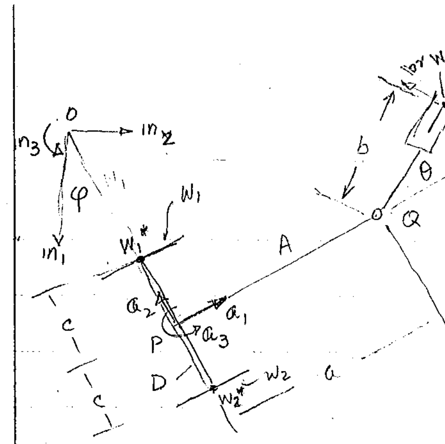
Take home exam #1

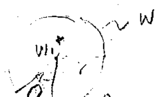
Acutel



define Reference frame R with fixed vectors n_1, n_2, n_3 : R being the surface that the system rolls on.

Let P be the point where A joins the cross axis

let S be the point directly below P on R



then $r^S = x\,n_1 + y\,n_2$

$r^P = x n_1 + y n_2 + r n_3$ $\quad\boxed{{}^R\omega^D = \dot\varphi n_3 = \dot\varphi a_3}$

let W_1^* be the center of the wheel W_1 & W_2^* be the center of the wheel W_2, W_3^* be the center of wheel W_3

Then $r^{W_1^*} = r^P + c\,a_2$ $\qquad r^{W_2^*} = r^P - c\,a_2$

then ${}^R v^{W_1^*} = {}^R v^P + c\,\frac{{}^R d a_2}{dt} = \dot x n_1 + \dot y n_2 + c\frac{{}^R d a_2}{dt} = \dot x n_1 + \dot y n_2 + c[{}^R\omega^D \times a_2]$

$= \dot x n_1 + \dot y n_2 - c\dot\varphi a_1$

${}^R v^P = \dot x n_1 + \dot y n_2$ $\qquad {}^R v^Q = {}^R v^P + {}^R\omega^A \times r^{Q/P} = {}^R v^P + {}^R\omega^D \times a(a_1) = {}^R v^P + \dot\varphi a\,a_2$

$\boxed{{}^R\omega^A = {}^R\omega^D}$ since A is fixed ⊥ D and the axis A is at an angle $\varphi + \pi/2$ to the n_1 axis $\quad r^{Q/P}$ = the radius vector from P to Q.

Thus ${}^R v^Q = \dot x n_1 + \dot y n_2 + \dot\varphi a\,a_2$

$= -\dot x[s\varphi\,a_1 + c\varphi\,a_2] + \dot y[c\varphi a_1 + s\varphi a_2] + \dot\varphi a\,a_2$

	n_1	n_2	n_3
a_1	$-s\varphi$	$c\varphi$	0
a_2	$-c\varphi$	$-s\varphi$	0
a_3	0	0	1

${}^R v^Q = (-\dot x s\varphi + \dot y c\varphi)a_1 + (\dot\varphi a - \dot x c\varphi - \dot y s\varphi)a_2$

Since $r^{W_1^*} = r^P + c\,a_2$; then is W_1^o is the point on W_1 in contact with R

then $r^{W_1^o} = r^{W_1^*} - r\,a_3$ and let $\boxed{{}^D\omega^{W_1} = \dot\Omega_1 a_2}$

rolling condition $0 = {}^R v^{W_1^o} = {}^R v^{W_1^*} + {}^R\omega^{W_1} \times r^{W_1^o/W_1^*}$ $\qquad {}^R\omega^{W_1} = {}^R\omega^D + {}^D\omega^{W_1} = \dot\varphi a_3 + \dot\Omega_1 a_2$

$= \dot x n_1 + \dot y n_2 - c\dot\varphi a_1 + [\dot\varphi a_3 + \dot\Omega_1 a_2] \times (-r a_3)$

$= \dot x n_1 + \dot y n_2 - c\dot\varphi a_1 - \dot\Omega r\,a_1$

$= -\dot x[s\varphi a_1 + c\varphi a_2] + \dot y[c\varphi a_1 - s\varphi a_2] - c\dot\varphi a_1 - \dot\Omega_1 r a_1$

$= [-\dot x s\varphi + \dot y c\varphi - c\dot\varphi - r\dot\Omega_1]a_1 + [-\dot x c\varphi - \dot y s\varphi]a_2 = 0$

$\therefore \quad \boxed{-\dot x s\varphi + \dot y c\varphi - c\dot\varphi - r\dot\Omega_1 = 0} \quad \boxed{-\dot x c\varphi - \dot y s\varphi = 0}$

Since $r^{W_2^*} = r^P - c\,a_2$ $\qquad$ ~~I will assume~~ (since system has only 2 degrees of freedom) ~~all wheels have same magnitude of angular velocity~~

$\boxed{{}^D\omega^{W_2} = \dot\Omega_2 a_2}$

define W_2^o to be the point of W_2 in contact with R ; then

$r^{W_2^o} = r^{W_2^*} - r\,a_3$

thus $0 = {}^R v^{W_2^o} = {}^R v^{W_2^*} + {}^R\omega^{W_2} \times r^{W_2^o/W_2^*}$; since the only difference between ${}^R v^{W_2^*}$ & ${}^R v^{W_1^*}$ is the replacement of c by −c then ${}^R v^{W_2^*} = \dot x n_1 + \dot y n_2 + c\dot\varphi a_1$

and ${}^R v^{W_2^o} = \dot x n_1 + \dot y n_2 + c\dot\varphi a_1 + [\dot\varphi a_3 + \dot\Omega_2 a_2] \times (-r a_3) = \dot x n_1 + \dot y n_2 + c\dot\varphi a_1 - r\dot\Omega_2 a_1$

$${}^{R}\mathbb{V}^{W_2^o} = 0 = [-\dot{x}\, s_\varphi + \dot{y}\, c_\varphi + c\dot{\varphi} - r\dot{\Omega}_2]\, a_1 + [-\dot{x}\, c_\varphi - \dot{y}\, s_\varphi]\, a_2$$

thus the constraints are $\boxed{-\dot{x}\, s_\varphi + \dot{y}\, c_\varphi + c\dot{\varphi} - r\dot{\Omega}_2 = 0}$ and $\boxed{-\dot{x}\, c_\varphi - \dot{y}\, s_\varphi = 0}$ again

now $\boxed{{}^{A}\omega^{B} = \dot{\theta}\, a_3}$ $\quad {}^{R}\mathbb{V}^{W_3^*} = {}^{R}\mathbb{V}^{Q} + {}^{R}\omega^{B} \times (b\, b_1) = {}^{R}\mathbb{V}^{Q} + {}^{R}\omega^{B} \times r^{W_3^*/Q}$

$${}^{R}\omega^{B} = {}^{R}\omega^{A} + {}^{A}\omega^{B} = \boxed{\dot{\varphi}\, a_3 + \dot{\theta}\, a_3 = {}^{R}\omega^{B}} \qquad b_1 = c_\theta\, a_1 + s_\theta\, a_2$$

$$\therefore\ {}^{R}\mathbb{V}^{W_3^*} = {}^{R}\mathbb{V}^{Q} + b[\dot{\varphi} + \dot{\theta}]\, a_3 \times (c_\theta\, a_1 + s_\theta\, a_2)$$

$$= {}^{R}\mathbb{V}^{Q} + b[\dot{\varphi} + \dot{\theta}]\, c_\theta\, a_2 - b[\dot{\varphi} + \dot{\theta}]\, s_\theta\, a_1$$

$$= [-\dot{x}\, s_\varphi + \dot{y}\, c_\varphi - b(\dot{\varphi} + \dot{\theta})\, s_\theta]\, a_1 + [\dot{\varphi} a - \dot{x}\, c_\varphi - \dot{y}\, s_\varphi + b(\dot{\varphi} + \dot{\theta})\, c_\theta]\, a_2$$

now ${}^{B}\omega^{W_3} = \dot{\Omega}_3\, b_2 = \dot{\Omega}_3 [c_\theta\, a_2 - s_\theta\, a_1]$ ~~since I assume all wheels have same~~
magnitude of angular velocity

$$0 = {}^{R}\mathbb{V}^{W_3^o} = {}^{R}\mathbb{V}^{W_3^*} + {}^{R}\omega^{W_3} \times (-r\, a_3) \qquad \text{since } r^{W_3^o/W_3^*} = -r\, a_3$$

$$= {}^{R}\mathbb{V}^{W_3^*} + [{}^{R}\omega^{B} + {}^{B}\omega^{W_3}] \times (-r\, a_3)$$

$$= {}^{R}\mathbb{V}^{W_3^*} + [(\dot{\varphi} + \dot{\theta})\, a_3 + \dot{\Omega}_3 (c_\theta\, a_2 - s_\theta\, a_1)] \times (-r\, a_3)$$

$$= {}^{R}\mathbb{V}^{W_3^*} + [-\dot{\Omega}_3\, c_\theta\, r\, a_1 - \dot{\Omega}_3\, s_\theta\, r\, a_2]$$

thus
$$0 = \{-\dot{x}\, s_\varphi + \dot{y}\, c_\varphi - b(\dot{\varphi} + \dot{\theta})\, s_\theta - \dot{\Omega}_3\, c_\theta\, r\}\, a_1 + [\dot{\varphi} a - \dot{x}\, c_\varphi - \dot{y}\, s_\varphi + b(\dot{\varphi} + \dot{\theta})\, c_\theta - \dot{\Omega}_3\, s_\theta\, r]\, a_2$$

thus our constraint eqns are

$$\{-\dot{x}\, s_\varphi + \dot{y}\, c_\varphi - b(\dot{\varphi} + \dot{\theta})\, s_\theta - \dot{\Omega}_3\, c_\theta\, r\} = 0 \quad \& \quad [\dot{\varphi} a - \dot{x}\, c_\varphi - \dot{y}\, s_\varphi + b(\dot{\varphi} + \dot{\theta})\, c_\theta - \dot{\Omega}_3\, s_\theta\, r] = 0$$

Collecting our constraint eqns.

$$-\dot{x}\, s_\varphi + \dot{y}\, c_\varphi - c\dot{\varphi} - r\dot{\Omega}_1 = 0 \qquad (1)$$

$$-\dot{x}\, c_\varphi - \dot{y}\, s_\varphi = 0 \qquad (2)$$

$$-\dot{x}\, s_\varphi + \dot{y}\, c_\varphi + c\dot{\varphi} - r\dot{\Omega}_2 = 0 \qquad (3)$$

$$-\dot{x}\, s_\varphi + \dot{y}\, c_\varphi - b(\dot{\varphi} + \dot{\theta})\, s_\theta - \dot{\Omega}_3\, c_\theta\, r = 0 \qquad (4)$$

$$\dot{\varphi} a - \dot{x}\, c_\varphi - \dot{y}\, s_\varphi + b(\dot{\varphi} + \dot{\theta})\, c_\theta - \dot{\Omega}_3\, s_\theta\, r = 0 \qquad (5)$$

this is 5 eqns in 7 unknowns $(\dot{x}, \dot{y}, \dot{\varphi}, \dot{\Omega}_1, \dot{\Omega}_2, \dot{\theta}, \dot{\Omega}_3)$

now $${}^{R}\mathbb{V}^{Q} = (-\dot{x}\, s_\varphi + \dot{y}\, c_\varphi)\, a_1 + (\dot{\varphi} a - \dot{x}\, c_\varphi - \dot{y}\, s_\varphi)\, a_2$$

subtract (3) from (2) then $-2c\dot{\varphi} - r\dot{\Omega}_1 + r\dot{\Omega}_2 = 0$

$$\therefore \quad \boxed{\dot{\varphi} = r(\dot{\Omega}_2 - \dot{\Omega}_1)/2c} \quad (6)$$

sub (6) into (1) to get

$$-\dot{x}\,S\varphi + \dot{y}\,C\varphi = r(\dot{\Omega}_2 - \dot{\Omega}_1)/2 + \frac{2r\dot{\Omega}_1}{2} = r(\dot{\Omega}_2 + \dot{\Omega}_1)/2 \quad (7)$$

using this and (2) in ${}^R\mathbf{V}^Q$

$${}^R\mathbf{V}^Q = \frac{r}{2}(\dot{\Omega}_2 + \dot{\Omega}_1)\mathbf{a}_1 + \dot{\varphi}\,a\,\mathbf{a}_2 = \frac{r}{2}(\dot{\Omega}_2 + \dot{\Omega}_1)\mathbf{a}_1 + \frac{ar}{2c}(\dot{\Omega}_2 - \dot{\Omega}_1)\mathbf{a}_2 \quad (8)$$

$$\Longrightarrow \quad \boxed{u_1 = \frac{r}{2}(\dot{\Omega}_2 + \dot{\Omega}_1)} \ (9) \quad \boxed{u_2 = \frac{ar}{2c}(\dot{\Omega}_2 - \dot{\Omega}_1)} \ (10)$$

now ${}^R\omega^A = \dot{\varphi}\,\mathbf{a}_3 = r(\dot{\Omega}_2 - \dot{\Omega}_1)/2c\,\mathbf{a}_3 = \frac{u_2}{a}\mathbf{a}_3$

$$\therefore \quad \boxed{{}^R\omega^A_{u_1} = 0 \qquad {}^R\omega^A_{u_2} = \mathbf{a}_3/a}$$

$${}^R\omega^B = {}^R\omega^A + {}^A\omega^B = \frac{u_2}{a}\mathbf{a}_3 + \dot{\theta}\,\mathbf{a}_3$$

when $\theta = 180°$ (4) becomes $-\dot{x}\,S\varphi + \dot{y}\,C\varphi = -\dot{\Omega}_3 r = r(\dot{\Omega}_2 + \dot{\Omega}_1)/2$

$$\therefore \quad \dot{\Omega}_3 = -(\dot{\Omega}_2 + \dot{\Omega}_1)/2 = -u_1/r \quad (11)$$

when $\theta = 180°$ (5) becomes (using (2))

$$\dot{\varphi}a + b(\dot{\varphi} + \dot{\theta}) = 0 \quad \therefore \quad \dot{\theta} = \dot{\varphi}\frac{(a-b)}{b} = \frac{r(\dot{\Omega}_2 - \dot{\Omega}_1)}{2c}\left(\frac{a-b}{b}\right)$$

$$\therefore \quad \boxed{\dot{\theta} = \frac{u_2}{a}\left(\frac{a-b}{b}\right)}$$

$${}^R\omega^B = \frac{u_2}{a}\mathbf{a}_3 + \frac{u_2}{a}\left(\frac{a-b}{b}\right)\mathbf{a}_3 = u_2\,\mathbf{a}_3\left[\frac{b}{ab} + \frac{a-b}{ab}\right] = \frac{u_2}{b}\mathbf{a}_3$$

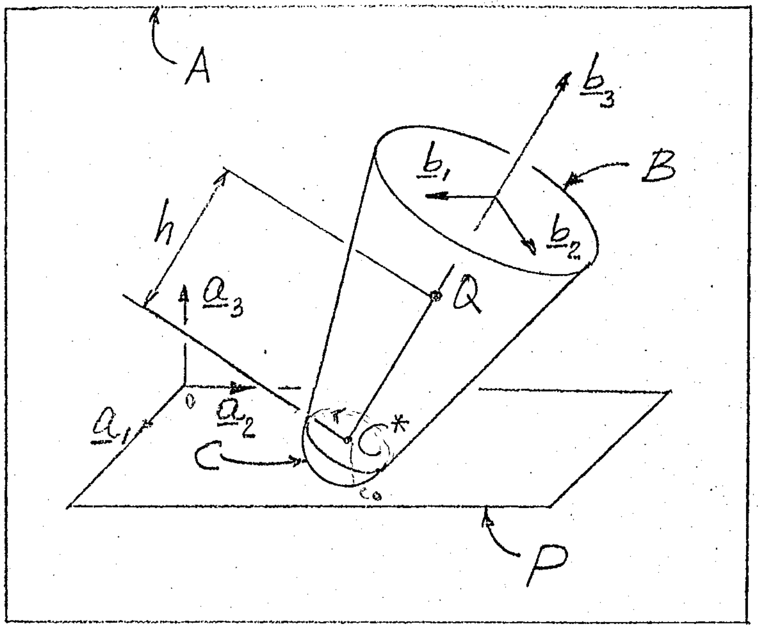
$$\boxed{{}^R\omega^B_{u_1} = 0 \qquad {}^R\omega^B_{u_2} = \frac{\mathbf{a}_3}{b}}$$

$${}^R\omega^{W_1} = {}^R\omega^D + {}^D\omega^{W_1} = {}^R\omega^A + {}^D\omega^{W_1} = \frac{u_2}{a}\mathbf{a}_3 + \dot{\Omega}_1\mathbf{a}_2$$

$$u_1\frac{a}{c} = \frac{ra}{2c}(\dot{\Omega}_2 + \dot{\Omega}_1)$$
$$u_2 = \frac{ra}{2c}(\dot{\Omega}_2 - \dot{\Omega}_1)$$
$$u_1 a/c - u_2 = \frac{ra}{2c}\cdot 2\dot{\Omega}_1 = \frac{ra}{c}\dot{\Omega}_1$$

$$\therefore \quad \dot{\Omega}_1 = \frac{u_1 a - u_2 c}{c}\cdot\frac{c}{ra} = \frac{u_1 a - u_2 c}{ra}$$

$${}^R\omega^{W_1} = \frac{u_2}{a}\mathbf{a}_3 + \frac{u_1 a - u_2 c}{ra}\mathbf{a}_2 \qquad \boxed{{}^R\omega^{W_1}_{u_1} = \frac{1}{r}\mathbf{a}_2 \qquad {}^R\omega^{W_1}_{u_2} = \frac{\mathbf{a}_3}{a} - \frac{c}{ra}\mathbf{a}_2}$$



A conical top B has a hemispherical base C that remains in contact with a plane P . (In general, C slips on P .) P is fixed in a reference frame A , and $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ are mutually perpendicular unit vectors fixed in A , with $\underline{a}_3$ normal to P and $\underline{a}_3 = \underline{a}_1 \times \underline{a}_2$. $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ are mutually perpendicular unit vectors fixed in B , with $\underline{b}_3 = \underline{b}_1 \times \underline{b}_2$ and parallel to the axis of B . The distance from the center C* of C to a certain point Q that is fixed on the axis of B is equal to h , and C has a radius r .

To bring B into a general orientation in A , one can align $\underline{b}_i$ with $\underline{a}_i$ (i=1,2,3) and then subject B successively to right-handed rotations of amounts q_1 , q_2 , and q_3 radians about lines parallel to $\underline{b}_1$, $\underline{b}_2$, and $\underline{b}_1$ (note the last subscript), respectively.

Defining generalized speeds u_1 , ..., u_5 as $u_i \triangleq \underline{\omega} \cdot \underline{b}_i$ (i=1,2,3) , $u_4 \triangleq \underline{v} \cdot \underline{a}_1$, $u_5 \triangleq \underline{v} \cdot \underline{a}_2$, where $^A\underline{\omega}^B$ is the angular velocity of B in A and $^A\underline{v}^P$ is the velocity in A of that point of C that is in contact with P , determine the partial velocity of Q in A with respect to u_1 . Express the results in terms of r , h , $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$, and trigonometric functions of q_1 , q_2 , q_3 , using the notations $s_i \triangleq \sin q_i$, $c_i \triangleq \cos q_i$ (i=1,2,3).

Figure 1 is a schematic representation of a spacecraft A that carries a boom B . B is attached to A by means of a screw S , so that B can rotate and translate relative to A simultaneously. S has a pitch p (i.e., when B performs one revolution relative to A , B advances a distance p); and the axes of B and S are perpendicular to each other.

At a certain time t , the velocity of point A* , the acceleration of point A* the angular velocity of A , and the angular acceleration of A , all in a certain reference frame N , are equal to $3\underline{a}_1$ ms^{-1} , 0 , $4\underline{a}_2$ s^{-1} , and $5\underline{a}_3$ s^{-2} , where $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ are mutually perpendicular unit vectors directed as shown; and the axis of B is parallel to $\underline{a}_1$, as indicated in Fig. 2. Assuming that the angular velocity of B in A is at all times equal to $6\underline{a}_3$ s^{-1} , determine the velocity and the acceleration of P in N at time t , P being the tip of B . (B has a length of 7m , and the distance from A* to B* , the base of B , is equal to 8m at time t .)

$^A\omega^B = 6\underline{a}_3$ find $^N v^P$.

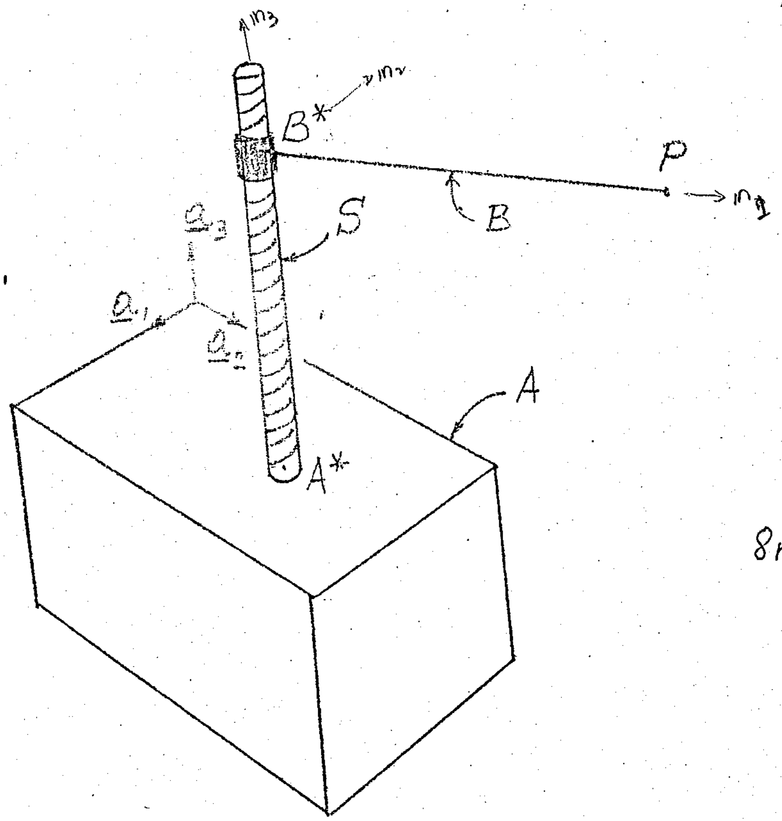


Fig. 1

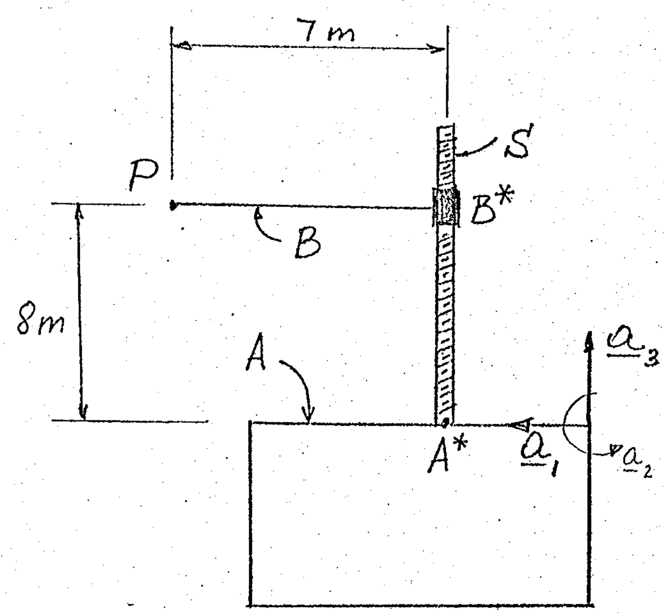
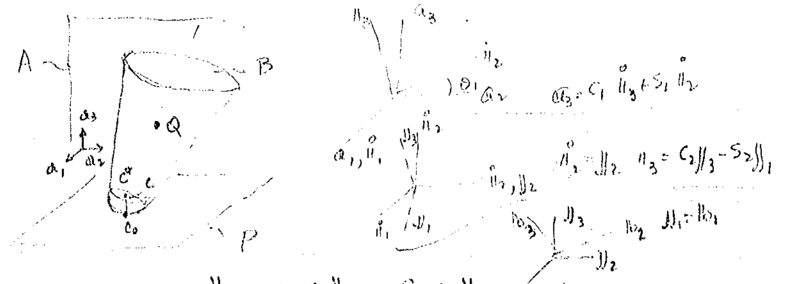


Fig. 2



Since $u_i = {}^A\omega^B \cdot \mathbb{b}_i$ ${}^A\omega^B = u_i \mathbb{b}_i$

$${}^A\mathbb{V}^{C_0} = u_4 \mathbb{a}_1 + u_5 \mathbb{a}_2$$

$$\mathbb{a}_1 = c_2 \mathbb{b}_1 + s_2 s_3 \mathbb{b}_2 + s_2 c_3 \mathbb{b}_3$$

$$\mathbb{a}_2 = s_2 s_1 \mathbb{b}_1 + (c_1 c_3 - s_1 c_2 s_3)\mathbb{b}_2 - (c_1 s_3 + s_1 c_2 c_3)\mathbb{b}_3$$

$${}^A\mathbb{V}^Q = {}^A\mathbb{V}^{C_0} + {}^A\omega^B \times \mathbb{r}^{Q/C_0}$$

$$\mathbb{r}^{Q/C_0} = h\,\mathbb{b}_3 + r\,\mathbb{a}_3$$

$$\mathbb{a}_3 = -c_1 s_2 \mathbb{b}_1 + (s_1 c_3 + c_2 c_1 s_3)\mathbb{b}_2 + (c_2 c_1 s_3 - s_1 s_3)\mathbb{b}_3$$

$${}^A\mathbb{V}^Q_{u_1} = {}^A\mathbb{V}^{C_0}_{u_1} + {}^A\omega^B_{u_1} \times \mathbb{r}^{Q/C_0}$$

$\to 0$, since ${}^A\mathbb{V}^{C_0} \neq f(u_1)$

$$= \mathbb{b}_1 \times \left[h\,\mathbb{b}_3 + r\left\{-c_1 s_2 \mathbb{b}_1 + (s_1 c_3 + c_2 c_1 s_3)\mathbb{b}_2 + (c_2 c_1 s_3 - s_1 s_3)\mathbb{b}_3\right\}\right]$$

$$= -h\,\mathbb{b}_2 + r(s_1 c_3 + c_2 c_1 s_3)\mathbb{b}_3 - r(c_2 c_1 s_3 - s_1 s_3)\mathbb{b}_2$$

now ${}^A\omega^B = \dot q_1 \mathbb{l}_1 + \dot q_2 \mathbb{j}_2 + \dot q_3 \mathbb{b}_1$

	a_1	a_2	a_3
l_1	1	0	0
l_2	0	c_1	s_1
l_3	0	$-s_1$	c_1

	l_1	l_2	l_3
j_1	c_2	0	$-s_2$
j_2	0	1	0
j_3	s_2	0	c_2

	j_1	j_2	j_3
b_1	1	0	0
b_2	0	c_3	s_3
b_3	0	$-s_3$	c_3

$${}^A\mathbb{V}^{C_0} = {}^A\mathbb{V}^{\bar A} + {}^B\mathbb{V}^{C_0}$$

$${}^B\mathbb{V}^{C_0} = \frac{{}^Bd}{dt}(-r\mathbb{a}_3) = -\dot r \mathbb{a}_3 - r\left({}^B\omega^A \times (\mathbb{a}_3)\right) = -\dot r\mathbb{a}_3 - r\left({}^B\omega^A \times \mathbb{a}_3\right) = -\dot r \mathbb{a}_3 + r({}^A\omega^B \times \mathbb{a}_3)$$

$${}^A\mathbb{V}^{\bar A} = \dot x \mathbb{a}_1 + \dot y \mathbb{a}_2 \qquad \mathbb{r}^{\bar A} = x\mathbb{a}_1 + y\mathbb{a}_2$$

$${}^A\omega^B = (\dot q_1 c_2 + \dot q_3)\mathbb{b}_1 + (\dot q_1 s_2 s_3 + \dot q_2 c_3)\mathbb{b}_2 + (\dot q_1 c_3 s_2 - \dot q_2 s_3)\mathbb{b}_3 = u_i \mathbb{b}_i$$

$$\boxed{u_1 = \dot q_1 c_2 + \dot q_3} \quad \boxed{u_2 = \dot q_1 s_2 s_3 + \dot q_2 c_3} \quad \boxed{u_3 = \dot q_1 s_2 c_3 - \dot q_2 s_3}$$

$${}^A\mathbb{V}^{C_0} = \dot x \mathbb{a}_1 + \dot y \mathbb{a}_2 - \dot r \mathbb{a}_3 + r\,({}^A\omega^B \times \mathbb{a}_3)$$

$${}^A\mathbb{V}^{C_0}\cdot\mathbb{a}_1 = \dot x + r\,({}^A\omega^B \times \mathbb{a}_3 \cdot \mathbb{a}_1)$$

$$\begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix} = \omega_2$$

$$= \boxed{\dot x + r\,(\dot q_2 c_1 + s_2 s_1 \dot q_3) = u_4}$$

$${}^A\omega^B = \dot q_1 \mathbb{a}_1 + \dot q_2(c_1\mathbb{a}_2 + s_1\mathbb{a}_3) + \dot q_3\left(c_2\mathbb{a}_1 - s_2[-s_1\mathbb{a}_2 + c_1\mathbb{a}_3]\right)$$

$$= (\dot q_1 + \dot q_3 c_2)\mathbb{a}_1 + (\dot q_2 c_1 + s_2 s_1 \dot q_3)\mathbb{a}_2 + (\dot q_2 s_1 - \dot q_3 s_2 c_1)\mathbb{a}_3$$

$${}^A\mathbb{V}^{C_0}\cdot\mathbb{a}_2 = \dot y + r\,({}^A\omega^B \times \mathbb{a}_3 \cdot \mathbb{a}_2) = \boxed{\dot y - r\,(\dot q_1 + \dot q_3 c_2) = u_5}$$

$$\begin{vmatrix} \omega_1 & \omega_2 & \omega_3 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -\omega_1$$

$${}^A\mathbb{V}^Q = {}^A\mathbb{V}^{C_0} + {}^A\omega^B \times \mathbb{r}^{Q/C_0} = \dot x\mathbb{a}_1 + \dot y\mathbb{a}_2 - \dot r\mathbb{a}_3 + r\,({}^A\omega^B \times \mathbb{a}_3) + {}^A\omega^B \times h\,\mathbb{b}_3$$

$$\mathbb{a}_1 = c_2\mathbb{b}_1 + s_2 s_3 \mathbb{b}_2 + s_2 c_3 \mathbb{b}_3$$

$$\mathbb{a}_2 = s_1 s_2 \mathbb{b}_1 + (c_1 c_3 - s_1 s_3 c_2)\mathbb{b}_2 + (-c_1 s_3 - s_1 c_2 c_3)\mathbb{b}_3$$

$$\mathbb{a}_3 = -c_1 s_2 \mathbb{b}_1 + (s_1 c_3 + c_2 c_1 s_3)\mathbb{b}_2 + (c_2 c_1 s_3 - s_1 s_3)\mathbb{b}_3$$

$${}^A\mathbb{V}^Q = \Big\{\dot x c_2 + \dot y s_1 s_2 - \dot r(-c_1 s_2) + r\Big[\text{[illegible]} + (\dot q_1 s_2 s_3 + \dot q_2 c_3)(c_2 c_1 s_3 - s_1 s_3)$$

$$- (\dot q_1 c_3 s_2 - \dot q_2 s_3)(s_1 c_3 + c_2 c_1 s_3)\Big] + h\left[\dot q_1 s_2 s_3 + \dot q_2 c_3\right]\Big\}\mathbb{b}_1 +$$

$$\Big\{ \dot{x}\, s_2 s_3 + \dot{y}\,(c_1 c_3 - s_1 s_3 c_2) - \dot{r}\,(s_1 c_3 + c_2 c_1 s_3) + r\Big[-(\dot{q}_1 c_2 + \dot{q}_3)(c_2 c_1 s_3 - s_1 s_3) - (\dot{q}_1 c_3 s_2 - \dot{q}_2 s_3)(c_2 s_2)\Big] + h\Big[-(\dot{q}_1 c_2 + \dot{q}_3)\Big]\Big\}\, \mathbb{b}_2 +$$

$$\Big\{ \dot{x}\, s_2 c_3 + \dot{y}\,(-c_1 s_3 - s_1 c_2 c_3) - \dot{r}\,(c_2 c_1 s_3 - s_1 s_3) + r\Big[(\dot{q}_1 c_2 + \dot{q}_3)(s_1 c_3 + c_2 c_1 s_3) + (\dot{q}_1 s_2 s_3 + \dot{q}_2 c_3)\, c_2 s_2\Big]\Big\}\, \mathbb{b}_3$$

$$\hat{\mathbb{V}}^{Q}_{u_1} = + r^2 (c_2^2 s_2 - s_1 s_3)\, \mathbb{b}_1 - r\,(c_2 c_1 s_3 - s_1 s_3)\, \mathbb{b}_2 - h\, \mathbb{b}_2 + r\,(s_1 c_3 + c_2 c_1 s_3)\, \mathbb{b}_3$$

1. Given ${}^N\omega^A = 4a_2$ ${}^N\alpha^A = 5a_3$ ${}^N\mathbf{v}^{A^*} = 3a_1$ ${}^N\mathbf{a}^{A^*} = 0$

$n_1 = a_1$ ${}^A\omega^B = 6a_3$ at time t

find ${}^N\mathbf{v}^P$ and ${}^N\mathbf{a}^P$ P is moving in rigid body A

$${}^N\mathbf{v}^{B^*} = {}^N\mathbf{v}^{\bar{A}} + {}^A\mathbf{v}^{B^*}$$

$${}^N\mathbf{v}^{\bar{A}} = {}^N\mathbf{v}^{A^*} + {}^N\omega^A \times \mathbf{r}^{\bar{A}/A^*}$$

$${}^N\mathbf{v}^{\bar{A}} = 3a_1 + 4a_2 \times (8a_3) = 35a_1$$

if ${}^A\omega^B = \dot{\theta} a_3 = 6a_3$ $\dot{\theta} = 6$ $\mathbf{r}^{B^*} = \frac{\theta p}{2\pi} a_3$ ${}^A\mathbf{v}^{B^*} = \frac{\dot{\theta} p}{2\pi} a_3 = \frac{6p}{2\pi} a_3$

$${}^N\mathbf{v}^{B^*} = 35a_1 + \frac{6p}{2\pi} a_3$$

$${}^N\mathbf{v}^P = {}^N\mathbf{v}^{B^*} + {}^N\omega^B \times \mathbf{r}^{P/B^*} = 35a_1 + \frac{6p}{2\pi} a_3 + {}^N\omega^B \times 7a_1$$ $\boxed{\mathbf{r}^{P/B^*} = 7a_1}$

$\boxed{{}^N\omega^B = {}^N\omega^A + {}^A\omega^B = 4a_2 + 6a_3}$ ${}^N\mathbf{v}^P = {}^N\mathbf{v}^{\bar{A}} + {}^A\mathbf{v}^{B^*} + {}^N\omega^B \times \mathbf{r}^{P/B}$

$${}^N\mathbf{v}^P = 35a_1 + \frac{6p}{2\pi} a_3 + [4a_2 + 6a_3] \times 7a_1$$

$$= 35a_1 + \frac{6p}{2\pi} a_3 + (-28a_3 + 42a_2) = 35a_1 + \left(\frac{6p}{2\pi} - 28\right) a_3 + 42a_2$$

$${}^N\mathbf{a}^P = {}^N\mathbf{a}^{B^*} + {}^N\alpha^B \times \mathbf{r}^{P/B^*} + {}^N\omega^B \times \frac{{}^N d\, \mathbf{r}^{P/B^*}}{dt}$$

$$\frac{{}^N d\, \mathbf{r}^{P/B^*}}{dt} = \frac{{}^B d\, \mathbf{r}^{P/B^*}}{dt}_{\to 0} + {}^N\omega^B \times \mathbf{r}^{P/B^*} = (4a_2 + 6a_3) \times 7a_1 = -28a_3 + 42a_2$$ $\boxed{{}^N\mathbf{v}^{P/B^*} = 42a_2 - 28a_3}$

$${}^N\alpha^B = \frac{{}^N d\, {}^N\omega^B}{dt} = \frac{{}^N d\, {}^N\omega^A}{dt} + \frac{{}^N d\, {}^A\omega^B}{dt} = {}^N\alpha^A + \frac{{}^A d\, {}^A\omega^B}{dt} + {}^N\omega^A \times {}^A\omega^B$$

$$= 5a_3 + 4a_2 \times 6a_3$$

$$\boxed{{}^N\alpha^B = 5a_3 + 24a_1}$$

to find ${}^N\mathbf{a}^{B^*}$ $\Rightarrow$ ${}^N\mathbf{a}^{B^*} = {}^A\mathbf{a}^{B^*} + {}^N\mathbf{a}^{\bar{A}} + 2\,{}^N\omega^A \times {}^A\mathbf{v}^{B^*}$

$$= \frac{\ddot{\theta} p}{2\pi} a_3 + {}^N\mathbf{a}^{\bar{A}} + 2[4a_2] \times \frac{6p}{2\pi} a_3$$

$${}^N\mathbf{a}^{\bar{A}} = {}^N\mathbf{a}^{A^*} + {}^N\alpha^A \times \mathbf{r}^{\bar{A}/A^*} + {}^N\omega^A \times \frac{{}^N d\, \mathbf{r}^{\bar{A}/A^*}}{dt}$$ ${}^N\omega^A \times \mathbf{r}^{\bar{A}/A^*}$

$$= 0 + 5a_3 \times 8a_3 + 4a_2 \times 4a_2 \times 8a_3$$

$${}^N\mathbf{a}^{\bar{A}} = \quad 4a_2 \times (+32a_1) = -128a_3$$

$$\therefore {}^N\mathbf{a}^{B^*} = \frac{\ddot{\theta} p}{2\pi} a_3 - 128a_3 + \frac{48p}{2\pi} a_1$$

$$\therefore {}^N\mathbf{a}^P = \frac{\ddot{\theta} p}{2\pi} a_3 - 128a_3 + \frac{48p}{2\pi} a_1 + 35a_2 + (4a_2 + 6a_3) \times (42a_2 - 28a_3)$$

$$-112a_1 - 252a_1$$

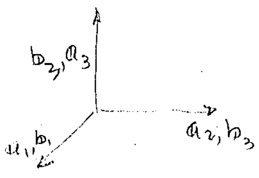
$$= \frac{\ddot{\theta} p}{2\pi} a_3 + 35a_2 - 128a_3 - 112a_1 - 252a_1 + \frac{48p}{2\pi} a_1 = \left(\frac{48p}{2\pi} - 364\right) a_1 + 35a_2 + \left(\frac{\ddot{\theta} p}{2\pi} - 128\right) a_3$$

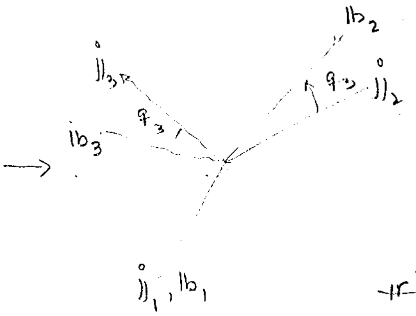
${}^N\mathbf{a}^P = {}^N\mathbf{a}^{\bar{A}} + {}^A\mathbf{a}^{B^*} + 2\,{}^N\omega^A \times {}^A\mathbf{v}^{B^*} + {}^N\alpha^B \times \mathbf{r}^{P/B^*} + {}^N\omega^B \times ({}^N\omega^B \times \mathbf{r}^{P/B^*})$ where ${}^N\mathbf{a}^{\bar{A}} = {}^N\mathbf{a}^{A^*} + {}^N\alpha^A \times \mathbf{r}^{\bar{A}/A^*} + {}^N\omega^A \times ({}^N\omega^A \times \mathbf{r}^{\bar{A}/A^*})$

poor

LEVY

1\.





	i_1	i_2	i_3
a_1	1	0	0
a_2	0	c_1	$-s_1$
a_3	0	s_1	c_1

	i_1	i_2	i_3
j_1	c_2	0	$-s_2$
j_2	0	1	0
j_3	$+s_2$	0	c_2

	j_1	j_2	j_3
b_1	1	0	0
b_2	0	c_3	s_3
b_3	0	$-s_3$	c_3

} not necessary

1\. $^A\omega^B = {}^A\omega^I + {}^I\omega^J + {}^J\omega^B$

$= \dot{q}_1 i_1 + \dot{q}_2 j_2 + \dot{q}_3 b_1$ ✓

2: Need to find $^A\underline{v}^P$, $^A v^Q$

~~$r^{c^*} = x a_1 + y a_2 + r a_3$~~ where x, y is location of common pt of P & C

~~$^A v^{c^*} = \dot{x} a_1 + \dot{y} a_2 + \dot{r} a_3$~~

~~$r^P = x a_1 + y a_2$~~

Let the point Q be the fixed point in B

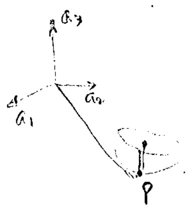
then $r^{c^*/Q} = -h b_3$ where $r^{A/B}$ denotes vector from B to A

thus $^A v^{c^*} = {}^A\omega^c \times r^{c^*/Q}$ X

$= (\dot{q}_1 i_1 + \dot{q}_2 j_2 + \dot{q}_3 b_1) \times -h b_3$

now $^A r^P = {}^A r^{c^*} - r a_3$

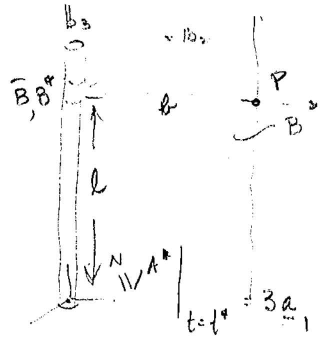
$\therefore$ $^A v^P = {}^A v^{c^*}$ since $r a_3$ is a fixed vector in A



now $^A v^Q = {}^A v^{c^*} + {}^A\omega^C \times r^{Q/c^*}$ ✓

poor

Gwen



Let a plane which contain P & ⊥ to A have fixed unit vectors b_1, b_2, b_3

How can something be perpendicular to a spacecraft?

$${}^N v^{A^*}\Big|_{t=t^*} = 3 a_1 \qquad {}^N a^{A^*}\Big|_{t=t^*} = 0$$

$${}^N \omega^A = 4 a_2 \qquad {}^N \alpha^A = 5 a_3$$

given that ${}^A \omega^B = 6 a_3$: find ${}^N v^P\Big|_t$ & ${}^N a^P\Big|_t$

$${}^N \omega^B = {}^N \omega^A + {}^A \omega^B = 4a_2 + 6a_3 \quad \text{at time } t$$ ✓

~~$${}^A r^{B^*} = {}^A r^{A^*} + {}^A r^{B^*/A} = l b_3$$~~

is ${}^A r^{B^*}$ the position vector of B^* relative to some point? Which point? What is ${}^A r^{B^*/A}$?

~~$${}^A v^{B^*} = {}^A v^{A^*} + l b_3 + {}^A\omega^B l b_3$$~~ ?@ $t = t^*$

~~$${}^A v^P = {}^A v^{B^*} + {}^A \omega^B \times b b_1$$~~

~~[illegible]~~

$${}^N v^{B^*} = {}^N v^{A^*} + {}^N \omega^B \times (l b_3)\Big|_{t=t^*} \qquad {}^N \omega^B = {}^N \omega^A + {}^A \omega^B = 4a_2 + 6a_3$$

$${}^N v^{B^*} = 3a_1 + \text{~~}4a_3 \times 8a_3\text{~~} \; (4a_2 + 6a_3) \times 8 a_3$$

$$= 3a_1 + 32 a_1 = 35 a_1$$

$${}^N v^P = {}^N v^{B^*} + {}^N \omega^B \times r \Big|_{t=t}$$

$$= 35 a_1 + [4a_2 + 6a_3] \times 7 b_1$$

Excellent

B+

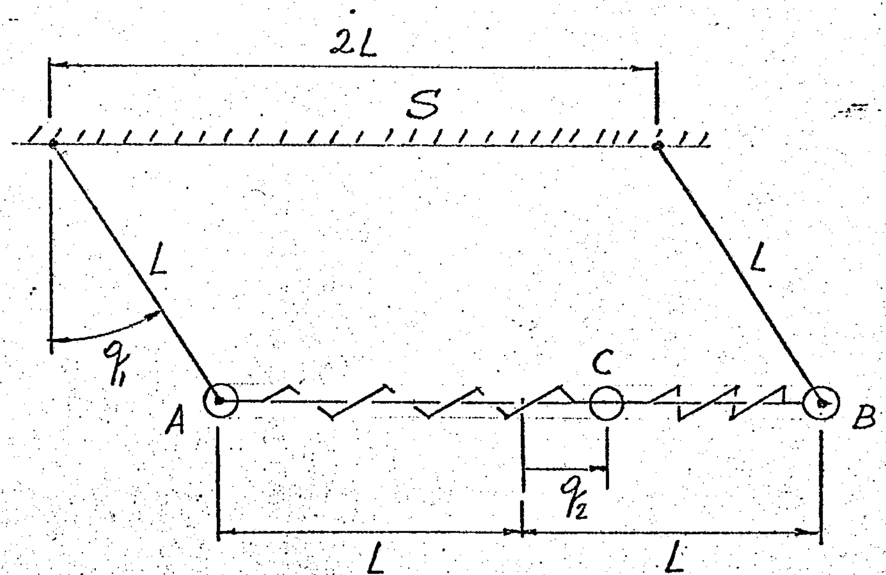
791202

(28)

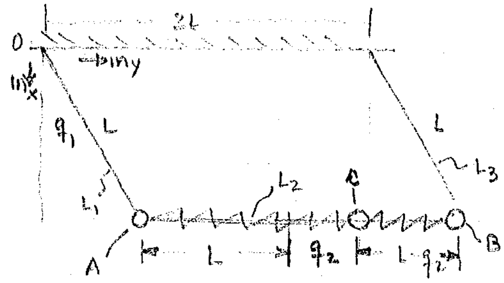
LEVY

In the sketch, A, B and C designate particles of equal mass m. A and B are attached to light rods of length L, and these rods are pinned to a horizontal support S. C can slide on a smooth, light rod of length 2L, which is pin-connected to A and B. Two light, linear springs, each of modulus k and natural length L, connect A and B to C.

Letting q_1 and q_2 measure an angle and a distance as indicated in the sketch, form two differential equations of motion for this system.



2.



1. Since L_1, L_2, L_3 are pin connected at their common joints and L_1 & L_3 are pin connected to S, the contribution of the forces at the joints to the generalized forces are zero.
2. Since C slides on a smooth rod, the contact force of C on the rod L_2 (or vise versa) contributes nothing to the generalized active force.

We must get $F_r + F_r^* = 0$ (1)

To accomplish this we first assume that L_1, L_2, L_3 are inextensible and have <u>no mass</u>. We also assume that the springs have no mass. Thus

define $\cos q_1 \triangleq c_1$ $\sin q_1 \triangleq s_1$. Let R be reference frame & O be the reference point

$\therefore$ $\mathbf{r}^A = Lc_1\mathbf{n}_x + Ls_1\mathbf{n}_y$ (2)

$\mathbf{v}^A = -L\dot{q}_1 s_1\mathbf{n}_x + L\dot{q}_1 c_1\mathbf{n}_y$ (3)

$\mathbf{a}^A = -L\mathbf{n}_x(\ddot{q}_1 s_1 + \dot{q}_1^2 c_1) + L\mathbf{n}_y(\ddot{q}_1 c_1 - \dot{q}_1^2 s_1)$ (4)

$\mathbf{r}^B = Lc_1\mathbf{n}_x + (Ls_1 + 2L)\mathbf{n}_y$ (5)

$\mathbf{v}^B = -L\dot{q}_1 s_1\mathbf{n}_x + L\dot{q}_1 c_1\mathbf{n}_y$ (6)

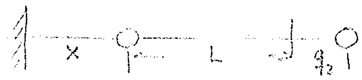
$\mathbf{a}^B = -L\mathbf{n}_x(\ddot{q}_1 s_1 + \dot{q}_1^2 c_1) + L\mathbf{n}_y(\ddot{q}_1 c_1 - \dot{q}_1^2 s_1)$ (7)

$\mathbf{r}^C = Lc_1\mathbf{n}_x + (Ls_1 + L + q_2)\mathbf{n}_y$ (8)

$\mathbf{v}^C = -L\dot{q}_1 s_1\mathbf{n}_x + (L\dot{q}_1 c_1 + \dot{q}_2)\mathbf{n}_y$ (9)

$\mathbf{a}^C = -L\mathbf{n}_x(\ddot{q}_1 s_1 + \dot{q}_1^2 c_1) + (L\ddot{q}_1 c_1 - L\dot{q}_1^2 s_1 + \ddot{q}_2)\mathbf{n}_y$ (10)

to find the force, on C due to A $\mathbf{F} = -k\Delta x\mathbf{n}$, where Δx is the change of length wrt rest length & $\mathbf{n}$ is in direction of motion

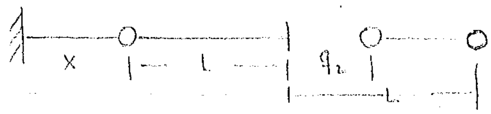


$\therefore$ $k\Delta x = k(x + L + q_2) - kx = k(L + q_2)$
but $kL = 0$ since L is the rest length $\therefore |\mathbf{F}_1^C| = kq_2$

the direction of $\mathbf{F}^C$ is toward A. Thus $\mathbf{F}_1^C = -kq_2\mathbf{n}_y$ (11)

to find the force on A due to C: $\mathbf{F}^A = -\mathbf{F}_1^C = kq_2\mathbf{n}_y$ (12)

to find the force on B due to C



$k\Delta x = k(x + 2L) - k(x + L + q_2) = k(L - q_2)$
but again $kL = 0$ since L is the rest length

$\therefore$ $|\mathbf{F}^B| = kq_2$. The direction of $\mathbf{F}^B$ is toward B. Thus $\mathbf{F}^B = kq_2\mathbf{n}_y$ (13)

to find the forces on C due to B: $\mathbf{F}_2^C = -\mathbf{F}^B = -kq_2\mathbf{n}_y$ (14)

Thus $\mathbf{F}^C = -2kq_2\mathbf{n}_y$ (15)

Now $F_r = (F_r)_I + (F_r)_G$ (16)

$$(F_r)_I = \mathbb{V}_{\dot q_r}^{A} \cdot \mathbb{F}^{A} + \mathbb{V}_{\dot q_r}^{B} \cdot \mathbb{F}^{B} + \mathbb{V}_{\dot q_r}^{C} \cdot \mathbb{F}^{C} \quad (17)$$

$$(F_r)_G = \mathbb{V}_{\dot q_r}^{A} \cdot W\mathbb{n}_x + \mathbb{V}_{\dot q_r}^{B} \cdot W\mathbb{n}_x + \mathbb{V}_{\dot q_r}^{C} \cdot W\mathbb{n}_x \quad (18)$$

Now using (3), (6), (9) we obtain

$$\mathbb{V}_{\dot q_1}^{A} = -Ls_1\mathbb{n}_x + Lc_1\mathbb{n}_y \qquad \mathbb{V}_{\dot q_2}^{A} = 0 \quad (19a,b)$$

$$\mathbb{V}_{\dot q_1}^{B} = -Ls_1\mathbb{n}_x + Lc_1\mathbb{n}_y \qquad \mathbb{V}_{\dot q_2}^{B} = 0 \quad (20a,b)$$

$$\mathbb{V}_{\dot q_1}^{C} = -Ls_1\mathbb{n}_x + Lc_1\mathbb{n}_y \qquad \mathbb{V}_{\dot q_2}^{C} = \mathbb{n}_y \quad (21a,b)$$

$$(F_1)_I = kq_2Lc_1 + kq_2Lc_1 - 2kLc_1q_2 = 0 \quad (22)$$

$$(F_2)_I = 0 + 0 - 2kq_2 = -2kq_2 \quad (23)$$

$$(F_1)_G = -Ls_1W - Ls_1W - WLs_1 = -3WLs_1 \quad (24)$$

$$(F_2)_G = 0 + 0 + 0 = 0 \quad (25)$$

$$F_1 = (F_1)_I + (F_1)_G \underset{(22,24)}{=} -3WL\sin q_1 \quad (26)$$

$$F_2 = (F_2)_I + (F_2)_G \underset{(23,25)}{=} -2kq_2 \quad (27)$$

Now

$$F_r^* = \sum \mathbb{V}_{\dot q_r}^{P_i} \cdot \mathbb{F}_i^* \quad \text{where } \mathbb{F}_i^* = -m_i\,\mathbb{a}^{P_i} \quad (28)$$

$$\therefore \quad F_1^* = \mathbb{V}_{\dot q_1}^{A} \cdot \mathbb{F}^{A*} + \mathbb{V}_{\dot q_1}^{B} \cdot \mathbb{F}^{B*} + \mathbb{V}_{\dot q_1}^{C} \cdot \mathbb{F}^{C*}$$

$$= -3m\left[-Ls_1(-L\ddot q_1 s_1 - \dot q_1^2 Lc_1) + Lc_1(\ddot q_1 Lc_1 - \dot q_1^2 Ls_1)\right] - mLc_1\ddot q_2 \quad (29)$$

$$F_2^* = \mathbb{V}_{\dot q_2}^{A} \cdot \mathbb{F}^{A*} + \mathbb{V}_{\dot q_2}^{B} \cdot \mathbb{F}^{B*} + \mathbb{V}_{\dot q_2}^{C} \cdot \mathbb{F}^{C*}$$

$$= 0 + 0 - m(L\ddot q_1 c_1 - L\dot q_1^2 s_1 + \ddot q_2) \quad (30)$$

Now F_1^* reduces to $F_1^* = -3mL^2\ddot q_1 - mL\cos q_1\,\ddot q_2$

Thus (1) becomes

$$F_1 + F_1^* = -3WL\sin q_1 - 3mL^2\ddot q_1 - mL\cos q_1\,\ddot q_2 = 0 \quad (31)$$

$$F_2 + F_2^* = -2kq_2 - m(L\ddot q_1\cos q_1 - L\dot q_1^2\sin q_1 + \ddot q_2) = 0 \quad (32)$$

Excellent

791203-1

(28)

C. LEVY

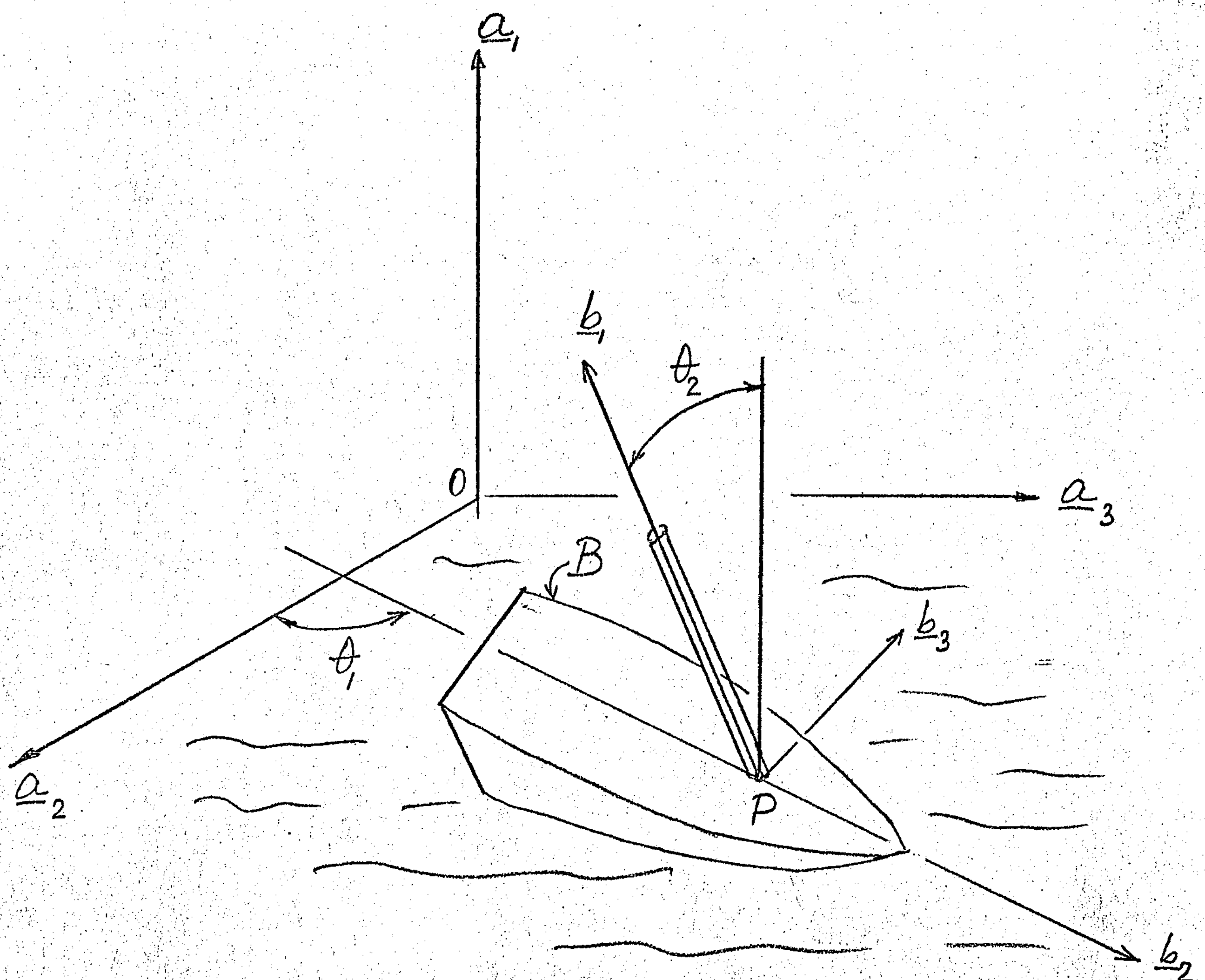
To analyze motions of a sailboat B approximately one may proceed as follows: Letting $\underline{a}_1$, $\underline{a}_2$, $\underline{a}_3$ be a dextral set of orthogonal unit vectors, with $\underline{a}_1$ pointing vertically upward, and letting $\underline{b}_1$, $\underline{b}_2$, $\underline{b}_3$ be a similar set fixed in B , with $\underline{b}_1$ parallel to the main mast and $\underline{b}_2$ pointing forward, align $\underline{b}_i$ with $\underline{a}_i$ (i = 1, 2, 3), and then subject B successively to a dextral $\underline{b}_1$ rotation of amount θ_1 (yaw) and a dextral $\underline{b}_2$ rotation of amount θ_2 (roll). Next, letting O be a point fixed in the surface of the water and P a point lying in the surface of the water and fixed in the plane of symmetry of B , express the position vector of P relative to O as $y\underline{a}_2 + z\underline{a}_3$. The quantities θ_1 , θ_2 , y , and z then characterize motions during which pitching and vertical translation are negligible.

The action of the keel (or centerboard) of B may be taken into account by assuming that the velocity of P is always parallel to $\underline{b}_2$. Under these circumstances the four quantities θ_1 , θ_2 , y , and z may be regarded as generalized coordinate q_1 , ..., q_4 , respectively, of a nonholonomic system possessing <u>three</u> degrees of freedom.

Taking the system of all contact and body forces acting on B to be equivalent to a couple of torque $\underline{R}$ together with a force $\underline{S}$ applied to P , with

$$\underline{R} = R_1\underline{b}_1 + R_2\underline{b}_2 + R_3\underline{b}_3 , \qquad \underline{S} = S_1\underline{b}_1 + S_2\underline{b}_2 + S_3\underline{b}_3$$

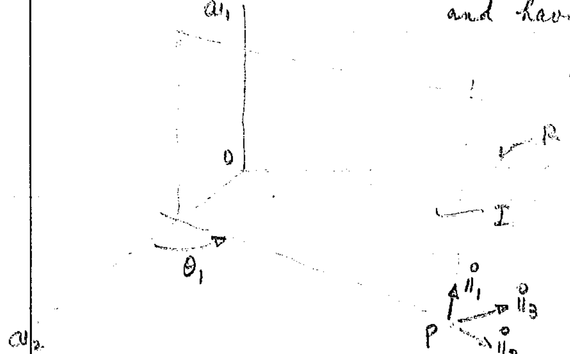
determine the generalized forces F_1 , F_2 , F_3 associated with q_1 , q_2 , q_3 , respectively.


a1
b1
θ2
0
a3
B
b3
θ1
a2
P
b2

C. LEVY
ME 231A Dynamics
Prof. Kane Fall 79

Final Examination

1. With the information given let us define the auxiliary frame I, which is ⊥ to the plane of the water surface, containing the point P as shown below and having unit vectors $\mathbb{I}_1^o, \mathbb{I}_2^o, \mathbb{I}_3^o$ forming a right handed triad fixed in I. Thus



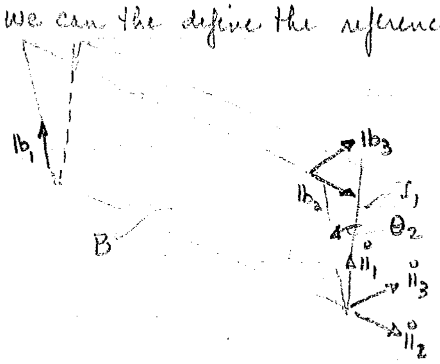
$${}^{R}\omega^{I} = \dot{\theta}_1 a_1 \quad (1)$$

	a_1	a_2	a_3
$\mathbb{I}_1^o$	1	0	0
$\mathbb{I}_2^o$	0	c_1	s_1
$\mathbb{I}_3^o$	0	$-s_1$	c_1

define $c_i \triangleq \cos\theta_i$; $s_i \triangleq \sin\theta_i$

$$\mathbb{I}_1^o \times \mathbb{I}_2^o = \mathbb{I}_3^o$$

We can the define the reference frame B as follows



Thus $${}^{I}\omega^{B} = \dot{\theta}_2 b_2 \quad (2)$$

and

	$\mathbb{I}_1^o$	$\mathbb{I}_2^o$	$\mathbb{I}_3^o$
b_1	c_2	0	$-s_2$
b_2	0	1	0
b_3	s_2	0	c_2

Since v^P is parallel to b_2 then $v^P = (v^P \cdot b_2)\, b_2$ (3)

Thus starting with $r^P = y\, a_2 + z\, a_3$ then $\frac{{}^{R}d r^P}{dt} = v^P = \dot{y}\, a_2 + \dot{z}\, a_3$ (4)

But using the definition of $\mathbb{I}_1^o, \mathbb{I}_2^o, \mathbb{I}_3^o$ and b_1, b_2, b_3 we can convert the a_i bases in terms of the b_i bases. Hence

$$v^P = \dot{y}\, a_2 + \dot{z}\, a_3 = (\dot{y} c_1 + \dot{z} s_1)\mathbb{I}_2^o + (-\dot{y} s_1 + c_1 \dot{z})\mathbb{I}_3^o$$
$$= (\dot{y} c_1 + \dot{z} s_1) b_2 + (\dot{y} s_1 s_2 - \dot{z} c_1 s_2) b_1 + (-\dot{y} s_1 c_2 + c_1 c_2 \dot{z}) b_3 \quad (5)$$

but by (3) we know that $\dot{y} s_1 s_2 - \dot{z} c_1 s_2 = 0$ (6a) and

$$-\dot{y} s_1 c_2 + c_1 c_2 \dot{z} = 0 \quad (6b)$$

~~For nontrivial solutions of $\dot{y}$ & $\dot{z}$ we must have the determinant of the coefficients be zero. The determinant $= s_1 s_2 c_1 c_2 - s_1 s_2 c_1 c_2 \equiv 0 \quad \forall\ \theta_1$ and θ_2~~

We can then solve either (6a) or (6b) for $\dot{z}$ in terms of $\dot{y}$. Solving (6b) we find

$$\dot{z} = \frac{s_1}{c_1}\dot{y} \quad (7)$$

Now from (5) $v^P = (\dot{y} c_1 + \dot{z} s_1) b_2$; using (7) in this relation gives

$$v^P = (\dot{y} c_1 + \dot{y} s_1^2/c_1) b_2 = \dot{y}/c_1\, b_2 \quad (8)$$

we also have from (1) and (2) that ${}^{R}\omega^{B} = {}^{R}\omega^{I} + {}^{I}\omega^{B} = \dot{\theta}_1 \mathbb{a}_1 + \dot{\theta}_2 \mathbb{b}_2$ (9)

To find the F_r's we employ the relationship for a rigid body

$$F_r = {}^{R}\mathbb{W}^{P}_{\dot{q}_r} \cdot \mathbb{S} + {}^{R}\omega^{B}_{\dot{q}_r} \cdot \mathbb{R} \qquad (10)$$

Since we only have 3 degrees of freedom but 4 generalized coordinates it was necessary to find a relationship between 2 of the generalized coordinates, i.e. eqn (7)

Therefore from (8) and (9) and employing $\theta_1 = q_1$, $\theta_2 = q_2$, $y = q_3$

$${}^{R}\mathbb{W}^{P}_{\dot{q}_1} = 0 \qquad {}^{R}\mathbb{W}^{P}_{\dot{q}_2} = 0 \qquad {}^{R}\mathbb{W}^{P}_{\dot{q}_3} = \mathbb{b}_2/c_1 \qquad (11a,b,c)$$

and

$${}^{R}\omega^{B}_{\dot{q}_1} = \mathbb{a}_1 \qquad {}^{R}\omega^{B}_{\dot{q}_2} = \mathbb{b}_2 \qquad {}^{R}\omega^{B}_{\dot{q}_3} = 0 \qquad (12a,b,c)$$

Hence with the definitions

$$\mathbb{R} \stackrel{\wedge}{=} \sum_{i=1}^{3} R_i \mathbb{b}_i \qquad \mathbb{S} \stackrel{\wedge}{=} \sum_{i=1}^{3} S_i \mathbb{b}_i \qquad (13a,b)$$

Then

$$F_1 \underset{(11a,12a,13)}{=} {}^{R}\omega^{B}_{\dot{q}_1} \cdot \mathbb{R} + {}^{R}\mathbb{W}^{P}_{\dot{q}_1} \cdot \mathbb{S} = R_1 \cos\theta_2 + R_3 \sin\theta_2 \qquad (14)$$

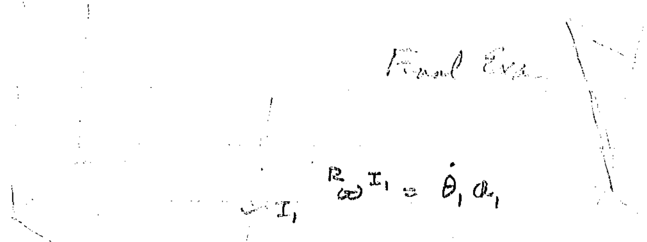
here we've used implicitly the relation that $\mathbb{a}_1 = \mathbb{i}^0_1 = c_2 \mathbb{b}_1 + s_2 \mathbb{b}_3$

$$F_2 \underset{(11b,12b,13)}{=} {}^{R}\omega^{B}_{\dot{q}_2} \cdot \mathbb{R} + {}^{R}\mathbb{W}^{P}_{\dot{q}_2} \cdot \mathbb{S} = R_2 \qquad (15)$$

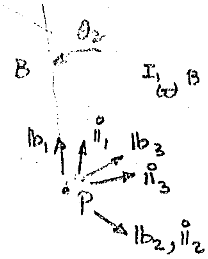
and

$$F_3 \underset{(11c,12c,13)}{=} {}^{R}\omega^{B}_{\dot{q}_3} \cdot \mathbb{R} + {}^{R}\mathbb{W}^{P}_{\dot{q}_3} \cdot \mathbb{S} = \frac{S_2}{c_1} = S_2 \sec\theta_1 = S_2 \sqrt{1+\tan^2\theta_1} \qquad (16)$$

Final Exa



$^{R}\omega^{I_1} = \dot{\theta}_1 a_1$



$^{I_1}\omega^{B} = +\dot{\theta}_2 b_2$

$\therefore\ ^{R}\omega^{B} = \dot{\theta}_1 a_1 + \dot{\theta}_2 b_2$

	a_1	a_2	a_3
i_1	1	0	0
i_2	0	c_1	s_1
i_3	0	$-s_1$	c_1

	i_1	i_2	i_3
b_1	c_2	0	$-s_2$
b_2	0	1	0
b_3	s_2	0	c_2

$r^P = y a_2 + z a_3$ no pitching, vertical translation

$v^P = b\, b_2 \Rightarrow \theta_1 = q_1,\ \theta_2 = q_2,\ y = q_3,\ z = q_4$ but there are only 3 degrees of freedom. The fourth is removed by the fact that $v^P \parallel b_2$.

All contact & body forces can be made equivalent to $R = \sum R_i b_i$ & $S = \sum S_i b_i$ acting at P where R is a torque & S is the force

What we use: $F_r = {}^{R}\omega^{B}_{\dot{q}_r} \cdot R + {}^{R}v^{P}_{\dot{q}_r} \cdot S$

we must find ${}^{R}\omega^{B}_{\dot{q}_r}$, ${}^{R}v^{P}_{\dot{q}_r}$ in terms of b_i for simplicity

$^{R}v^P = \dot{y} a_2 + \dot{z} a_3 = \dot{y}[c_1 i_2 - s_1 i_3] + \dot{z}[s_1 i_2 + c_1 i_3]$

$= (\dot{y} c_1 + \dot{z} s_1) i_2 + (-\dot{y} s_1 + c_1 \dot{z}) i_3 = (\dot{y} c_1 + \dot{z} s_1) b_2 + (-\dot{y} s_1 + c_1 \dot{z}) \cdot (-s_2 b_1 + c_2 b_3)$

$\therefore (\dot{y} s_1 s_2 - c_1 \dot{z} s_2) = 0$ and $(-\dot{y} s_1 c_2 + c_1 \dot{z} c_2) = 0$

$\therefore \begin{bmatrix} s_1 s_2 & -c_1 s_2 \\ -s_1 c_2 & c_1 c_2 \end{bmatrix} \begin{pmatrix} \dot{y} \\ \dot{z} \end{pmatrix} = 0 \Rightarrow s_1 c_1 s_2 c_2 - s_1 c_1 s_2 c_2 = 0$ for nontriv $\dot{y}, \dot{z}$

$\therefore \boxed{\dot{y} = \dfrac{c_1 c_2 \dot{z}}{s_1 c_2}}$ or $\boxed{\dot{z} = \dfrac{s_1 c_2 \dot{y}}{c_1 c_2}}$

$\therefore {}^{R}v^{P} = \left(\dot{y} c_1 + \dfrac{s_1 c_2}{c_1 c_2} \cdot s_1 \dot{y}\right) b_2 = \dot{y}\left(c_1 + \dfrac{s_1^2 c_2}{c_1 c_2}\right) b_2 = \dot{y} c_2 \left(\dfrac{c_1^2 + s_1^2}{c_1 c_2}\right) b_2 = \dot{y} \dfrac{b_2}{c_1}$

${}^{R}v^{P}_{\dot{q}_1} = 0 \quad {}^{R}v^{P}_{\dot{q}_2} = 0 \quad {}^{R}v^{P}_{\dot{q}_3} = \dfrac{b_2}{c_1}$

${}^{R}\omega^{B} = \dot{\theta}_1 a_1 + \dot{\theta}_2 b_2 \quad \therefore {}^{R}\omega^{B}_{\dot{q}_1} = a_1 \quad {}^{R}\omega^{B}_{\dot{q}_2} = +b_2 \quad {}^{R}\omega^{B}_{\dot{q}_3} = 0$

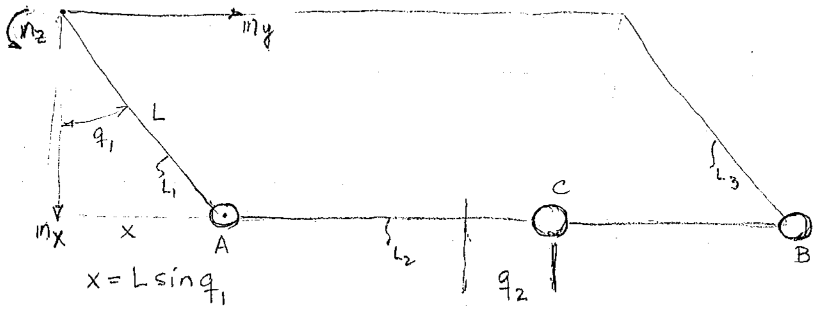
${}^{R}\omega^{B}_{\dot{q}_1} = i_1 = c_2 b_1 + s_2 b_3$

$\therefore F_1 = [c_2 b_1 + s_2 b_3] \cdot [R_1 b_1 + R_2 b_2 + R_3 b_3] + [0] \cdot \sum S_i b_i = {}^{R}\omega^{B}_{\dot{q}_1} \cdot R + {}^{R}v^{P} \cdot S$

$F_1 = c_2 R_1 + s_2 R_3 = R_1 \cos\theta_2 + R_3 \sin\theta_2$

$$F_2 = (+\mathbb{b}_2)\cdot\mathbb{R} + \underset{\sim}{0}\cdot\mathbb{S} = +R_2 = {}^{R}\omega^{B_1}_{\dot{q}_2}\cdot\mathbb{R} + {}^{R}\mathbb{V}^{P}_{\dot{q}_2}\cdot\mathbb{S}$$

$$F_3 = {}^{R}\omega^{B}_{\dot{q}_3}\cdot\mathbb{R} + {}^{R}\mathbb{V}^{P_1}_{\dot{q}_3}\cdot\mathbb{S} = \underset{\sim}{0}\cdot\Sigma R_i\mathbb{b}_i + \frac{\mathbb{b}_2}{c_1}\cdot(S_1\mathbb{b}_1 + S_2\mathbb{b}_2 + S_3\mathbb{b}_3) = \frac{S_2}{c_1} = S_2\sec\theta_1$$



$x = L\sin q_1$

Since C slides on a smooth rod the contact force of c on the rod (or vice versa) contributes nothing to the generalized active force F_r

$r^A = L\cos q_1\, n_x + L\sin q_1\, n_y$ $\qquad r^{L_1^*} = r^A/2 \Rightarrow v^{L_1^*} = \frac{v^A}{2} \qquad a^{L_1^*} = \frac{a^A}{2}$

$v^A = -L\sin q_1\, \dot q_1\, n_x + L\cos q_1\, \dot q_1\, n_y$

$r^C = L\cos q_1\, n_x + (L\sin q_1 + L + q_2)\, n_y$ $\qquad r^{L_2^*} = r^A + L n_x \qquad v^{L_2^*} = v^A;\ a^{L_2^*} = a^A$

$v^C = -\dot q_1 L\sin q_1\, n_x + (\dot q_1 L\cos q_1 + \dot q_2)\, n_y$

$r^B = L\cos q_1\, n_x + (2L + L\sin q_1)\, n_y$ $\qquad r^{L_3^*} = r^{L_1^*} + 2L n_x \qquad v^{L_3^*} = v^{L_1^*} = \frac{v^A}{2};$

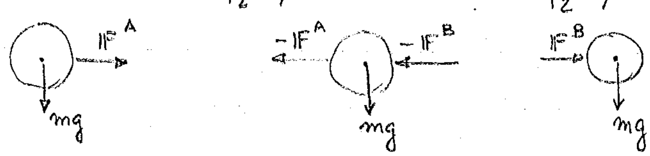
$v^B = -L\dot q_1 \sin q_1\, n_x + \dot q_1 L\cos q_1\, n_y$ $\qquad a^{L_3^*} = \frac{a^A}{2}$

$a^A = -(\ddot q_1 L\sin q_1 + L\cos q_1\, \dot q_1^2)\, n_x + (\ddot q_1 L\cos q_1 - \dot q_1^2 L\sin q_1)\, n_y$

$a^C = -(\ddot q_1 L\sin q_1 + \dot q_1^2 L\cos q_1)\, n_x + (\ddot q_1 L\cos q_1 - \dot q_1^2 L\sin q_1 + \ddot q_2)\, n_y$

$a^B = -(L\ddot q_1 \sin q_1 + L\dot q_1^2 \cos q_1)\, n_x + (\ddot q_1 L\cos q_1 - \dot q_1^2 L\sin q_1)\, n_y$

the force on A is $F^A = +k q_2 n_y \qquad F^C = -2k q_2 n_y \qquad F^B = k q_2 n_y$



Since L_1, L_2, L_3 are pin connected at their common joints and L_1 & L_3 are pin connected to S, the contribution of the forces at the joints to the generalized forces are zero,

Now $(F_r)_I = \sum \tilde{v}_{\dot q_r}^{P_i} \cdot F_i = \tilde{v}_{\dot q_r}^{A} \cdot F_A + \tilde{v}_{\dot q_r}^{B} \cdot F_B + \tilde{v}_{\dot q_r}^{C} \cdot F_C$

$(F_r)_G = \sum_i \tilde{v}_{\dot q_r}^{*} \cdot m_i g\, n_x = v_{\dot q_r}^{L_1^*} \cdot Mg n_x + v_{\dot q_r}^{L_2^*} \cdot 2Mg n_x + v_{\dot q_r}^{L_3^*} \cdot Mg n_x$
$\qquad + v_{\dot q_r}^{A} \cdot mg n_x + v_{\dot q_r}^{B} \cdot mg n_x + v_{\dot q_r}^{C} \cdot mg n_x$

$\tilde{v}_{\dot q_1}^{A} = -L\sin q_1\, n_x + L\cos q_1\, n_y \qquad \tilde{v}_{\dot q_2}^{A} = 0 \qquad v_{\dot q_1}^{L_1^*} = \frac{1}{2}\tilde{v}_{\dot q_1}^{A} \qquad \tilde{v}_{\dot q_2}^{A^*} = 0$

$\tilde{v}_{\dot q_1}^{B} = -L\sin q_1\, n_x + L\cos q_1\, n_y \qquad \tilde{v}_{\dot q_2}^{B} = 0 \qquad v_{\dot q_1}^{L_2^*} = v_{\dot q_1}^{A} \qquad v_{\dot q_2}^{L_2^*} = 0$

$\tilde{v}_{\dot q_1}^{C} = -L\sin q_1\, n_x + L\cos q_1\, n_y \qquad \tilde{v}_{\dot q_2}^{C} = n_y \qquad v_{\dot q_1}^{L_3^*} = \frac{1}{2} v_{\dot q_1}^{A} \qquad v_{\dot q_2}^{L_3^*} = 0$

Assume rods are massless $M = 0$

$(F_1)_I = kLq_2\cos q_1 + kLq_2\cos q_1 - 2kLq_2\cos q_1 = 0$

$(F_2)_I = -2kq_2$

$(F_1)_G = -\frac{1}{2}MgL\sin q_1 + (-2LMg\sin q_1) + -\frac{L}{2}Mg\sin q_1 - mgL\sin q_1 - mgL\sin q_1 - mgL\sin q_1$
$\qquad -3MLg\sin q_1 - 3mgL\sin q_1$

$$(F_2)_{G_1} = 0+0+0+0+0+0 = 0$$

$$\therefore \quad F_1 = -3MLg\sin q_1 - 3mLg\sin q_1 \qquad\qquad F_1 = -3mLg\sin q_1$$

$$F_2 = -2kq_2 \qquad\qquad F_2 = -2kq_2$$

$$F_r^* = \sum \tilde{\mathbb{V}}_{\dot{q}_r}^{P_i} \cdot \mathbb{F}_i^* \quad \text{where} \quad \mathbb{F}_i^* = -m_i \mathbb{a}_i^{P_i} \qquad \mathbb{F}^{A*} = -m\mathbb{a}^A \quad \mathbb{F}^{B*} = -m\mathbb{a}^B$$

$$\mathbb{F}^{C*} = -m\mathbb{a}^C$$

$$F_1^* = \mathbb{V}_{\dot{q}_1}^{A} \cdot \mathbb{F}^{A*} + \mathbb{V}_{\dot{q}_1}^{B} \cdot \mathbb{F}^{B*} + \mathbb{V}_{\dot{q}_1}^{C} \cdot \mathbb{F}^{C*}$$

$$= -m\left[-L\sin q_1(-\ddot{q}_1 L\sin q_1 + L\cos q_1\,\dot{q}_1^2) + L\cos q_1(\ddot{q}_1 L\cos q_1 - \dot{q}_1^2 L\sin q_1)\right]$$

$$-m\left[-L\sin q_1(-L\ddot{q}\sin q_1 + L\dot{q}_1^2\cos q_1) + L\cos q_1(\ddot{q}_1 L\cos q_1 - \dot{q}_1^2 L\sin q_1)\right]$$

$$-m\left[-L\sin q_1(-\ddot{q}_1 L\sin q_1 - L\dot{q}_1^2\cos q_1) + L\cos q_1(\ddot{q}_1 L\cos q_1 - \dot{q}_1^2 L\sin q_1 + \ddot{q}_2)\right]$$

$$F_2^* = \mathbb{V}_{\dot{q}_2}^{A} \cdot \mathbb{F}^{A*} + \mathbb{V}_{\dot{q}_2}^{B} \cdot \mathbb{F}^{B*} + \mathbb{V}_{\dot{q}_2}^{C} \cdot \mathbb{F}^{C*}$$

$$= 0 + 0 + (-m)(\ddot{q}_1 L\cos q_1 - \dot{q}_1^2 L\sin q_1 + \ddot{q}_2)$$

$$F_2 + F_2^* = -2kq_2 - m\ddot{q}_1 L\cos q_1 + m\dot{q}_1^2 L\sin q_1 - m\ddot{q}_2 = 0$$

$$F_1 + F_1^* = -3mLg\sin q_1 - 3mL^2\ddot{q}_1 - mL\cos q_1\,\ddot{q}_2 = 0$$

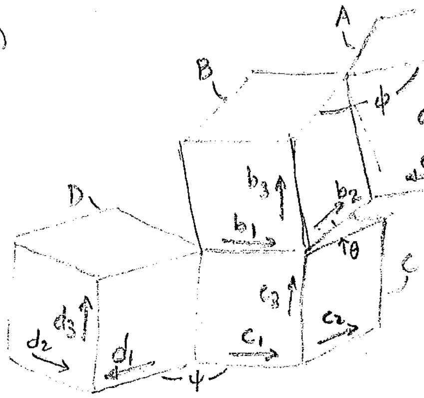
$$F_1^* = \left[3mL\sin q_1(-\ddot{q}_1 L\sin q_1 - L\cos q_1\,\dot{q}_1^2) - 3mL\cos q_1(\ddot{q}_1 L\cos q_1 - \dot{q}_1^2 L\sin q_1)\right] - mL\cos q_1\,\ddot{q}_2$$

$$-3mL^2\ddot{q}_1\sin^2 q_1 - 3mL^2\sin q_1\cos q_1\,\dot{q}_1^2 - 3mL\cos^2 q_1\,\ddot{q}_1 + 3mL^2\dot{q}_1^2\sin q_1\cos q_1 - mL\cos q_1\,\ddot{q}_2$$

$$= -3mL^2\ddot{q}_1 - mL\cos q_1\,\ddot{q}_2$$

Problem Set #1

1(a)



- $^B\underset{\sim}{a}_1 = \cos\phi\, \underset{\sim}{b}_1 + \sin\phi\, \underset{\sim}{b}_2 \quad \therefore \quad \frac{^B\partial a_1}{\partial\phi} = -\sin\phi\, \underset{\sim}{b}_1 + \cos\phi\, \underset{\sim}{b}_2$

  and $\left|\frac{^B\partial a_1}{\partial \phi}\right| = (\cos^2\phi + \sin^2\phi)^{1/2} = 1$
- $^B\underset{\sim}{b}_1 = \underset{\sim}{b}_1 \quad \therefore \quad \frac{^B\partial b_1}{\partial\phi} = 0 \quad \therefore \quad \left|\frac{^B\partial b_1}{\partial\phi}\right| = 0$
- $^B\underset{\sim}{a}_3 = \underset{\sim}{b}_3 \quad \therefore \quad \frac{^B\partial a_3}{\partial\phi} = \frac{^B\partial b_3}{\partial\phi} = 0 \quad \therefore \quad \left|\frac{^B\partial b_3}{\partial\phi}\right| = 0$ (a_3 not in ref frame)
- $^B\underset{\sim}{b}_2 = \underset{\sim}{b}_2 \quad \therefore \quad \frac{^B\partial b_2}{\partial\theta} = 0 \quad \therefore \quad \left|\frac{^B\partial b_2}{\partial\theta}\right| = 0$ { b_2 changes dir w/r C but not B
- $^C\underset{\sim}{b}_2 = \cos\theta\, \underset{\sim}{c}_2 + \sin\theta\, \underset{\sim}{c}_3 \quad \therefore \quad \frac{^C\partial b_2}{\partial\theta} = -\sin\theta\, \underset{\sim}{c}_2 + \cos\theta\, \underset{\sim}{c}_3$

  and $\left|\frac{^C\partial b_2}{\partial\theta}\right| = (\sin^2\theta + \cos^2\theta)^{1/2} = 1$
- $^D\underset{\sim}{c}_2 = \cos\psi\, \underset{\sim}{d}_2 - \sin\psi\, \underset{\sim}{d}_1 \qquad ^D\underset{\sim}{c}_3 = \underset{\sim}{d}_3$

  $\therefore \quad ^D\underset{\sim}{b}_2 = \cos\theta\cos\psi\, \underset{\sim}{d}_2 - \cos\theta\sin\psi\, \underset{\sim}{d}_1 + \sin\theta\, \underset{\sim}{d}_3$

  $\therefore \quad \frac{^D\partial b_2}{\partial\theta} = -\sin\theta\, \underset{\sim}{c}_2 + \cos\theta\, \underset{\sim}{d}_3 \implies \left|\frac{^D\partial b_2}{\partial\theta}\right| = 1$
- $^C\underset{\sim}{b}_2 = \cos\theta\, \underset{\sim}{c}_2 + \sin\theta\, \underset{\sim}{c}_3 \qquad \frac{^C\partial b_2}{\partial\psi} = 0 \quad \therefore \quad \left|\frac{^C\partial b_2}{\partial\psi}\right| = 0$
- $^D\underset{\sim}{b}_2 = \cos\theta\cos\psi\, \underset{\sim}{d}_2 - \cos\theta\sin\psi\, \underset{\sim}{d}_1 + \sin\theta\, \underset{\sim}{d}_3 \quad \therefore \quad \frac{^D\partial b_2}{\partial\psi} = -\cos\theta\sin\psi\, \underset{\sim}{d}_2 - \cos\theta\cos\psi\, \underset{\sim}{d}_1 = -\cos\theta\, ^D\underset{\sim}{b}_1$

  thus $\left|\frac{^D\partial b_2}{\partial\psi}\right| = |\cos\theta|(\sin^2\psi + \cos^2\psi)^{1/2} = |\cos\theta|$
- $^D\underset{\sim}{b}_1 = \cos\psi\, \underset{\sim}{d}_1 + \sin\psi\, \underset{\sim}{d}_2 = {^D\underset{\sim}{c}_1}$ Now $^D a_1 = \cos\phi\, ^D\underset{\sim}{b}_1 + \sin\phi\, ^D\underset{\sim}{b}_2 = \cos\phi\cos\psi\, \underset{\sim}{d}_1 + \cos\phi\sin\psi\, \underset{\sim}{d}_2$

  $- \sin\phi\cos\theta\sin\psi\, \underset{\sim}{d}_1 + \sin\phi\cos\theta\cos\psi\, \underset{\sim}{d}_2$

  $+ \sin\phi\sin\theta\, \underset{\sim}{d}_3$

$$\frac{^D\partial a_1}{\partial\psi} = (-\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi)\underset{\sim}{d}_1 + (\cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi)\underset{\sim}{d}_2$$

$$\left|\frac{^D\partial a_1}{\partial\psi}\right| = (\cos^2\phi\cdot 1 + \sin^2\phi\cos^2\theta\cdot 1)^{1/2}$$

1(b) $^B\underset{\sim}{a}_1 = \cos\phi\, \underset{\sim}{b}_1 + \sin\phi\, \underset{\sim}{b}_2 \qquad ^C\underset{\sim}{b}_1 = \underset{\sim}{c}_1 \qquad ^C\underset{\sim}{b}_2 = \cos\theta\, \underset{\sim}{c}_2 + \sin\theta\, \underset{\sim}{c}_3$

$\therefore \, ^B\underset{\sim}{a}_1 = \cos\phi\, \underset{\sim}{c}_1 + \sin\phi\, (\cos\theta\, \underset{\sim}{c}_2 + \sin\theta\, \underset{\sim}{c}_3) \qquad \frac{^C\partial a_1}{\partial\theta} = \sin\phi\, (-\sin\theta\, \underset{\sim}{c}_2 + \cos\theta\, \underset{\sim}{c}_3)$

$= \sin\phi\, ^C\underset{\sim}{b}_3 = \sin\phi\, ^C\underset{\sim}{a}_3$

$\therefore \quad \frac{^C\partial a_1}{\partial\theta} = \omega_1 \underset{\sim}{a}_1 + \omega_2 \underset{\sim}{a}_2 + \omega_3 \underset{\sim}{a}_3 \implies \omega_1 = \omega_2 = 0 \quad \omega_3 = \sin\phi$

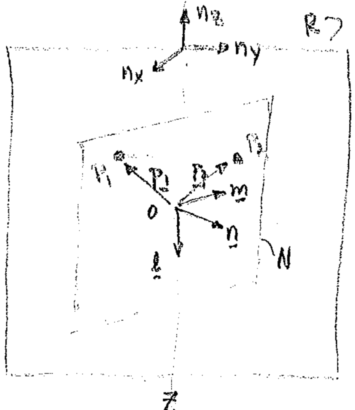
1(c) $\left.\frac{^C da_1}{dt}\right|_{t^*} = \frac{^C\partial a_1}{\partial\theta}\left.\frac{\partial\theta}{\partial t}\right|_{t^*} + \frac{^C\partial a_1}{\partial\phi}\left.\frac{\partial\phi}{\partial t}\right|_{t^*} + \frac{^C\partial a_1}{\partial\psi}\left.\frac{\partial\psi}{\partial t}\right|_{t^*}$ since $^C a_1 = \cos\phi\, \underset{\sim}{c}_1 + \sin\phi\cos\theta\, \underset{\sim}{c}_2 + \sin\phi\sin\theta\, \underset{\sim}{c}_3$

$\frac{^C\partial a_1}{\partial\theta} = \sin\phi\, ^C\underset{\sim}{a}_3 \qquad \frac{^C\partial a_1}{\partial\phi} = -\sin\phi\, \underset{\sim}{c}_1 + \cos\phi\, (^C\underset{\sim}{b}_2)$

but $\underset{\sim}{c}_1 = {^C\underset{\sim}{b}_1} = \cos\phi\, \underset{\sim}{a}_1 - \sin\phi\, \underset{\sim}{a}_2 \qquad ^C\underset{\sim}{b}_2 = \cos\phi\, \underset{\sim}{a}_2 + \sin\phi\, \underset{\sim}{a}_1 \quad \therefore \quad \frac{^C\partial a_1}{\partial\phi} = \underset{\sim}{a}_2$

and $^C a_1 \neq$ fn of $\psi \implies \frac{^C da_1}{dt} = (\sin\phi\, ^C\underset{\sim}{a}_3)\cdot 6\,\text{rad/sec} + (\underset{\sim}{a}_2)\cdot 4\,\text{rad/sec} = \frac{1}{2}\cdot 6\, \underset{\sim}{a}_3 + 4\, \underset{\sim}{a}_2 = 3\underset{\sim}{a}_3 + 4\underset{\sim}{a}_2$

1(d)



$\underset{\sim}{\ell} = -\underset{\sim}{n}_z$

$\underset{\sim}{n} = \underset{\sim}{n}_x \cos\omega t + \underset{\sim}{n}_y \sin\omega t$

$\underset{\sim}{m} = \underset{\sim}{n}_y \cos\omega t - \underset{\sim}{n}_x \sin\omega t$

	n_x	n_y	n_z
ℓ	0	0	-1
n	$\cos\omega t$	$\sin\omega t$	0
m	$-\sin\omega t$	$\cos\omega t$	0

$\underset{\sim}{p_1} = a_{11}\underset{\sim}{\ell} + a_{12}\underset{\sim}{m} = -a_{11}\underset{\sim}{n}_z + a_{12}\cos\omega t\, \underset{\sim}{n}_y - a_{12}\sin\omega t\, \underset{\sim}{n}_x$

$\underset{\sim}{p_2} = a_{21}\underset{\sim}{\ell} + a_{22}\underset{\sim}{m} = -a_{21}\underset{\sim}{n}_z + a_{22}\cos\omega t\, \underset{\sim}{n}_y - a_{22}\sin\omega t\, \underset{\sim}{n}_x$

Now $\underset{\sim}{p_1} - \underset{\sim}{p_2} = (a_{11} - a_{21})\underset{\sim}{\ell} + (a_{12} - a_{22})\underset{\sim}{m}$

$x_1 = -a_{12}\sin\omega t \quad y_1 = a_{12}\cos\omega t \quad z_1 = -a_{11}$

$x_2 = -a_{22}\sin\omega t \quad y_2 = a_{22}\cos\omega t \quad z_2 = -a_{21}$

$\therefore |\underset{\sim}{p_1} - \underset{\sim}{p_2}|^2 = (a_{11} - a_{21})^2 + (a_{12} - a_{22})^2 = L^2$

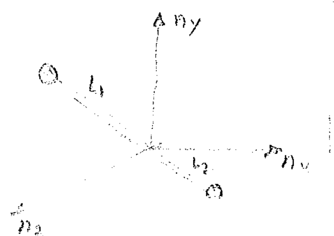
Now x_1 $\boxed{x_1\cos\omega t + y_1\sin\omega t = 0 = f_1(x_1, y_1, z_1, x_2, y_2, z_2, t)}$ $\quad p_1 \cdot n = 0$

$\boxed{x_2\cos\omega t + y_2\sin\omega t = 0 = f_2(x_1, y_1, z_1, x_2, y_2, z_2, t)}$ $\quad p_2 \cdot n = 0$

$|\underset{\sim}{p_1} - \underset{\sim}{p_2}|^2 = (a_{11} - a_{21})^2 + (a_{12} - a_{22})^2 = L^2$

$\boxed{f_3 = |\underset{\sim}{p_1} - \underset{\sim}{p_2}|^2 - L^2 = (z_1 - z_2)^2 + (x_1 - x_2)^2 + (y_1 - y_2)^2 - L^2 = 0}$

1(e)



since P_1 and P_2 move in the plane of n_y & n_x then

$\boxed{z_1 = z_2 = 0}$ and $\underset{\sim}{p_i} = x_i\underset{\sim}{n}_x + y_i\underset{\sim}{n}_y \quad |p_i|^2 = L_i^2$

$\therefore \boxed{x_i^2 + y_i^2 - L_i^2 = 0 \quad i = 1, 2}$

also $\underset{\sim}{p_1} - \underset{\sim}{p_2} = (x_1 - x_2)\underset{\sim}{n}_x + (y_1 - y_2)\underset{\sim}{n}_y \quad \therefore |\underset{\sim}{p_1} - \underset{\sim}{p_2}|^2 = (L_1 + L_2)^2$

$\therefore (x_1 - x_2)^2 + (y_1 - y_2)^2 = (L_1 + L_2)^2 \Rightarrow \cancel{x_1^2 + x_2^2} - 2x_1x_2 + \cancel{y_1^2 + y_2^2} - 2y_1y_2 = \cancel{L_1^2} + 2L_1L_2 + \cancel{L_2^2}$

using the 3rd & 4th constraint $\quad \therefore \boxed{x_1x_2 + y_1y_2 + L_1L_2 = 0} \quad p_1 \cdot p_2 = -|p_1||p_2|\cos\pi$

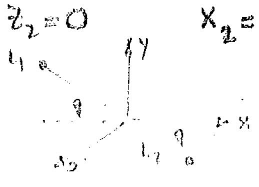
1(f) What are the degrees of freedom of $S = (P_1, P_2)$ in R in problem 1(d)

P_1 & P_2 have 3 degrees of freedom in R (in the n_x, n_y, n_z directions)

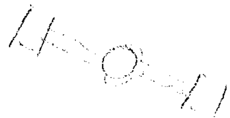
3 · 2 part − 3 constr = 6 − 3 = 3.

1(g) Using problem 1(e) let q be the angle between $\underset{\sim}{p_2}$ and $\underset{\sim}{n}_x$ and rewrite the 5 constraint eqns in terms of q

$z_1 = z_2 = 0 \quad x_2 = L_2\cos q \quad y_2 = -L_2\sin q \quad x_1 = -L_1\sin(90 - q) = -L_1\cos q \quad y_1 = L_1\sin q$



1 (R) Determine the number of degrees of freedom of the following

(a)  since the ball & socket can rotate in any of 3 directions
6 degrees of freedom for each particle & socket gives 3 conds $12 - 3 = 9$

(b) ~~point O is sun; 3 particles earth, satellite & rotor, 3 constraints~~ 7 degrees of freedom (6 for satel, 1 for rotor φ); 1 constraint $\varphi - f(t) = 0$

$n = 3 \cdot 3 - 3 = 6$

(c) Same as (b) yet no 3rd constraint $n = 3 \cdot 3 - 2 = 7$

(d) $2 \cdot 3 - 5 = 1$ particles P_1 & P_2 of problem 1(c)