function, Φ, in terms of its relationship to the stresses, that is,

$$\left.\begin{aligned} \sigma_x &= \frac{\partial^2\Phi}{\partial y^2} \\ \sigma_y &= \frac{\partial^2\Phi}{\partial x^2} \\ \tau_{xy} &= \frac{-\partial^2\Phi}{\partial x\partial y} \end{aligned}\right\} \quad \text{(116)}$$

Substituting Eq 116 into Eq 115 leads to:

$$\nabla^4\Phi = \nabla^2(\nabla^2\Phi) = 0 \quad \text{(117)}$$

In order to solve a problem, the stress function, Φ, must satisfy Eq 117 and the boundary conditions of that problem.

Choosing the stress function, Φ, to be:

$$\Phi = \psi_1 + x\psi_2 + y\psi_3 \quad \text{(118)}$$

it will automatically satisfy Eq 117 if the ψ_i are each harmonic, that is,

$$\nabla^2\psi_i = 0 \quad \text{(119)}$$

Define a complex variable, z, by

$$z = x + iy \quad \text{(120)}$$

Functions of that complex variable, $\bar{\bar{Z}}(z)$, and its derivatives,

$$\bar{Z} = \frac{d\bar{\bar{Z}}}{dz}, \qquad Z = \frac{d\bar{Z}}{dz}, \qquad Z' = \frac{dZ}{dz} \quad \text{(121)}$$

have harmonic real and imaginary parts, if the function is analytic, for example, if $\bar{Z} = \text{Re}\,\bar{Z} + i\,\text{Im}\,\bar{Z}$, then

$$\nabla^2(\text{Re}\,\bar{Z}) = \nabla^2(\text{Im}\,\bar{Z}) = 0 \quad \text{(122)}$$

This is a result of the Cauchy-Riemann conditions, that is,

$$\left.\begin{aligned} \frac{\partial \text{Re}\,\bar{Z}}{\partial x} &= \frac{\partial \text{Im}\,\bar{Z}}{\partial y} = \text{Re}\,Z \\ \frac{\partial \text{Im}\,\bar{Z}}{\partial x} &= -\frac{\partial \text{Re}\,\bar{Z}}{\partial y} = \text{Im}\,Z \end{aligned}\right\} \quad \text{(123)}$$

Equations 123 may be used to differentiate these functions $\bar{\bar{Z}}$ through Z.

First Mode:

In conformity with Eqs 118–123, Westergaard (8) defined an Airy stress function, Φ, by

$$\Phi_\text{I} = \text{Re}\,\bar{\bar{Z}}_\text{I} + y\,\text{Im}\,\bar{Z}_\text{I} \quad \text{(124)}$$

which as a consequence automatically satisfies equilibrium and compatability, Eqs 114 and 117.

Using Eqs 116 and 123, the stresses resulting from Φ, as defined in Eq 124, are

$$\left.\begin{aligned} \sigma_x &= \text{Re}\,Z_\text{I} - y\,\text{Im}\,Z_\text{I}' \\ \sigma_y &= \text{Re}\,Z_\text{I} + y\,\text{Im}\,Z_\text{I}' \\ \tau_{xy} &= -y\,\text{Re}\,Z_\text{I}' \end{aligned}\right\} \quad \text{(125)}$$

Now any function, Z_I, which is analytic in the region except for a particular branch cut along a portion of the x-axis will have the form

$$Z_\text{I} = \frac{g(z)}{[(z+b)(z-a)]^{1/2}} \quad \text{(126)}$$

This will solve crack problems for a crack along the x-axis from $x = -b$ to $x = a$, ($y = 0$), if $g(z)$ is well behaved, since the stresses, σ_y and τ_{xy}, along that interval are zero, provided that

$$\text{Im}\,g(x) = 0 \text{ (for } -b < x < a) \quad \text{(127)}$$

For example, if the function

$$Z_\text{I} = \frac{\sigma z}{(z^2 - a^2)^{1/2}} \quad \text{(128)}$$

is examined, it solves the problem of a stress-free crack at $-a < x < a$, $y = 0$, and leads to boundary conditions of uniform biaxial stress, σ, at infinity (see Fig. 3).

Now, reverting to the more general case, Eq 126, a substitution of variable

$$\zeta = z - a \quad \text{(129)}$$

leads to

$$Z_\text{I} = \frac{f(\zeta)}{\zeta^{1/2}} \quad \text{(130)}$$

where, from Eqs 126 and 127, $f(\zeta)$ is well behaved for small $|\zeta|$ (that is, near the crack tip at $x = a$). Moreover, in that region, as $|\zeta| \to 0$, f may be replaced by a real constant, or Eq 130 may be written

$$Z_\text{I}\,\big|_{|\zeta|\to 0} = \frac{K_\text{I}}{(2\pi\zeta)^{1/2}} \quad \text{(131)}$$

Other stress functions, Z_I, for crack problems, such as Eq 16, will also always lead