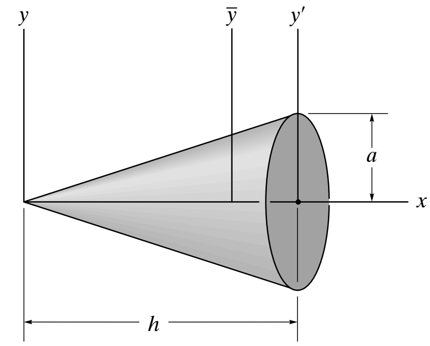
•21–1. Show that the sum of the moments of inertia of a body, $I_{xx} + I_{yy} + I_{zz}$, is independent of the orientation of the x, y, z axes and thus depends only on the location of its origin.

$$I_{xx} + I_{yy} + I_{zz} = \int_m (y^2 + z^2)dm + \int_m (x^2 + z^2)dm + \int_m (x^2 + y^2)dm$$

$$= 2\int_m (x^2 + y^2 + z^2)dm$$

However, $x^2 + y^2 + z^2 = r^2$, where r is the distance from the origin O to dm. Since $|r|$ is constant, it does not depend on the orientation of the x, y, z axis. Consequently, $I_{xx} + I_{yy} + I_{zz}$ is also indepenent of the orientation of the x, y, z axis. **Q.E.D.**

21–2. Determine the moment of inertia of the cone with respect to a vertical $\overline{y}$ axis that passes through the cone's center of mass. What is the moment of inertia about a parallel axis y' that passes through the diameter of the base of the cone? The cone has a mass m.



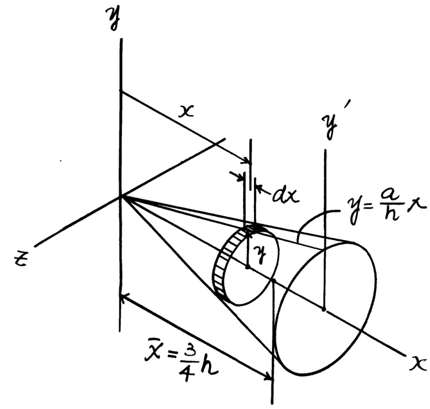
The mass of the differential element is $dm = \rho dV = \rho(\pi y^2)\,dx = \dfrac{\rho\pi a^2}{h^2}x^2 dx$.

$$dI_v = \frac{1}{4}dmy^2 + dmx^2$$

$$= \frac{1}{4}\left[\frac{\rho\pi a^2}{h^2}x^2\,dx\right]\left(\frac{a}{h}x\right)^2 + \left(\frac{\rho\pi a^2}{h^2}x^2\right)x^2\,dx$$

$$= \frac{\rho\pi a^2}{4h^4}(4h^2 + a^2)\,x^4\,dx$$

$$I_v = \int dI_v = \frac{\rho\pi a^2}{4h^4}(4h^2 + a^2)\int_0^h x^4 dx = \frac{\rho\pi a^2 h}{20}(4h^2 + a^2)$$



However,

$$m = \int_m dm = \frac{\rho\pi a^2}{h^2}\int_0^h x^2\,dx = \frac{\rho\pi a^2 h}{3}$$

Hence,

$$I_y = \frac{3m}{20}(4h^2 + a^2)$$

Using the parallel axis theorem:

$$I_v = I_v + md^2$$

$$\frac{3m}{20}(4h^2 + a^2) = I_v + m\left(\frac{3h}{4}\right)^2$$

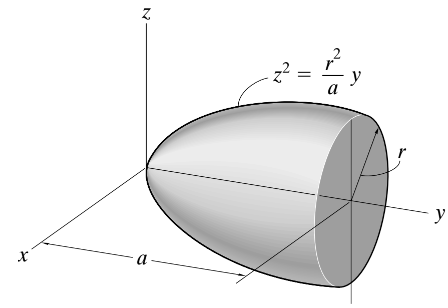
$$I_v = \frac{3m}{80}(h^2 + 4a^2) \qquad \textbf{Ans.}$$

$$I_v = I_y + md^2$$

$$= \frac{3m}{80}(h^2 + 4a^2) + m\left(\frac{h}{4}\right)^2$$

$$= \frac{m}{20}(2h^2 + 3a^2) \qquad \textbf{Ans.}$$

21–3. Determine the moments of inertia I_x and I_y of the paraboloid of revolution. The mass of the paraboloid is m.

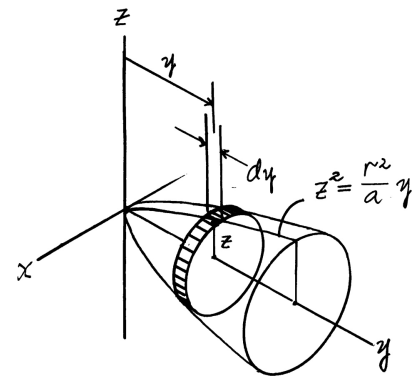


$$m = \rho \int_0^a \pi z^2\,dy = \rho\pi \int_0^a \left(\frac{r^2}{a}\right) y\,dy = \rho r\left(\frac{r^2}{2}\right)a$$

$$I_v = \int_m \frac{1}{2}\,dm\,z^2 = \frac{1}{2}\rho\pi \int_0^a z^4 dy = \frac{1}{2}\rho\pi\left(\frac{r^4}{a^2}\right)\int_0^a v^2 dy = \rho\pi\left(\frac{r^4}{6}\right)a$$

Thus,

$$I_x = \frac{1}{3}mr^2 \qquad \textbf{Ans.}$$

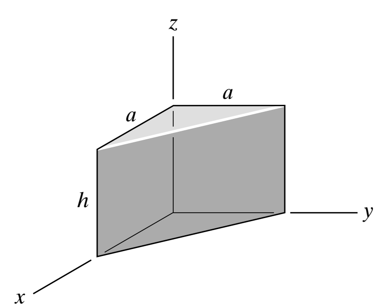


$$I_x = \int_m \left(\frac{1}{4}\,dm\,z^2 + dm\,y^2\right) = \frac{1}{4}\rho\pi \int_0^a z^4\,dy + \rho \int_0^a \pi z^2 y^2\,dy$$

$$= \frac{1}{4}\rho\pi\left(\frac{r^4}{a^2}\right)\int_0^a y^2 dy + \rho\pi\left(\frac{r^2}{a}\right)\int_0^a y^3 dy = \frac{\rho\pi r^4 a}{12} + \frac{\rho\pi r^2 a^3}{4} = \frac{1}{6}mr^2 + \frac{1}{2}ma^2$$

$$I_x = \frac{m}{6}(r^2 + 3a^2) \qquad \textbf{Ans.}$$

***21–4.** Determine by direct integration the product of inertia I_{yz} for the homogeneous prism. The density of the material is ρ. Express the result in terms of the total mass m of the prism.



The mass of the differential element is $dm = \rho dV = \rho hx\,dy = \rho h(a - y)dy$.

$$m = \int_m dm = \rho h \int_0^a (a - y)dy = \frac{\rho a^2 h}{2}$$

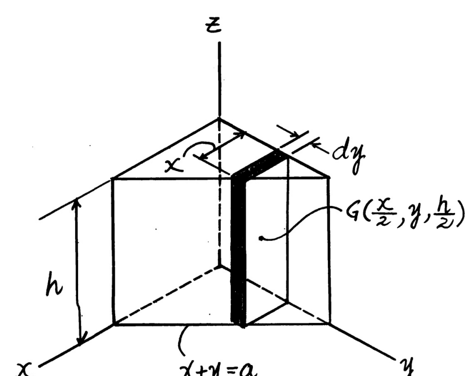
Using the parallel axis theorem:

$$dI_{yz} = (dI_{y'z'})_G + dm\,y_G z_G$$

$$= 0 + (\rho hx\,dy)(y)\left(\frac{h}{2}\right)$$

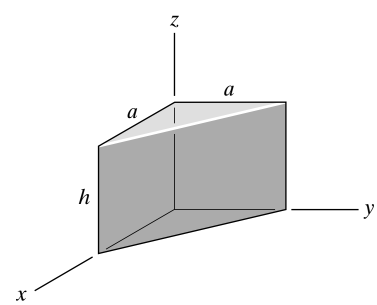
$$= \frac{\rho h^2}{2}\,xy\,dy$$

$$= \frac{\rho h^2}{2}(ay - y^2)\,dy$$



$$I_{yz} = \frac{\rho h^2}{2}\int_0^a (ay - y^2)\,dy = \frac{\rho a^3 h^2}{12} = \frac{1}{6}\left(\frac{\rho a^2 h}{2}\right)(ah) = \frac{m}{6}\,ah \qquad \textbf{Ans.}$$

•21–5. Determine by direct integration the product of inertia I_{xy} for the homogeneous prism. The density of the material is ρ. Express the result in terms of the total mass m of the prism.



The mass of the differential element is $dm = \rho dV = \rho hx dy = \rho h(a - y)dy$.

$$m = \int_m dm = \rho h \int_0^a (a - y)dy = \frac{\rho a^2 h}{2}$$

Using the parallel axis theorem:

$$dI_{xy} = (dI_{x'y'})_G + dm x_G y_G$$

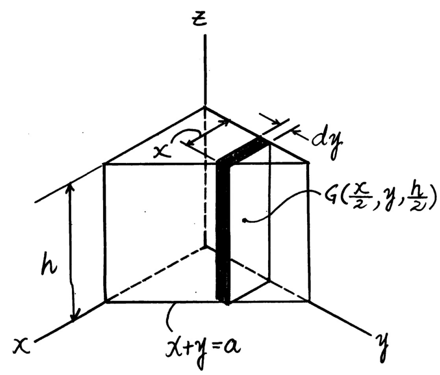
$$= 0 + (\rho hx dy)\left(\frac{x}{2}\right)(y)$$

$$= \frac{\rho h^2}{2} x^2 y dy$$

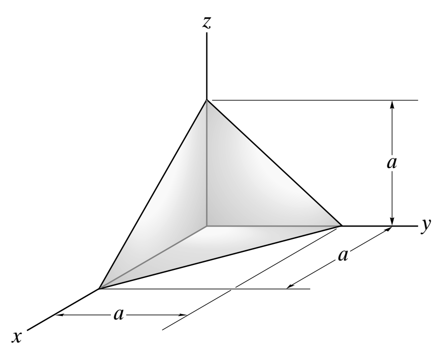
$$= \frac{\rho h^2}{2}(y^3 - 2ay^2 + a^2 y)\, dy$$

$$I_{xy} = \frac{\rho h}{2}\int_0^a (y^3 - 2ay^2 + a^2 y)\, dy$$

$$= \frac{\rho a^4 h}{24} = \frac{1}{12}\left(\frac{\rho a^2 h}{2}\right)a^2 = \frac{m}{12}a^2$$ **Ans.**



21–6. Determine the product of inertia I_{xy} for the homogeneous tetrahedron. The density of the material is ρ. Express the result in terms of the total mass m of the solid. *Suggestion:* Use a triangular element of thickness dz and then express dI_{xy} in terms of the size and mass of the element using the result of Prob. 21–5.



$$dm = \rho\, dV = \rho\left[\frac{1}{2}(a - z)(a - z)\right]dz = \frac{\rho}{2}(a - z)^2\, dz$$

$$m = \frac{\rho}{2}\int_0^a (a^2 - 2az + z^2)dz = \frac{\rho a^3}{6}$$

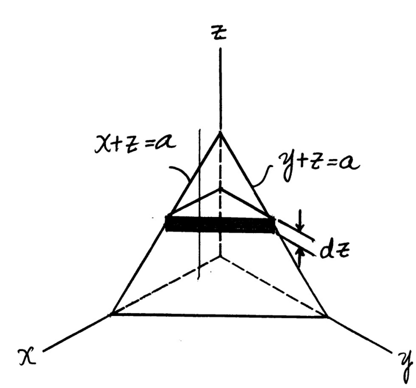
From Prob. 21–5 the product of inertia of a triangular prism with respect to the xz and yz planes is $I_{xy} = \frac{\rho a^4 h}{24}$. For the element above, $dI_{xy} = \frac{\rho dz}{24}(a - z)^4$. Hence,

$$I_{xy} = \frac{\rho}{24}\int_0^a (a^4 - 4a^3 z + 6z^2 a^2 - 4az^3 + z^4)dz$$

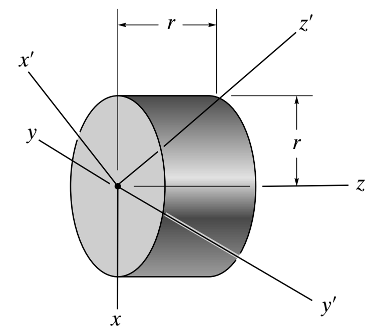
$$I_{xy} = \frac{\rho a^5}{120}$$

or,

$$I_{xy} = \frac{ma^2}{20}$$ **Ans.**



21–7. Determine the moments of inertia for the homogeneous cylinder of mass m about the x', y', z' axes.



Due to symmetry

$$I_{xy} = I_{yz} = I_{zx} = 0$$

$$I_y = I_x = \frac{1}{12}m(3r^2 + r^2) + m\left(\frac{r}{2}\right)^2 = \frac{7mr^2}{12} \qquad I_z = \frac{1}{2}mr^2$$

For x',

$$u_x = \cos 135° = -\frac{1}{\sqrt{2}}, \qquad u_y = \cos 90° = 0, \qquad u_z = \cos 135° = -\frac{1}{\sqrt{2}}$$

$$I_x = I_x u_x^2 + I_y u_y^2 + I_z u_z^2 - 2I_{xy}u_x u_y - 2I_{yz}u_y u_z - 2I_{zx}u_z u_x$$

$$= \frac{7mr^2}{12}\left(-\frac{1}{\sqrt{2}}\right)^2 + 0 + \frac{1}{2}mr^2\left(-\frac{1}{\sqrt{2}}\right)^2 - 0 - 0 - 0$$

$$= \frac{13}{24}mr^2$$ **Ans.**

For y',

$$I_{y'} = I_y = \frac{7mr^2}{12}$$ **Ans.**

For z',

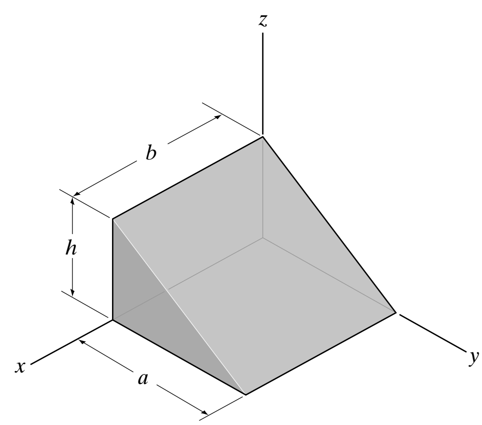
$$u_x = \cos 135° = -\frac{1}{\sqrt{2}}, \qquad u_y = \cos 90° = 0, \qquad u_z = \cos 45° = \frac{1}{\sqrt{2}}$$

$$I_{z'} = I_x u_x^2 + I_y u_y^2 + I_z u_z^2 - 2I_{xy}u_x u_y - 2I_{yz}u_y u_z - 2I_{zx}u_z u_x$$

$$= \frac{7mr^2}{12}\left(-\frac{1}{\sqrt{2}}\right)^2 + 0 + \frac{1}{2}mr^2\left(-\frac{1}{\sqrt{2}}\right)^2 - 0 - 0 - 0$$

$$= \frac{13}{24}mr^2$$ **Ans.**

***21–8.** Determine the product of inertia I_{xy} of the homogeneous triangular block. The material has a density of ρ. Express the result in terms of the total mass m of the block.

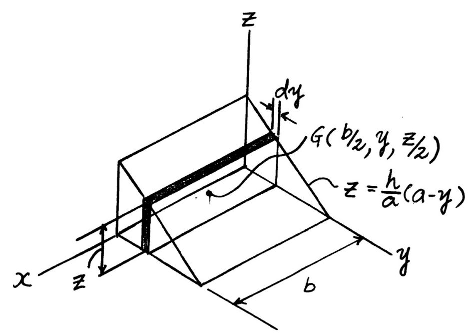


The mass of the differential rectangular volume element shown in Fig. a is $dm = \rho dV = \rho bz dy$. Using the parallel - plane theorem,

$$dI_{xy} = dI_{x'y'} + dm x_G y_G$$

$$= 0 + [\rho bz dy]\left(\frac{b}{2}\right)y$$

$$= \frac{\rho b^2}{2} zy dy$$



However, $z = \frac{h}{a}(a - y)$. Then

$$dI_{xy} = \frac{\rho b^2}{2}\left[\frac{h}{a}(a - y)y\right]dy = \frac{\rho b^2 h}{2a}\left(ay - y^2\right)dy$$

Thus,

$$I_{xy} = \int dI_{xy} = \frac{\rho b^2 h}{2a}\int_0^a \left(ay - y^2\right)dy$$

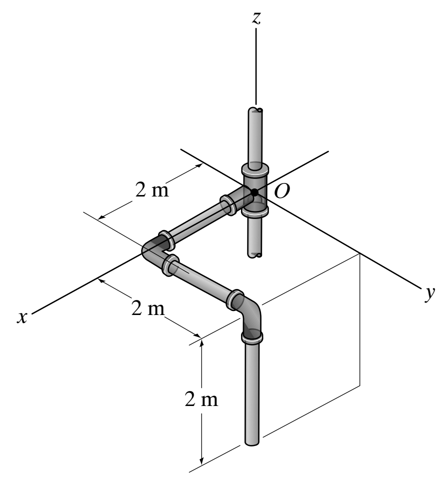
$$= \frac{\rho b^2 h}{2a}\left(\frac{ay^2}{2} - \frac{y^3}{3}\right)\Bigg|_0^a$$

$$= \frac{1}{12}\rho a^2 b^2 h$$

However, $m = \rho V = \left(\frac{1}{2}ahb\right) = \frac{1}{2}\rho abh$. Then

$$I_{xy} = \frac{1}{12}\rho a^2 b^2 h\left(\frac{m}{\frac{1}{2}\rho abh}\right) = \frac{1}{6}mab \qquad \textbf{Ans.}$$

•21–9. The slender rod has a mass per unit length of 6 kg/m. Determine its moments and products of inertia with respect to the x, y, z axes.



The mass of segments (1), (2), and (3) shown in Fig. a is $m_1 = m_2 = m_3 = 6(2) = 12$ kg. The mass moments of inertia of the bent rod about the x, y, and z axes are

$$I_x = \Sigma \bar{I}_{x'} + m\left(y_G{}^2 + z_G{}^2\right)$$

$$= \left(0 + 0\right) + \left[\frac{1}{12}(12)\left(2^2\right) + 12\left(1^2 + 0^2\right)\right] + \left[\frac{1}{12}(12)\left(2^2\right) + 12\left[2^2 + \left(-1\right)^2\right]\right]$$

$$= 80 \text{ kg} \cdot \text{m}^2$$ **Ans.**

$$I_y = \Sigma \bar{I}_{y'} + m\left(x_G{}^2 + z_G{}^2\right)$$

$$= \left[\frac{1}{12}(12)\left(2^2\right) + 12\left(1^2 + 0^2\right)\right] + \left[0 + 12\left(2^2 + 0^2\right)\right] + \left[\frac{1}{12}(12)\left(2^2\right) + 12\left[2^2 + \left(-1\right)^2\right]\right]$$

$$= 128 \text{ kg} \cdot \text{m}^2$$ **Ans.**

$$I_z = \Sigma \bar{I}_{z'} + m\left(x_G{}^2 + y_G{}^2\right)$$

$$= \left[\frac{1}{12}(12)\left(2^2\right) + 12\left(1^2 + 0^2\right)\right] + \left[\frac{1}{2}(12)\left(2^2\right) + 12\left(2^2 + 1^2\right)\right] + \left[0 + 12\left(2^2 + 2^2\right)\right]$$

$$= 176 \text{ kg} \cdot \text{m}^2$$ **Ans.**

Due to symmetry, the products of inertia of segments (1), (2), and (3) with respect to their centroidal planes are equal to zero. Thus,

$$I_{xy} = \Sigma \bar{I}_{x'y'} + m x_G y_G$$

$$= \left[0 + 12(1)(0)\right] + \left[0 + 12(2)(1)\right] + \left[0 + 12(2)(2)\right]$$

$$= 72 \text{ kg} \cdot \text{m}^2$$ **Ans.**

$$I_{yz} = \Sigma \bar{I}_{y'z'} + m y_G z_G$$

$$= \left[0 + 12(0)(0)\right] + \left[0 + 12(1)(0)\right] + \left[0 + 12(2)(-1)\right]$$

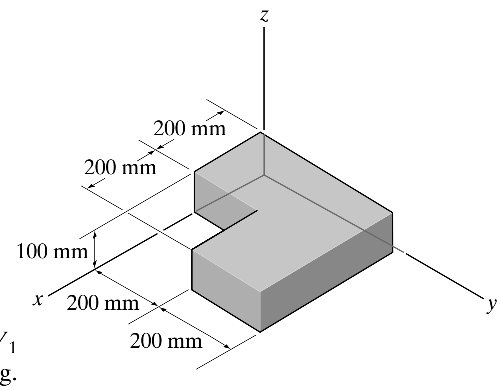
$$= -24 \text{ kg} \cdot \text{m}^2$$ **Ans.**

$$I_{xz} = \Sigma \bar{I}_{x'z'} + m x_G z_G$$

$$= \left[0 + 12(1)(0)\right] + \left[0 + 12(2)(0)\right] + \left[0 + 12(2)(-1)\right]$$

$$= -24 \text{ kg} \cdot \text{m}^2$$ **Ans.**

21–10. Determine the products of inertia I_{xy}, I_{yz}, and I_{xz} of the homogeneous solid. The material has a density of $7.85\ \text{Mg/m}^3$.



The masses of segments (1) and (2) shown in Fig. *a* are $m_1 = \rho V_1 = 7850(0.4)(0.4)(0.1) = 125.6\ \text{kg}$ and $m_2 = \rho V_2 = 7850(0.2)(0.2)(0.1) = 31.4\ \text{kg}$. Due to symmetry $\bar{I}_{x'y'} = \bar{I}_{y'z'} = \bar{I}_{x'z'} = 0$ for segment (1) and $\bar{I}_{x''y''} = \bar{I}_{y''z''} = \bar{I}_{x''z''} = 0$ for segment (2). Since segment (2) is a hole, it should be considered as a negative segment. Thus

$$I_{xy} = \Sigma \bar{I}_{x'y'} + m x_G y_G$$

$$= \left[0 + 125.6(0.2)(0.2)\right] - \left[0 + 31.4(0.3)(0.1)\right]$$

$$= 4.08\ \text{kg}\cdot\text{m}^2 \qquad \textbf{Ans.}$$

$$I_{yz} = \Sigma \bar{I}_{y'z'} + m y_G z_G$$

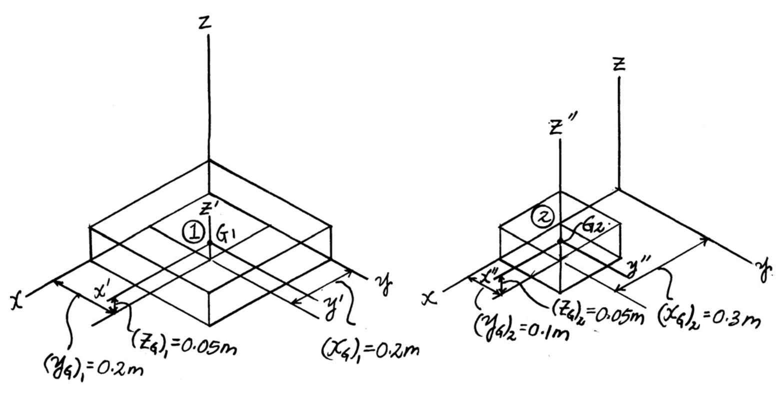
$$= \left[0 + 125.6(0.2)(0.05)\right] - \left[0 + 31.4(0.1)(0.05)\right]$$

$$= 1.10\ \text{kg}\cdot\text{m}^2 \qquad \textbf{Ans.}$$

$$I_{xz} = \Sigma \bar{I}_{x'z'} + m x_G z_G$$

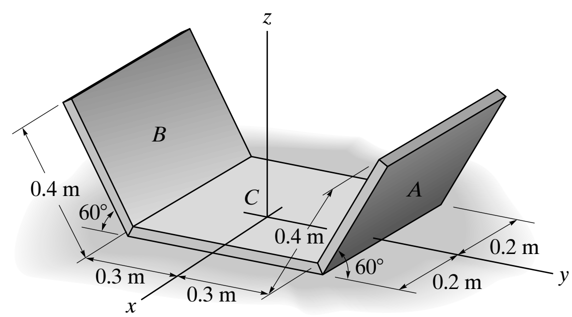
$$= \left[0 + 125.6(0.2)(0.05)\right] - \left[0 + 31.4(0.3)(0.05)\right]$$

$$= 0.785\ \text{kg}\cdot\text{m}^2 \qquad \textbf{Ans.}$$



(a)

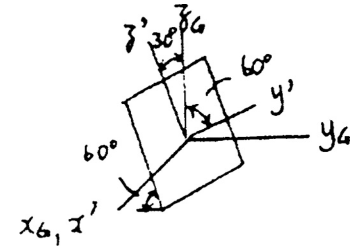
21–11. The assembly consists of two thin plates A and B which have a mass of 3 kg each and a thin plate C which has a mass of 4.5 kg. Determine the moments of inertia I_x, I_y and I_z.



$$I_{x'} = I_{z'} = \frac{1}{12}(3)(0.4)^2 = 0.04\ \text{kg}\cdot\text{m}^2$$

$$I_{y'} = \frac{1}{12}(3)[(0.4)^2 + (0.4)^2] = 0.08\ \text{kg}\cdot\text{m}^2$$

$$I_{x'y'} = I_{z'y'} = I_{z'x'} = 0$$



For z_G,

$$u_{x'} = 0$$

$$u_{y'} = \cos 60° = 0.50$$

$$u_{z'} = \cos 30° = 0.8660$$

$$I_{z_G} = 0 + 0.08(0.5)^2 + 0.04(0.866)^2 - 0 - 0 - 0$$

$$= 0.05\ \text{kg}\cdot\text{m}^2$$

$$I_{x_G} = I_{x'} = 0.04\ \text{kg}\cdot\text{m}^2$$

For y_G,

$$u_{x'} = 0$$

$$u_{y'} = \cos 30° = 0.866$$

$$u_{z'} = \cos 120° = -0.50$$

$$I_{y_G} = 0 + 0.08(0.866)^2 + 0.04(-0.5)^2 - 0 - 0 - 0$$

$$= 0.07\ \text{kg}\cdot\text{m}^2$$

$$I_x = \frac{1}{12}(4.5)(0.6)^2 + 2[0.04 + 3\{(0.3 + 0.1)^2 + (0.1732)^2\}]$$

$I_x = 1.36\ \text{kg}\cdot\text{m}^2$ **Ans.**

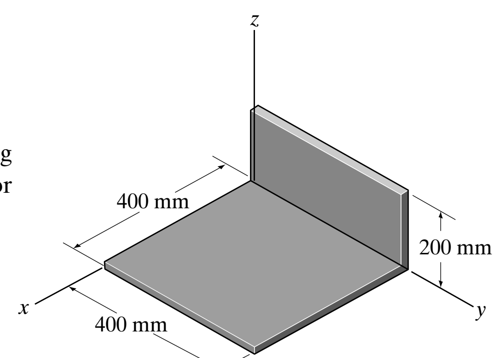
$$I_y = \frac{1}{12}(4.5)(0.4)^2 + 2[0.07 + 3(0.1732)^2]$$

$I_y = 0.380\ \text{kg}\cdot\text{m}^2$ **Ans.**

$$I_z = \frac{1}{12}(4.5)[(0.6)^2 + (0.4)^2] + 2[0.05 + 3(0.3 + 0.1)^2]$$

$I_z = 1.26\ \text{kg}\cdot\text{m}^2$ **Ans.**

***21–12.** Determine the products of inertia I_{xy}, I_{yz}, and I_{xz}, of the thin plate. The material has a density per unit area of $50\ \text{kg/m}^2$.



The masses of segments (1) and (2) shown in Fig. *a* are $m_1 = 50(0.4)(0.4) = 8$ kg and $m_2 = 50(0.4)(0.2) = 4$ kg. Due to symmetry $\bar{I}_{x'y'} = \bar{I}_{y'z'} = \bar{I}_{x'z'} = 0$ for segment (1) and $\bar{I}_{x''y''} = \bar{I}_{y''z''} = \bar{I}_{x''z''} = 0$ for segment (2).

$$I_{xy} = \Sigma \bar{I}_{x'y'} + m x_G y_G$$

$$= \left[0 + 8(0.2)(0.2)\right] + \left[0 + 4(0)(0.2)\right]$$

$$= 0.32\ \text{kg}\cdot\text{m}^2 \qquad \textbf{Ans.}$$

$$I_{yz} = \Sigma \bar{I}_{y'z'} + m y_G z_G$$

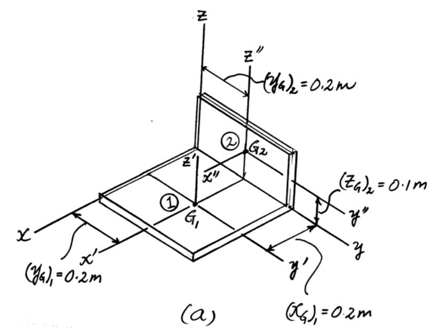
$$= \left[0 + 8(0.2)(0)\right] + \left[0 + 4(0.2)(0.1)\right]$$

$$= 0.08\ \text{kg}\cdot\text{m}^2 \qquad \textbf{Ans.}$$

$$I_{xz} = \Sigma \bar{I}_{x'z'} + m x_G z_G$$

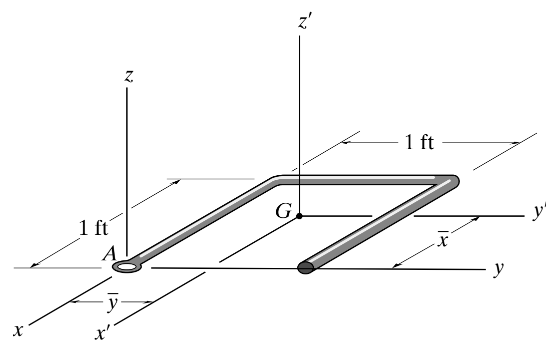
$$= \left[0 + 8(0.2)(0)\right] + \left[0 + 4(0)(0.1)\right]$$

$$= 0 \qquad \textbf{Ans.}$$



(a)

•21–13. The bent rod has a weight of 1.5 lb/ft. Locate the center of gravity $G(\bar{x}, \bar{y})$ and determine the principal moments of inertia $I_{x'}$, $I_{y'}$, and $I_{z'}$ of the rod with respect to the x', y', z' axes.



Due to symmetry

$$\bar{y} = 0.5\ \text{ft} \qquad \textbf{Ans.}$$

$$\bar{x} = \frac{\Sigma \bar{x} W}{\Sigma w} = \frac{(-1)(1.5)(1) + 2\left[(-0.5)(1.5)(1)\right]}{3\left[1.5(1)\right]} = -0.667\ \text{ft} \qquad \textbf{Ans.}$$

$$I_{x'} = 2\left[\left(\frac{1.5}{32.2}\right)(0.5)^2\right] + \frac{1}{12}\left(\frac{1.5}{32.2}\right)(1)^2$$

$$= 0.0272\ \text{slug}\cdot\text{ft}^2 \qquad \textbf{Ans.}$$

$$I_{y'} = 2\left[\frac{1}{12}\left(\frac{1.5}{32.2}\right)(1)^2 + \left(\frac{1.5}{32.2}\right)(0.667 - 0.5)^2\right] + \left(\frac{1.5}{32.2}\right)(1 - 0.667)^2$$

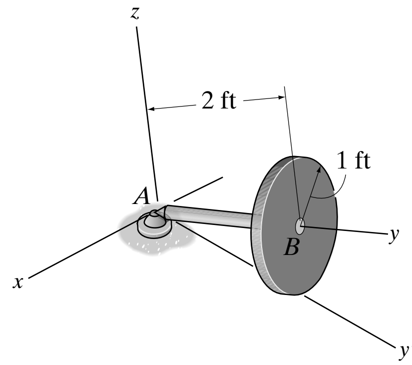
$$= 0.0155\ \text{slug}\cdot\text{ft}^2 \qquad \textbf{Ans.}$$

$$I_{z'} = 2\left[\frac{1}{12}\left(\frac{1.5}{32.2}\right)(1)^2 + \left(\frac{1.5}{32.2}\right)(0.5^2 + 0.1667^2)\right]$$

$$+ \frac{1}{12}\left(\frac{1.5}{32.2}\right)(1)^2 + \left(\frac{1.5}{32.2}\right)(0.3333)^2$$

$$= 0.0427\ \text{slug}\cdot\text{ft}^2 \qquad \textbf{Ans.}$$

21–14. The assembly consists of a 10-lb slender rod and a 30-lb thin circular disk. Determine its moment of inertia about the y' axis.



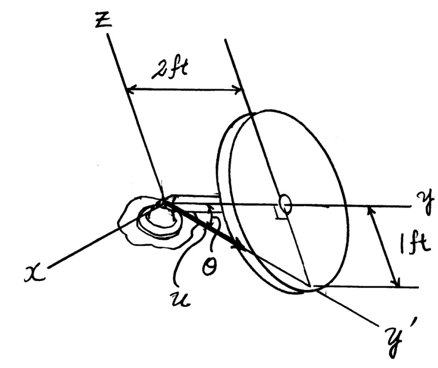
The mass of moment inertia of the assembly about the x, y, and z axes are

$$I_x = I_z = \left[\frac{1}{4}\left(\frac{30}{32.2}\right)(1^2) + \frac{30}{32.2}(2^2)\right] + \left[\frac{1}{12}\left(\frac{10}{32.2}\right)(2^2) + \frac{10}{32.2}(1^2)\right]$$

$$= 4.3737 \text{ slug} \cdot \text{ft}^2$$

$$I_y = \frac{1}{2}\left(\frac{30}{32.2}\right)(1^2) + 0 = 0.4658 \text{ slug} \cdot \text{ft}^2$$

Due to symmetry, $I_{xy} = I_{yz} = I_{xz} = 0$. From the geometry shown in Fig. a, $\theta = \tan^{-1}\left(\frac{1}{2}\right) = 26.57°$. Thus, the direction of the y' axis is defined by the unit vector



(a)

$$\mathbf{u} = \cos 26.57°\mathbf{j} - \sin 26.57°\mathbf{k} = 0.8944\mathbf{j} - 0.4472\mathbf{k}$$

Thus,

$$u_x = 0 \qquad u_y = 0.8944 \qquad u_z = -0.4472$$

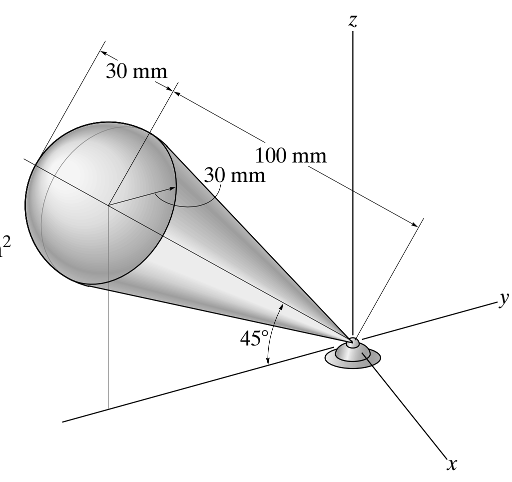
Then

$$I_{y'} = I_x u_x{}^2 + I_y u_y{}^2 + I_z u_z{}^2 - 2I_{xy}\, u_x u_y - 2I_{yz}\, u_y u_z - 2I_{xz} u_x u_z$$

$$= 4.3737(0) + 0.4658(0.8944)^2 + 4.3737(-0.4472)^2 - 0 - 0 - 0$$

$$= 1.25 \text{ slug} \cdot \text{ft}^2$$ **Ans.**

21–15. The top consists of a cone having a mass of 0.7 kg and a hemisphere of mass 0.2 kg. Determine the moment of inertia I_z when the top is in the position shown.



$$I_{x'} = I_{y'} = \frac{3}{80}(0.7)\left[(4)(0.3)^2 + (0.1)^2\right] + (0.7)\left[\frac{3}{4}(0.1)\right]^2$$

$$+ \left(\frac{83}{320}\right)(0.2)(0.03)^2 + (0.2)\left[\frac{3}{8}(0.03) + (0.1)\right]^2 = 6.816\left(10^{-3}\right)\text{kg} \cdot \text{m}^2$$

$$I_{z'} = \left(\frac{3}{10}\right)(0.7)(0.03)^2 + \left(\frac{2}{5}\right)(0.2)(0.03)^2$$

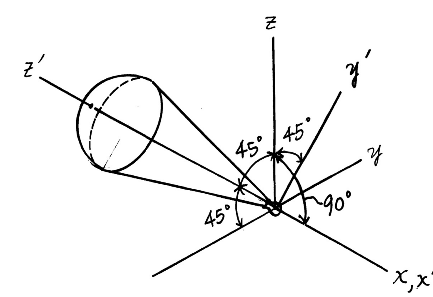
$$I_z = 0.261\left(10^{-3}\right) \text{ kg} \cdot \text{m}^2$$

$$u_x = \cos 90° = 0, \qquad u_{y'} = \cos 45° = 0.7071, \qquad u_{z'} = \cos 45° = 0.7071$$

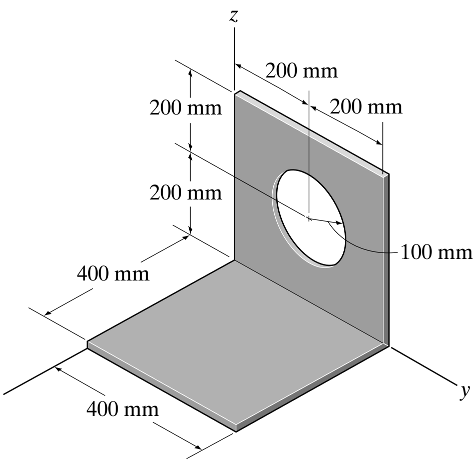
$$I_z = I_{x'} u_{x'}^2 + I_{y'} u_{y'}^2 + I_{z'} u_{z'}^2 - 2I_{x'y'} u_{x'} u_{y'} - 2I_{y'z'} u_{y'} u_{z'} - 2I_{x'z'} u_{x'} u_{z'}$$

$$= 0 + 6.816\left(10^{-3}\right)(0.7071)^2 + (0.261)\left(10^{-3}\right)(0.7071)^2 - 0 - 0 - 0$$

$$I_z = 3.54\left(10^{-3}\right) \text{ kg} \cdot \text{m}^2$$ **Ans.**



***21–16.** Determine the products of inertia I_{xy}, I_{yz}, and I_{xz} of the thin plate. The material has a mass per unit area of 50 kg/m^2.



The masses of segments (1), (2), and (3) shown in Fig. *a* are $m_1 = m_2 = 50(0.4)(0.4) = 8$ kg and $m_3 = 50\left[\pi(0.1)^2\right] = 0.5\pi$ kg.

Due to symmetry $\bar{I}_{x'y'} = \bar{I}_{y'z'} = \bar{I}_{x'z'} = 0$ for segment (1), $\bar{I}_{x''y''} = \bar{I}_{y''z''} = \bar{I}_{x''z''} = 0$ for segment (2), and $\bar{I}_{x'''y'''} = \bar{I}_{y'''z'''} = \bar{I}_{x'''z'''} = 0$ for segment (3). Since segment (3) is a hole, it should be considered as a negative segment. Thus

$$I_{xy} = \Sigma\bar{I}_{x'y'} + m x_G y_G$$

$$= \left[0 + 8(0.2)(0.2)\right] + \left[0 + 8(0)(0.2)\right] - \left[0 + 0.5\pi(0)(0.2)\right]$$

$$= 0.32 \text{ kg} \cdot \text{m}^2$$ **Ans.**

$$I_{yz} = \Sigma\bar{I}_{y'z'} + m y_G z_G$$

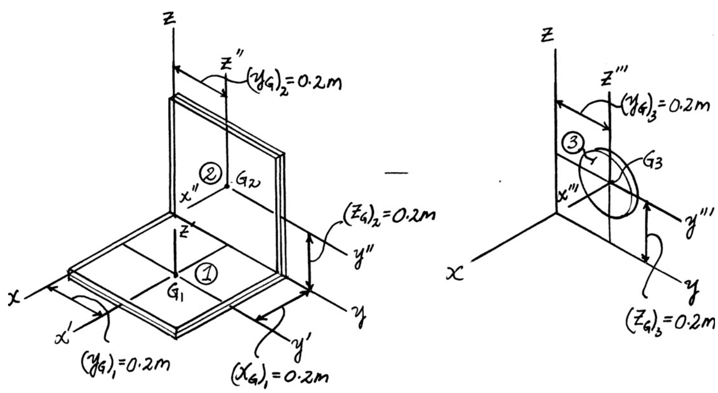
$$= \left[0 + 8(0.2)(0)\right] + \left[0 + 8(0.2)(0.2)\right] - \left[0 + 0.5\pi(0.2)(0.2)\right]$$

$$= 0.257 \text{ kg} \cdot \text{m}^2$$ **Ans.**

$$I_{xz} = \Sigma\bar{I}_{x'z'} + m x_G z_G$$

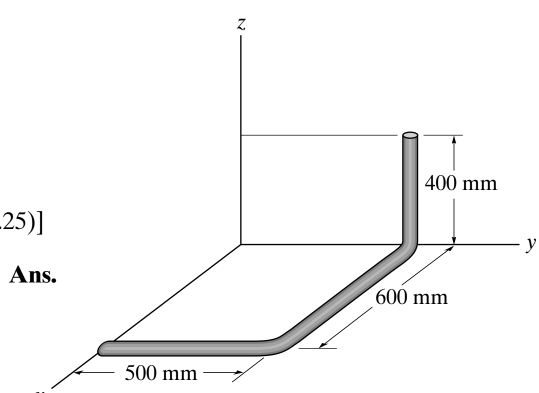
$$= \left[0 + 8(0.2)(0)\right] + \left[0 + 8(0)(0.2)\right] - \left[0 + 0.5\pi(0)(0.2)\right]$$

$$= 0 \text{ kg} \cdot \text{m}^2$$ **Ans.**



(a)

•21–17. Determine the product of inertia I_{xy} for the bent rod. The rod has a mass per unit length of 2 kg/m.



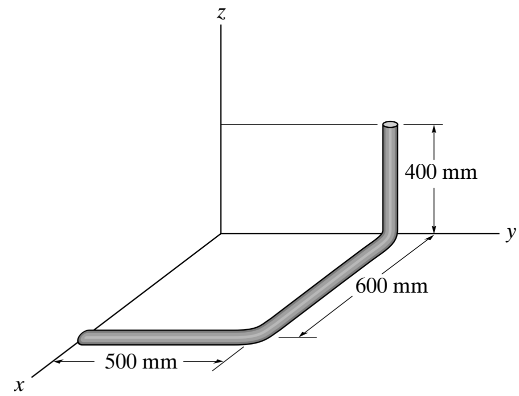
Product of Inertia: Applying Eq. 21–4. we have

$$I_{xy} = \Sigma\left(I_{x'y'}\right)_G + m x_G\, y_G$$

$$= [0 + 0.4\,(2)\,(0)\,(0.5)] + [0 + 0.6\,(2)\,(0.3)\,(0.5)] + [0 + 0.5\,(2)\,(0.6)\,(0.25)]$$

$$= 0.330 \text{ kg} \cdot \text{m}^2$$ **Ans.**

21–18. Determine the moments of inertia I_{xx}, I_{yy}, I_{zz} for the bent rod. The rod has a mass per unit length of 2 kg/m.



Moments of Inertia: Applying Eq. 21–3, we have

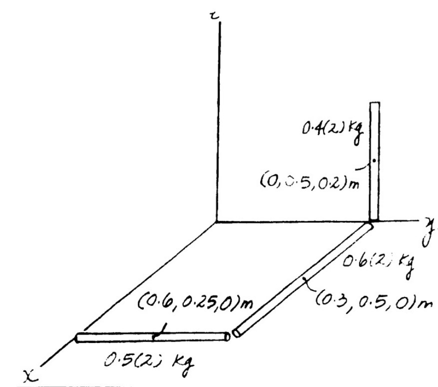
$$I_{xx} = \Sigma(I_{x'x'})_G + m(y_G^2 + z_G^2)$$

$$= \left[\frac{1}{12}(0.4)(2)\left(0.4^2\right) + 0.4(2)\left(0.5^2 + 0.2^2\right)\right]$$

$$+ \left[0 + 0.6(2)\left(0.5^2 + 0^2\right)\right]$$

$$+ \left[\frac{1}{12}(0.5)(2)\left(0.5^2\right) + 0.5(2)\left(0.25^2 + 0^2\right)\right]$$

$$= 0.626 \text{ kg}\cdot\text{m}^2 \qquad \textbf{Ans.}$$



$$I_{xy} = \Sigma(I_{y'y'})_G + m(x_G^2 + z_G^2)$$

$$= \left[\frac{1}{12}(0.4)(2)\left(0.4^2\right) + 0.4(2)\left(0^2 + 0.2^2\right)\right]$$

$$+ \left[\frac{1}{12}(0.6)(2)\left(0.6^2\right) + 0.6(2)\left(0.3^2 + 0^2\right)\right]$$

$$+ \left[0 + 0.5(2)\left(0.6^2 + 0^2\right)\right]$$

$$= 0.547 \text{ kg}\cdot\text{m}^2 \qquad \textbf{Ans.}$$

$$I_{zz} = \Sigma(I_{z'z'})_G + m(x_G^2 + y_G^2)$$

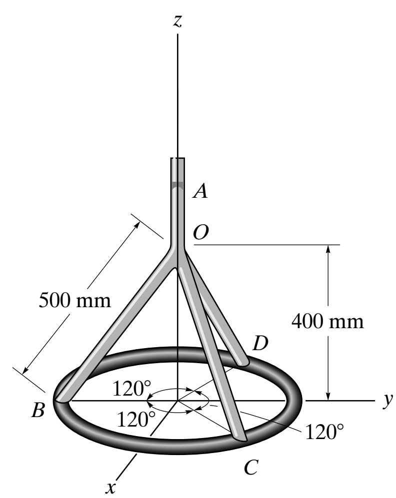
$$= \left[0 + 0.4(2)\left(0^2 + 0.5^2\right)\right]$$

$$+ \left[\frac{1}{12}(0.6)(2)\left(0.6^2\right) + 0.6(2)\left(0.3^2 + 0.5^2\right)\right]$$

$$+ \left[\frac{1}{12}(0.5)(2)\left(0.5^2\right) + 0.5(2)\left(0.6^2 + 0.25^2\right)\right]$$

$$= 1.09 \text{ kg}\cdot\text{m}^2 \qquad \textbf{Ans.}$$

21–19. Determine the moment of inertia of the rod-and-thin-ring assembly about the z axis. The rods and ring have a mass per unit length of 2 kg/m.



For the rod,

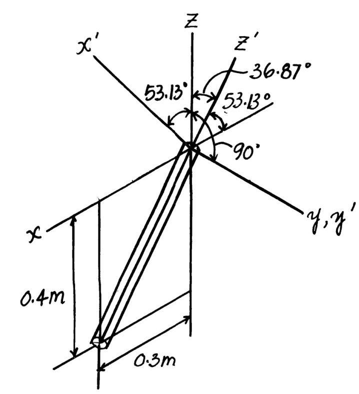
$$u_{x'} = 0.6, \qquad u_{y'} = 0, \qquad u_{z'} = 0.8$$

$$I_x = I_y = \frac{1}{3}[(0.5)(2)](0.5)^2 = 0.08333 \text{ kg} \cdot \text{m}^2$$

$$I_{x'} = 0$$

$$I_{x'y'} = I_{y'z'} = I_{x'z'} = 0$$

From Eq. 21–5,



$$I_z = 0.08333(0.6)^2 + 0 + 0 - 0 - 0 - 0$$

$$I_z = 0.03 \text{ kg} \cdot \text{m}^2$$

For the ring,

The radius is $r = 0.3$ m

Thus,

$$I_z = mR^2 = [2\,(2\pi)(0.3)](0.3)^2 = 0.3393 \text{ kg} \cdot \text{m}^2$$

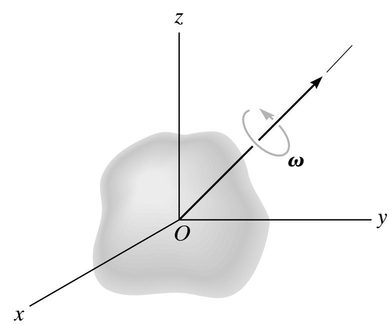
Thus the moment of inertia of the assembly is

$$I_z = 3(0.03) + 0.339 = 0.429 \text{ kg} \cdot \text{m}^2$$ **Ans.**

***21–20.** If a body contains *no planes of symmetry,* the principal moments of inertia can be determined mathematically. To show how this is done, consider the rigid body which is spinning with an angular velocity $\boldsymbol{\omega}$, directed along one of its principal axes of inertia. If the principal moment of inertia about this axis is I, the angular momentum can be expressed as $\mathbf{H} = I\boldsymbol{\omega} = I\omega_x\mathbf{i} + I\omega_y\mathbf{j} + I\omega_z\mathbf{k}$. The components of $\mathbf{H}$ may also be expressed by Eqs. 21–10, where the inertia tensor is assumed to be known. Equate the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components of both expressions for $\mathbf{H}$ and consider ω_x, ω_y, and ω_z to be unknown. The solution of these three equations is obtained provided the determinant of the coefficients is zero. Show that this determinant, when expanded, yields the cubic equation

$$I^3 - (I_{xx} + I_{yy} + I_{zz})I^2 + (I_{xx}I_{yy} + I_{yy}I_{zz} + I_{zz}I_{xx} - I_{xy}^2 - I_{yz}^2 - I_{zx}^2)I - (I_{xx}I_{yy}I_{zz} - 2I_{xy}I_{yz}I_{zx} - I_{xx}I_{yz}^2 - I_{yy}I_{zx}^2 - I_{zz}I_{xy}^2) = 0$$

The three positive roots of I, obtained from the solution of this equation, represent the principal moments of inertia I_x, I_y, and I_z.



$$\mathbf{H} = I\omega = I\omega_x\,\mathbf{i} + I\omega_y\,\mathbf{j} + I\omega_z\,\mathbf{k}$$

Equating the $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ components to the scalar equations (Eq. 21–10) yields

$$(I_{xx} - I)\,\omega_x - I_{xy}\,\omega_y - I_{xz}\,\omega_z = 0$$

$$-I_{xx}\,\omega_x + (I_{xy} - I)\,\omega_y - I_{yz}\,\omega_z = 0$$

$$-I_{zx}\,\omega_z - I_{zy}\,\omega_y + (I_{zz} - I)\,\omega_z = 0$$

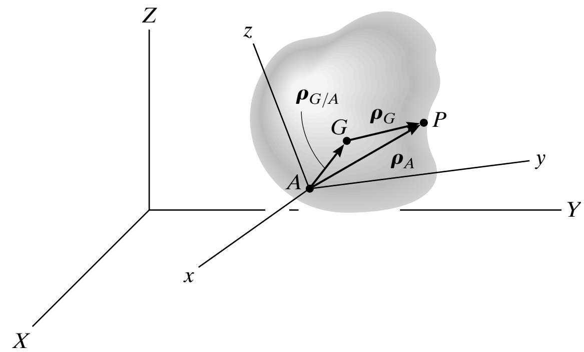
Solution for ω_x, ω_y, and ω_z requires

$$\begin{vmatrix} (I_{xx} - I) & -I_{xy} & -I_{xz} \\ -I_{yx} & (I_{yy} - I) & -I_{yz} \\ -I_{zx} & -I_{zy} & (I_{zz} - I) \end{vmatrix} = 0$$

Expanding

$$I^3 - (I_{xx} + I_{yy} + I_{zz})I^2 + \left(I_{xx}I_{yy} + I_{yy}I_{zz} + I_{zz}I_{xx} - I_{xy}^2 - I_{yz}^2 - I_{zx}^2\right)I - \left(I_{xx}I_{yy}I_{zz} - 2I_{xy}I_{yz}I_{zx} - I_{xx}I_{yz}^2 - I_{yy}I_{zx}^2 - I_{zz}I_{xy}^2\right) = 0 \quad \textbf{Q.E.D.}$$

•**21–21.** Show that if the angular momentum of a body is determined with respect to an arbitrary point A, then $\mathbf{H}_A$ can be expressed by Eq. 21–9. This requires substituting $\boldsymbol{\rho}_A = \boldsymbol{\rho}_G + \boldsymbol{\rho}_{G/A}$ into Eq. 21–6 and expanding, noting that $\int \boldsymbol{\rho}_G\, dm = \mathbf{0}$ by definition of the mass center and $\mathbf{v}_G = \mathbf{v}_A + \boldsymbol{\omega} \times \boldsymbol{\rho}_{G/A}$.



$$\mathbf{H}_A = \left(\int_m \rho_A\, dm\right) \times \mathbf{v}_A + \int_m \rho_A \times (\omega \times \rho_A) dm$$

$$= \left(\int_m (\rho_G + \rho_{G/A})\, dm\right) \times \mathbf{v}_A + \int_m (\rho_G + \rho_{G/A}) \times \left[\omega \times \rho_G + \rho_{G/A})\right] dm$$

$$= \left(\int_m \rho_G\, dm\right) \times \mathbf{v}_A + (\rho_{G/A} \times \mathbf{v}_A) \int_m dm + \int_m \rho_G \times (\omega \times \rho_G)\, dm$$

$$+ \left(\int_m \rho_G dm\right) \times (\omega \times \rho_{G/A}) + \rho_{G/A} \times \left(\omega \times \int_m \rho_G\, dm\right) + \rho_{G/A} \times (\omega \times \rho_{G/A}) \int_m dm$$

Since $\int_m \rho_G\, dm = 0$ and from Eq. 21–8 $\mathbf{H}_G = \int_m \rho_G \times (\omega \times \rho_G) dm$

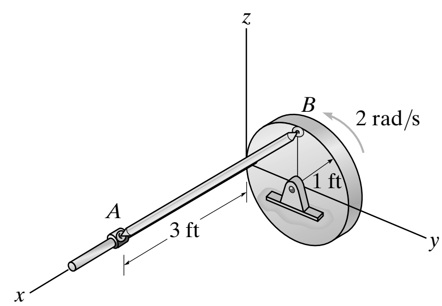
$$\mathbf{H}_A = (\rho_{G/A} \times \mathbf{v}_A) m + \mathbf{H}_G + \rho_{G/A} \times (\omega \times \rho_{G/A}) m$$

$$= \rho_{G/A} \times (\mathbf{v}_A + (\omega \times \rho_{G/A})) m + \mathbf{H}_G$$

$$= (\rho_{G/A} \times m\mathbf{v}_G) + \mathbf{H}_G$$

Q.E.D.

21–22. The 4-lb rod AB is attached to the disk and collar using ball-and-socket joints. If the disk has a constant angular velocity of 2 rad/s, determine the kinetic energy of the rod when it is in the position shown. Assume the angular velocity of the rod is directed perpendicular to the axis of the rod.



$$\mathbf{v}_A = \mathbf{v}_B + \omega \times \mathbf{r}_{A/B}$$

$$v_A\mathbf{i} = -(1)(2)\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 3 & -1 & -1 \end{vmatrix}$$

Expand and equate components:

$$v_A = -\omega_y + \omega_z \qquad \textbf{(1)}$$

$$2 = \omega_x + 3\,\omega_z \qquad \textbf{(2)}$$

$$0 = -\omega_x - 3\,\omega_y \qquad \textbf{(3)}$$

Also:

$$\omega \cdot \mathbf{r}_{A/B} = 0$$

$$3\omega_x - \omega_y - \omega_z = 0 \qquad \textbf{(4)}$$

Solving Eqs. (1)–(4):

$$\omega_x = 0.1818 \text{ rad/s}$$

$$\omega_y = -0.06061 \text{ rad/s}$$

$$\omega_z = 0.6061 \text{ rad/s}$$

$$v_A = 0.667 \text{ ft/s}$$

ω is perpendicular to the rod.

$$\omega_z = 0.1818 \text{ rad/s}, \qquad \omega_y = -0.06061 \text{ rad/s}, \qquad \omega_z = 0.6061 \text{ rad/s}$$

$$\mathbf{v}_B = \{-2\mathbf{j}\} \text{ ft/s}$$

$$\mathbf{r}_{A/B} = \{3\mathbf{i} - 1\mathbf{j} - 1\mathbf{k}\} \text{ ft}$$

$$\mathbf{v}_G = \mathbf{v}_B + \omega \times \frac{\mathbf{r}_{A/B}}{2}$$

$$\mathbf{v}_G = -2\mathbf{j} + \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0.1818 & -0.06061 & 0.6061 \\ 3 & -1 & -1 \end{vmatrix}$$

$$\mathbf{v}_G = \{0.333\mathbf{i} - 1\mathbf{j}\} \text{ ft/s}$$

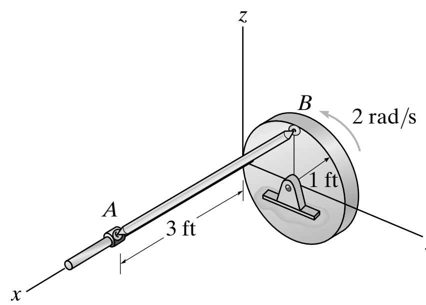
$$v_G = \sqrt{(0.333)^2 + (-1)^2} = 1.054 \text{ ft/s}$$

$$\omega = \sqrt{(0.1818)^2 + (-0.06061)^2 + (0.6061)^2} = 0.6356 \text{ rad/s}$$

$$T = \left(\frac{1}{2}\right)\left(\frac{4}{32.2}\right)(1.054)^2 + \left(\frac{1}{2}\right)\left[\frac{1}{12}\left(\frac{4}{32.2}\right)(3.3166)^2\right](0.6356)^2$$

$$T = 0.0920 \text{ ft} \cdot \text{lb} \qquad \textbf{Ans.}$$

21–23. Determine the angular momentum of rod AB in Prob. 21–22 about its mass center at the instant shown. Assume the angular velocity of the rod is directed perpendicular to the axis of the rod.



$$\mathbf{v}_A = \mathbf{v}_B + \omega \times \mathbf{r}_{A/B}$$

$$v_A\mathbf{i} = -(1)(2)\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ 3 & -1 & -1 \end{vmatrix}$$

Expand and equate components:

$$v_A = -\omega_y + \omega_z \qquad \textbf{(1)}$$

$$2 = \omega_x + 3\,\omega_z \qquad \textbf{(2)}$$

$$0 = -\omega_x - 3\,\omega_y \qquad \textbf{(3)}$$

Also:

$$\omega \cdot \mathbf{r}_{A/B} = 0$$

$$3\omega_x - \omega_y - \omega_z = 0 \qquad \textbf{(4)}$$

Solving Eqs. (1)–(4):

$$\omega_x = 0.1818 \text{ rad/s}$$

$$\omega_y = -0.06061 \text{ rad/s}$$

$$\omega_z = 0.6061 \text{ rad/s}$$

$$v_A = 0.667 \text{ ft/s}$$

ω is perpendicular to the rod.

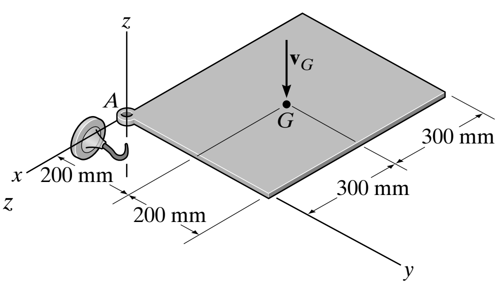
$$r_{A/B} = \sqrt{(3)^2 + (-1)^2 + (-1)^2} = 3.3166 \text{ ft}$$

$$I_G = \left(\frac{1}{12}\right)\left(\frac{4}{32.2}\right)(3.3166)^2 = 0.1139 \text{ slug} \cdot \text{ft}^2$$

$$\mathbf{H}_G = I_G\,\omega = 0.1139\,(0.1818\mathbf{i} - 0.06061\mathbf{j} + 0.6061\mathbf{k})$$

$$\mathbf{H}_G = \{0.0207\mathbf{i} - 0.00690\mathbf{j} + 0.0690\mathbf{k}) \text{ slug} \cdot \text{ft}^2/\text{s} \qquad \textbf{Ans.}$$

***21–24.** The uniform thin plate has a mass of 15 kg. Just before its corner A strikes the hook, it is falling with a velocity of $\mathbf{v}_G = \{-5\mathbf{k}\}$ m/s with no rotational motion. Determine its angular velocity immediately after corner A strikes the hook without rebounding.



Referring to Fig. a, the mass moments of inertia of the plate about the x, y, and z axes are

$$I_x = I_{x'} + m\left(y_G{}^2 + z_G{}^2\right) = \frac{1}{12}(15)\left(0.4^2\right) + 15\left(0.2^2 + 0^2\right) = 0.8\ \text{kg}\cdot\text{m}^2$$

$$I_y = I_{y'} + m\left(x_G{}^2 + z_G{}^2\right) = \frac{1}{12}(15)\left(0.6^2\right) + 15\left[(-0.3)^2 + 0^2\right] = 1.8\ \text{kg}\cdot\text{m}^2$$

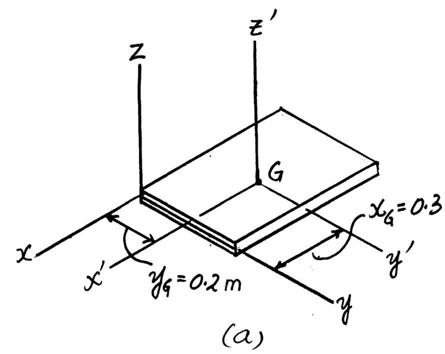
$$I_z = I_{z'} + m\left(x_G{}^2 + y_G{}^2\right) = \frac{1}{12}(15)\left(0.4^2 + 0.6^2\right) + 15\left[(-0.3)^2 + 0.2^2\right] = 2.6\ \text{kg}\cdot\text{m}^2$$

Due to symmetry, $I_{x'y'} = I_{y'z'} = I_{x'z'} = 0$. Thus,

$$I_{xy} = I_{x'y'} + mx_Gy_G = 0 + 15(-0.3)(0.2) = -0.9\ \text{kg}\cdot\text{m}^2$$

$$I_{yz} = I_{y'z'} + my_Gz_G = 0 + 15(0.2)(0) = 0$$

$$I_{xz} = I_{x'z'} + mx_Gz_G = 0 + 15(-0.3)(0) = 0$$



Since the plate falls without rotational motion just before the impact, its angular momentum about point A is

$$(\mathbf{H}_A)_1 = \mathbf{r}_{G/A} \times m\mathbf{v}_G = (-0.3\mathbf{i} + 0.2\mathbf{j}) \times 15(-5\mathbf{k})$$

$$= [-15\mathbf{i} - 22.5\mathbf{j}]\ \text{kg}\cdot\text{m}^2/\text{s}$$

Since the plate rotates about point A just after impact, the components of its angular momentum at this instant can be determined from

$$\left[(H_A)_2\right]_x = I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$

$$= 0.8\omega_x - (-0.9)\omega_y - 0(\omega_z)$$

$$= 0.8\omega_x + 0.9\omega_y$$

$$\left[(H_A)_2\right]_y = -I_{xy}\omega_x + I_y\omega_y - I_{yz}\omega_z$$

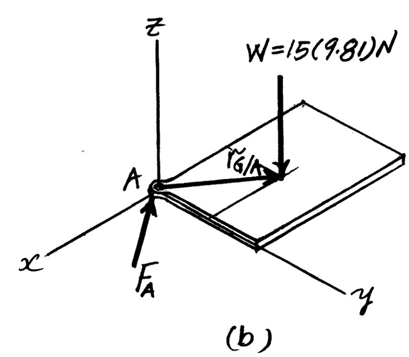
$$= -(-0.9)\omega_x + 1.8\omega_y - 0(\omega_z)$$

$$= 0.9\omega_x + 1.8\omega_y$$

$$\left[(H_A)_2\right]_z = -I_{xz}\omega_x + I_{yz}\omega_y - I_z\omega_z$$

$$= 0(\omega_x) - 0(\omega_y) + 2.6\omega_z$$

$$= 2.6\omega_z$$



Thus,

$$(\mathbf{H}_A)_2 = (0.8\omega_x + 0.9\omega_y)\mathbf{i} + (0.9\omega_x + 1.8\omega_y)\mathbf{j} + 2.6\omega_z\mathbf{k}$$

Referring to the free-body diagram of the plate shown in Fig. b, the weight $\mathbf{W}$ is a nonimpulsive force and the impulsive force $\mathbf{F}_A$ acts through point A. Therefore, angular momentum of the plate is conserved about point A. Thus,

$$(\mathbf{H}_A)_1 = (\mathbf{H}_A)_2$$

$$-15\mathbf{i} - 22.5\mathbf{j} = (0.8\omega_x + 0.9\omega_y)\mathbf{i} + (0.9\omega_x + 1.8\omega_y)\mathbf{j} + 2.6\omega_z\mathbf{k}$$

Equating the **i**, **j**, and **k** components,

$$-15 = 0.8\omega_x + 0.9\omega_y \qquad \textbf{(1)}$$

$$-22.5 = 0.9\omega_x + 1.8\omega_y \qquad \textbf{(2)}$$

$$0 = 2.6\omega_z \qquad \textbf{(3)}$$

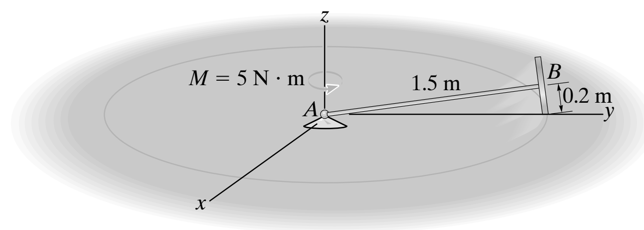
Solving Eqs. (1) through (3),

$\omega_x = -10.71$ rad/s $\qquad \omega_y = -7.143$ rad/s $\qquad \omega_z = 0$

Thus,

$$\omega = [-10.7\mathbf{i} - 7.14\mathbf{j}]\ \text{rad/s} \qquad \textbf{Ans.}$$

•21–25. The 5-kg disk is connected to the 3-kg slender rod. If the assembly is attached to a ball-and-socket joint at A and the 5-N · m couple moment is applied, determine the angular velocity of the rod about the z axis after the assembly has made two revolutions about the z axis starting from rest. The disk rolls without slipping.

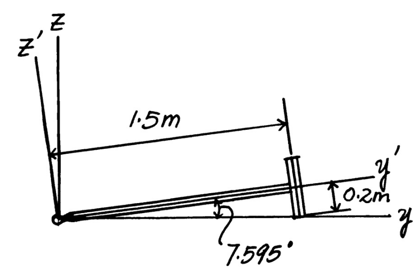


$$I_x = I_z = \frac{1}{4}(5)(0.2)^2 + 5(1.5)^2 + \frac{1}{3}(3)(1.5)^2 = 13.55$$

$$I_y = \frac{1}{2}(5)(0.2)^2 = 0.100$$

$$\omega = -\omega_y\,\mathbf{j}' + \omega_z\,\mathbf{k} = -\omega_{y'}\,\mathbf{j}' + \omega_z \sin 7.595°\mathbf{j}' + \omega_z \cos 7.595°\mathbf{k}'$$

$$= (0.13216\omega_z - \omega_{y'})\mathbf{j}' + 0.99123\,\omega_z\,\mathbf{k}'$$



Since points A and C have zero velocity,

$$\mathbf{v}_C = \mathbf{v}_A + \omega \times \mathbf{r}_{C/A}$$

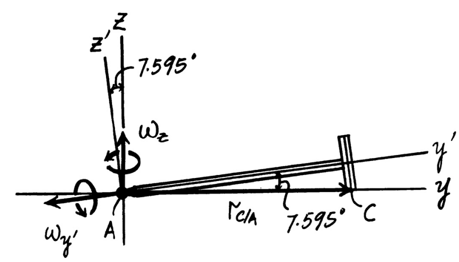
$$\mathbf{0} = \mathbf{0} + \left[(0.13216\,\omega_z - \omega_{y'})\mathbf{j}' + 0.99123\omega_z\,\mathbf{k}'\right] \times (1.5\mathbf{j}' - 0.2\mathbf{k}')$$

$$0 = -1.48684\omega_z - 0.026433\,\omega_z + 0.2\,\omega_{y'}$$

$$\omega_{y'} = 7.5664\,\omega_z$$

Thus,

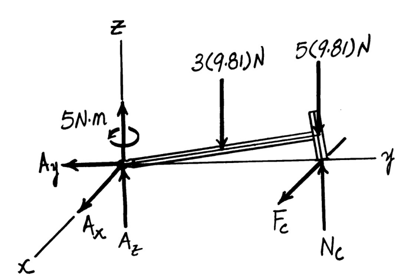
$$\omega = -7.4342\,\omega_z\,\mathbf{j}' + 0.99123\,\omega_z\,\mathbf{k}'$$



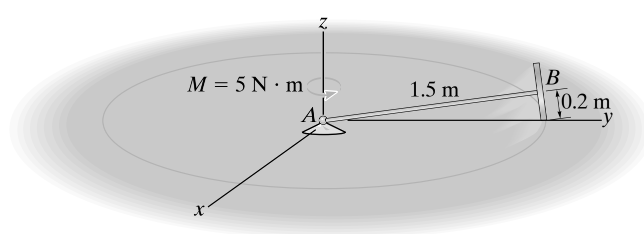
$$T_1 + \Sigma U_{1-2} = T_2$$

$$0 + 5(2\pi)(2) = 0 + \frac{1}{2}(0.100)(-7.4342\,\omega_z)^2 + \frac{1}{2}(13.55)(0.99123\,\omega_z)^2$$

$$\omega_z = 2\ 58 \text{ rad/s}$$ **Ans.**



21–26. The 5-kg disk is connected to the 3-kg slender rod. If the assembly is attached to a ball-and-socket joint at A and the 5-N·m couple moment gives it an angular velocity about the z axis of $\omega_z = 2$ rad/s, determine the magnitude of the angular momentum of the assembly about A.

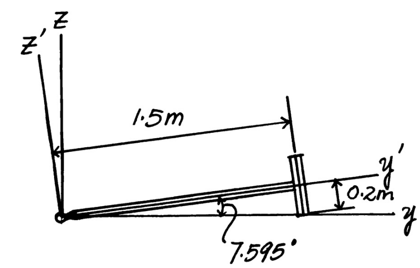


$$I_x = I_z = \frac{1}{4}(5)(0.2)^2 + 5(1.5)^2 + \frac{1}{3}(3)(1.5)^2 = 13.55$$

$$I_x = \frac{1}{2}(5)(0.2)^2 = 0.100$$

$$\omega = -\omega_y\,\mathbf{j}' + \omega_z\,\mathbf{k} = -\omega_{y'}\,\mathbf{j}' + \omega_z \sin 7.595°\mathbf{j}' + \omega_z \cos 7.595°\mathbf{k}'$$

$$= (0.13216\omega_z - \omega_y)\mathbf{j}' + 0.99123\,\omega_z\,\mathbf{k}'$$



Since points A and C have zero velocity,

$$\mathbf{v}_C = \mathbf{v}_A + \omega \times \mathbf{r}_{C/A}$$

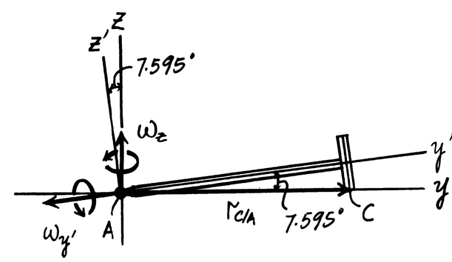
$$\mathbf{0} = \mathbf{0} + \left[(0.13216\,\omega_z - \omega_y\,)\,\mathbf{j}' + 0.99123\,\omega_z\,\mathbf{k}'\right] \times (1.5\mathbf{j}' - 0.2\mathbf{k}')$$

$$0 = -1\;\;48684\omega_z - 0.26433\omega_z + 0.2\omega_y$$

$$\omega_{y'} = -7.5664\,\omega_z$$

Thus,

$$\omega = -7.4342\,\omega_z\,\mathbf{j}' + 0.99123\,\omega_z\,\mathbf{k}'$$



Since $\omega_z = 2$ rad/s

$$\omega = -14.868\mathbf{j}' + 1\;\;9825\mathbf{k}'$$

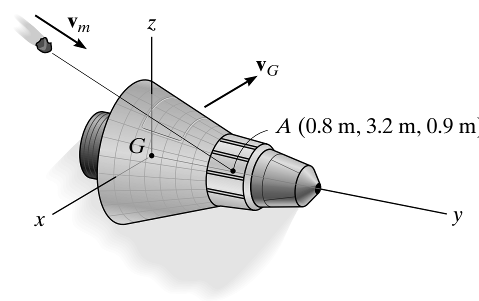
So that,

$$\mathbf{H}_A = I_{x'}\,\omega_x\,\mathbf{i}' + I_{y'}\,\omega_{y'}\,\mathbf{j} + I_{z'}\,\omega_{z'}\,\mathbf{k}' = \mathbf{0} + 0.100(-14.868)\mathbf{j}' + 13.55(1.9825)\,\mathbf{k}'$$

$$= -1.4868\mathbf{j}' + 26.862\mathbf{k}'$$

$$H_A = \sqrt{(-1.4868)^2 + (26.862)^2} = 26.9 \text{ kg}\cdot\text{m}^2/\text{s}$$ **Ans.**

21–27. The space capsule has a mass of 5 Mg and the radii of gyration are $k_x = k_z = 1.30$ m and $k_y = 0.45$ m. If it travels with a velocity $\mathbf{v}_G = \{400\mathbf{j} + 200\mathbf{k}\}$ m/s, compute its angular velocity just after it is struck by a meteoroid having a mass of 0.80 kg and a velocity $\mathbf{v}_m = \{-300\mathbf{i} + 200\mathbf{j} - 150\mathbf{k}\}$ m/s. Assume that the meteoroid embeds itself into the capsule at point A and that the capsule initially has no angular velocity.



Conservation of Angular Momentum: The angular momentum is conserved about the center of mass of the space capsule G. Neglect the mass of the meteroid after the impact.

$$(H_G)_1 = (H_G)_2$$

$$\mathbf{r}_{GA} \times m_m \mathbf{v}_m = I_G \omega$$

$$(0.8\mathbf{i} + 3.2\mathbf{j} + 0.9\mathbf{k}) \times 0.8(-300\mathbf{i} + 200\mathbf{j} - 150\mathbf{k})$$
$$= 5000\left(1.30^2\right)\omega_x\,\mathbf{i} + 5000\left(0.45^2\right)\omega_y\,\mathbf{j} + 5000\left(1.30^2\right)\omega_z\,\mathbf{k}$$

$$-528\mathbf{i} - 120\mathbf{j} + 896\mathbf{k} = 8450\omega_x\,\mathbf{i} + 1012.5\omega_y\,\mathbf{j} + 8450\,\omega_z\,\mathbf{k}$$

Equating **i**, **j** and **k** components, we have

$$-528 = 8450\omega_x \qquad \omega_x = -0.06249 \text{ rad/s}$$

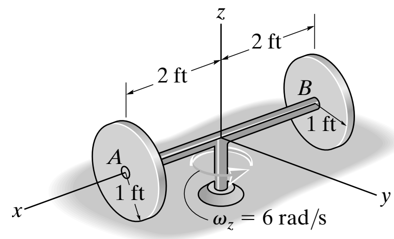
$$-120 = 1012.5\omega_y \qquad \omega_y = -0.11852 \text{ rad/s}$$

$$896 = 8450\omega_z \qquad \omega_z = 0.1060 \text{ rad/s}$$

Thus,

$$\omega = \{-0.0625\mathbf{i} - 0.119\mathbf{j} + 0.106\mathbf{k}\} \text{ rad/s} \qquad \textbf{Ans.}$$

***21–28.** Each of the two disks has a weight of 10 lb. The axle AB weighs 3 lb. If the assembly rotates about the z axis at $\omega_z = 6$ rad/s, determine its angular momentum about the z axis and its kinetic energy. The disks roll without slipping.

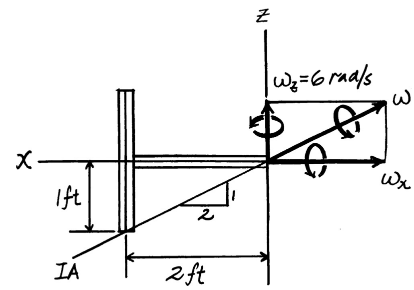


$$\frac{6}{\omega_x} = \frac{1}{2} \qquad \omega_x = 12 \text{ rad/s}$$

$$\omega_A = \{-12\mathbf{i}\} \text{ rad/s} \qquad \omega_B = \{12\mathbf{i}\} \text{ rad/s}$$

$$\mathbf{H}_z = \left[\frac{1}{2}\left(\frac{10}{32.2}\right)(1)^2\right](12\mathbf{i}) + \left[\frac{1}{2}\left(\frac{10}{32.2}\right)(1)^2\right](-12\mathbf{i})$$

$$+ \mathbf{0} + \left\{2\left[\frac{1}{4}\left(\frac{10}{32.2}\right)(1)^2 + \frac{10}{32.2}(2)^2\right](6) + \frac{1}{12}\left(\frac{3}{32.2}\right)(4)^2(6)\right\}\mathbf{k}$$



$$\mathbf{H}_z = \{16.6\mathbf{k}\} \text{ slug} \cdot \text{ft}^2/\text{s}$$ **Ans.**

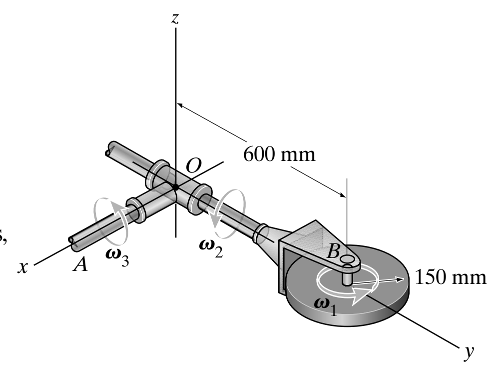
$$T = \frac{1}{2} I_x \omega_x^2 + \frac{1}{2} I_y \omega_y^2 + \frac{1}{2} I_z \omega_z^2$$

$$= \frac{1}{2}\left[2\left(\frac{1}{2}\left(\frac{10}{32.2}\right)(1)^2\right)\right](12)^2 + 0$$

$$+ \frac{1}{2}\left\{2\left[\frac{1}{4}\left(\frac{10}{32.2}\right)(1)^2 + \frac{10}{32.2}(2)^2\right] + \frac{1}{12}\left(\frac{3}{32.2}\right)(4)^2\right\}(6)^2$$

$$= 72.1 \text{ lb} \cdot \text{ft}$$ **Ans.**

•21–29. The 10-kg circular disk spins about its axle with a constant angular velocity of $\omega_1 = 15$ rad/s. Simultaneously, arm OB and shaft OA rotate about their axes with constant angular velocities of $\omega_2 = 0$ and $\omega_3 = 6$ rad/s, respectively. Determine the angular momentum of the disk about point O, and its kinetic energy.

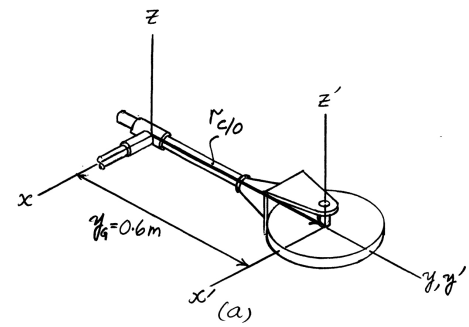


The mass moments of inertia of the disk about the centroidal x', y', and z' axes, Fig. *a*, are

$$I_{x'} = I_{y'} = \frac{1}{4}mr^2 = \frac{1}{4}(10)\left(0.15^2\right) = 0.05625\ \text{kg}\cdot\text{m}^2$$

$$I_{z'} = \frac{1}{2}mr^2 = \frac{1}{2}(10)\left(0.15^2\right) = 0.1125\ \text{kg}\cdot\text{m}^2$$

Due to symmetry, the products of inertia of the disk with respect to its centroidal planes are equal to zero.



$$I_{x'y'} = I_{y'z'} = I_{x'z'} = 0$$

Here, the angular velocity of the disk can be determined from the vector addition of ω_1 and ω_3. Thus,

$$\omega = \omega_1 + \omega_2 = [6\mathbf{i} + 15\mathbf{k}]\ \text{rad/s}$$

The angular momentum of the disk about its mass center G can be obtained by applying

$$H_x = I_{x'}\omega_x = 0.05625(6) = 0.3375\ \text{kg}\cdot\text{m}^2/\text{s}$$

$$H_y = I_{y'}\omega_y = 0.05625(0) = 0$$

$$H_z = I_{z'}\omega_z = 0.1125(15) = 1.6875\ \text{kg}\cdot\text{m}^2/\text{s}$$

Thus,

$$\mathbf{H}_G = [0.3375\mathbf{i} + 1.6875\mathbf{k}]\ \text{kg}\cdot\text{m}^2/\text{s}$$

Since the mass center G rotates about the x axis with a constant angular velocity of $\omega_3 = [6\mathbf{i}]$ rad/s, its velocity is

$$\mathbf{v}_G = \omega_3 \times \mathbf{r}_{C/O} = (6\mathbf{i}) \times (0.6\mathbf{j}) = [3.6\mathbf{k}]\ \text{m/s}$$

Since the disk does not rotate about a fixed point O, its angular momentum must be determined from

$$\mathbf{H}_O = \mathbf{r}_{C/O} \times m\mathbf{v}_C + \mathbf{H}_G$$

$$= (0.6\mathbf{j}) \times 10(3.6\mathbf{k}) + (0.3375\mathbf{i} + 1.6875\mathbf{k})$$

$$= [21.9375\mathbf{i} + 1.6875\mathbf{k}]\ \text{kg}\cdot\text{m}^2/\text{s}$$

$$= [21.9\mathbf{i} + 1.69\mathbf{k}]\ \text{kg}\cdot\text{m}^2/\text{s}$$ **Ans.**

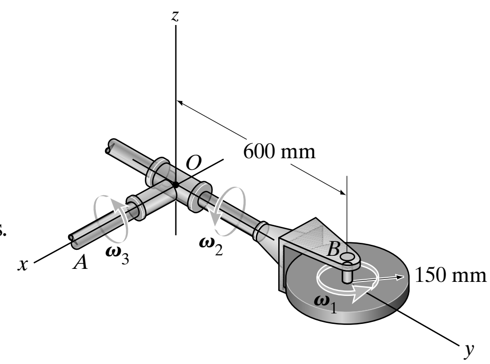
The kinetic energy of the disk is therefore

$$T = \frac{1}{2}\omega\cdot\mathbf{H}_O$$

$$= \frac{1}{2}(6\mathbf{i} + 15\mathbf{k})\cdot(21.9375\mathbf{i} + 1.6875\mathbf{k})$$

$$= 78.5\ \text{J}$$ **Ans.**

21–30. The 10-kg circular disk spins about its axle with a constant angular velocity of $\omega_1 = 15$ rad/s. Simultaneously, arm OB and shaft OA rotate about their axes with constant angular velocities of $\omega_2 = 10$ rad/s and $\omega_3 = 6$ rad/s, respectively. Determine the angular momentum of the disk about point O, and its kinetic energy.



The mass moments of inertia of the disk about the centroidal x', y', and z' axes. Fig. a, are

$$I_{x'} = I_{y'} = \frac{1}{4}mr^2 = \frac{1}{4}(10)\left(0.15^2\right) = 0.05625 \text{ kg}\cdot\text{m}^2$$

$$I_{z'} = \frac{1}{2}mr^2 = \frac{1}{2}(10)\left(0.15^2\right) = 0.1125 \text{ kg}\cdot\text{m}^2$$

Due to symmetry, the products of inertia of the disk with respect to its centroidal planes are equal to zero.

$$I_{x'y'} = I_{y'z'} = I_{x'z'} = 0$$

Here, the angular velocity of the disk can be determined from the vector addition of ω_1, ω_2, and ω_3. Thus,

$$\omega = \omega_1 + \omega_2 + \omega_3 = [6\mathbf{i} + 10\mathbf{j} + 15\mathbf{k}] \text{ rad/s}$$

The angular momentum of the disk about its mass center G can be obtained by applying

$$H_x = I_{x'}\omega_x = 0.05625(6) = 0.3375 \text{ kg}\cdot\text{m}^2$$

$$H_y = I_{y'}\omega_y = 0.05625(10) = 0.5625 \text{ kg}\cdot\text{m}^2$$

$$H_z = I_{z'}\omega_z = 0.1125(15) = 1.6875 \text{ kg}\cdot\text{m}^2$$

Thus,

$$\mathbf{H}_G = [0.3375\mathbf{i} + 0.5625\mathbf{j} + 1.6875\mathbf{k}] \text{ kg}\cdot\text{m}^2$$

Since the mass center G rotates about the fixed point O with an angular velocity of $\Omega = \omega_2 + \omega_3 = [6\mathbf{i} + 10\mathbf{j}]$, its velocity is

$$\mathbf{v}_G = \Omega \times \mathbf{r}_{G/O} = (6\mathbf{i} + 10\mathbf{j}) \times (0.6\mathbf{j}) = [3.6\mathbf{k}] \text{ m/s}$$

Since the disk does not rotate about a fixed point O, its angular momentum must be determined from

$$\mathbf{H}_O = \mathbf{r}_{C/O} \times m\mathbf{v}_G + \mathbf{H}_G$$

$$= (0.6\mathbf{j}) \times 10(3.6\mathbf{k}) + (0.3375\mathbf{i} + 0.5625\mathbf{j} + 1.6875\mathbf{k})$$

$$= [21.9375\mathbf{i} + 0.5625\mathbf{j} + 1.6875\mathbf{k}] \text{ kg}\cdot\text{m}^2/\text{s}$$

$$= [21.9\mathbf{i} + 0.5625\mathbf{j} + 1.69\mathbf{k}] \text{ kg}\cdot\text{m}^2/\text{s}$$ **Ans.**

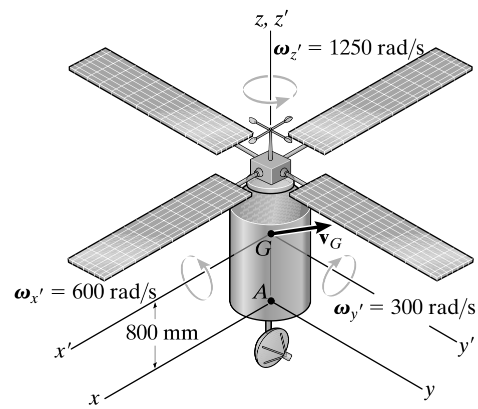
The kinetic energy of the disk is therefore

$$T = \frac{1}{2}\omega\cdot\mathbf{H}_O$$

$$= \frac{1}{2}(6\mathbf{i} + 10\mathbf{j} + 15\mathbf{k})\cdot(21.9375\mathbf{i} + 0.5625\mathbf{j} + 1.6875\mathbf{k})$$

$$= 81.3\text{J}$$ **Ans.**

21–31. The 200-kg satellite has its center of mass at point G. Its radii of gyration about the z', x', y' axes are $k_{z'} = 300$ mm, $k_{x'} = k_{y'} = 500$ mm, respectively. At the instant shown, the satellite rotates about the x', y', and z' axes with the angular velocity shown, and its center of mass G has a velocity of $\mathbf{v}_G = \{-250\mathbf{i} + 200\mathbf{j} + 120\mathbf{k}\}$ m/s. Determine the angular momentum of the satellite about point A at this instant.



The mass moments of inertia of the satellite about the x', y', and z' axes are

$$I_{x'} = I_{y'} = 200\left(0.5^2\right) = 50 \text{ kg}\cdot\text{m}^2$$

$$I_{z'} = 200\left(0.3^2\right) = 18 \text{ kg}\cdot\text{m}^2$$

Due to symmetry, the products of inertia of the satellite with respect to the x', y', and z' coordinate system are equal to zero.

$$I_{x'y'} = I_{y'z'} = I_{x'z'} = 0$$

The angular velocity of the satellite is

$$\omega = [600\mathbf{i} + 300\mathbf{j} + 1250\mathbf{k}] \text{ rad/s}$$

Thus,

$$\omega_{x'} = 600 \text{ rad/s} \qquad \omega_{y'} = -300 \text{ rad/s} \qquad \omega_{z'} = 1250 \text{ rad/s}$$

Then, the components of the angular momentum of the satellite about its mass center G are

$$(H_G)_{x'} = I_{x'}\omega_{x'} = 50(600) = 30\,000 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_G)_{y'} = I_{y'}\omega_{y'} = 50(-300) = 15\,000 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$(H_G)_{z'} = I_{z'}\omega_{z'} = 18(1250) = 22\,500 \text{ kg}\cdot\text{m}^2/\text{s}$$

Thus,

$$\mathbf{H}_G = [30\,000\mathbf{i} + 15\,000\mathbf{j} + 22\,500\mathbf{k}] \text{ kg}\cdot\text{m}^2/\text{s}$$

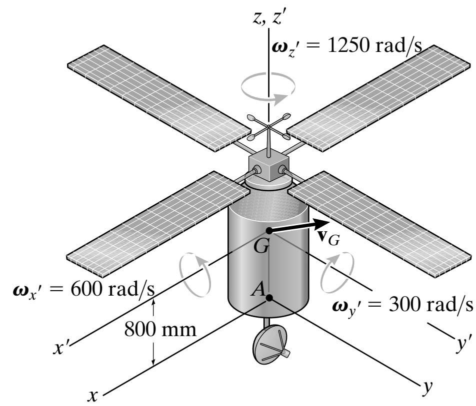
The angular momentum of the satellite about point A can be determined from

$$\mathbf{H}_A = \mathbf{r}_{G/A} \times m\mathbf{v}_G + \mathbf{H}_G$$

$$= (0.8\mathbf{k}) \times 200(-250\mathbf{i} + 200\mathbf{j} + 120\mathbf{k}) + (30\,000\mathbf{i} + 15\,000\mathbf{j} + 22\,500\mathbf{k})$$

$$= [-2000\mathbf{i} - 25\,000\mathbf{j} + 22\,500\mathbf{k}] \text{ kg}\cdot\text{m}^2/\text{s}$$ **Ans.**

***21–32.** The 200-kg satellite has its center of mass at point G. Its radii of gyration about the z', x', y' axes are $k_{z'} = 300$ mm, $k_{x'} = k_{y'} = 500$ mm, respectively. At the instant shown, the satellite rotates about the x', y', and z' axes with the angular velocity shown, and its center of mass G has a velocity of $\mathbf{v}_G = \{-250\mathbf{i} + 200\mathbf{j} + 120\mathbf{k}\}$ m/s. Determine the kinetic energy of the satellite at this instant.



The mass moments of inertia of the satellite about the x', y', and z' axes are

$$I_{x'} = I_{y'} = 200\left(0.5^2\right) = 50\ \text{kg}\cdot\text{m}^2$$

$$I_{z'} = 200\left(0.3^2\right) = 18\ \text{kg}\cdot\text{m}^2$$

Due to symmetry, the products of inertia of the satellite with respect to the x', y', and z' coordinate system are equal to zero.

$$I_{x'y'} = I_{y'z'} = I_{x'z'} = 0$$

The angular velocity of the satellite is

$$\omega = [600\mathbf{i} - 300\mathbf{j} + 1250\mathbf{k}]\ \text{rad/s}$$

Thus,

$$\omega_{x'} = 600\ \text{rad/s} \qquad \omega_{y'} = -300\ \text{rad/s} \qquad \omega_{z'} = 1250\ \text{rad/s}$$

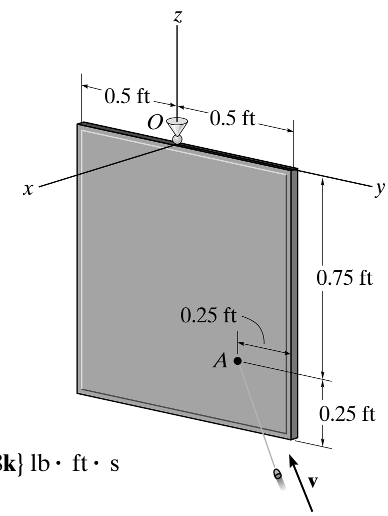
Since $v_G{}^2 = (-250)^2 + 200^2 + 120^2 = 116\,900\ \text{m}^2/\text{s}^2$, the kinetic energy of the satellite can be determined from

$$T = \frac{1}{2}mv_G{}^2 + \frac{1}{2}I_{x'}\omega_{x'}{}^2 + \frac{1}{2}I_{y'}\omega_{y'}{}^2 + \frac{1}{2}I_{z'}\omega_{z'}{}^2$$

$$= \frac{1}{2}(200)(116\,900) + \frac{1}{2}(50)\left(600^2\right) + \frac{1}{2}(50)(-300)^2 + \frac{1}{2}(18)\left(1250^2\right)$$

$$= 37.0025\left(10^6\right)\text{J} = 37.0\text{MJ}$$ **Ans.**

•21–33. The 25-lb thin plate is suspended from a ball-and-socket joint at O. A 0.2-lb projectile is fired with a velocity of $\mathbf{v} = \{-300\mathbf{i} - 250\mathbf{j} + 300\mathbf{k}\}$ ft/s into the plate and becomes embedded in the plate at point A. Determine the angular velocity of the plate just after impact and the axis about which it begins to rotate. Neglect the mass of the projectile after it embeds into the plate.



Angular momentum about point O is conserved.

$$(\mathbf{H}_O)_2 = (\mathbf{H}_O)_1 = \mathbf{r}_{OA} \times m_p \mathbf{v}_p$$

$$(\mathbf{H}_O)_1 = (0.25\mathbf{j} - 0.75\mathbf{k}) \times \left(\frac{0.2}{32.2}\right)(-300\mathbf{i} - 250\mathbf{j} + 300\mathbf{k}) = \{-0.6988\mathbf{i} + 1.3975\mathbf{j} + 0.4658\mathbf{k}\}\ \text{lb}\cdot\text{ft}\cdot\text{s}$$

$$I_x = \left(\frac{1}{12}\right)\left(\frac{25}{32.2}\right)\left[(1)^2 + (1)^2\right] + \left(\frac{25}{32.2}\right)(0.5)^2 = 0.3235\ \text{slug}\cdot\text{ft}^2$$

$$I_y = \left(\frac{1}{12}\right)\left(\frac{25}{32.2}\right)(1)^2 + \left(\frac{25}{32.2}\right)(0.5)^2 = 0.2588\ \text{slug}\cdot\text{ft}^2$$

$$I_z = \left(\frac{1}{12}\right)\left(\frac{25}{32.2}\right)(1)^2 = 0.06470\ \text{slug}\cdot\text{ft}^2$$

$$(\mathbf{H}_O)_1 = (\mathbf{H}_O)_2$$

$$-0.6988\mathbf{i} + 1.3975\mathbf{j} + 0.4658\mathbf{k} = 0.3235\omega_x\,\mathbf{i} + 0.2588\omega_y\,\mathbf{j} + 0.06470\omega_z\,\mathbf{k}$$

$$\omega_x = \frac{-0.6988}{0.3235} = -2.160\ \text{rad/s}$$

$$\omega_y = \frac{1.3975}{0.2588} = 5.400\ \text{rad/s}$$

$$\omega_z = \frac{0.4658}{0.06470} = 7.200\ \text{rad/s}$$

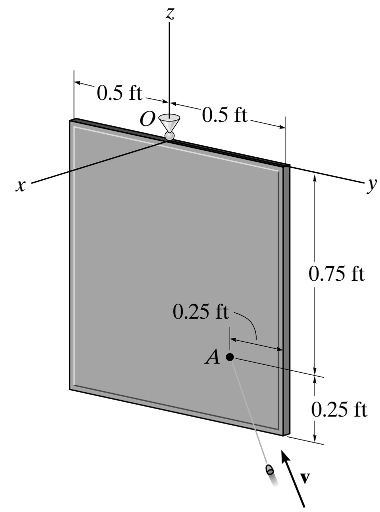
$\omega = \{-2.16\mathbf{i} + 5.40\mathbf{j} + 7.20\mathbf{k}\}$ rad/s **Ans.**

Axis of rotation line is along ω:

$$\mathbf{u}_O = \frac{-2.160\mathbf{i} + 5.400\mathbf{j} + 7.200\mathbf{k}}{\sqrt{(-2.160)^2 + (5.400)^2 + (7.200)^2}}$$

$= -0.233\mathbf{i} + 0.583\mathbf{j} + 0.778\mathbf{k}$ **Ans.**

21–34. Solve Prob. 21–33 if the projectile emerges from the plate with a velocity of 275 ft/s in the same direction.



$$\mathbf{u_v} = \left(\frac{-300}{492.4}\right)\mathbf{i} - \left(\frac{250}{492.4}\right)\mathbf{j} + \left(\frac{300}{492.4}\right)\mathbf{k} = -0.6092\mathbf{i} - 0.5077\mathbf{j} + 0.6092\mathbf{k}$$

$$I_x = \left(\frac{1}{12}\right)\left(\frac{25}{32.2}\right)\left[(1)^2 + (1)^2\right] + \left(\frac{25}{32.2}\right)(0.5)^2 = 0.32350 \text{ slug}\cdot\text{ft}^2$$

$$I_y = \left(\frac{1}{12}\right)\left(\frac{25}{32.2}\right)(1)^2 + \left(\frac{25}{32.2}\right)(0.5)^2 = 0.25880 \text{ slug}\cdot\text{ft}^2$$

$$I_z = \left(\frac{1}{12}\right)\left(\frac{25}{32.2}\right)(1)^2 = 0.06470 \text{ slug}\cdot\text{ft}^2$$

$$\mathbf{H}_1 + \Sigma\int \mathbf{M}_O\,dt = \mathbf{H}_2$$

$$(0.25\mathbf{j} - 0.75\mathbf{k}) \times \left(\frac{0.2}{32.2}\right)(-300\mathbf{i} - 250\mathbf{j} + 300\mathbf{k}) + \mathbf{0} = 0.32350\omega_x\,\mathbf{i} + 0.25880\omega_y\,\mathbf{j} + 0.06470\omega_z\,\mathbf{k}$$

$$+(0.25\mathbf{j} - 0.75\mathbf{k}) \times \left(\frac{0.2}{32.2}\right)(275)(-0.6092\mathbf{i} - 0.5077\mathbf{j} + 0.6092\mathbf{k})$$

Expanding, the **i**, **j**, **k**, components are:

$$-0.6988 = 0.32350\omega_x - 0.390215$$

$$1.3975 = 0.25880\omega_y + 0.78043$$

$$0.4658 = 0.06470\omega_z + 0.26014$$

$$\omega_x = -0.9538, \qquad \omega_y = 2.3844, \qquad \omega_z = 3.179$$

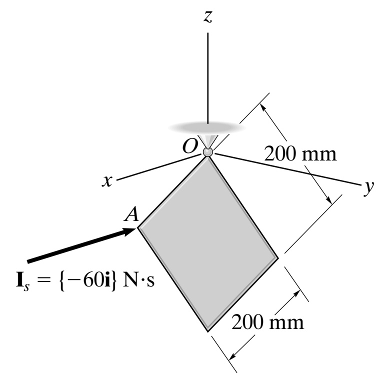
$$\omega = \{-0.954\mathbf{i} + 2.38\mathbf{j} + 3.18\mathbf{k}\} \text{ rad/s}$$ **Ans.**

Axis of rotation is along ω:

$$\mathbf{u}_A = \frac{-0.954\mathbf{i} + 2.38\mathbf{j} + 3.18\mathbf{k}}{\sqrt{(-0.954)^2 + (2.38)^2 + (3.18)^2}}$$

$$\mathbf{u}_A = -0.233\mathbf{i} + 0.583\mathbf{j} + 0.778\mathbf{k}$$ **Ans.**

21–35. A thin plate, having a mass of 4 kg, is suspended from one of its corners by a ball-and-socket joint O. If a stone strikes the plate perpendicular to its surface at an adjacent corner A with an impulse of $\mathbf{I}_s = \{-60\mathbf{i}\}\ \text{N}\cdot\text{s}$, determine the instantaneous axis of rotation for the plate and the impulse created at O.



$$(\mathbf{H}_O)_1 + \Sigma\int \mathbf{M}_O\,dt = (\mathbf{H}_O)_2$$

$$\mathbf{0} + \mathbf{r}_{A/O}\times\mathbf{I}_S = (\mathbf{H}_O)_2$$

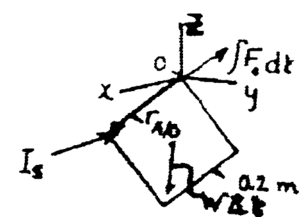
$$\mathbf{0} + (-0.2(0.7071)\mathbf{j} - 0.2(0.7071)\mathbf{k})\times(-60\mathbf{i}) = (I_O)_x\,\omega_x\,\mathbf{i} + (I_O)_y\,\omega_y\,\mathbf{j} + (I_O)_z\,\omega_z\,\mathbf{k}$$

Expand and equate components:

$$0 = (I_O)_x\,\omega_x \qquad (1)$$

$$8.4853 = (I_O)_y\,\omega_y \qquad (2)$$

$$-8.4853 = (I_O)_z\,\omega_z \qquad (3)$$



$$I_{x'y'} = 0, \qquad I_{y'z'} = 0, \qquad I_{x'z'} = 0$$

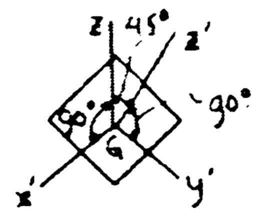
$$I_{y'} = \left(\frac{1}{12}\right)(4)(0.2)^2 = 0.01333, \qquad I_{z'} = \left(\frac{1}{12}\right)(4)(0.2)^2 = 0.01333$$

$$u_{x'} = \cos 90^\circ = 0, \qquad u_{y'} = \cos 135^\circ = -0.7071, \qquad u_{z'} = \cos 45^\circ = 0.7071$$

$$(I_G)_z = I_{x'}u_{x'}^2 + I_{y'}u_{y'}^2 + I_{z'}u_{z'}^2 - 2I_{x'y'}u_{x'}u_{y'} - 2I_{y'z'}u_{y'}u_{z'} - 2I_{z'x'}u_{z'}u_{x'}$$

$$= 0 + (0.01333)(-0.7071)^2 + (0.01333)(0.7071)^2 - 0 - 0 - 0$$

$$(I_G)_z = (I_O)_z = 0.01333$$



For $(I_O)_y$, use the parallel axis theorem.

$$(I_O)_y = 0.01333 + 4\left[0.7071(0.2)\right]^2, \qquad (I_O)_y = 0.09333$$

Hence, from Eqs. (1) and (2):

$$\omega_x = 0, \qquad \omega_y = 90.914, \qquad \omega_z = -636.340$$

The instantaneous axis of rotation is thus,

$$\mathbf{u}_{IA} = \frac{90.914\mathbf{j} - 636.340\mathbf{k}}{\sqrt{(90.914)^2 + (-636.340)^2}} = 0.141\mathbf{j} - 0.990\mathbf{k} \qquad \textbf{Ans.}$$

The velocity of G just after the plate is hit is

$$\mathbf{v}_G = \omega\times\mathbf{r}_{G/O}$$

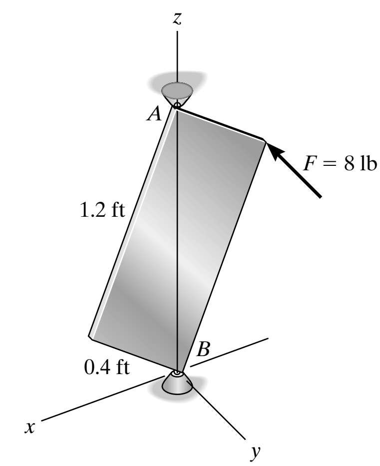
$$\mathbf{v}_G = (90.914\mathbf{j} - 636.340\mathbf{k})\times(-0.2(0.7071)\mathbf{k}) = -12.857\mathbf{i}$$

$$m(\mathbf{v}_G)_1 + \Sigma\int \mathbf{F}\,dt = m(\mathbf{v}_G)_2$$

$$\mathbf{0} - 60\mathbf{i} + \int \mathbf{F}_O\,dt = -4(12.857)\mathbf{i}$$

$$\int \mathbf{F}_O\,dt = \{8.57\mathbf{i}\}\ \text{N}\cdot\text{s} \qquad \textbf{Ans.}$$

***21–36.** The 15-lb plate is subjected to a force $F = 8$ lb which is always directed perpendicular to the face of the plate. If the plate is originally at rest, determine its angular velocity after it has rotated one revolution (360°). The plate is supported by ball-and-socket joints at A and B.



Due to symmetry

$$I_{x'y'} = I_{y'z'} = I_{z'x'} = 0$$

$$I_{x'} = \frac{1}{12}\left(\frac{15}{32.2}\right)(1.2)^2 = 0.05590 \text{ slug}\cdot\text{ft}^2$$

$$I_{y'} = \frac{1}{12}\left(\frac{15}{32.2}\right)(1.2^2 + 0.4^2) = 0.06211 \text{ slug}\cdot\text{ft}^2$$

$$I_{z'} = \frac{1}{12}\left(\frac{15}{32.2}\right)(0.4)^2 = 0.006211 \text{ slug}\cdot\text{ft}^2$$

For z axis

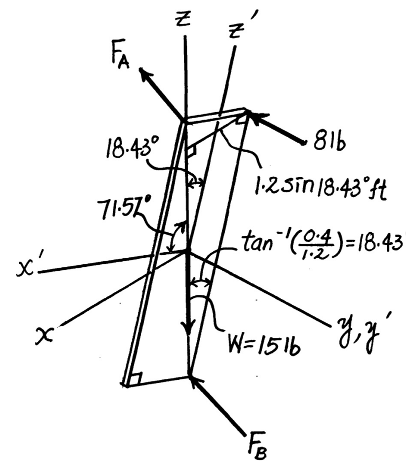
$u_{x'} = \cos 71.57° = 0.3162 \qquad u_{y'} = \cos 90° = 0$

$u_{z'} = \cos 18.43° = 0.9487$

$$I_z = I_{x'}u_{x'}^2 + I_{y'}u_{y'}^2 + I_{z'}u_{z'}^2 - 2I_{x'y'}u_{x'}u_{y'} - 2I_{y'z'}u_{y'}u_{z'} - 2I_{z'x'}u_{z'}u_{x'}$$

$$= 0.05590(0.3162)^2 + 0 + 0.006211(0.9487)^2 - 0 - 0 - 0$$

$$= 0.01118 \text{ slug}\cdot\text{ft}^2$$



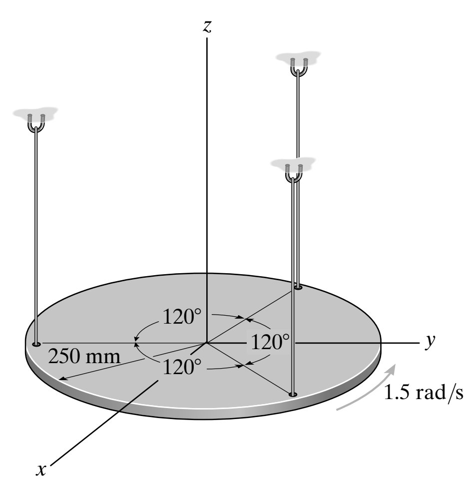
Principle of work and energy:

$$T_1 + \Sigma U_{1-2} = T_2$$

$$0 + 8(1.2 \sin 18.43°)(2\pi) = \frac{1}{2}(0.01118)\omega^2$$

$$\omega = 58.4 \text{ rad/s} \qquad \textbf{Ans.}$$

•21–37. The plate has a mass of 10 kg and is suspended from parallel cords. If the plate has an angular velocity of 1.5 rad/s about the z axis at the instant shown, determine how high the center of the plate rises at the instant the plate momentarily stops swinging.



Consevation Energy: Datum is set at the initial position of the plate. When the plate is at its final position and its mass center is located h *above* the datum. Thus, its gravitational potential energy at this position is $10(9.81)h = 98.1h$. Since the plate momentarily stops swinging, its final kinetic energy $T_2 = 0$. Its initial kinetic energy i

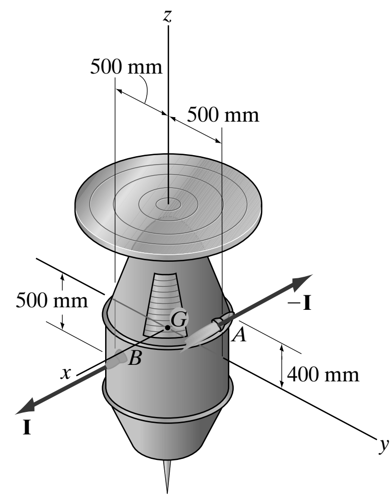
$$T_1 = \frac{1}{2}I_G\,\omega^2 = \frac{1}{2}\left[\frac{1}{2}(10)\left(0.25^2\right)\right]\left(1.5^2\right) = 0.3516 \text{ J}$$

$$T_1 + V_1 = T_2 + V_2$$

$$0.3516 + 0 = 0 + 98.1h$$

$$h = 0.00358 \text{ m} = 3.58 \text{ mm} \qquad \textbf{Ans.}$$

21–38. The satellite has a mass of 200 kg and radii of gyration of $k_x = k_y = 400$ mm and $k_z = 250$ mm. When it is not rotating, the two small jets A and B are ignited simultaneously, and each jet provides an impulse of $I = 1000\ \text{N}\cdot\text{s}$ on the satellite. Determine the satellite's angular velocity immediately after the ignition.



The mass moments of inertia of the satellite about the x, y, and z axes are

$$I_x = I_y = 200\left(0.4^2\right) = 32\ \text{kg}\cdot\text{m}^2$$

$$I_z = 200\left(0.25^2\right) = 12.5\ \text{kg}\cdot\text{m}^2$$

Due to symmetry,

$$I_{xy} = I_{yz} = I_{xz} = 0$$

Thus, the angular momentum of the satellite about its mass center G is

$$H_x = I_x\omega_x = 32\omega_x \qquad H_y = I_y\omega_y = 32\omega_y \qquad H_z = I_z\omega_z = 12.5\omega_z$$

Applying the principle of angular impulse and momentum about the x, y, and z axes,

$$(H_x)_1 + \Sigma\int_{t_1}^{t_2} M_x\,dt = (H_x)_2$$

$$0 + 0 = 32\omega_x$$

$$\omega_x = 0$$

$$\left(H_y\right)_1 + \Sigma\int_{t_1}^{t_2} M_y\,dt = \left(H_y\right)_2$$

$$0 - 1000(0.4) - 1000(0.5) = 32\omega_y$$

$$\omega_y = -28.125\ \text{rad/s}$$

$$(H_z)_1 + \Sigma\int_{t_1}^{t_2} M_y\,dt = (H_z)_2$$

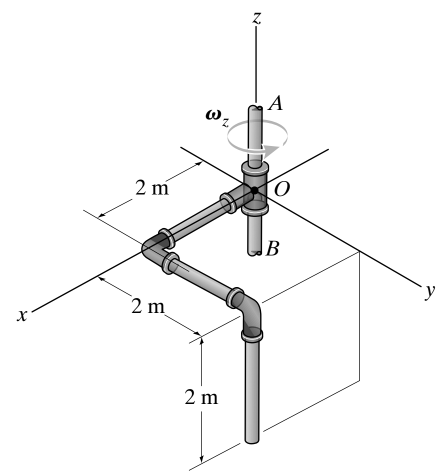
$$0 + 1000(0.5) + 1000(0.5) = 12.5\omega_z$$

$$\omega_z = 80\ \text{rad/s}$$

Thus

$$\omega = \{-28.1\mathbf{j} + 80\mathbf{k}\}\ \text{rad/s}$$ **Ans.**

21–39. The bent rod has a mass per unit length of 6 kg/m, and its moments and products of inertia have been calculated in Prob. 21–9. If shaft AB rotates with a constant angular velocity of $\omega_z = 6$ rad/s, determine the angular momentum of the rod about point O, and the kinetic energy of the rod.



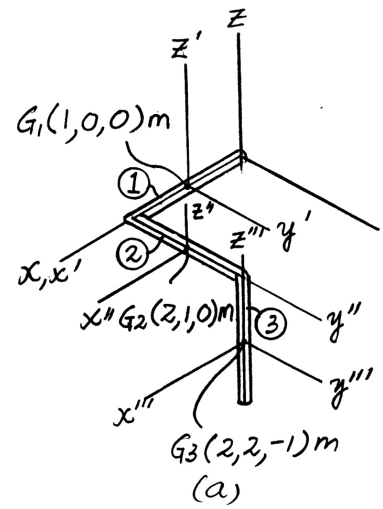
Here, the angular velocity of the rod is

$$\omega = [6\mathbf{k}] \text{ rad/s}$$

Thus,

$$\omega_x = \omega_y = 0 \qquad \omega_z = 6 \text{ rad/s}$$

The rod rotates about a fixed point O. Using the results of Prob. 20–91



$$H_x = I_x\omega_x - I_{xy}\omega_y - I_{xz}\omega_z$$

$$= 80(0) - 72(0) - (-24)(6) = 144 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$H_y = -I_{xy}\omega_x + I_y\omega_y - I_{yz}\omega_z$$

$$= -72(0) + 128(0) - (-24)(6) = 144 \text{ kg}\cdot\text{m}^2/\text{s}$$

$$H_z = -I_{xz}\omega_x - I_{yz}\omega_y + I_z\omega_z$$

$$= -(-24)(0) - (-24)(0) + 176(6) = 1056 \text{ kg}\cdot\text{m}^2/\text{s}$$

Thus,

$$\mathbf{H}_O = [144\mathbf{i} + 144\mathbf{j} + 1056\mathbf{k}] \text{ kg}\cdot\text{m}^2/\text{s}$$ **Ans.**

The kinetic energy of the rod can be determined from

$$T = \frac{1}{2}\omega\cdot\mathbf{H}_O$$

$$= \frac{1}{2}(6\mathbf{k})\cdot(144\mathbf{i} + 144\mathbf{j} + 1056\mathbf{k})$$

$$= 3168 \text{ J} = 3.17 \text{ kJ}$$ **Ans.**

***21–40.** Derive the scalar form of the rotational equation of motion about the x axis if $\mathbf{\Omega} \neq \boldsymbol{\omega}$ and the moments and products of inertia of the body are *not constant* with respect to time.

In general

$$\mathbf{M} = \frac{d}{dt}(H_x\,\mathbf{i} + H_y\,\mathbf{j} + H_z\,\mathbf{k})$$

$$= \left(\dot{H}_x\,\mathbf{i} + \dot{H}_y\mathbf{j} + \dot{H}_z\,\mathbf{k}\right)_{xyz} + \Omega \times (H_x\,\mathbf{i} + H_y\mathbf{j} + H_z\,\mathbf{k})$$

Substitute $\Omega = \Omega_x\,\mathbf{i} + \Omega_y\,\mathbf{j} + \Omega_z\,\mathbf{k}$ and expanding the cross product yields

$$\mathbf{M} = \left(\left(\dot{H}_x\right)_{xyz} - \Omega_z H_y + \Omega_y H_z\right)\mathbf{i} + \left(\left(\dot{H}_y\right)_{xyz} - \Omega_x H_z + \Omega_z H_x\right)\mathbf{j} + \left(\left(\dot{H}_z\right)_{xyz} - \Omega_y H_x + \Omega_x H_y\right)\mathbf{k}$$

Subsitute H_x, H_y and H_z using Eq. 21–10. For the **i** component

$$\Sigma M_x = \frac{d}{dt}(I_x\,\omega_x - I_{xy}\,\omega_y - I_{xz}\,\omega_z) - \Omega_z(I_y\,\omega_y - I_{yz}\,\omega_z - I_{yx}\omega_x) + \Omega_y(I_z\,\omega_z - I_{zx}\,\omega_x - I_{zy}\,\omega_y)$$ **Ans.**

One can obtain y and z components in a similar manner.

•21–41. Derive the scalar form of the rotational equation of motion about the x axis if $\mathbf{\Omega} \neq \boldsymbol{\omega}$ and the moments and products of inertia of the body are *constant* with respect to time.

In general

$$\mathbf{M} = \frac{d}{dt}(H_x\,\mathbf{i} + H_y\,\mathbf{j} + H_z\,\mathbf{k})$$

$$= \left(\dot{H}_x\,\mathbf{i} + \dot{H}_y\mathbf{j} + \dot{H}_z\,\mathbf{k}\right)_{xyz} + \Omega \times (H_x\,\mathbf{i} + H_y\mathbf{j} + H_z\,\mathbf{k})$$

Substitute $\Omega = \Omega_x\,\mathbf{i} + \Omega_y\,\mathbf{j} + \Omega_z\,\mathbf{k}$ and expanding the cross product yields

$$\mathbf{M} = \left(\left(\dot{H}_x\right)_{xyz} - \Omega_z H_y + \Omega_y H_z\right)\mathbf{i} + \left(\left(\dot{H}_y\right)_{xyz} - \Omega_x H_z + \Omega_z H_x\right)\mathbf{j} + \left(\left(\dot{H}_z\right)_{xyz} - \Omega_y H_x + \Omega_x H_y\right)\mathbf{k}$$

Substitute H_x, H_y and H_z using Eq. 21–10. For the **i** component

$$\Sigma M_x = \frac{d}{dt}(I_x\,\omega_x - I_{xy}\,\omega_y - I_{xz}\,\omega_z) - \Omega_z(I_y\,\omega_y - I_{yz}\,\omega_z - I_{yx}\omega_x) + \Omega_y(I_z\,\omega_z - I_{zx}\,\omega_x - I_{zy}\,\omega_y)$$

For constant inertia, expanding the time derivative of the above equation yields

$$\Sigma M_x = (I_x\,\dot{\omega}_x - I_{xy}\,\dot{\omega}_y - I_{xz}\dot{\omega}_z) - \Omega_z(I_y\,\omega_y - I_{yz}\,\omega_z - I_{yx}\,\omega_x) + \Omega_y(I_z\,\omega_z - I_{zx}\omega_x - I_{zy}\,\omega_y)$$ **Ans.**

One can obtain y and z components in a similar manner.

21–42. Derive the Euler equations of motion for $\mathbf{\Omega} \neq \boldsymbol{\omega}$, i.e., Eqs. 21–26.

In general

$$\mathbf{M} = \frac{d}{dt}(H_x\,\mathbf{i} + H_y\,\mathbf{j} + H_z\,\mathbf{k})$$

$$= \left(\dot{H}_x\,\mathbf{i} + \dot{H}_y\mathbf{j} + \dot{H}_z\,\mathbf{k}\right)_{xyz} + \Omega \times (H_x\,\mathbf{i} + H_y\mathbf{j} + H_z\,\mathbf{k})$$

Substitute $\Omega = \Omega_x\,\mathbf{i} + \Omega_y\,\mathbf{j} + \Omega_z\,\mathbf{k}$ and expanding the cross product yields

$$\mathbf{M} = \left(\left(\dot{H}_x\right)_{xyz} - \Omega_z\,H_y + \Omega_y H_z\right)\mathbf{i} + \left(\left(\dot{H}_y\right)_{xyz} - \Omega_x\,H_z + \Omega_z H_x\right)\mathbf{j} + \left(\left(\dot{H}_z\right)_{xyz} - \Omega_y\,H_x + \Omega_x\,H_y\right)\mathbf{k}$$

Substitute H_x, H_y and H_z using Eq. 21–10. For the **i** component

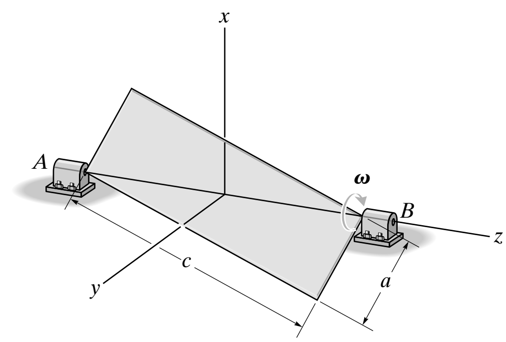
$$\Sigma M_x = \frac{d}{dt}(I_x\,\omega_x - I_{xy}\,\omega_y - I_{xz}\,\omega_z) - \Omega_z\,(I_y\,\omega_y - I_{yz}\,\omega_z - I_{yx}\omega_x) + \Omega_y\,(I_z\,\omega_z - I_{zx}\,\omega_x - I_{zy}\,\omega_y)$$

Set $I_{xy} = I_{yz} = I_{zx} = 0$ and require I_x, I_y, I_z to be constant. This yields

$$\Sigma M_x = I_x\,\dot{\omega}_x - I_y\,\Omega_z\,\omega_y + I_z\,\Omega_y\omega_z \qquad \textbf{Ans.}$$

One can obtain y and z components in a similar manner.

21–43. The uniform rectangular plate has a mass of $m = 2$ kg and is given a rotation of $\omega = 4$ rad/s about its bearings at A and B. If $a = 0.2$ m and $c = 0.3$ m, determine the vertical reactions at A and B at the instant the plate is vertical as shown. Use the x, y, z axes shown and note that $I_{zx} = -\left(\frac{mac}{12}\right)\left(\frac{c^2 - a^2}{c^2 + a^2}\right)$.

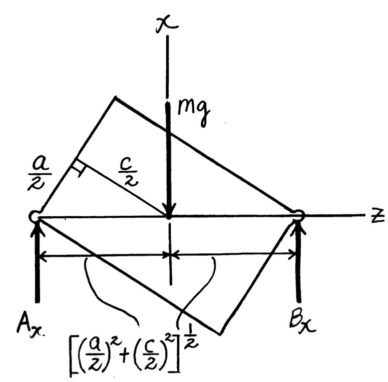


$$\omega_x = 0, \qquad \omega_y = 0, \qquad \omega_z = -4$$

$$\dot{\omega}_x = 0, \qquad \dot{\omega}_y = 0, \qquad \dot{\omega}_z = 0$$

$$\Sigma M_y = I_{yy}\,\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\,\omega_x - I_{yz}\left(\dot{\omega}_z - \omega_x\,\omega_y\right) - I_{zx}\left(\omega_z^2 - \omega_x^2\right) - I_{xy}\left(\dot{\omega}_x + \omega_y\,\omega_z\right)$$

$$B_x\left[\left(\frac{a}{2}\right)^2 + \left(\frac{c}{2}\right)^2\right]^{\frac{1}{2}} - A_x\left[\left(\frac{a}{2}\right)^2 + \left(\frac{c}{2}\right)^2\right]^{\frac{1}{2}} = -I_{zx}\,(\omega)^2$$



$$B_x - A_x = \left(\frac{mac}{6}\right)\left(\frac{c^2 - a^2}{\left[a^2 + c^2\right]^{\frac{3}{2}}}\right)\omega^2$$

$$\Sigma F_x = m(a_G)_x\,; \qquad A_x + B_x - mg = 0$$

Substitute the data,

$$B_x - A_x = \frac{2(0.2)(0.3)}{6}\left[\frac{(0.3)^2 - (0.2)^2}{\left[(0.3)^2 + (0.2)^2\right]^{\frac{3}{2}}}\right](-4)^2 = 0.34135$$

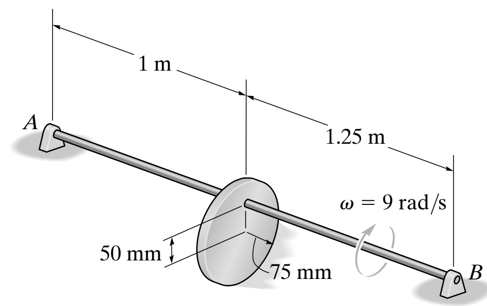
$$A_x + B_x = 2(9.81)$$

Solving:

$A_x = 9.64$ N **Ans.**

$B_x = 9.98$ N **Ans.**

***21–44.** The disk, having a mass of 3 kg, is mounted eccentrically on shaft AB. If the shaft is rotating at a constant rate of 9 rad/s, determine the reactions at the journal bearing supports when the disk is in the position shown.



$\omega_x = 0, \qquad \omega_y = -9, \qquad \omega_z = 0$

$\Sigma M_x = I_x \dot{\omega}_x - (I_y - I_z)\omega_y \omega_z$

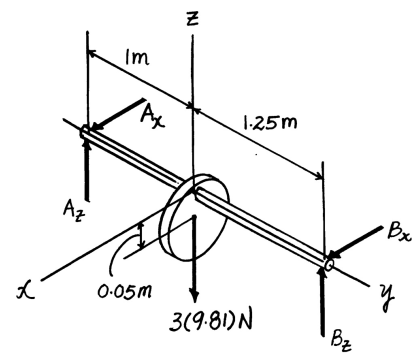
$B_z(1.25) - A_z(1) = 0 - 0$

$\Sigma M_z = I_z \dot{\omega}_z - (I_x - I_y)\omega_x \omega_y$

$A_x(1) - B_x(1.25) = 0 - 0$

$\Sigma F_x = ma_x; \qquad A_x + B_x = 0$

$\Sigma F_z = ma_z; \qquad A_z + B_z - 3(9.81) = 3(9)^2(0.05)$



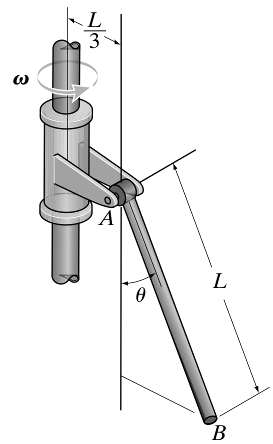
Solving,

$A_x = B_x = 0$ **Ans.**

$A_z = 23.1 \text{ N}$ **Ans.**

$B_z = 18.5 \text{ N}$ **Ans.**

•21–45. The slender rod AB has a mass m and it is connected to the bracket by a smooth pin at A. The bracket is rigidly attached to the shaft. Determine the required constant angular velocity of $\boldsymbol{\omega}$ of the shaft, in order for the rod to make an angle of θ with the vertical.



The rotating xyz frame is set with its origin at the rod's mass center, Fig. *a*. This frame will be attached to the rod so that its angular velocity is $\Omega = \omega$ and the x, y, z axes will always be the principal axes of inertia. Referring to Fig. *b*,

$$\omega = -\omega\cos\theta\mathbf{j} + \omega\sin\theta\mathbf{k}$$

Thus,

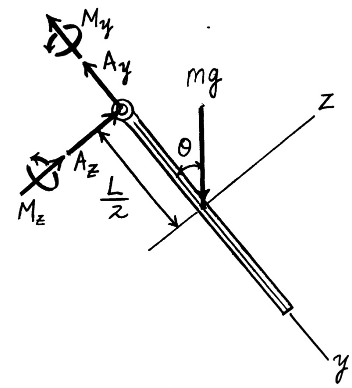
$$\omega_x = 0 \qquad \omega_y = -\omega\cos\theta \qquad \omega_z = \omega\sin\theta$$

Since both the direction and the magnitude is constant $\dot{\omega} = \mathbf{0}$. Also, since $\Omega = \omega$, $\left(\dot{\omega}_{xyz}\right) = \dot{\omega} = 0$. Thus,

$$\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$$

The mass moments of inertia of the rod about the x, y, z axes are

$$I_x = I_z = \frac{1}{12}mL^2 \qquad I_y = 0$$

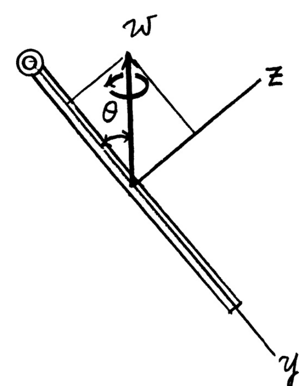


(a)

Applying the equation of motion and referring to the free-body diagram of the rod, Fig. *a*,

$$\Sigma M_x = I_x\dot{\omega}_x - \left(I_y - I_z\right)\omega_y\omega_z; \quad -A_z\left(\frac{L}{2}\right) = 0 - \left(0 - \frac{1}{12}mL^2\right)(-\omega\cos\theta)(\omega\sin\theta)$$

$$A_z = \frac{m\omega^2 L}{6}\sin\theta\cos\theta \qquad \textbf{(1)}$$



(b)

The acceleration of the mass center of the rod can be determined from $a_G = \omega^2 r = \omega^2\left(\frac{L}{2}\sin\theta + \frac{L}{3}\right) = \frac{\omega^2 L}{6}(3\sin\theta + 2)$ and is directed as shown in Fig. *c*. Thus,

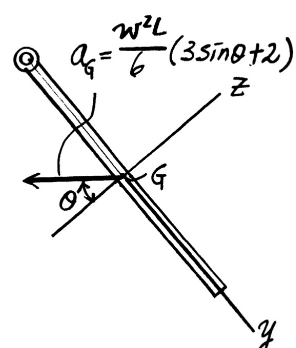
$$\Sigma F_z = m(a_G)_z; \qquad A_z - mg\sin\theta = -\frac{m\omega^2 L}{6}(3\sin\theta + 2)\cos\theta$$

$$A_z = mg\sin\theta = -\frac{m\omega^2 L}{6}(3\sin\theta + 2)\cos\theta \quad \textbf{(2)}$$

Equating Eqs. (1) and (2),

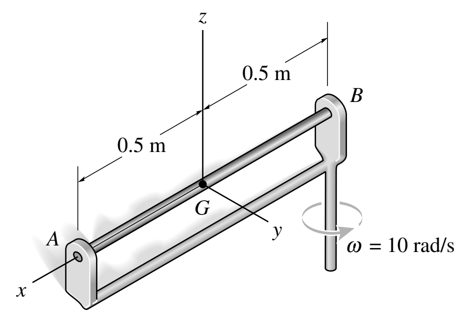
$$\frac{m\omega^2 L}{6}\sin\theta\cos\theta = mg\sin\theta - \frac{m\omega^2 L}{6}(3\sin\theta + 2)\cos\theta$$

$$\omega = \sqrt{\frac{3g\tan\theta}{L(2\sin\theta + 1)}} \qquad \textbf{Ans.}$$



(c)

21–46. The 5-kg rod AB is supported by a rotating arm. The support at A is a journal bearing, which develops reactions normal to the rod. The support at B is a thrust bearing, which develops reactions both normal to the rod and along the axis of the rod. Neglecting friction, determine the x, y, z components of reaction at these supports when the frame rotates with a constant angular velocity of $\omega = 10$ rad/s.



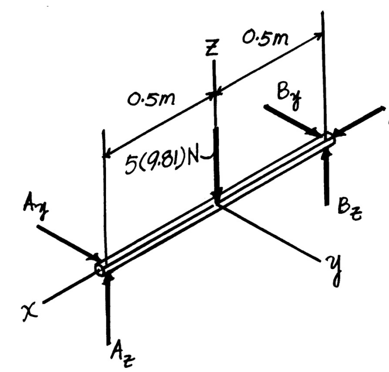
$$I_y = I_z = \frac{1}{12}(5)(1)^2 = 0.4167\ \text{kg}\cdot\text{m}^2 \qquad I_x = 0$$

Applying Eq. 21–25 with $\omega_x = \omega_y = 0$ $\omega_z = 10$ rad/s $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$

$\Sigma M_x = I_x \dot{\omega}_x - (I_y - I_z)\omega_y \omega_z;$ $\quad 0 = 0$

$\Sigma M_y = I_y \dot{\omega}_y - (I_z - I_x)\omega_z \omega_x;$ $\quad B_z(0.5) - A_z(0.5) = 0$ **(1)**

$\Sigma M_z = I_z \dot{\omega}_z - (I_x - I_y)\omega_x \omega_y;$ $\quad A_y(0.5) - B_y(0.5) = 0$ **(2)**



Also,

$\Sigma F_x = m(a_G)_x;$ $\quad B_x = -5(10)^2(0.5)$ $\quad B_x = -250\text{N}$ **Ans.**

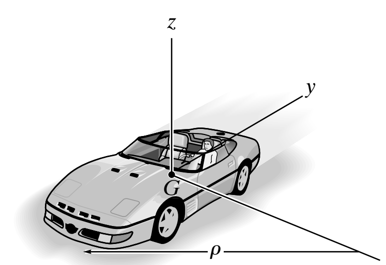
$\Sigma F_y = m(a_G)_y;$ $\quad A_y + B_y = 0$ **(3)**

$\Sigma F_z = m(a_G)_z;$ $\quad A_z + B_z - 5(9.81) = 0$ **(4)**

Solving Eqs. (1) to (4) yields:

$$A_y = B_y = 0 \qquad A_z = B_z = 24.5\ \text{N}$$ **Ans.**

21–47. The car travels around the curved road of radius ρ such that its mass center has a constant speed v_G. Write the equations of rotational motion with respect to the x, y, z axes. Assume that the car's six moments and products of inertia with respect to these axes are known.



Applying Eq. 21–24 with $\omega_x = 0,\quad \omega_y = 0,\quad \omega_z = \dfrac{v_G}{\rho},$

$\omega_x = \omega_y = \omega_z = 0$

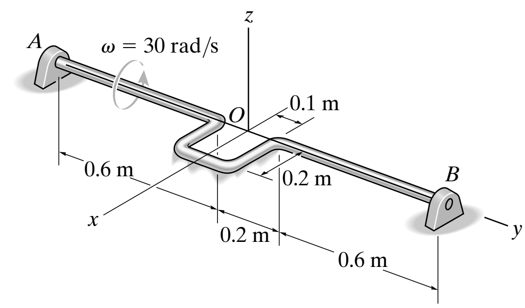
$$\Sigma M_x = -I_{yz}\left[0 - \left(\frac{v_G}{\rho}\right)^2\right] = \frac{I_{yz}}{\rho^2}v_G^2$$ **Ans.**

$$\Sigma M_y = -I_{zx}\left[\left(\frac{v_G}{\rho}\right)^2 - 0\right] = -\frac{I_{zx}}{\rho^2}v_G^2$$ **Ans.**

$\Sigma M_z = 0$ **Ans.**

Note: This result indicates the normal reactions of the tires on the ground are not all necessarily equal. Instead, they depend upon the speed of the car, radius of curvature, and the products of inertia I_{yz} and I_{zx}. (See Example 13–6.)

***21–48.** The shaft is constructed from a rod which has a mass per unit length of 2 kg/m. Determine the x, y, z components of reaction at the bearings A and B if at the instant shown the shaft spins freely and has an angular velocity of $\omega = 30$ rad/s. What is the angular acceleration of the shaft at this instant? Bearing A can support a component of force in the y direction, whereas bearing B cannot.

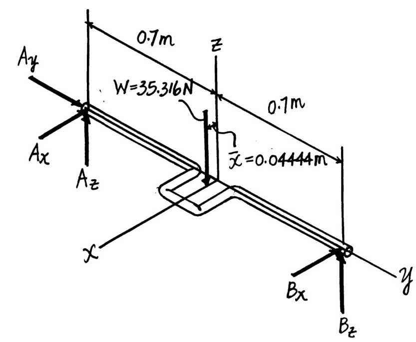


$$\Sigma W = \left[3(0.2) + 1.2\right](2)(9.81) = 35.316 \text{ N}$$

$$\Sigma \bar{x}W = 0\left[1.2(2)(9.81)\right] + 0.1\left[0.4(2)(9.81)\right] + 0.2\left[0.2(2)(9.81)\right] = 1.5696 \text{ N} \cdot \text{m}$$

$$\bar{x} = \frac{\Sigma \bar{x}W}{\Sigma W} = \frac{1.5696}{35.316} = 0.04444 \text{ m}$$

$$I_x = 2\left[\frac{1}{3}\left[0.2(2)\right](0.2)^2\right] + \left[0.2(2)\right](0.2)^2 = 0.02667 \text{ kg} \cdot \text{m}^2$$



Applying Eq. 21–25 with $\omega_x = \omega_z = 0$ $\omega_y = 30$ rad/s $\dot{\omega}_x = \dot{\omega}_z = 0$

$\Sigma M_x = I_x \dot{\omega}_x - (I_y - I_z)\omega_y \omega_z;$ $\quad B_z(0.7) - A_z(0.7) = 0$ **(1)**

$\Sigma M_y = I_y \dot{\omega}_y - (I_z - I_x)\omega_z \omega_x;$ $\quad 35.316(0.04444) = 0.02667\dot{\omega}_y$

$\dot{\omega}_y = 58.9 \text{ rad/s}^2$ **Ans.**

$\Sigma M_z = I_z \dot{\omega}_z - (I_x - I_y)\omega_x \omega_y;$ $\quad B_x(0.7) - A_x(0.7) = 0$ **(2)**

Also,

$\Sigma F_x = m(a_G)_x;$ $\quad -A_x - B_x = -1.8(2)(0.04444)(30)^2$ **(3)**

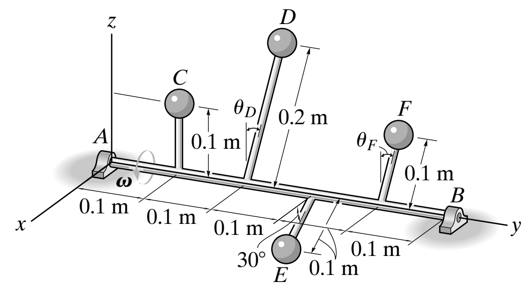
$\Sigma F_y = m(a_G)_y;$ $\quad A_y = 0$ **Ans.**

$\Sigma F_z = m(a_G)_z;$ $\quad A_z + B_z - 35.316 = -1.8(2)(0.04444)(58.9)$ **(4)**

Solving Eqs. (1) to (4) yields:

$A_x = B_x = 72.0 \text{ N} \qquad A_z = B_z = 12.9 \text{ N}$ **Ans.**

•21–49. Four spheres are connected to shaft AB. If $m_C = 1$ kg and $m_E = 2$ kg, determine the mass of spheres D and F and the angles of the rods, θ_D and θ_F, so that the shaft is dynamically balanced, that is, so that the bearings at A and B exert only vertical reactions on the shaft as it rotates. Neglect the mass of the rods.



For $\bar{x} = 0$; $\Sigma \bar{x}_1 m_1 = 0$

$$(0.1 \cos 30°)(2) - (0.1 \sin \theta_F)m_F - (0.2 \sin \theta_D)m_D = 0 \qquad \textbf{(1)}$$

For $\bar{z} = 0$; $\Sigma \bar{z}_1 m_1 = 0$

$$(0.1)(1) - (0.1 \sin 30°)(2) + (0.2 \cos \theta_D)m_D + (0.1 \cos \theta_F)m_F = 0 \qquad \textbf{(2)}$$

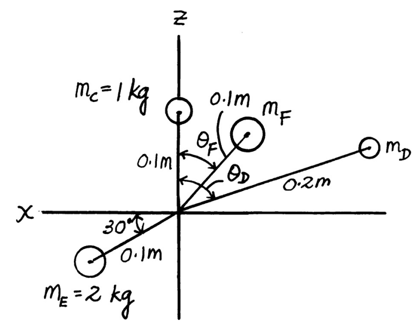
For $I_{xz} = 0$; $\Sigma \bar{x}_1 \bar{z}_1 m_1 = 0$

$$-(0.2)(0.2 \sin \theta_D)m_D + (0.3)(0.1 \cos 30°)(2) - (0.4)(0.1 \sin \theta_F)m_F = 0 \qquad \textbf{(3)}$$

For $I_{xy} = 0$; $\Sigma \bar{x}_1 \bar{y}_1 m_1 = 0$

$$(0.1)(0.1)(1) + (0.2)(0.2 \cos \theta_D)m_D - (0.3)(0.1 \sin 30°)(2)$$

$$+ (0.1 \cos \theta_F)(0.4)(m_F) = 0 \qquad \textbf{(4)}$$



Solving,

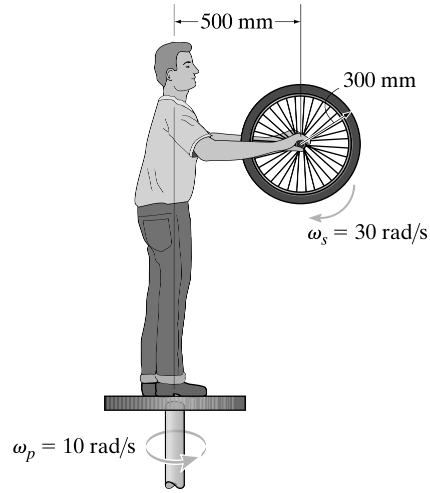
$\theta_D = 139°$ **Ans.**

$m_D = 0.661$ kg **Ans.**

$\theta_F = 40.9°$ **Ans.**

$m_F = 1.32$ kg **Ans.**

21–50. A man stands on a turntable that rotates about a vertical axis with a constant angular velocity of $\omega_p = 10$ rad/s. If the wheel that he holds spins with a constant angular speed of $\omega_s = 30$ rad/s, determine the magnitude of moment that he must exert on the wheel to hold it in the position shown. Consider the wheel as a thin circular hoop (ring) having a mass of 3 kg and a mean radius of 300 mm.



The rotating xyz frame will be set with an angular velocity of $\Omega = \omega_P = [10\mathbf{k}]$ rad/s. Since the wheel is symmetric about its spinning axis, the x, y, and z axes will remain as the principle axes of inertia. Thus,

$$I_y = I_z = \frac{1}{2}mr^2 = \frac{1}{2}(3)\left(0.3^2\right) = 0.135\ \text{kg}\cdot\text{m}^2$$

$$I_x = mr^2 = 3\left(0.3^2\right) = 0.27\ \text{kg}\cdot\text{m}^2$$

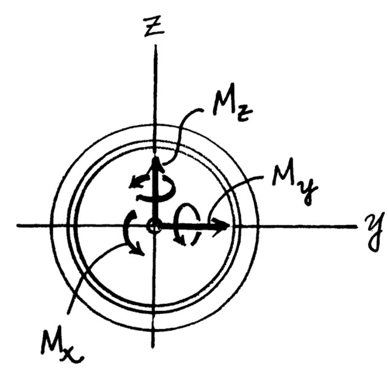
The angular velocity of the wheel is $\omega = \omega_s + \omega_P = [-30\mathbf{i} + 10\mathbf{k}]$ rad/s. Thus,

$$\omega_x = -30\ \text{rad/s} \qquad \omega_y = 0 \qquad \omega_z = 10\ \text{rad/s}$$

Since the directions of ω_s and ω_p do not change with respect to the xyz frame and their magnitudes are constant, $\dot{\omega}_{xyz} = 0$. Thus,

$$\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$$

Applying the equations of motion and referring to the free-body diagram shown in Fig. *a*,



(a)

$$\Sigma M_x = I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z; \qquad M_x = 0$$

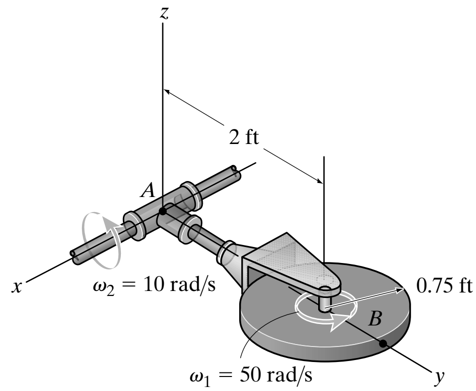
$$\Sigma M_y = I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x; \qquad M_y = 0 - 0 + 0.27(10)(-30) = -81.0\ \text{N}\cdot\text{m}$$

$$\Sigma M_z = I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y; \qquad M_z = 0$$

Thus,

$$M = \sqrt{{M_x}^2 + {M_y}^2 + {M_z}^2} = \sqrt{0^2 + (-81.0)^2 + 0^2} = 81.0\ \text{N}\cdot\text{m} \qquad \textbf{Ans.}$$

21–51. The 50-lb disk spins with a constant angular rate of $\omega_1 = 50$ rad/s about its axle. Simultaneously, the shaft rotates with a constant angular rate of $\omega_2 = 10$ rad/s. Determine the x, y, z components of the moment developed in the arm at A at the instant shown. Neglect the weight of arm AB.



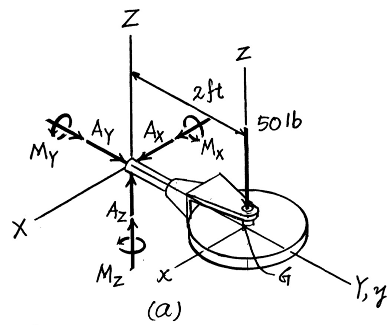
The rotating xyz frame is established as shown in Fig. a. This frame will be set to have an angular velocity of $\Omega = \omega_2 = [10\mathbf{i}]$ rad/s. Since the disk is symmetric about its spinning axis, the x, y, and z axes will remain as the principle axes of inertia. Thus,

$$I_x = I_y = \frac{1}{4}\left(\frac{50}{32.2}\right)\left(0.75^2\right) = 0.2184 \text{ slug} \cdot \text{ft}^2$$

$$I_z = \frac{1}{2}\left(\frac{50}{32.2}\right)\left(0.75^2\right) = 0.4367 \text{ slug} \cdot \text{ft}^2$$

The angular velocity of the disk is $\omega = \omega_s + \omega_p = [10\mathbf{i} + 50\mathbf{k}]$ rad/s. Thus,

$$\omega_x = 10 \text{ rad/s} \qquad \omega_y = 0 \qquad \omega_z = 50 \text{ rad/s}$$



Since the directions of ω_1 and ω_2 do not change with respect to the xyz frame and their magnitudes are constant, $\dot{\omega}_{xyz} = \mathbf{0}$. Thus,

$$\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$$

Applying the equations of motion and referring to the free-body diagram shown in Fig. a,

$\Sigma M_x = I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z;\qquad M_X - A_Z(2) = 0$ **(1)**

$\Sigma M_y = I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x;\qquad M_Y = 0 - 0.4367(10)(50) + 0$

$M_Y = -218.36 \text{ lb} \cdot \text{ft} = -218 \text{ lb} \cdot \text{ft}$ **Ans.**

$\Sigma M_z = I_z\dot{\omega}_z - I_x\Omega_y\omega_{zx} + I_y\Omega_x\omega_y;\qquad M_Z = 0 - 0 + 0$

$M_Z = 0$ **Ans.**

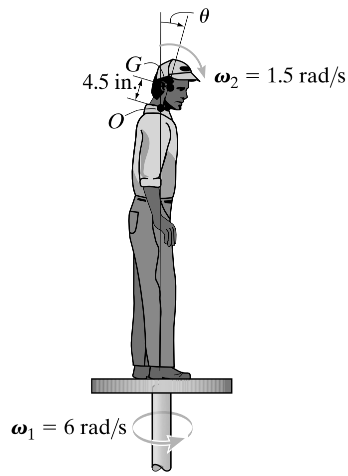
Since the mass center of the disk rotates about the X axis with a constant angular velocity of $\omega_1 = [10\mathbf{i}]$ rad/s, its acceleration is $\mathbf{a}_G = \dot{\omega}_2 \times \mathbf{r}_G - \omega^2\mathbf{r}_G = \mathbf{0} - 10^2(2\mathbf{j}) = [-200\mathbf{j}]$ ft/s^2. Thus,

$$\Sigma F_Z = m(a_G)_Z;\qquad A_Z - 50 = \frac{50}{32.2}(0)\quad A_Z = 50 \text{ lb}$$

Substituting this result into Eq. (1), we have

$M_X = 100 \text{ lb} \cdot \text{ft}$ **Ans.**

***21–52.** The man stands on a turntable that rotates about a vertical axis with a constant angular velocity of $\omega_1 = 6$ rad/s. If he tilts his head forward at a constant angular velocity of $\omega_2 = 1.5$ rad/s about point O, determine the magnitude of the moment that must be resisted by his neck at O at the instant $\theta = 30°$. Assume that his head can be considered as a uniform 10-lb sphere, having a radius of 4.5 in. and center of gravity located at G, and point O is on the surface of the sphere.



The rotating xyz frame shown in Fig. a will be attached to the head so that it rotates with an angular velocity of $\Omega = \omega$, where $\omega = \omega_1 + \omega_2$. Referring to Fig. b, $\omega_1 = [6\cos 30°\,\mathbf{j} + 6\sin 30°\,\mathbf{k}]$ rad/s $= [5.196\mathbf{j} + 3\mathbf{k}]$ rad/s. Thus, $\omega = [-1.5\mathbf{i} + 5.196\mathbf{j} + 3\mathbf{k}]$ rad/s. Then

$$\omega_x = -1.5 \text{ rad/s} \qquad \omega_y = 5.196 \text{ rad/s} \qquad \omega_z = 3 \text{ rad/s}$$

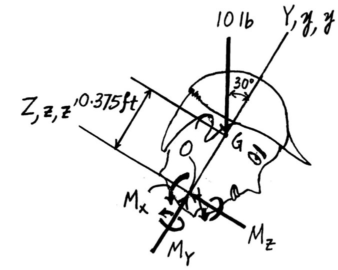
The angular acceleration of the head $\dot{\omega}$ with respect to the XYZ frame can be obtained by setting another $x'y'z'$ frame having an angular velocity of $\Omega' = \omega_1 = [5.196\mathbf{j} + 3\mathbf{k}]$ rad/s. Thus

$$\dot{\omega} = \left(\dot{\omega}_{x'y'z'}\right) + \Omega' \times \omega$$

$$= (\dot{\omega}_1)_{x'y'z'} + (\dot{\omega}_2)_{x'y'z'} + \Omega' \times \omega_1 + \Omega' \times \omega_2$$

$$= 0 + 0 + 0 + (5.196\mathbf{j} + 3\mathbf{k}) \times (-1.5\mathbf{i})$$

$$= [-4.5\mathbf{j} + 7.794\mathbf{k}] \text{ rad/s}^2$$



(a)

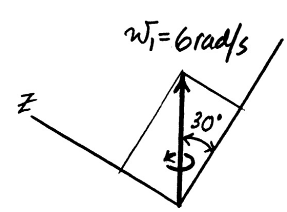
Since $\Omega = \omega$, $\dot{\omega}_{x'y'z'} = \dot{\omega} = [-4.5\mathbf{j} + 7.794\mathbf{k}]$ rad/s². Thus,

$$\dot{\omega}_x = 0 \qquad \dot{\omega}_y = -4.5 \text{ rad/s}^2 \qquad \dot{\omega}_z = 7.794 \text{ rad/s}^2$$

Also, the x, y, z axes will remain as principal axes of inertia. Thus,

$$I_x = I_z = \frac{2}{5}\frac{10}{32.2}\left(0.375^2\right) + \left(\frac{10}{32.2}\right)\left(0.375^2\right) = 0.06114 \text{ slug}\cdot\text{ft}^2$$

$$I_y = \frac{2}{5}\left(\frac{10}{32.2}\right)\left(0.375^2\right) = 0.01747 \text{ slug}\cdot\text{ft}^2$$



(b)

Applying the moment equations of motion and referring to the free-body diagram shown in Fig. a,

$$\Sigma M_x = I_x\dot{\omega}_x - \left(I_y - I_z\right)\omega_y\omega_z; \qquad M_x - 10\sin 30°(0.375) = 0 - (0.01747 - 0.06114)(5.196)(3)$$

$$M_x = 2.556 \text{ lb}\cdot\text{ft}$$

$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x; \qquad M_y = 0.01747(-4.5) - 0 \qquad M_y = -0.07861 \text{ lb}\cdot\text{ft}$$

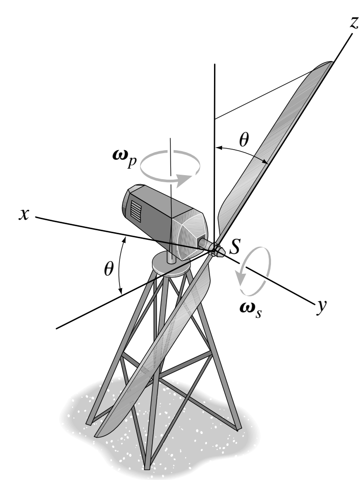
$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y; \qquad M_z = 0.06114(7.794) - (0.06114 - 0.01747)(-1.5)(5.196)$$

$$= 0.8161 \text{ lb}\cdot\text{ft}$$

Thus,

$$M_A = \sqrt{M_x^{\,2} + M_y^{\,2} + M_z^{\,2}} = \sqrt{2.556^2 + (-0.07861)^2 + 0.8161^2} = 2.68 \text{ lb}\cdot\text{ft} \qquad \textbf{Ans.}$$

•21–53. The blades of a wind turbine spin about the shaft S with a constant angular speed of ω_s, while the frame precesses about the vertical axis with a constant angular speed of ω_p. Determine the x, y, and z components of moment that the shaft exerts on the blades as a function of θ. Consider each blade as a slender rod of mass m and length l.

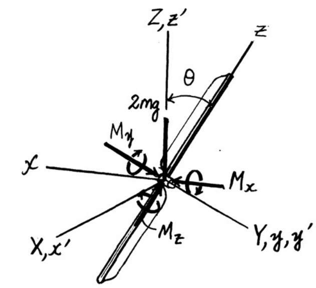


The rotating xyz frame shown in Fig. a will be attached to the blade so that it rotates with an angular velocity of $\Omega = \omega$, where $\omega = \omega_s + \omega_p$. Referring to Fig. b $\omega_p = \omega_p \sin\theta\mathbf{i} + \omega_p \cos\theta\mathbf{k}$. Thus, $\omega = \omega_p \sin\theta\mathbf{i} + \omega_s\mathbf{j} + \omega_p\cos\theta\mathbf{k}$. Then

$$\omega_x = \omega_p \sin\theta \qquad \omega_y = \omega_s \quad \omega_z = \omega_p\cos\theta$$

The angular acceleration of the blade $\dot{\omega}$ with respect to the XYZ frame can be obtained by setting another $x'y'z'$ frame having an angular velocity of $\Omega' = \omega_p = \omega_p \sin\theta\mathbf{i} + \omega_p\cos\theta\mathbf{k}$. Thus,

$$\begin{aligned}\dot{\omega} &= \left(\dot{\omega}_{x'y'z'}\right) + \Omega' \times \omega \\ &= (\dot{\omega}_1)_{x'y'z'} + (\dot{\omega}_2)_{x'y'z'} + \Omega' \times \omega_S + \Omega' \times \omega_P \\ &= \mathbf{0} + \mathbf{0} + \left(\omega_p \sin\theta\mathbf{i} + \omega_p\cos\theta\mathbf{k}\right) \times (\omega_s\mathbf{j}) + \mathbf{0} \\ &= -\omega_s\omega_p\cos\theta\mathbf{i} + \omega_s\omega_p\sin\theta\mathbf{k}\end{aligned}$$



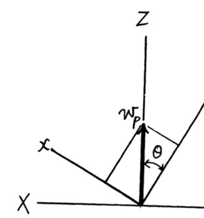
(a)

Since $\Omega = \omega$, $\dot{\omega}_{x'y'z'} = \dot{\omega}$. Thus,

$$\dot{\omega}_x = -\omega_s\omega_p\cos\theta \qquad \dot{\omega}_y = 0 \qquad \dot{\omega}_z = \omega_s\omega_p\sin\theta$$

Also, the x, y, and z axes will remain as principle axes of inertia for the blade. Thus,

$$I_x = I_y = \frac{1}{12}(2m)(2l)^2 = \frac{2}{3}ml^2 \qquad I_z = 0$$



(b)

Applying the moment equations of motion and referring to the free-body diagram shown in Fig. a,

$$\Sigma M_x = I_x\dot{\omega}_x - \left(I_y - I_z\right)\omega_y\omega_z; \qquad M_x = \frac{2}{3}ml^2(-\omega_s\omega_p\cos\theta) - \left(\frac{2}{3}ml^2 - 0\right)(\omega_s)(\omega_p\cos\theta)$$

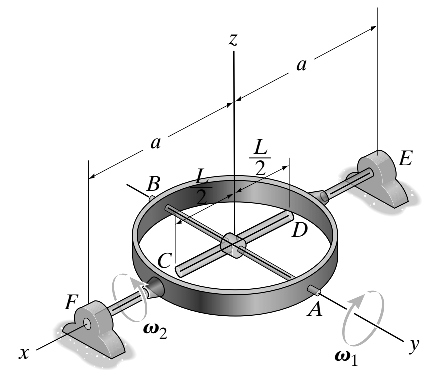
$$= -\frac{4}{3}ml^2\,\omega_s\omega_p\cos\theta \qquad \textbf{Ans.}$$

$$\Sigma M_y = I_y\dot{\omega}_y - \left(I_z - I_x\right)\omega_z\omega_x; \qquad M_y = 0 - \left(0 - \frac{2}{3}ml^2\right)(\omega_p\cos\theta)(\omega_p\sin\theta)$$

$$= \frac{1}{3}ml^2\,\omega_p{}^2\sin 2\theta \qquad \textbf{Ans.}$$

$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y; \qquad M_z = 0 - 0 = 0 \qquad \textbf{Ans.}$$

21–54. Rod CD of mass m and length L is rotating with a constant angular rate of ω_1 about axle AB, while shaft EF rotates with a constant angular rate of ω_2. Determine the X, Y, and Z components of reaction at thrust bearing E and journal bearing F at the instant shown. Neglect the mass of the other members.



The rotating xyz frame shown in Fig. a will be attached to the rod so that it rotates with an angular velocity of $\Omega = \omega$, where $\omega = \omega_1 + \omega_2 = \omega_2\mathbf{i} - \omega_1\mathbf{j}$. Thus,

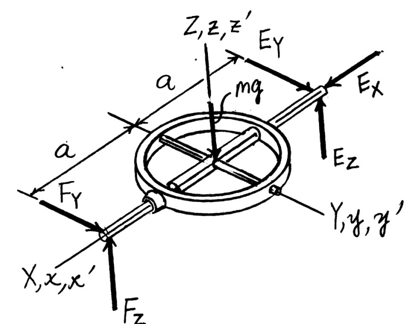
$$\omega_x = \omega_2 \qquad \omega_y = -\omega_1 \qquad \omega_z = 0$$

The angular acceleration of the rod $\dot{\omega}$ with respect to the XYZ frame can be obtained by using another rotating $x'y'z'$ frame having an angular velocity of $\Omega' = \omega_2 = \omega_2\mathbf{i}$. Fig. a. Thus,

$$\begin{aligned}\dot{\omega} &= \left(\dot{\omega}_{x'y'z'}\right) + \Omega' \times \omega \\ &= (\dot{\omega}_1)_{x'y'z'} + (\dot{\omega}_2)_{x'y'z'} + \Omega' \times \omega_1 + \Omega' \times \omega_2 \\ &= 0 + 0 + \left(\omega_2\mathbf{i}\right) \times (-\omega_1\mathbf{j}) + \mathbf{0} \\ &= -\omega_1\omega_2\mathbf{k}\end{aligned}$$

Since $\Omega = \omega$, $\dot{\omega}_{x'y'z'} = \dot{\omega}$. Thus,

$$\dot{\omega}_x = \dot{\omega}_y = 0 \qquad \dot{\omega}_z = -\omega_1\omega_2$$



(a)

Also, the x, y, and z axes will remain as principal axes of inertia for the rod. Thus,

$$I_x = 0 \qquad I_y = I_z = \frac{1}{12}mL^2$$

Applying the equations of motion and referring to the free-body diagram shown in Fig. a,

$$\Sigma M_x = I_x\dot{\omega}_x - \left(I_y - I_z\right)\omega_y\omega_z; \quad 0 = 0$$

$$\Sigma M_y = I_y\dot{\omega}_y - \left(I_z - I_x\right)\omega_z\omega_x; \quad E_Z(a) - F_Z(a) = 0 - 0$$

$$E_Z - F_Z = 0 \qquad \textbf{(1)}$$

$$\Sigma M_z = I_z\dot{\omega}_z - (I_x - I_y)\omega_x\omega_y; \quad F_Y(a) - E_Y(a) = \frac{1}{12}mL^2(-\omega_1\omega_2) - \left(0 - \frac{1}{12}mL^2\right)(\omega_2)(-\omega_1)$$

$$F_Y - E_Y = -\frac{mL^2\omega_1\omega_2}{6a} \qquad \textbf{(2)}$$

Since the mass center G does not move, $\mathbf{a}_G = \mathbf{0}$. Thus,

$$\Sigma F_X = m(a_G)_X; \qquad E_X = 0 \qquad \textbf{Ans.}$$

$$\Sigma F_Y = m(a_G)_Y; \qquad F_Y + E_Y = 0 \qquad \textbf{(3)}$$

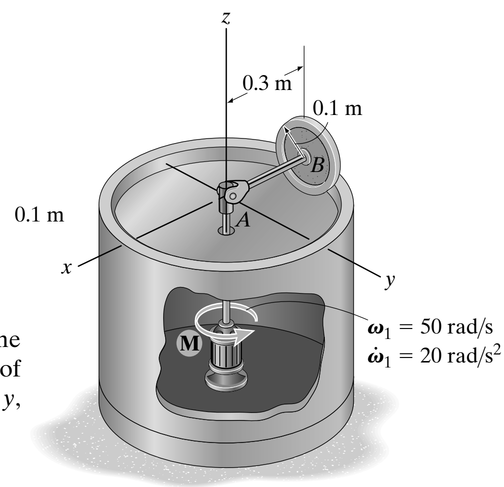
$$\Sigma F_Z = m(a_G)_Z; \qquad F_Z + E_Z - mg = 0 \qquad \textbf{(4)}$$

Solving Eqs. (1) through (4),

$$F_Y = -\frac{mL^2\omega_1\omega_2}{12a} \qquad E_Y = \frac{mL^2\omega_1\omega_2}{12a} \qquad \textbf{Ans.}$$

$$E_Z = F_Z = \frac{mg}{2} \qquad \textbf{Ans.}$$

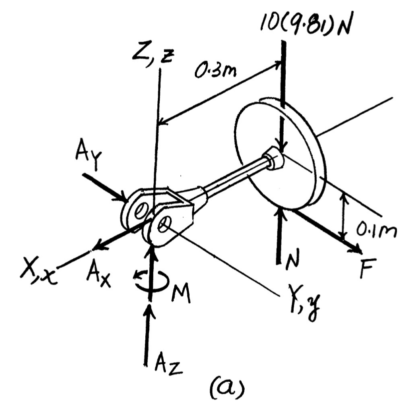
21–55. If shaft AB is driven by the motor with an angular velocity of $\omega_1 = 50$ rad/s and angular acceleration of $\dot{\omega}_1 = 20$ rad/s^2 at the instant shown, and the 10-kg wheel rolls without slipping, determine the frictional force and the normal reaction on the wheel, and the moment **M** that must be supplied by the motor at this instant. Assume that the wheel is a uniform circular disk.



The rotating xyz frame is established to coincide with the fixed XYZ frame at the instant considered, Fig. a. This frame will be set to have an angular velocity of $\Omega = \omega_1 = [50\mathbf{k}]$ rad/s. Since the wheel is symmetric about its spinning axis, the x, y, z axes will remain as the principal axes of inertia. Thus,

$$I_z = I_y = \frac{1}{4}(10)\left(0.1^2\right) + 10\left(0.3^2\right) = 0.925 \text{ kg}\cdot\text{m}^2$$

$$I_x = \frac{1}{2}(10)\left(0.1^2\right) = 0.05 \text{ kg}\cdot\text{m}^2$$



Since the wheel rolls without slipping, the instantaneous axis of zero velocity is shown in Fig. b. Thus

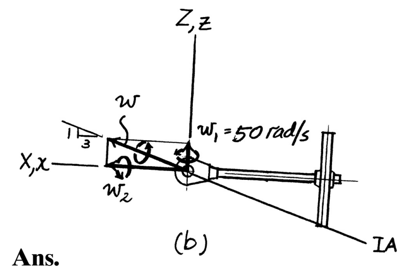
$$\frac{\omega_2}{\omega_1} = \frac{3}{1} \qquad\qquad \omega_2 = 3\omega_1 = 3(50) = 150 \text{ rad/s}$$

The angular velocity of the wheel is $\omega = \omega_1 + \omega_2 = [150\mathbf{i} + 50\mathbf{k}]$ rad/s. Then,

$$\omega_x = 150 \text{ rad/s} \qquad\qquad \omega_y = 0 \qquad\qquad \omega_z = 50 \text{ rad/s}$$

Since the directions of ω_1 and ω_2 do not change with respect to the xyz frame, $\dot{\omega}_{xyz} = (\dot{\omega}_1)_{xyz} + (\dot{\omega}_2)_{xyz}$ where $(\dot{\omega}_2)_{xyz} = 3(\dot{\omega}_1)_{xyz} = 3(20) = 60$ rad/s^2. Thus, $\dot{\omega}_{xyz} = [60\mathbf{i} + 20\mathbf{k}]$ rad/s^2, so that

$$\dot{\omega}_x = 60 \text{ rad/s}^2 \qquad \dot{\omega}_y = 0 \qquad\qquad \dot{\omega}_z = 20 \text{ rad/s}^2$$



Applying the equations of motion and referring to the free-body diagram shown in Fig. a,

$\Sigma M_x = I_x\dot{\omega}_x - I_y\Omega_z\omega_y + I_z\Omega_y\omega_z;$ $\qquad F(0.1) = 0.05(60) - 0 + 0 \qquad F = 30\text{N}$ **Ans.**

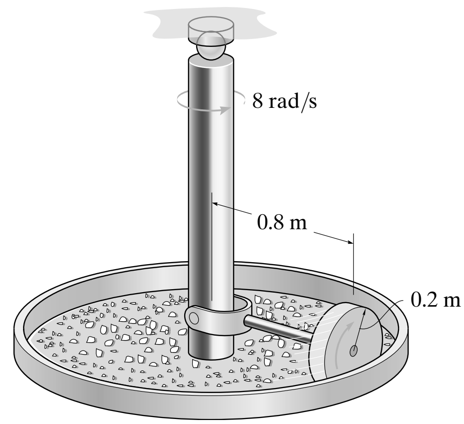
$\Sigma M_y = I_y\dot{\omega}_y - I_z\Omega_x\omega_z + I_x\Omega_z\omega_x;$ $\qquad N(0.3) - 10(9.81)(0.3) = 0 - 0 + 0.05(50)(150)$

$$N = 1348.1 \text{ N} = 1.35 \text{ kN}$$ **Ans.**

$\Sigma M_z = I_z\dot{\omega}_z - I_x\Omega_y\omega_x + I_y\Omega_x\omega_y;$ $\qquad M - 30(0.3) = 0.925(20) - 0 + 0$

$$M = 27.5 \text{ N}\cdot\text{m}$$ **Ans.**

***21–56.** A stone crusher consists of a large thin disk which is pin connected to a horizontal axle. If the axle rotates at a constant rate of 8 rad/s, determine the normal force which the disk exerts on the stones. Assume that the disk rolls without slipping and has a mass of 25 kg. Neglect the mass of the axle.



$$I_x = I_z = \frac{1}{4}(25)(0.2)^2 + 25(0.8)^2 = 16.25\ \text{kg}\cdot\text{m}^2$$

$$I_y = \frac{1}{2}(25)(0.2)^2 = 0.5\ \text{kg}\cdot\text{m}^2$$

$\omega = -\omega_y\,\mathbf{j} + \omega_z\,\mathbf{k}$, where $\omega_z = 8$ rad/s

$$v = 0.8\omega_z = (0.8)(8) = 6.4\ \text{m/s}$$

$$\omega_y = -\frac{6.4}{0.2} = -32\ \text{rad/s}$$

Thus,

$$\omega = -32\mathbf{j} + 8\mathbf{k}$$

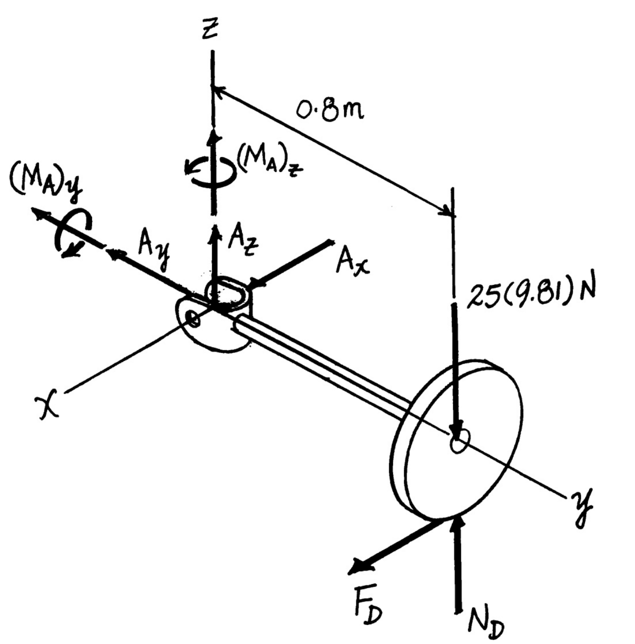
$$\dot{\omega} = \dot{\omega}_{xyz} + \Omega \times \omega = \mathbf{0} + (8\mathbf{k}) \times (-32\mathbf{j} + 8\mathbf{k}) = 256\mathbf{i}$$

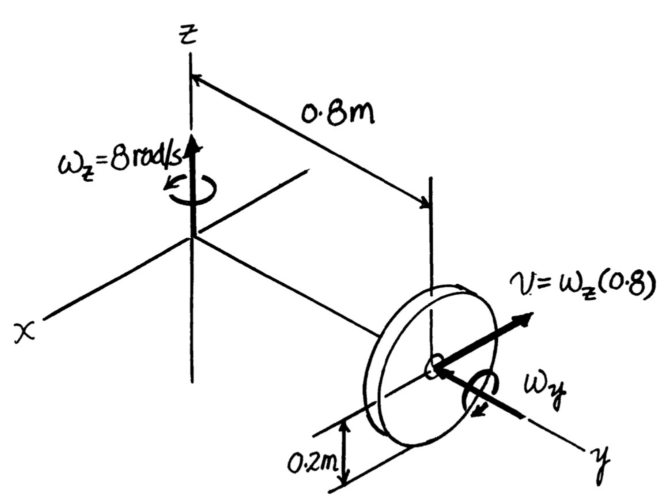
$$\omega_x = 256\ \text{rad/s}^2$$

$$\Sigma M_x = I_x\,\omega_x - (I_y - I_z)\,\omega_y\,\omega_z$$

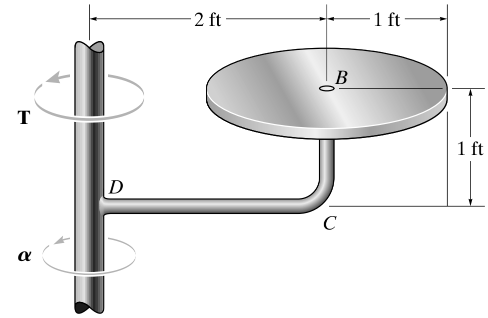
$$N_D(0.8) - 25(9.81)(0.8) = (16.25)(256) - (0.5 - 16.25)(-32)(8)$$

$N_D = 405$ N **Ans.**





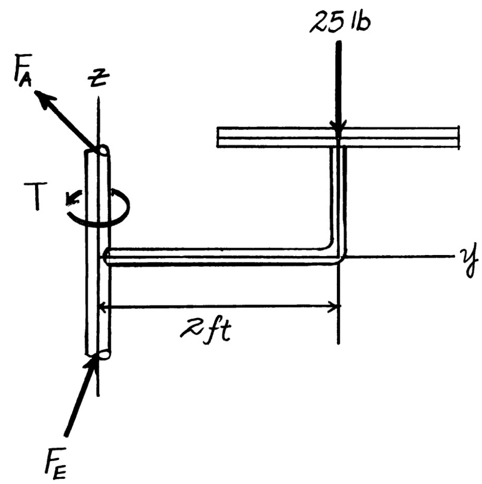
•21–57. The 25-lb disk is *fixed* to rod BCD, which has negligible mass. Determine the torque **T** which must be applied to the vertical shaft so that the shaft has an angular acceleration of $\alpha = 6 \text{ rad/s}^2$. The shaft is free to turn in its bearings.



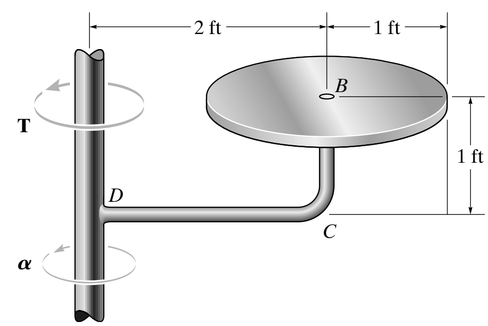
$$I_z = \frac{1}{2}\left(\frac{25}{32.2}\right)(1)^2 + \left(\frac{25}{32.2}\right)(2)^2 = 3.4938 \text{ slug} \cdot \text{ft}^2$$

Applying the third of Eq. 21–25 with $I_x = I_y, \omega_x = \omega_y = 0, \dot{\omega}_z = 6 \text{ rad/s}^2$

$\Sigma M_z = I_z \dot{\omega}_z - (I_x - I_y)\omega_x \omega_y; \qquad T = 3.4938(6) = 21.0 \text{ lb} \cdot \text{ft}$ **Ans.**



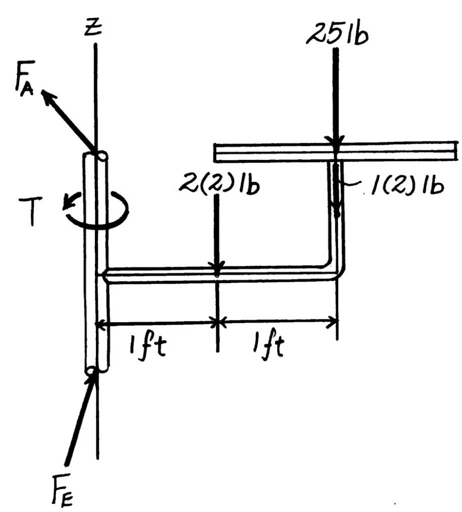
21–58. Solve Prob. 21–57, assuming rod BCD has a weight per unit length of 2 lb/ft.



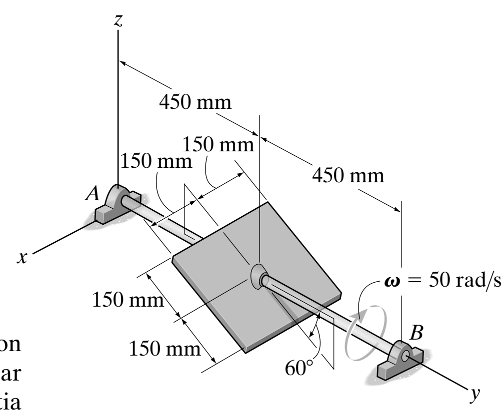
$$I_z = \frac{1}{2}\left(\frac{25}{32.2}\right)(1)^2 + \left(\frac{25}{32.2}\right)(2)^2 + \frac{1}{3}\left(\frac{2(2)}{32.2}\right)(2)^2 + \left(\frac{1(2)}{32.2}\right)(2)^2 = 3.9079 \text{ slug} \cdot \text{ft}^2$$

Applying the third of Eq. 21–25 with $I_x = I_y, \omega_x = \omega_y = 0, \dot{\omega}_z = 6 \text{ rad/s}^2$

$\Sigma M_z = I_z \dot{\omega}_z - (I_x - I_y)\,\omega_x \omega_y; \qquad T = 3.9079(6) = 23.4 \text{ lb} \cdot \text{ft}$ **Ans.**

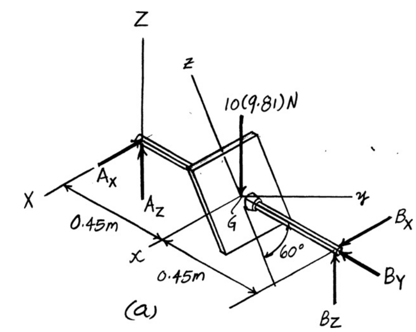


21–59. If shaft AB rotates with a constant angular velocity of $\omega = 50$ rad/s, determine the X, Y, Z components of reaction at journal bearing A and thrust bearing B at the instant shown. The thin plate has a mass of 10 kg. Neglect the mass of shaft AB.



The rotating xyz frame is set with its origin at the plate's mass center as shown on the free-body diagram, Fig. a. This frame will be fixed to the plate so that its angular velocity is $\Omega = \omega$ and the x, y, and z axes will always be the principal axes of inertia of the plate. Referring to Fig. b,

$$\omega = [-50 \sin 60° \, \mathbf{j} + 50 \cos 60° \, \mathbf{k}] \text{ rad/s} = [-43.30\mathbf{j} + 25\mathbf{k}] \text{ rad/s}$$



Thus,

$$\omega_x = 0 \qquad \omega_y = -43.30 \text{ rad/s} \qquad \omega_z = 25 \text{ rad/s}$$

Since ω is always directed towards the $-Y$ axis and has a constant magnitude, $\dot{\omega} = 0$. Also, since $\Omega = \omega$, $\dot{\omega}_{xyz} = \dot{\omega} = 0$. Thus,

$$\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$$

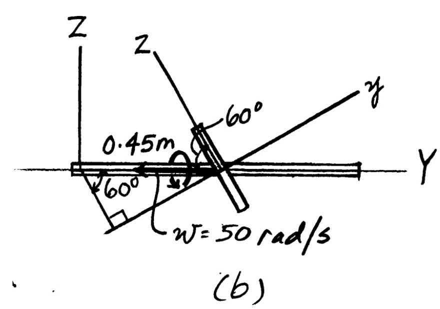
The mass moments of inertia of the plate about the x, y, and z axes are

$$I_x = I_z = \frac{1}{12}(10)\left(0.3^2\right) = 0.075 \text{ kg}\cdot\text{m}^2 \qquad I_y = \frac{1}{12}(10)\left(0.3^2 + 0.3^2\right) = 0.15 \text{ kg}\cdot\text{m}^2$$

Applying the equations of motion,

$$\Sigma M_x = I_x\dot{\omega}_x - \left(I_y - I_z\right)\omega_y\omega_z; \quad B_Z(0.45) - A_Z(0.45) = 0 - (0.15 - 0.075)(-43.30)(25)$$

$$B_Z - A_Z = 180.42 \qquad \textbf{(1)}$$



$$\Sigma M_y = I_y\dot{\omega}_y - (I_z - I_x)\omega_z\omega_x; \quad -A_X(0.45 \cos 60°) - B_X(0.45 \cos 60°) = 0 - 0$$

$$A_X = -B_X \qquad \textbf{(2)}$$

$$\Sigma M_z = I_z\dot{\omega}_z - \left(I_x - I_y\right)\omega_x\omega_y; \quad -A_X(0.45 \sin 60°) - B_X(0.45 \sin 60°) = 0 - 0$$

$$A_X = -B_X$$

$$\Sigma F_X = m(a_G)_X; \qquad B_X - A_X = 0 \qquad \textbf{(3)}$$

$$\Sigma F_Y = m(a_G)_Y; \qquad B_Y = 0 \qquad \textbf{Ans.}$$

$$\Sigma F_Z = m(a_G)_Z; \qquad A_Z + B_Z - 10(9.81) = 0 \qquad \textbf{(4)}$$

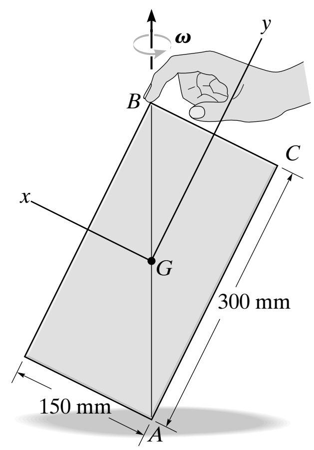
Solving Eqs. (1) through (4),

$$A_Z = -41.16\text{N} = -41.6 \text{ N} \qquad B_Z = 139.26 \text{ N} = 139 \text{ N} \qquad \textbf{Ans.}$$

$$A_X = B_X = 0 \qquad \textbf{Ans.}$$

The negative sign indicates that $\mathbf{A}_Z$ acts in the opposite sense to that shown on the free-body diagram.

***21–60.** A thin uniform plate having a mass of 0.4 kg spins with a constant angular velocity $\boldsymbol{\omega}$ about its diagonal AB. If the person holding the corner of the plate at B releases his finger, the plate will fall downward on its side AC. Determine the necessary couple moment **M** which if applied to the plate would prevent this from happening.



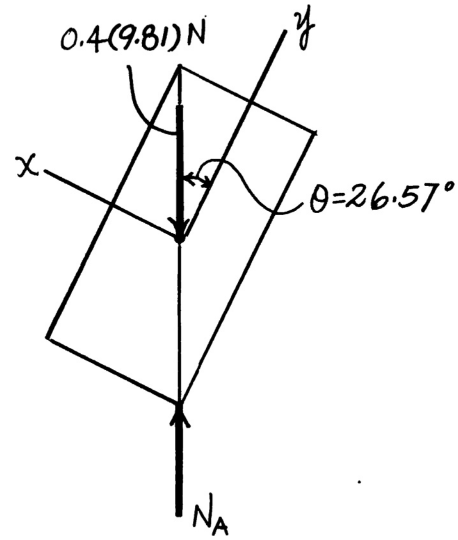
Using the principal axis shown,

$$I_x = \frac{1}{12}(0.4)(0.3)^2 = 3\left(10^{-3}\right)\ \text{kg}\cdot\text{m}^2$$

$$I_y = \frac{1}{12}(0.4)(0.15)^2 = 0.75\left(10^{-3}\right)\ \text{kg}\cdot\text{m}^2$$

$$I_z = \frac{1}{12}(0.4)\left[(0.3)^2 + (0.15)^2\right] = 3.75\left(10^{-3}\right)\ \text{kg}\cdot\text{m}^2$$

$$\theta = \tan^{-1}\left(\frac{75}{150}\right) = 26.57°$$



$$\omega_x = \omega \sin 26.57°, \qquad \dot{\omega}_x = 0$$

$$\omega_y = \omega \cos 26.57°, \qquad \omega_y = 0$$

$$\omega_z = 0, \qquad \dot{\omega}_z = 0$$

$$\Sigma M_x = I_x \omega_x - (I_y - I_z)\omega_y \omega_z$$

$$M_x = 0$$

$$\Sigma M_y = I_y \omega_y - (I_z - I_x)\omega_z \omega_z$$

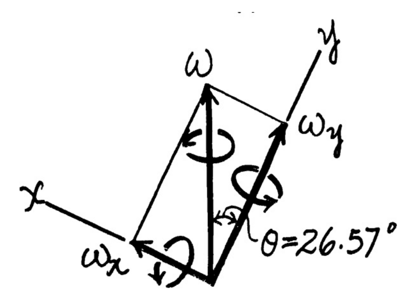
$$M_y = 0$$

$$\Sigma M_z = I_z \omega_z - (I_x - I_y)\omega_x \omega_y$$

$$M_z = 0 - \left[3\left(10^{-3}\right) - 0.75\left(10^{-3}\right)\right]\omega^2 \sin 26.57° \cos 26.57°$$

$$M_z = -0.9\left(10^{-3}\right)\omega^2\ \text{N}\cdot\text{m} = -0.9\omega^2\ \text{mN}\cdot\text{m}$$ **Ans.**

The couple acts outward, perpendicular to the face of the plate.



21–61. Show that the angular velocity of a body, in terms of Euler angles ϕ, θ, and ψ, can be expressed as $\omega = (\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi)\mathbf{i} + (\dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi)\mathbf{j} + (\dot{\phi} \cos\theta + \dot{\psi})\mathbf{k}$, where **i**, **j**, and **k** are directed along the x, y, z axes as shown in Fig. 21–15d.

From Fig. 21–15b. due to rotation ϕ, the x, y, z components of $\dot{\phi}$ are simply $\dot{\phi}$ along z axis.

From Fig 21–15c, due to rotation θ, the x, y, z components of $\dot{\phi}$ and $\dot{\theta}$ are $\dot{\phi}\sin\theta$ in the y direction, $\dot{\phi}\cos\theta$ in the z direction, and $\dot{\theta}$ in the x direction.

Lastly, rotation ψ. Fig. 21–15d, produces the final components which yields

$$\omega = \left(\dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi\right)\mathbf{i} + \left(\dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi\right)\mathbf{j} + \left(\dot{\phi} \cos\theta + \dot{\psi}\right)\mathbf{k} \quad \textbf{Q.E.D.}$$

21–62. A thin rod is initially coincident with the Z axis when it is given three rotations defined by the Euler angles $\phi = 30°, \theta = 45°$, and $\psi = 60°$. If these rotations are given in the order stated, determine the coordinate direction angles α, β, γ of the axis of the rod with respect to the X, Y, and Z axes. Are these directions the same for any order of the rotations? Why?

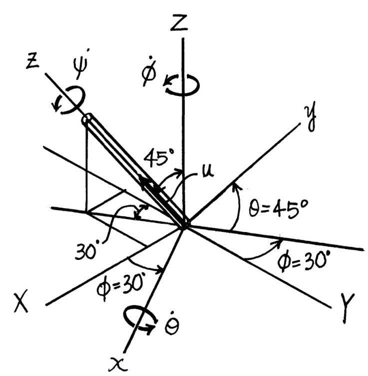
$$\mathbf{u} = (1 \sin 45°) \sin 30° \,\mathbf{i} - (1 \sin 45°) \cos 30° \mathbf{j} + 1 \cos 45° \,\mathbf{k}$$

$$\mathbf{u} = 0.3536\mathbf{i} - 0.6124\mathbf{j} + 0.7071\mathbf{k}$$

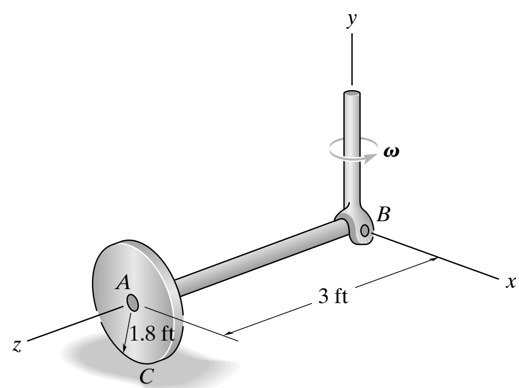
$\alpha = \cos^{-1} 0.3536 = 69.3°$ **Ans.**

$\beta = \cos^{-1}(-0.6124) = 128°$ **Ans.**

$\gamma = \cos^{-1}(0.7071) = 45°$ **Ans.**



21–63. The 30-lb wheel rolls without slipping. If it has a radius of gyration $k_{AB} = 1.2$ ft about its axle AB, and the vertical drive shaft is turning at 8 rad/s, determine the normal reaction the wheel exerts on the ground at C. Neglect the mass of the axle.

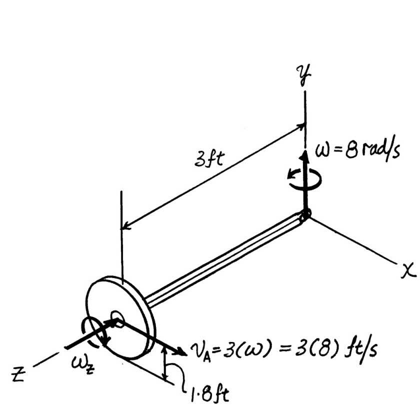


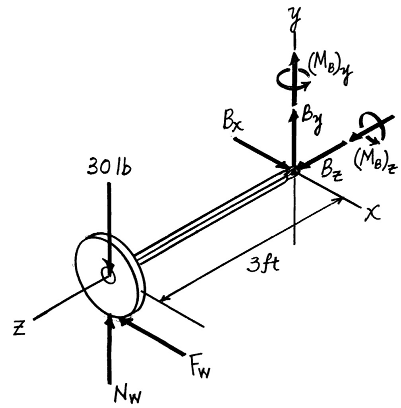
$$\Omega_y = \omega = 8 \text{ rad/s}$$

$$\omega_z = -\frac{3(8)}{1.8} = -13.33 \text{ rad/s}$$

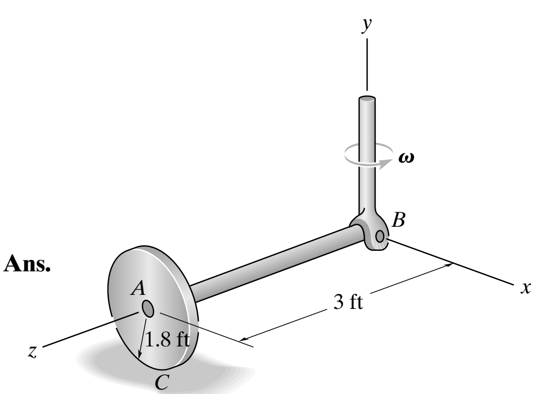
$$\Sigma M_z = I_z \Omega_y \omega_z; \qquad 30(3) - N_w(3) = \left[\left(\frac{30}{32.2}\right)(1.2)^2\right](8)(-13.33)$$

$$N_w = 77.7 \text{ lb}$$ **Ans.**



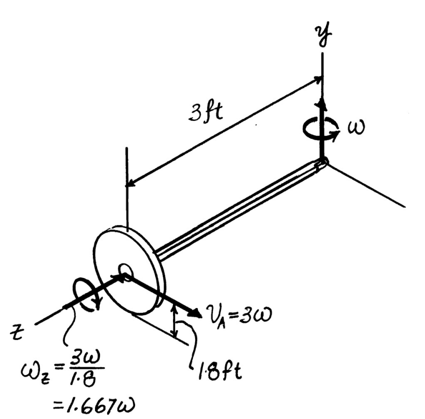


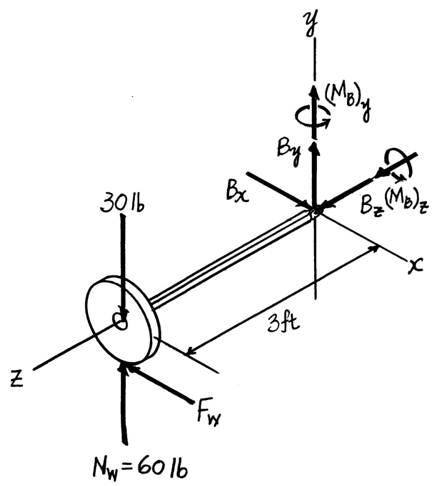
***21–64.** The 30-lb wheel rolls without slipping. If it has a radius of gyration $k_{AB} = 1.2$ ft about its axle AB, determine its angular velocity $\boldsymbol{\omega}$ so that the normal reaction at C becomes 60 lb. Neglect the mass of the axle.



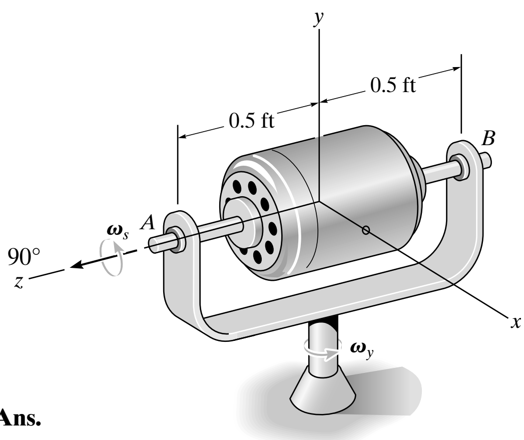
$$\Sigma M_x = I_x \Omega_y \omega_z; \qquad 30(3) - 60(3) = \left[\frac{30}{32.2}(1.2)^2\right]\omega(-1.667\omega)$$

$$\omega = 6.34 \text{ rad/s}$$ **Ans.**





•21–65. The motor weighs 50 lb and has a radius of gyration of 0.2 ft about the z axis. The shaft of the motor is supported by bearings at A and B, and spins at a constant rate of $\boldsymbol{\omega}_s = \{100\mathbf{k}\}$ rad/s, while the frame has an angular velocity of $\boldsymbol{\omega}_y = \{2\mathbf{j}\}$ rad/s. Determine the moment which the bearing forces at A and B exert on the shaft due to this motion.



Applying Eq. 21–30: For the coordinate system shown $\theta = 90°$ $\phi = 90°$ $\dot{\theta} = 0$ $\dot{\phi} = 2$ rad/s $\dot{\psi} = 100$ rad/s.

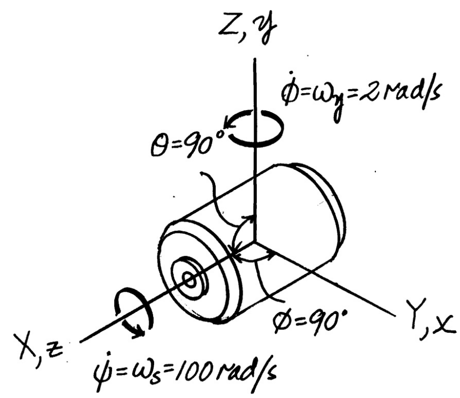
$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})$ reduces to

$$\Sigma M_x = I_z\dot{\phi}\dot{\psi}; \qquad M_x = \left[\left(\frac{50}{32.2}\right)(0.2)^2\right](2)(100) = 12.4 \text{ lb}\cdot\text{ft} \qquad \textbf{Ans.}$$

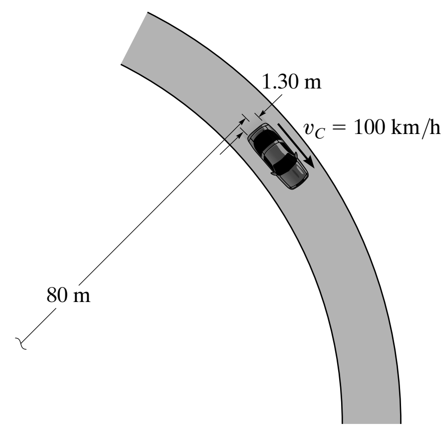
Since $\omega_x = 0$

$\Sigma M_y = 0; \quad M_y = 0$ **Ans.**

$\Sigma M_z = 0; \quad M_z = 0$ **Ans.**



21–66. The car travels at a constant speed of $v_C = 100$ km/h around the horizontal curve having a radius of 80 m. If each wheel has a mass of 16 kg, a radius of gyration $k_G = 300$ mm about its spinning axis, and a radius of 400 mm, determine the difference between the normal forces of the rear wheels, caused by the gyroscopic effect. The distance between the wheels is 1.30 m.



$I = 2[16(0.3)^2] = 2.88 \text{ kg}\cdot\text{m}^2$

$$\omega_s = \frac{100(1000)}{3600(0.4)} = 69.44 \text{ rad/s}$$

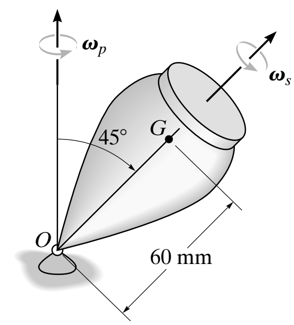
$$\omega_p = \frac{100(1000)}{80(3600)} = 0.347 \text{ rad/s}$$

$M = I\,\omega_s\,\omega_p$

$\Delta F(1.30) = 2.88(69.44)(0.347)$

$\Delta F = 53.4$ N **Ans.**

21–67. The top has a mass of 90 g, a center of mass at G, and a radius of gyration $k = 18$ mm about its axis of symmetry. About any transverse axis acting through point O the radius of gyration is $k_t = 35$ mm. If the top is connected to a ball-and-socket joint at O and the precession is $\omega_p = 0.5$ rad/s, determine the spin $\boldsymbol{\omega}_s$.



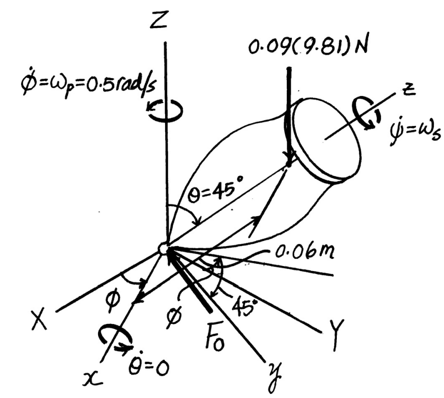
$\omega_p = 0.5$ rad/s

$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta \cos\theta + I_z\dot{\phi}\sin\theta\left(\dot{\phi}\cos\theta + \dot{\psi}\right)$$

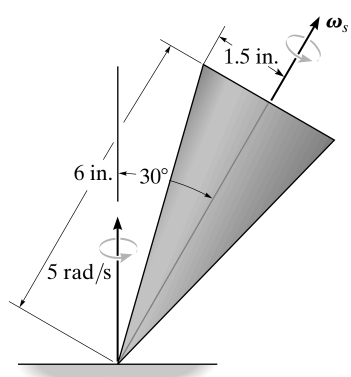
$$0.090(9.81)(0.06)\sin 45° = -0.090(0.035)^2(0.5)^2(0.7071)^2$$

$$+\ 0.090(0.018)^2(0.5)(0.7071)\left[0.5(0.7071) + \dot{\psi}\right]$$

$$\omega_s = \psi = 3.63\left(10^3\right)\text{ rad/s}$$ **Ans.**



***21–68.** The top has a weight of 3 lb and can be considered as a solid cone. If it is observed to precess about the vertical axis at a constant rate of 5 rad/s, determine its spin.



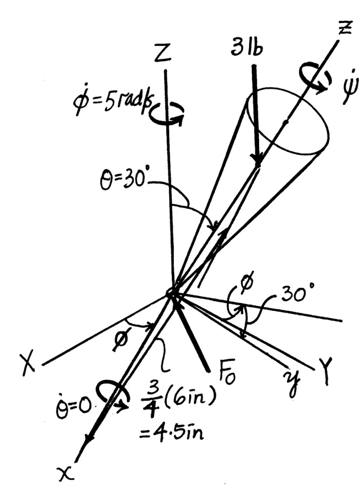
$$I = \frac{3}{80}\left(\frac{3}{32.2}\right)\left[4\left(\frac{1.5}{12}\right)^2 + \left(\frac{6}{12}\right)^2\right] + \frac{3}{32.2}\left(\frac{4.5}{12}\right)^2 = 0.01419\text{ slug}\cdot\text{ft}^2$$

$$I_z = \frac{3}{10}\left(\frac{3}{32.2}\right)\left(\frac{1.5}{12}\right)^2 = 0.43672\left(10^{-3}\right)\text{ slug}\cdot\text{ft}^2$$

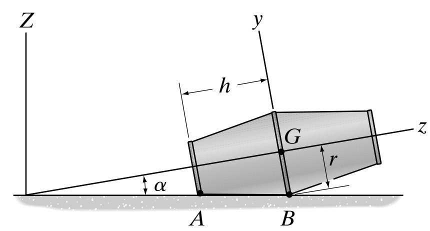
$$\Sigma M_x = -I\dot{\phi}^2\sin\theta\cos\theta + I_z\,\dot{\phi}\sin\theta\left(\dot{\phi}\cos\theta + \dot{\psi}\right)$$

$$(3)\left(\frac{4.5}{12}\right)(\sin 30°) - (0.01419)(5)^2\sin 30°\cos 30° + 0.43672\left(10^{-3}\right)(5)\sin 30°\left(5\cos 30° + \dot{\psi}\right)$$

$$\psi = 652\text{ rad/s}$$ **Ans.**



•21–69. The empty aluminum beer keg has a mass of m, center of mass at G, and radii of gyration about the x and y axes of $k_x = k_y = \frac{5}{4}r$, and about the z axis of $k_z = \frac{1}{4}r$, respectively. If the keg rolls without slipping with a constant angular velocity, determine its largest value without having the rim A leave the floor.



Since the beer keg rolls without slipping, the instantaneous axis of zero velocity is indicated in Fig. a. Thus, $\omega_p = \omega_s \sin\alpha$.

Since $\dot{\psi} = \omega_s$, $\dot{\phi} = -\omega_p = -\omega_s \sin\alpha$, and $\theta = 90° - \alpha$ are constant, the beer keg undergoes steady precession.

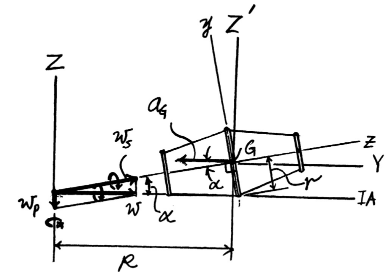
$I = I_x = I_y = m\left(\frac{5}{4}r\right)^2 = \frac{25}{16}mr^2$ and $I_z = m\left(\frac{1}{4}r\right)^2 = \frac{1}{16}mr^2$. Referring to the free-body diagram of the beer keg in Fig. b,

$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi});$$

$$F_B\cos\alpha(r) - N_B\sin\alpha(r) = -\frac{25}{16}mr^2(-\omega_s\sin\alpha)^2\sin(90° - \alpha)\cos(90° - \alpha)$$

$$+\frac{1}{16}mr^2(-\omega_s\sin\alpha)\sin(90°-\alpha)[(-\omega_s\sin\alpha)\cos(90°-\alpha) + \omega_s]$$

$$F_B\cos\alpha - N_B\sin\alpha = \frac{1}{16}mr\,\omega_s^2\sin\alpha\cos\alpha\left(26\sin^2\alpha - 1\right) \qquad \textbf{(1)}$$



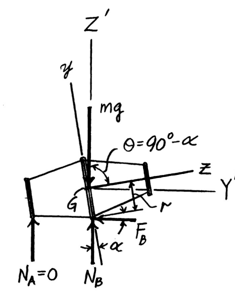
(a)

Since the mass center G of the beer keg rotates about the Z axis, Fig. a, its acceleration can be found from $a_G = \omega_p^2 R = (-\omega_s\sin\alpha)^2\left(\frac{r\cos^2\alpha}{\sin\alpha}\right)$ $= \omega_s^2 r\sin\alpha\cos^2\alpha$ and it is directed towards the negative Y' axis. Fig. a. Since the mass center does not move along the Z', $(a_G)_{Z'} = 0$. Thus,

$$\Sigma F_{Y'} = m(a_G)_{Y'}; \qquad -F_B = -m\left(\omega_s^2 r\sin\alpha\cos^2\alpha\right)$$

$$F_B = m\omega_s^2 r\sin\alpha\cos^2\alpha$$

$$\Sigma F_{Z'} = m(a_G)_{Z'}; \qquad N_B - mg = 0 \qquad N_B = mg$$

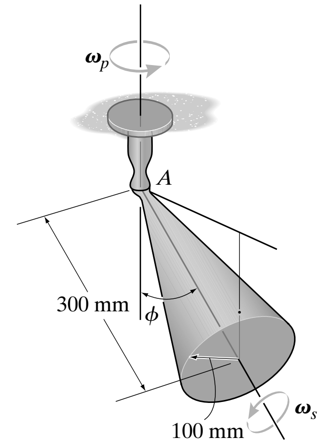


(b)

Substituting these results into Eq. (1),

$$\omega_S = \sqrt{\frac{16g}{r\cos\alpha\left(16\cos^2\alpha - 26\sin^2\alpha + 1\right)}} \qquad \textbf{Ans.}$$

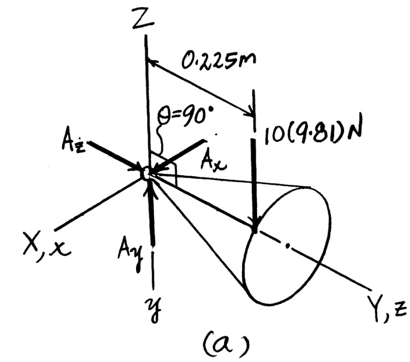
21–70. The 10-kg cone spins at a constant rate of $\omega_s = 150$ rad/s. Determine the constant rate ω_p at which it precesses if $\phi = 90°$.



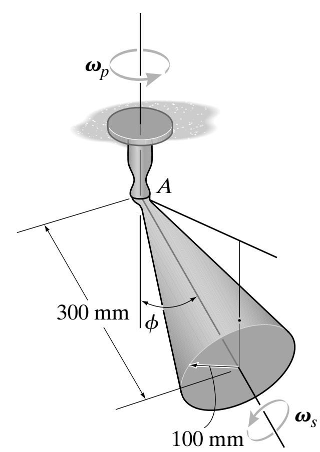
Here, $\theta = 180° - 90° = 90°$, $\dot{\psi} = \omega_s = -150$ rad/s, and $\dot{\phi} = \omega_p$ are constants. Thus, this is a special case of steady precession. $I_z = \frac{3}{10}(10)(0.1^2) = 0.03\ \text{kg}\cdot\text{m}^2$, $\Omega_y = -\dot{\phi} = -\omega_p$, and $\omega_z = \dot{\psi} = -150$ rad/s. Thus,

$$\Sigma M_x = I_z\Omega_y\omega_z; \qquad -10(9.81)(0.225) = 0.03(-\omega_p)(-150)$$

$$\omega_p = -4.905 \text{ rad/s} \qquad \textbf{Ans.}$$



21–71. The 10-kg cone is spinning at a constant rate of $\omega_s = 150$ rad/s. Determine the constant rate ω_p at which it precesses if $\phi = 30°$.



Since $\dot{\psi} = \omega_s = -150$ rad/s, $\dot{\phi} = \omega_p$, and $\theta = 180° - 30° = 150°$ are constants, the cone undergoes steady precession. $I_z = \frac{3}{10}(10)(0.2^2) = 0.03\ \text{kg}\cdot\text{m}^2$ and $I = I_x = I_y = \frac{3}{80}(10)\left[4(0.1^2) + 0.3^2\right] + 10(0.225^2) = 0.555\ \text{kg}\cdot\text{m}^2$.

Thus,

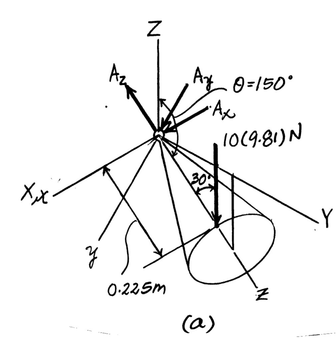
$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\sin\theta(\dot{\phi}\cos\theta + \dot{\psi})$$

$$-10(9.81)\sin 30°(0.225) = -0.555\omega_p^2 \sin 150° \cos 150° + 0.03\omega_p \sin 150°\left[\omega_p \cos 150° + (-150)\right]$$

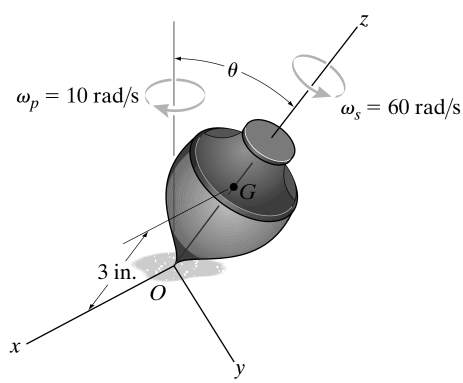
$$0.2273\omega_p^2 - 2.25\omega_p + 11.036 = 0$$

Solving,

$$\omega_p = 13.5 \text{ rad/s or } 3.60 \text{ rad/s} \qquad \textbf{Ans.}$$



***21–72.** The 1-lb top has a center of gravity at point G. If it spins about its axis of symmetry and precesses about the vertical axis at constant rates of $\omega_s = 60$ rad/s and $\omega_p = 10$ rad/s, respectively, determine the steady state angle θ. The radius of gyration of the top about the z axis is $k_z = 1$ in., and about the x and y axes it is $k_x = k_y = 4$ in.



Since $\dot{\psi} = \omega_s = 60$ rad/s and $\dot{\phi} = \omega_p = -10$ rad/s and θ are constant, the top undergoes steady precession.

$$I_z = \left(\frac{1}{32.2}\right)\left(\frac{1}{12}\right)^2 = 215.67\left(10^{-6}\right) \text{ slug} \cdot \text{ft}^2 \quad \text{and} \quad I = I_x = I_y = \left(\frac{1}{32.2}\right)\left(\frac{4}{12}\right)^2$$

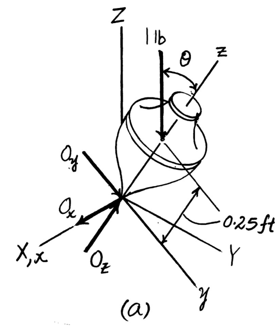
$$= 3.4507\left(10^{-3}\right) \text{ slug} \cdot \text{ft}^2.$$

Thus,

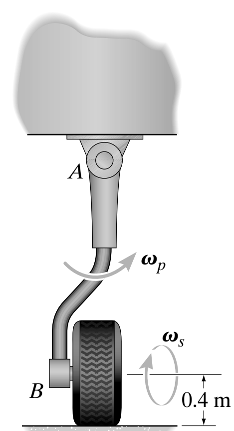
$$\Sigma M_x = -I\dot{\phi}^2 \sin\theta\cos\theta + I_z\dot{\phi}\sin\theta\left(\dot{\phi}\cos + \dot{\psi}\right)$$

$$-1\sin\theta(0.25) = -3.4507\left(10^{-3}\right)(-10)^2\sin\theta\cos\theta + 215.67\left(10^{-6}\right)(-10)\sin\theta[(-10)\cos\theta + 60]$$

$$\theta = 68.1^\circ \qquad \textbf{Ans.}$$

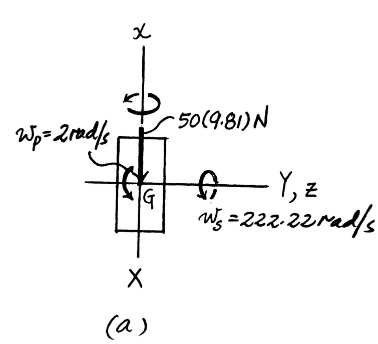


•21–73. At the moment of take off, the landing gear of an airplane is retracted with a constant angular velocity of $\omega_p = 2$ rad/s, while the wheel continues to spin. If the plane takes off with a speed of $v = 320$ km/h, determine the torque at A due to the gyroscopic effect. The wheel has a mass of 50 kg, and the radius of gyration about its spinning axis is $k = 300$ mm.

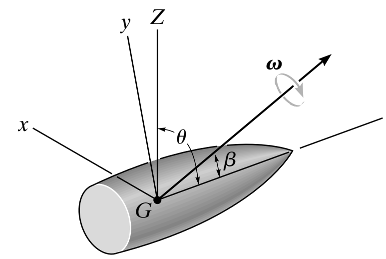


When the plane travels with a speed of $v = \left(320\,\frac{\text{km}}{\text{h}}\right)\left(\frac{1000\text{ m}}{1\text{ km}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 88.89$ m/s, its wheel spins with a constant angular velocity of $\omega_s = \frac{v}{r} = \frac{88.89}{0.4} = 222.22$ rad/s. Here, $\theta = 90^\circ$, $\Omega_y = \omega_p = 2$ rad/s and $\omega_z = \omega_s = 222.22$ rad/s are constants. This is a special case of steady precession. $I_z = 50\left(0.3^2\right) = 4.5 \text{ kg} \cdot \text{m}^2$. Thus,

$$\Sigma M_x = I_z\Omega_y\omega_z; \qquad M_x = 4.5(2)(222.22) = 2000 \text{ N} \cdot \text{m} = 2\text{kN} \cdot \text{m} \qquad \textbf{Ans.}$$



21–74. The projectile shown is subjected to torque-free motion. The transverse and axial moments of inertia are I and I_z, respectively. If θ represents the angle between the precessional axis Z and the axis of symmetry z, and β is the angle between the angular velocity $\boldsymbol{\omega}$ and the z axis, show that β and θ are related by the equation $\tan\theta = (I/I_z)\tan\beta$.



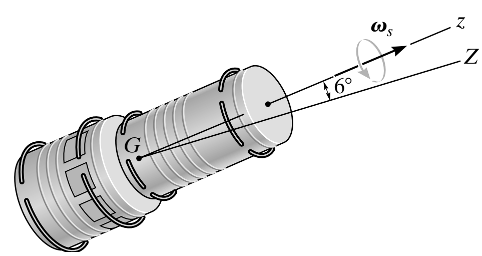
From Eq. 21–34 $\omega_y = \dfrac{H_G \sin\theta}{I}$ and $\omega_z = \dfrac{H_G\cos\theta}{I_z}$ Hence $\dfrac{\omega_y}{\omega_z} = \dfrac{I_z}{I}\tan\theta$

However, $\omega_y = \omega\sin\beta$ and $\omega_z = \omega\cos\beta$

$$\frac{\omega_y}{\omega_z} = \tan\beta = \frac{I_z}{I}\tan\theta$$

$$\tan\theta = \frac{I}{I_z}\tan\beta \qquad \textbf{Q.E.D.}$$

21–75. The space capsule has a mass of 3.2 Mg, and about axes passing through the mass center G the axial and transverse radii of gyration are $k_z = 0.90$ m and $k_t = 1.85$ m, respectively. If it spins at $\omega_s = 0.8$ rev/s, determine its angular momentum. Precession occurs about the Z axis.



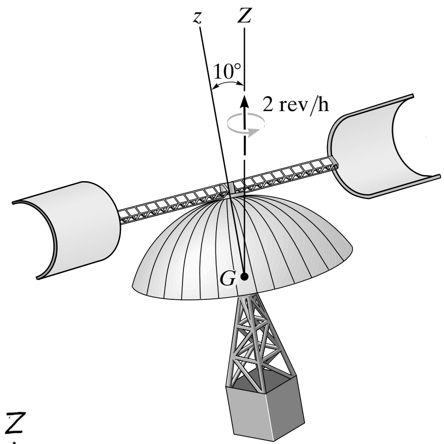
Gyroscopic Motion: Here, the spinning angular velocity $\psi = \omega_s = 0.8(2\pi) = 1.6\pi$ rad/s. The moment of inertia of the satelite about the z axis is $I_z = 3200(0.9^2) = 2592\ \text{kg}\cdot\text{m}^2$ and the moment of inertia of the satelite about its transverse axis is $I = 3200(1.85^2) = 10\,952\ \text{kg}\cdot\text{m}^2$. Applying the third of Eq. 21–36 with $\theta = 6°$, we have

$$\dot{\psi} = \frac{I - I_z}{II_z}H_G\cos\theta$$

$$1.6\pi\left[\frac{10\,952 - 2592}{10\,952(2592)}\right]H_G\cos 6°$$

$$H_G = 17.16(10^3)\ \text{kg}\cdot\text{m}^2/\text{s} = 17.2\ \text{Mg}\cdot\text{m}^2/\text{s} \qquad \textbf{Ans.}$$

***21–76.** The radius of gyration about an axis passing through the axis of symmetry of the 2.5-Mg satellite is $k_z = 2.3$ m, and about any transverse axis passing through the center of mass G, $k_t = 3.4$ m. If the satellite has a steady-state precession of two revolutions per hour about the Z axis, determine the rate of spin about the z axis.



$$I_z = 2500(2.3)^2 = 13\,225 \text{ kg} \cdot \text{m}^2$$

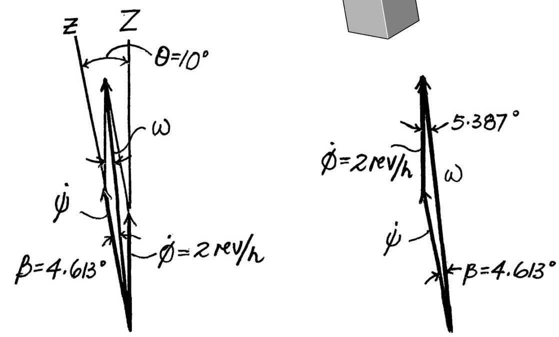
$$I = 2500(3.4)^2 = 28\,900 \text{ kg} \cdot \text{m}^2$$

Use the result of Prob. 21–74.

$$\tan\theta = \left(\frac{I}{I_z}\right)\tan\beta$$

$$\tan 10° = \left(\frac{28\,900}{13\,225}\right)\tan\beta$$

$$\beta = 4.613°$$

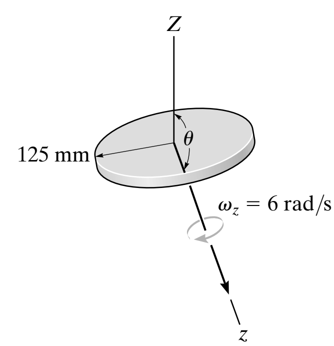


From the law of sines,

$$\frac{\sin 5.387°}{\psi} = \frac{\sin 4.613°}{2}$$

$$\psi = 2.33 \text{ rev/h} \qquad \textbf{Ans.}$$

•21–77. The 4-kg disk is thrown with a spin $\omega_z = 6$ rad/s. If the angle θ is measured as 160°, determine the precession about the Z axis.



$$I = \tfrac{1}{4}(4)(0.125)^2 = 0.015625 \text{ kg} \cdot \text{m}^2 \qquad I_z = \tfrac{1}{2}(4)(0.125)^2 = 0.03125 \text{ kg} \cdot \text{m}^2$$

Applying Eq. 21–36 with $\theta = 160°$ and $\dot{\psi} = 6$ rad/s

$$\dot{\psi} = \frac{I - I_z}{II_z} H_O \cos\theta$$

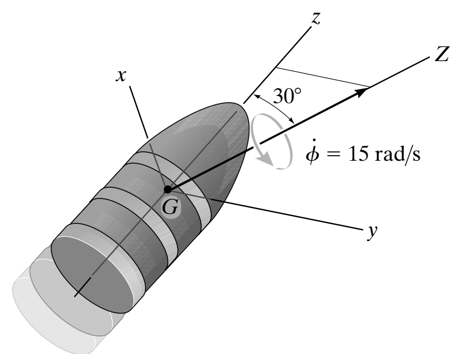
$$6 = \frac{0.015625 - 0.03125}{0.015625(0.03125)} H_O \cos 160°$$

$$H_G = 0.1995 \text{ kg} \cdot \text{m}^2/\text{s}$$

$$\phi = \frac{H_G}{I} = \frac{0.1995}{0.015625} = 12.8 \text{ rad/s} \qquad \textbf{Ans.}$$

Note that this is a case of retrograde precession since $I_z > I$.

21–78. The projectile precesses about the Z axis at a constant rate of $\dot{\phi} = 15$ rad/s when it leaves the barrel of a gun. Determine its spin $\dot{\psi}$ and the magnitude of its angular momentum $\mathbf{H}_G$. The projectile has a mass of 1.5 kg and radii of gyration about its axis of symmetry (z axis) and about its transverse axes (x and y axes) of $k_z = 65$ mm and $k_x = k_y = 125$ mm, respectively.



Since the only force that acts on the projectile is its own weight, the projectile undergoes torque-free motion. $I_z = 1.5\left(0.065^2\right) = 6.3375\left(10^{-3}\right)\ \text{kg}\cdot\text{m}^2$, $I = I_x = I_y = 1.5\left(0.125^2\right) = 0.0234375\ \text{kg}\cdot\text{m}^2$, and $\theta = 30°$. Thus,

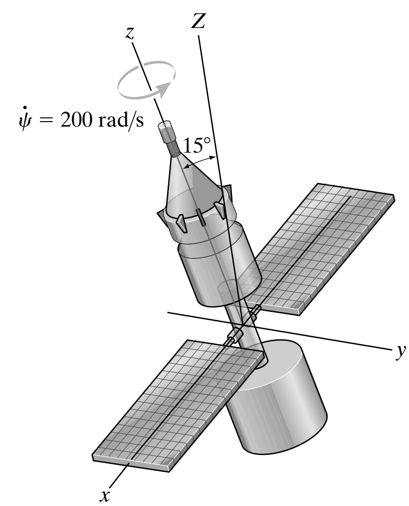
$$\dot{\phi} = \frac{H_G}{I};\quad H_G = I\dot{\phi} = 0.0234375(15) = 0.352\ \text{kg}\cdot\text{m}^2/\text{s} \qquad \textbf{Ans.}$$

$$\dot{\psi} = \frac{I - I_z}{I_z}\dot{\phi}\cos\theta$$

$$= \frac{0.0234375 - 6.3375\left(10^{-3}\right)}{6.3375\left(10^{-3}\right)}(15)\cos 30°$$

$$= 35.1\ \text{rad/s} \qquad \textbf{Ans.}$$

21–79. The satellite has a mass of 100 kg and radii of gyration about its axis of symmetry (z axis) and its transverse axes (x or y axis) of $k_z = 300$ mm and $k_x = k_y = 900$ mm, respectively. If the satellite spins about the z axis at a constant rate of $\dot{\psi} = 200$ rad/s, and precesses about the Z axis, determine the precession $\dot{\phi}$ and the magnitude of its angular momentum $\mathbf{H}_G$.



Since the weight is the only force acting on the satellite, it undergoes torque-free motion.

Here, $I_z = 100\left(0.3^2\right) = 9\ \text{kg}\cdot\text{m}^2$, $I = I_x = I_y = 100\left(0.9^2\right) = 81\ \text{kg}\cdot\text{m}^2$, and $\theta = 15°$. Then,

$$\dot{\psi} = \frac{I - I_z}{I_z}\dot{\phi}\cos\theta$$

$$200 = \left(\frac{81 - 9}{9}\right)\dot{\phi}\cos 15°$$

$$\dot{\phi} = 25.88\ \text{rad/s}$$

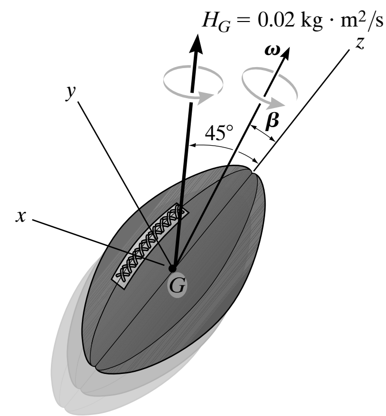
Using this result,

$$\dot{\phi} = \frac{H_G}{I}$$

$$25.88 = \frac{H_G}{81}$$

$$H_G = 2096\ \text{kg}\cdot\text{m}^2/\text{s} = 2.10\ \text{Mg}\cdot\text{m}^2/\text{s} \qquad \textbf{Ans.}$$

***21–80.** The football has a mass of 450 g and radii of gyration about its axis of symmetry (z axis) and its transverse axes (x or y axis) of $k_z = 30$ mm and $k_x = k_y = 50$ mm, respectively. If the football has an angular momentum of $H_G = 0.02\ \text{kg}\cdot\text{m}^2/\text{s}$, determine its precession $\dot{\phi}$ and spin $\dot{\psi}$. Also, find the angle β that the angular velocity vector makes with the z axis.



Since the weight is the only force acting on the football, it undergoes torque-free motion. $I_z = 0.45(0.03^2) = 0.405(10^{-3})\ \text{kg}\cdot\text{m}^2$, $I = I_x = I_y = 0.45(0.05^2) = 1.125(10^{-3})\ \text{kg}\cdot\text{m}^2$, and $\theta = 45°$.

Thus,

$$\dot{\phi} = \frac{H_G}{I} = \frac{0.02}{1.125(10^{-3})} = 17.78\ \text{rad/s} = 17.8\ \text{rad/s} \qquad \textbf{Ans.}$$

$$\dot{\psi} = \frac{I - I_z}{I I_z} H_G \cos\theta = \frac{1.125(10^{-3}) - 0.405(10^{-3})}{1.125(10^{-3})(0.405)(10^{-3})}(0.02)\cos 45°$$

$$= 22.35\ \text{rad/s} = 22.3\ \text{rad/s} \qquad \textbf{Ans.}$$

Also,

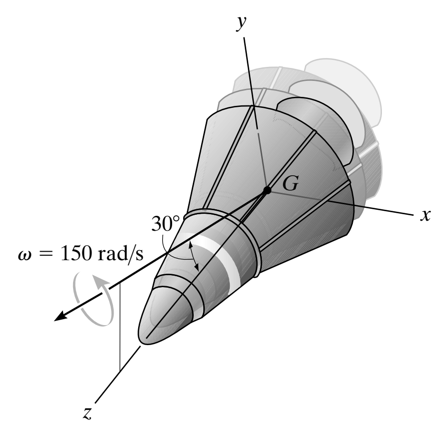
$$\omega_y = \frac{H_G \sin\theta}{I} = \frac{0.02 \sin 45°}{1.125(10^{-3})} = 12.57\ \text{rad/s}$$

$$\omega_z = \frac{H_G \cos\theta}{I_z} = \frac{0.02 \cos 45°}{0.405(10^{-3})} = 34.92\ \text{rad/s}$$

Thus,

$$\beta = \tan^{-1}\left(\frac{\omega_y}{\omega_z}\right) = \tan^{-1}\left(\frac{12.57}{34.92}\right) = 19.8° \qquad \textbf{Ans.}$$

•21–81. The space capsule has a mass of 2 Mg, center of mass at G, and radii of gyration about its axis of symmetry (z axis) and its transverse axes (x or y axis) of $k_z = 2.75$ m and $k_x = k_y = 5.5$ m, respectively. If the capsule has the angular velocity shown, determine its precession $\dot{\phi}$ and spin $\dot{\psi}$. Indicate whether the precession is regular or retrograde. Also, draw the space cone and body cone for the motion.



The only force acting on the space capsule is its own weight. Thus, it undergoes torque-free motion. $I_z = 2000(2.75^2) = 15\,125\ \text{kg}\cdot\text{m}^2$, $I = I_x = I_y = 2000(5.5^2) = 60\,500\ \text{kg}\cdot\text{m}^2$. Thus,

$$\omega_y = \frac{H_G \sin\theta}{I}$$

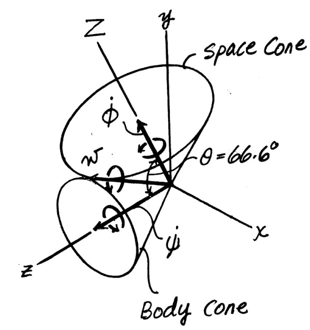
$$150 \sin 30° = \frac{H_G \sin\theta}{60\,500}$$

$$H_G \sin\theta = 4\,537\,500 \qquad \textbf{(1)}$$

$$\omega_z = \frac{H_G \cos\theta}{I_z}$$

$$150 \cos 30° = \frac{H_G \cos\theta}{15\,125}$$

$$H_G \cos\theta = 1\,964\,795.13 \qquad \textbf{(2)}$$



Solving Eqs. (1) and (2),

$$H_G = 4.9446(10^6)\ \text{kg}\cdot\text{m}^2/\text{s} \qquad \theta = 66.59°$$

Using these results,

$$\dot{\phi} = \frac{H_G}{I} = \frac{H_G}{60\,500} = \frac{4.9446(10^6)}{60\,500} = 81.7\ \text{rad/s} \qquad \textbf{Ans.}$$

$$\dot{\psi} = \frac{I - I_z}{I I_z} H_G \cos\theta = \left[\frac{60\,500 - 15\,125}{60\,500(15125)}\right] 4.9446(10^6) \cos 30°$$

$$= 212\ \text{rad/s} \qquad \textbf{Ans.}$$

Since $I > I_z$, the motion is *regular precession*. **Ans.**