})$

$\therefore$ $\mathbb{L}_3$ 0 0 1

$$\cos(M^{G/A}, \mathbb{L}_3) = \frac{M^{G/A}\cdot \mathbb{L}_3}{|M^{G/A}||\mathbb{L}_3|} = \frac{+162.2}{229.4} = +.707$$

$$\cos(M^{F/A}, \mathbb{L}_3) = \frac{M^{F/A}\cdot \mathbb{L}_3}{|M^{F/A}||\mathbb{L}_3|} = \frac{-300}{403.6} = -.743$$

4b. $r^{F/O} = x_i m_{x_i}$ $\mathbb{F} = F_{x_j} m_{x_j}$ $M = r^{F/O} \times \mathbb{F} = x_i F_{x_j} e_{ijk} m_{x_k}$

where e_{ijk} = +1 for cyclic, −1 for anticyc. when we have a rh system and −1 for cycl, +1 for anticyc for a lh system

thus $M_{RHS} = M_{LHS}$ $M_{RHS} = M_{x_k} m_{x_k}$ $M_{y_{RHS}} = zF_x - xF_z$ $\therefore$ $M_{y_{LHS}} = -(zF_x - xF_z)$

4c. for 4a

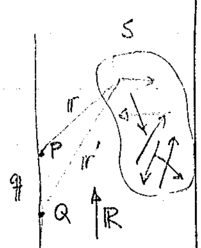
	m_1	m_2	m_3
$\mathbb{F}$	0	-30	0
$\mathbb{G}$	32.77	+16.22	-16.22
$\mathbb{H}$	-32.77	+13.78	+16.22
$r^{F/B}$	0	+0	-9

; we will take moments about B $|\mathbb{H}| = 39.07$

now $\mathbb{T} = \sum r^{F/B} \times \mathbb{F} + r^{G/B} \times \mathbb{G} + r^{H/B} \times \mathbb{H} = -270$ in-lb direction is m_1

magnitude of $\mathbb{T} = |\mathbb{T}| = 270$ in-lb

$$\cos(\mathbb{T}, \mathbb{H}) = \frac{\mathbb{T}\cdot\mathbb{H}}{|\mathbb{T}||\mathbb{H}|} = \frac{-270 \cdot -32.77}{270 \cdot 39.07} = +.839 \qquad \theta = 33^\circ$$

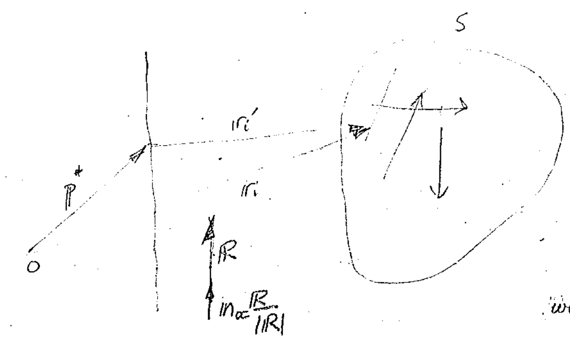


4d. Suppose we have a set of bound vectors $\{\$_i\}$ with vector distances to a point P on line L $\{r_i\}$ then $M^P = r_i \times \$_i$. Suppose they have vector distances to a point Q on line L $\{r_i'\}$ such that $r_i = q + r_i'$ $\therefore$ $M^Q = r_i' \times \$_i$

or $M^P = \sum q \times \$_i + r_i' \times \$_i = \sum q \times \$_i + M^Q$ but $\sum q \times \$_i = q \times \sum \$_i = q \times \mathbb{R} = 0$ since $q \parallel \mathbb{R}$

thus $M^P = M^Q$

4e)



$$\mathbb{M} = \mathbb{M}^* + \mathbb{p}^* \times \mathbb{R}$$

now $\mathbb{M}^* = \mathbb{M} - \mathbb{p}^* \times \mathbb{R}$ or $\mathbb{M} + \mathbb{R} \times \mathbb{p}^*$

we need some way to find $\mathbb{p}^*$

we can decompose $\mathbb{p}^*$ in the direction of $\mathbb{R}$, $\mathbb{M}$, and $\mathbb{M} \times \mathbb{R}$.

$\therefore$ let $\mathbb{p}^* = \beta \mathbb{M} + \gamma \mathbb{R} + \delta\, \mathbb{M} \times \mathbb{R}$

then $\mathbb{M}^* = \mathbb{M} + \beta \mathbb{R} \times \mathbb{M} + \gamma \cancelto{0}{\mathbb{R} \times \mathbb{R}} + \delta\, \mathbb{R} \times (\mathbb{M} \times \mathbb{R})$

$$\mathbb{M}^* \cdot \mathbb{M}^* = \mathbb{M} \cdot \mathbb{M} + 2\beta\, \cancelto{0}{\mathbb{R} \times \mathbb{M} \cdot \mathbb{M}} + 2\delta\, \mathbb{M} \cdot \mathbb{R} \times (\mathbb{M} \times \mathbb{R}) + \beta^2 (\mathbb{R} \times \mathbb{M})^2 + 2\beta\delta\, \cancelto{0}{\mathbb{R} \times (\mathbb{M} \times \mathbb{R}) \cdot (\mathbb{R} \times \mathbb{M})} + \delta^2 (\mathbb{R} \times (\mathbb{M} \times \mathbb{R}))^2$$

For minimizing $\mathbb{M}^{*2}$ take $\frac{\partial \mathbb{M}^{*2}}{\partial \beta} = \frac{\partial \mathbb{M}^{*2}}{\partial \delta} = 0$:

$\Rightarrow 2\beta (\mathbb{R} \times \mathbb{M}^*)^2 = 0$ and $\mathbb{M} \cdot \mathbb{R} \times (\mathbb{M} \times \mathbb{R}) + 2\delta\, (\mathbb{R} \times (\mathbb{M} \times \mathbb{R}))^2 = 0$

$\Rightarrow \beta = 0$ and $\delta = -\frac{\mathbb{M} \cdot \mathbb{R} \times (\mathbb{M} \times \mathbb{R})}{[\mathbb{R} \times (\mathbb{M} \times \mathbb{R})]^2} = -\frac{1}{\mathbb{R}^2}$

$$\therefore \mathbb{p}'^* = \gamma \mathbb{R} - \frac{\mathbb{M} \cdot [\mathbb{R} \times (\mathbb{M} \times \mathbb{R})]}{[\mathbb{R} \times (\mathbb{M} \times \mathbb{R})]^2} (\mathbb{M} \times \mathbb{R})$$

[now $\mathbb{R} \times (\mathbb{M} \times \mathbb{R}) = \mathbb{R}^2 \mathbb{M} - (\mathbb{M} \cdot \mathbb{R})\mathbb{R}$ and $[\mathbb{R} \times (\mathbb{M} \times \mathbb{R})]^2 = \mathbb{R}^4 \mathbb{M}^2 - 2\mathbb{R}^2 (\mathbb{M} \cdot \mathbb{R})^2 + (\mathbb{M} \cdot \mathbb{R})^2 \mathbb{R}^2$
$= \mathbb{R}^4 \mathbb{M}^2 - \mathbb{R}^2 (\mathbb{M} \cdot \mathbb{R})^2$
$= \mathbb{R}^2 [\mathbb{R}^2 \mathbb{M}^2 - (\mathbb{M} \cdot \mathbb{R})^2]$

$\mathbb{M} \cdot [\mathbb{R} \times (\mathbb{M} \times \mathbb{R})] = \mathbb{R}^2 \mathbb{M}^2 - (\mathbb{M} \cdot \mathbb{R})^2$

$\mathbb{p}^* = \gamma \mathbb{R} - (\mathbb{M} \times \mathbb{R})/\mathbb{R}^2 = \gamma \mathbb{R} + (\mathbb{R} \times \mathbb{M})/\mathbb{R}^2$

$\therefore \mathbb{M}^* = \mathbb{M} + \mathbb{R} \times \mathbb{p}'^* = \mathbb{M} + \cancel{\gamma \mathbb{R} \times \mathbb{R}} - \frac{1}{\mathbb{R}^2} (\mathbb{R} \times (\mathbb{M} \times \mathbb{R})) = \mathbb{M} - \frac{1}{\mathbb{R}^2} [\mathbb{R}^2 \mathbb{M} - (\mathbb{M} \cdot \mathbb{R})\mathbb{R}]$

$$\boxed{\mathbb{M}^* = + \frac{(\mathbb{M} \cdot \mathbb{R})\mathbb{R}}{\mathbb{R}^2}}$$

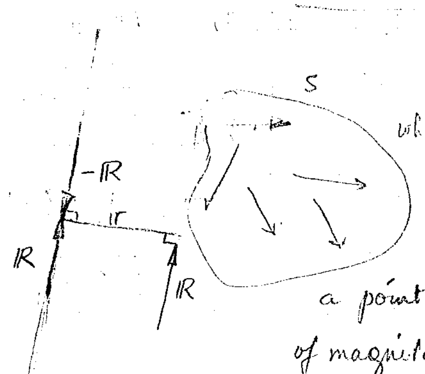
now $\mathbb{M} = \sum \mathbb{r}_i \times \$_i = \sum \mathbb{r}_i' \times \$_i + \sum \mathbb{p}^* \times \mathbb{R} = \mathbb{p}^* \times \mathbb{R} + \mathbb{M}^*$

now $\mathbb{R} \times \mathbb{M} = \mathbb{R} \times (\mathbb{p}^* \times \mathbb{R}) + \mathbb{R} \times \mathbb{M}^* = (\mathbb{R} \cdot \mathbb{R})\mathbb{p}^* - (\mathbb{p}^* \cdot \mathbb{R})\mathbb{R} + \cancelto{0}{\mathbb{R} \times \frac{(\mathbb{M} \cdot \mathbb{R})\mathbb{R}}{\mathbb{R}^2}}$

now $\mathbb{R} \times \mathbb{M}$ is $\perp$ to $\mathbb{R}$ $\therefore \mathbb{p}^* \cdot \mathbb{R} = 0$ (or $\mathbb{p}^*$ is $\perp$ to $\mathbb{R}$)

$\therefore \boxed{\frac{\mathbb{R} \times \mathbb{M}}{\mathbb{R}^2} = \mathbb{p}^*}$ as required

4f)



when $\sum \$_i = 0$ $\quad \sum \mathbb{r}_i \times \$_i = \mathbb{T}$ where $\mathbb{r}_i$ is the vector to any point.

Suppose $\mathbb{R}$ is the resultant of the set S. Pick a point on L^* and draw vectors // to $\mathbb{R}$, equal & opposite, of magnitude same as $\mathbb{R}$. Now the vectors $-\mathbb{R}$ & $\mathbb{R}$ form a [illegible] $\mathbb{T} = \mathbb{r} \times \mathbb{R}$. Hence we can replace the set of vectors S

having resultant $\mathbb{R}$ with an equivalent system of torque Π and another vector $\parallel$ to $\mathbb{R}$ acting at the point on L^*. ~~Now since we've replaced it with~~ The equivalent system consisting of Π, $\mathbb{R}$ is a wrench. Since Π can be placed anywhere (its value is the same) then it must be a minimum, thus $\Pi = \mathbb{M}^*$.

4g.

$\mathbb{F}$	0	-30	0
$\mathbb{G}$	32.77	$+16.22$	-16.22
$\mathbb{R}$	32.77	-13.78	-16.22

we now apply the results of 4e with pt A being pt "O" of 4e

$\mathbf{r}^{F/A}$	10	0	-9
$\mathbf{r}^{G/A}$	0	$-\frac{7\sqrt{2}}{2}$	$\frac{7\sqrt{2}}{2}$

$$\therefore \mathbb{M}_A = \mathbf{r}^{F/A} \times \mathbb{F} + \mathbf{r}^{G/A} \times \mathbb{G} = \begin{pmatrix} 10 & 0 & -9 \\ 0 & -30 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -\frac{7\sqrt{2}}{2} & \frac{7\sqrt{2}}{2} \\ 32.77 & +16.22 & -16.22 \end{pmatrix}$$

$$= -270\mathbf{m}_1 - 300\mathbf{m}_3 + 162.2\mathbf{m}_2 + 162.2\mathbf{m}_3$$

$$= -270\mathbf{m}_1 + 162.2\mathbf{m}_2 - 137.8\mathbf{m}_3$$

$\mathbb{R}^2 = 1526.85$

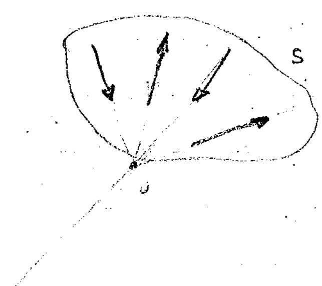
$\mathbb{R}\cdot\mathbb{M} = -8874.9$ $\qquad \mathbb{M}^* = \frac{\mathbb{R}\cdot\mathbb{M}}{\mathbb{R}^2}\mathbb{R} = -5.813 \left[32.77\mathbf{m}_1 - 13.78\mathbf{m}_2 - 16.22\mathbf{m}_3\right]$

$|\mathbb{M}^*| = 5.813\,(39.07) = 227.1$ in-lb

$\mathbb{P}^* = \frac{\mathbb{R}\times\mathbb{M}}{\mathbb{R}^2}$ $\qquad \mathbb{R}\times\mathbb{M} = \begin{pmatrix} 32.77 & -13.78 & -16.22 \\ -270 & 162.2 & -137.8 \end{pmatrix} = 4529.76\mathbf{m}_1 + 8895.1\mathbf{m}_2 + 1594.7\mathbf{m}_3$

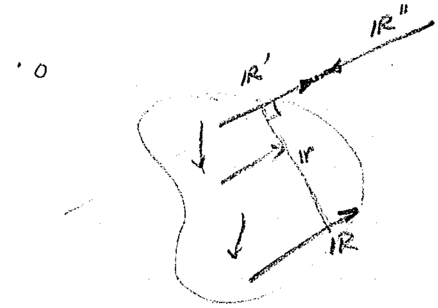
$\mathbb{P}^* = 2.97\mathbf{m}_1 + 5.83\mathbf{m}_2 + 1.04\mathbf{m}_3$ $\qquad |\mathbb{P}^*| = 6.62$

4h.



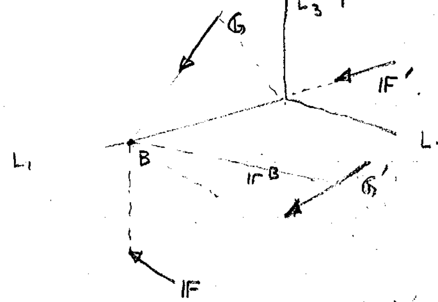
Since they all pass through the point O (and have in general a non zero resultant) then $\mathbb{M}_O = 0$. Since $|\mathbb{M}_O|$ is min about O then O must be a point on central axis. By 4f we can replace S by an equivalent set S' about any pt on central axis which form a wrench; here $\Pi = \mathbb{M}_O = 0$. Hence if we pick the point of the central axis to be O then we have shown that if a set of vectors are concurrent at a point O we can replace that set with the resultant of that set acting at the pt. O.

4i) Since the set S are coplanar then $\Rightarrow$ that $\mathbb{R}$ must also be coplanar. Look at a point O on the plane of S. Now $\mathbb{M}_O$ is a vector $\perp$ to the plane and $\mathbb{M}^* = \frac{\mathbb{R}\cdot\mathbb{M}}{\mathbb{R}^2}\mathbb{R} = \underset{\sim}{0}$ since $\mathbb{R} \perp \mathbb{M}$. also $\mathbb{P}^* = \underset{\sim}{0}$. Hence the min $\mathbb{M} = \mathbb{M}^* \equiv 0$. But by 4f we can replace S by the vector $\mathbb{R}$



acting on the central axis and a torque $\mathbb{T} = \mathbb{M}^*$. But here $\mathbb{M}^* = \underline{0} \Rightarrow \mathbb{T} = \underline{0}$. Hence we can replace S by a vector $= \mathbb{R}$ on the central axis. QED.

4k. for equivalence the sets must have equal resultants and equal moments about at least one point.



Since the line of action of $\mathbb{F}'$ is L_1 then $\mathbb{F}' = F'\mathbf{n}_1$ and $\mathbb{F} + \mathbb{G} = \mathbb{F}' + \mathbb{G}'$; but as shown previously in 4a:

	$\mathbf{n}_1$	$\mathbf{n}_2$	$\mathbf{n}_3$
$\mathbb{F}$	0	−30	0
$\mathbb{G}$	32.77	+16.22	−16.22

and $\mathbb{M}_B = \mathbf{r}^{F/B} \times \mathbb{F} + \mathbf{r}^{G/B} \times \mathbb{G} = -270\,\mathbf{n}_1 = \mathbf{r}^{F'/B} \times \mathbb{F}' + \mathbf{r}^{G'/B} \times \mathbb{G}' = \mathbf{r}^{G'/B} \times \mathbb{G}'$ since $\mathbf{r}^{F'/B}$ is // $\mathbb{F}'$. Since the requirement that the resultants be the same and the moment about a point be the same, This results in 6 equations. But there are 7 unknowns (3 coords of $\mathbf{r}^B$, 3 components of $\mathbb{G}'$ and 1 comp of $\mathbb{F}'$), This is an underdetermined system. We can thus take another condition to remove the indeterminacy - take $\mathbf{r}^B \cdot \mathbb{G}' = 0$.

now take $\mathbb{G}' \times \mathbb{M}_B = \mathbb{G}' \times (\mathbf{r}^B \times \mathbb{G}') = \mathbb{G}^2 \mathbf{r}^B - (\mathbf{r} \cdot \mathbb{G}')\mathbb{G}' = \mathbb{G}'^2 \mathbf{r}^B$

and $\mathbf{r}^B = \dfrac{\mathbb{G}' \times \mathbb{M}_B}{\mathbb{G}'^2}$. Because of the set up of $\mathbb{F}'$ $\Rightarrow \mathbb{G}' = a\mathbf{n}_1 - 13.78\,\mathbf{n}_2 - 16.22\,\mathbf{n}$

now $\mathbb{M}_B \cdot \mathbb{G}' = -270a = \mathbb{G}' \cdot (\mathbf{r}^B \times \mathbb{G}') = 0 \Rightarrow a = 0 \quad \therefore \mathbb{G}' = -13.78\,\mathbf{n}_2 - 16.22\,\mathbf{n}_3$

and from $\mathbb{F}' + \mathbb{G}' = \mathbb{F} + \mathbb{G} \Rightarrow \mathbb{F}' = 32.77\,\mathbf{n}_1 \qquad |\mathbb{G}'| = 21.28 \qquad |\mathbb{F}'| = 32.77$

and $\mathbf{r}^B = \dfrac{[+13.78(-270)\,\mathbf{n}_3 - 270 \times (-16.22)\,\mathbf{n}_2]}{(21.28)^2} = -\dfrac{270}{(21.28)^2}[13.78\,\mathbf{n}_3 - 16.22\,\mathbf{n}_2]$

$|\mathbf{r}^B| = +\dfrac{270}{(21.28)^2}(21.28) = 12.69$ in

4j. if $\mathbb{V}_1 \ldots, \mathbb{V}_m$ are // to $\mathbf{n}$ then $\mathbb{V}_i = k_i\mathbf{n}$. Then $\mathbb{R} = \sum k_i\,\mathbf{n}$ now this is a special case of 4(i). Here they are not only coplanar they are parallel. By 4(i) we can replace S by the resultant applied on the central axis. Since $\mathbb{R}$ lies on the central axis and all $\mathbb{V}_i$'s have the same unit vector $\Rightarrow$ all vectors $\mathbb{V}_i$ must lie on the central axis. Since P_i lie on line of action of $\mathbb{V}_i$ then for the gravitational force between particles must lie along $\mathbf{n}$. This means that the line of action of the mass center must be in the direction

of $\mathbf{n}_1$. The force of particle p_i on $p_j = \dfrac{G m_i m_j}{l_{ij}^2}\mathbf{n}_1$ where m_i, m_j are the ~~dcats~~ masses of each particle and l_{ij} is the distance between the particles similarly p_j on $p_i = -\dfrac{G m_j m_i}{(l_{ji})^2}\mathbf{n}_1$

Thus for a particle p_i, the force of p_i on all particles is $\sum_{j=1}^{n} \dfrac{G m_i m_j \mathbf{n}_1}{l_{ij}^2}$

thus $\mathbb{F}_i = m_i \mathbf{n}_1 \sum_{j=1}^{n-1} \dfrac{G m_j}{l_{ij}^2}$ define $k = \sum_{j \neq i} \dfrac{G m_j}{l_{ij}^2}$ then $\mathbb{F}_i = k m_i \mathbf{n}_1$

Thus $\mathbb{R} = \sum_i \mathbb{F}_i$ and since the mass center will be that pt where we can replace the forces of each particle on the others by the force at the mass center & the Torque about the mass center, then because the radius from the center to each point is along $\mathbf{n}_1$ the torque $= 0$. Thus we can apply $\mathbb{R}$ at the mass center and the mass center must lie on the central axis.

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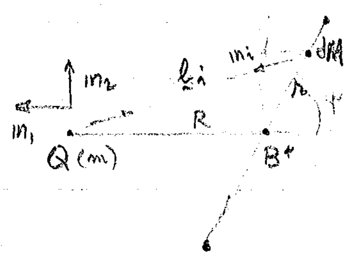


the mass $dM = \frac{M}{2L}dr_i$ has a force on $Q = \dfrac{G m\, dr_i}{l_i^2}\mathbf{n}_i \cdot \dfrac{M}{2L}$

$\mathbf{n}_i = \dfrac{\mathbf{l}_i}{|\mathbf{l}_i|}$ $\quad \mathbf{l}_i = (r\cos\psi + R)\,\mathbf{n}_1 - r\sin\psi\,\mathbf{n}_2$

$$\therefore \quad \mathbb{F} = \int_{-L}^{L} \frac{GMm}{2L}\frac{dr_i}{|\mathbf{l}_i|^3}\mathbf{l}_i = \frac{GMm}{2L}\int_{-L}^{L}\frac{[(R + r\cos\psi)\mathbf{n}_1 - r\sin\psi\,\mathbf{n}_2]\,dr}{[(r + R\cos\psi)^2 + R^2\sin^2\psi]^{3/2}}$$

$$\frac{GmM}{2L}(\cos\psi\,\mathbf{n}_1 - \sin\psi\,\mathbf{n}_2)\int_{-L}^{L}\frac{r\,dr}{[(r + \cos\psi \cdot R)^2 + R^2\sin^2\psi]^{3/2}} + \frac{GmM}{2L}R\,\mathbf{n}_1\int_{-L}^{L}\frac{dr}{[\quad]^{3/2}}$$

to get 1st integ.: let $s = r + R\cos\psi$ $\quad r = s - R\cos\psi$ $\qquad \displaystyle\int \frac{r\,dr}{[\quad]} = \int_{-L+b}^{L+b}\frac{(s - b)\,ds}{(s^2 + a^2)^{3/2}}$ $\qquad b = R\cos\psi$, $a = R\sin\psi$.

$$\int_{b-L}^{b+L}\frac{s\,ds}{(s^2 + a^2)^{3/2}} = -(s^2 + a^2)^{-1/2}\Big|_{b-L}^{b+L} \;; \qquad -b\int_{b-L}^{b+L}\frac{ds}{(s^2 + a^2)^{3/2}} = -\frac{b}{a^2}\frac{s}{\sqrt{s^2 + a^2}}\Big|_{b-L}^{b+L}$$

to get 2nd integ: let $s = r + R\cos\theta$; $r = s - R\cos\psi$; $dr = ds$

$$\int_{-L}^{L}\frac{dr}{[\quad]^{3/2}} = \int_{b-L}^{L+b}\frac{ds}{(s^2 + a^2)^{3/2}} = \frac{s}{a^2\sqrt{s^2 + a^2}}\Big|_{b-L}^{b+L}$$

$$\therefore \frac{GmM}{2L}(\cos\psi\,\mathbf{n}_1 - \sin\psi\,\mathbf{n}_2)\left[\frac{-1}{(s^2 + a^2)^{1/2}} - \frac{b}{a^2}\frac{s}{(s^2 + a^2)^{1/2}}\right]_{b-L}^{L+b} + \frac{GmMR\,\mathbf{n}_1}{2L}\left[\frac{s}{a^2(s^2 + a^2)^{1/2}}\right]_{b-L}^{b+L}$$

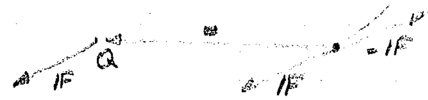
$$\frac{GmM}{2LR}\mathbf{n}_1\left[\frac{s}{(s^2 + a^2)^{1/2}} - \frac{R\cos\psi}{(s^2 + a^2)^{1/2}}\right] + \frac{GmM\,\mathbf{n}_2}{2La^2}\left[\frac{(a^2 + bs)\sin\psi}{[r^2 + 2rR\cos\psi + R^2]^{1/2}}\right]$$

$$\frac{GmM\,\mathbf{n}_1}{2LR}\frac{r}{[r^2 + 2rR\cos\psi + R^2]^{1/2}}\Big|_{r=-L}^{r=L} + \frac{GmM\,\mathbf{n}_2}{2LR\sin\psi}\left[\frac{R + r\cos\psi}{[r^2 + 2rR\cos\psi + R^2]^{1/2}}\right]_{-L}^{L}$$

thus $\dfrac{GmM\mathbf{m}_1}{2LR}\left\{\dfrac{L/R}{\left[(L/R)^2+2(L/R)\cos\psi+1\right]^{1/2}} + \dfrac{L/R}{\left[(L/R)^2-2(L/R)\cos\psi+1\right]^{1/2}}\right\}$

$$+\frac{GmM\mathbf{m}_2}{2LR\sin\psi}\left\{\frac{1+L/R\cos\psi}{\left[(L/R)^2+2(L/R)\cos\psi+1\right]^{1/2}} - \frac{(1-L/R\cos\psi)}{\left[(L/R)^2-2(L/R)\cos\psi+1\right]^{1/2}}\right\} = \mathbb{F}$$

Since all elemental $d\mathbb{F}$ are concurrent to pt. Q then by Varignon's theorem we can replace $\int d\mathbb{F}$ by $\mathbb{F}$ at point Q. We can now use the fact that



$\uparrow \mathbb{T}$. Since $\mathbb{M}_{B^*} = \mathbb{M}_Q + R\mathbf{m}_1 \times \mathbb{F}$

but by Varignon's Theorem $\mathbb{M}_Q = 0 \Rightarrow \mathbb{M}_{B^*} = \mathbb{T} = R\mathbf{m}_1 \times \mathbb{F}$

$$\therefore\ \mathbb{T} = \frac{GmM}{2L\sin\psi}\left\{\frac{1+L/R\cos\psi}{\left[1+2(L/R)\cos\psi+(L/R)^2\right]^{1/2}} - \frac{(1-L/R\cos\psi)}{\left[1-2(L/R)\cos\psi+(L/R)^2\right]^{1/2}}\right\}\mathbf{m}_1\times\mathbf{m}_2$$

now $\lim\limits_{L/R\to 0} \mathbb{F} = \dfrac{GmM}{2LR}\left\{\dfrac{2L}{R}\mathbf{m}_1 + \left(\dfrac{1}{1}-\dfrac{1}{1}\right)\dfrac{\mathbf{m}_2}{\sin\psi}\right\} = \dfrac{GmM}{R^2}\mathbf{m}_1$

$\lim\limits_{L/R\to 0} \mathbb{T}$ is harder $\therefore$ if $L/R \ll 1$ $\left[1+2(L/R)\cos\psi+(L/R)^2\right]^{-1/2} = 1-(L/R)\cos\psi - \frac{1}{2}\left(\frac{L}{R}\right)^2 + \frac{3}{8}(2xc+x^2)^2$

$\left[1-2(L/R)\cos\psi+(L/R)^2\right]^{-1/2} = 1+(L/R)\cos\psi - \frac{1}{2}(L/R)^2 + \frac{3}{8}(2xc+x^2)^2$

Let $L/R = x$ $\cos\psi = c$,

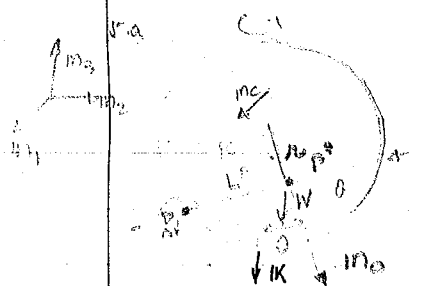
add up   add up

$\therefore$ we get $\dfrac{GmM}{2L\sin\psi}\mathbf{m}_1\times\mathbf{m}_2\Big\{1 - xc - \frac{1}{2}x^2 + xc - x^2c^2 + \frac{1}{2}x^3c\ldots - \left[1 + xc - \frac{1}{2}x^2 - xc - x^2c^2 + \frac{1}{2}x^3c\ldots\right]\Big\}$

$+\frac{3}{8}\left[4x^2c^2 + 4x^3c + x^4\right] + \ldots - \frac{3}{8}\left[4x^2c^2 - 4x^3c + x^4\right] + \frac{3}{8}\left[4x^3c^3 + 4x^4c^2 + x^5c\right]$

add up $+\frac{3}{8}\left[4x^3c^3 - 4x^4c^2 + x^5c\right]$

$$\therefore\ \mathbb{T} \sim \frac{GmM}{2L\sin\psi}\mathbf{m}_1\times\mathbf{m}_2\left(3x^3c - x^3c\right) = 2x^3c\,\frac{GmM\,\mathbf{m}_1\times\mathbf{m}_2}{2L\sin\psi} = \frac{L^2GmM}{R^3}\frac{\cos\psi}{\sin\psi}\mathbf{m}_1\times\mathbf{m}_2$$

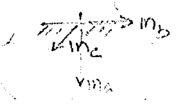


$${}^{R}\omega^{wire} = \Omega\, n_3 = -\Omega\, \mathbb{K}$$

$${}^{wire}\omega^{Rod} = \dot\theta\, n_c$$

$${}^{R}\omega^{Rod} = \dot\theta\, n_c - \Omega\, \mathbb{K}$$

fixed vectors in the plane of wire



$n_c = \cos\Omega t\, n_1 + \sin\Omega t$

$n_b = \cos\Omega t\, n_2 - \sin\Omega t\, n$

since it is smooth velocity is $\perp$ to force & contributes nothing. Only force is body force

By Section 3.5 $\quad (F_r)_G = mg\underset{\sim}{k}\cdot \tilde{\mathbb{V}}_{\dot q_r}^{p^*}$ where $\tilde{\mathbb{V}}_{\dot q_r}^{p^*}$ is the partial velocity of center of mass. velocity of mass center is $\mathbb{V}^{p^*} = {}^{R}\omega^{Rod} \times \mathbb{r}$ where ${}^{R}\omega^{rod} = \dot\theta\, n_c - \Omega$

$$\mathbb{r} = (R^2 - L^2)^{1/2}\, n_0 \qquad n_0 = \cos\theta\, \mathbb{K} + \sin\theta\,[\overbrace{\cos\Omega t\, n_2 + \sin\Omega t\, n_1}^{n_b}]$$

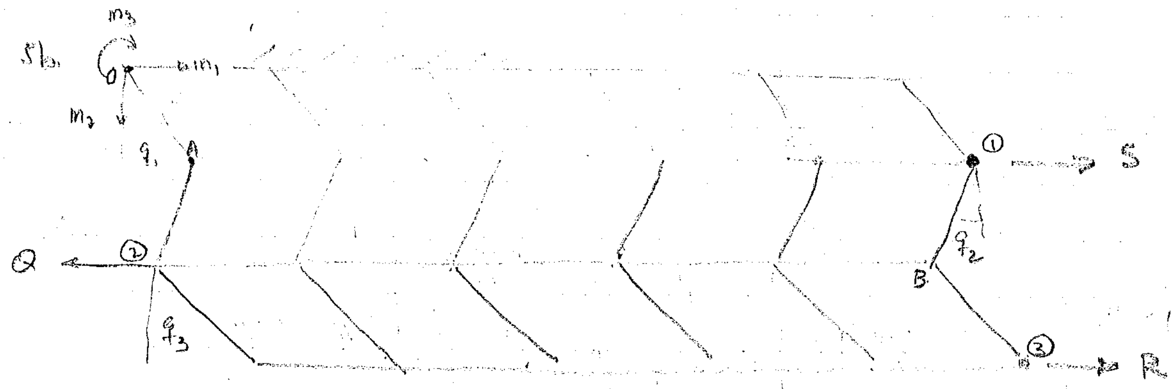
$$\tilde{\mathbb{V}}_{\dot\theta}^{p^*} = (R^2 - L^2)^{1/2}\left\{\cos\theta\, n_b - \sin\theta\, \mathbb{K}\right\} = \cos\theta\, \mathbb{K} + \sin\theta\, n_b$$

$$(F_1)_G = mg\mathbb{K}\cdot \tilde{\mathbb{V}}_{\dot\theta}^{p^*} = -W(R^2 - L^2)^{1/2}\sin\theta$$

$$n_0 = c\mathbb{K} + s\, n_b$$

$$\omega\times\mathbb{r} = (R^2 - L^2)^{1/2}\left[\dot\theta\, n_c\times n_0 - \Omega\, \mathbb{K}\times n_0\right] \qquad n_c\times n_0 = \cos\theta\, n_b - \sin\theta\, n_a \quad n_a = \mathbb{K}$$

$$\mathbb{V}^{p^*} = (R^2 - L^2)^{1/2}\left[\dot\theta\cos\theta\, n_b - \dot\theta\sin\theta\, \mathbb{K} - \Omega s\, n_c\right] \qquad \mathbb{K}\times n_0 = s\, n_c$$



$${}^{R}\mathbb{V}^{①} = \omega\times\mathbb{r} \quad \text{where} \quad \omega^{①} = -\dot q_1 n_3 \quad \mathbb{r} = (5L + L\sin q_1)\, n_1 + (L\cos q_1)\, n_2$$

$${}^{R}\mathbb{V}^{②} = \mathbb{V}^{A} + {}^{①}\omega^{②}\times\mathbb{r}^{②/A} \qquad {}^{①}\omega^{②} = \dot q_2 n_3 \qquad \mathbb{V}^{A} = \omega^{①}\times\mathbb{r}^{A} = -\dot q_1 n_3\times[L\cos q_1\, n_2 + L\sin q_1\, n_1]$$

$$\mathbb{r}^{②/A} = L\cos q_2\, n_2 - L\sin q_2\, n_1 \;;\; {}^{R}\mathbb{V}^{②} = [+\dot q_1 L\cos q_1\, n_1 - \dot q_1 L\sin q_1\, n_2] + [-\dot q_2 L\cos q_2\, n_1 - \dot q_2 L\sin$$

$${}^{R}\mathbb{V}^{②} = \dot q_1 (L\cos q_1\, n_1 - L\sin q_1\, n_2) - \dot q_2 (L\cos q_2\, n_1 + L\sin q_2\, n_2)$$

$${}^{R}\mathbb{V}^{①} = +\dot q_1\left[-(5L + L\sin q_1)\, n_2 + L\cos q_1\, n_1\right]$$

$${}^{R}\mathbb{V}^{③} = {}^{R}\mathbb{V}^{B} + {}^{③}\omega^{②}\times\mathbb{r}^{③/B} \;;\; {}^{R}\mathbb{V}^{B} = {}^{R}\mathbb{V}^{①} + {}^{①}\omega^{②}\times\mathbb{r}^{B/①} = {}^{R}\mathbb{V}^{①} + {}^{①}\omega^{②}\times\mathbb{r}^{②/A} \;;\; {}^{③}\omega^{②} = -\dot q_3 n_3 \;\&\; \mathbb{r}^{③/B} = L\cos q_3\, n_2 + L\sin q_3$$

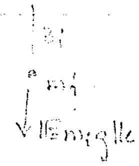
$$= \dot q_1\left[-(5L + L\sin q_1)\, n_2 + L\cos q_1\, n_1\right] - \dot q_2 (L\cos q_2\, n_1 + L\sin q_2\, n_2) + \dot q_3 (L\cos q_3\, n_1 - L\sin q_3\, n_2)$$

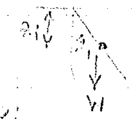
$\therefore$ generalized forces due to contact forces $S n_1$, $-Q n_1$, $R n_1$

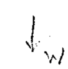
$$F_1 = SL\cos q_1 - QL\cos q_1 + RL\cos q_1 = \sum \mathbb{V}_{\dot q_i}^{①}\cdot \text{contact forces}$$

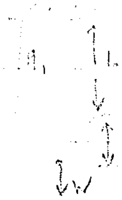
$$F_2 = S\cdot 0 + QL\cos q_2 - RL\cos q_2 \quad ; \quad F_3 = S\cdot 0 + Q\cdot 0 + RL\cos q_3$$

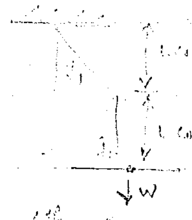
To get the generalized force due to the body forces, $(F_i)_G = \frac{\partial}{\partial q_i}\left(\sum_{i=1}^{N} m g z_i\right)$

where 

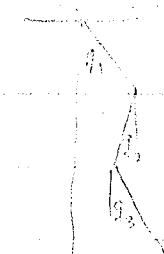
for first 6 rods  $z_i = \frac{L}{2}\cos q_1$

for 2nd 6 rods  $z_i = L\cos q_1$

for 3rd 6 rods  $\therefore z_i = L\left[\cos q_1 + \frac{\cos q_2}{2}\right]$

for 4th 6 rods  $z_i = L\left[\cos q_1 + \cos q_2\right]$

for 5th 6 rods  $z_i = L\left[\cos q_1 + \cos q_2 + \frac{\cos q_3}{2}\right]$

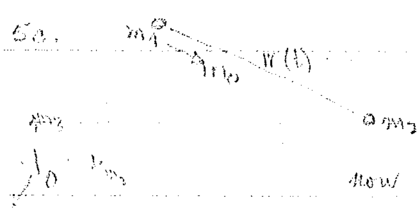
for 6th 6 rods  $z_i = L\left[\cos q_1 + \cos q_2 + \cos q_3\right]$

$$\therefore \sum m_i g z_i = WL\left\{\frac{6}{2}\cos q_1 + 5\cos q_1 + 6\left(\cos q_1 + \frac{\cos q_2}{2}\right) + 5(\cos q_1 + \cos q_2) + 6\left(\cos q_1 + \cos q_2 + \frac{\cos q_3}{2}\right) + 5(\cos q_1 + \cos q_2 + \cos q_3)\right\} = WL\left\{30\cos q_1 + 19\cos q_2 + 8\cos q_3\right\}$$

$$\therefore (F_1)_G = \frac{\partial}{\partial q_1} 30WL\cos q_1 = -30WL\sin q_1 \;;\; (F_2)_G = -19WL\sin q_2 \;;\; (F_3)_G = -8WL\sin q_3$$

$$\therefore F_1 = F_{1_C} + F_{1_G} = L\cos q_1(S - Q + R) - 30WL\sin q_1 \qquad F_3 = F_{3_C} + F_{3_G} = RL\cos q_3 - 8WL\sin q_3$$

$$F_2 = F_{2_C} + F_{2_G} = L\cos q_2(Q - R) - 19WL\sin q_2$$

5a. 

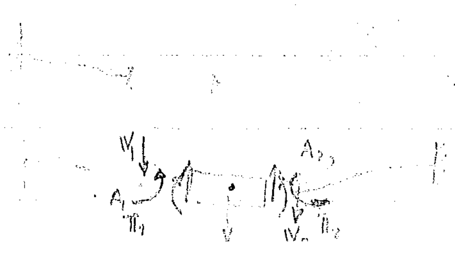
$\mathbb{R}^{m_1} = x\mathbb{n}_1 + y\mathbb{n}_2 + z\mathbb{n}_3 \qquad \mathbb{V}^{m_1} = \dot{x}\mathbb{n}_1 + \dot{y}\mathbb{n}_2 + \dot{z}\mathbb{n}_3$

$\mathbb{R}^{m_2} = \mathbb{R}^{m_1} + r\mathbb{n}_0 \qquad \mathbb{V}^{m_2} = \mathbb{V}^{m_1} + \dot{r}\mathbb{n}_0 + r\frac{d\mathbb{n}_0}{dt}$

now $\mathbb{F}^{m_1} = \frac{G m_1 m_2}{r^2}\mathbb{n}_0 \qquad \mathbb{F}^{m_2} = -\frac{G m_1 m_2}{r^2}\mathbb{n}_0$

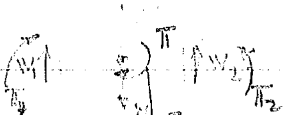
$$\therefore (F_r)_G = \tilde{\mathbb{V}}_r^{m_1}\cdot\mathbb{F}^{m_1} + \tilde{\mathbb{V}}_r^{m_2}\cdot\mathbb{F}^{m_2} \;;\; \tilde{\mathbb{V}}_r^{m_1} = 0 \quad \tilde{\mathbb{V}}_r^{m_2} = \mathbb{n}_0$$

$$= 0\cdot\frac{G m_1 m_2}{r^2}\mathbb{n}_0 - \frac{G m_1 m_2}{r^2}\mathbb{n}_0\cdot\mathbb{n}_0 = -\frac{G m_1 m_2}{r^2}$$

5b. 

${}^B\mathbb{V}^{A_1} = \dot{q}_1\mathbb{n}_2 \qquad {}^B\mathbb{V}^{A_2} = \dot{q}_2\mathbb{n}_2 \qquad \omega^B = \frac{(-\dot{q}_1 + \dot{q}_2)}{2W}\mathbb{n}_3$

$\mathbb{V}^{B^*} = \mathbb{V}^{A_1} + \omega^B \times W\mathbb{n}_1 = \frac{\dot{q}_2 + \dot{q}_1}{2}\mathbb{n}_2$

replace the load cycle with moment & force at mass center.  $\mathbb{R} = W m_2 - \mathbb{V}_1 - \mathbb{V}_2$

$\mathbb{T} = \mathbb{T}_1 + \mathbb{V}_1 w - \mathbb{V}_2 w - \mathbb{T}_2$

Thus $F_1 = \mathbb{V}^{B^*}_{\dot q_1} \cdot \mathbb{R} + \omega^{B}_{\dot q_1} \cdot \mathbb{T}$

$EI w^{IV} = 0 \quad w(0) = 0 \quad w'(0) = 0; \quad EI w'''(L) = V_1 \quad -EI w''(L) = T_1$

$w = Ax^3 + Bx^2 + Cx + D$

$w(0) = 0 \Rightarrow D = 0 \qquad w'''(L) = 6A = V_1/EI \qquad A = V_1/6EI$

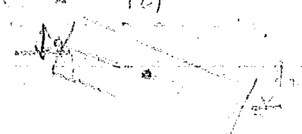
$w'(0) = 0 \Rightarrow C = 0 \qquad w''(L) = 6AL + 2B = -T_1/EI \qquad B = -\frac{T_1}{2EI} - \frac{V_1 L}{2EI}$

$\therefore \quad w = \frac{V_1}{6EI} x^3 - \left(\frac{T_1}{2EI} + \frac{V_1 L}{2EI}\right) x^2 \qquad w(L) = q_1 = \frac{V_1 L^3}{6EI} - \left(\frac{T_1}{2EI} + \frac{V_1 L}{2EI}\right) L^2 = -\frac{V_1 L^3}{3EI} - \frac{T_1 L^2}{2EI}$

$w' = \frac{V_1}{2EI} x^2 - \left(\frac{2T_1}{2EI} + \frac{2V_1 L}{2EI}\right) x \qquad w'(L) = \theta_1 = \frac{V_1 L^2}{2EI} - \frac{2T_1 L}{2EI} - \frac{2V_1 L^2}{2EI} = -\frac{V_1 L^2}{2EI} - \frac{T_1 L}{EI}$

$(L\theta_1 - 2q_1) = \frac{V_1 L^3}{6EI} \quad \therefore \quad V_1 = \frac{6EI}{L^3}(L\theta_1 - 2q_1) \qquad T_1 = \frac{12EI}{L^2}\left(q_1/2 - \frac{L\theta_1}{3}\right)$

similarly $V_2 = \frac{6EI}{L^3}(L\theta_2 - 2q_2) \qquad T_2 = \frac{12EI}{L^2}\left(q_2/2 - \frac{L\theta_2}{3}\right)$

but  $\theta_2 = \frac{q_2 - q_1}{2w} \qquad \theta_2 = -\theta_1$

$\therefore \quad \therefore \quad V_2 = \frac{6EI}{L^3}\left[L \frac{q_2 - q_1}{2w} - 2q_2\right] \qquad T_2 = \frac{12EI}{L^3}\left[q_2/2 + \frac{L}{3}\frac{q_2 - q_1}{2w}\right]$

$\mathbb{V}_2 = V_2 m_2 \qquad \mathbb{T}_2 = T_2 m_3 \qquad \mathbb{V}_1 = V_1 m_2 \qquad \mathbb{T}_1 = -T_1 m_3$

$\mathbb{V}^{B^*}_{\dot q_1} = \frac{m_2}{2} \qquad \mathbb{V}^{B^*}_{\dot q_2} = \frac{m_2}{2} \qquad \omega^{B}_{\dot q_1} = \frac{-m_3}{2w} \qquad \omega^{B}_{\dot q_2} = \frac{m_3}{2w}$

$\mathbb{T} = \left[(\mathbb{T}_1 - \mathbb{T}_2) + (\mathbb{V}_1 - \mathbb{V}_2) w\right] m_3 = \left[\frac{12EI}{L^2}\left(\frac{q_1 - q_2}{2} - \frac{L}{3}\frac{q_2 - q_1}{w}\right) + \frac{12EI}{L^3} w \left(L \frac{q_2 - q_1}{2w} - 2(q_1 - q_2)\right)\right] m_3$

$\mathbb{R} = \left[W m_2 - (\mathbb{V}_1 + \mathbb{V}_2)\right] \cdot W m_2 \quad \frac{6EI}{L^3}\left[L \cdot 0 + 2q_1 - 2q_2\right] m_2 = \left[W + \frac{12EI}{L^3}(q_1 + q_2)\right] m_2$

Thus $\mathbb{T} = \frac{12EI}{L^2}\left[-\frac{L}{3}\frac{q_2 - q_1}{w} - \frac{w}{L}(q_1 - q_2)\right] m_3$

$\therefore \quad F_1 = \frac{W}{2} + \frac{12EI}{L^3}\frac{q_1 + q_2}{2} + \frac{12EI}{L^2}\left(\frac{L}{6}\frac{q_2 - q_1}{w^2} + \frac{1}{2L}(q_1 - q_2)\right)$

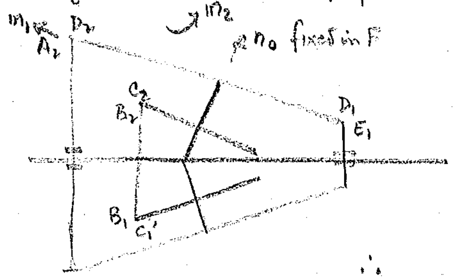
$= \frac{W}{2} + \frac{12EI}{L^3} q_1 \left[1 - \frac{L^2}{6w^2}\right] + \frac{12EI}{L^3} q_2 \left[\frac{L^2}{6w^2}\right]$

← there's a mistake perhaps in defn. of θ_1 & θ_2.

$F_2 = \frac{W}{2} + \frac{12EI}{L^3}\left(\frac{q_1 + q_2}{2}\right) + \frac{12EI}{L^2}\left[-\frac{L}{6}\frac{(q_2 - q_1)}{w^2} - \frac{1}{2L}(q_1 - q_2)\right]$

$= \frac{W}{2} + \frac{12EI}{L^3} q_1 \left(\frac{L^2}{6w^2}\right) + \frac{12EI}{L^3} q_2 \left(1 - \frac{L^2}{6w^2}\right)$

Problem 5g $\qquad m_1$ fixed in F



Given $\mathbb{T}^B = T_b\, n \qquad \mathbb{T}^E = T_e\, n$

$${}^A\omega^B = \dot q_1\, n$$

In a suitable reference frame we have rolling contact which contribute nothing to F_1

$\therefore \quad {}^F\mathbb{V}^{A_2} = {}^F\mathbb{V}^{D_2} \Rightarrow$ if ${}^F\omega^A = {}^F\omega^A n$ then ${}^F\omega^A a = {}^F\omega^D d$

or ${}^F\omega^A = {}^F\omega^D \frac{d}{a} n$

also $\quad {}^F\omega^B = {}^F\omega^A + {}^A\omega^B = \left({}^F\omega^D \frac{d}{a} + \dot q_1\right) n = {}^F\omega^B n \qquad (1)$

but $\quad {}^F\mathbb{V}^{B_2} = {}^F\mathbb{V}^{B_2} \Rightarrow {}^F\omega^B b = {}^F\omega^C c \quad$ but ${}^F\omega^C = {}^F\omega^D \quad \therefore \quad {}^F\omega^D = {}^F\omega^B \frac{b}{c} \quad (2)$

or $\quad {}^F\omega^D \frac{d}{a} + \dot q_1 = \frac{c}{b}\, {}^F\omega^D \quad$ from (1) & (2) $\Rightarrow \boxed{{}^F\omega^D = \frac{ab}{ac - bd}\dot q_1} = -60\dot q_1$

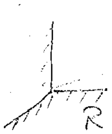
but $\quad {}^F\mathbb{V}^{D_1} = {}^F\mathbb{V}^{E_1} \Rightarrow -{}^F\omega^D d = {}^F\omega^E e \quad$ or ${}^F\omega^E = -{}^F\omega^D \frac{d}{e} = 183\dot q_1 = -(-60\dot q_1)\left(\frac{61}{20}\right)$

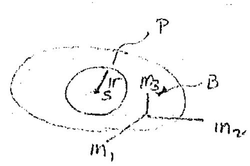
${}^F\omega^E = 183\dot q_1 n \qquad$ also $\quad {}^F\omega^A = {}^F\omega^D \frac{d}{a} n = -60\dot q_1 \left(\frac{61}{60}\right) n = -61\dot q_1 n \Rightarrow {}^A\omega^F = 61\dot q_1 n$

and $\quad {}^A\omega^E = {}^A\omega^F + {}^F\omega^E = (183 + 61)\dot q_1 n = 244\dot q_1 n$

Now $\qquad F_1 = {}^A\omega^B_{\dot q_1} \cdot \mathbb{T}^B + {}^A\omega^E_{\dot q_1} \cdot \mathbb{T}^E = 1 \cdot T_b + 244 \cdot T_e$

5h.





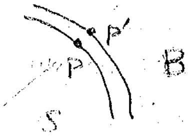
Given S is mass center of B and also that of S
9 degrees of freedom ie 6 to describe motion of B
6 to describe motion of S but 3 constraint eqns of motion of S in B.

Given $\quad {}^R\omega^B = u_1 m_1 + u_2 m_2 + u_3 m_3$
$\quad {}^R\omega^S = u_4 m_1 + u_5 m_2 + u_6 m_3 \quad$ where m_1, m_2, m_3 are unit vectors fixed in B
$\quad {}^R\mathbb{V}^{S^*} = u_7 m_1 + u_8 m_2 + u_9 m_3$

if $d\mathbb{F}^S = -c\,{}^B\mathbb{V}^P dA$ is force exerted by fluid on S and $d\mathbb{F}^B = -d\mathbb{F}^S$ find $F_2, \dots$

now if P is a point on S & P′ is a point on B then

$${}^R\mathbb{V}^P = {}^R\mathbb{V}^{S^*} + {}^S\mathbb{V}^P + {}^R\omega^S \times \mathbb{r}$$
$$= {}^R\mathbb{V}^{S^*} + {}^B\mathbb{V}^P + {}^R\omega^B \times \mathbb{r}, \quad \text{thus } {}^B\mathbb{V}^P = {}^S\mathbb{V}^P + {}^B\omega^S \times \mathbb{r}$$



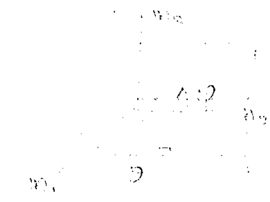
Now $\quad F_i = \int {}^R\mathbb{V}^P_{u_i} \cdot d\mathbb{F}^S + {}^R\mathbb{V}^{P'}_{u_i} \cdot d\mathbb{F}^B = \int \left({}^R\mathbb{V}^P_{u_i} - {}^R\mathbb{V}^{P'}_{u_i}\right) \cdot d\mathbb{F}^S$

but $\quad {}^R\mathbb{V}^{P'} = {}^R\mathbb{V}^{S^*} + {}^B\mathbb{V}^{P'} + {}^R\omega^B \times \mathbb{r} \quad \Rightarrow {}^R\mathbb{V}^P_{u_i} - {}^R\mathbb{V}^{P'}_{u_i} = \left({}^S\mathbb{V}^P_{u_i} - {}^B\mathbb{V}^{P'}_{u_i}\right) + {}^B\omega^S_{u_i} \times \mathbb{r}$

But ${}^S\mathbb{V}^P = {}^B\mathbb{V}^{P'} = 0$ since P & P′ are fixed in S & B respectively

now $d\mathbb{F}^S = -c\,{}^B\mathbb{V}^P dA = -c\,{}^B\omega^S \times \mathbb{r}\, dA$ since ${}^B\mathbb{V}^P = {}^B\mathbb{V}^{S^*} + {}^B\omega^S \times \mathbb{r}$

But $dA = r^2 \cos\varphi\, d\varphi\, d\theta$ if $|\varphi| \le \pi/2$, $0 \le \theta \le 2\pi$



then $F_i = -c\int ({}^B\omega^S_{u_i} \times \mathbb{r}) \cdot ({}^B\omega^S \times \mathbb{r})\, dA$

$${}^B\omega^S = (u_4 - u_1)\mathbb{m}_1 + (u_5 - u_2)\mathbb{m}_2 + (u_6 - u_3)\mathbb{m}_3$$

$$\mathbb{r} = r\sin\varphi\,\mathbb{m}_3 + r\cos\varphi\cos\theta\,\mathbb{m}_1 + r\cos\varphi\sin\theta\,\mathbb{m}_2$$

thus $${}^B\omega^S \times \mathbb{r} = -(u_4 - u_1) r s_\varphi \mathbb{m}_2 + (u_4 - u_1) r c_\varphi s_\theta \mathbb{m}_3 + (u_5 - u_2) r s_\varphi \mathbb{m}_1 - (u_5 - u_2) r c_\varphi c_\theta \mathbb{m}_3$$
$$+ (u_6 - u_3) r c_\varphi c_\theta \mathbb{m}_2 - (u_6 - u_3) r c_\varphi s_\theta \mathbb{m}_1$$

$${}^B\omega^S_{u_2} \times \mathbb{r} = -r s_\varphi \mathbb{m}_1 + r c_\varphi c_\theta \mathbb{m}_3$$
$${}^B\omega^S_{u_4} \times \mathbb{r} = -r s_\varphi \mathbb{m}_2 + r c_\varphi s_\theta \mathbb{m}_3$$

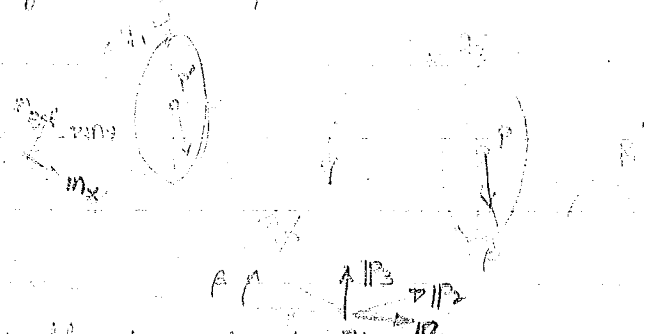
$$\therefore \frac{-F_2}{c} = \left\{ -(u_5 - u_2) r^4 \frac{\sin^3\varphi}{3}\Big|_{-\pi/2}^{\pi/2} \cdot 2\pi - (u_5 - u_2) r^4 \int_{-\pi/2}^{\pi/2} \cos^3\varphi\, d\varphi \int_0^{2\pi} \cos^2\theta\, d\theta \right\} = -\frac{8}{3}\pi r^4 (u_5 - u_2)$$

$$\therefore F_2 = \frac{8}{3}\pi c r^4 (u_5 - u_2)$$

$$\frac{-F_4}{c} = \left\{ (u_4 - u_1) r^4 \frac{\sin^3\varphi}{3}\Big|_{-\pi/2}^{\pi/2} \cdot 2\pi + (u_4 - u_1) r^4 \int_{-\pi/2}^{\pi/2} \cos^3\varphi\, d\varphi \int_0^{2\pi} \sin^2\theta\, d\theta \right\} = \frac{8}{3}\pi r^4 (u_4 - u_1)$$

$$\therefore F_4 = \frac{8}{3}\pi c r^4 (u_1 - u_4).$$

5e. Two sharp edged disks of weight w & disks have radius r connected to shaft of weight 2W, length 2L



Since the disks are sharp edged there are no sidewise motion and hence we have rolling. Rolling contributes nothing to the generalized force. Only forces are the weight

This problem is similar to 3h: $\mathbb{P}_1$

$$\text{Now}\quad \begin{array}{c|ccc} & \mathbb{P}_1 & \mathbb{P}_2 & \mathbb{P}_3 \\ m_x & \cos\beta & 0 & -\sin\beta \\ m_y & 0 & 1 & 0 \\ m_z & \sin\beta & 0 & \cos\beta \end{array}$$

from 3h we had shown that

$${}^{R'}W^{P'} = \tfrac{r}{2}(\dot q_1 + \dot q_2)\sin\phi\, m_x - \tfrac{r}{2}(\dot q_1 + \dot q_2)\cos\phi\, m_y + \tfrac{r}{2}(\dot q_1 - \dot q_2)(-\sin\phi\, m_x + \cos\phi\, m_y) = r\dot q_2 \sin\phi\, m_x - r\dot q_2\cos\phi\, m_y$$

$${}^{R'}W^{P'} = \tfrac{r}{2}(\dot q_1 + \dot q_2)\sin\phi\, m_x - \tfrac{r}{2}(\dot q_1 + \dot q_2)\cos\phi\, m_y - \tfrac{r}{2}(\dot q_1 - \dot q_2)(-\sin\phi\, m_x + \cos\phi$$

and $${}^{R'}W^{Q} = \dot x\, m_x + \dot y\, m_y = \tfrac{r}{2}(\dot q_1 + \dot q_2)\sin\phi\, m_x - \tfrac{r}{2}(\dot q_1 + \dot q_2)\cos\phi\, m_y$$

but $${}^{R'}W^{P} = r\dot q_2(\sin\phi\, m_x - \cos\phi\, m_y) = r\dot q_2(\sin\phi\cos\beta\,\mathbb{P}_1 + \sin\phi\sin\beta\,\mathbb{P}_3 - \cos\phi\,\mathbb{P}_2) = {}^{R}W^{P}$$

$${}^{R'}W^{P'} = r\dot q_1(\sin\phi\, m_x - \cos\phi\, m_y) = r\dot q_1\,(\cancel{\sin\phi\,\mathbb{P}_1 - \cos\phi\cos\beta\,\mathbb{P}_2 + \cos\phi\ldots}) = {}^{R}W^{P'}$$

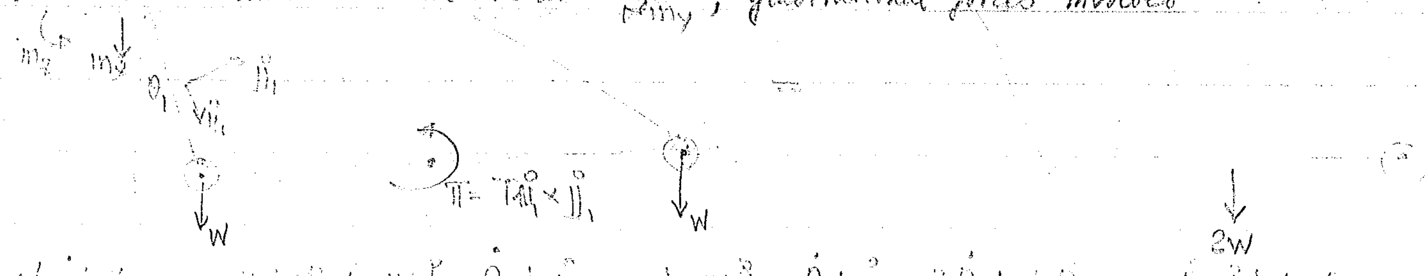
$${}^{R'}W^{Q} = \tfrac{r}{2}\dot q_1(\sin\phi\, m_x - \cos\phi\, m_y) + \tfrac{r}{2}\dot q_2(\sin\phi\, m_x - \cos\phi\, m_y) = \ldots = {}^{R}W^{Q}$$

$$F_1 = -w\mathbb{P}_3\cdot {}^{R'}W^{P}_{\dot q_1} - w\mathbb{P}_3\cdot {}^{R'}W^{P'}_{\dot q_1} - 2W\mathbb{P}_3\cdot {}^{R}W^{Q}_{\dot q_1} = +wr\overset{\sin\phi}{\cancel{\cos\phi}}\sin\beta + Wr\overset{\sin\phi}{\cancel{\cos\phi}}\sin\beta$$

$$= +r(w+W)\overset{\sin\phi}{\cancel{\cos\phi}}\sin\beta$$

$$F_2 = -w\mathbb{P}_3\cdot {}^{R}W^{P}_{\dot q_2} - w\mathbb{P}_3\cdot {}^{R}W^{P'}_{\dot q_2} - 2W\mathbb{P}_3\cdot {}^{R}W^{Q}_{\dot q_2} = +wr\overset{\sin\phi}{\cancel{\cos\phi}}\sin\beta + Wr\overset{\sin\phi}{\cancel{\cos\phi}}\sin\beta = r(w+W)\overset{\sin\phi}{\cancel{\cos\phi}}\sin\beta$$

5f. smooth interactions at hinges pin; gravitational forces involved



we have shown in 4j that $W^{P} = \dot\theta_1 L\, \mathbb{j}_1^{\circ}$ & $W^{Q} = \dot\theta_1 L\mathbb{j}_1^{\circ} - 3\dot\theta_3 L\sin\theta_3 m_x + \dot\theta_3 3L\cos\theta_3 m_y$

$$= \dot\theta_1 L(\cos\theta_1 m_y - \sin\theta_1 m_x) - 3L\dot\theta_3(\sin\theta_3 m_x - \cos\theta_3 m_y)$$

$\mathbb{j}_1^{\circ} = \cos\theta_1 m_y - \sin\theta_1 m_x$ & $\dot{\mathbb{j}}_1 = \mathbb{j}_1^{\circ} \cdot m_z$

$$= L\mathbb{j}_2\,\frac{\sin(\theta_3-\theta_1)}{\sin(\theta_3-\theta_2)}\,\dot\theta_1$$

Now $\omega^T = \dot\theta_3 \mathbf{n}_z$ and since
$$2\cos\theta_2\,\dot\theta_2 = \cos\theta_1\,\dot\theta_1 + 3\cos\theta_3\,\dot\theta_3$$
$$2\sin\theta_2\,\dot\theta_2 = \sin\theta_1\,\dot\theta_1 + 3\sin\theta_3\,\dot\theta_3$$

Then $2\cos\theta_2\cos\theta_3\,\dot\theta_2 + 2\sin\theta_2\sin\theta_3\,\dot\theta_2 = \cos\theta_1\cos\theta_3\,\dot\theta_1 + \sin\theta_1\sin\theta_3\,\dot\theta_1 + 3\dot\theta_3$

$$\frac{1}{3}\left[2\dot\theta_2\left[\cos(\theta_2-\theta_3)\right] - \dot\theta_1\cos(\theta_1-\theta_3)\right] = \dot\theta_3 \quad \text{or since } 2\dot\theta_2 = \frac{\sin(\theta_3-\theta_1)}{\sin(\theta_3-\theta_2)}\dot\theta_1$$

$$\frac{1}{3}\left[\dot\theta_1\frac{\sin(\theta_3-\theta_1)\cos(\theta_2-\theta_3)}{\sin(\theta_3-\theta_2)}\right] - \frac{1}{3}\dot\theta_1\cos(\theta_1-\theta_3) = \dot\theta_3$$

Then
$$F_1 = \mathbf{v}^P_{\dot\theta_1}\cdot W\mathbf{n}_x + \mathbf{v}^Q_{\dot\theta_1}\cdot W\mathbf{n}_x + \omega^T_{\dot\theta_1}\cdot \overline{T}$$
$$= -L\sin\theta_1 W - L\sin\theta_1 W - 3L\sin\theta_3\left\{\frac{\sin(\theta_3-\theta_1)\cos(\theta_2-\theta_3)}{3\sin(\theta_3-\theta_2)} - \frac{1}{3}\cos(\theta_1-\theta_3)\right\}W$$
$$+ T/3\left\{\frac{\sin(\theta_3-\theta_1)\cos(\theta_2-\theta_3)}{\sin(\theta_3-\theta_2)} - \cos(\theta_1-\theta_3)\right\}$$

$$= -2L\sin\theta_1 W - L\sin\theta_3 W\left\{\frac{\sin(\theta_2-\theta_1)}{\sin(\theta_3-\theta_2)}\right\} + T/3\left\{\frac{\sin(\theta_2-\theta_1)}{\sin(\theta_3-\theta_2)}\right\}$$

$$= (T/3)\sin(\theta_2-\theta_1) + WL\,\frac{\left\{2\sin(\theta_2-\theta_3)\sin\theta_1 - \sin(\theta_2-\theta_1)\sin\theta_3\right\}}{\sin(\theta_3-\theta_2)}$$

$$= (T/3)\sin(\theta_2-\theta_1) + WL\,\frac{\left\{\sin(\theta_2-\theta_3)\sin\theta_1 + \left[\sin(\theta_2-\theta_3)\sin\theta_1 - \sin(\theta_2-\theta_1)\sin\theta_3\right]\right\}}{\sin(\theta_3-\theta_2)}$$

Now $\sin\theta_2\cos\theta_3\sin\theta_1 - \cos\theta_2\sin\theta_3\sin\theta_1 - \sin\theta_2\cos\theta_1\sin\theta_3 + \sin\theta_1\cos\theta_2\sin\theta_3 =$
$$\sin\theta_2\left[\sin(\theta_1-\theta_3)\right] = -\sin\theta_2\sin(\theta_3-\theta_1)$$
$$F_1 = (T/3)\sin(\theta_2-\theta_1) + WL\,\frac{\left\{\sin\theta_1\sin(\theta_2-\theta_3) - \sin\theta_2\sin(\theta_3-\theta_1)\right\}}{\sin(\theta_3-\theta_2)} \quad \text{or}$$

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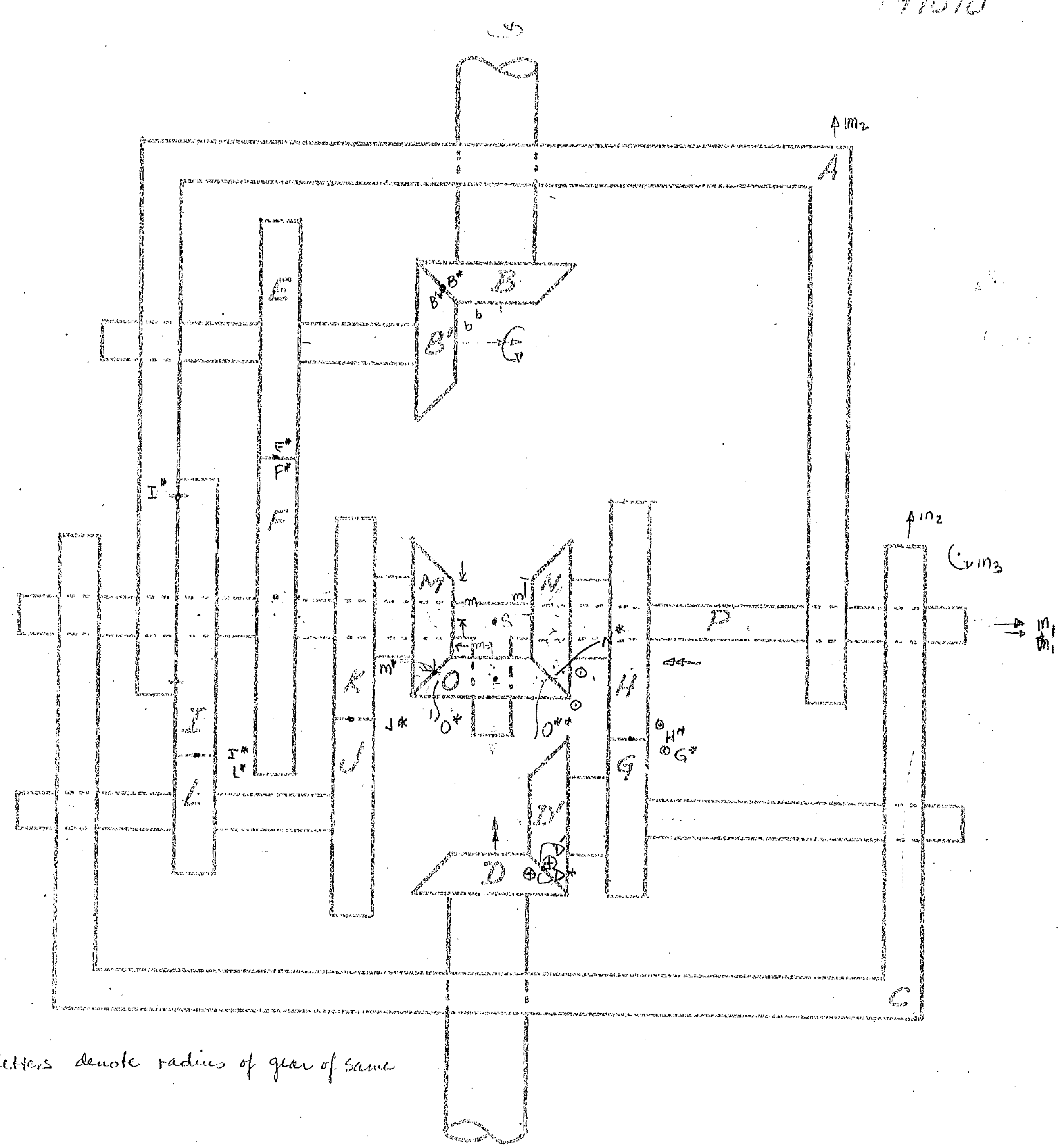


Small letters denote radius of gear of same letter

$b = b'$, $d = d'$, $m = n = 0$

If ij/kl is chosen suitably, then ${}^A\omega^B/{}^C\omega^P$ is independent of ${}^A\omega^C$ for all values of the angle between the axes of B and D.

Determine ij/kl and ${}^A\omega^B/{}^C\omega^P$.

Let A be moving relative to C w/ $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ fixed in A. Notation $\mathbb{V}^{A/B}$ means velocity of pt A wrt pt B.

Let C be fixed w/ $\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3$ fixed in C then ${}^{C}\omega^{A} = {}^{C}\omega^{A}\mathbf{m}_1$

$${}^{A}\omega^{B} = {}^{A}\omega^{B}\mathbf{m}_2 \quad \& \quad {}^{A}\mathbb{V}^{B*} = {}^{A}\omega^{B} \times \mathbf{r}^{B*/B} = +b\,{}^{A}\omega^{B}\mathbf{m}_3 \;;\quad {}^{A}\mathbb{V}^{B*} = {}^{A}\mathbb{V}^{B'} = {}^{A}\omega^{B'} \times \mathbf{r}^{B*/B'} = {}^{A}\omega^{B'}\, b\,\mathbf{m}_3$$

$$\therefore \left|\, {}^{A}\omega^{B'} = {}^{A}\omega^{B} = {}^{A}\omega^{C} \,\right| \;;\quad {}^{A}\omega^{E} = {}^{A}\omega^{B}\mathbf{m}_1 \;;\quad {}^{A}\mathbb{V}^{E*} = {}^{A}\omega^{E} \times \mathbf{r}^{E*/E} = -e\,{}^{A}\omega^{B}\mathbf{m}_3 \;;\quad {}^{A}\mathbb{V}^{F*} = {}^{A}\mathbb{V}^{E*} = {}^{A}\omega^{F} \times \mathbf{r}^{F*/F} = {}^{A}\omega^{F}\cdot f\,\mathbf{m}_3$$

$$\therefore \boxed{{}^{A}\omega^{F} = -{}^{A}\omega^{B}\, e/f = {}^{A}\omega^{P}} \qquad {}^{A}\omega^{P} = {}^{A}\omega^{P}\mathbf{m}_1$$

Now since A & I are pinned ${}^{A}\omega^{I} = 0$ $\therefore$ ${}^{C}\omega^{I} = {}^{C}\omega^{A} = {}^{C}\omega^{A}\mathbf{m}_1$; ${}^{C}\mathbb{V}^{I*} = {}^{C}\omega^{A} \times \mathbf{r}^{I*/I} = -i\,{}^{C}\omega^{A}\mathbf{m}_3$

$${}^{C}\mathbb{V}^{L*} = {}^{C}\mathbb{V}^{I*} = {}^{C}\omega^{L}\, l\,\mathbf{m}_3 \Rightarrow {}^{C}\omega^{L} = -\tfrac{i}{l}\,{}^{C}\omega^{A} \;;\quad {}^{C}\omega^{J} = {}^{C}\omega^{L} = -\tfrac{i}{l}\,\omega\,\mathbf{m}_1 \;;\quad {}^{C}\mathbb{V}^{J*} = {}^{C}\omega^{J} \times \mathbf{r}^{J*/J} = -\tfrac{ij}{l}\,\omega^{CA}\mathbf{m}_3$$

$${}^{C}\mathbb{V}^{J*} = {}^{C}\mathbb{V}^{K*} = {}^{C}\omega^{K} \times \mathbf{r}^{K*/K} = -\omega^{K} k\,\mathbf{m}_3 \quad \therefore \quad \left|\, {}^{C}\omega^{K} = \frac{ij}{kl}\,{}^{C}\omega^{A} = {}^{C}\omega^{m} \,\right| \;;\quad {}^{C}\omega^{m} = {}^{C}\omega^{m}\mathbf{m}_1$$

$${}^{C}\mathbb{V}^{m*} = {}^{C}\omega^{m} \times \mathbf{r}^{m*/m} = -\frac{ij}{kl}\,{}^{C}\omega^{A}\, m\,\mathbf{m}_3 \;;\quad \text{now}\quad {}^{C}\mathbb{V}^{m*} = {}^{P}\mathbb{V}^{m*} + {}^{C}\omega^{P} \times \mathbf{r} = {}^{P}\mathbb{V}^{m*} + \left[{}^{C}\omega^{A} + {}^{A}\omega^{P}\right] \times (-m\,\mathbf{m}_2)$$

if A & C lay flat thus $\boxed{({}^{C}\omega^{A}\mathbf{m}_1 - {}^{A}\omega^{B} e/f\, \mathbf{m}_1)} = {}^{C}\omega^{P} = {}^{C}\omega^{P}\mathbf{m}_1$ $\therefore$ ${}^{P}\mathbb{V}^{m*} = -\frac{ij}{kl}\,{}^{C}\omega^{A} m\,\mathbf{m}_3 + m\,{}^{C}\omega^{P}\mathbf{m}_3 = {}^{P}\mathbb{V}^{O*} = {}^{P}\omega^{O} \times \mathbf{r}^{O*/O}$

Now ${}^{C}\omega^{D} = {}^{C}\omega^{D}\mathbf{m}_2$; ${}^{C}\mathbb{V}^{D*} = -{}^{C}\omega^{D} d\,\mathbf{m}_3 = {}^{C}\mathbb{V}^{D'*} = {}^{C}\omega^{D'} \times \mathbf{r}^{D*/D} = -{}^{C}\omega^{D'} d\,\mathbf{m}_3 \Rightarrow {}^{C}\omega^{D} = {}^{C}\omega^{D'}$

$${}^{C}\omega^{D'} = {}^{C}\omega^{G} = {}^{C}\omega^{D}\mathbf{m}_1 \;;\quad {}^{C}\mathbb{V}^{G*} = {}^{C}\omega^{G} \times \mathbf{r} = +{}^{C}\omega^{D} g\,\mathbf{m}_3 = {}^{C}\mathbb{V}^{H*} = {}^{C}\omega^{H} \times \mathbf{r} = {}^{C}\omega^{H}\mathbf{m}_1 \times (-h\,\mathbf{m}_2) = -h\,{}^{C}\omega^{H}\mathbf{m}_3$$

$${}^{C}\omega^{H} = {}^{C}\omega^{H}\mathbf{m}_1 = -{}^{C}\omega^{D}\frac{h}{g}\mathbf{m}_1 \;;\quad {}^{C}\omega^{H} = {}^{C}\omega^{N} = -{}^{C}\omega^{D}\frac{h}{g}\mathbf{m}_1 \;;\quad {}^{C}\mathbb{V}^{N*} = {}^{C}\omega^{N} \times \mathbf{r}^{N*/N} = -{}^{C}\omega^{D}\frac{h}{g}\mathbf{m}_1 \times -m\,\mathbf{m}_2 = {}^{C}\omega^{D}\frac{hm}{g}\mathbf{m}_3$$

$$\text{Now}\quad {}^{P}\mathbb{V}^{O*} = \left[m\left({}^{C}\omega^{A} - {}^{A}\omega^{B} e/f\right) - \frac{ij}{kl}\,{}^{C}\omega^{A} m\right]\mathbf{m}_3 = {}^{P}\omega^{O}\mathbf{m}_2 \times (-m\,\mathbf{m}_1) = {}^{P}\omega^{O} m\,\mathbf{m}_3$$

$$\therefore \quad {}^{P}\omega^{O} m = \left(m - \frac{ij}{kl}m\right){}^{C}\omega^{A} - \frac{me}{f}\,{}^{A}\omega^{B} \qquad {}^{P}\omega^{O} = \left(1 - \frac{ij}{kl}\right){}^{C}\omega^{A} - \frac{e}{f}\,{}^{A}\omega^{B}$$

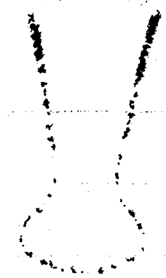
$${}^{C}\mathbb{V}^{N*} = {}^{P}\mathbb{V}^{N*} + {}^{C}\omega^{P} \times \mathbf{r} = {}^{P}\mathbb{V}^{N*} + \left[{}^{C}\omega^{A} + {}^{A}\omega^{P}\right] \times (-m\,\mathbf{m}_2) = {}^{C}\omega^{D}\frac{hm}{g}\mathbf{m}_3$$

$$\therefore \quad {}^{P}\mathbb{V}^{N*} = {}^{C}\omega^{D}\frac{hm}{g}\mathbf{m}_3 + m\left[{}^{C}\omega^{A} - {}^{A}\omega^{B} e/f\right]\mathbf{m}_3 = {}^{P}\mathbb{V}^{O**} = {}^{P}\omega^{O} \times \mathbf{r}^{O**/O} = {}^{P}\omega^{O}\mathbf{m}_2 \times m\,\mathbf{m}_1$$
$$= -{}^{P}\omega^{O} m\,\mathbf{m}_3$$

$$\therefore \quad {}^{C}\omega^{D}\frac{hm}{g}\mathbf{m}_3 + m\left({}^{C}\omega^{A} - {}^{A}\omega^{B} e/f\right)\mathbf{m}_3 = \left[\frac{me}{f}\,{}^{A}\omega^{B} - m\,{}^{C}\omega^{A} + \frac{ijm}{kl}\,\omega^{CA}\right]\mathbf{m}_3$$

$${}^{C}\omega^{D}\frac{hm}{g} + {}^{C}\omega^{A}\left[2m - \frac{ijm}{kl}\right] - 2\frac{me}{f}\,{}^{A}\omega^{B} = 0 \qquad \left|\; {}^{A}\omega^{B}/{}^{C}\omega^{D} = \frac{2eg}{fh} \;\right|$$

$$\frac{ij}{kl} = 2$$



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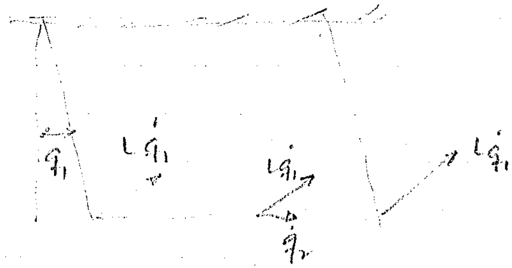
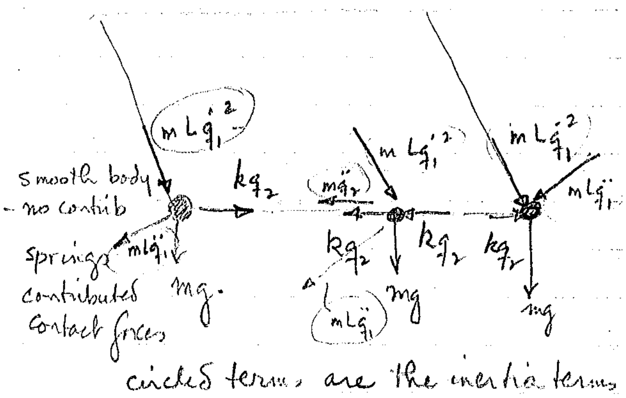
Roger Chen – TA Hrs same as before
Midterm: possibly take-home

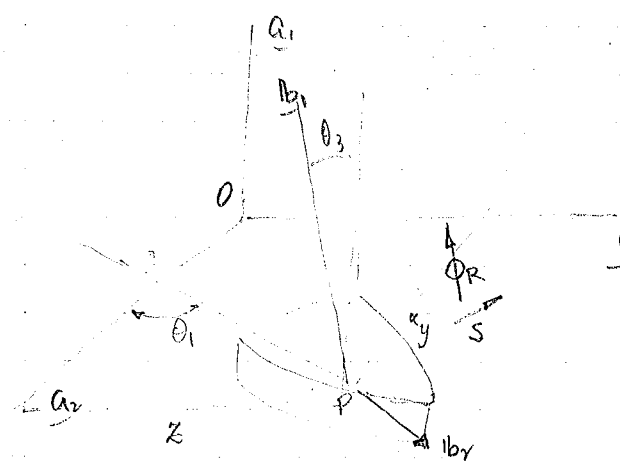
Review of Final:

(1)





$$-3mL^2\ddot{q}_1 - 3mgLs_1 - mLc_1\ddot{q}_2 = 0$$
$$-mL\ddot{q}_1c_1 - m\ddot{q}_2 - 2kq_2 + mL\dot{q}_1^2 s_1 = 0$$



$P = y a_2 + z a_3$
$v = \dot{y} a_2 + \dot{z} a_3$
$v \cdot b_1 = 0 \quad v \cdot b_3 = 0$ constraints

	a_1	a_2	a_3
b_1	c_2	s_1s_2	$-c_1s_2$
b_2	0	c_1	s_1
b_3	s_2	$-s_1c_2$	c_1c_2

the constraint eq.n $\dot{z} = \dot{y}\, s_1/c_1$ from

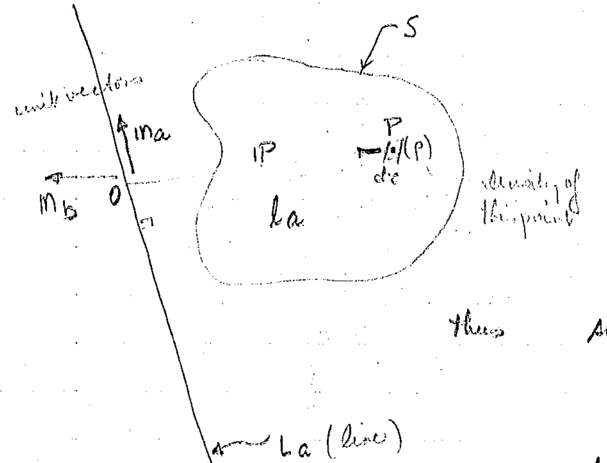
$$\therefore\ v = \dot{q}_3\left[a_2 + (s_1/c_1)a_3\right]$$
$$= (\dot{q}_3/\cos\theta_1)\, b_2$$
$$\omega = \dot{q}_1(c_2 b_1 + s_2 b_3) + \dot{q}_2 b_2$$

$q_1 \triangleq \theta_1,\ q_2 \triangleq \theta_2,\ q_3 \triangleq y_1,\ q_4 \triangleq z$

Continuation from the last regular class
Sums of the form ______ should be studied

$$\sum_{i=1}^{N} m_i r_i \times (n \times r_i) \implies \sum s_i v_i \times (n \times v_i)$$

(unit vector; general scalar; general vector)



$$\mathbb{I}_a^{S/O} = \int \rho\, \mathbb{p} \times (\mathbb{n}_a \times \mathbb{p})\, d\tau \Longleftrightarrow \sum m_i \mathbb{r}_i \times (\mathbb{n} \times \mathbb{r}_i)$$

thus second moment of S (solid) relative to O for $\mathbb{n}_a$

we can define a scalar $I_{ab}^{S/O} \triangleq \mathbb{I}_a^{S/O} \cdot \mathbb{n}_b$

the product of inertia of S relative to O for $\mathbb{n}_a$ & $\mathbb{n}_b$

i's $\quad I_{ab} = I_{ba}$

$$\int \rho\, \mathbb{p} \times (\mathbb{n}_a \times \mathbb{p}) \cdot \mathbb{n}_b\, d\tau = \boxed{I_{ab} \quad \underline{\text{can be negative}}}$$

$$\int \rho (\mathbb{p} \times \mathbb{n}_a) \cdot (\mathbb{p} \times \mathbb{n}_b)\, d\tau = \int \rho\, \mathbb{p} \times (\mathbb{n}_b \times \mathbb{p}) \cdot \mathbb{n}_a\, d\tau = I$$

Now $\quad I_a^{S/O} \triangleq I_{aa}^{S/O} = \mathbb{I}_a^{S/O} \cdot \mathbb{n}_a \quad$ moment of inertia of S about line l_a

it can be shown

$$\boxed{I_a^{S/O} = \int \rho\, l_a^2\, d\tau} \quad \boxed{\underline{\underline{\text{never negative}}}}$$

Example

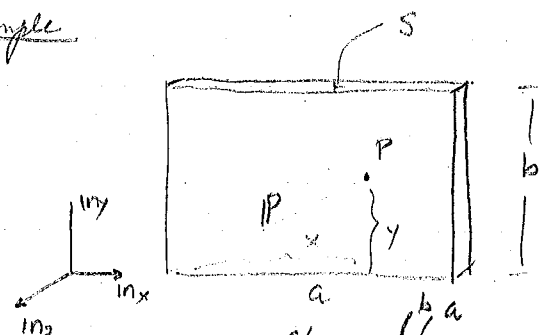


plate has total mass m

homogeneous & uniform

$\rho = \frac{m}{ab} \qquad \mathbb{p} = x\,\mathbb{n}_x + y\,\mathbb{n}_y$

$$I_{xy}^{S/O} = \int_0^b\!\int_0^a \rho (\mathbb{p} \times \mathbb{n}_x) \cdot (\mathbb{p} \times \mathbb{n}_y)\, dx\, dy = \int \rho (\mathbb{p} \times \mathbb{n}_x) \cdot (\mathbb{p} \times \mathbb{n}_y)\, d\tau$$

$$= \frac{m}{ab} \int_0^b \int_0^a (-y\,\mathbb{n}_z) \cdot (x\,\mathbb{n}_z)\, dx\, dy = -\frac{m}{ab} \int_0^b \int_0^a y x\, dx\, dy$$

$$-\frac{m}{ab} \int_0^b y\, dy \int_0^a x\, dx$$

$$-\frac{m}{ab} \left(\frac{y^2}{2} \cdot \frac{x^2}{2}\right) = \frac{a^2 b^2}{4}$$

$$-\frac{m}{ab}$$

thus $I_{xy}^{S/O} = -\frac{mab}{4}$

$$\mathbb{I}_z^{S/O} = \int \rho\, \mathbb{p} \times (\mathbb{n}_z \times \mathbb{p})\, d\tau = \frac{m}{ab}\int_0^b \int_0^a (x^2+y^2)\,dx\,dy\ \mathbb{n}_z = \frac{m}{3}(a^2+b^2)\mathbb{n}_z$$

$$I_x^{S/O} = mb^2/3 \leftarrow I_x^{S/O} = \mathbb{I}_x^{S/O}\cdot \mathbb{n}_x$$

$$\mathbb{I}_x^{S/O} = \int \rho\, \mathbb{p} \times (\mathbb{n}_x \times \mathbb{p})\, d\tau = mb\left(\frac{b}{3}\mathbb{n}_x - \frac{a}{4}\mathbb{n}_y\right)$$

Consequences

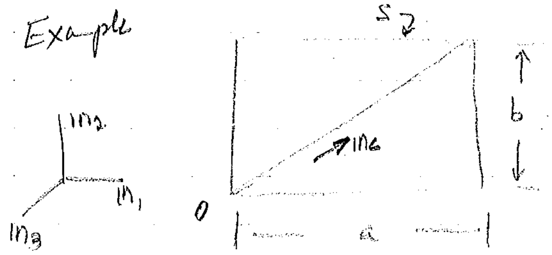
1. $I_{ab}^{S/O} = I_{ba}^{S/O}$

2. I_{ab} can be expressed in terms of I_{jk} where (j,k) refer to any 3 mutually $\perp$ unit vectors $\mathbb{n}_1, \mathbb{n}_2, \mathbb{n}_3$

$$\boxed{I_{ab} = \sum_{j=1}^{3}\sum_{k=1}^{3} a_j I_{jk} b_k} \quad \text{where} \quad \boxed{\begin{aligned} a_i &\triangleq \mathbb{n}_a \cdot \mathbb{n}_i \\ b_i &\triangleq \mathbb{n}_b \cdot \mathbb{n}_i \end{aligned}}$$

What does this mean.

Example



I_{jk}	1	2	3
1	$mb^2/3$	$-mab/4$	0
2	$-mab/4$	$ma^2/3$	0
3	0	0	$m(a^2+b^2)/3$

to find mom of inertia about the diagonal $I_c^{S/O}$ $\quad \tilde{\mathbb{n}}_c = a\mathbb{n}_1 + b\mathbb{n}_2 \quad \mathbb{n}_c = \dfrac{\tilde{\mathbb{n}}_c}{|\tilde{\mathbb{n}}_c|}$

let $c_i \triangleq \mathbb{n}_c \cdot \mathbb{n}_i \qquad c_1 = a(a^2+b^2)^{-1/2} \qquad c_2 = b(a^2+b^2)^{-1/2} \qquad c_3 = 0$

$$I_c^{S/O} = c_1^2 I_{11} + c_2^2 I_{22} + c_3^2 I_{33} + 2(c_1c_2I_{12} + c_2c_3I_{23} + c_3c_1I_{31})$$

$$= \frac{ma^2b^2}{6(a^2+b^2)}$$

1/10/80

1. Matrix Representation
2. Inertia Properties about other points
3. Principal Moments of Inertia

Matrix Representation

Inertia Matrix - imply a particular basis

$$\underline{I} \triangleq \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix}$$

$a \triangleq [a_1 \quad a_2 \quad a_3]$

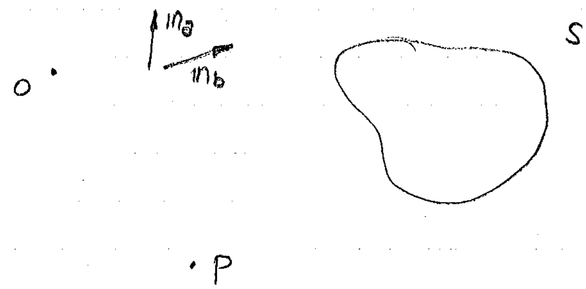
$b \triangleq [b_1 \quad b_2 \quad b_3]$

coeffs of n_a, n_b wrt same basis

$a\underline{I} = [a_1 I_{11} + a_2 I_{21} + a_3 I_{31}, \; \ldots \quad]$ a 1×3 matrix

$a\underline{I}b^T = I_{ab} = [(a_1 I_{11} + a_2 I_{21} + a_3 I_{2})b_1 + \ldots \quad]$ a scalar

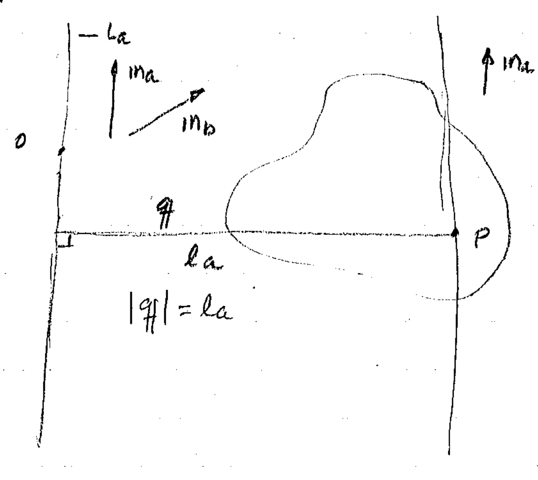
About other points



$$\boxed{I_{ab}^{S/O} = I_{ab}^{S/P} + I_{ab}^{P/O}}$$ (Parallel axis Theorem.)

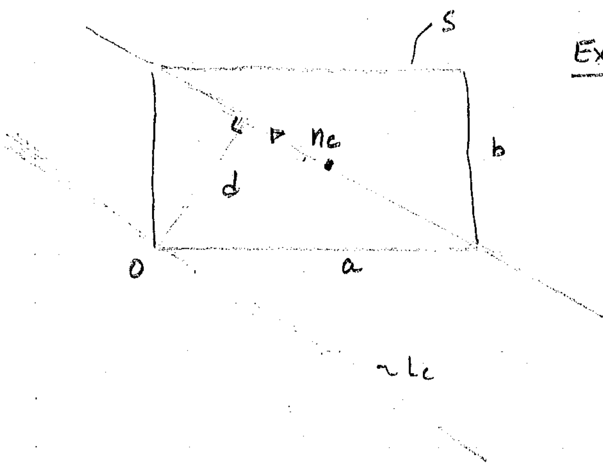
[only when P is at mass center]

$I_{ab}^{P/O}$ is really meaningless (mom. of inertia of a point about another point) unless we define a mass for Point P (the total mass of S). This implies that P must be the mass center.



$$I_{aa}^{P/O} = m l_a^2$$

$I_{ab}^{P/O} = m(q \times n_a)\cdot(q \times n_b)$ operationally effective way to get $I_{ab}^{P/O}$



<u>Example:</u> area of triangle

$$d\sqrt{a^2+b^2} = ab$$

Find $I_c^{S/O}$

$$I_c^{S/O} = I_c^{S/S^*} + I_c^{S^*/O}$$

$$I_c^{S/S^*} = \frac{ma^2b^2}{6(a^2+b^2)}$$

$$I_S^{S^*/O} = md^2 = \frac{m(a^2b^2)}{a^2+b^2}$$

$$I_c^{S/O} = \tfrac{7}{6}\,\frac{ma^2b^2}{a^2+b^2}$$

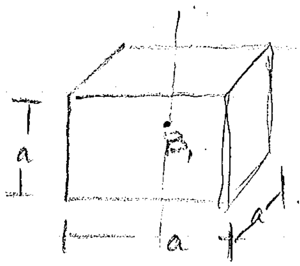
Rotation $\quad I_{ab} = \sum_{j=1}^{3}\sum_{k=1}^{3} a_j I_{jk} a_k$

Translation $\quad I_{ab}^{S/O} = I_{ab}^{S/S^*} + I_{ab}^{S^*/O}$

where $n_a = \Sigma a_j n_j \quad n_b = \Sigma a_k n_k$ will give you all inertia properties for any rotation of axis

Rotate first then translate

<u>Thin walled bodies</u> (not in book)



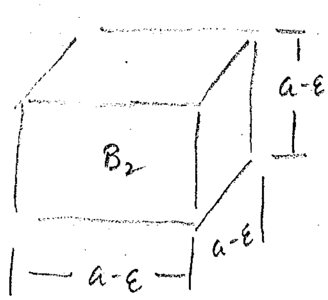
solid cube

$$m_1 = \rho a^3$$

$$I_1 = \frac{\rho a^5}{6}$$ about an axis ∥ to one edge & passing through mass center.

Radius of gyration $k_1^2 = I/m = a^2/6 \cdot \quad k_1 = \frac{a}{\sqrt{6}}$

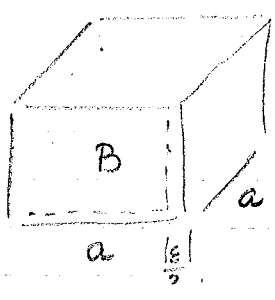
consider a solid cube



solid cube

$$m_2 = \rho(a-\varepsilon)^3$$

$$I_2 = \rho\frac{(a-\varepsilon)^5}{6}$$



thin walled box of thickness $\varepsilon/2$

$$m = \rho\left(a^3 - (a-\varepsilon)^3\right)$$

$$I = I_1 - I_2 = \rho/6\left[a^5 - (a-\varepsilon)^5\right]$$

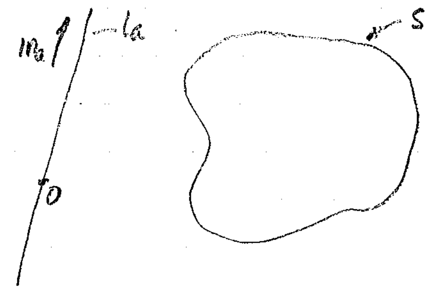
$$k^2 = I/m = \frac{1}{6}\left[\frac{a^5-(a-\varepsilon)^5}{a^3-(a-\varepsilon)^3}\right]$$

$$\lim_{\epsilon \to 0} k^2 = \lim_{\epsilon \to 0} \frac{1}{6}\,\frac{5(a-\epsilon)^4}{3(a-\epsilon)^2} = \frac{5}{18}a^2 \qquad k = \sqrt{\frac{5}{18}}\,a$$

Alternative to dealing with differential; assume $d\rho = 0$

$$dm_1 = 3\rho a^2 da$$
$$dI_1 = \frac{5\rho a^4}{6} da \qquad k^2 = \frac{dI_1}{dm_1} = \frac{5}{18}a^2 \qquad k = \sqrt{\frac{5}{18}}\,a$$

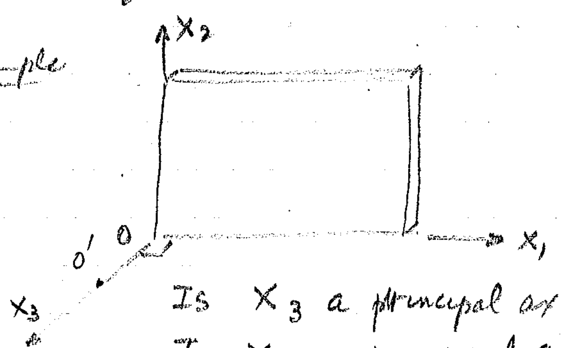
Principal axes & moments of inertia



we defines $\mathbb{I}_a^{S/O}$ (is this $\parallel$ to n_a?)
if $\mathbb{I}_a^{S/O}$ is $\parallel$ to n_a then L_a is called a principal axis of S for O, and $I_a^{S/O}$ is a principal moment of inertia of S for O.

Is "O" important? Can L_a be a principal axis of S for O but not for O'? Yes "O" is important

Example



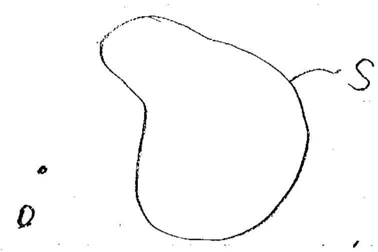
Is X_3 a principal axes of S for O? yes $\mathbb{I}_3^{S/O} = \frac{m}{3}(a^2+b^2)\,n_3$
Is X_1 a principal axis of S for O? no,

$$\mathbb{I}_1^{S/O} = mb\left(\frac{b}{3}n_1 - \frac{a}{4}n_2\right)$$

Is X_2 a principal axis of S for O? no.

Is X_3 a principal axis of S for O'? no.
A line can be a principal axis for a point but not for another point.

Can a line be a principal axis of S for more than one point? Yes



Is there a straightforward procedure to find the principal axis? Do they exist. Yes there must exist at least 3. If there are only 3 then they are mutually $\perp$.

Procedure

Suppose $\quad \mathbb{I}_a^{S/O} = \lambda\, \mathbf{n}_a$

Solve the Char eqn through a determinant

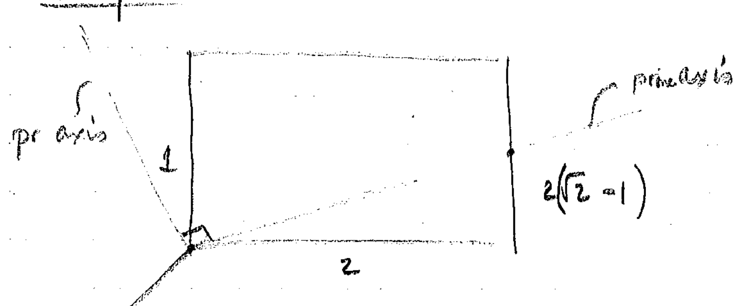
$$\begin{vmatrix} I_{11} - I_a & I_{12} & I_{13} \\ I_{21} & I_{22} - I_a & I_{33} \\ I_{31} & I_{32} & I_{33} - I_a \end{vmatrix} = 0$$

Solve for I_a. Let $\mathbf{n}_a = a_1 \mathbf{n}_1 + a_2 \mathbf{n}_2 + a_3 \mathbf{n}_3 \qquad \sum a_i a_i = 1$ is used to find a_is

plus two of the following eqns.

$$a_1 (I_{11} - I_a) + a_2 I_{13} + a_3 I_{13} = 0$$
$$a_1 I_{21} + a_2 (I_{22} - I_a) + a_3 I_{23} = 0$$
$$a_1 I_{31} + a_2 I_{32} + a_3 (I_{33} - I_a) = 0$$

Example



Why do we care about moments of inertia & principal axis

1. Computational simplification $\quad I_{ab} = \sum_j \sum_k a_j I_{jk} b_k \Rightarrow I_{ab} = \sum_i a_i I_{ii} b_i$

if $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are all $\parallel$ to prin axes

2. Physically significant

3. Have significance w/ reference to max. & min mom. of inertia

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6a. Generalized Force problem

6b. Was discussed in class

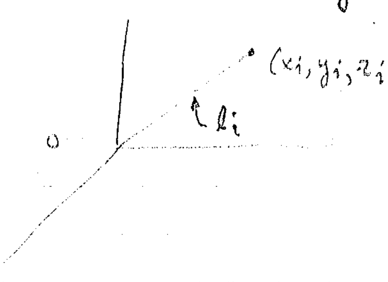
6c. rhs, lhs doesn't matter; if coord system is not mutually ⊥ it makes a difference
define $I_{yz} = -\sum m_i y_i z_i$ from $m_z \cdot \mathbb{I}_y = I_{yz}$ is definition

6d. Figures for which I can exist: Solids, Surfaces, Curves
normally if $I=0 \Rightarrow k=0$

6e. Invariants of 2nd Rank tensors. Use it for checking. Hint use 6(c) eqn

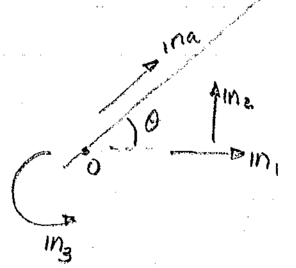
$$A = \sum m_i (y_i^2 + z_i^2) \qquad B = \sum m_i (z_i^2 + x_i^2) \qquad C = \sum (x_i^2 + y_i^2)$$

$$A + B + C = 2 \sum m_i (x_i^2 + y_i^2 + z_i^2) = 2 \sum m_i l_i^2$$



independent of axes since l_i is a distance that doesn't depend on orientation.

6f.



$m_a = \sum m_i a_i$ most general.

$a_1 = \cos\theta = c$.

if we make $a_2 = \sin\theta = s$

$m_a \parallel$ to m_1, m_2: $a_3 = 0$

$$I_a = I_1 a_1^2 + I_2 a_2^2 + I_3 a_3^2 + 2(a_1 a_2 I_{12} + a_2 a_3 I_{23} + a_3 a_1 I_{31})$$

$$I_a = I_1 c^2 + I_2 s^2 + 2sc\, I_{12}$$

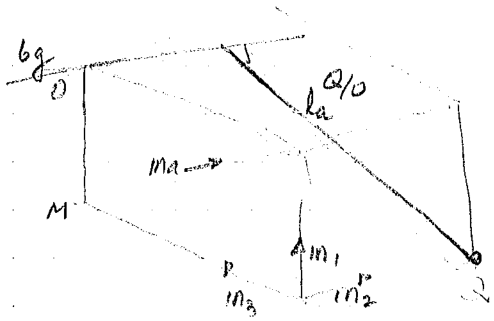
to minim take $\frac{d}{d\theta} I_a = 0$: $\quad -2cs I_1 + 2sc I_2 + 2\cos 2\theta\, I_{12}$

$$(I_2 - I_1)\sin 2\theta + 2\cos 2\theta\, I_{12} = 0 \Rightarrow (\tan 2\theta)^{-1} = \frac{I_1 - I_2}{2 I_{12}}$$

$\left[\left(\frac{I_1 - I_2}{2}\right)^2 + I_{12}^2\right]^{1/2} = \rho$

2θ, I_{12}, $\frac{I_1 - I_2}{2}$

max, min are ⊥ to each other



mass center

$$I_a^M = I_a^Q + m\left(l_a^{Q/M}\right)^2 \qquad I_a^Q = I_a^O - m\left(l_a^{Q/O}\right)^2$$

$$I_a^M = I_a^O + m\left[\left(l_a^{Q/M}\right)^2 - \left(l_a^{Q/O}\right)^2\right]$$

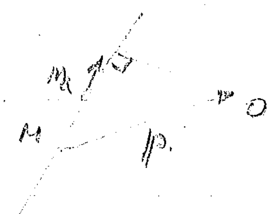
$$I_a^O = \sum\sum a_j \underbrace{I_{jk}^O}_{\text{given}} a_k$$

$a_1 = 3/5 \quad a_2 = 0 \quad a_3 = 4/5$

$m_a = \sum a_i m_i$

$$= \frac{8500}{25}$$

$$\left(l_a^{Q/M}\right)^2 = \left(p^{Q/M} \times m_a\right)^2$$

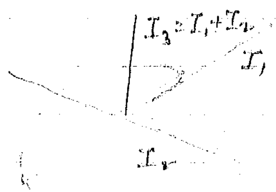


$$= \frac{244}{25}$$

$$\left(l_a^{Q/O}\right)^2 = \frac{676}{25} \qquad I_a = \frac{8500 + 12\,[244 - 676]}{25}$$

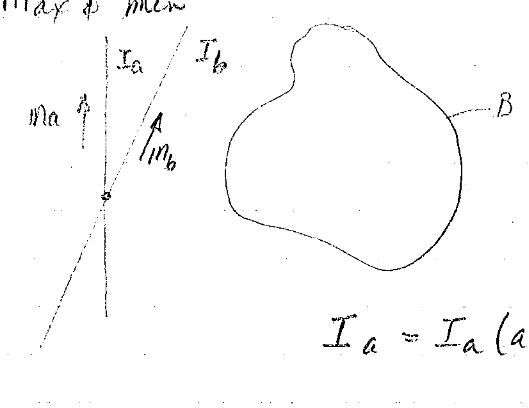
6h Use composite body & parallel axis theorem to get answer

6i Lamina only :



use results of 6c.

Max & min



one will be bigger than the other.
How can we find the line that gives max

How to find Optimum values of moments of inertia

$$I_a = I_a(a_1, a_2, a_3) \qquad m_a = \sum a_i m_i$$

$$I_a(a_1,a_2,a_3) = a_1^2 I_1 + a_2^2 I_2 + a_3^3 I_3 + 2(a_1 a_2 I_{12} + \dots) \qquad (1)$$

$$\text{subject to } a_1^2 + a_2^2 + a_3^2 - 1 = 0 \qquad (2)$$

Determine a_1, a_2, a_3 $\ni$ we optimize I_a

---

generalization: determine a_1, a_2, a_3 $\ni$ we optimize $f(a_1,a_2,a_3)$ subject to constraint $g(a_1,a_2,a_3) = 0$. (3)

let $f \equiv (1)$ $\quad g \equiv (2)$

Mundane way.

Regard a_3 as a fn of a_1, a_2. $\quad g(a_1,a_2,a_3) = \bar{g}(a_1,a_2)$ (4).

and $\bar{g}(a_1,a_2) = 0$. $\quad a_3 = h(a_1,a_2) \Rightarrow g(a_1,a_2,h(a_1,a_2)) = \bar{g}(a_1,a_2)$ (5)

Diff (5) wrt a_1, a_2 letting $\bar{g}_i \triangleq \dfrac{\partial \bar{g}}{\partial a_i}$ $\quad (i=1,2)$ (6)

This gives $\quad \bar{g}_1 \underset{(5)}{=} 0 \qquad \bar{g}_2 \underset{(5)}{=} 0$ (7)

But from (4). $\quad \bar{g}_1 = g_1 + g_3 \dfrac{\partial a_3}{\partial a_1} \underset{(7)}{=} 0 \quad$ & $\quad \bar{g}_2 = g_2 + g_3 \dfrac{\partial a_3}{\partial a_2} \underset{(7)}{=} 0$ (8, 9)

where $g_i \triangleq \dfrac{\partial g}{\partial a_i} \qquad \dfrac{\partial a_3}{\partial a_i} \triangleq \dfrac{\partial h(a_1,a_2)}{\partial a_i}$, $\quad (i=1,2)$ (10)

Hence $\quad \dfrac{\partial a_3}{\partial a_1} \underset{8}{=} -g_1/g_3 \qquad \dfrac{\partial a_3}{\partial a_2} = -g_2/g_3$ (11)

With a_3, regarded as a fn of a_1, a_2

$$f(a_1,a_2,a_3) = \bar{f}(a_1,a_2) \qquad (12)$$

~~$\bar{f}$ is stationary~~ when $\bar{f}_1 = \bar{f}_2 = 0 \qquad \bar{f}_i \triangleq \dfrac{\partial \bar{f}}{\partial a_i}$ (13) (14)

Now $\quad \bar{f}_1 \underset{(12)}{=} f_1 + f_3 \dfrac{\partial a_3}{\partial a_1} \qquad \bar{f}_2 = f_2 + f_3 \dfrac{\partial a_3}{\partial a_2}$ (15)

$$f_i \triangleq \frac{\partial f}{\partial a_i} \qquad (16)$$

Hence f_a is stationary if $\quad f_1 + f_3 \dfrac{\partial a_3}{\partial a_1} = 0 \quad$ & $\quad f_2 + f_3 \dfrac{\partial a_3}{\partial a_2} = 0$ (17)

& a_1, a_2, a_3 are found by solving $\quad f_1 - g_1/g_3 f_3 = 0$ (18)

$f_2 - g_2/g_3 f_3 = 0$ (19)

$g(a_1, a_2, a_3) = 0$ (20)

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Continuation of last class

Lagrange Multipliers Method

Suppose $L(a_1, a_2, a_3) \triangleq f(a_1, a_2, a_3) - \lambda g(a_1, a_2, a_3)$ λ = a scalar (21)

and one seeks a_1, a_2, a_3 and λ $\ni$ $g(a_1, a_2, a_3) = 0$
and that $\frac{\partial L}{\partial a_i} = L_i = 0$ $(i = 1, 2, 3)$

Then $\underset{(21)}{L_i} = \underset{(16)}{f_i} - \lambda \underset{(11)}{g_i}$ $i = 1, 2, 3$ (24)

with $\underline{i = 3}$ $\lambda \underset{(24)}{=} f_3/g_3$ (25)

for $i = 1, 2$ $f_1 - \frac{f_3}{g_3} g_1 = 0$ use w/(24) & (25) (26) = (18)

$f_2 - \frac{f_3}{g_3} g_2 = 0$ (27) = (19)

$\frac{\partial L}{\partial \lambda} = -g(a_1, a_2, a_3) = 0$ (20)

For inertia case

$L = I_a - \lambda (a_1^2 + a_2^2 + a_3^2 - 1)$

$\frac{\partial L}{\partial a_1} = 2a_1 I_1 + 2a_2 I_{12} + 2a_3 I_{13} - \lambda \cdot 2a_1 = 0 \Rightarrow 2(a_1 I_{11} + a_2 I_{12} + a_3 I_{13} - \lambda a_1) = 0$

$\frac{\partial L}{\partial a_2} = 2(a_2 I_{22} + a_3 I_{23} + a_1 I_{31} - \lambda a_2) = 0$

$\frac{\partial L}{\partial a_3} = 2(a_3 I_{33} + a_1 I_{31} + a_2 I_{32} - \lambda a_3) = 0$

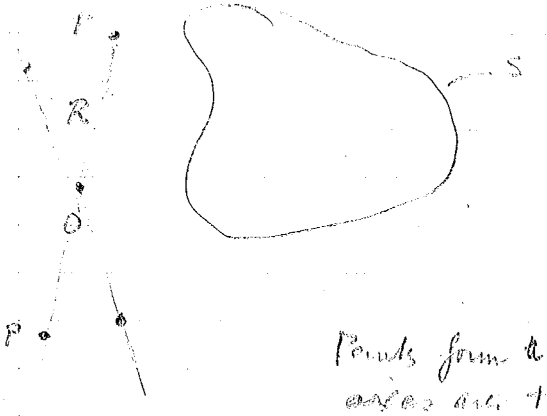
$a_1^2 + a_2^2 + a_3^2 - 1 = 0$

Since a_1, a_2, a_3 cannot be all $= 0$ (cannot satisfy constraint)
then det of $(\underline{I} - \lambda I) = 0$

$$\begin{vmatrix} I_{11} - \lambda & I_{12} & I_{13} \\ I_{21} & I_{22} - \lambda & I_{23} \\ I_{31} & I_{32} & I_{33} - \lambda \end{vmatrix} = 0$$

this λ is the principal moment of inertia — this links optimum mom. of inertia w/ principal moments of inertia!

Inertia Ellipsoid of S for a point O. Another physical method of finding optimum mom.

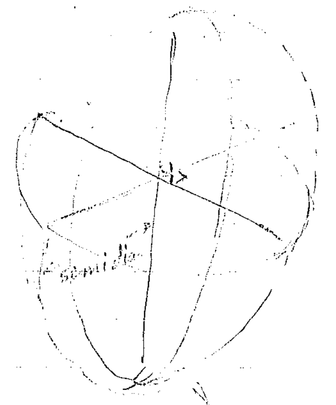


Construct the locus of points P, Q.

$R = C I^{-1/2}$

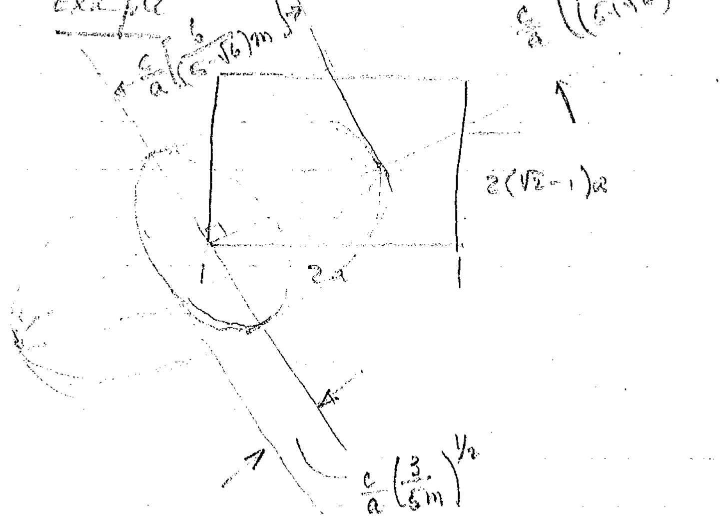
I is moment of inertia of S about line OP.
C is a constant w/ units $M^{1/2}L^2$

Points form a closed surface. This locus is an ellipsoid whose axes are the principal axes of inertia of S for point O.



Ellipsoids have different length principal axes
The semi diameters of ellipsoid have lengths
$cI_1^{-1/2}$, $cI_2^{-1/2}$, $cI_3^{-1/2}$ (I_i are principal moments of inertia).

Example



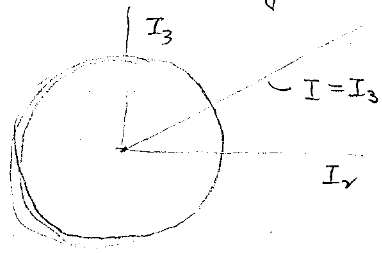
Why are they important

Theorem

1) Axes of min & max mom. of inertia are principal axes

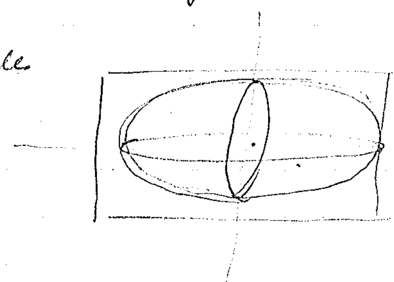
2) If two principal mom of inertia are equal, then the mom of inertia about all axes lying in the plane determined by the assoc principal axes are equal to each other

We have an ellipsoid of revolution there for those two axes since $\mathbb{R}$ is dependent of $I^{-1/2}$ then for any other axis in the plane it will be of same distance R & hence must have the same I



Sometimes true The central inertia ellipsoid (ellipsoid about point center of mass) tends to simulate the body.

Example



Inertia dyadics

$$\mathbb{n}_a = a_1\mathbb{n}_1 + a_2\mathbb{n}_2 + a_3\mathbb{n}_3$$
$$\mathbb{I}_a^{S/O} = a_1\mathbb{I}_1^{S/O} + a_2\mathbb{I}_2^{S/O} + a_3\mathbb{I}_3^{S/O}$$

$$= \mathbb{n}_a\cdot\mathbb{n}_1\,\mathbb{I}_1^{S/O} + \mathbb{n}_a\cdot\mathbb{n}_2\,\mathbb{I}_2^{S/O} + \mathbb{n}_a\cdot\mathbb{n}_3\,\mathbb{I}_3^{S/O}$$
$$= \mathbb{n}_a\cdot\left(\mathbb{n}_1\mathbb{I}_1^{S/O} + \mathbb{n}_2\mathbb{I}_2^{S/O} + \mathbb{n}_3\mathbb{I}_3^{S/O}\right) = \mathbb{n}_a\cdot\mathbb{I}^{S/O}$$

If $\underbrace{\mathbb{I}^{S/O}}_{\text{DYADIC}} \triangleq \left(\mathbb{n}_1\mathbb{I}_1^{S/O} + \mathbb{n}_2\mathbb{I}_2^{S/O} + \mathbb{n}_3\mathbb{I}_3^{S/O}\right)$ and if $\mathbb{n}_a\cdot\mathbb{I}^{S/O} \triangleq \mathbb{n}_a\cdot\left(\mathbb{n}_1\mathbb{I}_1^{S/O} + \mathbb{n}_2\mathbb{I}_2^{S/O} + \mathbb{n}_3\mathbb{I}_3^{S/O}\right)$

then $\mathbb{I}_a^{S/O} = \mathbb{n}_a\cdot\mathbb{I}^{S/O}$

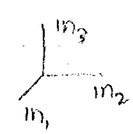
DYADIC – is a quantity that when ~~mult~~ dot mult by a vector gives another vector.

If $\mathbb{D} \triangleq \underbrace{\mathbb{a}_1\mathbb{b}_1}_{\text{DYAD}} + \mathbb{a}_2\mathbb{b}_2 + \cdots \mathbb{a}_n\mathbb{b}_n$ $\qquad \mathbb{V}\cdot\mathbb{D} \triangleq (\mathbb{V}\cdot\mathbb{a}_1)\mathbb{b}_1 + \cdots + (\mathbb{V}\cdot\mathbb{a}_n)\mathbb{b}_n$

DYAD. – juxtaposition of 2 vectors ; Sum of dyads is a dyadic

$$\mathbb{D}\cdot\mathbb{V} \triangleq \mathbb{a}_1(\mathbb{b}_1\cdot\mathbb{V}) + \cdots + \mathbb{a}_n(\mathbb{b}_n\cdot\mathbb{V})$$

Suppose $\bar{U} \triangleq \mathbb{n}_1\mathbb{n}_1 + \mathbb{n}_2\mathbb{n}_2 + \mathbb{n}_3\mathbb{n}_3$



$\mathbb{V}\cdot\bar{U} \triangleq \mathbb{V}$ $\qquad U$ – is the idemfactor (unit dyadic)

$\mathbb{I}$ is very basis dependent form.

$$\mathbb{I} = \mathbb{n}_1\, \mathbb{I}_1 + \mathbb{n}_2\, \mathbb{I}_2 + \mathbb{n}_3\, \mathbb{I}_3 = \mathbb{n}_1 \sum m_i\, \mathbb{p}_i \times (\mathbb{n}_1 \times \mathbb{p}_i) + \mathbb{n}_2 \sum m_i\, \mathbb{p}_i \times (\mathbb{n}_2 \times \mathbb{p}_i)$$

$\mathbb{p}_i^2 \mathbb{n}_1 - (\mathbb{p}_i \cdot \mathbb{n}_1)\mathbb{p}_i$

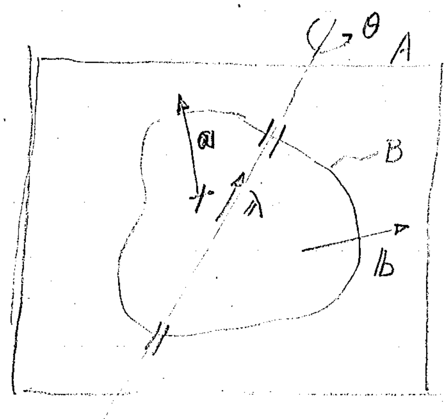
$$+ \mathbb{n}_3 \sum m_i\, \mathbb{p}_i \times (\mathbb{n}_3 \times \mathbb{p}_i).$$

$$= \sum m_i \Big[\mathbb{n}_1 \big(\underline{\mathbb{p}_i^2 \mathbb{n}_1} - \overbrace{(\mathbb{p}_i \cdot \mathbb{n}_1)\mathbb{p}_i}^{(\mathbb{p}_i \cdot \mathbb{n}_1)\mathbb{n}_1 \mathbb{p}_i} \big) + \mathbb{n}_2 \big(\underline{\mathbb{p}_i^2 \mathbb{n}_2} - \overbrace{(\mathbb{p}_i \cdot \mathbb{n}_2)\mathbb{p}_i}^{(\mathbb{p}_i \cdot \mathbb{n}_2)\mathbb{n}_2 \mathbb{p}_i} \big)$$

$$+ \mathbb{n}_3 \big(\underline{\mathbb{p}_i^2 \mathbb{n}_3} - \overbrace{(\mathbb{p}_i \cdot \mathbb{n}_3)\mathbb{p}_i}^{(\mathbb{p}_i \cdot \mathbb{n}_3)\mathbb{n}_3 \mathbb{p}_i} \big) \Big]$$

$$\boxed{\mathbb{I} = \sum m_i \left[\underline{\mathbb{p}_i^2 \bar{U}} - \mathbb{p}_i \mathbb{p}_i \right]}$$

basis independent form of matrix dyadic



rotate B in A : keeping a fixed in B how is b related to a after rotation

$\mathbb{b} = \mathbb{a} \cdot \mathbb{C}$

$\mathbb{C}$ (rotation dyadic) $= \bar{U}\cos\theta - \bar{U} \times \lambda \sin\theta + \lambda\lambda(1 - \cos\theta)$

λ is the unit vector

$(\mathbb{a}\mathbb{b}) \times \mathbb{c} = \mathbb{a}(\mathbb{b} \times \mathbb{c})$

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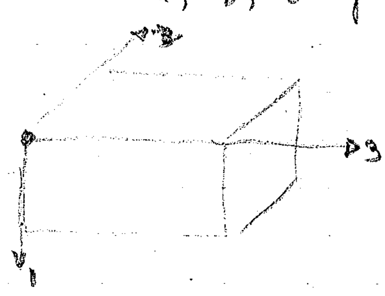
Problems in Set 7   7b, c

Principal Direction - products of inertia are zero, min-max

$\mathbb{I}_a \parallel \mathbb{n}_a$, $\mathbb{I}_a \times \mathbb{n}_a = 0$, $\mathbb{I}_a = \lambda \mathbb{n}_a$

7a. $\parallel$ axis theorem & $\sum$

7b. unique min exist. Since line through mass center gives mins (any other line $= I_{\text{masscenter}} + Ad^2$ [d : $\perp$ distance from mass center line to new line] $> I_{\text{mass centers}}$). Pick lines corresponding to what body shapes find I_{ab}, I_{ac}, I_{bc}, I_a, I_b, I_c using #(6c). This gives the idea of $I_{min} \Rightarrow \det(\mathbb{I}_a - \tilde{I}_a \mathbb{I}) = 0$
$\Rightarrow \tilde{I}_a, \tilde{I}_b, \tilde{I}_c$ gives max, min mom of inertia.



I s/s#

15	1.5	3
1.5	18.75	2
3	2	9.75

$\Rightarrow$ cubic

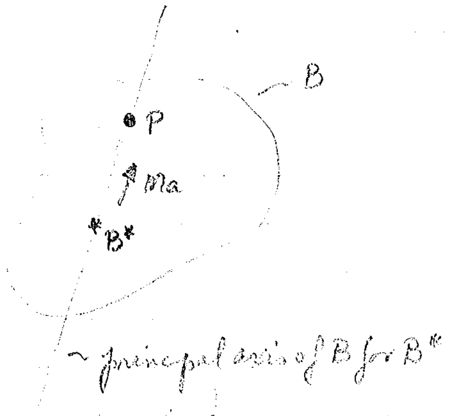
$0 = \lambda^3 - 48.5\lambda^2 + 595.0625\lambda - 2509.$

to find min use min value of diag of first guess - it must be less $I_a = 8.243$

value of I will be less than 8.5

define $I = (\Sigma m) r^2 \qquad r = \sqrt{\frac{I}{\Sigma m}}$

7c i)

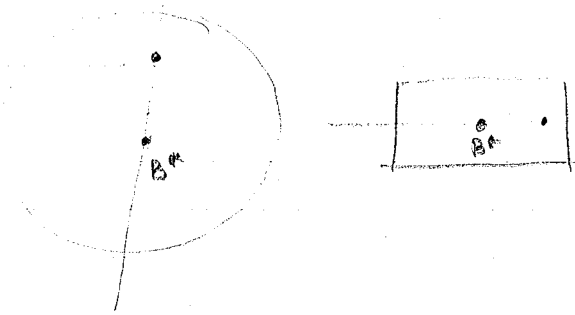


to find $\mathbb{I}_a^{B/P} \times m_a \overset{?}{=} 0$

$$\mathbb{I}_a^{B/P} = \mathbb{I}_a^{B/B^*} + \underbrace{m\, p^{B^*/P} \times (m_a \times p^{B^*/P})}_{\mathbb{I}^{B^*/P}}$$

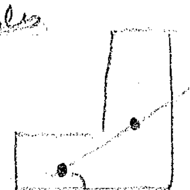
$$\mathbb{I}_a \times m_a = \underbrace{\mathbb{I}_a^{B/B^*} \times m_a}_{\substack{0 \\ \mathbb{I}_a \parallel m_a}} + \underbrace{m(p^{B^*/P} \times m_a \times p^{B^*/P}) \times m_a}_{0}$$

Examples



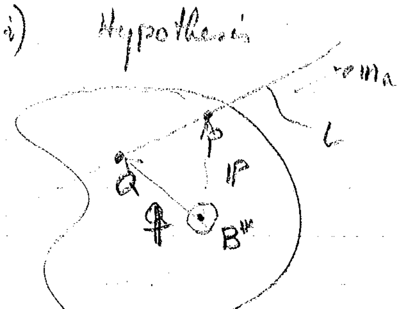
7c ii)

Examples



if line principal axis for this pt.

7c iii) Hypothesis



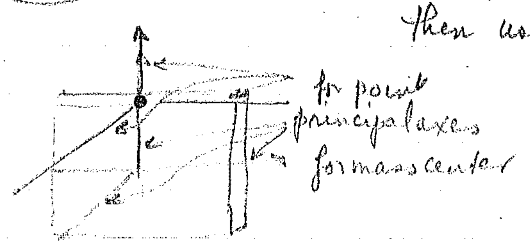
$\mathbb{I}_a^{B/P} \times m_a = 0$

$\mathbb{I}_a^{B/Q} \times m_a = 0$

show p, q are $\parallel m_a$ & hence B^* must lie on L

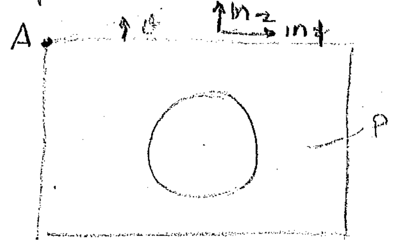
then use 7c i)

7c iv)



7c v) Degenerate Ellipsoid discussion gives result

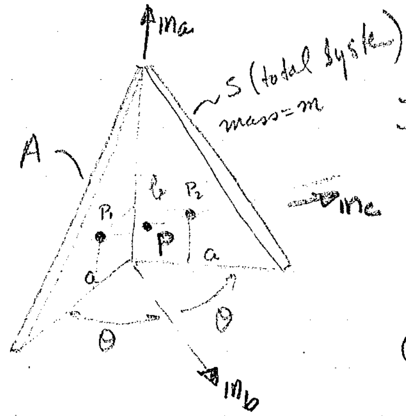
7d) Straight forward – subtraction of mass using parallel axes theorem.



$$I_{11}^{P/A} = 114.65\rho \quad I_{22}^{P/A} = 258.94\rho \quad I_{12}^{P/A} = 125.15\rho$$

$$\tan 2\theta = \frac{2 I_{12}}{I_{11} - I_{22}} = -30.02^\circ$$

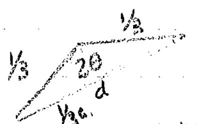
7e)



Ideas (1) k_{min} is one of k_x, k_y, k_z principal radii of gyration.

P is mass center of whole place ; P_1, P_2 is mass center of each.

(2) One principal axis of S for P is parallel to n_c & the other two are normal to n_c



$$d = \left(\frac{2}{9} + \frac{2}{9}\cos 2\theta\right)^{1/2}$$

$$d^2 = \frac{2}{9}(1+\cos 2\theta)$$

$c^2+s^2+c^2-s^2$

$$\left(\frac{4c^2}{9}\right)^{1/2} = \frac{2}{3}c = d$$

(3) k_x, k_y, k_z can be found

$$2mk_x^2 = I_c^{S/P}$$

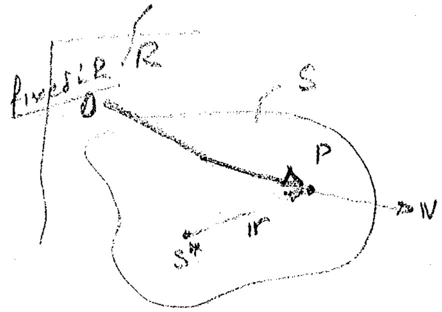
$$2mk_{y,z}^2 = \frac{I_a^{S/P} + I_b^{S/P}}{2} \pm \sqrt{\left(\frac{I_a^{S/P} - I_B^{S/P}}{2}\right)^2 + \left(I_{ab}^{S/P}\right)^2}$$

(4)
$$\left.\begin{aligned} I_a^{S/P} &= 2I_a^{A/P} \\ I_b^{S/P} &= 2I_b^{A/P} \\ I_c^{S/P} &= 2I_c^{A/P} \\ I_{ab}^{S/P} &= 2I_{ab}^{A/P} \end{aligned}\right\}$$
use ∥ axis theorem & the inertia properties of triangle wrt some given lines & a point

7f. $\mathbb{I}_a^{S/P} = \mathbb{I}_a^{S/S^*} + \mathbb{I}_a^{S^*/P} \Rightarrow \mathbb{I}^{S/P} = \mathbb{I}^{S/S^*} + \mathbb{I}^{S^*/P}$

7g. $I_{ab} = n_a \cdot \mathbb{I} \cdot n_b$

7h did in class in different manner



Angular Momentum.

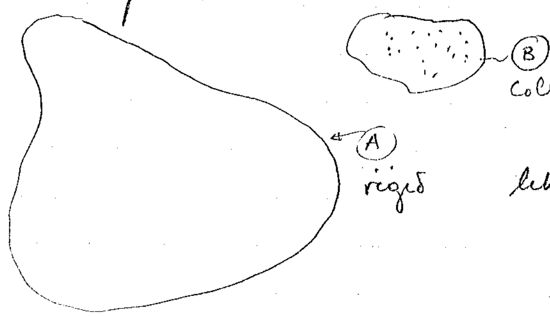
$$\mathbb{H} = \int \rho\, \mathbb{r}^{P/S^*} \times {}^{R}\mathbb{V}^{P}\, d\tau$$

$${}^{R}\mathbb{H}^{S/S^*} = \int \rho\, \mathbb{r}^{P/S^*} \times {}^{R}\mathbb{V}^{P}\, d\tau$$

$${}^{R}\mathbb{V}^{P} = \frac{{}^{R}d\,\mathbb{r}^{P/O}}{dt}$$

1/24/80

Soln of Cubic
Angular Momentum
Principal Moments of Inertia – derivatives



collection of Particles non rigid

rigid

let Ⓒ = Ⓑ ∪ Ⓐ collection of both systems

define D: rigid body that has same mass distrib as C @ any instant but has the same motion as A.

R: any reference frame whatsoever

*: signifies mass center ex: C^*, D^*

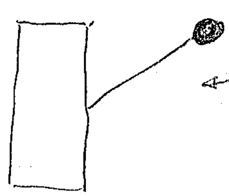
$${}^{R}\mathbb{H}^{C/C^*} \text{ (of body C wrt mass center) } = \underbrace{{}^{R}\mathbb{H}^{D/D^*}}_{\text{assumed rigid body}} + {}^{A}\mathbb{H}^{B/C^*}$$

of collection of B wrt C^* in reference A.

C^* & D^* are in the same point in space but different velocities

See if you can prove it. Why do we need this. See reverse for proof.

Example:



Setup which moves about in space.

Derivatives of Principal moments of inertia

See handout

Prove $\quad {}^{R}\mathbb{H}^{C/C^*} = {}^{R}\mathbb{H}^{D/D^*} + {}^{A}\mathbb{H}^{B/C^*} \quad$ where $\quad {}^{R}\mathbb{H}^{D/D^*} = \mathbb{I}^{C/C^*} \cdot {}^{R}\boldsymbol{\omega}^{A} \qquad {}^{A}\mathbb{H}^{B/C^*} = \sum_{i=1}^{n_B} m_i \mathbb{r}_i^{P_i/C^*} \times {}^{A}\mathbb{V}^{P_i}$

$${}^{R}\mathbb{H}^{C/C^*} = \sum^{n_C} m_i \mathbb{r}_i^{P_i/C^*} \times {}^{R}\mathbb{V}^{P_i} = \sum_{i=1}^{n_C} m_i \mathbb{r}_i^{P_i/C^*} \times \left({}^{A}\mathbb{V}^{P_i} + {}^{R}\boldsymbol{\omega}^{A} \times \mathbb{r}^{P_i/O} \right)$$ where O is fixed in R

$$= \sum^{n_C} m_i \mathbb{r}_i^{P_i/C^*} \times {}^{A}\mathbb{V}^{P_i} + \sum_{i=1}^{n_C} m_i \mathbb{r}_i^{P_i/C^*} \times {}^{R}\boldsymbol{\omega}^{A} \times \left(\mathbb{r}^{P_i/C^*} + \mathbb{R}^{C^*/O} \right).$$

$$+ \underbrace{\sum^{n_C} m_i \mathbb{r}_i^{P_i/C^*} \times {}^{R}\boldsymbol{\omega}^{A} \times \mathbb{r}^{P_i/C^*}}_{\mathbb{I}^{C/C^*} \cdot {}^{R}\boldsymbol{\omega}^{A}} + \underbrace{\cancel{\sum^{n_C} m_i \mathbb{r}_i^{P_i/C^*} \times \left({}^{R}\boldsymbol{\omega}^{A} \times \mathbb{R}^{C^*/O} \right)}}_{=0}$$

since $\sum^{n_C} m_i \mathbb{r}_i^{P_i/C^*} = 0$

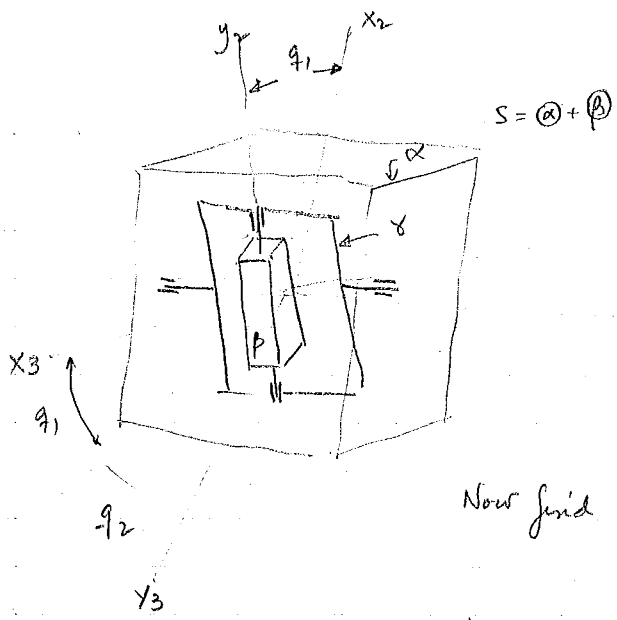
$$= \sum_{i=1}^{N_C} m_i \mathbb{r}_i^{P_i/C^*} \times {}^{A}\mathbb{V}^{P_i} + \mathbb{I}^{C/C^*} \cdot {}^{R}\boldsymbol{\omega}^{A}$$

$$= \sum_{i=1}^{n_B} m_i \mathbb{r}_i^{P_i/C^*} \times {}^{A}\mathbb{V}^{P_i} + \sum_{i=1}^{n_A} m_i \mathbb{r}_i^{P_i/C^*} \times {}^{A}\mathbb{V}_i^{P_i} + \mathbb{I}^{C/C^*} \cdot {}^{R}\boldsymbol{\omega}^{A}$$

but since A is a rigid body ${}^{A}\mathbb{V}^{P_i} \equiv 0$ $\therefore$ middle sum drops out and

$${}^{R}\mathbb{H}^{C/C^*} = \mathbb{I}^{C/C^*} \cdot {}^{R}\boldsymbol{\omega}^{A} + \sum_{i=1}^{n_B} m_i \mathbb{r}_i^{P_i/C^*} \times {}^{A}\mathbb{V}^{P_i} = {}^{R}\mathbb{H}^{D/D^*} + {}^{A}\mathbb{H}^{B/C^*}$$

QED.



2 degrees of freedom.
X's are principal axes of α
Y's are " " " β.
when q's are $= 0$ then axes are coincident

Now where are the axes of the collection $\alpha + \beta$ if $q_1, q_2 \neq 0$

Now find I_1, I_2, I_3 of collection with A_1, A_2, A_3 } principal mom. of each. B_1, B_2, B_3 }

find I_s' of α wrt x_1, x_2, x_3 now take $I_s + B_1, B_2, B_3$ for given q_1, q_2

now form cubic – cannot solve in general.

Now need to find $\left.\frac{\partial I_1}{\partial q_1}\right|_{q_1 = q_2 = 0}$; why to check stability we need to check minimum.

given : $f(x,y)$ is a min @ $x=y=0$? what are conditions

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0 \quad @\ x = y = 0$$

and $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} > 0$ w/ $f_{xx} \geq 0$

To find $\frac{\partial I_1}{\partial q_1}$ – For general problem

Define $q_1, \dots, q_n$: coords governing the relative pos. & orientation of parts of S.

A_1, A_2, A_3 : mutually $\perp$ central principal axes of inertia of S
a_1, a_2, a_3 : unit vectors // A_i
A : reference frame that fixed

B_1, B_2, B_3, b_1, b_2, b_3, B) any mutually $\perp$ axes (but subject to following) passing through S* and oriented relative to S so that B_1, B_2, B_3 coords of every particle of S are singlevalued functions of $q_1, \dots, q_n$ and B_i coincides w/ A_i when $q_1, \dots, q_n = 0$.

It follows $\tilde{a}_i = \tilde{b}_i$ $(i = 1, 2, 3)$ when $q_1, \dots, q_n = 0$. (1) $\tilde{a}_i = a_i$ when $q_i = 0$

Let $\mathbb{I}$: the central inertia dyadic of S in any state.

$$I_k = \mathbb{a}_k \cdot \mathbb{I} \cdot \mathbb{a}_k \qquad (k = 1, 2, 3) \qquad (2)$$

are principal moments of inertia — hard to find since we don't know $\mathbb{a}_k$

$$J_{kl} = \mathbb{b}_k \cdot \mathbb{I} \cdot \mathbb{b}_l \qquad (k, l = 1, 2, 3) \qquad (3)$$

central product of inertia — easy to find since we define $\mathbb{b}_k$

We want to find $\left.\dfrac{\partial I_k}{\partial q_r}\right|_{q_1 = q_2 = \cdots = q_n = 0} = \tilde{I}_{k,r}$

$$\left.\frac{\partial^2 I_k}{\partial q_r \partial q_s}\right|_{q_1 = \cdots = q_n = 0} = \tilde{I}_{k,rs}$$

Take I_j (principal mom) expand as a fn by McLaurin

$$I_j = \tilde{I}_j + \underbrace{\tilde{I}_{j,r}\, q_r}_{\text{Einstein Notation } \sum_r} + \underbrace{\tilde{I}_{j,rs} \frac{q_r q_s}{2!}}_{\sum_s \sum_r} + \cdots \qquad (4)$$

$$a_{jk} \triangleq \mathbb{a}_j \cdot \mathbb{b}_k \qquad \left(\tilde{a}_{jk} = \tilde{\mathbb{a}}_j \cdot \tilde{\mathbb{b}}_k \underset{(1)}{=} \tilde{\mathbb{b}}_j \cdot \tilde{\mathbb{b}}_k = \delta_{jk}\right) \qquad (5)$$

Now $\mathbb{a}_1 \underset{(5)}{=} a_{11}\mathbb{b}_1 + a_{12}\mathbb{b}_2 + a_{13}\mathbb{b}_3 = a_{jk}\mathbb{b}_k$

now expand a_{11}, a_{12}, a_{13} as fn of q_r, q_s

$$= \left(1 + \tilde{a}_{11,r} q_r + 2^{-1}\tilde{a}_{11,rs} q_r q_s + \cdots\right)\mathbb{b}_1 + \left(0 + \tilde{a}_{12,r} q_r + \tilde{a}_{12,rs}\frac{q_r q_s}{2} + \cdots\right)\mathbb{b}_2$$

$$+ \left(0 + \tilde{a}_{13,r} q_r + \tilde{a}_{13,rs}\, q_r q_s/2 + \cdots\right)\mathbb{b}_3 \qquad (6)$$

Look at inertia dyadic

$$\mathbb{I} \underset{(3)}{=} J_1 \mathbb{b}_1\mathbb{b}_1 + J_{12}\mathbb{b}_1\mathbb{b}_2 + J_{13}\mathbb{b}_1\mathbb{b}_3 + J_{21}\mathbb{b}_2\mathbb{b}_1 + \cdots \qquad (7)$$

Now we observe - since $\mathbb{a}_1$ is $\parallel$ to a principal axis then

$$(\mathbb{I} - I_1 \bar{U}) \cdot \mathbb{a}_1 = 0 \qquad \text{defn of Characteristic Polynomial} \qquad (8)$$

now take $\mathbb{b}_K \cdot (\mathbb{I} - I_1 \bar{U}) \cdot \mathbb{a}_1 = 0 \qquad (9)$

$$\mathbb{b}_K \cdot \mathbb{I} \cdot \mathbb{a}_1 - I_1\, \mathbb{b}_K \cdot \mathbb{a}_1 = 0 \qquad k = 1, 2, 3 \qquad (10)$$

now let $k = 1$

$$\mathbb{b}_1 \cdot \mathbb{I} \cdot \mathbb{a}_1 - I_1\, \mathbb{b}_1 \cdot \mathbb{a}_1 \underset{(6,7)}{=}$$

$\tilde{J}_{12} = 0$

$$\left(\tilde{J}_1 + \tilde{J}_{1,r} q_r + 2^{-1}\tilde{J}_{1,rs} q_r q_s + \cdots\right) \cdot \left(1 + \tilde{a}_{11,t}\, q_t + 2^{-1}\tilde{a}_{11,tu}\, q_t q_u + \cdots\right)$$

to keep notation straight

$$+ \left(\tilde{J}_{12,r} q_r + \cdots\right)\left(\tilde{a}_{12,t}\, q_t + \cdots\right) + \cdots$$

$$- \left(\tilde{I}_1 + \tilde{I}_{1,r} q_r + 2^{-1}\tilde{I}_{1,rs} q_r q_s + \cdots\right) \cdot \left(1 + \tilde{a}_{11,t} q_t + \cdots\right) = 0 \qquad (11)$$

(4) (6)

Now collect order terms & use fact that $(\ ) + (\ )q + (\ )q + (\ )q\,q$

and each coeff of q must be zero

^{0}th order is q_r^{0} : $\tilde{J}_1 - \tilde{I}_1 = 0 \qquad \tilde{I}_1 = \tilde{J}_1$ (12)

also for $\tilde{J}_2 - \tilde{I}_2 = 0$

$\tilde{J}_3 - \tilde{I}_3 = 0$

1st order is q_r^{1} : $\tilde{J}_{1,r} + \cancel{\tilde{J}_1 \tilde{a}_{11,t}} - \cancel{\tilde{I}_1 \tilde{a}_{11,t}} - \tilde{I}_{1,r} = 0$

$$\tilde{J}_{1,r} - \tilde{I}_{1,r} = 0 \qquad \tilde{J}_{1,r} = \tilde{I}_{1,r} \qquad (13)$$

etc. ....

$q_r q_s$: $2^{-1}\tilde{J}_{1,rs} + \tilde{J}_{1,r}\tilde{a}_{11,s} + \dots = 0$ (14)

Consider for $k=2$

$$\mathbb{b}_2 \cdot \mathbb{I} \cdot \mathbb{a}_1 - I_1 \mathbb{b}_2 \cdot \mathbb{a}_1 \underset{(6,7)}{=} \dots \qquad (15)$$

from (14) & (15)

$$\tilde{a}_{12,r} = \frac{\tilde{J}_{21,r}}{\tilde{J}_1 - \tilde{J}_2} \qquad \text{gives info about orientation of axes} \qquad (16)$$

for $k=2$ we find

$$\tilde{a}_{13,r} = \frac{\tilde{J}_{31,r}}{\tilde{J}_1 - \tilde{J}_3} \qquad (17)$$

put this into (14)

$$\tilde{I}_{1,rs} \underset{(14)}{=} \tilde{J}_{1,rs} + 2\left(\frac{\tilde{J}_{12,r}\tilde{J}_{21,s}}{\tilde{J}_1 - \tilde{J}_2} + \frac{\tilde{J}_{13,r}\tilde{J}_{31,s}}{\tilde{J}_1 - \tilde{J}_3}\right) \qquad (18)$$

(19)

(20)

now by cyclic permutation we can find $\tilde{I}_{2,rs}$, $\tilde{I}_{3,rs}$

Now $a_{11}^2 + a_{12}^2 + a_{13}^2 = 1$ now $\frac{\partial}{\partial r}$ gives $a_{11}a_{11,r} + a_{12}a_{12,r} + a_{13}a_{13,r} =$

now $\tilde{a}_{11}\tilde{a}_{11,r} + \tilde{a}_{12}\tilde{a}_{12,r} + \tilde{a}_{13}\tilde{a}_{13,r} = 0$

$\tilde{a}_{11} = 1$, $\tilde{a}_{12} = 0$, $\tilde{a}_{13} = 0$

and $\Rightarrow \boxed{\tilde{a}_{11,r} = 0}$ (21)

Jan 29, 1980

Inertia of non deformable bodies   NASA TM X-2934 (1973)

Tues 2/5 in class midterm 2 questions & 2 takehome problems due 2/7

From last time we had shown

$$\tilde{I}_1 = \tilde{J}_1 \; ; \; \tilde{I}_2 = \tilde{J}_2 \; ; \; I_3 = \tilde{J}_3 \qquad (12)$$

$$\tilde{I}_{1,r} = \tilde{J}_{1,r} \; ; \; \tilde{I}_{2,r} = \tilde{J}_{2,r} \; ; \; \tilde{I}_{3,r} = \tilde{J}_{3,r} \qquad (13)$$

$$\tilde{I}_{1,rs} = \tilde{\tilde{J}}_{1,rs} + 2\left[\frac{\tilde{\tilde{J}}_{12,r}\tilde{\tilde{J}}_{21,s}}{\tilde{J}_1 - \tilde{J}_2} + \frac{\tilde{\tilde{J}}_{13,r}\tilde{\tilde{J}}_{31,s}}{\tilde{J}_1 - \tilde{J}_3}\right] \text{ etc.} \qquad (18-20)$$

$$\tilde{a}_{11,r} = 0 \;(21) \text{ also } \tilde{a}_{12,r} = \frac{\tilde{\tilde{J}}_{21,r}}{\tilde{J}_1 - \tilde{J}_2} \qquad \tilde{a}_{13,r} = \frac{J_{31,r}}{\tilde{J}_1 - \tilde{J}_3} \qquad (16,17)$$

– Why do we do all the inertia material –

Remember that $F_r^* = \sum_{i=1}^{n} \tilde{\mathbb{V}}_{\dot{q}_r}^{P_i} \cdot \mathbb{F}_i^*$ w/ $\mathbb{F}_i^* = -m_i a_i$

now this eqn was good for a set of particles but for a rigid body $n \gg$ large

thus we ended up looking at $\sum_{i=1}^{N} m_i \mathbb{r}_i \times (\mathbb{n}_a \times \mathbb{r}_i)$. for rigid body

thus $(F^*)_B \underset{3.10}{=} {}^R\tilde{\mathbb{V}}_{\dot{q}_r}^{B^*} \cdot \mathbb{F}^* + {}^R\tilde{\omega}_{\dot{q}_r}^B \cdot \mathbb{T}^*$

rigid body contribution — of mass center — inertia force — angular velocity of B in R — inertia torque

where $\mathbb{F}^* = -m\,\mathbb{a}^{*R}$ – accel of mass center

(1) $\mathbb{T}^* = -(\mathbb{I}^* \cdot \alpha + \omega \times \mathbb{I}^* \cdot \omega)$

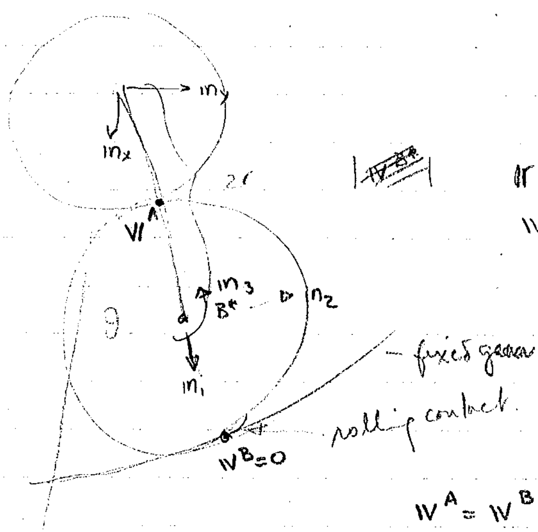
this is very good theoretically, but very hard to do numerically

(2) $$\mathbb{T}^* = -\Big\{\big[I_1\alpha_1 - (I_2 - I_3)\omega_2\omega_3\big]\mathbb{n}_1 + \big[I_2\alpha_2 - (I_3 - I_1)\omega_3\omega_1\big]\mathbb{n}_2 + \big[I_3\alpha_3 - (I_1 - I_2)\omega_1\omega_2\big]\mathbb{n}_3\Big\}$$

$\mathbb{n}_1, \mathbb{n}_2, \mathbb{n}_3$ parallel to central principal axis

(3) $\mathbb{T}^* = -(\dot{\omega}\,\mathbb{I}_a + \omega^2 \mathbb{n}_a \times \mathbb{I}_a)$ another form using 2nd moment vectors.

(4) $\mathbb{T}^* = -\dot{\omega}\, I_a \mathbb{n}_a - (\dot{\omega} I_{ab} - \omega^2 I_{ac})\mathbb{n}_b - (\dot{\omega} I_{ac} + \omega^2 I_{ab})\mathbb{n}_c$ bodies w/simple angular veloc



$$r^{B^*} = 2r\cos\theta\, n_x + 2r\sin\theta\, n_y$$

$$v^{B^*} = 2r(-\sin\theta\, n_x + \cos\theta\, n_y)\,\dot\theta$$

$$= 2r\dot\theta\, n_2$$

$$v^B = v^{B^*} + \omega \times r$$

$$= 2r\dot\theta\, n_2 + \omega^B n_3 \times r\, n_1$$

$$= 2r\dot\theta\, n_2 + \omega^B r\, n_2 = 0$$

$$\omega^B = -2\dot\theta \qquad \omega^B = -2\dot\theta\, n_3$$

$$v^A = v^{B^*} + \omega^B \times r$$

$$= 2r\dot\theta\, n_2 + (-2\dot\theta\, n_3) \times (-r\, n_1)$$

$$= 4r\dot\theta\, n_2 = \omega^A \times r$$

$$= \omega^A \times r\, n_1$$

$$\omega^A n_3 \times r\, n_1 \Rightarrow \omega^A = 4\dot\theta\, n_3 = 4\dot\theta\, n$$

$4 n_z \cdot J n_z$

$$-4\ddot\theta J n_z = T_A^*$$

$$\omega^A_{\dot q_r} \cdot T_A^* = -16\ddot\theta J$$

$$r = 2r(\cos\theta\, n_x + \sin\theta\, n_y)$$

$$v = \dot\theta\, 2r(-S\, n_x + C\, n_y) = \dot\theta\, 2r\, n_2$$

$$a = \ddot\theta\, 2r(-S\, n_x + C\, n_y) + \dot\theta(2r)\left[-C\, n_x - S\, n_y\right)$$

$$-m\, v_{\dot\theta} \cdot a = -m 4r^2\ddot\theta = v_{\dot\theta} \cdot F_B^*$$

$$\omega^B_{\dot q_r} \cdot T^* = -2n_z\,(+2\ddot\theta J n_z) = -4\ddot\theta J$$

$$-I \cdot \alpha$$

What other uses do we have for inertia

(1) Original purpose to get $\mathbb{F}_r^*$

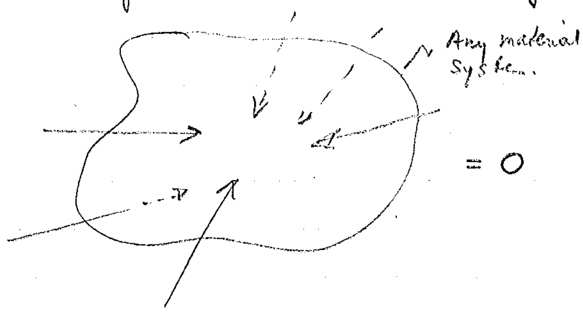
When the set of inertia forces ($\mathbb{F}_i^* = -m_i a_i \quad i = 1, 2, \ldots, N$) for a rigid body is replaced with a force $\mathbb{F}^*$ at B^* (mass center of B) and a couple torque $\mathbb{T}^*$, then

$$\mathbb{F}^* = -m a^*$$

$$\mathbb{T}^* = -(\mathbb{I}^* \cdot \alpha + \omega \times \mathbb{I}^* \cdot \omega)$$

Equivalence $\mathbb{F} = \Sigma \mathbb{F}_i \qquad m = \Sigma m_i$

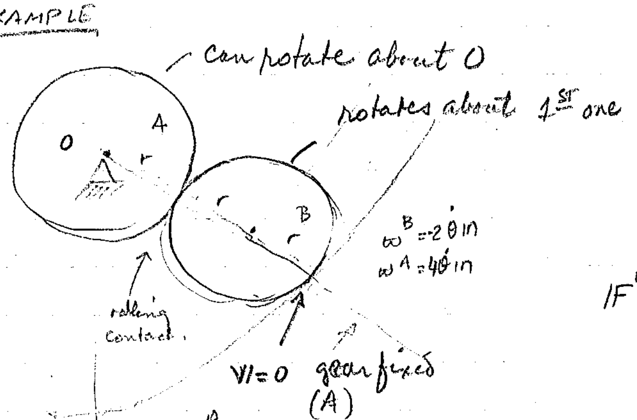
Useful in D'Alembert's Principle

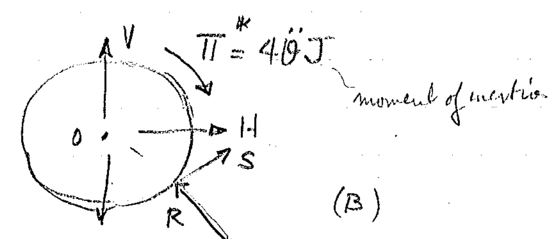


$= 0$

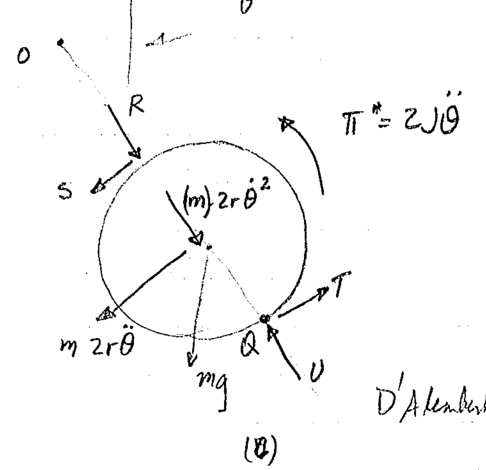
Solid arrows: body of contact forces.
dotted arrows: inertia forces.
this system of vectors have zero resultant & zero moment about any point in space

EXAMPLE





$\mathbb{F}^*$ for upper disk is 0 since $a^* = 0$
also $\omega \parallel \mathbb{I} \quad \therefore \omega \times \mathbb{I} = 0$



Refering to figure B & take moments about O.

D'Alembert $\quad \mathbb{T}^* - rS = 0 \Rightarrow 4\ddot{\theta}J - rS = 0$

Refer to figure C & take moments about Q

D'Alembert $\quad (2J\ddot{\theta} = \mathbb{T}^*) + 2rS + 2mr^2\ddot{\theta} + mg \sin\theta\, r = 0$

take it about Q to get rid of S since we want DE on $\ddot{\theta}$

$\Rightarrow \quad \ddot{\theta} + \frac{mgr}{\quad} \sin\theta = 0 \qquad$ equivalent length of compound pendulum $= \frac{mgr}{\ldots} = \frac{g}{\ldots}$

and $L = \dfrac{2(5J + mr^2)}{mr}$

How is this related to Lagrangian
from the generalized active force

$$F_\theta = -2mgr\sin\theta \quad \text{only terms since contact is by rolling.}$$

generalized inertia force

$$F_\theta^* = -16\ddot{\theta}J - 4mr^2\ddot{\theta} - 4J\ddot{\theta}$$

$$F_\theta + F_\theta^* = 0 \Rightarrow -20J\ddot{\theta} - 4m\ddot{\theta}r^2 - 2mgr\sin\theta = 0$$

$$\ddot{\theta} + \frac{mgr}{2(5J+mr^2)}\sin\theta = 0$$

1/31/80

800131 Handout: $\tilde{I}_{3,1} = 0 \quad \tilde{I}_{3,2} = 0$ must be enforced

$\tilde{I}_{3,11} > 0 \ , \quad \tilde{I}_{3,11}\tilde{I}_{3,22} - \tilde{I}_{3,12}^{\,2} > 0$

Choosing axes for B_1, B_2, B_3 can choose X_1, X_2, X_3 normally best

$$\tilde{I}_{3,1} = \tilde{J}_{33,1} \ ; \quad \tilde{I}_{3,2} = \tilde{J}_{3,2} \ ; \ \tilde{I}_{3,11} = \tilde{J}_{3,11} + 2\left(\frac{\tilde{J}_{31,1}^{\,2}}{\tilde{J}_3 - \tilde{J}_1} + \frac{\tilde{J}_{32,1}^{\,2}}{\tilde{J}_3 - \tilde{J}_2}\right)$$

J – referred to axes X_1, X_2, X_3 which are known.

$\tilde{I}_{3,22}$ = same but cyclic

$$\tilde{I}_{3,12} = \tilde{J}_{31,12} + 2\left(\frac{J_{31,1}J_{31,2}}{J_3 - J_1} + \frac{J_{32,1}J_{32,2}}{J_3 - J_2}\right)$$

Need $\tilde{J}_{11}, \tilde{J}_{22}$, J_{31}, J_{32}, J_3 only.
evaluated — functions that must be differentiated before evaluation

	X_1	X_2	X_3
Y_1	c_2	$s_2 s_1$	$-s_2 c_1$
Y_2	0	c_1	s_1
Y_3	s_2	$-c_2 s_1$	$c_2 c_1$

$\tilde{J}_1 = A_1 + B_1 \qquad \tilde{J}_2 = A_2 + B_2 \qquad J_{31} = -B_1 c_1 s_2 c_2 + B_3 c_1 s_2 c_2 = (B_3 - B_1)c_1 s_2 c_2$

$J_{32} = -B_1 s_2^2 s_1 c_1 + B_2 s_1 c_1 - B_3 s_1 c_1 c_2^2 \ ; \quad J_3 = A_3 + B_1 c_1^2 s_2^2 + B_3 s_1^2 + B_3 c_1^2 c_2^2$

$$\tilde{J}_3 = A_3 + B_3$$

$\tilde{J}_{31,1} = 0 \qquad \left.\dfrac{\partial J_{31}}{\partial q_1}\right|_{q_1=0} \qquad \tilde{J}_{32,1} = B_2 - B_3 \qquad \tilde{J}_{3,11} = 2(B_2 - B_3) \qquad \tilde{J}_{3,22} = 2(B_1 - B_3)$

Check $\tilde{I}_{3,1} = \tilde{I}_{3,2} = 0$ ok.

$$\tilde{I}_{3,11} = 2(B_2 - B_3) + \frac{2(B_2 - B_3)^2}{A_3 + B_3 - A_2 - B_2} = \frac{2(A_2 - A_3)(B_2 - B_3)}{(A_2 - A_3) + (B_2 - B_2)}$$

$$\tilde{I}_{3,22} = \frac{2(A_1 - A_3)(B_1 - B_3)}{(A_1 - A_3) + (B_1 - B_3)} \qquad \tilde{I}_{3,12} = 0$$

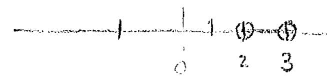
Now is $\tilde{I}_{3,11} \overset{?}{>} 0 \implies \dfrac{(A_2 - A_3)(B_2 - B_3)}{(A_2 - A_3) + (B_2 - B_3)} > 0 \qquad \dfrac{(2-3)(3-B_3)}{(2-3)+(3-B_3)} > 0$

Also. $\tilde{I}_{3,11}\,\tilde{I}_{3,22} - \tilde{I}_{12} = \dfrac{(A_1 - A_3)(B_1 - B_3)}{(A_1 - A_3) + (B_1 - B_3)} > 0 \qquad \dfrac{(4-3)(1.5 - B_3)}{4 - 3 + 1.5 - B_3} > 0$

from handout $A_2 = 2 \quad A_3 = 3 \quad B_1 = 1.5 \quad B_2 = 3$

from these 2. $\implies \left.\begin{array}{l} \dfrac{B_3 - 3}{2 - B_3} > 0 \quad (1) \\ \dfrac{1.5 - B_3}{2.5 - B_3} > 0 \end{array}\right\}$ there is also a third constraint $B_3 < B_1 + B_2 \implies B_3 < 4.5$

if $B_3 - 3 > 0$ & $2 - B_3 > 0$ impossible for 1 to be true
$B_3 - 3 < 0$ & $2 - B_3 < 0 \implies \boxed{2 < B_3 < 3}$

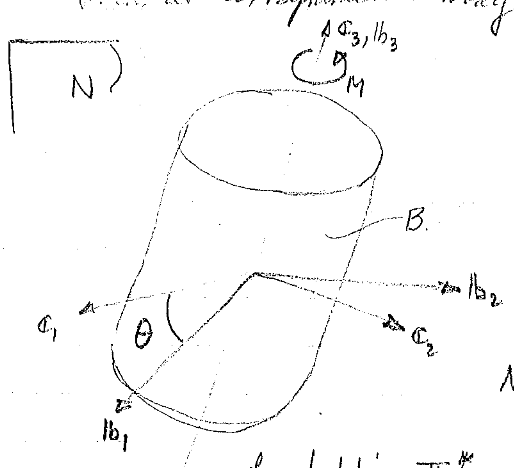


from $1.5 - B_3 > 0$ & $2.5 - B_3 > 0 \implies \boxed{B_3 < 1.5}$ impossible
$1.5 - B_3 < 0$ & $2.5 - B_3 < 0 \implies \boxed{B_3 > 2.5}$ ∴ OK

thus for both conditions must be satisfied for $2.5 < B_3 < 3$

New topics – D'Alembert point of view of Π^* (inertia torque)
– Axisymmetric Rigid Body.

Look at axisymmetric body under constant moment. (Spin-up problem)



c_i are not fixed in body.

D'Alembert (moments about mass center).
$\Pi^* + M c_3 = 0$
moment about mass centers of inertia forces.

Now we look for Π^*

from last time $\Pi^* = -\left(\mathbb{I} \cdot {}^N\alpha^B + {}^N\omega^B \times \mathbb{I} \cdot {}^N\omega^B\right) \quad (1)$

we want to get $\Pi^* \overset{?}{=} -\left(I\dot{\omega}_1 c_1 + I\dot{\omega}_2 c_2 + J\dot{\omega}_3 c_3\right)$

- Inertia ellipsoid for B^* must be a spheroid (since it is an axisym body)
- c_1, c_2, c_3 form a RHS (dextral set) of mutually ⊥ unit vectors w/ c_3 // to

symmetry axis, but $\mathbb{c}_1$ & $\mathbb{c}_2$ not necessarily fixed in B
- C is a reference frame in which $\mathbb{c}_1, \mathbb{c}_2, \mathbb{c}_3$ are fixed

Now $\mathbb{I} = I(\mathbb{c}_1\mathbb{c}_1 + \mathbb{c}_2\mathbb{c}_2) + J\mathbb{c}_3\mathbb{c}_3$ (2)
since 2 equal moments. J is the axial moment of inertia
I is the transverse.

$${}^N\boldsymbol{\omega}^B = {}^N\boldsymbol{\omega}^C + {}^C\boldsymbol{\omega}^B \qquad (3)$$

There exists a scalar s ∋. ${}^C\boldsymbol{\omega}^B = s\mathbb{c}_3$ (since $\mathbb{b}_3$ is $\parallel \mathbb{c}_3$ & (4) we have simple angular velocity then ${}^C\omega^B = s\mathbb{c}_3$)

$${}^N\boldsymbol{\omega}^B = {}^N\boldsymbol{\omega}^C + s\mathbb{c}_3 \qquad (5)$$

$${}^N\boldsymbol{\alpha}^B = \frac{d\,{}^N\boldsymbol{\omega}^B}{dt} = \frac{{}^C d\,{}^N\boldsymbol{\omega}^B}{dt} + {}^N\boldsymbol{\omega}^C \times {}^N\boldsymbol{\omega}^B$$

$$= \frac{{}^C d\,{}^N\boldsymbol{\omega}^B}{dt} + \underbrace{({}^N\boldsymbol{\omega}^B - s\mathbb{c}_3)}_{(5)} \times {}^N\boldsymbol{\omega}^B$$

$$= \frac{{}^C d\,{}^N\boldsymbol{\omega}^B}{dt} + s\,{}^N\boldsymbol{\omega}^B \times \mathbb{c}_3 \qquad (6)$$

Now let $\omega_i \triangleq {}^N\boldsymbol{\omega}^B \cdot \mathbb{c}_i \quad (i = 1, 2, 3)$ (7)

$${}^N\boldsymbol{\omega}^B = \Sigma\, \omega_i \mathbb{c}_i \qquad (8)$$

$${}^N\boldsymbol{\alpha}^B \underset{(6,8)}{=} \overbrace{\dot\omega_1 \mathbb{c}_1 + \dot\omega_2\mathbb{c}_2 + \dot\omega_3\mathbb{c}_3}^{\frac{{}^C d}{dt}({}^N\boldsymbol{\omega}^B)} + s(\omega_2\mathbb{c}_1 - \omega_1\mathbb{c}_2) \qquad (9)$$

$$\mathbb{I} \cdot {}^N\boldsymbol{\omega}^B \underset{(2,8)}{=} I(\omega_1\mathbb{c}_1 + \omega_2\mathbb{c}_2) + J\omega_3\mathbb{c}_3 \qquad (10)$$

$${}^N\boldsymbol{\omega}^B \times (\mathbb{I}\cdot{}^N\boldsymbol{\omega}^B) = (J - I)\,[\omega_2\omega_3\mathbb{c}_1 - \omega_3\omega_1\mathbb{c}_2] \qquad (11)$$

$$\mathbb{I} \cdot {}^N\boldsymbol{\alpha}^B \underset{(2,9)}{=} I\left[(\dot\omega_1 + s\omega_2)\mathbb{c}_1 + (\dot\omega_2 - s\omega_1)\mathbb{c}_2\right] + J\dot\omega_3\mathbb{c}_3 \qquad (12)$$

Now $\mathbb{T}^* = \left\{[(I-J)\omega_3 - Is]\omega_2 - I\dot\omega_1\right\}\mathbb{c}_1 - \left\{[(I-J)\omega_3 - Is]\omega_1 + I\dot\omega_2\right\}\mathbb{c}_2$
$- \left\{J\dot\omega_3\mathbb{c}_3\right\}$ (13)

Since s is a free constant take $s = \frac{I-J}{I}\omega_3 \Rightarrow$ (14)

$$\mathbb{T}^* = -\left\{I\dot\omega_1\mathbb{c}_1 + I\dot\omega_2\mathbb{c}_2 + J\dot\omega_3\mathbb{c}_3\right\} \qquad (15)$$

$${}^N\boldsymbol{\omega}^C \underset{(5,8,14)}{=} \omega_1\mathbb{c}_1 + \omega_2\mathbb{c}_2 + \frac{J}{I}\omega_3\mathbb{c}_3 \qquad (16)$$

$$-J\dot{\omega}_1 = 0 \qquad \dot{\omega}_1 = 0 \qquad \Rightarrow \omega_1 = \omega_1^* \text{ a constant}$$
$$-I\dot{\omega}_2 = 0 \qquad \dot{\omega}_2 = 0 \qquad \omega_2 = \omega_2^* \text{ a constant}$$
$$-J\dot{\omega}_3 + M = 0 \qquad \omega_3 = \omega_3^* + \frac{M}{J}t$$

$$\underset{(w)}{s} = \frac{I-J}{I}\omega_3 = \frac{I-J}{I}\left(\omega_3^* + \frac{M}{J}t\right) = \dot{\theta}$$

$$\theta = \frac{I-J}{I}\omega_3^* t + \frac{M}{2J}t^2 \qquad \text{w/ condition } \theta(0) = 0$$

$${}^N\omega^C = \omega_1^* c_1 + \omega_2^* c_2 + \frac{J}{I}\left(\omega_3^* + \frac{M}{J}t\right) c_3$$

for torque free motion $M = 0$

for wobble free motion we want ${}^N\omega^C$ to be $\parallel$ to c_3

2/7/80

Problems i Set 8 & Midterm will be discussed on Tuesday

Review of works

$$F_r^* = \sum_{i=1}^{n} \mathbb{V}_{u_r}^{P_i} \cdot \mathbb{F}_i^* \qquad \mathbb{F}_i^* = -m_i a_i$$

$$(F_r)^*_{\text{rigid body}} = \underset{\text{of mass center}}{\mathbb{V}_{u_r}} \cdot \mathbb{F}^* + \underset{\text{of body}}{\omega_{u_r}^B} \cdot \mathbb{T}^* \qquad \mathbb{F}^* = -\underset{\text{of body}}{m}\,\underset{\text{of mass center}}{a^*}$$

$$\mathbb{T}^* = -\mathbb{I} \cdot {}^{B/B^*}\alpha^{RB} - \omega \times (\mathbb{I}\cdot\omega)$$
$$= [\omega_2\omega_3(I_2 - I_3] - \alpha_1 I_1] b_1 + \dots$$
$$= \omega^2 \mathbb{I}_a \times n_a - \dot{\omega}\,\mathbb{I}_a \qquad \text{simple angular motion}$$
$$= -(I\dot{\omega}_1 c_1 + I\dot{\omega}_2 c_2 + J\dot{\omega}_3 c_3) \qquad \text{for thin plates}$$

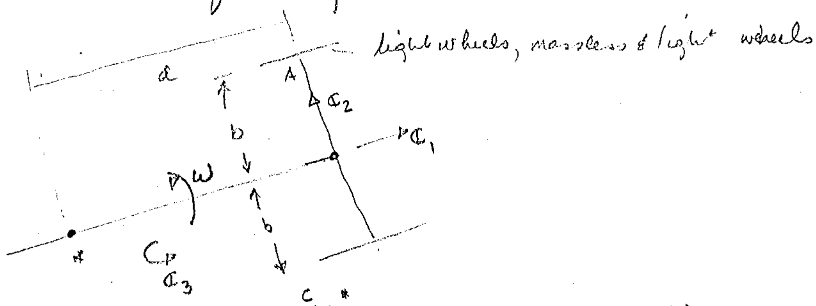
Why do we do this

Because $F_r + F_r^* = 0$. This implies a reference frame: (for our cases Earth is reference frame)

$$F_r^* = -\frac{\partial G}{\partial \dot{u}_r} \qquad G = \frac{1}{2}\sum_{i=1}^{N} m_i a_i^2$$
$$(G)_B = \frac{1}{2}\left(m a^2 + \alpha\cdot\mathbb{I}\cdot\alpha + 2\alpha\cdot\omega\times(\mathbb{I}\cdot\omega) + \omega^2\omega\cdot\mathbb{I}\cdot\omega\right)$$

normally last term is never involved in $\frac{\partial G}{\partial \dot{u}_r}$

Look at the following model



$$ {}^{C}\mathbb{V}^{*} = u_1 \mathbb{c}_1 + u_2 \mathbb{c}_2 \qquad (1)$$

$$ {}^{C}\mathbb{V}^{A} = \mathbb{V}^{*} + \omega \times \mathbb{r}^{A/*}$$

Constraint : motion of wheels in plane of motion $\quad u_2 + a\omega = 0 \Rightarrow \omega = -u_2/a \qquad (2)$

$$ {}^{R}\omega^{C} = -u_2/a\ \mathbb{c}_3 \qquad (3)$$

$$ {}^{R}\mathbb{a} \underset{(1)}{=} \dot{u}_1 \mathbb{c}_1 + \dot{u}_2 \mathbb{c}_2 + {}^{R}\omega^{C} \times \mathbb{V}^{*}$$

$$ \underset{(1,3)}{=} \left(\dot{u}_1 + \frac{u_2^2}{a}\right)\mathbb{c}_1 + \left(\dot{u}_2 - \frac{u_1 u_2}{a}\right)\mathbb{c}_2 \qquad (4)$$

$$ {}^{R}\alpha^{C} = -\frac{\dot{u}_2}{a}\mathbb{c}_3 \qquad (5) \qquad \text{since } a \text{ is fixed in } C.$$

$$ \mathbb{F}^{*} = -m\mathbb{a} \qquad \mathbb{T}^{*} = \frac{I\dot{u}_2}{a}\mathbb{c}_3 \qquad (6) \qquad \text{from simple angular motion}$$

Now $\quad \mathbb{V}_{u_1} \underset{(1)}{=} \mathbb{c}_1 \qquad \mathbb{V}_{u_2} \underset{(1)}{=} \mathbb{c}_2 \qquad \omega_{u_1} \underset{(3)}{=} 0 \qquad \omega_{u_2} \underset{(3)}{=} -\frac{\mathbb{c}_3}{a} \qquad (7)$

$$ F_1^{*} = \left[\mathbb{V}_{u_1}\cdot\mathbb{F}^{*} = \mathbb{c}_1\cdot(-m\mathbb{a})\right] + \left(\omega_{u_1}\cdot\mathbb{T}^{*} = 0\cdot\mathbb{T}^{*}\right) = -m\left(\dot{u}_1 + \frac{u_2^2}{a}\right) \qquad (8)$$

$$ F_2^{*} = m\left[\left(1 + \frac{I}{ma^2}\right)\dot{u}_2 - \frac{u_1 u_2}{a}\right] \qquad (9)$$

for generalized active force $\quad$ assume no friction then

$$ F_1 = F_2 = 0$$

$$ F_1 + F_1^{*} = 0 \Rightarrow \dot{u}_1 + \frac{u_2^2}{a} = 0 \qquad (10)$$

$$ F_2 + F_2^{*} = 0 \Rightarrow \left(1 + \frac{I}{ma^2}\right)\dot{u}_2 - \frac{u_1 u_2}{a} = 0 \qquad (11)$$

a particular soln : $u_1 = V$ and $u_2 = 0 \qquad (12)$

what are initial conditions

perturbed motion let $u_1 = V + u_1^{*}(t) \quad u_2 = 0 + u_2^{*}(t)$ and linearize in $u_1^{*}, u_2^{*} \qquad (13)$

put into (10) & (11)

$$ \dot{u}_1^{*} \underset{(10,13)}{=} 0 \qquad u_1^{*} = u_1^{*}(0) \qquad (14)$$

$$ \left(1 + \frac{I}{ma^2}\right)\dot{u}_2^{*} - [V + \underbrace{u_1^{*}(0)]u_2^{*}}_{\text{2nd order}} = 0 \qquad (15)$$

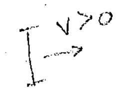
thus $u_2^* = u_2^*(0)\, e^{[V/a(1+ I/ma^2)]t}$ (16)

thus $u_1 = V + u_1^*(0)$

$u_2 = 0 + u_2^* = u_2^*(0)\, e^{[\quad]t}$

thus disturbance will propagate if $V>0$ unstable

dissipate $V<0$ stable

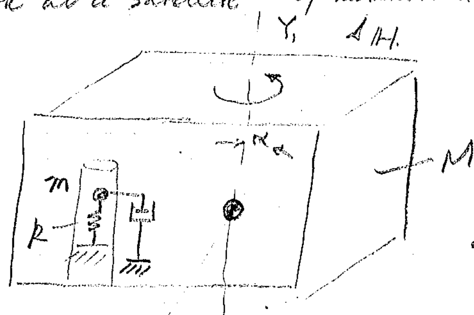


Experiment reveals it

actually $(1+ I/ma^2)\, u_2 \dot{u}_2 + u_1 \dot{u}_1 = 0$ or $\frac{(1+ I/ma^2)}{2}\frac{du_2^2}{dt} + \frac{du_1^2}{2\,dt} = 0$ or $(1+ I/ma^2)\frac{u_2^2}{2} + \frac{u_1^2}{2} = \text{const}$

Initial value problem.

Look at a satellite w/ nutation damper



desired motion $\omega \parallel Y_1$ & $H \parallel Y_1$

in real world

$\omega \cdot H = \cos\alpha$

want $\alpha \to 0$ as $t \to \infty$. What does α do a fn of & how can we optimize dissip

Thus we want plots of α vs. t for various initial conditions & values of system parameters

after solving problem get DE (see handout).

Thus we have coupled DE of dep. variables. Must solve de's of form.

$$\frac{dy_i}{dx} = F_i(z_1, \dots, z_m, y_1, \dots, y_n, x) \quad (i=1,\dots,n)$$

y - dependent variable.

x - independent variable ie t.

z - functions of $y_1, \dots, y_n$

if given $y_i(0)$ $i=1,\dots,n$ we can solve

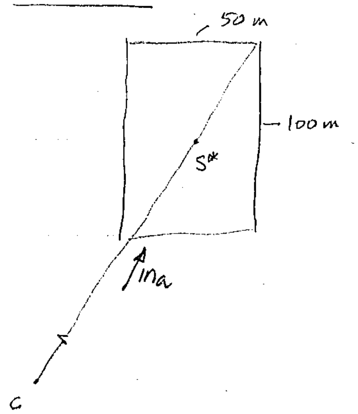
Job - find a problem of the physical world, solve the problem.

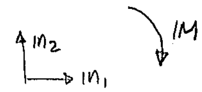
Problem: brief physical description; what ~~are~~ is initial value problem.

Feb 12, 1980

Discussion of Exam

Problem #1





$m_1 \times m_2 = m_3$

$m = 1000 kg$

Given $M = 3\Omega^2 m_a \times \mathbb{I}_a^{S/S^*}$

1) $m_a = \frac{1}{\sqrt{5}}(m_1 + 2m_2)$

2) $\mathbb{I}_a^{S/S^*} = a_1 I_1 m_1 + a_2 I_2 m_2$

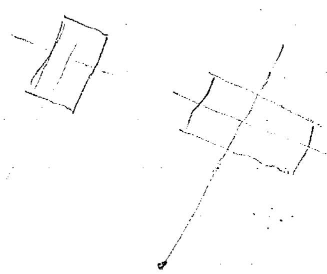
moments of inertia of plate about S^*

$I_1 = \frac{1000(100)^2}{12}$ $\quad I_2 = \frac{1000(50)^2}{12}$

3) $\Omega = \frac{2\pi}{90 \times 60}$ radians/sec.

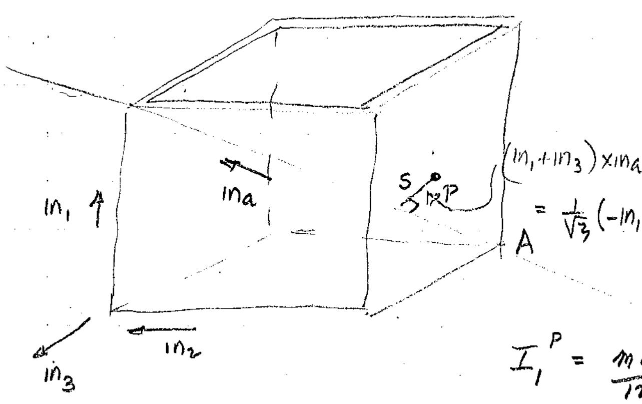
$M = -1.015\, Mm.\ m_3$

Implications no moment. if plate were lined up



if $\mathbb{I}_a$ is // to m_a then $m_a \times \mathbb{I}_a = 0$

Problem #2



$(m_1 + m_3) \times m_a = \frac{1}{\sqrt{3}}(-m_1 + m_3)$

Mom of inertia of each face about diag is same about the line.
Need mom of inertia of one face about that line

$k_1 = \sqrt{\frac{5}{18}}\, a$

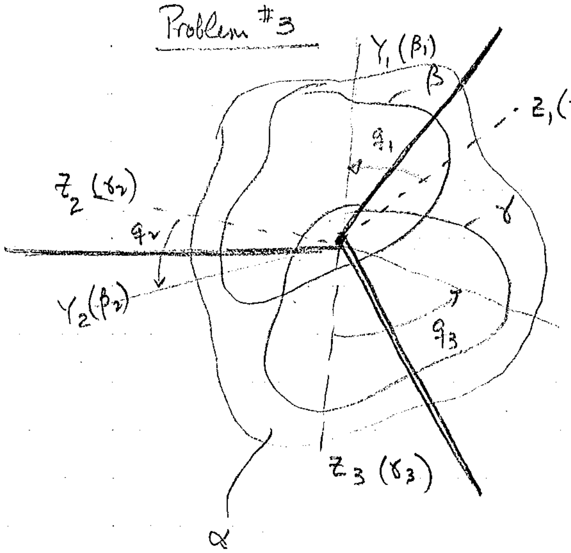
$I_1^P = \frac{ma^2}{12}$ for $I_2^P = \frac{ma^2}{6}$ $\quad I_3^P = \frac{ma^2}{12}$

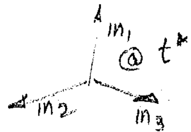
$m_a = \frac{1}{\sqrt{3}}(m_1 + m_2 + m_3)$

$I_{aa}^P = a_1^2 I_1 + a_2^2 I_2 + a_3^2 I_3 = \frac{ma^2}{3}\left(\frac{1}{12} + \frac{1}{6} + \frac{1}{12}\right) = \frac{ma^2}{9}$

$I_{aa}^A = I_{aa}^P + \frac{ma^2}{4}\left[(m_1 + m_2) \times m_a\right]^2 = \frac{5}{18} ma^2$

Problem #3





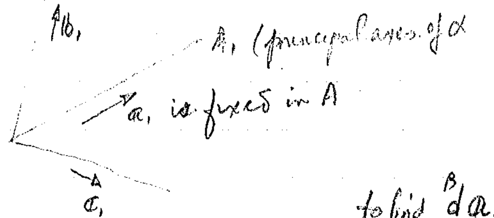
$${}^{\beta}\omega^{\gamma}\Big|_{t^*} = \omega_1 m_1 + \omega_2 m_3 + \omega_3 m_3$$

when $Z_1 \parallel Y_1$, $Z_2 \parallel Y_2$, $Z_3 \parallel Y_3$

what ${}^{\beta}\omega^{A}$

A is a reference frame of central axis of body α. α is a fictitious rigid body

$${}^{\beta}\omega^{A} = \left(\frac{{}^{\beta}\tilde{d}}{dt}a_2 \cdot \tilde{a}_3\right)\tilde{a}_1 + \left(\frac{{}^{\beta}\tilde{d}a_3}{dt} \cdot \tilde{a}_1\right)\tilde{a}_2 + \left(\frac{{}^{\beta}\tilde{d}a_1}{dt} \cdot \tilde{a}_2\right)\tilde{a}_3 \qquad (\tilde{\ }) = (\ )\ @\ t^*$$



let q_1, q_2, q_3 coordinates governing relative orientation of β & γ

$$\text{to find } \frac{{}^{\beta}da_2}{dt} = a_{21}\, b_1 + a_{22}\, b_2 + a_{23}\, b_3$$

$$\frac{{}^{\beta}\tilde{d}a_2}{dt} = \tilde{\dot{a}}_{21}\, b_1 + \tilde{\dot{a}}_{22} b_2 + \tilde{\dot{a}}_{23} b_3$$

$$\tilde{a}_3 = \tilde{b}_3\,, \quad \tilde{a}_1 = \tilde{b}_1$$

$$\text{thus } {}^{\beta}\omega^{A} = \tilde{\dot{a}}_{23}\,\tilde{b}_1 + \tilde{\dot{a}}_{31}\,\tilde{b}_2 + \tilde{\dot{a}}_{12}\,\tilde{b}_3$$

$$= \left(\tilde{a}_{23,r}\,\tilde{b}_1 + \tilde{a}_{31,r}\,\tilde{b}_2 + \tilde{a}_{12,r}\,\tilde{b}_3\right)\dot{q}_r$$

$$= \frac{\tilde{I}_{23,r}}{\tilde{I}_{22} - \tilde{I}_{33}}\,\dot{q}_r\, n_1 + \dots$$

using results obtained in class

Now	c_1	c_2	c_3
b_1	$1+\dots$	$-q_3+\dots$	$q_2+\dots$
b_2	$q_3+\dots$	$1+\dots$	$-q_1+\dots$
b_3	$-q_2+\dots$	$q_1+\dots$	$1+\dots$

need derivatives only.

$$\mathbb{I}^{\alpha} = \mathbb{I}^{\beta} + \mathbb{I}^{\gamma} \quad \text{due to common mass center.}$$

$$= \beta_1 b_1 b_1 + \dots + \gamma_1 c_1 c_1 + \dots$$

$$\tilde{I}_{22} = \beta_2 + \gamma_2\,, \quad \tilde{I}_{33} = \beta_3 + \gamma_3$$

$$I_{23} = b_2 \cdot \mathbb{I}^{\alpha} \cdot b_3 = \gamma_1(b_2 \cdot c_1)(c_1 \cdot b_3)$$

$$+ \gamma_2(b_2 \cdot c_2)(c_2 \cdot b_3) + \gamma_3(b_2 \cdot c_3)(c_3 \cdot b_3)$$

$$= \gamma_1 (q_3 + \dots)(-q_2 + \dots) + \gamma_2 (1 + \dots)(q_1 + \dots) + \gamma_3 (-q_1 + \dots)(1 + \dots)$$

$$= (\gamma_2 - \gamma_3) q_1 + \text{h.o.t.}$$

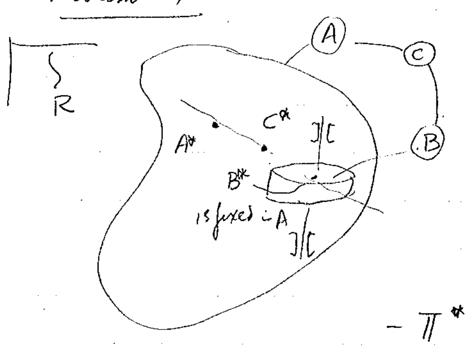
thus $\tilde{I}_{23,1} = \gamma_2 - \gamma_3 , \dots$ thus we can do this more for others.

now ${}^{B}\omega^{\gamma} = \left(\frac{{}^{B}d\tilde{c}_2}{dt} \cdot \tilde{c}_3 \right) \tilde{a}_1 + \dots$

using table $= \dot{\tilde{q}}_1 \tilde{b}_1 + \dot{\tilde{q}}_2 \tilde{b}_2 + \dot{\tilde{q}}_3 \tilde{b}_3$ thus $\dot{\tilde{q}}_1 = \omega_1 , \dots$

thus ${}^{B}\omega^{A} = \dfrac{\gamma_2 - \gamma_3}{\beta_2 + \gamma_2 - \beta_3 - \gamma_1} \omega_1 n_1 + \dots$

Problem # 4



Inertia torque of C in R

now $T^* = \sum r_i \times (-m_i a_i)$ wrt c^*

$$T^{*c} = -\frac{dH^{c/c^*}}{dt}$$

$$-T^* = \frac{{}^{R}d\,{}^{R}H^{c/c^*}}{dt} \qquad (1)$$

Now $${}^{R}H^{c/c^*} = {}^{R}H^{D/D^*} + {}^{A}H^{B/C^*} \qquad (2)$$

$${}^{R}H^{D/D^*} = I^{c/c^*} \cdot {}^{R}\omega^{A} \qquad (3)$$

$${}^{A}H^{B/c^*} = I^{B/B^*} \cdot {}^{A}\omega^{B} + \cancel{{}^{A}H^{B^*/C^*}} \to 0$$ since B^* is fixed in A (4)

Now $$-T^* = \frac{{}^{R}d}{dt}\left[{}^{R}H^{D/D^*} + {}^{A}H^{B/C^*} \right]$$

$$= \frac{{}^{A}d\,{}^{R}H^{D/D^*}}{dt} + {}^{R}\omega^{A} \times \frac{dH^{D/D^*}}{dt} + \frac{{}^{B}d\,{}^{A}H^{B/C^*}}{dt} + {}^{R}\omega^{B} \times {}^{A}H^{B/C^*}$$

Using (3) & (4) $$\frac{{}^{A}d}{dt}\left(I^{c/c^*} \cdot {}^{R}\omega^{A} \right) + {}^{R}\omega^{A} \times \left(I^{c/c^*} \cdot {}^{R}\omega^{A} \right) + \frac{{}^{B}d}{dt}\left(I^{B/B^*} \cdot {}^{A}\omega^{B} \right) + \left({}^{R}\omega^{A} + {}^{A}\omega^{B} \right) \times \left(I^{B/B^*} \cdot {}^{A}\omega^{B} \right)$$

$$\frac{{}^{A}d\, I^{c/c^*}}{dt} = 0 \qquad I^{c/c^*} \cdot {}^{R}\alpha^{A} + {}^{R}\omega^{A} \times \left(I^{c/c^*} \cdot {}^{R}\omega^{A} \right) + I^{B/B^*} \cdot {}^{A}\alpha^{B} + {}^{R}\omega^{A} \times \left(I \cdot {}^{A}\omega^{B} \right) + \underbrace{{}^{A}\omega^{B} \times \left(I^{B/B^*} \cdot {}^{A}\omega^{B} \right)}_{=0 \text{ for a symmetric body.}}$$

$I^{B/B^*} = J\, b_1 b_1 + I \left(b_2 b_2 + b_3 b_3 \right)$ for a symmetric body.

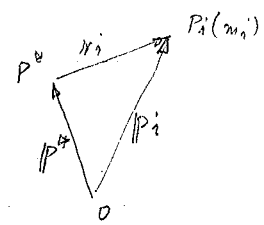
${}^{A}\omega^{B} = b_1 \cdot {}^{A}\omega^{B}$ then last term vanishes.

2/14/80

8a). Go back to definition of $F_r^* = -\int_\sigma \mathbf{a} \cdot \mathbf{v}_{\dot q_r} \, \rho \, d\sigma$

8b) $\mathbf{T}^* = -\sum_{i=1}^{N} m_i \mathbf{r}_i \times \mathbf{a}_i$

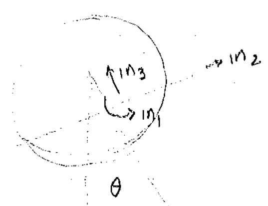
for non rigid body also



$\dot{\mathbf{H}} = \sum m_i \dot{\mathbf{r}}_i \times \mathbf{v}_i + \sum m_i \mathbf{r}_i \times \mathbf{a}$ $\qquad \mathbf{v}_i = \dot{\mathbf{P}}_i$

$= \sum m_i (\mathbf{v}_i - \mathbf{v}^*) \times \mathbf{v}_i - \mathbf{T}^* = \mathbf{v}^* \times \sum m_i \mathbf{v}_i - \mathbf{T}^* = \mathbf{v}^* \times M \mathbf{v}^* - \mathbf{T}^* = -\mathbf{T}^*$

8c) most direct - use eqn 3.44.



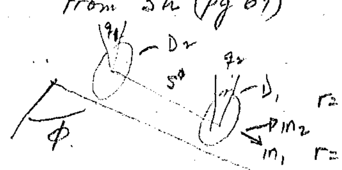
$\omega = \dot\theta \mathbf{n}_1 + \Omega(\sin\theta \, \mathbf{n}_2 + \cos\theta \, \mathbf{n}_3)$

$\omega_{\dot\theta} = \mathbf{n}_1 \quad \Rightarrow \alpha =$

$\mathbf{T}^* = -\frac{mL^2}{2}(\Omega^2 \sin\theta \cos\theta + \ddot\theta)\mathbf{n}_1 + \cdots$

$\mathbf{v}_{\dot\theta} = h\mathbf{n}_2 \quad \Rightarrow \mathbf{a} = -h[\omega \times (\omega \times \mathbf{n}_3) + \alpha \times \mathbf{n}_3]$

8d.) From 3h (pg 69)



	$\mathbf{v}_{\dot q_r}^{S^*}$	$\mathbf{v}_{\dot q_r}^{D_1^*}$	$\mathbf{v}_{\dot q_r}^{D_2^*}$	$\omega_{\dot q_r}^{D_1}$	$\omega_{\dot q_r}^{D_2}$	$\omega_{\dot q_r}^{S}$
$r=1$	$-\frac{r}{2}\mathbf{n}_2$			$\mathbf{n}_1 + \frac{r}{2L}\mathbf{n}_3$		$\frac{r}{2L}\mathbf{n}_3$
$r=2$	$-\frac{r}{2}\mathbf{n}_2$			$\mathbf{n}_1 - \frac{r}{2L}\mathbf{n}_3$		$-\frac{r}{2L}\mathbf{n}_3$

$\dot\phi = (r/2L)(\dot q_1 - \dot q_2)$ $\qquad \mathbf{v}^{S^*} = -\frac{r}{2}(\dot q_1 + \dot q_2)\mathbf{n}_2$

$\omega^S = (r/2L)(\dot q_1 - \dot q_2)\mathbf{n}_3$ $\qquad \mathbf{a}^S = -\frac{r}{2}(\ddot q_1 + \ddot q_2)\mathbf{n}_2 + \cdots$ $\qquad$ only items needed

$\omega^{D_1} = \dot q_1 \mathbf{n}_1 + \frac{r}{2L}(\dot q_1 - \dot q_2)\mathbf{n}_3$ $\qquad \mathbf{a}^{D^*} = -r\ddot q_1 \mathbf{n}_2 + \cdots$

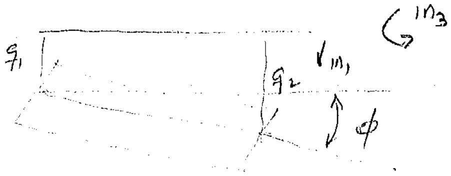
$\alpha^{D_1} = \ddot q_1 \mathbf{n}_1 + \frac{r}{2L}(\ddot q_1 - \ddot q_2)\mathbf{n}_3 + \cdots$

$\mathbf{T}^{D_1} = [(I_2^{D_1} - I_3^{D_1})\omega_2^{D_1}\omega_3^{D_1} - I_1^{D_1}\alpha_1^{D_1}]\mathbf{n}_1 + [\quad]\mathbf{n}_2 + [(I_1^{D_1} - I_2^{D_1})\omega_1\omega_2^{D_1} - I_3^{D_1}\alpha_3^{D_1}]$

$= -I_1^{D_1}\ddot q_1 \mathbf{n}_1 + (\quad)\mathbf{n}_2 - I_2^{D_1}\frac{r}{2L}(\ddot q_1 - \ddot q_2)\mathbf{n}_3$

$\mathbf{T}^S = -I_3^S \frac{r}{2L}(\ddot q_1 - \ddot q_2)\mathbf{n}_3$

8f.



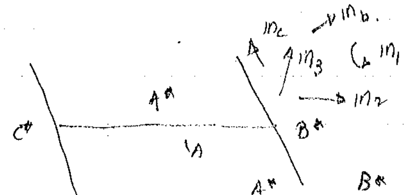
if q_1 & q_2 are small

$$v^{B*} = \frac{\dot{q}_1 + \dot{q}_2}{2} n_1 \qquad a^{B*} = \ldots$$

$$\omega^{B} = -\left(\frac{\dot{q}_2 - \dot{q}_1}{2W}\right) n_3 \qquad \sin\phi = \frac{q_2 - q_1}{2W} \text{ ; for small } q\text{'s}$$

$$\text{let } \dot{\phi} \sim \frac{\dot{q}_2 - \dot{q}_1}{2W}$$

8g.
$$F_r^* = -m\left(v^{A*}_{u_r} \cdot a^{A*} + v^{B*}_{u_r} \cdot a^{B*} + v^{C*}_{u_r} \cdot a^{C*}\right) + \omega^{A}_{u_r} \cdot T^{A} + \omega^{B}_{u_r} \cdot T^{B} + \omega^{C}_{u_r} \cdot T^{C} \qquad (1)$$



	$v^{A*}_{u_i}$	$v^{B*}_{u_i}$	$v^{C*}_{u_i}$	$\omega^{A}_{u_i}$	$\omega^{B}_{u_i}$	$\omega^{C}_{u_i}$
$i=1$	0	$\frac{L}{2} n_3$	$-\frac{L}{8} n_3$	n_1	n_1	n_1
$=4$	0	0	0	0	n_1	n_1
$=7$	n_3	n_3	n_3	0	0	0

$$F_1^* = -m\left(\frac{L}{2} n_3 \cdot a^{B*} - \frac{L}{2} n_3 \cdot a^{C*}\right) + n_1 \cdot T^{A} + n_1 T^{B} + n_1 T^{C}$$

$$F_4^* = n_1 \cdot T^{B} + n_1 \cdot T^{C}$$

$$F_7^* = -m\left(n_3 \cdot a^{A*} + n_3 \cdot a^{B*} + n_3 \cdot a^{C*}\right)$$

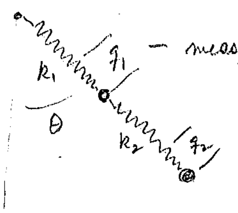
2/19/80

Energy

Deals with scalar quantities which serve 2 purposes

1. Facilitation of the evaluation of generalized forces. (active & inertia)
2. " of integration of equations of motion

Example Sect 3.2 p. 78



q_1 – measured from rest length.

$$(F_1)_{spring} = -k_1 q_1 + k_2(q_2 - q_1)$$

$$(F_2)_{springs} = -k_2(q_2 - q_1)$$

Let P be such that $P \triangleq \frac{1}{2}k_1 q_1^2 + \frac{1}{2}k_2(q_2 - q_1)^2$

$$\Rightarrow \quad -\frac{\partial P}{\partial q_1} = F_1 = -k_1 q_1 + k_2(q_2 - q_1)$$

$$-\frac{\partial P}{\partial q_2} = F_2 = -k_2(q_2 - q_1)$$

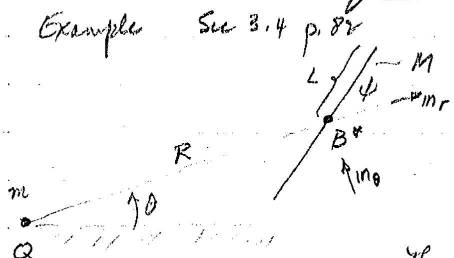
To generalize if $P(q_1, \dots, q_n; t)$ exists such that $\frac{\partial P}{\partial q_r} = -(F_r)_S$ $(r = 1, \dots, n)$ then P is called a potential function for S.

S: a system of forces that contribute to generalized active forces.

Where does one get the P.

1. Form F_r first (not best way if you want F_r)

Example Sec 3.4 p. 82



Generalized Coordinates for the rod R, θ, ψ.

Force system: $\mathbb{F} = -\frac{GmM}{R^2} n_r$

$$\mathbb{T} = \left(\frac{GmML^2}{2R^3} \sin 2\psi\right) n_\theta \times n_r$$

Thus we can get generalized forces

$$F_R = -GmM/R^2 \qquad F_\theta = -G(mML^2/2R^3)\sin 2\psi$$

$$F_\psi = -(GmML^2/2R^3)\sin 2\psi$$

$$F_R = -\frac{\partial P}{\partial R} \Rightarrow P = -\frac{GmM}{R} + f(\theta, \psi, t) \quad \text{There is } \underline{\text{no}}\ R \text{ in } f$$

$$F_\theta = -\frac{\partial P}{\partial \theta} \Rightarrow -\frac{\partial P}{\partial \theta} = -\frac{\partial f}{\partial \theta} \Rightarrow f = \frac{GmML^2}{2R^3}(\sin 2\psi)\theta + g(\psi, t)$$

but $f(\theta, \psi, t)$ has a fn of R $\Rightarrow$ P <u>doesn't exist!</u>

Suppose only ψ is a generalized coordinate ie θ & R are prescribed

$\Rightarrow$ F_ψ is not affected & $= -G(mML^2/2R^3)\sin 2\psi$

$$F_\psi = -\frac{\partial P}{\partial \psi} = -\frac{G m M L^2}{2R^3}\sin 2\psi \qquad P = \frac{-GmML^2\cos 2\psi}{4R^3} + f(t)$$

in this case P does exist

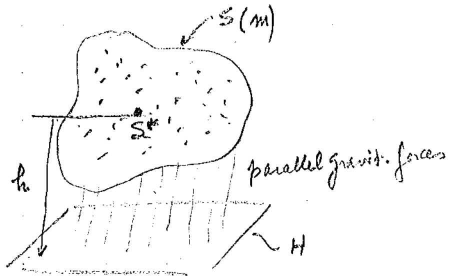
we may also write P in terms of moments of inertia

$P = \frac{3Gm}{2R^3} I_r + g(t)$ where I_R is the moment of inertia of B about line QB^*

Under what conditions does P exist -

Situations in which P is readily available

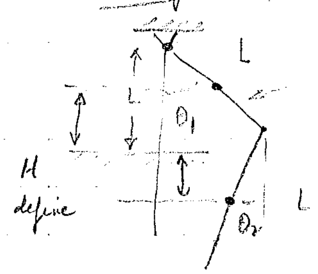
- Parallel Gravitational forces



$P = mgh + f(t)$ — g is the local acceleration of gravity, h is distance from masscenter to plane H

when S^* is above H $\Longleftrightarrow$ +

Example double pendulum



$$P_1 = mg\left(L - \tfrac{L}{2}\cos\theta_1\right)$$
$$P_2 = -mg\left(L\cos\theta_1 + \tfrac{L}{2}\cos\theta_2 - L\right)$$
$$P = P_1 + P_2 = mg\left(L - \tfrac{L}{2}\cos\theta_1 - L\cos\theta - \tfrac{L}{2}\cos\theta_2 + L\right)$$
$$= -\frac{mgL}{2}\left[3\cos\theta_1 + \cos\theta_2\right] + g(t)$$

we leave out of P constant portions

- Linear Springs

$P = \frac{1}{2}kx^2 + g(t)$ — x is the spring extension or contraction ie find current length and then subtract the natural length; take absolute value

Potential Energy not same as potential fn.

if P exists $\ni$ $F_r = -\frac{\partial P}{\partial q_r}$ $(r = 1, \ldots, n)$ then P is called a potential energy and the system under consideration is said to be conservative

Potential energy involves all the forces acting on the body

Dissipation fn.

If $\mathcal{F}(q_1,\dots,q_n,\dot q_1,\dots,\dot q_n,t)$ exists such that $(F_r)_S = -\frac{\partial \mathcal{F}}{\partial \dot q_r}$ $(r=1,\dots,n)$

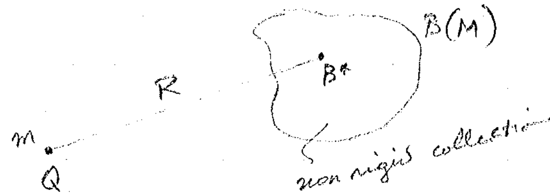
then $\mathcal{F}$ is called a dissipation function for S

Situations in which $\mathcal{F}$ is readily available

i.e. forces proportional to velocity $\mathbb{F}_i = -c\,\mathbb{V}_i$ $\quad \mathcal{F} = \frac{c}{2}\sum_{i=1}^{N} \mathbb{V}_i^2 + f(t)$

dashpots, damping

Now given P can one find actual forces – <u>yes</u>

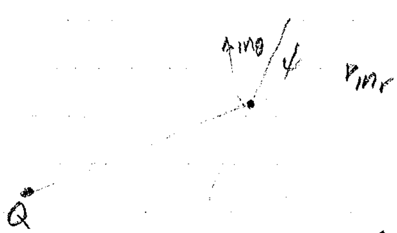


$$P = -Gm\left[\frac{M}{R} - \frac{3I-(A+B+C)}{2R^3}\right]$$

of B in field of m

I – mom. of inertia of B about line Q-B*

A, B, C are central principal moments of inertia of the body.



$I = \frac{ML^2}{3}\sin^2\psi \qquad A+B+C = \frac{2ML^2}{3}$ using the above

the worse this implies Force system: $\mathbb{G}$ at B^*: $\mathbb{G} = G_r m_r + G_\theta m_\theta$

Couple, torque $\mathbb{T}$: $T\, m_r \times m_\theta$

Now must find $\mathbb{T}$ & $\mathbb{G}$

$$\mathbb{V}^* = \dot R m_r + r\dot\theta m_\theta, \quad \omega = (\dot\theta + \dot\psi)\, m_r \times m_\theta$$

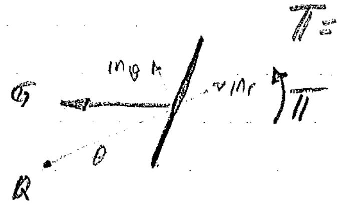
thus $F_R = \mathbb{V}^*_{\dot R}\cdot \mathbb{G} + \omega_{\dot R}\cdot \mathbb{T} = G_r = -\frac{\partial P}{\partial R} = Gm\left[-\frac{M}{R^2} + 3\,\frac{(3I-(A+B+C))}{2R^4}\right]$

$F_\theta = \mathbb{V}^*_{\dot\theta}\cdot \mathbb{G} + \omega_{\dot\theta}\cdot \mathbb{T} = RG_\theta + T = -\frac{\partial P}{\partial \theta} = 0 \Rightarrow G_\theta = -T/R$

$F_\psi = \mathbb{V}^*_{\dot\psi}\cdot \mathbb{G} + \omega_{\dot\psi}\cdot \mathbb{T} = T = -\frac{\partial P}{\partial \psi} = -\frac{GMmL^2}{2R^3}\sin 2\psi$ now integrate

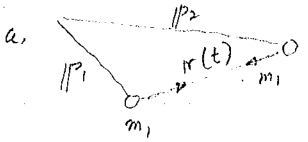
thus we can get $\mathbb{G} = -\frac{GmM}{R^2}\left[1 - 3\,\frac{L^2(\sin^2\psi - 2/3)}{2R^2}\right] m_r + \frac{GmML^2}{2R^4}\sin 2\psi\, m_\theta$

$$\mathbb{T} = -\frac{GmML^2}{2R^3}\sin 2\psi\, m_r \times m_\theta$$



here we get a contradiction to what we began the period with. However note that $\mathbb{G}$ is different than what we had

Problem Set # 9

a. 

$\mathbb{F}_1 = \frac{G m_1 m_2}{r^2}\ m_1 \qquad \mathbb{F}_2 = -\mathbb{F}_1$

must get generalized force

$$F_r = -\frac{G m_1 m_2}{r^2}\frac{\partial r}{\partial q_r} = -\frac{\partial P}{\partial q_r} \qquad q_r \text{ are generalized coords.}$$

$$\frac{\partial}{\partial q_r}\left(\frac{G m_1 m_2}{r}\right) = -\frac{\partial P}{\partial q_r}$$

b. Using replacement of system of forces : $\mathbb{T}$ & $\mathbb{F}$ at B^*

define Orientation of B : q_1, q_2, q_3 by either body fixed or space fixed axes.

Let $\frac{3GM}{R^3} = k$

$$F_r = k\, {}^R\omega^B_{\dot q_r} \cdot \mathbf{n}_a \times \mathbb{I}_a = \omega_{\dot q_r} \cdot \mathbb{T}$$

$$= k\, {}^R\omega^B_{\dot q_r} \times \mathbf{n}_a \cdot \mathbb{I}_a$$

Observation $\quad \frac{{}^R\partial \mathbf{n}_a}{\partial q_r} = 0 \qquad \frac{{}^R\partial \mathbf{n}_a}{\partial q_r} = \frac{{}^B\partial \mathbf{n}_a}{\partial q_r} + {}^R\omega^B_{\dot q_r} \times \mathbf{n}_a \Rightarrow F_r = -k\frac{{}^B\partial \mathbf{n}_a}{\partial q_r}\cdot \mathbb{I}_a$

since $\mathbf{n}_a$ is independent of q_r in R

let $\mathbf{n}_a = a_1\mathbf{n}_1 + a_2\mathbf{n}_2 + a_3\mathbf{n}_3$ w/ $\mathbf{n}_i$ being central principal axes of inertia

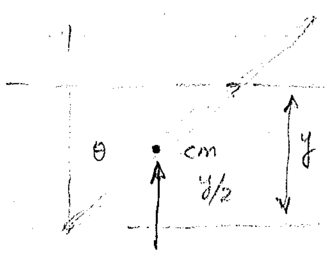
$$\mathbb{I}_a = \mathbb{I}\cdot\mathbf{n}_a = I_1 a_1 \mathbf{n}_1 + \cdots$$

$$\frac{\partial \mathbf{n}_a}{\partial q_r}\cdot\mathbb{I}_a = \frac{\partial a_1}{\partial q_r} a_1 I_1 + \frac{\partial a_2}{\partial q_r} a_2 I_2 + \frac{\partial a_3}{\partial q_r} a_3 I_3$$

$$= \frac{1}{2}\frac{\partial}{\partial q_r}\left(a_1^2 I_1 + a_2^2 I_2 + a_3^2 I_3\right)$$

$$I_a = \mathbf{n}_a\cdot\mathbb{I}\cdot\mathbf{n}_a \qquad \mathbb{I}^{B/B^*}$$

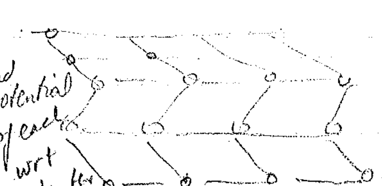
$$\therefore \quad F_r = -\frac{\partial}{\partial q_r}\left(\frac{kI}{2}\right)$$

c. 

use buoyancy concept - Archimedes principle

$|F| \sim$ fluid displaced

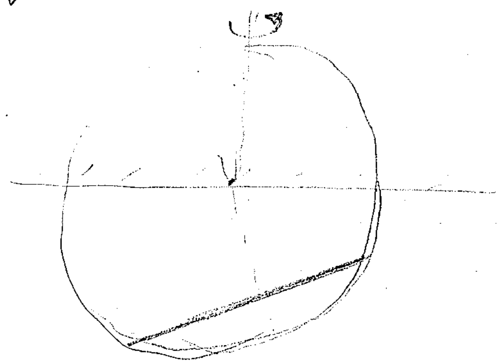
$$P = \frac{y}{2}\left(\underbrace{w\,\overbrace{A\,y\sec\theta}^{\text{volume of immersed rod.}}}_{\text{weight of displaced fluid}}\right)$$

d. (5 b) 

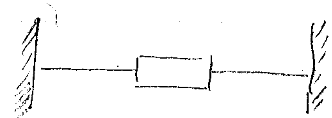
find potential for each rod wrt mass center then add

given F_1, F_2, F_3

(9e) fact that problem is driven doesn't hinder finding a potential energy. (5a)



(9f) (5d)



must go back to fundamentals.

$F_1 = a_1 q_1 + b_1 q_2 + c_1$

$F_2 = a_2 q_1 + b_2 q_2 + c_2$

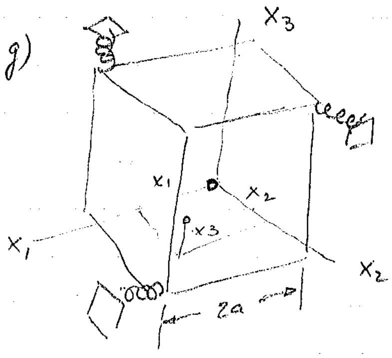
a_i, b_i, c_i are constants from (5d)

$\frac{\partial P}{\partial q_1} = -F_1 = -(a_1 q_1 + \dots)$ integrate

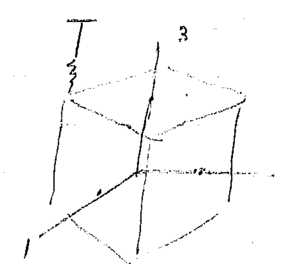
$P = -a_1 \frac{q_1^2}{2} - b_1 q_1 q_2 - c_1 q_1 + f(q_2)$; $-F_2 = \frac{\partial P}{\partial q_2} = -b_1 q_1 + f'(q_2) = -a_2 q_1 + b_2 q_2 + c_2$

works only if $b_1 = a_2$

(9g)



body 123 rotations + displacement of cm



"elegant" sol.
movement in 1 & 2 direct produce higher order terms but movement in 3 direction is $-x_3$.

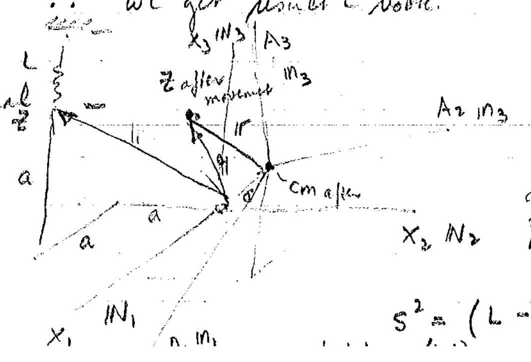
rotation about 3 gives higher order in 3 direct but rotation about 1 & 2 give $a\theta$ displacement.

thus for this spring total displ $d = -x_3 + a\theta_1 + a\theta_2$ thus $P = \frac{1}{2} k d^2$

now for other 3 we get similar results for cyclical permutation

∴ we get result in book.

More formal method



$\mathbb{p} = a\left[\mathbb{N}_1 - \mathbb{N}_2 + \left(1 + \frac{L}{a}\right)\mathbb{N}_3\right]$ (1)

$\mathbb{c} = x_1 \mathbb{N}_1 + x_2 \mathbb{N}_2 + x_3 \mathbb{N}_3$ (2)

distance to corner if fixed $\mathbb{r} = a(\mathbb{n}_1 - \mathbb{n}_2 + \mathbb{n}_3)$. (3)

$\mathbb{q} = \mathbb{c} + \mathbb{r} = \dots$ (2,3) (4)

$s^2 = (L - |\mathbb{q} - \mathbb{p}|)^2$

$$\mathbb{q}-\mathbb{p} = (x_1 - a)\,\mathbb{N}_1 + (x_2 + a)\,\mathbb{N}_3 + (x_3 - a - L)\,\mathbb{N}_3 + a(\mathbb{m}_1 - \mathbb{m}_2 + \mathbb{m}_2) \qquad (6)$$

let $\beta_{ij} = \mathbb{N}_i \cdot \mathbb{m}_j$ (7)

$$\mathbb{m}_1 = \beta_{11}\mathbb{N}_1 + \beta_{21}\mathbb{N}_2 + \beta_{31}\mathbb{N}_3$$

now using 6 & 7 gives

$$\mathbb{q}-\mathbb{p} = [x_1 + a(-1 + \beta_{11} - \beta_{12} + \beta_{13})]\mathbb{N}_1 + \cdots \qquad (8)$$ see problem 2a for β_{ij}

$$-1 + \beta_{11} + \beta_{12} + \beta_{13} = -1 + c_2 c_3 + c_2 s_3 + s_2 \qquad (9)$$ now using taylor's series

$$= \theta_3 + \theta_2 + O_2(\theta_2, \theta_3)$$

← terms of degree 2 in θ_2 & θ_3

$$1 + \beta_{21} - \ldots = \theta_3 - \theta_1 + O_2(\theta_1, \theta_2, \theta_3) \qquad (10)$$

$$-1 + \beta_{31} + \ldots = -\theta_2 - \theta_1 + O_2(\theta_2, \theta_3) \qquad (11)$$

$$\mathbb{q}-\mathbb{p} \underset{(8-10)}{=} -L\mathbb{N}_3 + a\left[\left(\frac{x_1}{a} + \theta_2 + \theta_3\right)\mathbb{N}_1 + \left(\frac{x_2}{a} + \theta_3 - \theta_1\right)\mathbb{N}_2 + \left(\frac{x_3}{a} - \theta_1 - \theta_2\right)\mathbb{N}_3\right]$$

$$+ O_2(\theta_1, \theta_2, \theta_3) \qquad (12)$$

$$(\mathbb{q}-\mathbb{p})^2 \underset{(12)}{=} L^2\left[1 - \frac{2a}{L}\left(\frac{x_3}{a} - \theta_1 - \theta_2\right) + O_2(x_1, x_2, x_3, \theta_1)\right] \qquad (13)$$

$$|\mathbb{q}-\mathbb{p}| = [(\mathbb{q}-\mathbb{p})^2]^{1/2} = L\left[1 - \frac{a}{L}\left(\frac{x_3}{a} - \theta_1 - \theta_2\right) + O_2(x_1, x_2, x_3, \theta_1)\right]$$ using binom (. Expansion

$$S = L - |\mathbb{q}-\mathbb{p}| = a\left(\frac{x_3}{a} - \theta_1 - \theta_2\right) + O_2(x_1, x_2, \ldots$$

$$S^2 = a^2\left(\frac{x_3}{a} - \theta_1 - \theta_2\right)^2 + O_3(\ldots\ldots) \rightarrow \text{drop}$$

2/26/80

Kinetic Energy of system of particles $P_1, \ldots, P_n$ in a reference frame R is defined as

$$K = \frac{1}{2}\sum_{i=1}^{N} m_i \mathbb{v}_i^2 \qquad {}^R\mathbb{v}^{P_i} \text{ - velocity of particles}$$

$\mathbb{v}$ for holonomic system $= f(q_1, \ldots, q_n, \dot{q}_1, \ldots, \dot{q}_n, t)$. has n degrees of freedom. It then satisfies

Theorem: $$F_r^* = \frac{\partial K}{\partial q_r} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_r}\right)$$

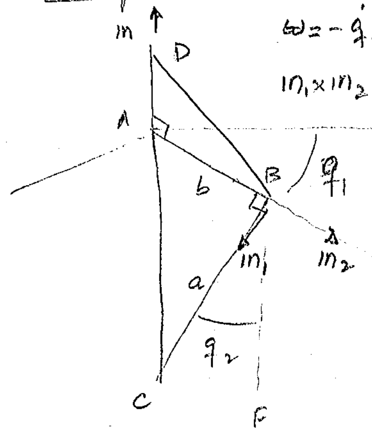
Proof: $$\mathbb{v}_{\dot{q}_r} \cdot \mathbb{a} = \frac{1}{2}\left(\frac{d}{dt}\frac{\partial \mathbb{v}^2}{\partial \dot{q}_r} - \frac{\partial \mathbb{v}^2}{\partial q_r}\right)$$

$$F_r^* = -\sum_{i=1}^{N} m_i \mathbb{v}_{\dot{q}_r}^{P_i} \cdot \mathbb{a}_i = -\sum_{i=1}^{N} \frac{m_i}{2}\left(\frac{d}{dt}\frac{\partial \mathbb{v}^2}{\partial \dot{q}_r} - \frac{\partial \mathbb{v}^2}{\partial q_r}\right)$$

$$= \frac{\partial}{\partial q_r}\left(\sum_i \frac{m_i}{2}\mathbb{v}_i^2\right) - \frac{d}{dt}\frac{\partial}{\partial \dot{q}_r}\left(\sum \frac{m_i}{2}\mathbb{v}_i^2\right)$$

$$\frac{\partial}{\partial q_r}K - \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_r}$$

Example : 8a



$\omega = -\dot q_1 n - \dot q_2 n_2$

$n_1 \times n_2 = n_3$

$$F_2^* = \frac{ma}{12}\left[2a\left(s_2 c_2 \dot q_1^2 - \ddot q_2\right) - 3b\, c_2 \ddot q_1\right]$$

$s_2 \triangleq \sin q_2 \qquad c_2 \triangleq \cos q_2$

$$\Longrightarrow \quad K = \frac{m}{2}\left[\left(\frac{b^2}{2} + \frac{a^2}{6}s_2^2\right)\dot q_1^2 + \frac{1}{2}ab\, c_2 \dot q_1 \dot q_2 + \frac{a^2}{6}\dot q_2^2\right]$$

to find F_2^*

$$\partial K/\partial q_2 = \frac{m}{2}\left[\frac{a^2}{6} 2 s_2 c_2 \dot q_1^2 - \cancel{\frac{1}{2}ab\, s_2 \dot q_1 \dot q_2}\right]$$

$$\frac{\partial K}{\partial \dot q_2} = \frac{m}{2}\left[\frac{1}{2}ab\, c_2 \dot q_1 + \frac{2a^2}{6}\dot q_2\right]$$

$$\frac{d}{dt}\left(\frac{\partial K}{\partial \dot q_2}\right) = \frac{m}{2}\left[\frac{1}{2}ab\, c_2 \ddot q_1 - \cancel{\frac{1}{2}ab\, s_2 \dot q_1 \dot q_2} + \frac{a^2}{3}\ddot q_2\right]$$

$$F_2^* = \frac{\partial K}{\partial q_2} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot q_2}\right) = \frac{m}{2}\left[\qquad\qquad\qquad\qquad\right] \checkmark$$

Thus for a particle

$$F_r^* = -m\, v_{\dot q_r}\cdot a = \frac{\partial K}{\partial q_r} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot q_r}\right)$$

To find KE for a rigid body.

Kinetic Energy of translation of B in R

$$K_v = \frac{1}{2} m\, v^2 \qquad {}^R v^B - \text{of mass center}$$

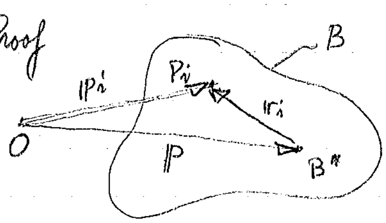
Kinetic Energy of Rotation of B in R

$$K_\omega = \frac{1}{2}\,\omega\cdot \mathbb{I}\cdot\omega \qquad {}^R\omega^B \text{ of body } B.$$

$\mathbb{I}$ inertia dyadic of B for B^*

Thm: For a rigid body $K = K_v + K_\omega$

Proof



$${}^R v^{P_i} = {}^R v^{B^*} + {}^R\omega^B \times r_i$$

$$K = \frac{1}{2}\sum m_i\, v^{P_i\,2}$$

$$= \frac{1}{2}\sum m_i\left[{}^R v^{B^*\,2} + 2\,{}^R v^{B^*}\cdot(\omega\times r_i) + (\omega\times r_i)^2\right]$$

$$= \frac{1}{2} m\, v^{B^*\,2} + \cancel{{}^R v^{B^*}\cdot\left(\omega\times\textstyle\sum m_i r_i\right)}^{\,0} + \frac{1}{2}\omega\cdot\mathbb{I}\cdot\omega$$

Alternate Expression for $K_\omega = \frac{1}{2}\omega \cdot \mathbb{I} \cdot \omega$

$K_\omega = \frac{1}{2} I \omega^2$ $\quad$ I – moment of inertia of B about a line through B^* & $\parallel$ to ω

$K_\omega = \frac{1}{2}\left[I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 \right]$ $\quad$ I_1, I_2, I_3 are moments of inertia about mutually $\perp$ axis passing through B^* and ω_i are the measure numbers of angular velocity.

$K_\omega = \frac{1}{2} \omega_j I_{jk} \omega_k$ $\quad$ einstein notation

Thm: if one point of Rigid body is fixed in R. Then $K = K_\omega$ where I is wrt point P

From previous example:

$$I_{ij}^A = \begin{bmatrix} \frac{mb^2}{2} & -\frac{mab}{4} & 0 \\ -\frac{mab}{4} & \frac{ma^2}{6} & 0 \\ 0 & 0 & \frac{m}{2}\left(\frac{a^2}{3} + b^2\right) \end{bmatrix} \qquad \omega_1 = c_2 \dot{q}_1 \quad \omega_2 = -\dot{q}_2 \quad \omega_3 = -s_2 \dot{q}_1$$

$$K = \frac{1}{2}\left[I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2 + 2 I_{12} \omega_1 \omega_2 \right] = \frac{1}{2}\left[\frac{mb^2}{2} c_2^2 \dot{q}_1^2 + \frac{ma^2}{6} \dot{q}_2^2 + \frac{m}{2}\left(\frac{a^2}{3} + b^2\right) s_2^2 \dot{q}_1^2 + -\frac{mab}{2}\left(-c_2 \dot{q}_1 \dot{q}_2\right) \right]$$

What is the character of Kinetic Energy

$$\underset{1\times n}{\dot{q}} = \left[\dot{q}_1, \dots, \dot{q}_n\right] \quad (1) \qquad \underset{1\times N}{\mathbb{V}} = \left[\mathbb{V}^{P_1}, \mathbb{V}^{P_2}, \dots, \mathbb{V}^{P_N} \right] \quad (2)$$

$$\underset{N\times N}{m} = \begin{bmatrix} m_1 & & 0 \\ & \ddots & \\ 0 & & m_N \end{bmatrix} \quad (3) \qquad \underset{1\times N}{\mathbb{V}_t} = \left[\mathbb{V}_t^{P_1}, \dots, \mathbb{V}_t^{P_N} \right] \quad (4)$$

$$\underset{n\times N}{\mathbb{V}_{\dot{q}}} = \begin{bmatrix} \mathbb{V}_{\dot{q}_1}^{P_1} & \dots, & \mathbb{V}_{\dot{q}_1}^{P_N} \\ \vdots & & \\ \mathbb{V}_{\dot{q}_n}^{P_1} & \dots & \mathbb{V}_{\dot{q}_n}^{P_N} \end{bmatrix} \quad (5) \qquad \Rightarrow \quad \mathbb{V} = \dot{q}\, \mathbb{V}_{\dot{q}} + \mathbb{V}_t \quad N \text{ eqns.} \quad (6)$$

$$2K = \mathbb{V} m \mathbb{V}^T \overset{(b)}{=} (\dot q \mathbb{V}_{\dot q} + \mathbb{V}_t) m (\mathbb{V}_{\dot q}^T \dot q^T + \mathbb{V}_t^T) = \dot q \mathbb{V}_{\dot q} m \mathbb{V}_{\dot q}^T \dot q^T + \underbrace{\dot q \mathbb{V}_{\dot q} m \mathbb{V}_t^T}_{1\times 1 \text{ matrix}} + \underbrace{\mathbb{V}_t m \mathbb{V}_{\dot q}^T \dot q^T}_{1\times 1 \text{ matrix}} + \mathbb{V}_t m \mathbb{V}_t^T$$

$$= \dot q \mathbb{V}_{\dot q} m \mathbb{V}_{\dot q}^T \dot q^T + 2 \dot q \mathbb{V}_{\dot q} m \mathbb{V}_t^T + \mathbb{V}_t m \mathbb{V}_t^T$$

$$K_0 = \tfrac{1}{2} \mathbb{V}_t m \mathbb{V}_t^T \qquad K_1 = \dot q \mathbb{V}_{\dot q} m \mathbb{V}_t^T \qquad K_2 = \tfrac{1}{2} \dot q \mathbb{V}_{\dot q} m \mathbb{V}_{\dot q}^T \dot q^T$$

homogeneous fns of $\dot q$ of order 0, 1, 2.

Homogeneous fn. has following prop $f(kx_1, \dots kx_n) = k^M f(x_1, \dots, x_n)$

Look at K_2

we define $M = \mathbb{V}_{\dot q} m \mathbb{V}_{\dot q}^T = [n \times n]$ $\qquad M_{ik}$ (are inertia coefficients) $= M_{ki}$

a generalized mass matrix

$$\frac{\partial K_2}{\partial \dot q} \dot q^T = 2K_2 \qquad \frac{\partial K_2}{\partial \dot q} = \left[\frac{\partial K_2}{\partial \dot q_1}, \dots, \frac{\partial K_2}{\partial \dot q_n} \right]$$

$$K_2 = \tfrac{1}{2} \dot q M \dot q^T = \sum_{s=1}^{n} \sum_{r=1}^{n} \frac{\dot q_r M_{rs} \dot q_s}{2}$$

$$\frac{\partial K_2}{\partial \dot q_j} = \tfrac{1}{2}\left[\sum_r \sum_s \frac{\partial \dot q_r}{\partial \dot q_j} M_{rs} \dot q_s + \dot q_r M_{rs} \frac{\partial \dot q_s}{\partial \dot q_j} \right]$$

$$= \tfrac{1}{2}\left[\sum_r \sum_s \delta_{rj} M_{rs} \dot q_s + \dot q_r M_{rs} \delta_{sj} \right]$$

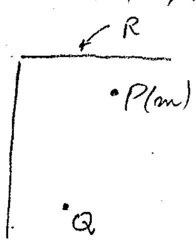
$$= \tfrac{1}{2}\left[\sum_s M_{js} \dot q_s + \sum_r \dot q_r M_{rj} \right] = \tfrac{1}{2}\left[\sum_z M_{jz} \dot q_z + M_{zj} \dot q_z \right]$$

$$= \sum_z M_{zj} \dot q_z = \dot q M$$

$$\frac{\partial K_2}{\partial \dot q_j} \dot q_j^T = \sum_j \sum_z M_{zj} \dot q_z \dot q_j = \dot q M \dot q^T$$

2/28/80

Relative & Absolute KE



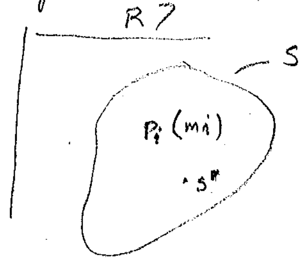
given particle P w/ mass m and a point Q

Absolute $\quad {}^R K^P \triangleq \frac{1}{2} m ({}^R\mathbb{V}^P)^2$ absolute velocity in R

but in reference frame R

Relative $\quad {}^R K^{P/Q} \triangleq \frac{1}{2} m ({}^R\mathbb{V}^{P/Q})^2$ relative velocity of P wrt Q

For a system



$${}^R K^S = \sum m_i ({\mathbb{V}}^{P_i})^2$$

$${}^R K^{S/S^*} + {}^R K^{S^*} = {}^R K^S \qquad {}^R K^{S^*} = \frac{(\sum m_i)}{2} (\mathbb{V}^{S^*})^2 = \frac{m}{2} ({}^R \mathbb{V}^{S^*})^2$$

This is Koenig's Theorem.

$${}^R \mathbb{V}^{P/Q} = \mathbb{V}^P - \mathbb{V}^Q = \frac{{}^R d\, \mathbb{r}^{P/Q}}{dt} \qquad \text{P-relative to Q}$$

Virtual work.

$$\underset{1\times N}{\mathbb{F}} \triangleq [\mathbb{F}_1, \mathbb{F}_2, \ldots, \mathbb{F}_N] \qquad \text{Active Force Matrix} \qquad (1)$$

$$\underset{1\times N}{\mathbb{F}^*} \triangleq [\mathbb{F}_1^*, \ldots, \mathbb{F}_N^*] \qquad \text{inertia forces} \qquad (2)$$

mutually compatible virtual displ matrix $\underset{1\times N}{\delta \mathbb{p}} \triangleq [\delta \mathbb{p}_1, \ldots, \delta \mathbb{p}_n]$ all virtual displacements are mutually compat (3)

$$\delta \mathbb{p}_i \underset{(2.38)}{=} \sum_{r=1}^{n-m} \tilde{\mathbb{V}}_{\dot q_r}^{P_i} \delta q_r \qquad \text{must be dimensional correct.}$$

arbitrary quantities matrix $\underset{1\times n-m}{\delta q} = [\delta q_1, \ldots, \delta q_{n-m}]$ (4)

generalized active force matrix $\underset{1\times(n-m)}{F} = [F_1, \ldots, F_{n-m}]$ (5)

generalized inertia force mat $\underset{1\times(n-m)}{F^*} = [F_1^*, \ldots, F_{n-m}^*]$ (6)

non holonomic partial rates of change. $\underset{(n-m)\times N}{\tilde{\mathbb{W}}_{\dot q}} = \begin{bmatrix} \tilde{\mathbb{V}}_{\dot q_1}^{P_1} & \cdots & \tilde{\mathbb{V}}_{\dot q_1}^{P_N} \\ \vdots & & \\ \tilde{\mathbb{V}}_{\dot q_{n-m}}^{P_1} & \cdots & \end{bmatrix}$ (7)

Using the above

$$\delta \mathbb{p} \underset{(2.38)}{=} \delta q\, \tilde{\mathbb{W}}_{\dot q} \qquad (8)$$

$$F \underset{(3.3)}{=} \mathbb{F}\, \tilde{\mathbb{W}}_{\dot q}^T \qquad (9)$$

$$F^* \underset{(3.8)}{=} \mathbb{F}^* \tilde{\mathbb{W}}_{\dot q}^T \qquad (10)$$

Definitions: $\mathbb{F}\, \delta \mathbb{p}^T \triangleq \delta W$ (virtual work of active forces) (11)

$\delta W^* \triangleq \mathbb{F}^* \delta \mathbb{p}^T$ (" " " inertia forces) (12)

Theorems:

(1) $\delta W = F\,\delta q^T$ (13)

Proof

$$\delta W \underset{(11)}{=} \mathbb{F}\cdot\Big(\delta q\,\underset{(8)}{\tilde{\mathbb{V}}_{\dot q}}\Big)^T$$

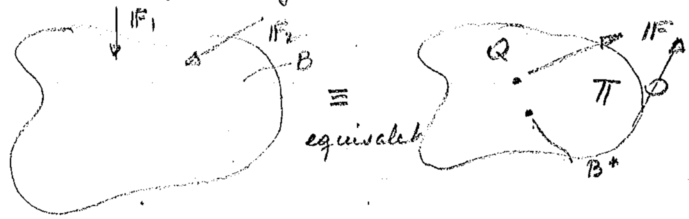
$$= \mathbb{F}\,\tilde{\mathbb{V}}_{\dot q}^{\,T}\,\delta q^T \underset{(9)}{=} F\,\delta q^T \quad \text{QED.}$$

(2) $\delta W^* = F^*\delta q^T$ (14)

$$= \mathbb{F}^*\big(\tilde{\mathbb{W}}_{\dot q}\,\delta q\big)^T = F^*\delta q^T$$

For rigid body

1. Rigid Body contribution to δW



$\mathbb{F} = \sum \mathbb{F}_i$

$\mathbb{T} = \sum \mathbb{r}_i \times \mathbb{F}_i$

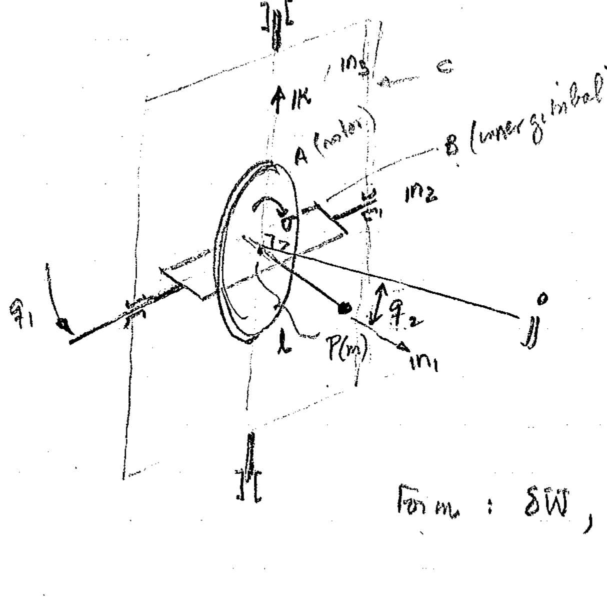
$$(\delta W)_B = \underbrace{\mathbb{F}\cdot\delta q}_{\text{virt disp of point Q}} + \mathbb{T}\cdot\underbrace{\delta\alpha}_{\text{virt rotation of body B}} \qquad (15)$$

$$(\delta W)_G = mg\,\mathbb{K}\cdot\underbrace{\delta p^*}_{\text{virtual disp of mass center}} \qquad (16)$$

Note: Forces of interaction depending on situation may or may not contribute

$$(\delta W^*)_B = \mathbb{F}^*\cdot\delta p^* + \mathbb{T}^*\cdot\delta\alpha \qquad (17)$$

Use of these results will be made on a gyroproblem,



$\sigma = {}^B\omega^A\cdot\mathbb{n}_1$

$\mathbb{j} = \mathbb{n}_2\times\mathbb{K}$

Motion of interest

$\dot q_1 = p \triangleq$ const

$q_2 = \bar q_2$ a constant

Find relationships between $\bar q_2, p, \sigma$, mgL and moments of inertia for steady precessional motion

Form: δW, δW^* $\Rightarrow$ $\delta W + \delta W^* = 0$ principle of virtual work

$\delta W_A = \mathbb{T} \cdot \delta \alpha^A$ $\qquad \delta W_B = -\mathbb{T} \cdot \delta \alpha^B$ (law of action-reaction) forces act at center & give no contrib ($\mathbb{V}^P = 0$).

$\delta W_C = 0$ $\qquad \delta W_P = -mg\,\mathbb{K} \cdot \delta \mathbb{p}$

$\delta W_A^* = \mathbb{T}_A^* \cdot \delta \alpha^A$ $\qquad \delta W_B^* = \mathbb{T}_B^* \cdot \delta \alpha^B$ $\qquad \delta W_C^* = \mathbb{T}_C^* \cdot \delta \alpha^C$

$$\delta \tilde{W}_P^* = mL\tilde{c}_2 p^2 \tilde{\mathbb{j}} \cdot \delta \tilde{\mathbb{p}} \quad \text{(during the motion of interest only)}$$

p moves about in a circle, $\mathbb{F} = -m a^P = -$ mass · radius · angular velo ^{2}.

$$\omega^A = \sigma \mathbb{n}_1 + \dot{q}_1 \mathbb{K} + \dot{q}_2 \mathbb{n}_2$$
$$\tilde{\omega}^A = \sigma \tilde{\mathbb{n}}_1 + p\mathbb{K} = (\sigma - p\tilde{s}_2)\tilde{\mathbb{n}}_1 + p\tilde{c}_2 \tilde{\mathbb{n}}_3$$
$$\omega^B = \dot{q}_2 \mathbb{n}_2 + \dot{q}_1 \mathbb{K} \Rightarrow \tilde{\omega}^B = -p\tilde{s}_2 \tilde{\mathbb{n}}_1 + p\tilde{c}_2 \tilde{\mathbb{n}}_3$$
$$\omega^C = \dot{q}_1 \mathbb{K}, \quad \tilde{\omega}^C = p\mathbb{K}$$

$$\delta \tilde{\alpha}^A = \delta q_1 \mathbb{K} + \delta q_2 \tilde{\mathbb{n}}_2 \qquad \delta \tilde{\alpha}^B = \delta q_1 \mathbb{K} + \delta q_2 \tilde{\mathbb{n}}_2$$
$$\delta \tilde{\alpha}^C = \delta q_1 \mathbb{K} \qquad \delta \tilde{\mathbb{p}} = \delta \tilde{\alpha}^B \times (L \tilde{\mathbb{n}}_1)$$
$$= L(\delta q_1 \tilde{c}_2 \tilde{\mathbb{n}}_2 - \delta q_2 \tilde{\mathbb{n}}_3)$$

$$\mathbb{T}_A^* = [\underbrace{(I_2^A - I_3^A)}_{=0 \text{ for circular disk}} \omega_2^A \omega_3^A - I_1^A \alpha_1^A]\mathbb{n}_1 + \dots [\quad]\mathbb{n}_2 + [\quad]\mathbb{n}_3$$

$$\mathbb{T}_B^* = \qquad\qquad , \qquad \mathbb{T}_C^* = -I_K^C \ddot{q}_1 \mathbb{K}$$

$$\tilde{\omega}_1^A = \sigma - p\tilde{s}_2 \qquad \tilde{\omega}_2^A = 0 \qquad \tilde{\omega}_3^A = p\tilde{c}_2$$
$$\tilde{\alpha}_1^A = 0 \qquad \tilde{\alpha}_2^A = \sigma p \tilde{c}_2 \qquad \tilde{\alpha}_3^A = 0$$

for body B

$$\tilde{\mathbb{T}}_C^* = 0$$

Then
$$\delta \tilde{W} + \delta \tilde{W}^* = \underbrace{0}_{(\delta \tilde{W}_A + \delta \tilde{W}_B)} + \underbrace{0}_{\delta \tilde{W}_C} \underbrace{- mgL(\delta q_1 \tilde{c}_2 \overbrace{\mathbb{K} \cdot \tilde{\mathbb{n}}_2}^{0} - \delta q_2 \underbrace{\mathbb{K} \cdot \tilde{\mathbb{n}}_3}_{c_2})}_{\delta \tilde{W}_P}$$
$$\underbrace{+ \delta q_2 [(I_3^A - I_1^A) p\tilde{c}_2 (\sigma - p\tilde{s}_2) - I_2^A \sigma p \tilde{c}_2]}_{\delta W_A^*}$$
$$\underbrace{- \delta q_2 (I_3^B - I_1^B) p^2 \tilde{c}_2 \tilde{s}_2}_{\delta W_B^*} + \underbrace{0}_{\delta W_C^*} + \underbrace{mL^2 \tilde{c}_2 p^2 (\delta q_1 \tilde{c}_2 \overbrace{\mathbb{j} \cdot \mathbb{n}_2}^{0} - \delta q_2 \overbrace{\mathbb{j} \cdot \mathbb{n}_3}^{\tilde{s}_2})}_{\delta W_P^*}$$

3/4/80

Today - talk about Gyro
Prob in set 9 (last part)
KE related to generalized inertia forces for Non holonomic

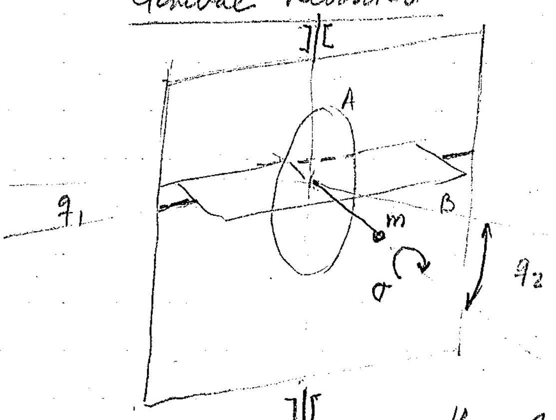
On March 13 turn in Initial Value Problem.
Final - Take Home given out on 13 March will be due by 9^{30} PM 20 March 80
would be nice to turn in before then
Exam will cover the material up to & including today.

I.V. Problem:

1. Must have title
2. Section 1 - problem Statement
   - (a) Introductory Sentence brief
   - (b) System description - explicit
     - Objects in System
     - talk about only those parameters that are relevant: Symbols (to be used) to indicate properties of system.
     - " " " " for coordinate variables
   - (c) Statement of Objective: what's to be found and why.
3. Section 2 - Results
   - (a) Introductory Sentence short
   - (b) Tables and/or plots
   - (c) Discussion (related to the why)
4. Section 3 - Analysis of how you got equations

Keep entire thing short!

Gimbal Revisited



we spent time finding $\tilde{\delta W} + \tilde{\delta W}^* = 0$

$\tilde{\delta W} + \tilde{\delta W}^* = \tilde{c}_2 \delta q_2 [mgl - I_1^A \sigma p + (I_1^A - I_3^A + I_1^B - I_3^B - mL^2)\tilde{s}_2 p^2] = 0$ by principal of virtual work.

either $[\quad] = 0$ or $\tilde{c}_2 \delta q_2 = 0$

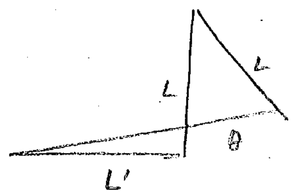
if $\tilde{c}_2 = 0 \Rightarrow q_2 = 90°$ $\delta q_2 \neq 0$. If we stay away from $q_2 = 90°$ then $[\quad] = 0$

thus Given p, we can find one σ

Given T, can find two p's since eqn is quadratic
- can only demonstrate the first value since that value is a stable result
2nd value is a very high precession & is unstable.

Problem #9H

now to get you back on track



$$P = -mgL\cos\theta + \tfrac{1}{2}ks^2 \qquad s = \text{stretch of spring}$$

$$s = \left[\left[(L' + L\sin\theta)^2 + (L - L\cos\theta)^2\right]^{1/2} - L'\right]$$

$$(1+x)^{1/2} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \dots$$

$$s = L'\left\{\left[1 + \frac{2L}{L'}\sin\theta + 2\left(\frac{L}{L'}\right)^2(1-\cos\theta)\right]^{1/2} - 1\right\} \quad \text{expand sin \& cos } \theta$$

$$s = L'\left\{\left[1 + \frac{2L}{L'}\left(\theta + \frac{1}{2}\frac{L}{L'}\theta^2 - \frac{1}{6}\theta^3\right) + \dots\right]^{1/2} - 1\right\} \quad \text{use binom}$$

$$= L'\left\{\left[1 + \frac{L}{L'}\left(\theta + \cancel{\frac{1}{2}\frac{L}{L'}\theta^2} - \frac{1}{6}\theta^3\right) - \cancel{\frac{1}{2}\left(\frac{L}{L'}\right)^2\left(\theta^2 + \frac{L}{L'}\theta^3\right)} + \cancel{\frac{1}{2}\left(\frac{L}{L'}\right)\theta^3} + \dots\right.\right.$$

$$= L\left(\theta - \frac{\theta^3}{6}\right) + \dots$$

$$s^2 = L^2\left(\theta^2 - \frac{\theta^4}{3} + \dots\right) \quad \text{now go back to your work from this point}$$

finally $P = mgL\left(3\theta^2 - \frac{7}{8}\theta^4\right)$

problem 9K, 9j were also discussed

Relation between F_r^* & K for non holonomic system

We already know $F_r^* = \dfrac{\partial K}{\partial q_r} - \dfrac{d}{dt}\left(\dfrac{\partial K}{\partial \dot q_r}\right) \quad (r = 1, \dots, n)$ holonomic systems. $\dot q_r$ are indep.

for non-holonomic we have $\dot q_r$ dependent.

Df: n - number of general. coord.
m " " const eqns
$p \triangleq n - m$

then solve constraint eqns for the last m $\dot q$'s

thus
$$\dot{q}_r = \sum_{i=1}^{p} c_{ri}\,\dot{q}_i + d_r \qquad (r = p+1, \ldots, n) \qquad (1)$$

Now $F_r^* = \frac{\partial K}{\partial q_r} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_r}\right) + \sum_{j=p+1}^{n}\left(\frac{\partial K}{\partial q_j} - \frac{d}{dt}\frac{\partial K}{\partial \dot{q}_j}\right) c_{jr} \quad (r = 1, \ldots, p)$

<u>Theorem</u> which appeared in print

Passerello & Huston J. Appl. Mech. Vol 40, No. 1 pp 101-104

here $K = K(q_1, \ldots, \dot{q}_n)$.

<u>Proof</u>:

$$\mathbb{V} \underset{(2.16)}{=} \sum_{r=1}^{n} \mathbb{V}_{\dot{q}_r}\dot{q}_r + \mathbb{V}_t$$
$$= \sum_{r=1}^{p} \mathbb{V}_{\dot{q}_r}\dot{q}_r + \sum_{r=p+1}^{n} \mathbb{V}_{\dot{q}_r}\dot{q}_r + \mathbb{V}_t$$
$$\underset{(1)}{=} \sum_{r=1}^{p} \mathbb{V}_{\dot{q}_r}\dot{q}_r + \sum_{r=p+1}^{n} \mathbb{V}_{\dot{q}_r}\left(\sum_{i=1}^{p} c_{ri}\dot{q}_i + d_r\right) + \mathbb{V}_t$$
$$= \quad '' \quad + \sum_{i=1}^{p}\sum_{r=p+1}^{n} \mathbb{V}_{\dot{q}_r} c_{ri}\,\dot{q}_i + \sum_{r=p+1}^{n} \mathbb{V}_{\dot{q}_r} d_r + \mathbb{V}_t$$
$$= \quad '' \quad + \sum_{r=1}^{p}\sum_{j=p+1}^{n} \mathbb{V}_{\dot{q}_j} c_{jr}\,\dot{q}_r + \sum \quad '' \quad ''$$ renaming dummy indices

Then
$$\mathbb{V} = \sum_{r=1}^{p}\left(\mathbb{V}_{\dot{q}_r} + \sum_{j=p+1}^{n} \mathbb{V}_{\dot{q}_j} c_{jr}\right)\dot{q}_r + \sum \quad '' \quad + \mathbb{V}_t \qquad (2)$$

now
$$\tilde{\mathbb{V}}_{\dot{q}_r} \underset{(2.28,2)}{=} \mathbb{V}_{\dot{q}_r} + \sum_{j=p+1}^{n} \mathbb{V}_{\dot{q}_j} c_{jr} \qquad r = 1, \ldots, p \qquad (3)$$

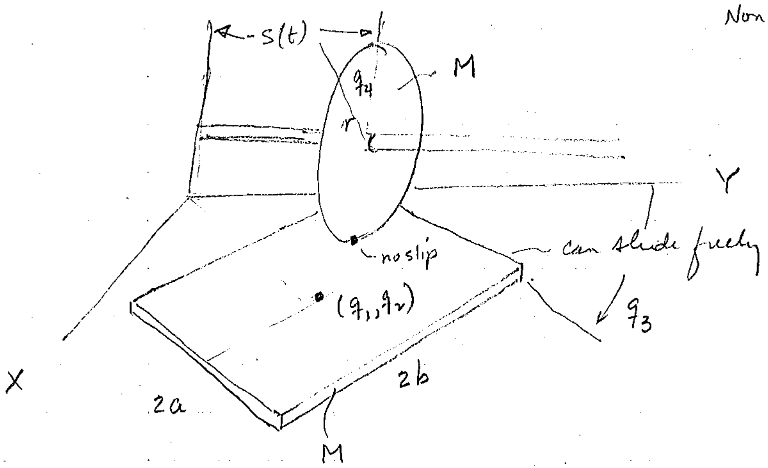
To get formula
$$\mathbb{V}_{\dot{q}_r}\cdot \mathbb{a} \underset{(2.25)}{=} \frac{1}{2}\left(\frac{d}{dt}\frac{\partial \mathbb{V}^2}{\partial \dot{q}_r} - \frac{\partial \mathbb{V}^2}{\partial q_r}\right) \qquad (r = 1, \ldots, n) \qquad (4)$$

$$\tilde{\mathbb{V}}_{\dot{q}_r}\cdot \mathbb{a} \underset{(3)}{=} \mathbb{V}_{\dot{q}_r}\cdot \mathbb{a} + \sum\left(\mathbb{V}_{\dot{q}_j}\, c_{jr}\cdot \mathbb{a}\right)$$ now plug in (4)

3/6/80

1. Example in applic of last work
2. Problems in set 10

<u>Example</u>: Section 3.24



Non holonomic

$n = 4 \quad m = 2 \quad p = n - m = 2$

Constraint eqns

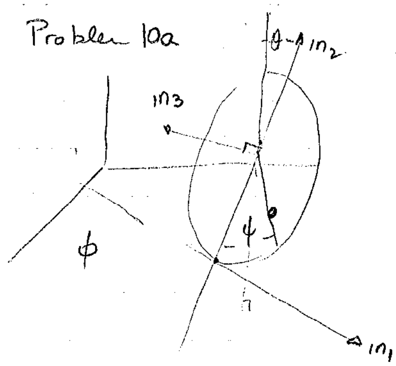
$$\dot q_2 = \dot s - q_1 \dot q_3$$

$$\dot q_4 = \frac{1}{r}\left[(q_2 - s)\dot q_3 - \dot q_1\right]$$

$$c_{31} = 0 \qquad c_{32} = -q_1 \qquad c_{41} = -\frac{1}{r} \qquad c_{42} = \frac{1}{r}(q_3 - s)$$

We shall return to this

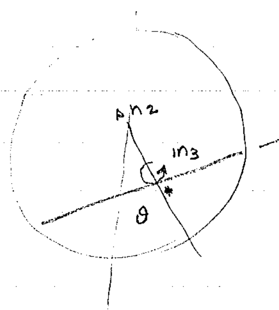
Problem 10a



$K = \frac{1}{2} m v^2$ Problem here is before we get KE we must do kinematics

$${}^R v^P \Big|_{\substack{\theta = 0 \\ \psi = \pi/2}} = -(v + r\dot\psi) n_1 + \frac{r\dot\psi}{2} n_2 - r\left(\dot\theta + \frac{\dot\phi}{2}\right) n_3$$

Prob. 10b



$$K = \frac{m}{2}\left[\left(R^2 - \frac{\Omega L^2}{3}\right)\dot\theta^2 + \Omega^2(R^2 - L^2)s^2 + \Omega^2\frac{L^2}{3}c^2\theta\right]$$

$$K = \frac{1}{2} m v^{*2} + \frac{1}{2}\left(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right)$$

Problem 10c KE of a plate with a fixed point Look at work in class when Mrs. Herrmann

Problem 10d

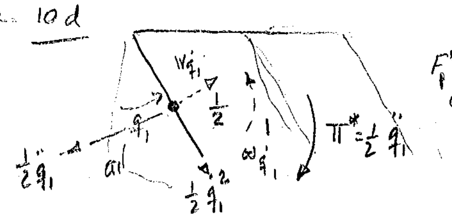
partial veloc

partial angular vel

inertia torque

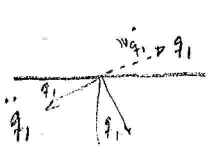
accel comp

$F^* \cdot v_{\dot q_1} \qquad T \cdot \omega_{\dot q_1}$

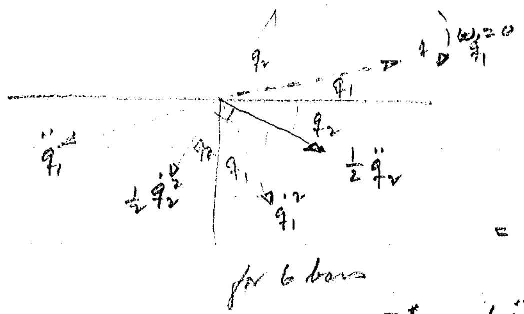


$$F^*_{1\ \text{contrib}} = -\frac{1}{4}\ddot q_1 - \frac{1}{12}\ddot q_1 = -\frac{1}{3}\ddot q_1 \quad \text{for each bar.}$$

for 6 bars : $-2\ddot q_1$



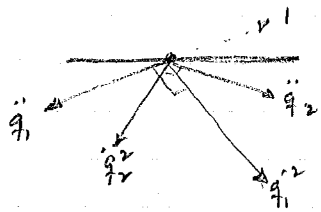
$$F^*_{1\ \text{contrib}} = -\ddot q_1 \qquad \text{for 5 bars : } -5\ddot q_1$$



$$c_1(-\ddot q_1 c_1 - \tfrac12 \dot q_2^2 s_2 + \dot q_1^2 s_1 + \tfrac12 \ddot q_2 c_2)$$
$$- s_1(\ddot q_1 s_1 + \tfrac12 \dot q_2^2 c_2 + \dot q_1^2 + \tfrac12 \ddot q_2 s_2)$$
$$= -\ddot q_1 - \tfrac12 \dot q_2^2 (c_1 s_2 + s_1 c_2) + \tfrac12 \ddot q_2 (c_1 c_2 - s_1 s_2)$$

for 6 bars

$$F^*_{1\,\text{contrib}} = -6\ddot q_1 - 3\dot q_2^2 (c_1 s_2 - s_1 c_2) + 3\ddot q_2 (c_1 c_2 - s_1 s_2)$$

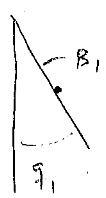


for 5 bars $\quad -5\ddot q_1 - 5\dot q_2^2 (c_1 s_2 + s_1 c_2) + 5\ddot q_2 (c_1 c_2 - s_1 s_2)$

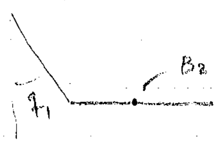
for the others do the same: then

$$F_1^* = \sum F_{1\,\text{contrib}}$$

To do it by KE method.

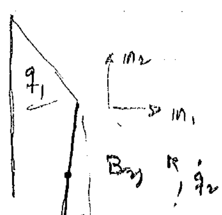


$$K_{B_1} = \frac12 \frac{W}{g} \frac{L^2}{3} \dot q_1^2 \quad \text{used fixed point idea} \quad = \frac12 I \omega^2$$



$$K_{B_2} = \frac12 \frac{W}{g} L^2 \dot q_1^2 = \frac12 m v^2$$

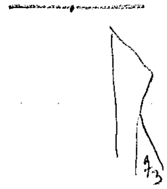
since $\omega = 0$



$$v^{B_3^*} = L\left[(\dot q_1 c_1 - \tfrac12 \dot q_2 c_2)\, n_1 + (\dot q_1 s_1 + \tfrac12 \dot q_2 s_2)\, n_2\right]$$

$$K_{B_3} = \frac12 \frac{W}{g} L^2 \left[(\dot q_1 c_1 - \tfrac12 \dot q_2 c_2)^2 + (\dot q_1 s_1 + \tfrac12 \dot q_2 s_2)^2 + \tfrac{1}{12} \dot q_2^2\right]$$

$$K_{B_4} = \frac12 \frac{W}{g} L^2 \left[(\dot q_1 c_1 - \dot q_2 c_2)^2 + (\dot q_1 s_1 + \dot q_2 s_2)^2\right]$$



$$K_{B_5} = \frac12 \frac{W}{g} L^2 \left[(\qquad) + \dots \qquad\right]$$

$$K = \frac{1}{2}\frac{W}{g}L^2\Big\{ 6\Big[\dot q_1^2/3 + (\dot q_1 c_1 - \tfrac{1}{2}\dot q_2 c_2)^2 + (\dot q_1 s_1 + \tfrac{1}{2}\dot q_2 s_2)^2 + \tfrac{1}{12}\dot q_2^2 + (\dot q_1 c_1 - \dot q_2 c_2 + \tfrac{1}{2}\dot q_3 c_3)^2$$
$$+ (\dot q_1 s_1 + \dot q_2 s_2 + \tfrac{1}{2}\dot q_3 s_3)^2 + \tfrac{1}{12}\dot q_3^2\Big] + 5\Big[\dot q_1^2 + (\dot q_1 c_1 - \dot q_2 c_2)^2 + (\dot q_1 s_1 + \dot q_2 s_2)^2$$
$$+ (\dot q_1 c_1 - \dot q_2 c_2 + \dot q_3 c_3)^2 + (\dot q_1 s_1 + \dot q_2 s_2 + \dot q_3 s_3)^2\Big]\Big\}$$

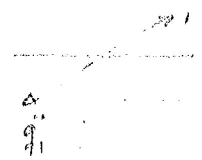
$$\frac{\partial K}{\partial \dot q_1} = \frac{W}{g}L^2\Big\{ 6\Big[\frac{\dot q_1}{3} + (\dot q_1 c_1 - \tfrac{1}{2}\dot q_2 c_2)c_1 + \cdots$$

How to linearize systems i.e. $s_1 \cong q_1$, $c_1 \cong 1$ $\quad s_2 \cong q_2$, $c_2 \cong 1$

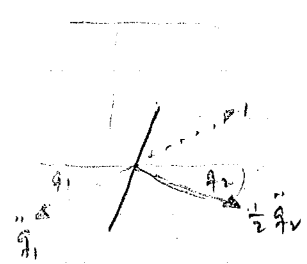
Must stay fully non linear through the formation of partials. Then linearize everything (including partials)

$$= -\tfrac{1}{4}\ddot q_1 - \tfrac{1}{12}\ddot q_1 = -\tfrac{1}{3}\ddot q_1$$

$$F^*_{1\,\text{contrib}} = 6\cdot\left(-\tfrac{1}{3}\ddot q_1\right)$$



$$F^*_{1\,\text{contrib}} = 5(-\ddot q_1)$$



$$F^*_{1\,\text{cont}} = +6\left[-\ddot q_1 + \tfrac{1}{2}\ddot q_2\right]$$

Similarly we do rest to get

$$F_1^* = \frac{WL^2}{g}\left(-29\ddot q_1 + 19\ddot q_2 - 8\ddot q_3\right)$$

10e non holonomic

10f Idea to be used $K = K_{cone} + 4K_s$

$$K_c = \tfrac{1}{2}J({}^R\omega^c)^2$$
$$K_s = \tfrac{1}{2}\left[m\,{}^R v^{p\,2} + \tfrac{2}{5}mr^2({}^R\omega^s)^2\right]$$

where notation is that of problem when discussed last quarter

$({}^R\mathbf{V}^P)^2 \underset{(1)}{=} {}^R V^{P\,2} \qquad ({}^R\omega^S)^2 \underset{(3)}{=} \omega_1^2+\omega_2^2+\omega_3^2 \underset{(14,15,16)}{=} 2\,({}^R V^P)^2/r^2$

$$K_S = \frac{7}{10} m \left({}^R V^P\right)^2$$

$${}^R V^P \underset{(17)}{=} \frac{a\,{}^R\omega^C}{1+\sin\theta+\cos\theta} \underset{(18)}{=} \frac{r\sin\theta}{\cos\theta-\sin\theta}\,{}^R\omega^C$$

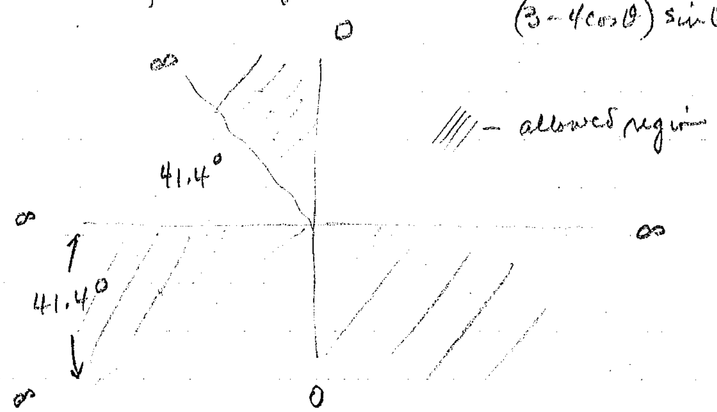
$$K = \left[\frac{1}{2}J + \frac{18}{5} m \frac{r^2\sin^2\theta}{(\cos\theta-\sin\theta)^2}\right]\left({}^R\omega^C\right)^2$$

for $\theta = 30^\circ$ we get results shown

10g. Since $K \sim I$ then $\frac{dK}{d\theta} = 0 \Rightarrow \frac{dI}{d\theta} = 0$!

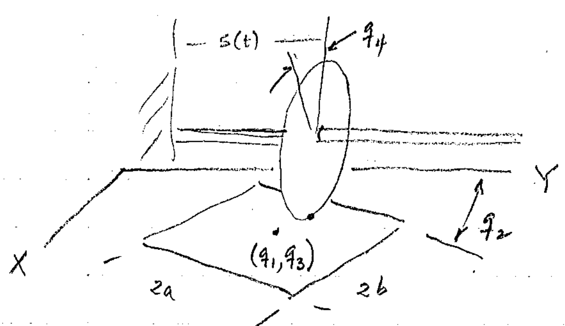
10h look at dynamic coupling section in book

10i. for θ general $N = \dfrac{6\cos\theta}{(3-4\cos\theta)\sin\theta}$. $N > 0$. since $N = \dfrac{L\omega^2}{g}$.



3/11/80

Return to first example of last time Pasarell–Huston Theory for non holonomic syste



No slip between plate and disk gives

$$\dot q_3 = \dot s - q_1\dot q_2 \ ; \quad \dot q_4 = \frac{1}{r}\left[(q_3-s)\dot q_2 - \dot q_1\right]$$

$$\dot q_3 = c_{31}\dot q_1 + c_{32}\dot q_2 + d_3 \qquad c_{32} = -q_1,\quad c_{31} = 0 \quad d_3 = \dot s$$

$$\dot q_4 = c_{41}\dot q_1 + c_{42}\dot q_2 + d_4 \qquad c_{41} = -\frac{1}{r} \quad c_{42} = \frac{1}{r}(q_3-$$

$$K = \frac{1}{2}M\left(\dot q_1^2 + \frac{a^2+b^2}{3}\dot q_2^2 + \dot q_3^2\right) + \frac{1}{4}mr^2\dot q_4^2 + \frac{1}{2}m\dot s^2$$

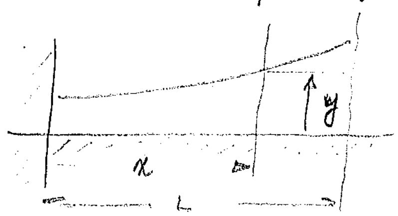
$$F_1^* = \frac{\partial K}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot q_1}\right) + \left(\frac{\partial K}{\partial q_3} - \frac{d}{dt}\frac{\partial K}{\partial \dot q_3}\right)c_{31} + \left(\frac{\partial K}{\partial q_4} - \frac{d}{dt}\frac{\partial K}{\partial \dot q_4}\right)c_{41}$$

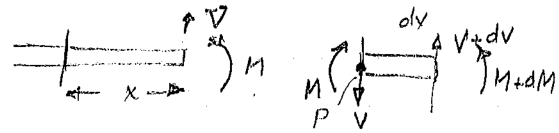
over

$$= 0 - M\ddot{q}_1 + (0 - M\ddot{q}_3)0 + \left(0 - \frac{mr^2}{2}\ddot{q}_4\right)\left(-\frac{1}{r}\right) = -M\ddot{q}_1 + \frac{mr}{2}\ddot{q}_4$$

for $F_2^* = -M\frac{(a^2+b^2)}{3}\ddot{q}_2 + M\ddot{q}_3 q_1 - \frac{mr}{2}\ddot{q}_4(q_3 - s)$

Finite Element Theory (for beams)





If we neglect rotatory inertia

Net Moment about point P = 0

$$\circlearrowright^{+} \quad -\cancel{M} + (V + dV)\,dx + \cancel{M} + dM = 0 \quad (1) \text{ taking limit}$$

$$V \underset{(1)}{=} -\partial M/\partial x \qquad (2)$$

Assume: M ~~is~~ prop to radius of curvature

$$M = EI \frac{\partial^2 y}{\partial x^2} \qquad (3)$$

$$V \underset{(2,3)}{=} -EI \frac{\partial^3 y}{\partial x^3} \qquad (4)$$

Net active force on beam element

$$-V + V + dV = dV \underset{(4)}{=} -EI \frac{\partial^4 y}{\partial x^4}\,dx \qquad (5)$$

Inertia force for a beam element,

$$-ma^* = -\frac{\partial^2 y}{\partial t^2}\,\rho\,dx \qquad (6)$$

Define Shape fn and generalized coordinate to help define partial velocity

Let $$y(x,t) = \underbrace{\psi(x)}_{\text{shape fn}}\,\underbrace{q(t)}_{\text{generalized coord.}} \qquad (7)$$

Speed fn. $$v(x,t) = \psi(x)\,\dot{q}(t) \qquad (8)$$

$$a(x,t) = \psi(x)\,\ddot{q}(t) \qquad (9)$$

partial velocity $$v_{\dot{q}} = \psi(x) \qquad (10)$$

Generalized active forces

$$dF = v_{\dot{q}} \cdot \left(-EI \frac{\partial^4 y}{\partial x^4}\,dx\right) = -\underset{(10)}{\psi(x)}\, \underbrace{EI\,\psi^{IV} q\,dx}_{(7)} \qquad (11)$$

$$F \underset{(11)}{=} -\int_0^l \psi(x)\,\psi^{IV} EI\,dx \cdot q = -qEI \int_0^l \psi(x)\,\psi^{IV}\,dx \quad \text{if } EI \neq f \text{ of } x$$

$$= -qEI\left[\psi\psi'''\Big|_0^l - \int_0^l \psi'\psi'''\,dx\right] = -qEI\left[(\psi\psi''' - \psi'\psi'')\Big|_0^l + \int_0^l (\psi'')^2\,dx\right]$$

Generalized inertia forces

$$dF^* = V_q\left(-\frac{\partial^2 y}{\partial t^2}\rho\, dx\right) = -\underset{(10)}{\psi}\ \underset{(7)}{\psi}\ddot{q}\,\rho\, dx \qquad (13)$$

$$F^* = -\ddot{q}\rho \int_0^l \psi^2 dx \qquad \text{if } \rho \neq \rho(x) \qquad (14)$$

∴ Eqn of Motion is

$F + F^* = 0 \Rightarrow$

$$\ddot{q} + \left\{EI\left[(\psi\psi''' - \psi'\psi'')_0^l + \int_0^l \psi''^2 dx\right] \Big/ \left[\rho\int_0^l \psi^2 dx\right]\right\} q = 0 \qquad (15)$$

or $\ddot{q} + p^2 q = 0$ where $p \triangleq \left\{EI\left[(\psi\psi''' - \psi'\psi'')_0^l + \int_0^l \psi''^2 dx\right] \Big/ \left[\rho\int_0^l \psi^2 dx\right]\right\}^{1/2}$ (16)

an equation for q but depends on p (which depends on ψ) which depends on BC

Restrained beam : Cantilever $y(0,t) = \frac{\partial y}{\partial x}(0,t) = 0 \Rightarrow \psi(0)\ \&\ \psi'(0) = 0$ (11,18)

$$V(L,t) = M(L,t) = 0 \Rightarrow \frac{\partial^3 y}{\partial x^3}(L,t) = \frac{\partial^2 y}{\partial x^2}(L,t) = 0$$

$$\psi'''(L) = \psi''(L) = 0 \qquad (19)$$

$\Rightarrow (\psi\psi''' - \psi'\psi'')_0^l = 0$ by boundary condary conditions

to take $\psi(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$ $\qquad \psi(0) = a_0 = 0$

$\psi'(x) = a_1 + 2a_2 x + 3a_3 x^2$ $\qquad \psi'(0) = a_1 = 0$

$\psi''(x) = 2a_2 + 6a_3 x$ $\qquad \psi''(L) = 2a_2 + 6a_3 L = 0$

$a_3 = -\frac{a_2}{3L}$

we will not satisfy $\psi'''(L) = 0$

$\psi(x) = a_2\left(x^2 - \frac{x^3}{3L}\right)$ $\qquad$ since $p \tilde{=} \int \psi''^2 dx \Big/ \int \psi^2 dx$ ∴ take $a_2 =$

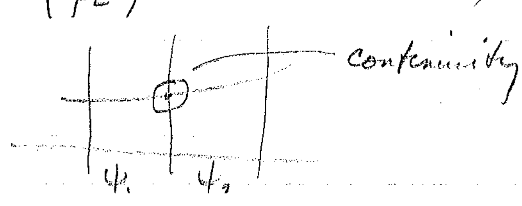
$\psi'' = 2a_2\left(1 - \frac{x}{L}\right)$

$$\int_0^l \psi^2 dx = 11 a_2^2 L^5/1680 \qquad \int_0^l \psi''^2 dx = a_2^2 L/12$$

$$p = \left(\frac{1680}{11\cdot 12}\right)^{1/2}\left(\frac{EI}{\rho L^4}\right)^{1/2} = 3.5675\left(\frac{EI}{\rho L^4}\right)^{1/2}$$ predicted of first mode

actual sol : $3.5160\left(\frac{EI}{\rho L^4}\right)^{1/2}$ (1.5% error)

For 2 elements assume



3/13/80

Final – place no. on each exam.

Next Qtrs work:

How does everything we did fit in & how to use it effectively.

Origin of $F_r + F_r^* = 0$

– $\mathbb{F} + \mathbb{F}^* = 0 \qquad (\mathbb{F}^* = -m\mathbb{a})$

(D'Alembert Principle for particle)

– $\mathbb{v}\cdot\mathbb{F} + \mathbb{v}\cdot\mathbb{F}^* = 0$

$A \triangleq \mathbb{v}\cdot\mathbb{F} \quad A^* = \mathbb{v}\cdot\mathbb{F}^*$ (activity of force)

$A + A^* = 0$

(Activity Principle)

$A^* = \mathbb{v}\cdot\mathbb{F}^* = \mathbb{v}\cdot(-m\mathbb{a}) = -m\,\mathbb{v}\cdot\frac{d\mathbb{v}}{dt}$

$= -\frac{m}{2}\frac{d}{dt}(\mathbb{v}^2) = -\frac{d}{dt}\left(\frac{m\mathbb{v}^2}{2}\right) = -\frac{dK}{dt}$

$\therefore \boxed{\frac{dK}{dt} = A}$ (Activity Energy Principle)

Severe limitation: only one degree of freedom.

$\mathbb{F} + \mathbb{F}^* = 0 \quad (\mathbb{F}^* = -m\mathbb{a})$

Dalembert.

$\mathbb{v}_{\dot q_r}\cdot\mathbb{F} + \mathbb{v}_{\dot q_r}\cdot\mathbb{F}^* = 0$

$F_r = \mathbb{v}_{\dot q_r}\cdot\mathbb{F}\ ; \quad F_r^* = \mathbb{v}_{\dot q_r}\cdot\mathbb{F}^* \quad (r = 1,\dots,n)$

$F_r + F_r^* = 0 \qquad (r = 1,\dots,n)$

(Lagrange's form of D'Alembert Principle)

$F_r^* = \mathbb{v}_{\dot q_r}\cdot\mathbb{F}^* = -m\,\mathbb{v}_{\dot q_r}\cdot\mathbb{a} = -m\,\mathbb{v}_{\dot q_r}\cdot\frac{d\mathbb{v}}{dt}$

$= -\frac{m}{2}\left[\frac{d}{dt}\left(\frac{\partial \mathbb{v}^2}{\partial \dot q_r}\right) - \frac{\partial \mathbb{v}^2}{\partial q_r}\right]$

$= -\left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot q_r}\frac{m\mathbb{v}^2}{2}\right) - \frac{\partial \frac{m\mathbb{v}^2}{2}}{\partial q_r}\right]$

$= -\frac{d}{dt}\frac{\partial K}{\partial \dot q_r} + \frac{\partial K}{\partial q_r}$

$\therefore \boxed{F_r = \frac{d}{dt}\frac{\partial K}{\partial \dot q_r} - \frac{\partial K}{\partial q_r} \qquad r = 1,\dots,n}$

How to use it effectively – systems at rest are important to consider
– constraint forces need to be determined
– Special cases of steady motions where $\dot q_s$ are constant.
– General methods for extracting information from Eqns of motion

Presentation of information from simulation

Exams Due by or before Thursday 20 March 1980 HW as well as exam