Dynamics

1. a vector can be expressed in any reference frame

2. Holonomic constraint : if it can be written as $f(x_i, y_i, z_i, t) = 0 \quad i = 1, \dots, N$

   Non holonomic " : if it cannot" " " " $f(x_i, y_i, z_i, \dot{x}_i, \dot{y}_i, \dot{z}_i, t) = 0$ simple nonhol.

   Rheonomic : holonomic constraint w/ explicit t

   Scleronomic : " " w/o explicit t

3. Holonomic System : given N particles and M holonomic constraints in R

   \# of degrees of freedom = \# generalized coords in R = 3N − M

   for rigid body : 6 degrees of freedom : 3 angular coords + 3 linear coords.

HOLONOMIC

4. Partial Rate of : ${}^R\omega^B = \sum_{r=1}^{n} {}^R\omega^B_{\dot{q}_r} \dot{q}_r + {}^R\omega^B_t$ $\quad \omega_{\dot{q}_r}$ = partial rate wrt q_r of orientation

   - if $\mathbb{r}$ is fixed in B then $\frac{{}^R d\, {}^B\mathbb{r}}{dt} = {}^R\omega^B \times {}^B\mathbb{r}$
   - ${}^R\omega^B_{\dot{q}_r} = \frac{\partial}{\partial q_r} {}^B\mathbb{b}_i \cdot {}^B\mathbb{b}_j\, {}^B\mathbb{b}_k\, e_{ijk}$ $\quad e_{ijk} = \begin{matrix} 1 & \text{if } ijk \text{ are cyclic} \\ 0 & \text{if not.} \end{matrix}$ $\quad \mathbb{b}_i$ are unit vectors fixed in B
   - ${}^R\omega^B = \frac{{}^R d\, {}^B\mathbb{b}_i}{dt} \cdot {}^B\mathbb{b}_j\, {}^B\mathbb{b}_k\, e_{ijk}$

5. Simple Angular Velocity if ∃ a vector in B & R independent of t ($\mathbb{k}$) $\quad {}^R\omega^B = \underbrace{{}^R\omega^B}_{\text{angular speed}} \mathbb{k}$

6. $\frac{{}^R d\, {}^S\mathbb{r}}{dt} = \frac{{}^S d\, {}^S\mathbb{r}}{dt} + {}^R\omega^S \times {}^S\mathbb{r}$ $\quad$ relates derivatives in 2 ref frames

7. ${}^R\omega^B = \sum_{i=0}^{n} {}^{R_i}\omega^{R_{i+1}}$ $\quad$ where $R_0 = R \quad R_n = B$ $\quad$ where ${}^{R_i}\omega^{R_{i+1}}$ is the angular velocity of frame i+1 wrt frame

   good w/ simple angular velocity

8. Angular accel : $\frac{{}^R d\, {}^R\omega^B}{dt} = {}^R\alpha^B$

9. Partial Rates of : ${}^R\mathbb{V} = \sum_{r=1}^{n} {}^R\mathbb{V}_{\dot{q}_r} \dot{q}_r + {}^R\mathbb{V}_t$

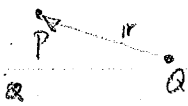
   - 2 pts on rigid body $\quad {}^R\mathbb{V}^P = {}^R\mathbb{V}^Q + {}^R\omega^B \times {}^B\mathbb{r}^{P/Q}$ $\quad$ where P, Q are in B & $\mathbb{r}$ is fixed in B
   - one pt. moving on a rigid body B : ${}^R\mathbb{V}^P = {}^R\mathbb{V}^{\bar{B}} + {}^B\mathbb{V}^P$ $\quad$ where ${}^R\mathbb{V}^{\bar{B}}$ is the velocity (wrt ref R) of the pt $\bar{B}$ of body B when P coincides w/ $\bar{B}$ at that instant
   - 2 pts of rigid body $\quad {}^R\mathbb{V}^P_{\dot{q}_r} = {}^R\mathbb{V}^Q_{\dot{q}_r} + {}^R\omega^B_{\dot{q}_r} \times {}^B\mathbb{r}$
   - accels of 2 pts on rigid body : ${}^R\mathbb{a}^P = {}^R\mathbb{a}^Q + {}^R\alpha^B \times {}^B\mathbb{r}^{P/Q} + {}^R\omega^B \times ({}^R\omega^B \times {}^B\mathbb{r}^{P/Q})$ $\quad$ $\mathbb{r}$ fixed in B, P & Q in B
   - One point moving on rigid body : ${}^R\mathbb{a}^P = {}^B\mathbb{a}^P + {}^R\mathbb{a}^{\bar{B}} + \underbrace{2\, {}^R\omega^B \times {}^B\mathbb{V}^P}_{\text{Coriolis accel.}}$ $\quad$ where P is on B & $\bar{B}$ is point of R that coincides w/ P

a couple: a set of vectors whose resultant $= 0$

Torque of a couple about any point $= 0$ All torques are moments; all moments need not be torques

$\mathbb{M}$ can only exist for bound vectors

$$\mathbb{M}^P = \mathbb{M}^Q + \mathbb{R} \times \mathbb{r}$$ where $\mathbb{R}$ is the resultant of all vectors



if $\mathbb{R}$ is replaced by ${}^A\omega^B$ then $\mathbb{M}^P$ & $\mathbb{M}^Q$ are replaced by ${}^A\mathbb{V}^P$ & ${}^A\mathbb{V}^Q$

Wrench – replacement of a force at point A by force & torque of couple at pt B

10. Accel & Partial Rates of change if ${}^R\mathbb{V}^P = f(q_i)$ ${}^R v^2 = f(q_i, \dot q_i, t)$ then

$${}^R\mathbb{V}^P_{\dot q_r}\cdot{}^R\mathbf{a}^P = \frac{1}{2}\left[\frac{d}{dt}\left(\frac{\partial\,{}^R v^2}{\partial \dot q_r}\right) - \frac{\partial\,{}^R v^2}{\partial q_r}\right]$$ only for holonomic system.

11. Nonholonomic Systems

first get holonomic constraints (n), then non holonomic (m) ⇒ you will more unknowns then eqs. : $n-m$

write m eqs in $\dot q_i$ in terms of $\dot q_j$ $j = m+1 \dots n$

Now $${}^R\mathbb{V} = \sum_{r=1}^{n-m} {}^R\tilde{\mathbb{V}}_{\dot q_r}\dot q_r + \tilde{\mathbb{V}}_t$$

$${}^R\omega^B = \sum_{r=1}^{n-m} {}^R\tilde{\omega}^B_{\dot q_r}\dot q_r + \tilde{\omega}_t$$

Also: $${}^R\tilde{\mathbb{V}}^P_{\dot q_r} = {}^R\tilde{\mathbb{V}}^Q_{\dot q_r} + {}^R\tilde{\omega}^B_{\dot q_r}\times \mathbf{r}^{P/Q}$$

12. we can also define fns $u_r = \sum_{j=1}^{n-m} A_{rj}\dot q_j + A_r$. $r = 1,\dots, n-m$

then $$\mathbb{V} = \sum_{r=1}^{n-m} \tilde{\mathbb{V}}_{u_r} u_r + \tilde{\mathbb{V}}_t'$$ if $m=0$ $u_r = \dot q_r$ & $\tilde{\mathbb{V}}_{u_r} = \mathbb{V}_{\dot q_r}$

$$\omega = \sum_{r=1}^{n-m} \tilde{\omega}_{u_r} u_r + \tilde{\omega}_t'$$

and $${}^R\tilde{\mathbb{V}}^P_{u_r} = {}^R\tilde{\mathbb{V}}^Q_{u_r} + {}^R\tilde{\omega}^B_{u_r}\times \mathbf{r}^{P/Q}$$

- gears – have same velocity at some pt in a suitable frame ~~about~~ in which each has a simple angular velocity.
- for top : use fixed pt to set up coords
- ~~pure~~ rolling : velocity of contact pts are the same

  pure " : " & angular velocity of contact pts are the same

0. $F_r = \sum \mathbb{V}_{\dot q_r}\cdot \mathbb{F}^i$
1. Forces exerted by a smooth body contributed nothing to generalized forces
2. forces of interaction (body forces & contact forces) with a rigid body contribute nothing to generalized forces. forces $\mathbb{F}_i$ (external) acting on particles P_i of the rigid body can be replaced by $\mathbb{F} = \sum \mathbb{F}_i$ & $\mathbb{T} = \sum \mathbf{r}_i \times \mathbb{F}_i$ at a pt Q where $\mathbf{r}_i = \overrightarrow{P_i Q}$

and $(F_r)_B = {}^R\tilde{\mathbb{V}}^Q_{\dot q_r}\cdot \mathbb{F} + {}^R\tilde{\omega}^B_{\dot q_r}\cdot \mathbb{T}$

Generalized active   single nonholon

$$F_r = \sum \mathbb{V}_{\dot q_r}^{P_i} \cdot \mathbb{F}^{P_i}$$

$\mathbb{F}^{P_i}$ is resultant of all forces acting on P_i

$\mathbb{V}_{\dot q_r}^{P_i}$ " partial rate of point P_i

for a body $\quad F_r = \mathbb{V}_{\dot q_r} \cdot \mathbb{F} + \omega_{\dot q_r} \cdot \overline{\Pi}$

where $\mathbb{V}_{\dot q_r}$ = partial vel of pt Q

$\mathbb{F} = \sum \mathbb{F}^i$ applied at Q

$\overline{\Pi} = \sum \mathbb{r} \times \mathbb{F}^i$ $\quad \mathbb{r}$ dist of Q relative to P_i

3. Gravitational forces: when the distance between 2 particles (or the char length of a body) is $\ll$ with diameter of the earth then $G = mg\mathbb{k}$ $\mathbb{K}\downarrow$ and the contribution to (F_r) of the grav. forces $(F_r)_G = mg\mathbb{k}\cdot\tilde{\mathbb{v}}_{\dot q_r}^{P^*}$ where P^* is the mass center & m is the mass of the body.

4. If a smooth body B remains in contact with a particle P, the total contribution of the contact force on B and on P contribute nothing to F_r

5. If a rigid body B of S ~~belonging~~ rolls upon another rigid body B' of R and motion of B' is prescribed then contact forces on B by B' don't contribute anything. If B' motion is not prescribed but B' belongs to S then the contact forces ~~contributes~~ of B & B' contribute nothing to F_r.

$$F_r + F_r^* = 0 \quad \leftarrow \text{newton's Law} \quad (\text{simple nonholonomic or holonomic})$$

A. Generalized ~~active~~ inertia forces.

$$F_r^* = \sum_{i=1}^{N} \mathbb{V}_{\dot q_i}^{P_i}\cdot \mathbb{F}_i^* \quad \text{for particles where } \mathbb{F}_i^* = -m\mathbb{a}_i \text{ \& } \mathbb{a}_i \text{ is accel of particle}$$

for nonholonomic $\mathbb{V}_{\dot q_i}$ is replaced by $\tilde{\mathbb{V}}_{\dot q_i}$

Rigid Body: $(F_r^*)_B = \mathbb{V}_{\dot q_r}\cdot\mathbb{F}^* + \omega_{\dot q_r}\cdot\mathbb{T}^*$
$= \tilde{\mathbb{V}}_{u_r}\cdot\mathbb{F}^* + \tilde{\omega}_{u_r}\cdot\mathbb{T}^*$

$\mathbb{V}_{\dot q_r}$ - velocity of mass center of B in R
ω - angular veloc. of B in R
$\mathbb{F}^* = -m\mathbb{a}^*$ accel of mass center
$\mathbb{T}^* = -\sum m_i \mathbb{r}_i^{P_i/B^*}\times \mathbb{a}_i^{B^*}$

B. Inertia $\mathbb{I}_a^O = \sum m_i \mathbb{r}_i \times (\mathbb{n}_a \times \mathbb{r}_i)$ where $\mathbb{r}_i$ is the vector from point O to P_i, $\mathbb{n}_a$ is a vector. For rigid Body $\mathbb{I}_a^{B/O} = \int_{\forall} \rho\,\mathbb{r}\times(\mathbb{n}_a\times\mathbb{r})\,d\forall$

$$\mathbb{I}_a = \mathbb{I}\cdot\mathbb{n}_a, \quad I_{ab} = \mathbb{n}_b\cdot\mathbb{I}\cdot\mathbb{n}_a \qquad I_a = \mathbb{n}_a\cdot\mathbb{I}\cdot\mathbb{n}_a$$

<u>Radius of Gyration:</u> $I_a = (\sum m_i)\, k_a^2$

<u>For rotation of axes</u> if I_{jk} are defined for $\mathbb{n}_j, \mathbb{n}_k$ & $\mathbb{n}_a = \sum a_j \mathbb{n}_j$ & $\mathbb{n}_b = \sum b_j \mathbb{n}_j$
then $I_{ab} = \sum\sum a_j I_{jk} b_k$ if $\mathbb{n}_j$ is $\perp$ $\mathbb{n}_k$

$$I_{ab} = \sum m_i (\mathbb{n}_a \times \mathbb{p}_i)\cdot(\mathbb{n}_b\times\mathbb{p}_i)$$

Parallel axis theorem

$$\underbrace{\mathbb{I}^{S/O}}_{\text{moment of inertia of } S \text{ wrt } O} = \underbrace{\mathbb{I}^{S/S^*}}_{\text{moment of inertia of } S \text{ wrt mass center}} + \underbrace{\mathbb{I}^{S^*/O}}_{\text{moment of inertia of point w/ mass } m \text{ located at mass center wrt } O}$$

$$\mathbb{I}^{S^*/O} = m\left[(\mathbb{p}\cdot\mathbb{p})\,\mathbb{U} - \mathbb{p}\mathbb{p}\right]$$

$$I_{ab}^{S^*/O} = m\left[\mathbb{p}^2\,\delta_{ab} - (\mathbb{p}\cdot\mathbb{n}_a)(\mathbb{p}\cdot\mathbb{n}_b)\right]$$

principal axes form $|\mathbb{I} - I_a \mathbb{U}| = 0$ to get principal moments

1. unequal principal moments have axes $\perp$ to each other.
2. If 2 moments are equal, every line in the plane of these 2 principal directions is also a principal direction. Ellipsoid degenerates.
3. $\mathbb{I}_a = \lambda m_a$ for principal axes.

see also pg 64 my notes.

$$\mathbb{T}^* = (\mathbb{I}^{B/B^*} \cdot {}^{R}\omega^{B}) \times {}^{R}\omega^{B} - \mathbb{I}^{B/B^*} \cdot {}^{R}\alpha^{B} \qquad \text{where} \quad \mathbb{I} = \mathbb{I}^{B/B^*} \ \& \ {}^{R}\omega^{B},\ {}^{R}\alpha^{B}.$$

if m_1, m_2, m_3 are unit vectors in principal direction $\mathbb{I} = I_1 m_1 m_1 + I_2 m_2 m_2 + I_3 m_3 m_3$

if these vectors are not fixed in B or R then

$$\mathbb{T}^* = [\omega_2\omega_3(I_2 - I_3) - \alpha_1 I_1] m_1 + [\omega_3\omega_1(I_3 - I_1) - \alpha_2 I_2] m_2 + [\omega_1\omega_2(I_1 - I_2) - \alpha_3 I_3] m_3$$

if m's are fixed in B or R then $\alpha_i \equiv \dot{\omega}_i$

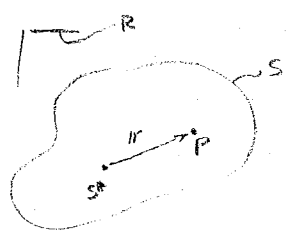
simple angular motion if $\omega = \omega m_a$

$$\mathbb{T}^* = \omega^2 \mathbb{I}_a \times m_a - \dot{\omega} \mathbb{I}_a \qquad \text{and} \quad m_a \cdot \mathbb{T}^* = -\dot{\omega} I_a$$

for laminate 1 principal will always be $\perp$ to plane of laminate to find principals for other 2 direction given I_a, I_b, I_{ab}

$$I = \frac{(I_a + I_b)}{2} \pm \sqrt{\left(\frac{I_a + I_b}{2}\right)^2 + I_{ab}^2}$$

$$\begin{bmatrix} I_a - I & I_{ab} \\ I_{ab} & I_b - I \end{bmatrix}$$

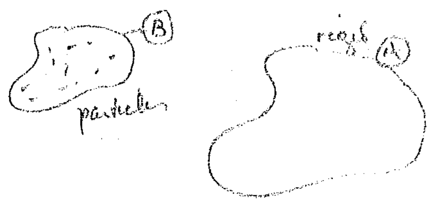


Angular Momentum

$${}^{R}\mathbb{H}^{S/S^*} = \int \rho \, \mathbb{r}^{P/S^*} \times {}^{R}\mathbb{V}^{P} d\tau$$

$$\mathbb{H}^{S/S^*} = \mathbb{I}^{S/S^*} \cdot {}^{R}\omega^{B}$$

$${}^{R}\mathbb{H}^{C/C^*} = {}^{R}\mathbb{H}^{D/D^*} + {}^{A}\mathbb{H}^{B/C^*}$$



$C = B \cup A$

for explanation

see pg 60. my notes also $-\dot{\mathbb{H}} = \mathbb{T}^*$

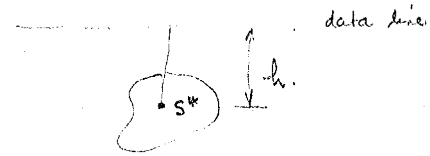
and $\mathbb{H}^{S/O} = \mathbb{H}^{S/S^*} + \mathbb{H}^{S^*/O}$ where $\mathbb{H}^{S/S^*} = \mathbb{I}^{S/S^*} \cdot {}^{R}\omega^{S} + m \mathbb{R} \times \mathbb{V}^{S^*}$

where $\mathbb{R}$ is distance to O relative to S'

Potential Functions: F_r' are contribs to generalized active forces F_r; P is a potential fn of $q_1, \ldots, q_n, t$

$$F_r' = -\frac{\partial P}{\partial q_r} \qquad r = 1, \ldots, n \qquad \text{for a holonomic system only}$$

Gravitational Potential $P = -mgh$



for spring (linear) $P = \frac{1}{2} k x^2$ $\quad x$ is the distance of stretch from the natural length.
for torsion spring $P = \frac{1}{2}\sigma\theta^2$ $\quad \theta$ is the angle of rotation of spring ; damper $\mathcal{F} = \frac{1}{2}\delta\dot{\theta}^2$
Potential Energy $\quad F_r = -\frac{\partial P}{\partial q_r} \quad r = 1, \dots, n$ where all contact & body forces that contribute to generalized active forces. If $\exists$ a P then P is the potential energy, for a holonomic system

Dissipation Function: $\quad F_r' = -\frac{\partial \mathcal{F}}{\partial \dot{q}_r} \quad r = 1, \dots, n$ holonomic system,
if $\quad \mathbb{F}_i = -c\mathbb{v}_i \quad i = 1, \dots, n$ " "
$\mathcal{F} = \frac{c}{2} \sum_{i=1}^{N} \mathbb{v}_i^2$ if $F_r' = \mathbb{F}_i \cdot \mathbb{v}_{\dot{q}_{r_i}}$

KE for a set of particles is $K = \frac{1}{2} \sum m_i v_i^2$ thus $F_r^* = \frac{\partial K}{\partial q_r} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_r}\right)$ holonomic only

KE for a rigid body $\quad K = K_{trans} + K_{rot} = \frac{1}{2} m {}^R\mathbb{v}^{*2} + \frac{1}{2} {}^R\omega^B \cdot \mathbb{I}^{B/B^*} \cdot {}^R\omega^B$
if $\mathbb{I} = I \mathbb{n}_1\mathbb{n}_1 + \dots$ & $\omega = \omega \mathbb{n}_1$ then $K_{rot} = \frac{1}{2}\omega^2 I$
if $\mathbb{I} = I_1\mathbb{n}_1\mathbb{n}_1 + I_2\mathbb{n}_2\mathbb{n}_2 + I_3\mathbb{n}_3\mathbb{n}_3$ & $\omega = \omega_i \mathbb{n}_i$ then $K_{rot} = \frac{1}{2}\left[I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2\right]$

if one point is fixed $\quad KE = \frac{1}{2}\omega \cdot \mathbb{I}^{B/O} \cdot \omega$ where O is the fixed point
$\mathbb{I}^{B/O} = \mathbb{I}^{B/B^*} + \mathbb{I}^{B^*/O} \qquad \omega \cdot \mathbb{I}^{B^*/O} \cdot \omega = m\left[(\mathbb{R}^{B^*/O} \cdot \mathbb{R}^{B^*/O})\,\omega\cdot\omega - (\mathbb{R}\cdot\omega)^2\right]$

Lagrange's Eqns of 1st kind
holonomic syst: $F_r + F_r^* = 0 \Rightarrow F_r = -F_r^* = -\left[\frac{\partial K}{\partial q_r} - \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_r}\right)\right] = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_r}\right) - \frac{\partial K}{\partial q_r}$
if S is also conservative $F_r = -\frac{\partial P}{\partial q_r}$
$\therefore \; F_r + F_r^* = \frac{\partial L}{\partial q_r} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_r}\right) = 0 \; ; \; L = K - P$ (Lagrangian)

also $\quad \mathbb{F} + \mathbb{F}^* = 0 \Rightarrow \mathbb{F}$ (Resultant of all contact & body forces) $= m\mathbb{a}^*$

ME 231A Kane Dynamics

Office: Durand 265 Hrs: 11-12 TTh always; but almost always there (X 7-2172)

Course Assistant: Roger Chen Rm 266 (MWF 2-4)

Text: Dynamics by Kane (same as last year)

References:

Whittaker: Analytical Dynamics tough reading but good book

Appell: Traité de Méchanique Rationelle 4 vol Fluid Elasticity

Hamel: Theoretische Mechanik

Berezkun: Lectures in Theoretical Mechanics

On Thursdays: lectures on new material – Tuesdays: problems

Grades: Based on Midterm & Final (Open Bk)/HW: will not be turned in except after final exams – HW will be used to upgrade not downgrade – use only one side of sheet

Objective:

To increase my effectiveness in solving physically meaningful problems.

Physically meaningful – something that can have a real counterpart but not necessarily useful.

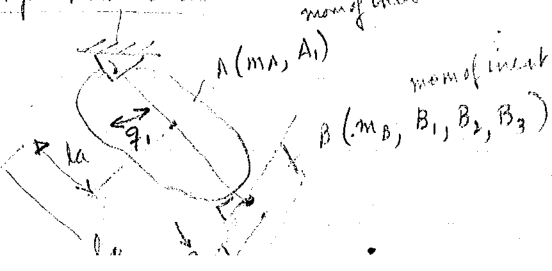
Three Facets of Problem Solving:

1. Generation of a mathematical model – using physics we derive a model

Concentrated Effort → 2. Use principles of mechanics to formulate equations governing quantities appearing in the math model

3. Extract desired information from the equations

Multiple Pendulum



Bodies: A, B

param: l_a, m_A, A_1; l_B, m_B, B_1, B_2, B_3

Unknowns: q_1, q_2

2. Use of Principles

$$\left(I + B_1 c_2^2 + B_2 s_2^2\right)\ddot{q}_1 + 2 J s_2 c_2 \dot{q}_1 \dot{q}_2 + K s_1 = 0 \qquad (1)$$

$$B_3 \ddot{q}_2 - J s_2 c_2 \dot{q}_1^2 = 0 \qquad (2)$$

$$I \triangleq m_A L_A^2 + m_B L_B^2 + A_1$$

$$J \triangleq B_2 - B_1$$

$$K \triangleq (m_A L_A + m_B L_B)\, g \quad \text{accel of grav.}$$

$$\sin q_2 = s_2 \qquad \cos q_2 = c_2 \qquad \sin q_1 = s_1$$

these are coupled nonlinear ODE. Thus eqns are difficult

3. Extraction phase

Exact Solution $q_1 = \tilde{q}_1 \qquad q_2 = 0$ where q_1 satisfies

$$(I + B_1)\ddot{\tilde{q}}_1 + K \sin \tilde{q}_1 \underset{(1)}{=} 0 \quad \text{comes from eqn (1)} \qquad (3)$$

Suppose $q_1 = \tilde{q}_1 + q_1^*(t)$ and $q_2 = q_2^*(t)$ assuming q_1^*, q_2^* are small and second degree terms in starred quantities are negligible

Then

$$B_3 \ddot{q}_2^* - J q_2^* \dot{\tilde{q}}_1^2 \underset{(2)}{=} 0 \qquad (4)$$

for small $\tilde{q}_1$ (3) can be replaced by

$$(I + B_1)\ddot{\tilde{q}}_1 + K \tilde{q}_1 \underset{(3)}{=} 0$$

and this has solution: $\tilde{q}_1 = Q \sin pt$ where $p \triangleq \left[K/(I+B_1)\right]^{1/2}$

now $\dot{\tilde{q}}_1 = pQ \cos pt$ in (4) and rewrite

$$\ddot{q}_2^* - \frac{J}{B_3} p^2 Q^2 \cos^2 pt \; q_2^* \underset{(4)}{=} 0 \qquad (5)$$

this if of form $\ddot{x} - f(t)x = 0$

$$\left.\text{let } x \triangleq 2pt \quad y = q_2^* \quad \alpha \triangleq \beta = -\frac{J}{B_3}\frac{Q^2}{8}\right\} \qquad (6)$$

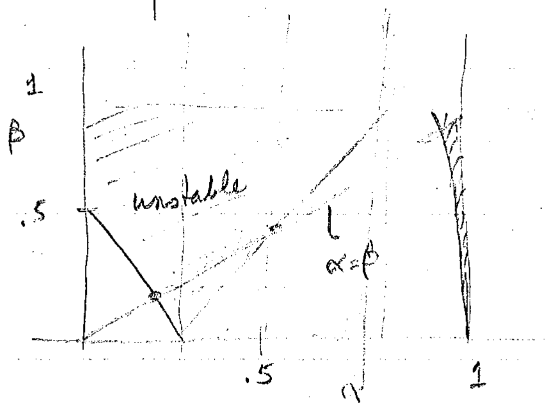
Then $$\frac{d^2 y}{dx^2} + (\alpha + \beta \cos x)\, y \underset{(5,6)}{=} 0 \qquad (7)$$

This is the Mathieu Eqn.

has solution $y = 0$ trivial soln $\Rightarrow q_2^* = 0 \Rightarrow q_2 = 0$ plate is not rotating

for certain α, β the soln $y=0$ is an unstable solution ie y cannot be kept

arbitrarily small if y & y' are small to start with. What are the α, β space where we have stability



$66° < Q < 115°$
$Q > 156°$ } unstable

based on $\alpha \hat{=} \beta = -\frac{J}{B_3}\frac{Q^2}{8}$

10/2/79

LOTS - there has been time been ALOTTED for this course

Overview of Course

I. Kinematics - to include differentiation of vector functions; partial velocities (linear and angular velocities)

II Force & Energy - Generalized forces, energy functions (PE, KE etc); inertia properties

III Laws of Motion - Formulation of Eqs of Motion & Solution

$$am - F = 0$$

accel. of points, mass centers, etc. — mass (moment of inertia) — force

Notation: $V = \underset{\sim}{V}$ vector

q: a scalar

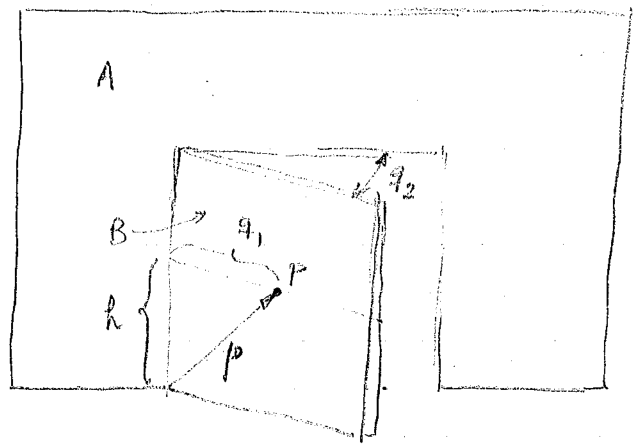
R: reference frame - any rigid body is a reference frame

→ "V is a function of q in R" ⟹ the magnitude &/or the direction of V in R depends on q. Direction < orientation — amount of parallelism; sense — where arrow is placed

Thus: V can be a function of q in A (another reference frame) but be

independent of q in B (another distinct reference frame)

Example



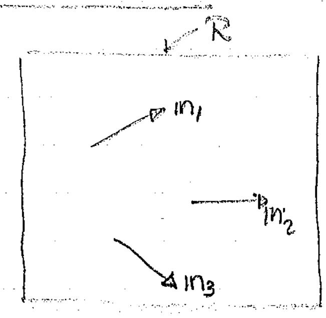
$\mathbf{p}$ is a function of q_1 and q_2 in A

if q_1 changes $\mathbf{p}$ mag changes
if q_2 changes $\mathbf{p}$ orient change

$\mathbf{p}$ is a fn of q_1, but is independent of q_2 in B

→ "$\mathbf{V}$ is fixed in R" $\Leftrightarrow$ the magnitude of $\mathbf{V}$ & direction of $\mathbf{V}$ in R do not dependent on any scalar variable

Vector Calculus



Given $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$: non coplanar, non parallel unit vectors fixed in R

$\mathbf{V} \;\exists\; V_1, V_2, V_3 \;\ni\; \mathbf{V} = V_i\mathbf{n}_i = V_1\mathbf{n}_1 + V_2\mathbf{n}_2 + V_3\mathbf{n}$

components of a vector are vectors. each scaler is a measure

Thus

$\mathbf{V}$ is a function of $q_1, \ldots, q_n$ in R $\Leftrightarrow$ V_1, V_2, V_3 are (scalar) fns of $q_1, \ldots\ldots, q_n$.

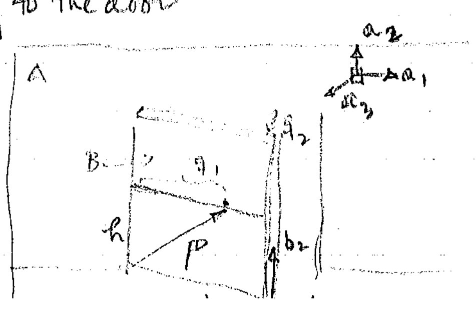
Defns: Starting w/ $\mathbf{V} = V_1\mathbf{n}_1 + V_2\mathbf{n}_2 + \ldots$

$$\frac{{}^R\partial \mathbf{V}}{\partial q_r} \triangleq \frac{\partial V_1}{\partial q_r}\mathbf{n}_1 + \frac{\partial V_2}{\partial q_r}\mathbf{n}_2 + \frac{\partial V_3}{\partial q_r}\mathbf{n}_3$$

the partial derivative of $\mathbf{V}$ wrt q_r in R since $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ are fixed in R

Review: If $\mathbf{V}$ is a fn of t (a scalar) in R then $\frac{{}^R d\mathbf{V}}{dt} \triangleq \frac{dV_1}{dt}\mathbf{n}_1 + \frac{dV_2}{dt}\mathbf{n}_2 + \ldots$

Returning to the door



	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$
$\mathbf{b}_1$	c_2	0	s_2
$\mathbf{b}_2$	0	1	0
$\mathbf{b}_3$	$-s_2$	0	c_2

$c_2 = \cos q_2$

Now we can write $\quad \mathbf{p} = q_1 \mathbf{b}_1 + h\,\mathbf{b}_2 \quad$ in B $\qquad (1)$

$\mathbf{p} = q_1 c_2 \mathbf{a}_1 + h\,\mathbf{a}_2 + q_1 s_2 \mathbf{a}_3 \quad$ in A $\qquad (2) \qquad$ substituting for $\mathbf{b}_1 = c_2\mathbf{a}_1 + s_2 \mathbf{a}_3$

$$ {}^A\frac{\partial \mathbf{p}}{\partial q_1}_{(2)} = c_2 \mathbf{a}_1 + s_2 \mathbf{a}_3 \qquad (3)$$

$$ {}^B\frac{\partial \mathbf{p}}{\partial q_1}_{(1)} = \mathbf{b}_1 = c_2 \mathbf{a}_1 + s_2 \mathbf{a}_3 \qquad (4)$$

$$ {}^A\frac{\partial \mathbf{p}}{\partial q_2}_{(2)} = -q_1 s_2 \mathbf{a}_1 + q_1 c_2 \mathbf{a}_3 = -q_1 \mathbf{b}_3 \qquad (5)$$

$$ {}^B\frac{\partial \mathbf{p}}{\partial q_2}_{(1)} = 0 \qquad (6)$$

are these the same — yes - since $\mathbf{p} \neq$ fn of $\mathbf{b}_3$ in reference frame B

$$ {}^B\frac{\partial}{\partial q_r}\left(\mathbf{v}_1 + \dots + \mathbf{v}_2\right) = \sum_i \frac{{}^B\partial \mathbf{v}_i}{\partial q_r} $$ differentiation of a sum of vectors

for Scalars: $\dfrac{\partial}{\partial q_r}(F_1 F_2 \dots F_n) = \dfrac{\partial F_1}{\partial q_r} F_2 \dots F_N + F_1 \dfrac{\partial F_2}{\partial q_r} F_3 \dots F_n + \dots \; F_1 \dots F_{n-1} \dfrac{\partial F_n}{\partial q_r}$

for Vectors: ${}^R\dfrac{\partial}{\partial q_r}(\mathbf{a}\cdot\mathbf{b}\,\mathbf{c}\times\mathbf{d}) = {}^R\dfrac{\partial \mathbf{a}}{\partial q_r}\cdot\mathbf{b}\,\mathbf{c}\times\mathbf{d} + \mathbf{a}\cdot{}^R\dfrac{\partial \mathbf{b}}{\partial q_r}\,\mathbf{c}\times\mathbf{d} + \mathbf{a}\cdot\mathbf{b}\,{}^R\dfrac{\partial \mathbf{c}}{\partial q_r}\times\mathbf{d} + \mathbf{a}\cdot\mathbf{b}\,\mathbf{c}\times{}^R\dfrac{\partial \mathbf{d}}{\partial q_r}$

Now Second derivatives: $\;{}^R\dfrac{\partial \mathbf{v}}{\partial q_r}\;$ can be differentiated in R or R' (another frame)

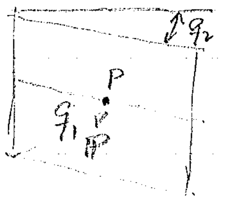
Example: ${}^B\dfrac{\partial \mathbf{p}}{\partial q_1} = \mathbf{b}_1 \implies {}^A\dfrac{\partial}{\partial q_2}\left({}^B\dfrac{\partial \mathbf{p}}{\partial q_1}\right) = {}^A\dfrac{\partial}{\partial q_2}(\mathbf{b}_1) = \dfrac{\partial}{\partial q_2}\left(c_2\mathbf{a}_1 + s_2\mathbf{a}_3\right) = -s_2\mathbf{a}_1 + c_2\mathbf{a}_3$

${}^A\dfrac{\partial \mathbf{p}}{\partial q_2} = -q_1\mathbf{b}_3 \implies {}^B\dfrac{\partial}{\partial q_1}\left({}^A\dfrac{\partial \mathbf{p}}{\partial q_2}\right) = -\mathbf{b}_3 = -s_2\mathbf{a}_1 + c_2\mathbf{a}_3$

<u>Total derivative in a reference frame</u> - let $q_1, \dots, q_n$ be fns of t

$\mathbf{v}$ regarded as a fn of $q_1, \dots, q_n$ <u>but not</u> t in R	$\mathbf{v}$ regarded as a fn of $q_1, \dots, q_n$ <u>and</u> t in R
${}^R\dfrac{d\mathbf{v}}{dt} = \displaystyle\sum_{r=1}^{n} {}^R\dfrac{\partial \mathbf{v}}{\partial q_r}\dot{q}_r \quad (a) \qquad \dot{q}_r = \dfrac{dq_r}{dt}$	${}^R\dfrac{d\mathbf{v}}{dt} = \displaystyle\sum_{r=1}^{n} {}^R\dfrac{\partial \mathbf{v}}{\partial q_r}\dot{q}_r + {}^R\dfrac{\partial \mathbf{v}}{\partial t} \quad (b)$

Return to the door



$q_1 = C\cos\omega t$

$q_2 = \omega t$

$\dot{q}_1 = -C\omega\sin\omega t \qquad (7)$

$\dot{q}_2 = \omega \qquad (8)$

let $\mathbb{P}$ be a fn of q_1, q_2 not t in A.
using the previous example

$$\mathbb{P} = q_1 c_2 a_1 + h a_2 + q_1 s_2 a_3 \qquad (2)$$

$$\frac{{}^A d\mathbb{P}}{dt} \underset{(a)}{=} \frac{{}^A\partial \mathbb{P}}{\partial q_1}\dot{q}_1 + \frac{{}^A\partial \mathbb{P}}{\partial q_2}\dot{q}_2 = (c_2 a_1 \underset{(3)}{+} s_2 a_3)\underset{(7)}{(-C\omega \sin\omega t)} + \underset{(5)}{(-q_1 s_2 a_1 + q_1 c_2 a_3)}\underset{(8)}{\omega}$$

$$\frac{{}^A d\mathbb{P}}{dt} \underset{(1,2)}{=} C\omega\left[-2\sin\omega t\cos\omega t\, a_1 + (\cos^2\omega t - \sin^2\omega t)\, a_3\right] \qquad (c)$$

if $\mathbb{P}$ is a fn of $q_1, q_2, \& t$ in A

$$\mathbb{P} \underset{(5)}{=} q_1 \underset{(2)}{\cos\omega t}\, a_1 + h a_2 + q_1 \sin q_2\, a_3$$

$$\frac{{}^A d\mathbb{P}}{dt} \underset{(b)}{=} \frac{{}^A\partial \mathbb{P}}{\partial q_1}\dot{q}_1 + \frac{{}^A\partial \mathbb{P}}{\partial q_2}\dot{q}_2 + \frac{{}^A\partial \mathbb{P}}{\partial t}$$

$$= \cos\omega t\, a_1 \underset{(7)}{(-C\omega \sin\omega t)} + \underset{(5)}{(-q_1 s_2 a_1 + q_1 c_2 a_3)}\underset{(8)}{\omega} - q_1\omega\sin\omega t\, a_1$$

$$= C\omega\left[-2\cos\omega t\sin\omega t\, a_1 + (\cos^2\omega t - \sin^2\omega t)\, a_3\right] \qquad (d)$$

Thus (c) = (d) as required

No homework is assigned - try to do as many as possible of first set. Read Book

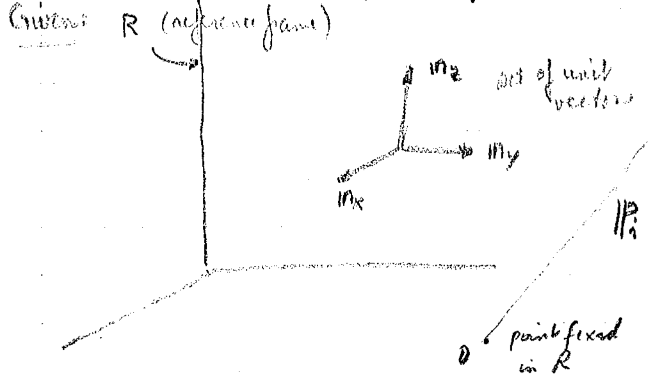
10/4/79

Read up to section 2.4
Today's lesson

1. Configuration
2. Constraint
3. Constraint Eqns
4. Holonomic & Non Holonomic Constraint Eqns.
5. Number of Degrees of Freedom
6. Generalized Coordinates

reference frames are distinct from axes embedded in reference frame

Given: R (reference frame)



P_i $i = 1, 2, \ldots, n$ (points in R where particle P_i is)

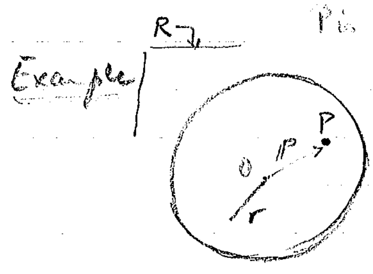
$\mathbb{P}_i$ = position vector $\Rightarrow$ $\mathbb{P}_i = x_i\, m_x + y_i\, m_y + z_i\, m_z$ wrt O

Configuration implies knowledge of position vectors of every particle

Constraints: restrictions on configurations (restrictions on position vectors)
restrictions on the manner in which configuration changes (position vectors change)

Holonomic Constraint Eqns: is any eqn of form $f(x_1, \ldots, x_n, y_1, \ldots, y_n, z_1, \ldots, z_n, t) = 0$

Example

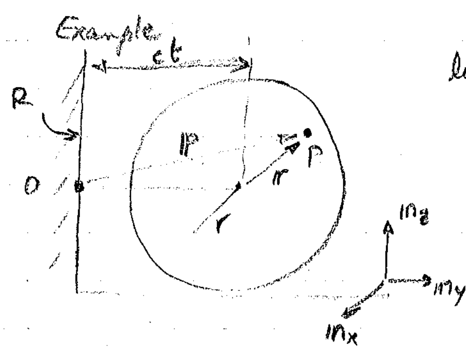


P is constrained to lie on sphere of radius r

$$\mathbb{P} = x\, m_x + y\, m_y + z\, m_z \quad (\text{wrt } O)$$
$$|\mathbb{P}| = (x^2 + y^2 + z^2)^{1/2}$$
$$|\mathbb{P}| = r \quad \text{is P is to remain on sphere}$$
$$\Rightarrow (x^2 + y^2 + z^2)^{1/2} = r$$

define $f(x, y, z, t) \triangleq (x^2 + y^2 + z^2)^{1/2} - r$ $\quad f = 0$ holonomic constraint Eqn.

Example



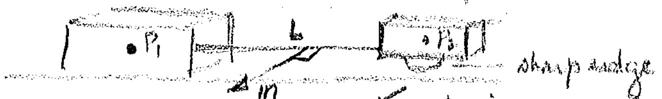
let sphere move in ref frame R. distance from line fixed in R to center is given by ct

$$\mathbb{P} = x\, m_x + y\, m_y + z\, m_z \quad (\text{wrt } O)$$
$$= ct\, m_y + \mathbb{r}$$
$$x\, m_x + (y - ct)\, m_y + z\, m_z = \mathbb{r}$$
$$|\mathbb{r}| = [(x^2) + (y - ct)^2 + z^2]^{1/2} = r$$
$$f(x, y, z, t) = [\qquad]^{1/2} - r = 0$$

when t appears explicitly it is rheonomic holonomic constraint eq
t does not " it is a scleronomic holonomic constraint eq

Non-Holonomic Eqns

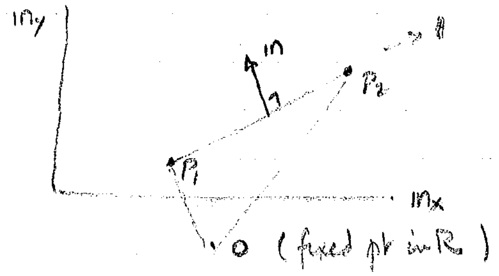
Given 2 blocks connected rigidly. One lies on sharp edge



constraint movement of sharp edge ; can't move in direction of $\mathbb{n}$

Constraint : veloc of P_2 has no component $\perp$ to line P_1P_2 : Due to resistance of sharp edge to motion in $\mathbb{n}$ direction

Thus $\quad \mathbb{v}^{P_2} \cdot \mathbb{n} = 0$ is math constraint representing real life



$\mathbb{l} = [(x_2 - x_1)\mathbb{n}_x + (y_2 - y_1)\mathbb{n}_y]/L$

$\mathbb{l} \cdot \mathbb{n} = 0 \Rightarrow \mathbb{n} = [-(y_2 - y_1)\mathbb{n}_x + (x_2 - x_1)\mathbb{n}_y]/L$

if $\mathbb{l} = a\mathbb{n}_x + b\,\mathbb{n}_y$

$\mathbb{n} = -b\mathbb{n}_x + a\mathbb{n}_y$

$\mathbb{P}_1 = x_1\mathbb{n}_x + y_1\mathbb{n}_y \qquad \mathbb{P}_2 = x_2\mathbb{n}_x + y_2\mathbb{n}_y \qquad$ (wrt O)

$\mathbb{v}^{P_2} = \dot{x}_2\mathbb{n}_x + \dot{y}_2\mathbb{n}_y \qquad \mathbb{n} = [-(y_2 - y_1)\mathbb{n}_x + (x_2 - x_1)\mathbb{n}_y]/L$

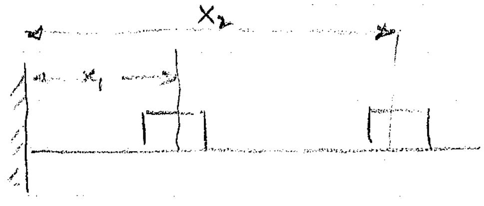
$\mathbb{v}^{P_2} \cdot \mathbb{n} = 0 \Rightarrow -\dot{x}_2(y_2 - y_1) + \dot{y}_2(x_2 - x_1) = 0 \qquad$ this is a differential constraint

Non holonomic since $\ \not\exists f \ni f(x_i, y_i, z_i, t) = 0 \qquad$ Nonholonomic eqn is anything that is not holonomic

The holonomic constraint due to rigid connection is $\ (x_2 - x_1)^2 + (y_2 - y_1)^2 - L^2 = 0$

Ex: $\ x\dot{x} + y\dot{y} + z\dot{z} = 0 \Rightarrow f(x,y,z,t) = x^2 + y^2 + z^2 - C = 0$ is a holonomic constr eqn. that can be brought into a holonomic constraint

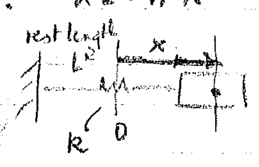
Ex:



Feedback Control Law - Given a block that moves & a sensor that finds $\dot{x}_1$, build an actuator that compensates

Build a sensor $\dot{x}_1$ (speedometer) ; build an actuator $x_2 = C\dot{x}_1 \qquad$ Nonholonomic constraints - since $f(\dot{x}_1, x_2) = 0$

Ex : $\ x = -n^2\ddot{x} \qquad$ Let $f(x,t) \triangleq x + A\sin(t/n) + B\cos(t/n)$



let $n^2 = \dfrac{m}{K}$

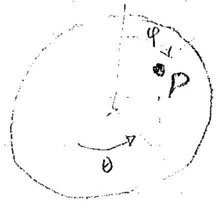
$x = -n^2\ddot{x}$ is the law of motion of a mass at end of spring w/ x measured from O

Number of degrees of freedom: n — for holonomic systems only

$n \triangleq$ count the no. of particles $\times 3 - M = 3N - M$

M = no. of holonomic constraint Eqns.

Example: Point constrained on surface of sphere



$N = 1$ $\quad (x^2+y^2+z^2)^{1/2} - r = 0$

$M = 1$ $\quad n = 2 = 3(1) - 1$ $\quad$ 2 degrees in θ, ϕ

Example: Point constrained on sphere where center moves with dist ct from wall in ref frame R

$N = 1$ $\quad [(x-ct)^2+y^2+z^2]^{1/2} - r = 0$

$M = 1$ $\quad n = 2$ $\quad$ 2 degrees in θ, ϕ

Generalized coordinates = # degrees of freedom in a holonomic system.

If each of $x_1, \dots, x_n$, $y_1, \dots, y_n$, $z_1, \dots, z_n$ can be expressed as a single valued fn. of t & n functions $q_1, \dots, q_n$ of t in such a way that all holon constraint eqns are satisfied $\forall\, q_1, \dots, q_n$ & t, then $q_1, \dots, q_n$ are called generalized coords of S in R.

S (set of particles)

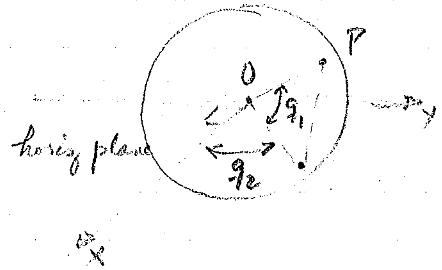
R (reference frame)

Ex: Ball rolling – same as second example above

$x = r\cos q_1 \cos q_2$ $\quad$ satisfies $\left[x^2+(y-ct)^2+z^2\right]^{1/2} - r = 0 \quad \forall\, q_1, q_2$

$y = ct + r\cos q_1 \sin q_2$

$z = r\sin q_1$ $\quad$ then q_1, q_2 are generalized coord of particles



q_1, q_2 are spherical coordinates

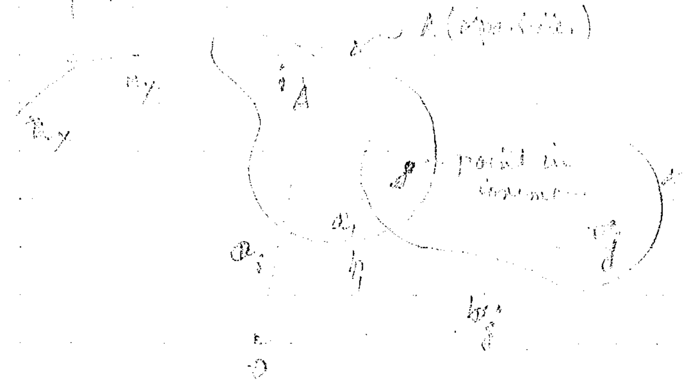
Ex: $x = q_1 \quad y = ct + q_2 \quad z = (r^2 - q_1^2 - q_2^2)^{1/2}$ also satisfies the constraint eqn.

q_1 & q_2 are x, y pos. measured relative to center O. (ie proj of P onto horiz plane)

10/9/95

Generalized coordinate systems is the minimum no. of coordinates that will characterize the motion of the body.

Problem 1(ii)a Pg 59 of Book



$1 \le i \le \alpha$

$1 \le j \le \beta$

$a_i = x_i^A n_x + y_i^A n_y + n_z z_i^A$

$b_j = x_j^B n_x + \dots$

Common point: $a_1 = b_1 \Rightarrow$ 3 scalar constraint eqns.
$x_1^A = x_1^B$, $y_1^A = y_1^B$, $z_1^A = z_1^B$

Rigidity of body: $(a_1 - a_2)^2 = L_{12}^2 = |a_1 a_2|^2$ by fixing 3 points & measuring all other points wrt these 3 pts

" " " $(b_1 - b_2)^2 = M_{12}^2 = |b_1 b_2|^2$

$(a_2 - a_3)^2 = |a_2 - a_3|^2$ $(a_3 - a_1)^2 = |a_3 - a_1|^2$

Thus for a_i $|a_i - a_1|^2 = L_{i1}^2$ $i = 4, \dots, \alpha$

$|a_i - a_2|^2 = L_{i2}^2$ $i = \dots$

$|a_i - a_3|^2 = L_{i3}^2$ $i = \dots$

No. of constraints of α

$$M = \underset{a_1 = b_1}{3} + (\underbrace{3}_{\text{build rigid triangle}} + \overbrace{3\alpha - 9}^{\text{other } \alpha - 3 \text{ eqns}}) = 6 + (3\alpha - 9) = 3\alpha - 3$$

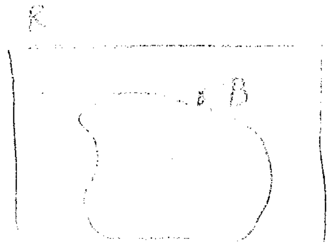
for No. of constr. of β

$$M = 3 + \overbrace{3\beta - 9}^{\dots} = 3\beta - 6$$

$$\therefore n = 3(\#\text{pts}) - M = 3(\alpha + \beta) - (3\alpha - 3 + 3\beta - 6) = 9$$

[if the no. of degrees of freedom of body S in ref A = n. & the no. of degrees of freedom of ref A wrt ref B = m then the no. of degrees of freedom of S in B is m+n]
only for a HOLONOMIC SYSTEM

Partial angular velocity



has variables $q_1, \dots, q_n$

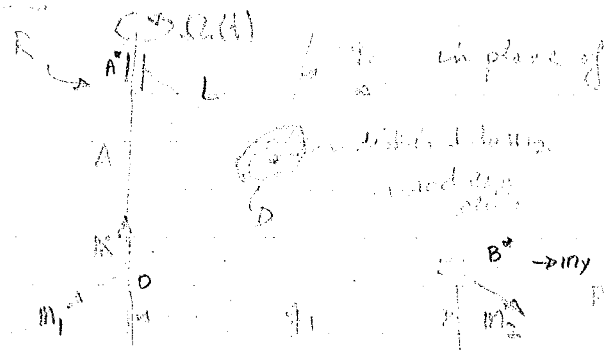
angular velocity $\left({}^R\omega^B\right)$

Evidence then ${}^R\omega^B = \sum_{r=1}^{n} {}^R\omega^B_{q_r}\,\dot q_r + {}^R\omega^B_t$

ω_{q_r} = fn of $(q_1, \dots, q_n, t)$ & so is ω_t

partial angular velocity

Example



$\mathbf{r}^{A^*} = \sqrt{L^2 - q_1^2}\,\mathbf{k}$ ${}^R\mathbf{v}^{A^*} = \frac{-2q_1\dot q_1\,\mathbf{k}}{\sqrt{}}$

${}^R\mathbf{v}^{A^*} = {}^R\mathbf{v}^{B^*} + {}^R\omega^{I_1} \times \mathbf{r}$

$\frac{-2q_1\dot q_1\,\mathbf{k}}{\sqrt{}} = \dot q_1\,\mathbf{n}_y + {}^R\omega^{I_1} \times (-L\,\mathbf{n}_2)$

$\Rightarrow {}^R\omega^{I_1} = \frac{\dot q_1}{L\sqrt{1 - q_1^2/L^2}}\,\mathbf{n}_1$

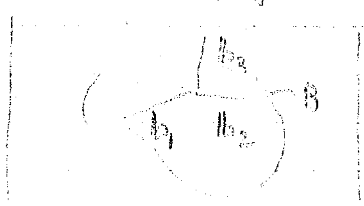
${}^R\omega^D = \frac{\dot q_1}{L\sqrt{1 - q_1^2/L^2}}\,\mathbf{n}_1 + \dot q_2\,\mathbf{n}_2 + \Omega(t)\,\mathbf{k}$

$\to \omega_{\dot q_1}$ $\omega_{\dot q_2} = \mathbf{n}_2$ ω_t

Angular velocity of Body B in ref A

Order Element

what is angular velocity



$${}^A\omega^B = \left(\frac{{}^A d\mathbf{b}_2}{dt}\cdot\mathbf{b}_3\right)\mathbf{b}_1 + \left(\frac{{}^A d\mathbf{b}_3}{dt}\cdot\mathbf{b}_1\right)\mathbf{b}_2 + \frac{{}^A d\mathbf{b}_1}{dt}\cdot\mathbf{b}_2\,\mathbf{b}_3$$

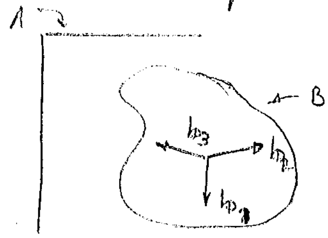
$\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are any 3 mutually $\perp$ unit vectors fixed in B & forming right handed triad

with this dfn: - how can we find angular velocity easily
- how did this eqn come about
- what is angular velocity good for.

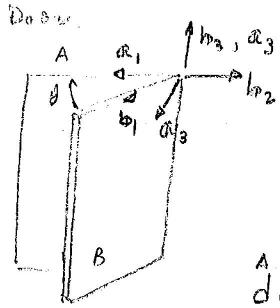
10/11/79

Final Exam Dec 13 7-10 PM

Continuation of last time



Recall Door



$b_1 = C a_1 + S a_2$

$b_2 = -S a_1 + C a_2$

$b_3 = a_3$

$$\frac{{}^A d b_1}{dt} = (-S a_1 + C a_2)\dot\theta = b_2 \dot\theta$$

$$\frac{{}^A d b_2}{dt} = -(C a_1 + S a_2)\dot\theta = -b_1 \dot\theta$$

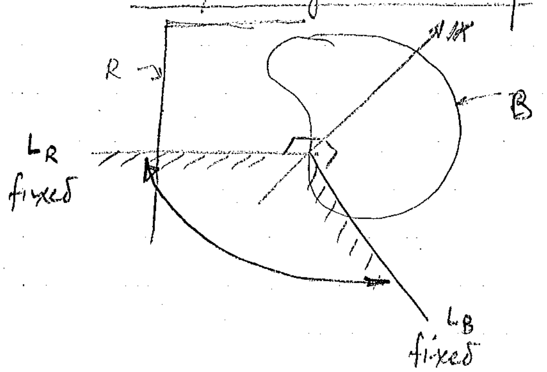
$$\frac{{}^A d b_3}{dt} = 0$$

$$\frac{{}^A d b_2}{dt} \cdot b_3 = 0 \qquad \frac{{}^A d b_3}{dt} \cdot b_1 = 0 \qquad \frac{{}^A d b_1}{dt} \cdot b_2 = \dot\theta$$

$${}^A\omega^B = 0 \cdot b_1 + 0 \cdot b_2 + \dot\theta b_3 = \dot\theta b_3$$ angular velocity of the door

This is too complicated — is there a better way

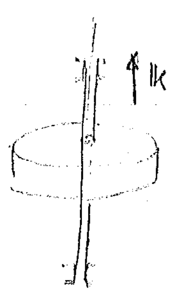
Simple Angular velocity



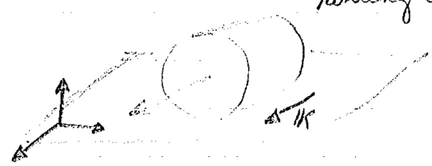
in general $\exists$ no unit vector $\mathbb{k}$ that remains fixed in both B & R as B moves in R. If, however, $\mathbb{k}$ does exist, then B has a simple angular velocity in R, and ${}^R\omega^B = \dot\theta \mathbb{k}$

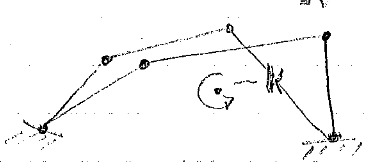
θ increases when B rotates relative to R in the right handed sense.

Other examples



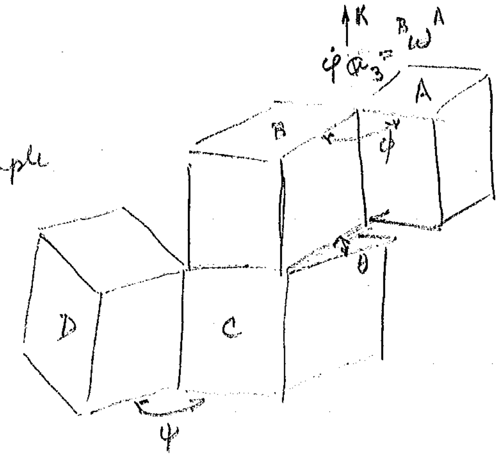
rolling cylinder





if system moves in plane of paper

Example



${}^{D}\omega^{A} \neq \mathbb{K}$

${}^{C}\omega^{A} \neq \mathbb{K}$

${}^{B}\omega^{A} = \dot\varphi a_3 = \dot\varphi b_3$

${}^{C}\omega^{B} = \dot\theta b_1 = \dot\theta c_1$

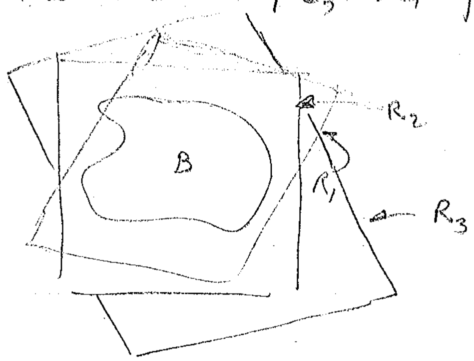
${}^{D}\omega^{C} = \dot\psi d_3 = \dot\psi c_3$

Chain Rule / Addition Thm / Auxiliary Reference Thm

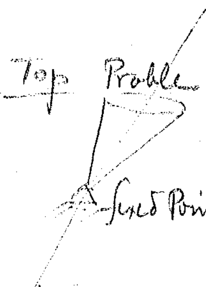
$$\boxed{{}^{R_n}\omega^{B} = {}^{R_n}\omega^{R_{n-1}} + {}^{R_{n-1}}\omega^{R_{n-2}} + \dots + {}^{R_1}\omega^{B}}$$

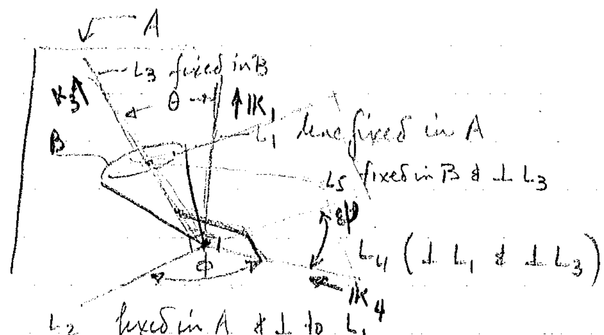
where ${}^{R_n}\omega^{R_{n-1}}$ are angular velocity of Reference frame R_{n-1} in Reference frame R_n. Can be simple or nonsimple angular velocity

Exaple ${}^{D}\omega^{A} = {}^{D}\omega^{C} + {}^{C}\omega^{B} + {}^{B}\omega^{A} = \dot\psi c_3 + \dot\theta b_1 + \dot\varphi a_3$



Top Problem





A: lines fixed in A — L_1 & L_2 physical reference frame $\mathbb{K}_1, \phi$ ${}^{A}\omega^{A_1} = \dot\phi\, \mathbb{K}_1$

A_1: lines L_1 & L_4 are fixed $\mathbb{K}_4, \theta$ ${}^{A_2}\omega^{A_1} = -\dot\theta\, \mathbb{K}_4$
(aux.)

A_2: L_3 & L_4 are fixed $\mathbb{K}_3, \psi$ ${}^{B}\omega^{A_2} = \dot\psi\, \mathbb{K}_3$
(aux.)

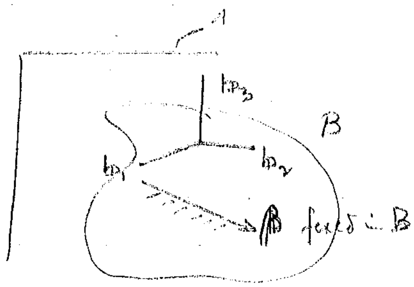
B: L_3 & L_5 physical reference frame

$${}^{B}\omega^{A} = {}^{B}\omega^{A_2} + {}^{A_2}\omega^{A_1} + {}^{A_1}\omega^{A} = \dot\psi\mathbb{K}_3 + \dot\phi\mathbb{K}_1 - \dot\theta\mathbb{K}_4$$

now we can define $\mathbb{K}_i$ in terms of vectors in the body B

$$\frac{{}^A d\mathbb{V}}{dt} = \frac{{}^B d\mathbb{V}}{dt} + {}^A\omega^B \times \mathbb{V}$$

given a vector $\mathbb{V}$ in Reference frames A, & B. The velocity of the vector in Ref. frame A, if it is known in Ref B, is given by :-)



Want to find $\frac{{}^A d\beta}{dt}$ =

$$\beta = \beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3 \qquad \beta_1, \beta_2, \beta_3 \text{ are constants} \quad (1)$$

$$\frac{{}^A d\beta}{dt} = \beta_1 \frac{{}^A db_1}{dt} + \beta_2 \frac{{}^A db_2}{dt} + \beta_3 \frac{{}^A db_3}{dt} \quad (2)$$

$$\frac{{}^A db_1}{dt} = \left(\frac{{}^A db_1}{dt}\cdot b_1\right) b_1 + \left(\frac{{}^A db_1}{dt}\cdot b_2\right) b_2 + \left(\frac{{}^A db_1}{dt}\cdot b_3\right) b_3 \quad (3)$$

$= 0$ since $b_1 \cdot b_1 = 1 \Rightarrow \frac{d}{dt}(\ ) = 0 \Rightarrow 2\, b_1 \cdot \dot{b}_1 = 0$

Similarly $\frac{{}^A db_2}{dt} =$ (4)

$$\frac{{}^A db_3}{dt} = \left(\frac{{}^A db_3}{dt}\cdot b_1\right) b_1 + \left(\frac{{}^A db_3}{dt}\cdot b_2\right) b_2 \quad (5)$$

$$\therefore \frac{{}^A d\beta}{dt} = \beta_1\left(\frac{{}^A db_1}{dt}\cdot b_2\, b_2 + \frac{{}^A db_1}{dt}\cdot b_3\, b_3\right) + \beta_2\left(\frac{{}^A db_2}{dt}\cdot b_1\ b_1 + \dots\right) + \beta_3\left(\frac{db_3}{dt}\cdot b_1\, b_1 + \dots\right)$$

$$= \left(\beta_2 \frac{{}^A db_2}{dt}\cdot b_1 + \beta_3 \frac{{}^A db_3}{dt}\cdot b_1\right) b_1 + \left(\beta_3 \frac{db_3}{dt}\cdot b_2 + \dots\right) b_2 + \left(\beta_1 \frac{db_1}{dt}\cdot b_3 + \dots\right) b_3 \quad (6)$$

Observations $\frac{{}^A db_2}{dt}\cdot b_1 + b_2 \cdot \frac{{}^A db_1}{dt} = 0 \Rightarrow \frac{{}^A d}{dt}(b_2 \cdot b_1) = 0$ (7) use these & subst for some of the deriv

$$\frac{{}^A d\beta}{dt} = \left(\frac{db_3}{dt}\cdot b_1\, \beta_3 - \frac{db_1}{dt}\cdot b_2\, \beta_2\right) b_1 + (\qquad) b_2 + (\qquad) b_3 \quad (8)$$

Abbrev. let $\alpha_1 \triangleq \frac{{}^A db_2}{dt}\cdot b_3 \qquad \alpha_2 \triangleq \frac{{}^A db_3}{dt}\cdot b_1 \qquad \alpha_3 = \frac{{}^A db_1}{dt}\cdot b_2$ (9)

$$\frac{{}^A d\beta}{dt} \underset{(8,9)}{=} (\alpha_2\beta_3 - \alpha_3\beta_2)\, b_1 + (\alpha_3\beta_1 - \beta_3\alpha_1)\, b_2 + (\alpha_1\beta_2 - \alpha_2\beta_1)\, b_3$$

$= \alpha \times \beta$ but $\alpha \triangleq \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3$

substit. $\alpha \underset{(9)}{=} \left(\frac{{}^A db_2}{dt}\cdot b_3\right) b_1 + \left(\frac{{}^A db_3}{dt}\cdot b_1\right) b_2 + \left(\frac{{}^A db_1}{dt}\cdot b_2\right) b_3$

$\alpha = {}^A\omega^B$

Thus $\therefore \boxed{\frac{{}^A d\beta}{dt} = {}^A\omega^B \times \beta}$

thus if β is fixed in B then $\frac{{}^A d\beta}{dt} = {}^A\omega^B \times \beta$

10/16/79

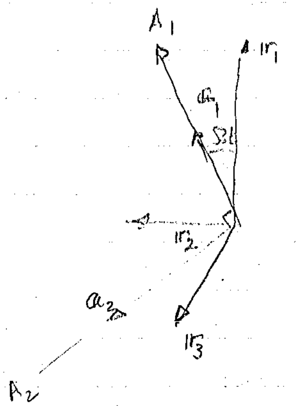
Problem #2a

let n_1, n_2, n_3 in R express m_1, m_2, m_3 (of B) in R and get

${}^R\omega^B = \frac{{}^R dm_2}{dt} \cdot m_3\ m_1$

Thus we can define

	a_1	a_2	a_3
m_1	$C_2 C_3$	$S_1 S_2 C_3 + S_3 C_1$	---
m_2			
m_3			



	a_1	a_2	a_3
r_1	$C(\Omega t)$	$-S(\Omega t)$	0
r_2	$S(\Omega t)$	$C(\Omega t)$	0
r_3	0	0	1

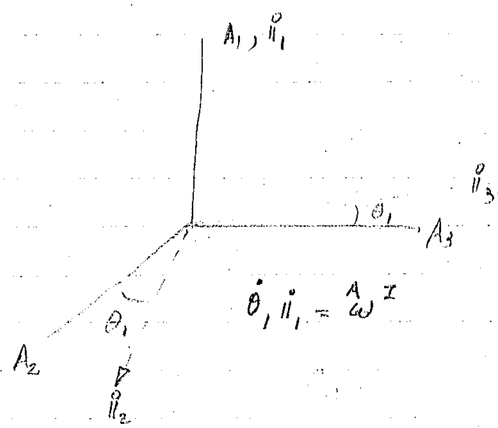
$$\Rightarrow \quad m_1 = (\quad) r_1 + (\quad) r_2 + (\quad) r_3$$

better way is to use addition theorem

$${}^R\omega^B = {}^R\omega^A + {}^A\omega^B \qquad {}^R\omega^A = \Omega a_3$$

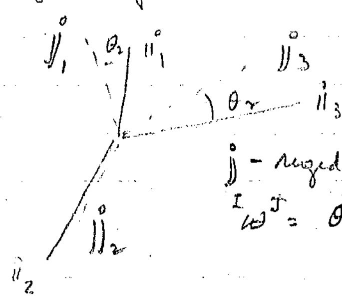
and $\quad {}^A\omega^B = {}^A\omega^I + {}^I\omega^J + {}^J\omega^K + {}^K\omega^B$

I, J, K are rigid bodies frames of reference (rigid body rotation



$\dot\theta_1 i_1 = {}^A\omega^I$

i – rigid body rotation by angle θ_1 about i



j – rigid body rotation by θ_2 about i_2

${}^I\omega^J = \dot\theta_2 \, i_2$

$${}^A\omega^B = \Omega a_3 + \dot\theta_1 i + \dot\theta_2 j_2 + \dot\theta_3 k_3 \qquad {}^K\omega^B = 0$$

must write ω as a fn m_1, m_2, m_3

but K & m are same $\quad K_3 = m_3$; using each picture we can get relation between the bases.

finally $\quad {}^R\omega^B = \omega_1 m_1 + \omega_2 m_2 + \omega_3 m_3$

$\omega_1 = f(\theta_i, \dot\theta_i) \quad$ nonlinear in θ_i but linear in $\dot\theta_i$

$\omega_2 =$

$\omega_3 =$

now solve for $\dot\theta_i$

2c. $\;{}^{B}\omega^{A} = {}^{B}\omega^{A} + {}^{A}\omega^{B} + {}^{B}\omega^{A}$
$= {}^{A}\omega^{B} + 2\,{}^{B}\omega^{A}$
$-{}^{B}\omega^{A} = {}^{A}\omega^{B}$

${}^{A}\omega^{A} = {}^{A}\omega^{B} + {}^{B}\omega^{A}$

fix a_1, a_2, a_3 in A

$\dfrac{{}^{A}da_1}{dt} = \dfrac{{}^{A}da_2}{dt} = \dfrac{{}^{A}da_3}{dt} = 0$

${}^{A}\omega^{A} = \left(\dfrac{da_1}{dt}\cdot a_2\right)a_3 + \dots = 0$

for any $\mathbb{V}$: if $\dfrac{{}^{A}d\mathbb{V}}{dt} = \dfrac{{}^{B}d\mathbb{V}}{dt} + {}^{A}\omega^{B}\times\mathbb{V}$; also have $\dfrac{{}^{B}d\mathbb{V}}{dt} = \dfrac{{}^{A}d\mathbb{V}}{dt} + {}^{B}\omega^{A}\times\mathbb{V}$ substitute for $\dfrac{{}^{B}d\mathbb{V}}{dt}$

$0 = \left({}^{A}\omega^{B} + {}^{B}\omega^{A}\right)\times\mathbb{V} \quad \forall\,\mathbb{V}$

$\Rightarrow {}^{A}\omega^{B} + {}^{B}\omega^{A} = 0$ QED

for 2nd part:
also prove. $\dfrac{{}^{A}d\,{}^{A}\omega^{B}}{dt} = \dfrac{{}^{B}d\,{}^{A}\omega^{B}}{dt} \Rightarrow {}^{A}\alpha^{B} \triangleq \dfrac{{}^{A}d\,{}^{A}\omega^{B}}{dt} = \dfrac{{}^{B}d\,{}^{A}\omega^{B}}{dt}$ angular accel.

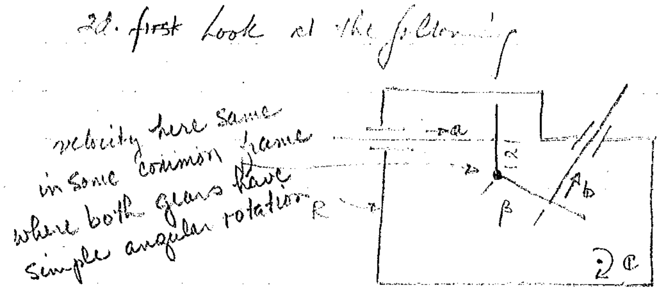
using $\mathbb{V} = {}^{A}\omega^{B}$ $\quad \dfrac{{}^{A}d\,\omega^{B}}{dt} = \dfrac{{}^{B}d\,{}^{A}\omega^{B}}{dt} + {}^{A}\omega^{B}\times\left({}^{A}\omega^{B}\right)$ but $\omega\times\omega = 0$

$\Rightarrow \dfrac{{}^{A}d\omega}{dt} = \dfrac{{}^{B}d\omega}{dt}$

To find acceleration use defn. ${}^{A}\alpha^{B} \triangleq \dfrac{{}^{A}d\omega}{dt} = \dfrac{{}^{B}d\omega}{dt}$

accelerations <u>do not</u> satisfy the addition theorem.

2d. first look at the following



${}^{R}\omega^{A} = {}^{R}\omega^{A}\,a$ or ${}^{R}\omega^{A} = {}^{R}\omega^{A}\cdot a$

${}^{R}\omega^{B} = {}^{R}\omega^{B}\,b$

${}^{R}\mathbb{V}^{A^*}$ $\quad A^*$ - that pt of A in contact w/ B

${}^{R}\mathbb{V}^{B^*}$ $\quad B^*$ - " " " B in contact w/ A

${}^{R}\mathbb{V}^{A^*} = \alpha\,{}^{R}\omega^{A}\,c$ $\quad {}^{R}\mathbb{V}^{B^*} = -\beta\,{}^{R}\omega^{B}\,c$ $\quad {}^{R}\mathbb{V}^{A^*} = {}^{R}\mathbb{V}^{B^*}$ due to gear contact

$\alpha\,{}^{R}\omega^{A} = -\beta\,{}^{R}\omega^{B}$

1. We want <u>contact</u> between little B w/ b of figure 2d on pg 61

now ${}^{c}\omega^{b}\,r = {}^{c}\omega^{B}R$ (1) $\quad {}^{c}\omega^{b} = {}^{c}\omega^{b}\,n \quad {}^{c}\omega^{B} = {}^{c}\omega^{B}\,N$

Contact between b & B' $\quad \omega \quad {}^{c}\omega^{b}r = -{}^{c}\omega^{B'}R$ (2)

E & G $\quad {}^{F}\omega^{G}a = {}^{F}\omega^{E}a = {}^{F}\omega^{G}b$ (3)

ω in problem.

2. Now use <u>chain rule</u> B, C, F

${}^{B}\omega^{C} = {}^{B}\omega^{F} + {}^{F}\omega^{C}$

$-\Omega$

${}^{B}\omega^{C} = {}^{B}\omega^{F} + {}^{F}\omega^{C}$ (4) if all are // to same vector

Also use chain rule for B', C, F

$$ {}^{R'}\omega^{C} = \underbrace{{}^{R'}\omega^{F}}_{=\Omega'} + {}^{F}\omega^{C} \qquad (5) $$

$$ \Omega \underset{(4)}{=} \omega \underset{(3)}{\cdot} b/a - {}^{B}\omega^{C} = \omega\frac{b}{a} + {}^{C}\omega^{B} \underset{(1)}{=} \omega\frac{b}{a} + {}^{C}\omega^{b}(r/R) \qquad (6) $$

$$ \Omega' \underset{(5)}{=} \omega(b/a) - {}^{B'}\omega^{C} \underset{(2)}{=} \omega(b/a) - {}^{C}\omega^{b}(r/R) \qquad (7) $$

$$ \Omega + \Omega' \underset{(6,7)}{=} 2\omega(b/a) $$

2e. idea is to differentiate $\mathbb{A}$ in 2 diff reference frames $\frac{{}^{R}d\mathbb{A}}{dt} + \frac{{}^{B}d\mathbb{A}}{dt} + {}^{R}\omega^{B} \times \mathbb{A}$
if $\mathbb{A}$ is in B

2f $\quad {}^{L}\omega^{B} = {}^{L}\omega^{D} + {}^{D}\omega^{B}$
$\quad = \dot\phi\, m_2 + \dot\theta\, m_3$

$$ {}^{L}\alpha^{B} = \frac{{}^{L}d\,{}^{L}\omega^{B}}{dt} = \ddot\phi\, m_2 + \dot\phi\frac{{}^{L}dm_2}{dt} + \ddot\theta\, m_3 + \dot\theta\frac{{}^{L}dm_3}{dt} $$

m_2 is fixed in L $\Rightarrow \frac{{}^{L}dm_2}{dt} = 0$

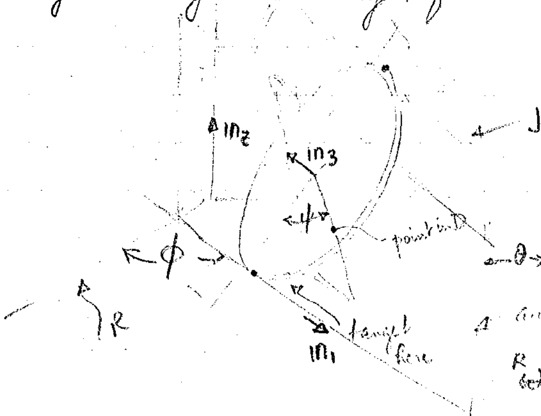
$$ \frac{{}^{L}dm_3}{dt} = {}^{L}\omega^{B} \times m_3 + \cancel{\frac{{}^{B}dm_3}{dt}} \qquad {}^{L}\omega^{B} = \dot\phi\, m_2 + \dot\theta\, m_3 $$

since m_3 is fixed in B

$$ \text{or} \quad {}^{L}\omega^{D} \times m_3 + \underbrace{\frac{{}^{D}dm_3}{dt}}_{=0} $$

since m_3 is also fixed in D

2g. use of auxiliary reference frame.



$$ {}^{J}\omega^{D} = \dot\psi\, m_3 $$

$$ {}^{I}\omega^{J} = \dot\theta(-m_1) $$

aux ref. I

$$ {}^{R}\omega^{I} = \dot\phi\, m_z $$

Now; $\quad {}^{R}\omega^{D} = {}^{R}\omega^{I} + {}^{I}\omega^{J} + {}^{J}\omega^{D} = \dot\phi\, m_z - \dot\theta\, m_1 + \dot\psi\, m_3 \qquad$ must write m_1, m_3 in R

$$ {}^{R}\alpha^{D} = \ldots \quad \frac{{}^{R}dm_1}{dt} \qquad \frac{{}^{R}dm_1}{dt} = \underbrace{\cancel{\frac{{}^{I}dm_1}{dt}}}_{=0} + {}^{R}\omega^{I} \times m_1 $$

since m_1 is fixed in I

$$ \frac{{}^{R}dm_3}{dt} = \underbrace{\cancel{\frac{{}^{J}dm_3}{dt}}}_{=0} + {}^{R}\omega^{J} \times m_3 $$

since m_3 is fixed in J

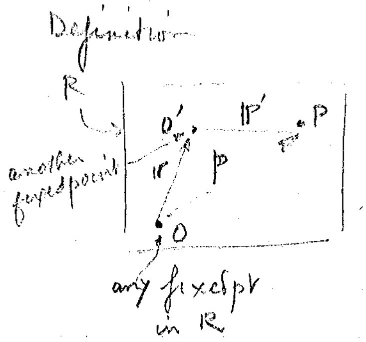
Velocity (Partial & Total) & Acceleration

Partial Velocity

Given $\mathbb{V} = \sum_{r=1}^{N} \mathbb{V}_{q_r} \dot{q}_r + \mathbb{V}_t$ in R

Given a particle P with N degrees of freedom expressed in $q_1, \dots, q_n$ generalized coordinates of a holonomic system S in R.

Definition



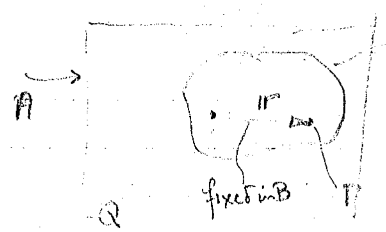
$${}^R\mathbb{V}^P \triangleq \frac{{}^R d\mathbb{p}}{dt}$$

$$\mathbb{p} = \mathbb{r} + \mathbb{p}' \qquad \frac{{}^R d\mathbb{p}}{dt} = \underbrace{\frac{{}^R d\mathbb{r}}{dt}}_{=0 \text{ since fixed in } R} + \frac{{}^R d\mathbb{p}'}{dt}$$

$${}^R\mathbb{a}^P \triangleq \frac{{}^R d\, {}^R\mathbb{V}^P}{dt}$$

Two important theorems

I. Two points both fixed on a rigid body B, when B is moving in A



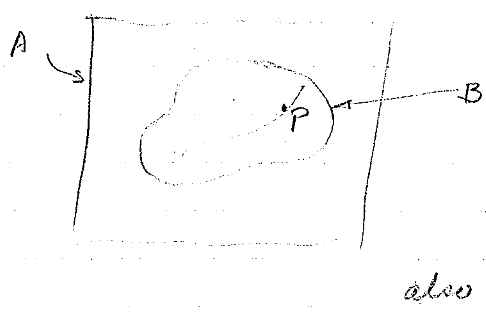
Eulers

$${}^A\mathbb{V}^P = {}^A\mathbb{V}^Q + {}^A\omega^B \times \mathbb{r} \qquad \frac{{}^A d\mathbb{r}}{dt} = \overbrace{\frac{{}^B d\mathbb{r}}{dt}}^{0} + {}^A\omega^B \times \mathbb{r} \quad (1)$$

$$\frac{{}^A d\, {}^A\mathbb{V}^P}{dt} = {}^A\mathbb{a}^P = {}^A\mathbb{a}^Q + {}^A\alpha^B \times \mathbb{r} + {}^A\omega^B \times ({}^A\omega^B \times \mathbb{r}) \quad (2)$$

Useful if you know what goes with pt Q & need info for p.

II. One point moving on a rigid body B, when B is moving in A



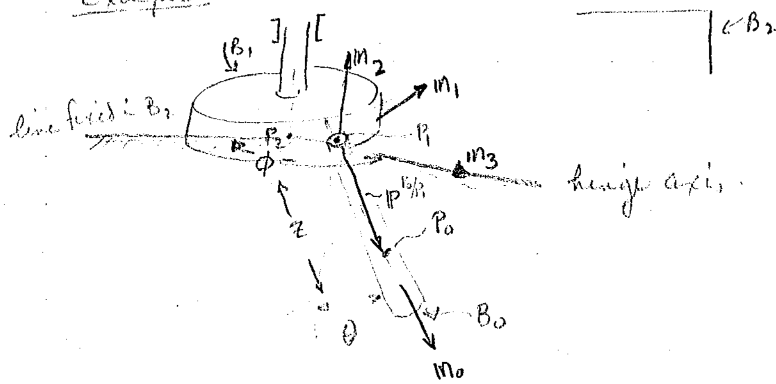
$${}^A\mathbb{V}^P = {}^B\mathbb{V}^P + {}^A\mathbb{V}^{\bar{B}} \quad (3)$$

When $\bar{B}$ is that pt of B in ref frame A, with which P coincides at the instant under consider

$\bar{B}$ is the coincident point or transport pt.

$$\text{also} \quad {}^A\mathbb{a}^P = \underbrace{{}^B\mathbb{a}^P}_{\text{relative accel}} + \underbrace{{}^A\mathbb{a}^{\bar{B}}}_{\text{transport accel}} + \underbrace{2\, {}^A\omega^B \times {}^B\mathbb{V}^P}_{\text{Coriolis accel.}} \quad (4)$$

Example



Suppose we have this rotating weight w/arm hinges in the weight and a sleeve on the arm moves downward

find ${}^{B_0}\mathbb{V}^{P_0}$, ${}^{B_1}\mathbb{V}^{P_0}$, ${}^{B_2}\mathbb{V}^{P_0}$, ${}^{B_0}\mathbb{a}^{P_0}$, ${}^{B_1}\mathbb{a}^{P_0}$, ${}^{B_2}\mathbb{a}^{P_0}$,

- $${}^{B_0}\mathbb{V}^{P_0} \underset{Def}{=} \frac{{}^{B_0}d}{dt}\left(\mathbb{p}^{P_0/P_1}\right) = \frac{{}^{B_0}d}{dt}(z\mathbb{n}_0) = \dot{z}\mathbb{n}_0 \qquad (a)$$
- $${}^{B_0}\mathbb{a}^{P_0} \underset{Def}{=} \frac{{}^{B_0}d\,{}^{B_0}\mathbb{V}^{P_0}}{dt} = \frac{{}^{B_0}d\,(\dot{z}\mathbb{n}_0)}{dt} = \ddot{z}\,\mathbb{n}_0 \qquad (b)$$

Since P_0 is moving on B_0

$${}^{B_1}\mathbb{V}^{P_0} \underset{(3)}{=} {}^{B_0}\mathbb{V}^{P_0} + {}^{B_1}\mathbb{V}^{\bar{B}_0} \quad (c) \; ; \text{ what is } {}^{B_0}\mathbb{V}^{P_0} \underset{(a)}{=} \dot{z}\mathbb{n}_0 \;(d) \; ; \; {}^{B_1}\mathbb{V}^{\bar{B}_0} \underset{(1)}{=} {}^{B_1}\mathbb{V}^{P_1} + {}^{B_1}\omega^{B_0} \times \mathbb{p}^{P_0/P_1} \quad (e)$$
$$= \dot{\theta}\mathbb{n}_3 \times z\mathbb{n}_0 \quad (h)$$

$${}^{B_1}\mathbb{V}^{P_1} \underset{def.}{=} \frac{{}^{B_1}d\,\mathbb{p}^{P_1/P_1}}{dt} = 0 \quad (f) \; ; \; {}^{B_1}\omega^{B_0} = \dot{\theta}\mathbb{n}_3 \quad (g)$$

since P_1 is fixed in B_1

$$\underset{(c)\;(d)\qquad(h)}{{}^{B_1}\mathbb{V}^{P_0} = \dot{z}\mathbb{n}_0 + \dot{\theta} z\mathbb{n}_3 \times \mathbb{n}_0} \qquad (i)$$

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Example Continued

$${}^{B_1}\mathbb{a}^{P_0} \underset{(4)}{=} {}^{B_0}\mathbb{a}^{P_0} + {}^{B_1}\mathbb{a}^{\bar{B}_0} + 2\,{}^{B_1}\omega^{B_0} \times {}^{B_0}\mathbb{V}^{P_0} \qquad (j)$$

$$\frac{{}^{B_0}d\,{}^{B_0}\mathbb{V}^{P_0}}{dt} = {}^{B_0}\mathbb{a}^{P_0} \underset{(b)}{=} \ddot{z}\mathbb{n}_0 \qquad (k) \; ;$$

$$\frac{{}^{B_1}d\,{}^{B_1}\mathbb{V}^{\bar{B}_0}}{dt} = {}^{B_1}\mathbb{a}^{\bar{B}_0} \underset{(2)}{=} \cancel{{}^{B_1}\mathbb{a}^{P_1}} + {}^{B_1}\alpha^{B_0} \times \mathbb{p}^{P_0/P_1} + {}^{B_1}\omega^{B_0} \times ({}^{B_1}\omega^{B_0} \times \mathbb{p}^{P_0/P_1})$$

$\to 0$ fixed (m)

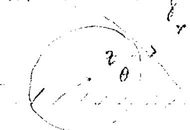
$$\ddot{\theta}\mathbb{n}_3 \times (z\mathbb{n}_0) + \dot{\theta}\mathbb{n}_3 \times (\dot{\theta}\mathbb{n}_3 \times z\mathbb{n}_0) \quad (n) \;(o)$$

$$2\,{}^{B_1}\omega^{B_0} \times {}^{B_0}\mathbb{V}^{P_0} = 2\dot{\theta}\mathbb{n}_3 \times \underset{(a)}{(\dot{z}\mathbb{n}_0)} \qquad (q)$$

$${}^{B_1}\mathbb{a}^{\bar{B}_0} = \ddot{\theta} z(\mathbb{n}_3 \times \mathbb{n}_0) - \dot{\theta}^2 z\mathbb{n}_0 \qquad (p)$$

In B_1 $\bar{B}_0$ moves on a circle of radius z centered at P_1 $\Rightarrow$ ${}^{B_1}\mathbb{a}^{\bar{B}_1} = z\ddot{\theta}\mathbb{n}_3 \times \mathbb{n}_0 - z\dot{\theta}^2\mathbb{n}_0$

without having used $l = 0$



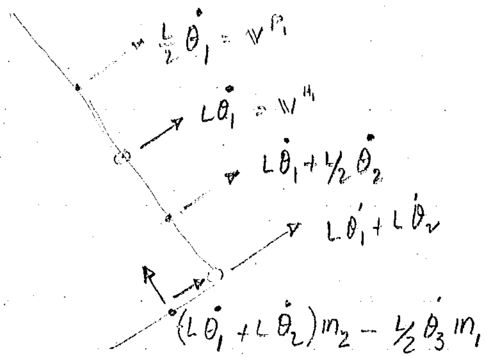
$$\underset{(j)}{{}^{B_1}\mathbb{a}^{P_1}} = \underset{(k)}{\ddot{z}\mathbb{n}_0} + \underset{(p)}{(\ddot{\theta} z\mathbb{n}_3 \times \mathbb{n}_0 - \dot{\theta}^2 z\mathbb{n}_0)} + \underset{(q)}{2\dot{\theta}\dot{z}\,\mathbb{n}_3 \times \mathbb{n}_0}$$

$${}^{B_2}\mathbb{V}^{P_0} = \dot z\, n_0 + \dot\theta z\, n_3 \times n_0 + \dot\phi\,(r n_1 + z\, n_2 \times n_0)$$

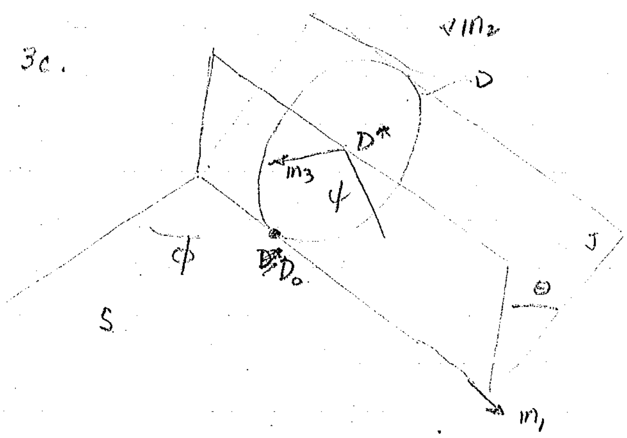
$${}^{B_2}a^{P_0} = \ddot z\, n_0 + (\ddot\theta z + 2\dot\theta \dot z)\, n_3 \times n_0 - \dot\theta^2 z\, n_0 + \ddot\phi\, n_2 \times (r n_3 + z\, n_0) + \dot\phi^2\, n_2 \times [n_2 \times (r n_3 + z\, n_0)]$$

Problem Set #3 1 Nov 79 Midterm

3a. Show sketch in its current configuration



3c.



we want to find

$$\omega = -\dot\theta n_1 + \dot\phi \cos\theta\, n_2 + (\dot\psi + \dot\phi \sin\theta)\, n_3$$

$${}^{R}\mathbb{V}^{D^*} = r\,{}^{R}\omega^{D} \times n_2 + \mathbb{V}^{D_0} \quad {}^{\mathbb{V}^{D_0} = 0} \text{ def of rolling}$$

$${}^{R}a^{D^*} = r\alpha \times n_2 + r\omega \times \frac{{}^{R}d n_2}{dt}$$

$$\left[({}^{R}\omega^{D} - \dot\psi n_3) \times n_2\right]$$

${}^{R}\omega^{J}$ in frame where n_2 is fixed

$$a^{D^*} = r\alpha \times n_2 + r(\omega \times \omega) \times n_2 - r\dot\psi\, \omega \times n_3 \times n_2$$

$$a = a^{D^*} - r\alpha \times n_2 - r\omega \times (\omega \times n_2) = r\dot\psi\, \omega \times n_1$$

$$|a| = r|\dot\psi|\,(\omega_2^2 + \omega_3^2)^{1/2}$$

$$= r|\dot\psi|\,\left[(\dot\phi\cos\theta)^2 + (\dot\psi + \dot\phi\sin\theta)^2\right]^{1/2}$$

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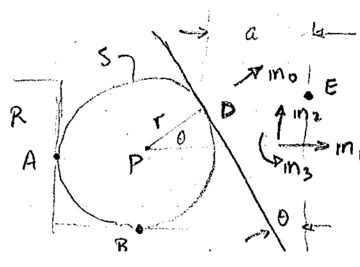
3d.



Restrictions on the motions due to construction of the device

$${}^{R}\mathbb{V}^{P} = {}^{R}V^{P} n_3 \qquad (1)$$

$${}^{R}\omega^{C} = {}^{R}\omega^{C} n_2 \qquad (2)$$

$${}^{R}\omega^{S} = \omega_1 n_1 + \omega_2 n_2 + \omega_3 n_3 \qquad (3)$$

(over)

define these wrt a known quantity

Rolling at A: ${}^R\mathbf{v}^P + {}^R\boldsymbol{\omega}^S \times (-r\,\mathbf{n}_1) = 0$ since $\mathbf{r}^{P/A}$ fixed in S (4) ${}^R\mathbf{v}^A = 0$ since A is fixed in R

B: ${}^R\mathbf{v}^P + {}^R\boldsymbol{\omega}^S \times (-r\,\mathbf{n}_2) = 0$ since $\mathbf{r}^{P/B}$ " " " (5) ${}^R\mathbf{v}^B = 0$

and ${}^R\mathbf{r}^D = {}^R\mathbf{r}^P + {}^R\mathbf{r}^{P/D}$ take $\frac{d}{dt}$ ↑ $\frac{{}^Rd\mathbf{r}^{P/D}}{dt} = \frac{{}^Sd\mathbf{r}^{P/D}}{dt} + {}^R\boldsymbol{\omega}^S \times \mathbf{r}^{P/D}$ ⇒ D: ${}^R\mathbf{v}^P + {}^R\boldsymbol{\omega}^S \times (r\,\mathbf{n}_0) = {}^R\boldsymbol{\omega}^C \times (-a\,\mathbf{n}_1)$ (6) ${}^R\mathbf{v}^D = {}^C\mathbf{v}^D$

since $\mathbf{r}^{P/D}$ is fixed in S since ${}^C\mathbf{r}^D$ is fixed in C.

Pure rolling at D:

$$ {}^C\boldsymbol{\omega}^S \cdot \mathbf{n}_0 = 0 \qquad (7)$$

Chain rule $${}^C\boldsymbol{\omega}^S = {}^R\boldsymbol{\omega}^S - {}^R\boldsymbol{\omega}^C \qquad (8)$$

$$\mathbf{n}_0 = \cos\theta\,\mathbf{n}_1 + \sin\theta\,\mathbf{n}_2 \qquad (9)$$

picture is good for all time

1, 3 → 4 (10) ${}^Rv^P\mathbf{n}_3 + \{\omega_2 r\,\mathbf{n}_3 - \omega_3 r\,\mathbf{n}_2\} = 0$

1, 3 → 5 (11) ${}^Rv^P\mathbf{n}_3 + \{-\omega_1 r\,\mathbf{n}_3 + \omega_3 r\,\mathbf{n}_1\} = 0$

1, 2, 3, 9 → 6 (12) ${}^Rv^P\mathbf{n}_3 + \{-\omega_2 r\cos\theta + \omega_1 r\sin\theta\}\mathbf{n}_3 + \omega_3(r\cos\theta\,\mathbf{n}_1 - r\sin\theta\,\mathbf{n}_1) = {}^R\omega^C a\,\mathbf{n}_3 \Rightarrow \omega_3 = 0$ (16)

8, 3, 2, 9 → 7 (13) ${}^C\boldsymbol{\omega}^S = (\omega_2 - {}^R\omega^C)\mathbf{n}_2 + \omega_1\mathbf{n}_1$ ${}^C\boldsymbol{\omega}^S\cdot\mathbf{n}_0 = \omega_1\cos\theta + (\omega_2 - {}^R\omega^C)\sin\theta = 0$

solve 10 $\omega_2 = -{}^Rv^P/r$ (15)

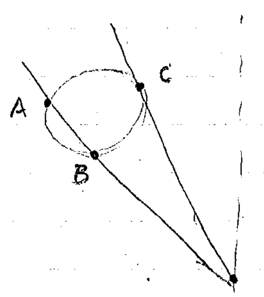
solve (11) $\omega_1 = {}^Rv^P/r$ (14)

${}^R\omega^C a = {}^Rv^P[1 + r\cos\theta + r\sin\theta]$ & ${}^R\omega^C \sin\theta = \frac{{}^Rv^P}{r}[\cos\theta - \sin\theta]$ (17)

Eliminate ${}^R\omega^C$ ⇒ $a = r\sin\theta\,(1 + \sin\theta + \cos\theta)/(\cos\theta - \sin\theta)$ (18)

13, 14, 15, 17

$$b = a + r\cos\theta = \frac{r(1+\sin\theta)}{\cos\theta - \sin\theta} \qquad (19)$$



3e. given $q_f = c(\Omega^2 t^2 - 1)\mathbb{n}_3$ find results $t = \frac{1}{\Omega}$ $\theta_1, \theta_2, \theta_3$ $q\big|_{t=\frac{1}{\Omega}} = 0$

$$ {}^R\!a^Q = {}^B\!a^Q + {}^R\!a^{\bar{B}} + 2\,{}^R\omega^B \times {}^B\mathbb{V}^Q $$

$$ {}^B\mathbb{V}^Q = \frac{{}^B d}{dt} q = c(2\Omega^2 t)\mathbb{n}_3 = 2c\Omega\mathbb{n}_3 \quad @ \quad t = t^* $$

$$ {}^B a^Q = \frac{{}^B d}{dt}{}^B\mathbb{V}^Q = 2c\Omega^2\mathbb{n}_3 \qquad \text{when } t = t^* $$

$$ {}^R\omega^B\Big|_{t=1/\Omega} = \dot\theta_1\mathbb{n}_1 + \dot\theta_2\mathbb{n}_2 + (\dot\theta_3 + \Omega)\mathbb{n}_3 \qquad \text{for previous work} $$

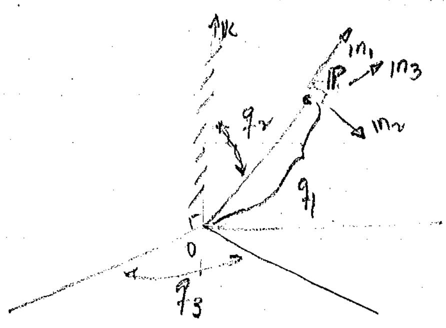
$\bar{B} = B^*$ $\quad {}^R a^{\bar{B}}\Big|_{t=1/\Omega} = -L\Omega^2\mathbb{n}_1$ $\quad B^*$ moves along in a circle whose radius is L

New material if $\mathbb{r} = \mathbb{r}(q_1, \dots, q_n)$ & $q_i = q_i(t)$

$$ \mathbb{V}_{\dot q_r} \cdot a = \frac{1}{2}\left(\frac{d}{dt}\frac{\partial}{\partial \dot q_r}\mathbb{V}^2 - \frac{\partial \mathbb{V}^2}{\partial q_r}\right) \quad (1) \qquad \mathbb{V}^2 = \mathbb{V}\cdot\mathbb{V} = |\mathbb{V}|^2 $$

q's are generalized coords of a point P

we can assume $\mathbb{V} = \mathbb{V}(q_1, \dots, q_n, \dot q_1, \dots, \dot q_n, t)$



Spherical

$$ {}^R\mathbb{p} = q_1\mathbb{n}_1 \qquad (2) $$

$$ {}^R\mathbb{V} = \dot q_1\mathbb{n}_1 + q_1\underbrace{\dot{\mathbb{n}}_1}_{\omega^N \times \mathbb{n}_1} \qquad (3) \qquad \omega^N - \mathbb{n}_1 \text{ fixed in } N $$

$$ \omega^N = \dot q_2\mathbb{n}_3 + \dot q_3\mathbb{k} \qquad (4) $$

$$ \omega^N \times \mathbb{n}_1 \underset{(4)}{=} \dot q_2\mathbb{n}_2 + \dot q_3 s_2\mathbb{n}_3 \quad (5) \qquad s_2 = s\,q_2 $$

$$ \mathbb{V} = \dot q_1\mathbb{n}_1 + q_1\dot q_2\mathbb{n}_2 + q_1\dot q_3 s_2\mathbb{n}_3 \qquad (6) $$

$$ \mathbb{V}_{\dot q_1} = \mathbb{n}_1 \qquad \mathbb{V}_{\dot q_2} = q_1\mathbb{n}_2 \qquad \mathbb{V}_{\dot q_3} = q_1 s_2\mathbb{n}_3 \qquad (7) $$

if $a = a_1\mathbb{n}_1 + a_2\mathbb{n}_2 + a_3\mathbb{n}_3$ then using (7) (8)

$$ \mathbb{V}_{\dot q_1}\cdot a = a_1 \qquad (9) $$

using the rhs of (1) $\mathbb{V}^2 = \dot q_1^2 + q_1^2\dot q_2^2 + q_1^2\dot q_3^2 s_2^2$ (10)

$$ \frac{\partial \mathbb{V}^2}{\partial \dot q_1} \underset{(10)}{=} 2\dot q_1 \qquad (11) $$

$$ \frac{\partial \mathbb{V}^2}{\partial q_1} \underset{(10)}{=} 2q_1\dot q_3^2 s_2^2 + 2q_1\dot q_2^2 \qquad (12) $$

$$ \frac{d}{dt}\left(\frac{\partial \mathbb{V}^2}{\partial \dot q_1}\right) = 2\ddot q_1 \qquad (13) $$

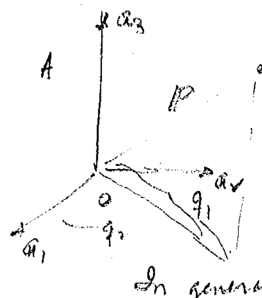
$$ a_1 \underset{(9,1)}{=} \underset{(13)}{\ddot q_1} - q_1\dot q_2^2 - q_1\dot q_3^2 s_2^2 \qquad (14) $$

(over)

$$a_2 = \frac{1}{q_1}\frac{d}{dt}\left(q_1^2 \dot{q}_2\right) - q_1 \dot{q}_3^2 s_2 c_2$$

$$a_3 = \frac{1}{q_1 s_2}\frac{d}{dt}\left(q_1^2 \dot{q}_3 s_2^2\right)$$

orthogonal curvilinear coords.



$$p = p_1 a_1 + p_2 a_2 + p_3 a_3 \qquad (1)$$

If $p_r = F_r(q_1, q_2, q_3) \qquad r = 1,2,3 \qquad (2)$

Then p is a function of q_1, q_2, q_3 in A; and q_1, q_2, q_3 are called curvilinear coordinates of P in A.

In general $\quad {}^A\partial p/\partial q_r = f_r(q_1, q_2, q_3)\, m_r \qquad (3)$

If $[m_1, m_2, m_3] = 1$ or -1 $\forall$ choices of q_1, q_2, q_3, then q_1, q_2, q_3 are called orthogonal. $\quad m_1 \times m_2 \cdot m_3$

Suppose $\quad p = \underbrace{q_1 \cos q_2}_{F_1} a_1 + \underbrace{q_1 \sin q_2}_{F_2} a_2 + \underbrace{q_3}_{F_3} a_3$

$$\frac{{}^A\partial p}{\partial q_1} = \cos q_2\, a_1 + \sin q_2\, a_2 = (1)\, m_1 \qquad F_1(q_1, q_2, q_3) = 1$$

$$\frac{{}^A\partial p}{\partial q_2} = -q_1 \sin q_2\, a_1 + q_1 \cos q_2\, a_2 = q_1(-\sin q_2\, a_1 + \cos q_2\, a_2) = q_1 m_2$$

$$F_2 = q_1$$

$$\frac{{}^A\partial p}{\partial q_3} = a_3 = (1) m_3 \qquad F_3 = 1$$

$$[m_1, m_2, m_3] = \begin{vmatrix} \cos q_2 & \sin q_2 & 0 \\ -\sin q_2 & \cos q_2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

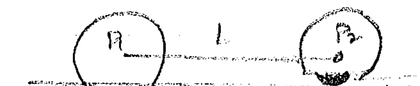
10/30/79

Midterm - Open Book (2 problems)

Today Non holonomic partials
Generalized speeds
Virtual quantities

Non holonomic partials

Systems 2 small spheres



Constraints on motion 1. constant distance between P_1 & P_2 must be maintained
2. contact with plane $(z_1 = z_2 = 0)$
3. velocity of P_2 has no component in the ⊥ direction to line $P_1 P_2$

holonomic constraint eqn.

1 eqn. $(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2=L^2$ from first constraint

2 eqns. $z_1=z_2=0$ contact with plane set origin of system on plane

Number of particles $=2=N$

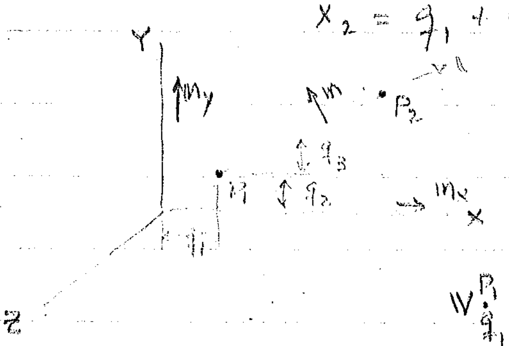
no. of holonomic constraints: $M=3$

no. of generalized coords $n=3N-M=6-3=3$ for holonomic system

Lets introduce q_1, q_2, q_3 (generalized coords) and define

$x_1=q_1 \quad y_1=q_2, \quad z_1=0$

$x_2=q_1+Lc_3, \quad y_2=q_2+LS_3, \quad z_2=0 \qquad c_3=\cos q_3$



lets form partial veloc: $v^{P_i}_{\dot{q}_j} = \dot{x}_i n_{x_i}$

$v^{P_1}=\dot{q}_1 n_x+\dot{q}_2 n_y \qquad \omega^{L}=\dot{q}_3 n_z \quad v^{P_2}=L\ell$

$v^{P_2}=\dot{q}_1 n_x+\dot{q}_2 n_y+L\dot{q}_3 n \quad v^{P_1}+\omega^{L}\times L\ell$

$v^{P_1}_{\dot{q}_1}=n_x \quad v^{P_1}_{\dot{q}_2}=n_y \quad v^{P_1}_{\dot{q}_3}=0$
$v^{P_2}_{\dot{q}_1}=n_x \quad v^{P_2}_{\dot{q}_2}=n_y \quad v^{P_2}_{\dot{q}_3}=Ln$ } holonomic partial velocities

Look at the non holonomic constraint

from last constraint: $v^{P_2}\cdot n=0$ w/ $n=-S_3 n_x+c_3 n_y$. $\ell=c_3 n_x+S_3 n_y$

Thus $\Rightarrow -\dot{q}_1 S_3+\dot{q}_2 c_3+L\dot{q}_3=0$

Number of nonholonomic eqs $=1=m$

Thus the no. of degrees of freedom differs from the no. of generalized coords by the no. of non holonomic eqs $\therefore n-m=(2)$ no. of degrees of freedom

Non holonomic partial velocities can be defined: $\overset{how?}{-}$ we can eliminate $\dot{q}_3$ in v^{P_2}

$\therefore \; \dot{q}_1 S_3/L - \dot{q}_2/L\, c_3=\dot{q}_3 \xRightarrow{\text{\& put into}} \tilde{v}^{P_1}=\dot{q}_1 n_x+\dot{q}_2 n_y$

$\tilde{v}^{P_2}=\dot{q}_1 n_x+\dot{q}_2 n_y+(\dot{q}_1 S_3-\dot{q}_2 c_3)n$

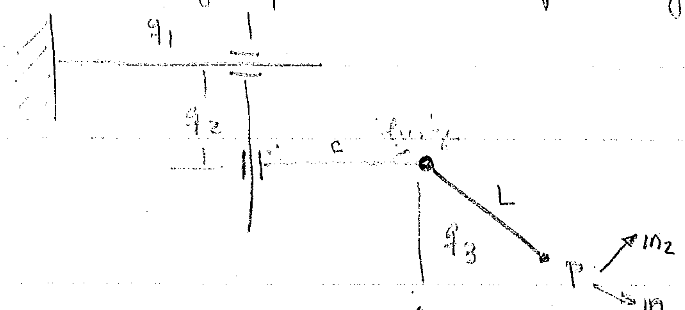
$\tilde{v}^{P_2}_{\dot{q}_1}=n_x+S_3 n \qquad \tilde{v}^{P_2}_{\dot{q}_2}=n_y-c_3 n \qquad \tilde{v}^{P_2}_{\dot{q}_3}=0$ non holonomic partial velocities

this is a nonunique substitution: we can eliminate $\dot{q}_1$ or $\dot{q}_2$

For simple nonholonomic systems the $\dot{q}_i$'s appear linearly in the constraint eqn

Generalized Speeds

Given a rod linkage system in the plane of the paper



$$\mathbf{v}^P = (\dot{q}_1 s_3 + \dot{q}_2 c_3)\,\mathbf{n}_1 + (\dot{q}_1 c_3 - \dot{q}_2 s_3 + L\dot{q}_3)\,\mathbf{n}_2$$

not correct

$$\mathbf{a}^P = (\ddot{q}_1 s_3 + \ddot{q}_2 c_3 - 2\dot{q}_2\dot{q}_3 s_3 - L\dot{q}_3^{\,2})\,\mathbf{n}_1 + (\ddot{q}_1 c_3 - \ddot{q}_2 s_3 + L\ddot{q}_3)\,\mathbf{n}_2$$

$$\mathbf{v}^P_{\dot{q}_1} = s_3\mathbf{n}_1 + c_3\mathbf{n}_2 \qquad \mathbf{v}^P_{\dot{q}_2} = c_3\mathbf{n}_1 - s_3\mathbf{n}_2 \qquad \mathbf{v}^P_{\dot{q}_3} = L\mathbf{n}_2 \qquad \mathbf{v}^P = \sum \mathbf{v}^P_{\dot{q}_i}\dot{q}_i$$

now let
$$\dot{q}_1 = u_1 s_3 + (u_2 - Lu_3)c_3$$
$$\dot{q}_2 = u_1 c_3 - (u_2 - Lu_3)s_3$$
$$\dot{q}_3 = u_3$$

the u_i's are new to the discussion & are <u>generalized speeds</u> can be real speeds or angular speed

Then $\mathbf{v}^P = u_1\mathbf{n}_1 + u_2\mathbf{n}_2 \qquad \mathbf{a}^P = (\dot{u}_1 - u_2u_3)\mathbf{n}_1 + (\dot{u}_2 + u_3u_1)\mathbf{n}_2$

Thus: $\tilde{\mathbf{v}}^P_{u_1} = \mathbf{n}_1 \qquad \tilde{\mathbf{v}}^P_{u_2} = \mathbf{n}_2 \qquad \tilde{\mathbf{v}}^P_{u_3} = 0 \qquad \therefore\ \mathbf{v}^P = \sum \tilde{\mathbf{v}}^P_{u_r}u_r$

to get ~~the~~ from $\mathbf{v}^P$ to the u_i's

$$\left.\begin{aligned} u_1 &= \dot{q}_1 s_3 + \dot{q}_2 c_3 \\ u_2 &= \dot{q}_1 c_3 - \dot{q}_2 s_3 + L\dot{q}_3 \\ u_3 &= \dot{q}_3 \end{aligned}\right\} \Rightarrow \mathbf{v}^P = u_1\mathbf{n}_1 + u_2\mathbf{n}_2$$

These generalized speeds simplify the eqs. Are the u_i's derivatives

if $u_j = f_j(q_i, t)$ exist then $\Rightarrow u_1 = \dfrac{df_1}{dt} = \dfrac{\partial f_1}{\partial q_1}\dot{q}_1 + \dfrac{\partial f_1}{\partial q_2}\dot{q}_2 + \dfrac{\partial f_1}{\partial q_3}\dot{q}_3 \quad \forall q_i$'s

$$\frac{\partial f}{\partial q_1} = s_3 \Rightarrow \frac{\partial^2 f}{\partial q_3 \partial q_1} = c_3 \qquad \text{"}0\text{"} \quad \frac{\partial^2 f_1}{\partial q_1 \partial q_3} = 0$$

in general the u_i are <u>not</u> derivatives of fns of time

Virtual velocity, angular velocity, displ, rotation

Definition

$$\hat{\mathbb{V}}^{P} \triangleq \sum_{r=1}^{n} \mathbb{V}_{\dot q_r} \hat{\dot q}_r \qquad \text{for non holon syst. } \sum_{r=1}^{n-m}$$

a virtual velocity of P

where $\hat{\dot q}_r$ is an arbitrary quantity restricted to having dimensions of $\dot q_r$

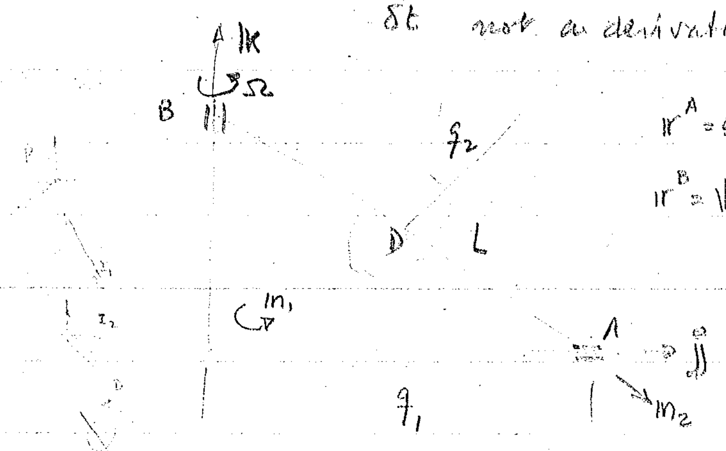
a virtual ang. velocity of P $\hat{\omega}^{P} = \sum_{r=1}^{n} \omega^{P}_{\dot q_r} \hat{\dot q}_r$

A virtual displacement of point P $\quad \delta p^{P} = \sum_{r=1}^{n} \mathbb{V}_{\dot q_r} \delta q_r \quad$ δq is an arbitrary quantity having dimension of q_r

A virtual rotation of point P $\quad \delta \alpha^{P} = \sum_{r=1}^{n} \omega_{\dot q_r} \delta q_r$

Are virtual velocities time deriv. of virtual displacements ie $\hat{\mathbb{V}}^{P} = \frac{d}{dt}(\delta p)$ NO!

But: $\hat{\mathbb{V}} = \frac{\delta p}{\delta t}$ where δt is a consequence of the dimension not a derivative



$q_1 \omega^{R} \times \mathbb{j} \qquad \Omega \mathbb{k} \times \mathbb{j} = -\Omega \mathbb{m}_1$

$$\mathbb{r}^{A} = q_1 \mathbb{j} \qquad \mathbb{V}^{A} = \dot q_1 \mathbb{j} - \Omega q_1 \mathbb{m}_1$$

$$\mathbb{r}^{B} = \sqrt{L^2 - q_1^2}\, \mathbb{k} \Rightarrow \mathbb{V}^{B} = \frac{-q_1 \dot q_1}{(L^2 - q_1^2)^{1/2}} \mathbb{k} \qquad \frac{d\mathbb{k}}{dt} = 0$$

$$\omega^{D} = \frac{\dot q_1}{(L^2 - q_1^2)^{1/2}} \mathbb{m}_1 + \dot q_1 \mathbb{m}_2 + \Omega \mathbb{k}$$

what is the virtual velocity of A $\quad \hat{\mathbb{V}}^{A} = \mathbb{V}^{A}_{\dot q_1} \hat{\dot q}_1 + \mathbb{V}^{A}_{\dot q_2} \hat{\dot q}_2 = \mathbb{j}\, \hat{\dot q}_1 + 0$

it's in the direction of $\hat{\mathbb{V}}^{A} = \mathbb{j}$

$$\hat{\mathbb{V}}^{B} = \frac{-q_1}{(L^2 - q_1^2)^{1/2}} \mathbb{k}\, \hat{\dot q}_1$$

These defns are good with defn of mutual compatability

$2\mathbb{j}$ ft/sec is a virtual velocity of A

$-\frac{3 q_1 \mathbb{k}}{(L^2 - q_1^2)^{1/2}}$ ft/sec is a virtual velocity of B

These two virtual velocities are non compatible with each other, but
$2\,\mathbf{j}$ and $-\dfrac{2\,q_1\,\mathbf{k}}{(L^2 - q_1^2)^{1/2}}$ are compatible with each other.

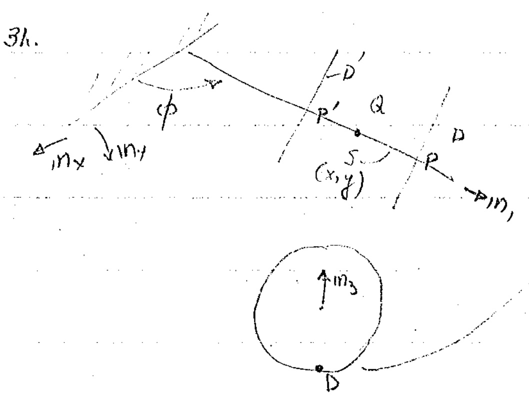
Midterm will not contain virtual items. All items to generalized speeds are going to be on exam.

11/6/79

1. Discussion of Exams
2. Virtual Disp/Velocity
3. HW problems

Take-Home Exam: Single Problem

31.



$${}^R\mathbf{v}^Q = \dot{x}\,\mathbf{n}_x + \dot{y}\,\mathbf{n}_y$$

Rolling of D: $\mathbf{v}^P$

$$\underbrace{{}^R\mathbf{v}^Q}_{\text{velocity of center}} + \omega^S \times (L\,\mathbf{n}_1) + \omega^D \times (-r\,\mathbf{n}_3) = 0$$

$$\omega^S = \dot{\phi}\,\mathbf{n}_3\,, \quad \omega^D = \dot{\phi}\,\mathbf{n}_3 + \dot{\theta}\,\mathbf{n}_1$$

$$\mathbf{v}^D = \dot{x}\,\mathbf{n}_x + \dot{y}\,\mathbf{n}_y + L\dot{\phi}\,\mathbf{n}_2 + r\dot{\theta}\,\mathbf{n}_2 = 0$$

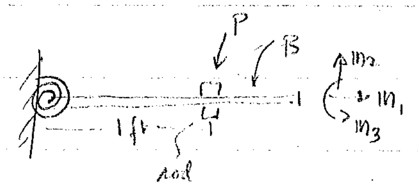
$$\mathbf{v}^{D'} = \dot{x}\,\mathbf{n}_x + \dot{y}\,\mathbf{n}_y - L\dot{\phi}\,\mathbf{n}_2 + r\dot{\theta}'\,\mathbf{n}_2 = 0$$

$$\mathbf{n}_2 = -\sin\phi\,\mathbf{n}_x + \cos\phi\,\mathbf{n}_y$$

from vector eqn

$$\left\{\begin{array}{ll} \dot{x} + (L\dot{\phi} + r\dot{\theta})(-\sin\phi) = 0 & \\ \dot{y} + (L\dot{\phi} + r\dot{\theta})\cos\phi = 0 & \dot{z} = 0 \\ \dot{x} + (-L\dot{\phi} + r\dot{\theta}')(-\sin\phi) = 0 & \\ \dot{y} + (-L\dot{\phi} + r\dot{\theta}')\cos\phi = 0 & \dot{z} = 0 \end{array}\right.$$

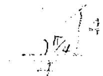
Virtual:



1: Suppose P experiences an actual displacement

$$\Delta p = 3n_1 + 4n_2 \text{ ft.}$$

find the associated actual rotation: $\Delta\alpha$. P will be at 4,4



$$\Delta\alpha = \frac{\pi}{4} n_3 \text{ radians}$$

2. Suppose P experiences a virtual disp δp

$$\delta p = 3n_1 + 4n_2$$

find the virtual rotation of B that is compatible with δp:

$$\delta p = 4n_3 \text{ radians}$$

3. Suppose P has an actual veloc. v

$$v = 3n_1 + 4n_2 \text{ ft/sec}$$

find the assoc. angular velocity of B $\quad \omega \times r = v \quad r = n_1 \quad (an_1 + bn_2 + cn_3) \times r = -bn_3 + cn_2 \quad c = 4 \quad b = -3$

$$\omega = 4n_3 \text{ rad./sec.}$$

4. P has a virtual veloc. $\hat{v}$

$$\hat{v} = 3n_1 + 4n_2 \text{ ft/sec}$$

Find the virtual angular velocity $\hat{\omega}^B$ that is compatible with $\hat{v}$:

$$\hat{\omega}^B = 4n_3 \text{ rad/sec.}$$

Topics to be covered

11/8/79

1. Bound Vectors, Free Vectors
2. Moment, Resultant, Equivalence
3. Couple, Torque

Start from scratch

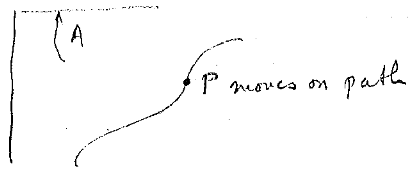
Bound Vector: vector that's assoc. with a point

Examples:

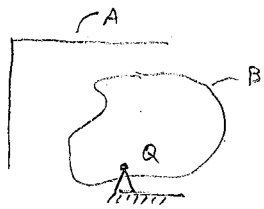
Free Vector: Vector not specifically tied to any point

Examples:

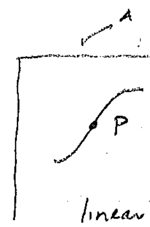
Given:



${}^{A}\mathbb{V}^{P}$ = free vector

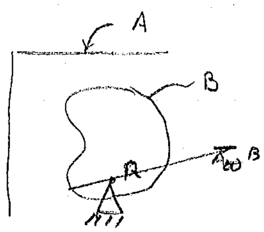


${}^{A}\omega^{B}$ - free vector

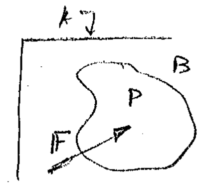


${}^{A}\mathbb{L}^{P} = m\,{}^{A}\mathbb{V}^{P}$ bound vector applied at P

linear momentum = $\mathbb{L}$



${}^{A}\omega^{B}$ - a bound vector applied at Q



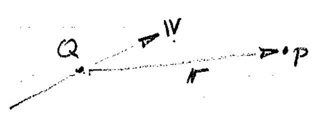
$\mathbb{F}$(applied) - bound vector applied at P

whether a vector is bound or free depends on the analyst and the context of the problem

Criteria to decide which vectors are bound or free

1. Moment ($\mathbb{M}$) of a bound vector ($\mathbb{V}$ applied at Q) about a point (P):

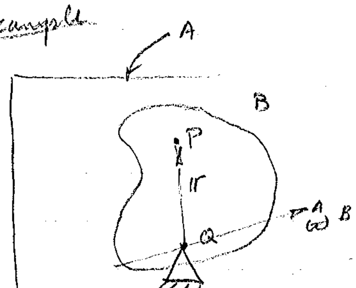
Given a vector $\mathbb{V}$ assoc. with pt. Q



$$\boxed{\mathbb{M} \triangleq \mathbb{V} \times \mathbb{r}}$$

Quest: does a free vector have a moment about a point? no! but they can be bound at any time

Example



Regard ${}^{A}\omega^{B}$ as applied at Q

Form $\mathbb{M}^{P}$, the moment about P of ${}^{A}\omega^{B}$

$$\mathbb{M}^{P} = {}^{A}\omega^{B} \times \mathbb{r}$$

since $\mathbb{r}$ is fixed in B: ${}^{A}\mathbb{V}^{P} = {}^{A}\omega^{B} \times \mathbb{r}$ is a moment of angular velocity

Question: are moments bound vectors? depends on the situation - if we want to take $\mathbb{M} \times \mathbb{r}$ we must tie the moment to a point. If not it is free

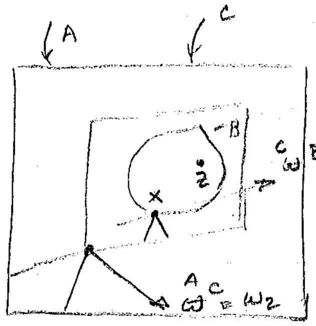
Sets of vectors : Resultant ($\mathbb{R}$) of a set of bound &/or free vectors ($\mathbb{V}_1, \mathbb{V}_2, \ldots, \mathbb{V}_n$)

$$\mathbb{R} \triangleq \mathbb{V}_1 + \mathbb{V}_2 + \ldots + \mathbb{V}_n$$

what is the physical significance of $\mathbb{R}$ - depends on context

Does $\mathbb{R}$ have a point of application? no! unless you tie it down ie (summation doesn't depend on localization to a pt.) D.O.T.S.

Given:



$\mathbb{R} = \omega_1 + \omega_2$
$= {}^{C}\omega^{B} + {}^{A}\omega^{C} = {}^{A}\omega^{B}$

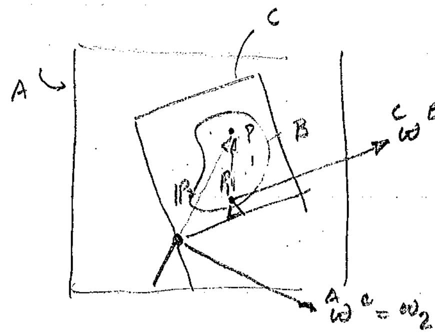
does it have an point of application : no
" " " a moment about a given point: no since $\mathbb{R}$ hasn't been given a point to be tied down to yet.
If I tie it down to z then I can find its moment about x.

Moment ($\mathbb{M}$) of a set of bound vectors ($\mathbb{V}_1, \dots, \mathbb{V}_n$) about a point ($P$):

Let $\boxed{\mathbb{M} = \sum_{i=1} \omega_i \times \mathbb{r}_i = \sum \mathbb{M}_i}$

resultant of the moments of indiv moments about point P

Example:



$\mathbb{M}^P = \omega_1 \times \mathbb{P}_1 + \omega_2 \times \mathbb{P}_2$

significantly

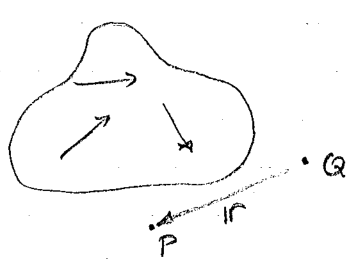
${}^{A}\mathbb{V}^{P} = {}^{C}\mathbb{V}^{P} + {}^{A}\mathbb{V}^{\bar{c}}$ $\quad$ $\bar{c}$ is in C

${}^{C}\mathbb{V}^{P} = {}^{C}\omega^{B} \times \mathbb{P}_1$

${}^{A}\mathbb{V}^{\bar{c}} = {}^{A}\omega^{C} \times \mathbb{P}_2$

$\Rightarrow {}^{A}\mathbb{V}^{P} = \omega_1 \times \mathbb{P}_1 + \omega_2 \times \mathbb{P}_2 = \mathbb{M}^P$

Relationship between the moments ($\mathbb{M}^P$ & $\mathbb{M}^Q$) of a set of vectors about two points (P and Q)



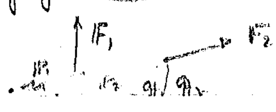
$\mathbb{M}^P = \sum_i \mathbb{V}_i \times \mathbb{r}_i^P$

$\mathbb{r}_i^P = \mathbb{r}_i^Q + \mathbb{r}$

$\mathbb{M}^Q = \sum_i \mathbb{V}_i \times \mathbb{r}_i^Q$

$\boxed{\mathbb{M}^P = \mathbb{M}^Q + \mathbb{R} \times \mathbb{r}^{Q/P}}$ where $\mathbb{R} = \sum_i \mathbb{V}_i$

Proof for two bound vectors $\mathbb{F}_1$ & $\mathbb{F}_2$



$\mathbb{M}^P = \mathbb{F}_1 \times \mathbb{P}_1 + \mathbb{F}_2 \times \mathbb{P}_2$

$\mathbb{M}^Q = \mathbb{F}_1 \times \mathbb{q}_1 + \mathbb{F}_2 \times \mathbb{q}_2$

$\mathbb{R} \triangleq \mathbb{F}_1 + \mathbb{F}_2$ Let $\mathbb{P}_1 = \mathbb{q}_1 + \mathbb{r}$ $\quad \mathbb{P}_2 = \mathbb{q}_2 + \mathbb{r}$

$$\mathbb{M}^P = \mathbb{F}_1 \times (\mathbb{q}_1 + \mathbb{r}) + \mathbb{F}_2 \times (\mathbb{q}_2 + \mathbb{r}) = \mathbb{F}_1 \times \mathbb{q}_1 + \mathbb{F}_2\, \mathbb{q}_2 + (\mathbb{F}_1 + \mathbb{F}_2) \times \mathbb{r}$$
$$= \mathbb{M}^Q + \mathbb{R} \times \mathbb{r} \qquad \text{QED}$$

Application:

if P & Q are points fixed on a rigid body B moving in a ref. frame A one can always introduce an intermediate ref frame C and angular velocities ${}^A\omega^C$, ${}^C\omega^B$ in such a way that

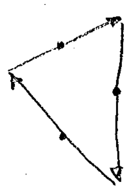
$${}^A\omega^B = \mathbb{R} \quad (\text{resultant of } {}^A\omega^C, {}^C\omega^B)$$
$${}^A\mathbb{v}^P = \mathbb{M}^P \; (\text{sum of moments of } {}^A\omega^C \;\&\; {}^C\omega^B \text{ about } P.)$$
$${}^A\mathbb{v}^Q = \mathbb{M}^Q \; (\text{" " " " } {}^A\omega^C \;\&\; {}^C\omega^B \text{ " } Q.)$$

$\Rightarrow$

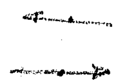
$${}^A\mathbb{v}^P = {}^A\mathbb{v}^Q + {}^A\omega^B \times \mathbb{r} \quad \text{since } \mathbb{M}^P = \mathbb{M}^Q + \mathbb{R} \times \mathbb{r}$$

Couple:

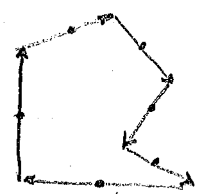
it is A set of bound vectors whose resultant is equal to zero.



Couple



Conventional use of the word: "Couple"



couple

is A couple a vector? no it is a set.

Torque of a couple: the moment of couple about any point.

$$\mathbb{M}^P = \mathbb{M}^Q + \mathbb{R} \times \mathbb{r} \qquad \mathbb{R} = 0 \Rightarrow \mathbb{M}^P = \mathbb{M}^Q$$

11/13/79

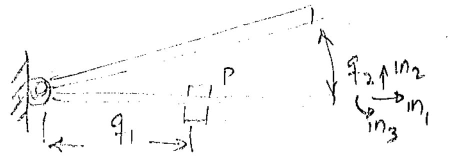
1. Odds & Ends
2. Midterms

Midterm

	${}^A\omega_{u_i}$	${}^B\omega_{u_i}$	${}^W\omega_{u_i}$
$i=1$	0	0	a_2/r
$i=2$	a_3/a	a_3/b	$-c/a \, \frac{a_2}{r} + \frac{a_3}{a}$

In general angular velocity of wheel does not only have component in direction of shaft.

See pg 26 notes of 11/6/79



define generalized coordinates q_1, q_2

$\delta p = 3n_1 + 4n_2$

$\delta\alpha = ?$

$v^P = \dot{q}_1 n_1 + \dot{q}_2 q_1 n_2$

$\omega^B = \dot{q}_2 n_3$

$v^P_{\dot{q}_1} = n_1 \quad v^P_{\dot{q}_2} = q_1 n_2 \quad \omega^B_{\dot{q}_1} = 0 \quad \omega^B_{\dot{q}_2} = n_3$

$\delta p = n_1\,\delta q_1 + q_1 n_1\,\delta q_2$ $\quad \delta q_{1,2}$ are totally arbitrary ∴ for us $\delta q_1 = 3 \quad \underset{1}{q_1}\,\underset{4}{\delta q_2} = 4$ from geom.

$\delta\alpha = \qquad \delta q_2\, n_3$

∴ $\delta\alpha = 4n_3$

Item 5. P experiences an actual displacement $\Delta p = \epsilon_1 n_1 + \epsilon_2 n_2$
Find the actual rotation with is actual displ

$$\Delta\alpha = \tan^{-1}\frac{\epsilon_2}{1+\epsilon_1}\; n_3$$

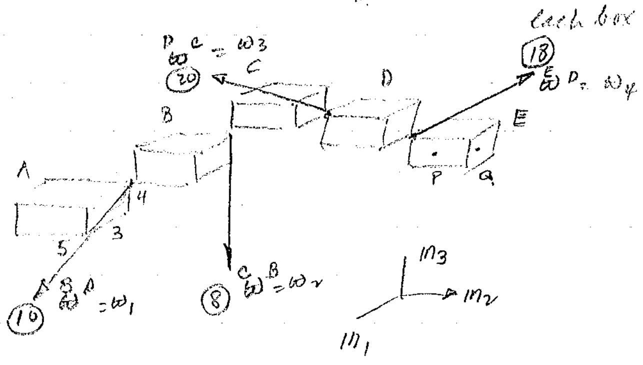
In some sense $\Delta\alpha \to \epsilon_2 n_2$ when $\epsilon_1, \epsilon_2 \to 0$

— P experiences a virtual displacement $\delta p = \epsilon_1 n_1 + \epsilon_2 n_2$
now find the virtual rot $\delta\alpha$ associated with δp.

Using the result above (replace ϵ_1 by 3 & ϵ_2 by 4). $\Rightarrow \delta\alpha = \epsilon_2 n_3$

Back to Moments & Couples

each box same size



all vectors are bound

$R = \sum \omega_i = {}^E\omega^A$

$= +6n_1 - 8n_3 + 8n_3 + 12n_1 + 16n_3 - 18n_1 = 0$

Torque of a couple (T): If P & Q are any 2 pts, then moments $M|^P$ and $M|^Q$ of any couple are equal

$$T \triangleq M|^P = M|^Q$$

all torques are moments, all moments are not torques

Kinematic Interp:

If P & Q are any 2 pts of A, their velocities in E are equal

${}^E\omega^A = 0 \quad ∴ \quad v^P = v^Q$

By $\quad {}^A v^P = {}^A v^Q + \underbrace{{}^A\omega^E \times r}_{=0 \text{ since } {}^A\omega^E = 0}$

ie E translates only in reference frame only

when velocity of points in body have same velocity we can talk of the velocity of body.

$^{E}\mathbb{V}^{A}$ is the torque of the couple formed by $\omega_1, \dots, \omega_4$

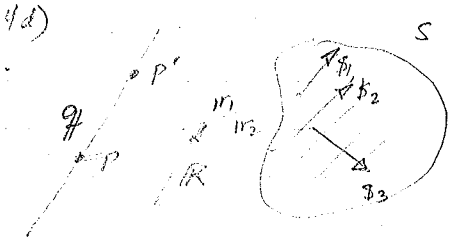
— To find $^{E}\mathbb{V}^{A}$, evaluate the sum of the moments of $\omega_1, \dots, \omega_4$ about any point

Two sets of bound vectors are equiv if they have same torque & same resultant about some point.

Prob. 4(a) Form tabulation

	m_1	m_2	m_3
$\mathbb{F}$	0	-30	0
$\mathbb{P}^{A/F}$	-10	0	9
$\mathbb{G}$	32.77	16.22	-16.28
$\mathbb{M}^{F/A}$	-270	0	-300.

Prob. (4d)

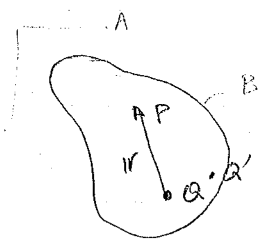


$$\mathbb{M}^{P} = +\sum \mathbb{r}_i \times \mathbb{s}_i$$
$$\mathbb{M}^{P'} = +\sum \mathbb{r}_i' \times \mathbb{s}_i$$
$$\mathbb{r}_i = \mathbb{r}_i' + \mathbb{q}$$
$$\mathbb{M}^{P} = \mathbb{M}^{P'} + \sum \mathbb{q} \times \mathbb{s}_i = \mathbb{M}^{P'} + \mathbb{q} \times \mathbb{R}$$

but $\mathbb{q}$ is $\parallel \mathbb{R}$ $\therefore$ $\mathbb{q} \times \mathbb{R} = 0$

What is kinematic interpretation:

(4e)



$$\mathbb{V}^{P} = \mathbb{V}^{Q} + {}^{A}\omega^{B} \times \mathbb{r}$$

if we find a point Q where $|\mathbb{V}^{P}|$ is smaller than at any other point, $\forall P$ then, we can find a line of points where where the velocity of points off that line are the smallest.

Suppose $\mathbb{a} = \mathbb{b} + \mathbb{c} \times \mathbb{r}$; given $\mathbb{b}$ and $\mathbb{c}$ find $\mathbb{r}$ $\ni$. mag $\mathbb{a}$ is min.
find $\mathbb{a}^*$, the $\mathbb{a}$ with min magnitude

Let $\mathbb{r} = \beta \mathbb{b} + \gamma \mathbb{c} + \delta\, \mathbb{b} \times \mathbb{c}$ where $\mathbb{b}$, $\mathbb{c}$, $\mathbb{b} \times \mathbb{c}$ are a non orthog, non unit

$$\mathbb{a} = \mathbb{b} + \beta\, \mathbb{c} \times \mathbb{b} + \delta\, \mathbb{c} \times (\mathbb{b} \times \mathbb{c})$$
$$\mathbb{a}^2 = \mathbb{b}^2 + \beta^2 (\mathbb{c} \times \mathbb{b})^2 + \delta^2 [\mathbb{c} \times (\mathbb{b} \times \mathbb{c})]^2 + 2\delta\, \mathbb{b} \cdot \mathbb{c} \times (\mathbb{b} \times \mathbb{c})$$

take $\dfrac{\partial \mathbb{a}^2}{\partial \beta} = 2\beta (\mathbb{c} \times \mathbb{b})^2 = 0 \implies \beta = 0$

note no γ dependences

$$\frac{\partial a^2}{\partial \delta} = 2\delta\left[\mathbf{c}\times(\mathbf{b}\times\mathbf{c})\right]^2 + 2\,\mathbf{b}\cdot\mathbf{c}\times(\mathbf{b}\times\mathbf{c}) = 0$$

$$\delta = \frac{-\,\mathbf{b}\cdot\mathbf{c}\times(\mathbf{b}\times\mathbf{c})}{\left[\mathbf{c}\times(\mathbf{b}\times\mathbf{c})\right]^2}$$

γ is unrestricted

$$\mathbf{r}^* = \gamma\mathbf{c} + \frac{-\,\mathbf{b}\cdot\mathbf{c}\times(\mathbf{b}\times\mathbf{c})}{\left[\mathbf{c}\times(\mathbf{b}\times\mathbf{c})\right]^2}(\mathbf{b}\times\mathbf{c}) = \gamma\mathbf{c} + \frac{(\mathbf{c}\times\mathbf{b})}{c^2}$$

$$a\Big|_{\mathbf{r}=\mathbf{r}^*} = \mathbf{b} + \frac{\mathbf{c}\times(\mathbf{c}\times\mathbf{b})}{c^2} = \frac{\mathbf{c}\cdot\mathbf{b}}{c^2}\,\mathbf{c}$$

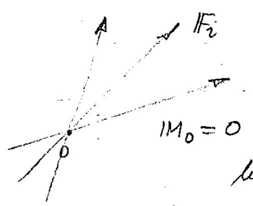
Problems 11/15/79

$$\mathbf{b}\cdot\mathbf{c}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{b}\times\mathbf{c})^2$$

$$\left[\mathbf{c}\times(\mathbf{b}\times\mathbf{c})\right]^2 = \mathbf{c}\cdot(\mathbf{b}\times\mathbf{c})\times\left[\mathbf{c}\times(\mathbf{b}\times\mathbf{c})\right] = \mathbf{c}\cdot\left\{(\mathbf{b}\times\mathbf{c})^2\mathbf{c} - \cancel{(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}}^{\,0}\,(\mathbf{b}\times\mathbf{c})\right\} = (\mathbf{b}\times\mathbf{c})^2 c^2$$

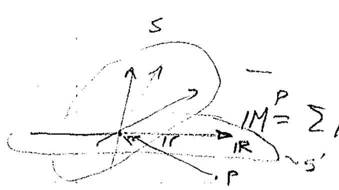
$$\therefore\ \mathbf{r}^* = \gamma\mathbf{c} + \frac{\mathbf{c}\times\mathbf{b}}{c^2}$$

4h.



$\mathbf{M}_0 = 0 \Rightarrow \mathbf{M}^* = 0$ & S is passing through O.

by 4f. we can replace a set of vectors w/ resultant by a wrench consisting of couple whose $\pi = \mathbf{M}^* = 0$ and $\mathbf{R}$ passing through O.

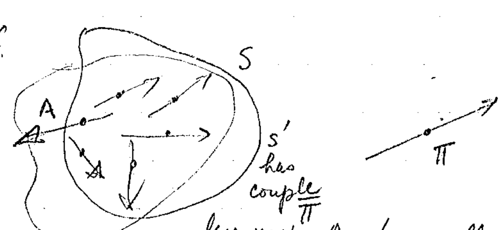


$\mathbf{M}^p = \sum \mathbf{M}_i = \mathbf{r}\times\mathbf{R}$ for S

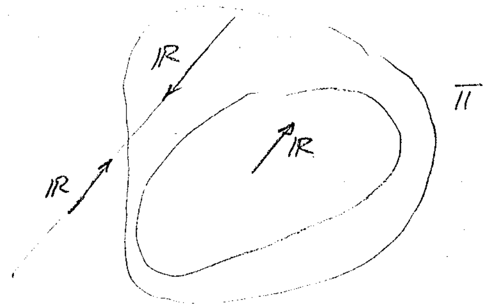
$\mathbf{M}'^p = \mathbf{r}\times\mathbf{R}$ for S'

but both sets same $\mathbf{R}$ resultant & moment $\Rightarrow$ equivalent & can replace one by the other

4f.



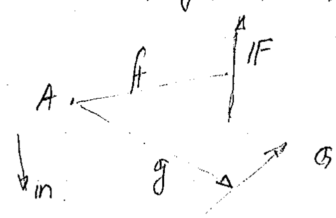
less vector A (normally A is not parallel) if it is $\parallel$ to π then we have a wrench



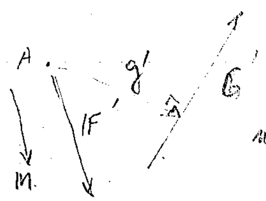
4i. Requires equivalence; equal $\mathbf{R}$ & equal $\mathbf{M}$ about 1 point

4k

Given A, $\mathbf{f}$, $\mathbf{g}$, $\mathbf{F}$, $\mathbf{G}$, $\mathbf{m}$ in some proble.



Find $\mathbf{F}'$ (a force applied at A & $\parallel$ to $\mathbf{m}$) and $\mathbf{G}'$, $\mathbf{g}'$ such that $(\mathbf{F},\mathbf{G}) \underset{equiv}{=} (\mathbf{F}',\mathbf{G}')$



now let

$$\begin{cases} \mathbf{g}'\cdot\mathbf{G}' = 0 & (1) \\ \mathbf{F}' \parallel \mathbf{m} \Rightarrow \mathbf{F}' = \mathbf{F}'\cdot\mathbf{m}\,\mathbf{m} & (2) \\ \text{Equivalence:} & \\ \mathbf{F}'+\mathbf{G}' = \mathbf{F}+\mathbf{G} & (3) \end{cases}$$

and $\mathbf{g}'\times\mathbf{G}' = \mathbf{f}\times\mathbf{F} + \mathbf{g}\times\mathbf{G}$ (4)

moments about A.

Now we must ~~[illegible]~~ solve (1) $\to$ (4) to find $\mathbf{F}'$, $\mathbf{G}'$, $\mathbf{g}'$

Introduce F', $\mathbf{R}$, $\mathbf{M}$:

Let $F' \triangleq \mathbf{F}'\cdot\mathbf{m}$ (5) $\Rightarrow$ (2) : $\mathbf{F}' = F'\mathbf{m}$ (8)

$\mathbf{R} \triangleq \mathbf{F}+\mathbf{G}$ (6) (3) : $F'\mathbf{m} + \mathbf{G}' = \mathbf{R}$ (9)

$\mathbf{M} = \mathbf{f}\times\mathbf{F} + \mathbf{g}\times\mathbf{G}$ (7) (4) : $\mathbf{g}'\times\mathbf{G}' = \mathbf{M}$ (10)

now $\mathbf{G}' = \mathbf{R} - F'\mathbf{m}$ (11)

$\mathbf{g}'\times\mathbf{G}' = \mathbf{g}'\times\mathbf{R} - F'\mathbf{g}'\times\mathbf{m}$ (12)

but $\mathbf{M} = \mathbf{g}'\times\mathbf{G}' = \mathbf{g}'\times\mathbf{R} - F'\mathbf{g}'\times\mathbf{m}$ (13)

$\mathbf{M}\cdot\mathbf{R} = 0 - F'(\mathbf{g}'\times\mathbf{m})\cdot\mathbf{R}$ (14)

take $\mathbf{M}\cdot\mathbf{m} = (\mathbf{g}'\times\mathbf{R}\cdot\mathbf{m}) - \underline{F'}\,\mathbf{g}'\times\mathbf{m}\cdot\mathbf{m}^{\,0}$

$$\therefore \quad F' = \frac{\mathbf{M}\cdot\mathbf{R}}{\mathbf{g}'\times\mathbf{R}\cdot\mathbf{m}} = \frac{\mathbf{M}\cdot\mathbf{R}}{\mathbf{M}\cdot\mathbf{m}} \quad (16)$$

$(\mathbf{g}'\times\mathbf{G}')\times\mathbf{G}' = \mathbf{M}\times\mathbf{G}'$ (17)

$$\underbrace{(\mathbf{g}'\cdot\mathbf{G}')}_{0\ (1)}\mathbf{G}' - \mathbf{g}'(\mathbf{G}')^2 \underset{17}{=} \mathbf{M}\times\mathbf{G}' \quad (18) \quad \Rightarrow \quad \mathbf{g}' = \frac{\mathbf{G}'\times\mathbf{M}}{(\mathbf{G}')^2} \quad (19)$$

algorith $\mathbf{R} \triangleq \mathbf{F}+\mathbf{G}$ (6) (7) $= \mathbf{M} = \mathbf{f}\times\mathbf{F} + \mathbf{g}\times\mathbf{G}$ (16) $F' = \dfrac{\mathbf{M}\cdot\mathbf{R}}{\mathbf{M}\cdot\mathbf{m}}$

(11) $\mathbb{G}' = \mathbb{R} - F'\mathbb{m}$ but $\mathbb{F}' = F'\mathbb{m}$ (8)

(19) $g' = \dfrac{\mathbb{G}' \times \mathbb{M}}{(\mathbb{G}')^2}$

Generalized Forces

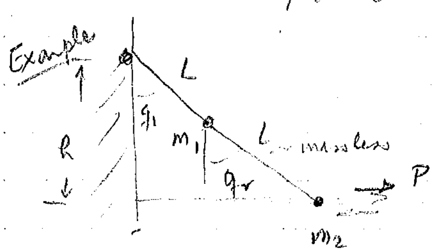
What are they?

- given a set of particles P_i $(i, \ldots, N)$
- Forces acting on each particle $\mathbb{F}_i$ when $\mathbb{F}_i$ is resultant of all contact and body forces acting on P_i. Body forces exerted on body without contact (at a distance) – gravity, electromagnetic etc.
- give Generalized coords $q_1, \ldots, q_n$
- For a non holonomic system $\dot{q}_1, \ldots, \dot{q}_{n-m}$ are independent.

then $\displaystyle\sum_{i=1}^{N} \mathbb{F}_i \cdot \tilde{\mathbb{V}}_{\dot{q}_r}^{P_i} = F_r$ generalized force. $r = 1, \ldots, n-m$

what are they good for?

Example



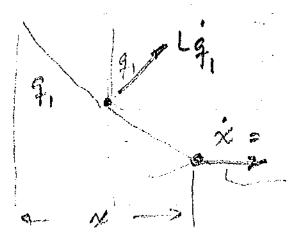
if problem is in equil what are q_1, q_2 if we know P

1. $L(c_1 + c_2) = h \Rightarrow c_2 = \frac{h}{L} - c_1$

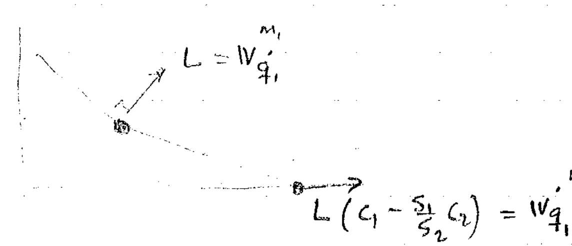
$s_2 = \left[1 - \left(\frac{h}{L} - c_1\right)^2\right]^{1/2}$

also $\frac{d}{dt}(1)$ $-(s_1 \dot{q}_1 + s_2 \dot{q}_2) = 0 \Rightarrow \dot{q}_2 = -\frac{s_1}{s_2}\dot{q}_1$

to get velocities



to get partials



$x = L(s_1 + s_2)$

$\dot{x} = L(c_1 \dot{q}_1 + c_2 \dot{q}_2) = L\left(c_1 - \frac{s_1}{s_2}c_2\right)\dot{q}_1$

Generalized forces F_1:

Body forces $m_1 g$, $m_2 g$ $F_1 = -m_1 g L s_1 + 0 + PL\left(c_1 - \frac{s_1}{s_2}c_2\right)$

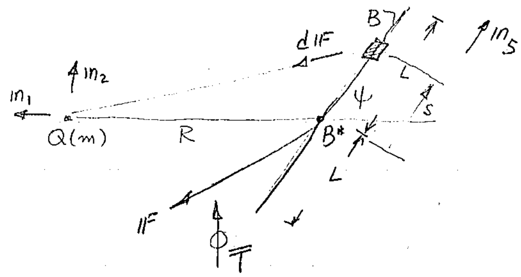
contact forces contribute nothing Generalized active force

Statics $F_r = 0$ $r = 1, \ldots, m-n$

$-m_1 g L s_1 + PL\left(c_1 - \frac{s_1}{s_2}c_2\right) = 0 \Rightarrow P = \dfrac{m_1 g s_1}{c_1 - s_1/s_2\, c_2}$

11/20/79

1. Problem 4(l)
2. Generalized Forces
   a. Active
   b. Inertia



... pass through B^* & must be sum of little forces

$$d\mathbf{F} = \int_{-L}^{L} \frac{GmM}{2L} \frac{(-s\mathbf{n}_s + R\mathbf{n}_1)\,ds}{[s^2+R^2+2sR\cos\psi]^{3/2}}$$

$\rho =$ mass density $\qquad \dfrac{\mathbf{r}\,ds}{r^2\,|\mathbf{r}|}$

$\mathbf{r} = -s\mathbf{n}_s + R\mathbf{n}_1$

$$I = \int_{-L}^{L} \frac{ds}{(s^2+R^2+2sR\cos\psi)^{3/2}} \qquad J = \int \frac{s\,ds}{(\quad)^{3/2}}$$

$$\mathbf{F} = \frac{GmM}{2L}\left(RI\mathbf{n}_1 - J\mathbf{n}_s\right) \qquad \mathbf{n}_s = -\mathbf{n}_1\cos\psi + \mathbf{n}_2\sin\psi$$

$$= \frac{GmM}{2L}\left[(RI + J\cos\psi)\mathbf{n}_1 \;\; J\sin\psi\,\mathbf{n}_2\right]$$

$$RI + J\cos\psi = \frac{1}{R}\left[1 + \frac{2L}{R}\cos\psi + \left(\frac{L}{R}\right)^2\right]^{-1/2} + \left[1 - \frac{2L}{R}\cos\psi + \left(\frac{L}{R}\right)^2\right]^{-1/2}\left(\frac{L}{R}\right)$$

$$-J\sin\psi = \frac{1}{R}\left[\left(1+\frac{L}{R}\cos\psi\right)\left[1 + \frac{2L}{R}\cos\psi + \left(\frac{L}{R}\right)^2\right]^{-1/2} - \left(1-\frac{L}{R}\cos\psi\right)\left(1 - \frac{2L}{R}\cos\psi + \left(\frac{L}{R}\right)^2\right)\right] \frac{1}{\sin\psi}$$

$\mathbf{T} = R\mathbf{n}_1 \times \mathbf{F}$ or $\mathbf{T} = \int_{-L}^{L} \mathbf{r} \times d\mathbf{F}$

$d\mathbf{F}$ form a set of concurrent vectors by Varignon's theorem $\Rightarrow$ replace by $\mathbf{F}$ at Q, $\mathbf{T}_Q = 0$

now $\mathbf{T}_{B^*} = \mathbf{T}_Q + \mathbf{R} \times \mathbf{F}$

let $c \triangleq \cos\psi \quad s \triangleq \sin\psi \quad \alpha \triangleq \dfrac{GmM}{2L}$

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2}z^2 + \dots \quad |z| < 1$$
$$(\text{for } n = -\tfrac{1}{2}) = 1 - \frac{1}{2}z + \frac{3}{8}z^2 - \frac{5}{16}z^3 + \dots$$

$$\mathbf{T} = T\mathbf{n}_1 \times \mathbf{n}_2\,;\; T = \frac{\alpha}{s}\left\{[1+cx][1+2cx+x^2]^{-1/2} - [1-cx][1-2cx+x^2]^{-1/2}\right\}; \quad x \triangleq \frac{L}{R}$$

If T is expressed as $T = T_0 + T_1 x + T_2 x^2 + \dots$ then

$T_0 = 0 \qquad$ Find $T_1, T_2, \dots$

$$T \approx \frac{\alpha}{s}\left\{[1+cx]\left[1 - cx - \frac{cx^2}{2} + \dots\right] - [1-cx]\left[1 + cx + \frac{cx^2}{2} + \dots\right]\right\}$$

$\cancel{2cx - 2cx} - \dfrac{cx^2}{2}[2cx]$ etc

$\cong \; 2\alpha c x^3$

Generalized forces:

Start from real forces (body, contact forces)

$$F_r = \sum_{i=1}^{N} \mathbb{F}_i \cdot \tilde{\mathbb{V}}^{P_i}_{\dot q_r} \qquad r = 1, \dots, n - m$$

n — # of generalized coords. m — nonholon const.

So what? Statics can be stated as $F_r = 0$.

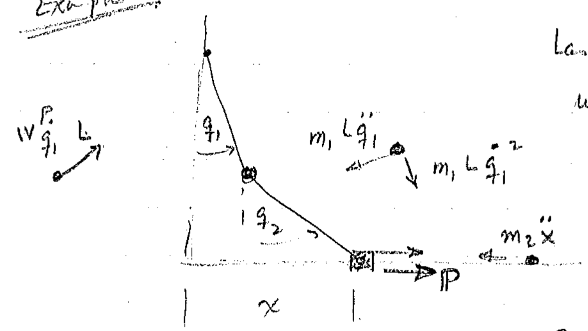
Where are we headed: Look at generalized inertia forces $F_r^* \triangleq \sum_{i=1}^{N} \mathbb{F}_i^* \cdot \tilde{\mathbb{V}}^{P_i}_{\dot q_r}$

$-m_i \mathbb{A}^{P_i} = \mathbb{F}_i^*$ inertia forces

What does this have to do with anything

all of dynamics can be stated as $F_r + F_r^* = 0 \qquad r = 1, \dots, n-m$

Example



Last time we found $\mathbb{V}^P_{\dot q}$

we need to find the accel.

this is inertia forces

$$\mathbb{V}^P_{\dot q_1} = \vec{L(c_1 - \frac{s_1}{s_2} c_2)}$$

Generalized inertia force

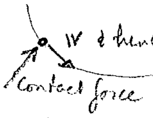
$$F_1^* = -m_1 L^2 \ddot q_1 - m_2 \ddot x L \left(c_1 - \frac{s_1}{s_2} c_2\right)$$

From previous F_1 was found it is the only generalized forces.

$$F_1 + F_1^* = 0 \implies -m_1 L^2 \ddot q_1 - m_2 \ddot x L \left(c_1 - \frac{s_1}{s_2} c_2\right) - m_1 g L s_1 + PL \left(c_1 - \frac{s_1}{s_2} c_2\right) = 0$$

To find Generalized Active Forces efficiently — 4 ideas involved

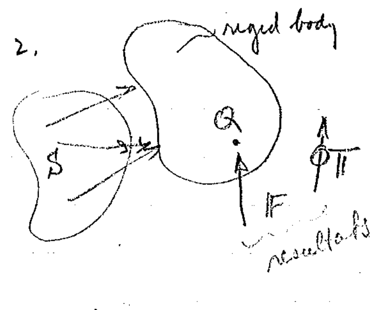
1) Smooth contacts — forces acting across smooth surfaces provide 0 contributions.



$\mathbb{V}$ & hence $\mathbb{V}_{\dot q}$ is tangential, smoothness $\implies$ no tangential component of force

$\therefore \mathbb{V}_{\dot q} \cdot \mathbb{F} = 0$

2.



We can replace effect of S by $\mathbb{F}$ and $\mathbb{T}$

then $\mathbb{F} \cdot \mathbb{V}^Q_{\dot q_r} + \mathbb{T} \cdot \omega_{\dot q_r}$ is contrib to generalized active force by set S

3. Gravitational forces near earth

Contrib to generalized active force is

$$m g \mathbb{k} \cdot \mathbb{V}^{P*}_{\dot q_r}$$

m — mass

g — accel of grav

$\mathbb{k}$ is vector acting down

$\mathbb{V}^{P*}_{\dot q_r}$ — partial veloc of mass

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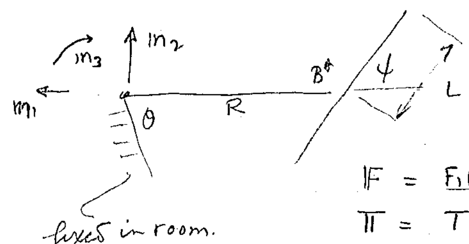
more about 4l



$$\mathbb{F} = F_1(R,\psi)\, m_1 + F_2(R,\psi)\, m_2$$
$$\mathbb{T} = T(R,\psi)\, m_3$$
$$a^{B^*} = (R\dot\theta^2 - \ddot R)\, m_1 + (R\ddot\theta - 2\dot R\dot\theta)\, m_2$$
$$\alpha^{B^*} = -(\ddot\theta + \ddot\psi)\, m_3$$

Suppose we apply newton's laws

$$\mathbb{F} = Ma \Rightarrow F_1(R,\psi) = M(R\dot\theta^2 - \ddot R)$$
$$F_2(R,\psi) = M(R\ddot\theta - 2\dot R\dot\theta)$$
$$\mathbb{T} = I\alpha\cdot m_3 \Rightarrow T(R,\psi) = -I(\ddot\theta + \ddot\psi)$$

} 3 DE

R, ψ, θ unknowns

The orbits will not be elliptical since body has lengths. If $\frac{L}{R} \ll 1$ then.

then $F_1(R,\psi) \approx \frac{GmM}{R^2}$, $F_2 \approx 0$, $T \approx \frac{GMmL^2}{2R^2}\sin 2\psi$

then using Newton's Law

$$\frac{GmM}{R^2} = M(R\dot\theta^2 - \ddot R) \quad (1)$$

$$0 = M(R\ddot\theta - 2\dot R\dot\theta) \quad (2)$$

$$\frac{GMmL^2}{2R^3}\sin 2\psi = -I(\ddot\theta + \ddot\psi) \quad (3)$$

These are eqns w/ particle motion $\Rightarrow$ we've uncoupled (3) from (1), (2)

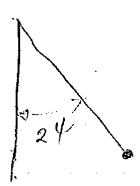
Solve (1) (2) for R & θ put into (3)

Look at circular orbit: $R = R_0$ $\dot\theta = \Omega$ = const.

(1) $\Rightarrow \frac{GmM}{R_0^2} = MR_0\Omega^2$ (1) $\Rightarrow \frac{GmM}{R_0^3 M} = \Omega^2$ & $\Omega = \left(\frac{Gm}{R_0^3}\right)^{1/2}$

(2) $\Rightarrow$ no info

(3) $\frac{GMmL^2}{2R_0^3}\sin 2\psi = -\frac{ML^2}{3}(\ddot\psi)$



$$\Rightarrow \frac{2\ddot\psi}{2} + \frac{3\Omega^2}{2}\sin 2\psi = 0$$

pendulum eqn.

ie for small 2ψ oscill motion about ℄

thus

oscill motion of rod.

Problems in Set #5

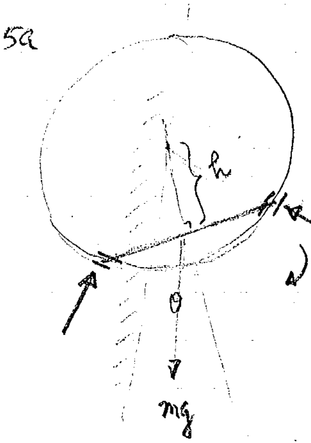
$$F_r = \sum_{i=1}^{N} \mathbb{F}_i \cdot \mathbb{V}^{P_i}_{q_r}$$

hardest to use – impractical

had 4 ideas to simplify evaluation

1. – Smooth contacts
2. – Rigid Bodies
3. – Gravity |
4. Rolling

5a



smooth contact – contact forces contribute nothing
replaces gravity force by ~~mass center~~ mg at mass center

$$\mathbb{V}_{\theta}^{*} = (h\, m_2)$$

5b. Smooth contacts at joints, gravity, rigid bodies treat rods as particles when dealing w/ gravity forces

Suppose z_i, m_i, $\mathbb{F} = m_i g \mathbb{k}$

contribution of all gravitational forces

$$(F_r)_G = g \frac{\partial}{\partial q_r}\left(\sum_{i=1}^{N} m_i z_i\right)$$

$z_i > 0$ if m_i below the reference plane

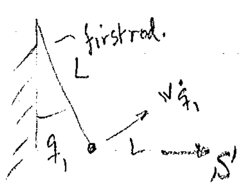
for the present system $\frac{W}{g} L \left[\frac{6}{2}\cos q_1 + 5\cos q_1 + 6(\cos q_1 + \frac{1}{2}\cos q_2) \right.$

6 rods of distance $\frac{1}{2}\cos q_1$ 5 rods of dist q_1 $+5(\cos q_1 + \cos q_2$

$\left. + 5(\cos q_1 + \cos q_2) + 6(\cos q_1 + \cos q_2 + \frac{1}{2}\cos q_3)\right]$

$$\frac{WL}{g}\left[30\cos q_1 + 19\cos q_2 + 8\cos q_3\right] = \sum m_i z_i$$

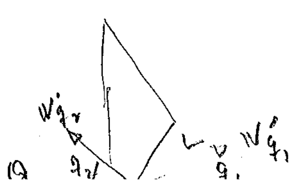
then $(F_1)_g = -30WL\sin q_1$ $(F_2)_G = -19\,WL\sin q_2$ $(F_3)_G = -8WL\sin q_3$



(F_1) S contrib $= LS\cos q_1$

$(F_2)_{S\,contr} = 0$

$(F_3)_{S\,contr} = 0$

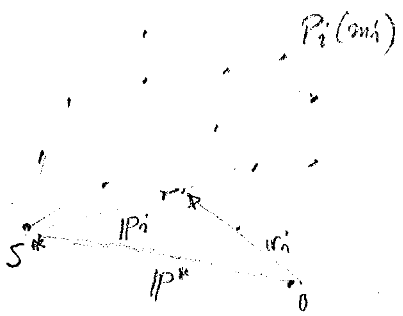


Q contr to $F_1 = -QL\cos q_1$

" $F_2 = QL\cos q_2$

" $F_3 = 0$

Mass Center – centroid point S^* $\ni$ $\sum m_i \mathbb{p}_i = 0$ where $\mathbb{p}_i$ is vector from S^* to particles



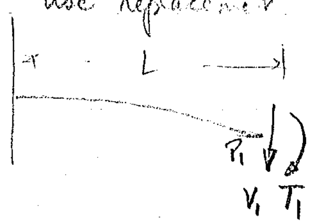
to find S^*: introduce O (known point)

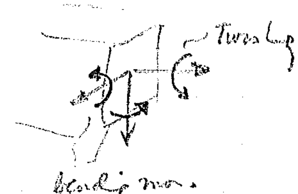
we want $\sum_{i=1}^{N} m_i \mathbb{p}_i = 0$ $\quad \mathbb{p}_i = \mathbb{r}_i - \mathbb{p}^*$

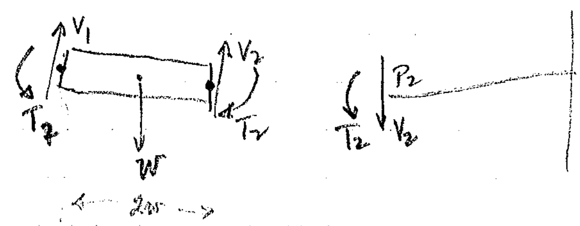
$$\sum m_i (\mathbb{r}_i - \mathbb{p}^*) = 0$$

$$\mathbb{p}^* = \frac{\sum m_i \mathbb{r}_i}{\sum m_i}$$

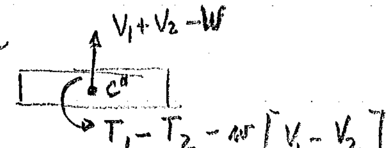
5d. Use replacement / rigid bodies







replace



define $\mathbb{a}$, $\mathbb{b}$

$$\mathbb{V}^{C^*} = -\left(\frac{\dot{q}_1 + \dot{q}_2}{2}\right)\mathbb{a} \qquad \mathbb{V}_{\dot{q}_1}^{C^*} = -\frac{\mathbb{a}}{2} \qquad \mathbb{V}_{\dot{q}_2}^{C^*} = -\frac{\mathbb{a}}{2}$$

we need no extensional forces since $\mathbb{V}_{\dot{q}_i}^{C^*} \cdot$ force (in dir $\perp$ to $\mathbb{a}$) $= 0$

$$\omega^{C^*} = \frac{\dot{q}_1 - \dot{q}_2}{2w}\mathbb{b} \qquad \omega_{\dot{q}_1}^{C^*} = \frac{\mathbb{b}}{2w} \qquad \omega_{\dot{q}_2}^{C^*} = -\frac{\mathbb{b}}{2w}$$

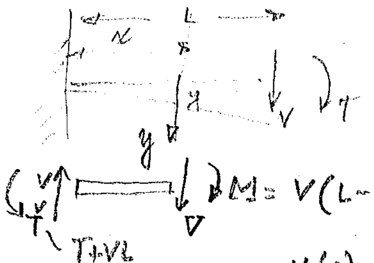
$$\mathbb{F} = (V_1 + V_2 - W)\,\mathbb{a} \qquad \mathbb{T} = \left[T_1 - T_2 - W(V_1 - V_2)\mathbb{b}\right]$$

$$F_1 = \mathbb{V}_{\dot{q}_1} \cdot \mathbb{F} + \omega_{\dot{q}_1} \cdot \mathbb{T} = \left(-V + \frac{1}{2w}(T_1 - T_2) + \frac{W}{2}\right)$$

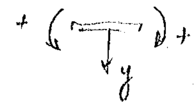
$F_2 =$

Beam Theory $\quad V_1 = \frac{12EI}{L^3}\left(q_1 - \frac{L}{2}\,\frac{q_2 - q_1}{2w}\right)$

$$T_1 = \frac{12EI}{L^2}\left(\frac{L}{3}\,\frac{q_2 - q_1}{} - q_1\right)$$



$M = V(L-x)+T$



$y(0)=0,\ y'(0)=0 \qquad EIy'' = M$

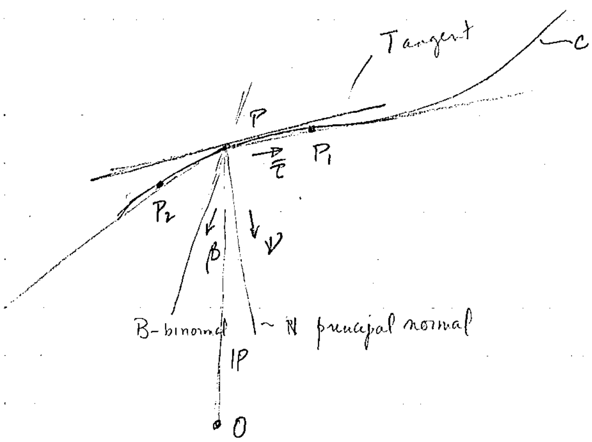
$$EIy'' = V(L-x)+T$$
$$EIy' = -V\frac{(L-x)^2}{2} + Tx + \frac{VL^2}{2}$$
$$EIy = +V\frac{(L-x)^3}{6} + \frac{Tx^2}{2} + \frac{VL^2}{2}x - \frac{VL^3}{6}$$
$$\left.\begin{aligned} EI\theta &= TL + \frac{VL^2}{2} \\ EI\delta &= \frac{TL^2}{2} + \frac{VL^3}{3} \end{aligned}\right\} \text{solve for } V \,\&\, T$$

Let $y(L)=\delta$
$y'(L)=\theta$

$$V = \frac{12EI}{L^3}\left(\delta - \frac{L}{2}\theta\right)$$
$$T = \frac{12EI}{L^2}\left(\frac{L}{3}\theta - \frac{\delta}{2}\right)$$

Final Exam : – Generalized Active Forces is final topic 11/29/79
No class : 6^{th} Dec 79
Final Exam : Hybrid Take Home Thursday 13^{th} at 7 – ~~10 PM~~ PM pick up exam at Prof. Kane's office
return it by 8^{30} AM on the 14^{th}

To work out problem 5(d)
Differential Geometry – Ref: Analytic Elements of Mechanics Vol 2 : Kane



z : a scalar variable governing the position of P on N.
Let P move ∋. $z \to z \pm h$, draw lines from PP_1 & PP_2 ; These lines ~~are~~ define a plane
- line ⊥ to PP_1–PP_2 plane is in the limit B

$N = -B \times T$

Define τ : a vector tangent to C at P
β : " " binormal " " " "
~~N~~ ν : <u>the</u> " principal normal " " "

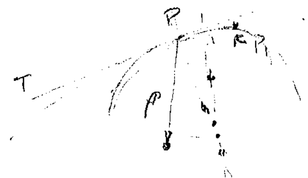
$\tau \triangleq \mathbb{P}'/|\mathbb{P}'|$ (1) $\quad ()' \triangleq \frac{d()}{dt}$ you will get either t depending on parameter of P

$\beta \triangleq \dfrac{\mathbb{P}' \times \mathbb{P}''}{|\mathbb{P}' \times \mathbb{P}''|}$ (2)

now $\tau \cdot \beta = 0 \Rightarrow \tau \perp \beta$

(3) $\psi = \beta \times \tau$ unique since we use a given β, τ

ρ = vector radius of curvature of C at P

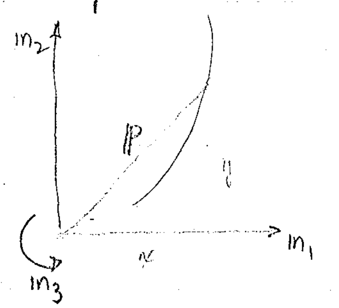


P_1 another point on C draw ⊥ bisector to PP_1 draw ⊥ to T through P. Intersection of 2 lines is in limit the radius of curvature

$$\rho = \frac{(p')^2}{(p' \times p'')^2} (p' \times p'') \times p' = \frac{(p')^2}{(p' \times p'')^2} |p' \times p''| |p'| \beta \times \tau \qquad (4)$$

$$\rho = \frac{|p'|^3}{|p' \times p''|} \psi \qquad (5)$$

Consider a planar curve



$$p = x m_1 + y(x) m_2 \qquad (6)$$

Let $x = z$

$$p' = m_1 + y' m_2 \qquad (7)$$

$$p'' = y'' m_2 \qquad (8)$$

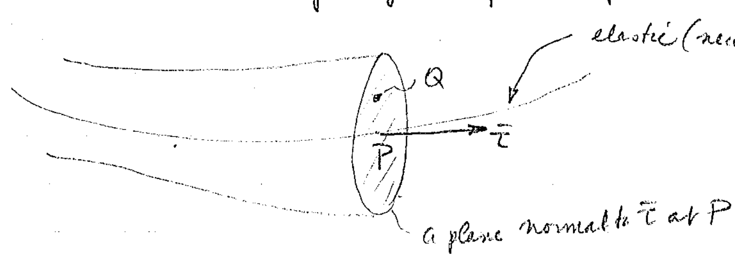
$$\boxed{\tau \underset{(1)}{=} \frac{m_1 + y' m_2}{(1+y'^2)^{1/2}} \quad (9)} \qquad \beta \underset{(2,7,8)}{=} \frac{(m_1 + y' m_2) \times y'' m_2}{|\quad|}$$

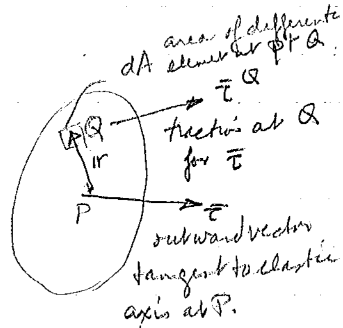
$$\boxed{\beta = \frac{y'' m_3}{|y''|} \quad (10)}$$

$$\boxed{\psi \underset{(3,9,10)}{=} \frac{y''}{|y''|} \frac{(-y' m_1 + m_2)}{(1+y'^2)^{1/2}} \quad (11)}$$

$$\rho \underset{(5,7,8,11)}{=} \frac{(1+y'^2)^{3/2}}{|y''|} \frac{y''}{|y''|} \frac{(-y' m_1 + m_2)}{(1+y'^2)^{1/2}} = \frac{(1+y'^2)}{y''} (-y' m_1 + m_2) \qquad (12)$$

Now we use beam & define generic pt Q in plane P.



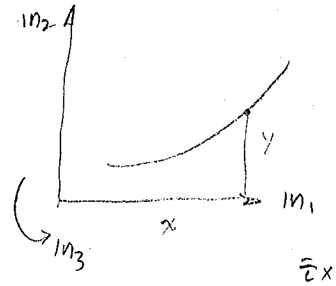


Dfn. $I \triangleq \int_A (r \times \beta)^2 dA$ (13) second moment of area

$M \triangleq \int (r \times \tau^Q \cdot \beta) \beta \, dA$ (14) component in binormal direction

Beam Law $\tau \times \rho \cdot M = EI$ — Young's Modulus (15)

To go back to 2D beam bending

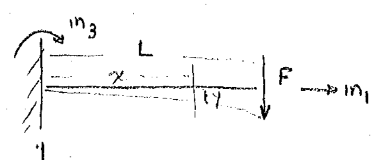


$$\bar{\bar{\varepsilon}} \times \rho \underset{(9,12)}{=} \frac{n_1 + y' n_2}{(1+y'^2)^{1/2}} \times (-y' n_1 + n_2) \left(\frac{1+y'^2}{y''}\right)$$

$$= \frac{(1+y'^2)^{3/2}}{y''} n_3 \qquad (16)$$

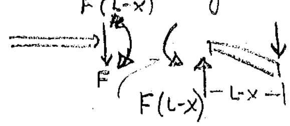
Using beam law $\quad n_3 \cdot \mathbb{M} \underset{(15,16)}{=} EI\,y''/(1+y'^2)^{3/2} \approx EI\,y''$ for $y' \ll 1$ $\quad (17)$

Consider



($y>>0 \Leftrightarrow$ downward) $\Leftrightarrow$ (n_2 down; n_3 in paper)

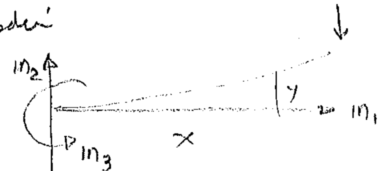
to get $\mathbb{M}$



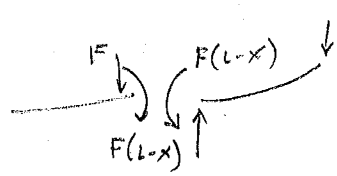
$$\mathbb{M} = F(L-x)\, n_3$$

$$n_3 \cdot \mathbb{M} \underset{(17)}{=} F(L-x) = EI\,y'' \qquad (18)$$

Consider



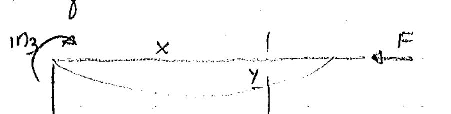
($y>0 \Leftrightarrow$ up) $\Leftrightarrow$ (n_2 up, n_3 out)



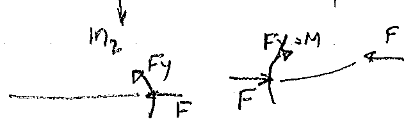
$$\mathbb{M} = F(L-x)(-n_3)$$

$$n_3 \cdot \mathbb{M} = -F(L-x) = EI\,y'' \qquad (19)$$

Buckling

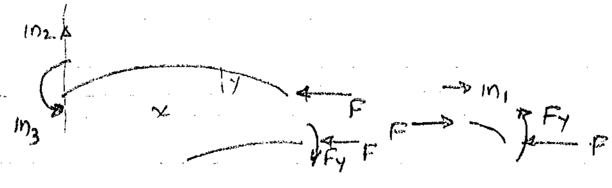


($y>0 \Leftrightarrow$ down) $\Leftrightarrow$ (n_2 down, n_3 in)



$$\mathbb{M} = Fy\,(-n_3)$$

$$-Fy = EI\,y'' \;!$$



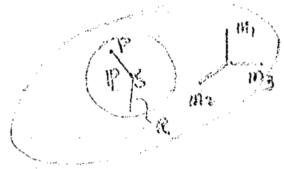
($y>0 \Leftrightarrow$ up) $\Leftrightarrow$ (n_2 up, n_3 out)

$$\mathbb{M} = Fy\,(-n_3)$$

one cannot tell from buckling problem what deflection is $\Rightarrow$ both eqns are possible based on 2 different definitions. We can only get buckling load since $EI y'' + Fy = 0$ $\quad y = A \sin \alpha x + B \cos \alpha x$

$$\alpha = \left(F/EI\right)^{1/2}$$

5h



$$\omega^B = u_1 m_1 + u_2 m_2 + u_3 m_3$$
$$\omega^S = u_4 m_1 + u_5 m_2 + u_6 m_3$$
$$v = u_7 m_1 + u_8 m_2 + u_9 m_3$$

Force on S at P:

$$dF = -c\, {}^B v^P dA = -c\, {}^B\omega^S \times p\, dA$$
$$= +c\,(\omega^B - \omega^S) \times p\, dA \qquad (1)$$

Force on B at P: $\quad -dF$

Generalized force $\quad dF_r = v^{P_S}_{u_r} \cdot dF + v^{P_B}_{u_r} \cdot (-dF) = (v^{P_S}_{u_r} - v^{P_B}_{u_r})\, dF \qquad (2)$

$$v^{P_S}_{u_r} = v_{u_r} + \omega^S_{u_r} \times p \qquad v^{P_B}_{u_r} = v_{u_r} + \omega^B_{u_r} \times p \qquad (3)$$

use in (2) to get

$$dF_r = (\omega^S_{u_r} - \omega^B_{u_r}) \times p \cdot dF$$
$$= c\,(\omega^S_{u_r} - \omega^B_{u_r}) \cdot p \times [(\omega^B - \omega^S) \times p]\, dA$$

$$F_5 = -c m_2 \cdot \int p \times [(\omega^B - \omega^S) \times p]\, dA$$

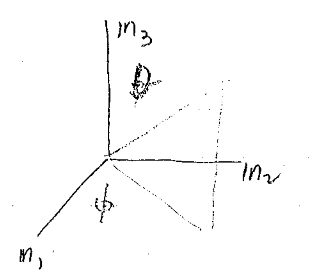
$$F_4 = c m_1 \cdot \int p \times [(\omega^B - \omega^S) \times p]\, dA$$

now look for $\int p \times [(\omega^B - \omega^S) \times p]\, dA$

$$= (u_1 - u_4) \int p \times (m_1 \times p)\, dA$$
$$+ (u_2 - u_5) \int p \times (m_2 \times p)\, dA$$
$$+ (u_3 - u_6) \int p \times (m_3 \times p)\, dA$$

$$\int p \times (m_i \times p)\, dA = \frac{8\pi R^4}{3} m_i$$

This is true since by principal of symmetry of sphere this integral has a preferred direction, namely m_i



$$p = R\,(\sin\theta \cos\phi\, m_1 + \sin\theta \sin\phi\, m_2 + \cos\theta\, m_3)$$
$$dA = R^2 \sin\theta\, d\theta\, d\phi$$

Do internal forces contribute to generalized active forces.
5h internals do $\quad$ 5b internals don't $\quad$ ∴ it depends on situation

Generalized Inertia Forces

$$F_r^* = \sum \mathbf{v}_{\dot q_r}^{P_i} \cdot \mathbf{F}_i^*$$

$$= -\sum_{i=1}^{N} m_i \, \mathbf{v}_{\dot q_r}^{P_i} \cdot \mathbf{a}^{P_i}$$

If one deals w/ rigid body 2 pts on rigid body

$$\mathbf{v}_{\dot q_r}^{P_i} = \mathbf{v}_{\dot q_r}^{B^*} + \omega_{\dot q_r} \times \mathbf{r}_i$$ if B^* is mass center

$$\mathbf{a}^{P_i} = \mathbf{a}^{B^*} + \omega \times (\omega \times \mathbf{r}_i) + \alpha \times \mathbf{r}_i$$

Now $$\mathbf{v}_{\dot q_r}^{P_i} \cdot \mathbf{a}^{P_i} = \mathbf{v}_{\dot q_r}^{B^*} \cdot \mathbf{a}^{B^*} + \mathbf{v}_{\dot q_r}^{B^*} \cdot [\omega \times (\omega \times \mathbf{r}_i)] + \mathbf{v}_{\dot q_r}^{B^*} \cdot (\alpha \times \mathbf{r}_i) + \omega_{\dot q_r} \times \mathbf{r}_i \cdot \mathbf{a}^{B^*}$$
$$+ (\omega_{\dot q_r} \times \mathbf{r}_i) \cdot [\omega \times (\omega \times \mathbf{r}_i)] + \omega_{\dot q_r} \times \mathbf{r}_i \cdot (\alpha \times \mathbf{r}_i)$$

Now subst.

$$-\mathbf{F}_r^* = \mathbf{v}_{\dot q_r}^{B^*} \cdot \mathbf{a}^{B^*} \sum^{N} m_i + \mathbf{v}_{\dot q_r} \cdot \Big[\omega \times \big(\omega \times \sum^{N} m_i \mathbf{r}_i\big)\Big] + \mathbf{v}_{\dot q_r}^{B^*} \cdot \big(\alpha \times \sum m_i \mathbf{r}_i\big)$$
$$+ \omega_{\dot q_r} \times \big(\sum m_i \mathbf{r}_i\big) \cdot \mathbf{a}^{B^*} + \sum m_i (\omega_{\dot q_r} \times \mathbf{r}_i) \cdot [\omega \times (\omega \times \mathbf{r}_i)] + \sum m_i (\omega_{\dot q_r} \times \mathbf{r}_i) \cdot (\alpha \times \mathbf{r}_i)$$

now let $\omega = \omega \mathbf{n}_0 \implies \sum_{i=1}^{N} m_i (\omega_{\dot q_r} \times \mathbf{r}_i) \cdot [\omega \times (\omega \times \mathbf{r}_i)] =$

$$= \omega^2 \sum m_i \big(\omega_{\dot q_r} \cdot \mathbf{r}_i \times [\mathbf{n}_0 \times (\mathbf{n}_0 \times \mathbf{r}_i)]\big)$$
$$= \omega^2 \, \omega_{\dot q_r} \cdot \mathbf{n}_0 \times \sum m_i \mathbf{r}_i \times [\mathbf{n}_0 \times \mathbf{r}_i]$$

now $\alpha = \alpha \mathbf{n}_a \implies \sum m_i (\omega_{\dot q_r} \times \mathbf{r}_i) \cdot (\alpha \times \mathbf{r}_i) = \alpha \sum m_i [\omega_{\dot q_r} \cdot \mathbf{r}_i \times (\mathbf{n}_a \times \mathbf{r}_i)]$

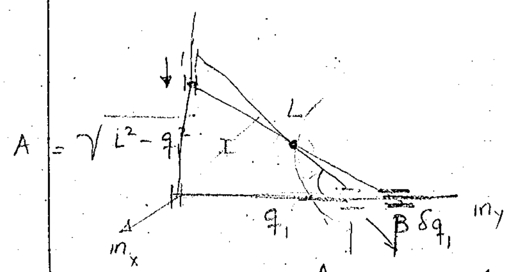
$$= \alpha \, \omega_{\dot q_r} \cdot \sum m_i \mathbf{r}_i \times (\mathbf{n}_a \times \mathbf{r}_i)$$

What's the point? $\sum m_i = M$ = total mass $\sum m_i \mathbf{r}_i = 0$ if B^* is mass center

Sums of the form $\sum_{i=1}^{N} m_i \mathbf{r}_i \times (\mathbf{n} \times \mathbf{r}_i)$, where $\mathbf{r}_i$ is a position vector of particle from mass center,

deserve attention – relates to moments of inertia

No class thursday put number 28 on exam.



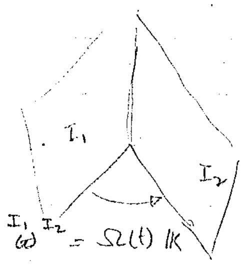
can be reduced to

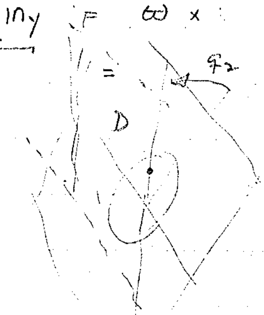
$$ {}^{R}\omega^{D} = {}^{R}\omega^{I_1} + {}^{I_1}\omega^{I_2} + {}^{I_2}\omega^{D} $$

now $A^2 + q_1^2 = L^2$

$$ 2A \frac{dA}{dt} + 2 q_1 \dot q_1 = 0 \qquad \frac{dA}{dt} = \frac{-q_1 \dot q_1}{\sqrt{L^2 - q_1^2}} \, \mathbb{K} $$

$$ {}^{A}\mathbb{V}^{1} = -q_1 \frac{\dot q_1}{\sqrt{}} n_y \qquad {}^{R}\omega^{A} \times $$





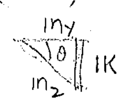
$$ {}^{I_2}\omega^{D} = \dot q_2 \, n_2 $$

$$ \mathbb{r}^{A} = \sqrt{L^2 - q_1^2}\, \mathbb{K} \qquad \frac{d\mathbb{r}}{dt} = \frac{-2 q_1 \dot q_1}{\sqrt{}} \mathbb{K} = {}^{R}\mathbb{V}^{A} $$

$$ {}^{R}\mathbb{V}^{A} = {}^{R}\mathbb{V}^{B} + {}^{R}\omega^{I_1} \times \mathbb{r} $$

$$ = \dot q_1 n_y + (\quad) \times L n_2 \qquad \omega \qquad \dot q_1 n_y $$

$$ \dot q_1 \left[\frac{-q_1}{\sqrt{}} \mathbb{K} - n_y \right] + (\quad) \times -L n_2 $$



$$ L n_2 = \left[+ \frac{q_1}{L} n_y - \frac{A}{L} \mathbb{K} \right] $$

$$ (\omega_x) n_x + (\omega_y) n_y + (\omega_z) $$

$$ \omega_x \frac{q_1}{L} \mathbb{K} + \omega_x \frac{A}{L} n_y - \omega_y \frac{A}{L} n_x $$

$$ -\omega_z \frac{q_1}{L} n_x = (\quad) \mathbb{K} - n_y $$

$$ \Rightarrow \omega_z = \omega_y = 0 $$

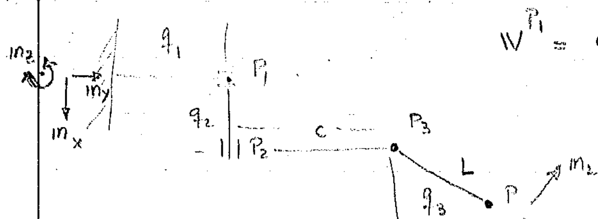
$$ \omega_x \frac{q_1}{L} = \frac{-\dot q_1 q_1}{A} \quad \Rightarrow \quad \boxed{\omega_x = -\dot q_1 \frac{L}{A}} $$

$$ = + \dot q_1 \frac{L}{L\sqrt{1 - q_1^2/L^2}} $$

must be true $\quad A = \sqrt{L^2 - q_1^2}$, $\quad q_1 = \frac{\sqrt{L^2 - A^2}}{L}$, $\quad \forall A \Rightarrow \omega$

$$ {}^{R}\mathbb{V}^{A} = {}^{R}\mathbb{V}^{B} + {}^{R}\omega^{I} \times \mathbb{r} $$

$$ 0 \qquad {}^{R}\omega^{I} \times (L \mathbb{K}) $$



	n_x	n_y	n_z
n_1	c_3	s_3	0
n_2	$-s_3$	c_3	0
n_3	0	0	1

$$ \frac{dn_1}{dt} = \dot q_3 n_z \times n_1 = \dot q_3 n_2 $$

$$ \frac{dn_2}{dt} = \dot q_3 n_z \times n_2 = -\dot q_3 n_1 $$

$$ \mathbb{V}^{P_1} = \dot q_1 n_y \qquad \mathbb{V}^{P_2} = \mathbb{V}^{P_1} + \mathbb{V}^{P_2/P_1} = \dot q_1 n_y + \dot q_2 n_x $$

$$ \mathbb{V}^{P_3} = \mathbb{V}^{P_2} + \underbrace{\mathbb{V}^{P_3/P_2}}_{=0} = \dot q_1 n_y + \dot q_2 n_x $$

$$ \mathbb{V}^{P} = \mathbb{V}^{P_3} + \omega \times \mathbb{r} = \mathbb{V}^{P_3} + \dot q_3 n_z \times L n_1 $$

$$ = \dot q_1 n_y + \dot q_2 n_x + \dot q_3 L n_2 $$

$$ \mathbb{V}^{P} = \dot q_1 (s_3 n_1 + c_3 n_2) + \dot q_2 (c_3 n_1 - s_3 n_2) + \dot q_3 L n_2 $$

$$ \frac{d\mathbb{V}^{P}}{dt} : \quad a^{P} = \ddot q_1 (s_3 n_1 + c_3 n_2) + \ddot q_2 (c_3 n_1 - s_3 n_2) + \ddot q_3 L n_2 $$

$$ + \dot q_1 (c_3 n_1 + s_3 n_2) \dot q_3 + \dot q_2 (-s_3 n_1 - c_3 n_2) \dot q_3 $$

$$ + \dot q_1 (s_3 n_2 - c_3 n_1) \dot q_3 + \dot q_2 (c_3 n_2 + s_3 n_1) \dot q_3 $$

$$ - L \dot q_3^2 n_1 $$

$$ = (\ddot q_1 s_3 + \ddot q_2 c_3 - L \dot q_3^2) n_1 + (\ddot q_1 c_3 - \ddot q_2 s_3 + \ddot q_3 L) n_2 $$

let $u_1 = \dot{q}_1 s_3 + \dot{q}_2 c_3$

$u_2 = \dot{q}_1 c_3 - \dot{q}_2 s_3 + L \cancel{\dot{q}_3} u_3$

$u_3 = \dot{q}_3$

$u_1 s_3 + u_2 c_3 = \dot{q}_1 \cancel{} + L u_3 c_3$

$\dot{q}_1 = u_1 s_3 + u_2 c_3 - L u_3 c_3$

$u_1 c_3 - u_2 s_3 = \dot{q}_2 - L u_3 s_3$

$$\mathbf{v}^P = u_1 \mathbf{n}_1 + u_2 \mathbf{n}_2$$

$$\mathbf{a}^P = \dot{u}_1 \mathbf{n}_1 + u_1 u_3 \mathbf{n}_2 + \dot{u}_2 \mathbf{n}_2 - u_2 u_3 \mathbf{n}_1$$

$$= (\dot{u}_1 - u_2 u_3)\mathbf{n}_1 + (\dot{u}_2 + u_1 u_3)\mathbf{n}_2$$

$\dot{u}_1 = \ddot{q}_1 s_3 + \cancel{\dot{q}_1 \dot{q}_3 c_3} + \ddot{q}_2 c_3 \mp \cancel{\dot{q}_2 s_3} \dot{q}_3$

$-u_2 u_3 = \cancel{-\dot{q}_1 \dot{q}_3 c_3} + \cancel{\dot{q}_2 \dot{q}_3 s_3} \mp L \dot{q}_3^2$

$$\dot{u}_1 - u_2 u_3 = \cancel{\ddot{q}_1 s_3} + \ddot{q}_2 c_3 - \dot{q}_1 \dot{q}_3 c_3 - L \dot{q}_3^2$$

$$= \ddot{q}_1 s_3 + \ddot{q}_2 c_3 + \dot{q}_1 \dot{q}_3 s_3 - \dot{q}_1 \dot{q}_3 c_3 - L \dot{q}_3^2$$

$\dot{u}_2 = \ddot{q}_1 c_3 \mp \dot{q}_1 s_3 \dot{q}_3 - \ddot{q}_2 s_3 - \dot{q}_2 c_3 \dot{q}_3 + L \ddot{q}_3$

$u_1 u_3 = \dot{q}_1 \dot{q}_3 s_3 + \dot{q}_2 \dot{q}_3 c_3$

$\ddot{q}_1 c_3 - \ddot{q}_2 s_3 - 2\dot{q}_2 \dot{q}_3 c_3 - 2\dot{q}_1 \dot{q}_3 s_3 + L \ddot{q}_3$

$${}^{D}\omega^{C} + {}^{C}\omega^{B}$$

$$\frac{{}^{D}d\,b_2}{dt} = \frac{{}^{B}d\,b_2}{dt} + {}^{D}\omega^{B} \times b_2 = \left[\dot{\psi}\, a_3 + \dot{\theta}\, b_1\right] \times b_2$$

$$\dot{\psi}\, b_1 + \dot{\theta}\, b_3 \qquad \frac{{}^{C}\partial b_2}{\partial \theta} = b_3 = a_3 \cos\theta - \sin\theta\, c_2$$

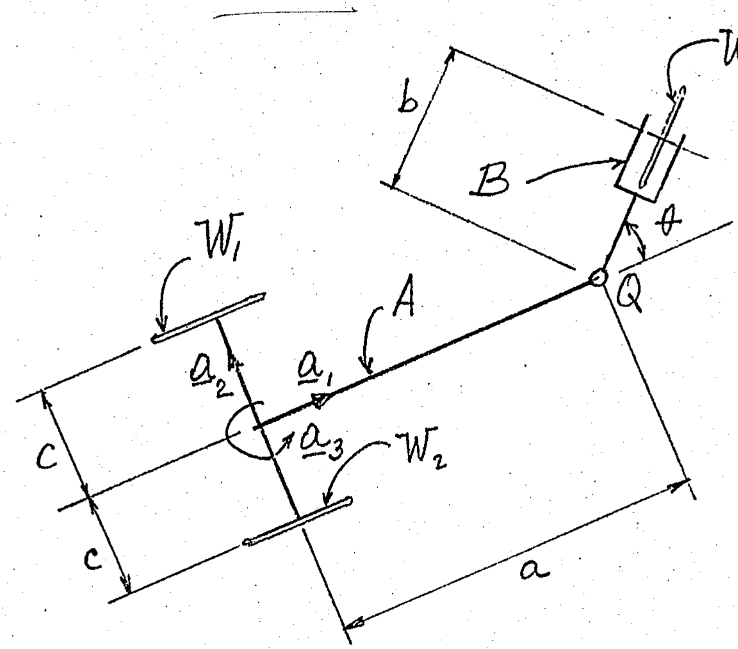
$$\frac{{}^{D}\partial b_2}{\partial \psi} = a_3 \times \frac{{}^{C}\partial b_2}{\partial t} + {}^{D}\omega^{C} \times b_2 \qquad a_3 \cos\theta - \sin\theta\, c_2$$

$$\frac{{}^{C}\partial b_2}{\partial t} + {}^{C}\omega^{B} \times b_2 = \dot{\theta}\, b_3 + \underbrace{\frac{{}^{C}\partial b_2}{\partial \psi}}_{=0} \dot{\psi}$$

Excellent

7911025 28 LEVY

Prof. Kane
ME 231A
Dynamics Fall 79

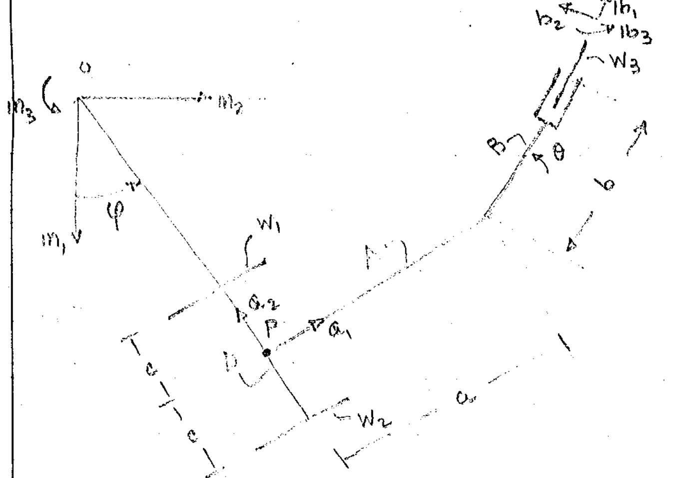


The sketch is a schematic representation of an aircraft landing gear consisting of two hinge-connected rigid bodies, A and B, and three identical wheels, W_1, W_2, and W_3, each with radius r. $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ are mutually perpendicular unit vectors fixed in A.

When this system rolls on a plane surface, it possesses two degrees of freedom; and two generalized speeds, u_1 and u_2, can be defined as $u_i = \underline{v} \cdot \underline{a}_i$ (i=1,2), where $\underline{v}^Q$ is the velocity of the hinge point Q.

Determine the partial angular velocities of A, B, and W_1 with respect to u_1 and u_2 for an instant at which θ (see the sketch) is equal to 180 degrees. Record your results in the table below.

	$\underline{\omega}^A_{u_i}$	$\underline{\omega}^B_{u_i}$	$\underline{\omega}^{W_1}_{u_i}$
$i = 1$	0	0	$\frac{1}{r}\underline{a}_2$
$i = 2$	$\underline{a}_3/a$	$\frac{\underline{a}_3}{b}$	$\frac{\underline{a}_3}{a} - \frac{c}{ra}\underline{a}_2$



Define a reference frame R with fixed vectors of unit length forming a right handed triad, R being the surface that the system rolls on.

Let P be the point where A joins the cross axis D, let S be the point directly below P on R. Let a_1, a_2, a_3 be unit vectors fixed in A and let b_1, b_2, b_3 be unit vectors fixed in B, each set forming a right handed triad.

Let the axis D be at an angle φ to the n_1 axis. Define superscript $(\ )_i^0$ to be the point on the wheel in contact with R and $(\ )_i^*$ to be the center of the wheel. Let (x,y) be the position of point S.

$$r^S = x n_1 + y n_2 \ ; \quad r^P = x n_1 + y n_2 + r n_3$$

Thus $\boxed{{}^R v^P = \dot{x} n_1 + \dot{y} n_2}$ (1)

Now $r^{W_1^*} = r^P + c a_2$ and $r^{W_2^*} = r^P - c a_2$ also $\boxed{{}^R\omega^D = \dot{\varphi} n_3 = \dot{\varphi} a_3}$ (2)

$${}^R v^{W_1^*} = {}^R v^P + c\left[{}^R\omega^D \times a_2\right] = {}^R v^P - c\dot{\varphi} a_1 \quad (3)$$

$${}^R v^{W_1^0} = {}^R v^{W_1^*} + {}^R\omega^{W_1} \times r^{W_1^0/W_1^*} \quad \text{where } \boxed{{}^R\omega^{W_1} = {}^R\omega^D + {}^D\omega^{W_1} = \underset{(2)}{\dot{\varphi}} a_3 + \dot{\Omega}_1 a_2} \quad (4)$$

But since W_1 rolls $\Rightarrow {}^R v^{W_1^0} = 0 \qquad r^{W_1^0/W_1^*} = -r a_3$

Also $\boxed{n_1 = -s\varphi\, a_1 + c\varphi\, a_2}$ (5) $\qquad \boxed{n_2 = c\varphi\, a_1 + s\varphi\, a_2}$ (6)

Thus $\boxed{\underset{(1,5,6)}{{}^R v^{W_1^0}} = \left[-\dot{x} s\varphi + \dot{y} c\varphi - c\dot{\varphi} - r\dot{\Omega}_1\right] a_1 + \left[-\dot{x} c\varphi - \dot{y} s\varphi\right] a_2 = 0}$ (7)

Similarly we can do the same thing for wheel W_2 if we replace in all the above equations c by $-c$ and $\dot{\Omega}_1$ by $\dot{\Omega}_2$. Thus our pertinent equations are

$$\boxed{{}^R\omega^{W_2} = {}^R\omega^D + {}^D\omega^{W_2} = \dot{\varphi} a_3 + \dot{\Omega}_2 a_2} \quad (8)$$

$${}^R v^{W_2^*} = {}^R v^P + c\dot{\varphi} a_1 \quad (9)$$

$${}^R v^{W_2^0} = {}^R v^{W_2^*} + {}^R\omega^{W_2} \times r^{W_2^0/W_2^*} \ ; \quad r^{W_2^0/W_2^*} = -r a_3$$

and since W_2 rolls: $\boxed{\underset{(1,5,6)}{{}^R v^{W_2^0}} = \left[-\dot{x} s\varphi + \dot{y} c\varphi + c\dot{\varphi} - r\dot{\Omega}_2\right] a_1 + \left[-\dot{x} c\varphi - \dot{y} s\varphi\right] a_2 = 0}$ (10)

Now $r^Q = r^P + a a_1$ thus

$${}^R v^Q = {}^R v^P + a\left({}^R\omega^A \times a_1\right) \ ; \quad \text{but since A is} \perp \text{to D} \ \boxed{{}^R\omega^A = {}^R\omega^D = \dot{\varphi} a_3} \ (11)$$

Thus $\boxed{{}^R v^Q = {}^R v^P + a\underset{(11)}{\dot{\varphi}} a_2}$ (12)

now

$$\boxed{\underset{(1,5,6,11)}{{}^R v^Q} = \left(-\dot{x} s\varphi + \dot{y} c\varphi\right) a_1 + \left(\dot{\varphi} a - \dot{x} c\varphi - \dot{y} s\varphi\right) a_2} \quad (13)$$

now $\boxed{{}^{A}\omega^{B} = \dot\theta\, a_3}$ (14)

and $r^{W_3^*} = r^{Q} + r^{W_3^*/Q}$, where $r^{W_3^*/Q} = b\, b_1$

$${}^{R}V^{W_3^*} = {}^{R}V^{Q} + {}^{R}\omega^{B} \times (b\, b_1)$$

now ${}^{R}\omega^{B} = {}^{R}\omega^{A} + {}^{A}\omega^{B} = \dot\varphi\, a_3 + \dot\theta\, a_3$ or $\boxed{{}^{R}\omega^{B} = (\dot\varphi + \dot\theta)\, a_3}$ (15)

and $\boxed{b_1 = c_\theta\, a_1 + s_\theta\, a_2}$ (16) $\boxed{b_2 = c_\theta\, a_2 - s_\theta\, a_1}$ (17)

now using (1, 5, 6, 15, 16)

$${}^{R}V^{W_3^*} = [-\dot x s_\varphi + \dot y c_\varphi - b(\dot\varphi + \dot\theta) s_\theta]\, a_1 + [\dot\varphi a - \dot x c_\varphi - \dot y s_\varphi + b(\dot\varphi + \dot\theta) c_\theta]\, a_2 \quad (18)$$

also $\boxed{{}^{B}\omega^{W_3} = \dot\Omega_3\, b_2 \underset{(17)}{=} \dot\Omega_3 [c_\theta a_2 - s_\theta a_1]}$ (19)

$${}^{R}V^{W_3^0} = {}^{R}V^{W_3^*} + {}^{R}\omega^{W_3} \times r^{W_3^0/W_3^*} \qquad r^{W_3^0/W_3^*} = -r\, a_3$$

since W_3 rolls: $\boxed{0 = {}^{R}V^{W_3^0} \underset{(18,19)}{=} [-\dot x s_\varphi + \dot y c_\varphi - b(\dot\varphi + \dot\theta) s_\theta - \dot\Omega_3 c_\theta r]\, a_1 + [\dot\varphi a - \dot x c_\varphi - \dot y s_\varphi + b(\dot\varphi + \dot\theta) c_\theta - \dot\Omega_3 s_\theta r]}$

(20)

From (7), (10), (20) we get 5 different scalar equations of constraint:

$$-\dot x s_\varphi + \dot y c_\varphi - c\dot\varphi - r\dot\Omega_1 \underset{(7)}{=} 0 \qquad (1')$$

$$-\dot x c_\varphi - \dot y s_\varphi \underset{(7,10)}{=} 0 \qquad (2')$$

$$-\dot x s_\varphi + \dot y c_\varphi + c\dot\varphi - r\dot\Omega_2 \underset{(10)}{=} 0 \qquad (3')$$

$$-\dot x s_\varphi + \dot y c_\varphi - b(\dot\varphi + \dot\theta) s_\theta - \dot\Omega_3 c_\theta r \underset{(20)}{=} 0 \qquad (4')$$

$$\dot\varphi a - \dot x c_\varphi - \dot y s_\varphi + b(\dot\varphi + \dot\theta) c_\theta - \dot\Omega_3 s_\theta r \underset{(20)}{=} 0 \qquad (5')$$

We have 7 unknowns $(\dot x, \dot y, \dot\varphi, \dot\theta, \dot\Omega_1, \dot\Omega_2, \dot\Omega_3)$. Thus we can write 5 of the in terms of any 2. I will write $\dot x, \dot y, \dot\varphi, \dot\theta, \dot\Omega_3$ in terms of $\dot\Omega_1, \dot\Omega_2$ (sort of).

Using (3') and (1') solve for $\dot\varphi$: $\boxed{\dot\varphi = r(\dot\Omega_2 - \dot\Omega_1)/2c}$ (6')

using (6') in (1') solve for $\boxed{-\dot x s_\varphi + \dot y c_\varphi = r(\dot\Omega_2 + \dot\Omega_1)/2}$ (7')

Now using (2'), (6'), (7') into ${}^{R}V^{Q}$ (eqn 20)

$${}^{R}V^{Q} = \frac{r}{2}(\dot\Omega_2 + \dot\Omega_1)\, a_1 + \frac{ar}{2c}(\dot\Omega_2 - \dot\Omega_1)\, a_2 \qquad (8')$$

Thus $\boxed{u_1 = \frac{r}{2}(\dot\Omega_2 + \dot\Omega_1)}$ $\boxed{u_2 = \frac{ar}{2c}(\dot\Omega_2 - \dot\Omega_1)}$ (9', 10')

now ${}^{R}\omega^{A} \underset{(11)}{=} \dot\varphi\, a_3 = \frac{r}{2c}(\dot\Omega_2 - \dot\Omega_1)\, a_3 = \frac{u_2}{a} a_3$ (11')

or $\boxed{{}^{R}\omega^{A}_{u_1} = 0 \qquad {}^{R}\omega^{A}_{u_2} = \frac{1}{a} a_3}$ (12', 13')

now use the condition that $\theta = 180^\circ$ in (4', 5') to get

$$-\dot\Omega_3 r = -\dot x s_\varphi + \dot y c_\varphi \Rightarrow \boxed{\dot\Omega_3 \underset{(7')}{=} -(\dot\Omega_2 + \dot\Omega_1)/2 \underset{(9')}{=} \frac{-u_1}{r}} \qquad (14')$$

and $\quad b\dot{\theta} = \dot{\varphi}(a-b) \Rightarrow \boxed{\dot{\theta} \underset{(6')}{=} \frac{(a-b)}{b} r(\dot{\Omega}_2 - \dot{\Omega}_1)/2c \underset{(10')}{=} \frac{(a-b)}{ab} u_2}$ (15')

Now $\quad {}^R\omega^B \underset{(15)}{=} (\dot{\varphi} + \dot{\theta})\,a_3 = \Big[\underset{(11')}{\frac{u_2}{a}} + \underset{(15')}{\frac{(a-b)}{ab}} u_2\Big] a_3 = \frac{u_2}{b} a_3$ (16')

$\boxed{{}^R\omega^B_{u_1} = 0} \quad \boxed{{}^R\omega^B_{u_2} = \frac{1}{b} a_3}$ (17', 18')

Now $\quad {}^R\omega^{W_1} \underset{(4)}{=} {}^R\omega^D + {}^D\omega^{W_1} \underset{(11)}{=} {}^R\omega^A + {}^D\omega^{W_1} \underset{(4)}{=} \underset{(11')}{\frac{u_2}{a}} a_3 + \dot{\Omega}_1 a_2$ (19')

Solving (9', 10') for $\dot{\Omega}_1$: $\dot{\Omega}_1 = \frac{u_1 a - u_2 c}{ra}$ (20')

Thus $\quad {}^R\omega^{W_1} \underset{(19',20')}{=} u_1\left(\frac{1}{r} a_2\right) + u_2\left(\frac{1}{a} a_3 - \frac{c}{ra} a_2\right)$ (21')

or $\boxed{{}^R\omega^{W_1}_{u_1} = \frac{1}{r} a_2} \quad \boxed{{}^R\omega^{W_1}_{u_2} = \frac{1}{a} a_3 - \frac{c}{ra} a_2}$ (22', 23')

$i =$	$\omega^A_{u_i}$	$\omega^B_{u_i}$	$\omega^{W_1}_{u_i}$
1	0 (12')	0 (17')	$\frac{1}{r} a_2$ (22')
2	$\frac{1}{a} a_3$ (13')	$\frac{1}{b} a_3$ (18')	$-\frac{c}{ra} a_2 + \frac{1}{a} a_3$ (23')

I know this is probably the long way to look at it, but I'm very comfortable with what I've done. ✓