seek a function $u(x, t)$, which is defined in the interior of a rectangle $ABCD$. This region is already determined by the statement of the problem, since the course of the heat propagation in the rod $0 \leqq x \leqq l$ during the time interval $t \leqq t = T$, in which the heat behavior of the boundary is known, was already investigated. Let $t_0 = 0$; we assume that $u(x, t)$ satisfies the heat-conduction equation only for $0 < x < l$, $0 < t \leqq T$, i.e., not for $t = 0$ (the side AB) or for $x = 0$, $x = l$ (the sides AD and BC). For $t = 0$, as well as $x = 0$ and $x = l$, the value of this function is given directly by the initial and boundary conditions. To require that the heat-conduction equation, for example, be satisfied also for $t = 0$ would imply that the derivative $\varphi'' = u_{xx}(x, 0)$ in this equation exists. Therefore, the generality of the physical phenomena to be investigated is limited, and thus the basic functions which do not satisfy this requirement are eliminated from consideration. The condition (3-1.3) loses its meaning when it is not required that $u(x, t)$ in the region $0 \leqq x \leqq l$, $0 \leqq t \leqq T$ (i.e., in the closed rectangle $ABCD$) be continuous or this requirement must be replaced by another appropriate assumption.[46] To understand the significance of this requirement we consider the function $v(x, t)$ defined by the following conditions:

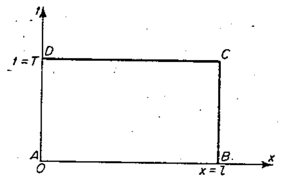

FIG. 37.

$$v(x, t) = C, \quad 0 < x < l, \quad 0 < t \leqq T$$
$$v(x, 0) = \varphi(x), \quad 0 \leqq x \leqq l$$
$$v(0, t) = \mu_1(t), \quad v(l, t) = \mu_2(t), \quad 0 \leqq t \leqq T$$

where C is an arbitrary constant. The function v obviously satisfies both condition (3-1.2) and the boundary conditions. However, this function in no case describes the course of the heat distribution in the rod with an initial temperature $\varphi(x) \neq C$ and boundary temperatures $\mu_1(t) \neq C$ and $\mu_2(t) \neq C$, since it is discontinuous for $t = 0$, $x = 0$, $x = l$.

The continuity of $u(x, t)$ for $0 < x < l$, $0 < t < T$ directly follows in that $u(x, t)$ satisfies the differential equation. Therefore, the requirement that $u(x, t)$ be continuous in $0 \leqq x \leqq l$, $0 \leqq t \leqq T$, is based essentially only on those points at which the boundary and the initial values are prescribed. In the following, by a solution of the equation which satisfies the boundary conditions, we shall always mean a function which satisfies the requirements (3-1.1), (3-1.2), and (3-1.3) and hence not repeat these each time, unless there are special conditions.

Correspondingly, this is the case for other boundary-value problems, in particular for problems of an infinite rod and problems without initial conditions.

[46] Later, boundary-value problems with discontinuous boundary and initial conditions will be considered. For these, the problems will be properly defined so that the boundary conditions are fulfilled.

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For problems with several independent geometric variables the above statements remain valid. In these problems, an initial temperature and boundary conditions determined on the surface of the body are prescribed for $t = 0$. We can also investigate problems for infinite domains.

With regard to all the problems discussed, the following problems exist[47]:

1. Are the solutions of the problems discussed uniquely determined?
2. Does a solution exist?
3. Do the solutions depend continuously on the auxiliary conditions?

If a problem admits of many solutions, then we naturally cannot speak of "the solution of the problem," and we must first prove the uniqueness. In practice, the second question above is the most important, since generally in proving the existence of a solution, we simultaneously find methods for its calculation.

As noted earlier (see Section 2-2 §3) we speak of a physically determined process when a small change in the initial or boundary conditions causes a small change in the solution. In the following, it will be shown that heat propagation is determined physically by the initial and boundary conditions, i.e., a small change in the initial or boundary conditions implies a small change in the solution.

5. The principle of the maximum

In the following we shall investigate differential equations with constant coefficients,

$$v_t = a^2 v_{xx} + \beta v_x + \gamma v. \qquad (3\text{-}1.34)$$

As already shown, these equations, by the substitution of

$$v = e^{\mu x + \lambda t} u \quad \text{with} \quad \mu = -\frac{\beta}{2a^2}, \quad \lambda = \gamma - \frac{\beta^2}{4a^2}$$

can be brought to the form

$$u_t = a^2 u_{xx}. \qquad (3\text{-}1.35)$$

The solutions of this equation have the following properties which will be denoted as the principle of the maximum.

A function $u(x, t)$ defined and continuous in the closed region $0 \leqq t \leqq T$, $0 \leqq x \leqq l$ and satisfying the heat-conduction equation

$$u_t = a^2 u_{xx} \qquad (3\text{-}1.35)$$

in the region $0 < t < T$, $0 < x < l$ assumes its maximum or minimum at the initial moment $t = 0$ or at the boundary points $x = 0$ or $x = l$.

Before we prove this, note that the function $u(x, t) = \text{const.}$ obviously satisfies the heat-conduction equation and assumes a maximum (minimum) at each point. However, this does not contradict our assertion, because it means only that when a maximum (minimum) is assumed in the interior of the region it is also (but not only) assumed for $t = 0$ or for $x = 0$ or $x = l$.

[47] *Cf.* Section 2-2.

The physical significance of this statement is immediately clear: if the temperature on the boundary and at the initial moment does not exceed a value M, then in the interior of the body no temperature higher than M can be attained. We shall limit ourselves to the proof of the statement of the maximum and give an indirect proof. We shall designate by M the maximum value of $u(x, t)$ for $t = 0$ $(0 \leqq x \leqq l)$ or for $x = 0$ or $x = l$ $(0 \leqq t \leqq T)$ and assume that the function $u(x, t)$ assumes its maximum at an interior point (x_0, t_0), $(0 < x_0 < l, 0 < t_0 \leqq T)$:[48]

$$u(x_0, t_0) = M + \varepsilon .$$

We now compare the signs in Eq. (3-1.35) at the point (x_0, t_0). Since the function at (x_0, t_0) assumes its maximum,[49] then necessarily

$$\frac{\partial u}{\partial x}(x_0, t_0) = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2}(x_0, t_0) \leqq 0 . \tag{3-1.36}$$

Also, since $u(x_0, t)$ for $t = t_0$ has a maximum,[50] then

$$\frac{\partial u}{\partial t}(x_0, t_0) \geqq 0 . \tag{3-1.37}$$

By comparison of the signs on the left and right sides of (3-1.35) it follows that both sides can be different. These considerations, however, still do not prove the correctness of our theorem; since the right and the left sides can simultaneously equal zero, it would signify no contradiction. We bring forth this consideration simply to emphasize the fundamental concepts of our proof. For the completion of the proof we shall seek more than one point (x_1, t_1) at which $\partial^2 u/\partial x^2 \leqq 0$ and $\partial u/\partial t > 0$. Therefore, we consider the auxiliary function

$$v(x, t) = u(x, t) + k(t_0 - t) , \tag{3-1.38}$$

where k is a constant. Obviously then

$$v(x_0, t_0) = u(x_0, t_0) = M + \varepsilon$$

and

$$k(t_0 - t) \leqq kT .$$

[48] If the continuity of $u(x, t)$ were assumed in the bounded region $0 \leqq x \leqq l$, $0 \leqq t \leqq T$, then the function $u(x, t)$ could not exceed its maximum, and further considerations would be contradictory. On the basis of the theorem that every continuous function in a bounded region attains its maximum, then (a) the function $u(x, t)$ attains a maximum within or on the boundaries which will be denoted by M; (b) if $u(x, t)$ also were to exceed M only at a point, then a point (x_0, t_0) would exist at which the function $u(x, t)$ assumes a maximum which is larger than M: $u(x_0, t_0) = M + \varepsilon$ $(\varepsilon > 0)$, where $0 < x_0 < l$, $0 < t_0 \leqq T$.

[49] As is known from analysis, for the existence of a relative minimum of a function $f(x)$ at an interior point x_0 of an interval $(0, l)$, the conditions

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0} = 0, \left.\frac{\partial^2 f}{\partial x^2}\right|_{x=x_0} > 0$$

are sufficient. If, therefore, at the point x_0 the function $f(x)$ has a maximum value, then (a) $f'(x_0) = 0$, and (b) $f''(x_0) > 0$ cannot hold; therefore $f''(x_0) \leqq 0$.

[50] Obviously, $\partial u/\partial t = 0$, in case $t_0 < T$, whereas for $t_0 = T$, then $\partial u/\partial t = 0$ must hold.

We now select $k > 0$ so that $kT < \varepsilon/2$, i.e., let $k < \varepsilon/2T$; then the maximum of $v(x, t)$ for $t = 0$ or for $x = 0, x = l$ does not exceed the value $M + \varepsilon/2$, i.e.,

$$v(x, t) \leqq M + \frac{\varepsilon}{2} \quad \text{for } t = 0 \text{ or } x = 0, x = l , \tag{3-1.39}$$

since for this argument the first summand of (3-1.38) is not larger than M, and the second is not larger than $\varepsilon/2$.

Now, $v(x, t)$ is a continuous function. Thus a point (x_1, t_1) exists at which it assumes its maximum. Then we have

$$v(x_1, t_1) \geqq v(x_0, t_0) = M + \varepsilon .$$

Therefore, $t_1 > 0$ and $0 < x_1 < l$, since for $t = 0$ or $x = 0$, $x = l$ the inequality (3-1.39) is valid. It follows that

$$v_{xx}(x_1, t_1) = u_{xx}(x_1, t_1) \leqq 0$$

and

$$v_t(x_1, t_1) = u_t(x_1, t_1) - k \geqq 0 \quad \text{or} \quad u_t(x_1, t_1) \geqq k > 0 .$$

By comparison of the signs on the right and the left sides in (3-1.35) at the point (x_1, t_1) we conclude that Eq. (3-1.35) at the point (x_1, t_1) cannot be satisfied, since the quantities on the right and left sides have different signs. Therefore, the first part of our proposition is proved. The statement for the minimum can be proved analogously, and it is sufficient to apply the first part to $u_1 = -u$.

6. The uniqueness theorem

We turn now to a series of consequences of the principle of the maximum. First, we prove the uniqueness theorem for the first boundary-value problem. If the functions $u_1(x, t)$ and $u_2(x, t)$, which are defined and continuous in a region $0 \leqq x \leqq l, 0 \leqq t \leqq T$, and which satisfy the heat-conduction equation

$$u_t = a^2 u_{xx} + f(x, t) \quad \text{for} \quad 0 < x < l, t > 0 \tag{3-1.35'}$$

as well as the same initial and boundary conditions

$$u_1(x, 0) = u_2(x, 0) = \varphi(x)$$
$$u_1(0, t) = u_2(0, t) = \mu_1(t)$$
$$u_1(l, t) = u_2(l, t) = \mu_2(t) ,$$

then necessarily[51]

$$u_1(x, t) \equiv u_2(x, t) .$$

For the proof of this theorem we consider the function

[51] Previously this theorem was refined and the continuity requirement at $t = 0$ was dropped.

$$v(x, t) = u_2(x, t) - u_1(x, t) .$$

Since $u_1(x, t)$ and $u_2(x, t)$ for

$$0 \leqq x \leqq l , \quad 0 \leqq t \leqq T$$

are continuous, their difference $v(x, t)$ in the same region is continuous. Further, $v(x, t)$ as the difference of two solutions of the heat-conduction equation for $0 < x < l$, $t > 0$ is similarly a solution of the heat-conduction equation in that region. Consequently, the principle of the maximum can also be applied to this function, and the maximum and the minimum of $v(x, t)$ for $t = 0$ or $x = 0$ or $x = l$ is assumed. According to the hypothesis we obtain

$$v(x, 0) = 0 , \quad v(0, t) = 0 , \quad v(l, t) = 0 .$$

Therefore, also

$$v(x, t) \equiv 0 ,$$

i.e.,

$$u_1(x, t) \equiv u_2(x, t) ,$$

from which the uniqueness of the solution of the first boundary-value problem follows.

We shall now prove a series of direct conclusions from the principle of the maximum. In the following discussion we shall refer to "the solution of the heat-conduction equation," instead of enumerating the properties of the function in detail which also satisfy the initial and boundary conditions.

1. If two solutions $u_1(x, t)$ and $u_2(x, t)$ of the heat-conduction equation satisfy the conditions

$$u_1(x, 0) \leqq u_2(x, 0) , \quad u_1(0, t) \leqq u_2(0, t) , \quad u_1(l, t) \leqq u_2(l, t) ,$$

then

$$u_1(x, t) \leqq u_2(x, t)$$

for all $0 \leqq x \leqq l, 0 \leqq t \leqq T$.

The difference $v(x, t) = u_2(x, t) - u_1(x, t)$ satisfies the conditions on which the principal of the maximum is based; also

$$v(x, 0) \geqq 0 \quad v(0, t) \geqq 0 \quad v(l, t) \geqq 0 .$$

Therefore

$$v(x, t) \geqq 0 \quad \text{for} \quad 0 < x < l,\ 0 < t \leqq T ,$$

since $v(x, t)$ in the region

$$0<x<l , \quad 0<t\leqq T$$

would otherwise have a negative value.

2. If three solutions

$$u(x, t) , \quad \underline{u}(x, t) , \quad \bar{u}(x, t)$$

of the heat-conduction equation satisfy the conditions

$$\underline{u}(x, t) \leqq u(x, t) \leqq \bar{u}(x, t) \quad \text{for} \quad t = 0 , \quad x = 0 , \quad x = l ,$$

then this inequality is fulfilled for all x in $0 \leqq x \leqq l$ and all t in $0 \leqq t \leqq T$.

This assertion represents an application of conclusion (1) to the functions

$$u(x, t) , \quad \bar{u}(x, t) \quad \text{and} \quad u(x, t) , \quad \underline{u}(x, t) .$$

3. If, for two solutions $u_1(x, t)$ and $u_2(x, t)$ of the heat conduction equation, the inequality

$$| u_1(x, t) - u_2(x, t) | \leqq \varepsilon , \quad \text{for} \quad t = 0 , \quad x = 0 , \quad x = l$$

is valid, then

$$| u_1(x, t) - u_2(x, t) | \leqq \varepsilon$$

for all x, t in

$$0 \leqq x \leqq l , \quad 0 \leqq t \leqq T$$

is satisfied.

This assertion results from conclusion (2), when we apply the heat-conduction equation to the solutions

$$\underline{u}(x, t) = -\varepsilon$$
$$u(x, t) = u_1(x, t) - u_2(x, t)$$
$$\bar{u}(x, t) = \varepsilon .$$

The question regarding the continuous dependence of the solution of the first boundary-value problem on the initial and boundary conditions is answered completely by conclusion (3). To understand this, we consider a solution $u(x, t)$ which satisfies other initial and boundary conditions, instead of the solution of the heat-conduction equation which corresponds to the initial and boundary conditions

$$u(x, 0) = \varphi(x) , \quad u(0, t) = \mu_1(t) , \quad u(l, t) = \mu_2(t) .$$

Let these be given by functions $\varphi^*(x)$, $\mu_1^*(t)$ and $\mu_2^*(t)$ which differ by less than ε from the functions $\varphi(x)$, $\mu_1(t)$, and $\mu_2(t)$:

$$| \varphi(x) - \varphi^*(x) | \leqq \varepsilon , \quad | \mu_1(t) - \mu_1^*(t)| \leqq \varepsilon , \quad | \mu_2(t) - \mu_2^*(t) | \leqq \varepsilon .$$

However, the function $u_1(x, t)$ according to conclusion (3) differs by less than ε from the function $u(x, t)$:

$$| u(x, t) - u_1(x, t) | \leqq \varepsilon .$$

Here the principle of the physical determination of a problem arises directly.

We have investigated in detail the question of the uniqueness and the physical determination of a problem in the case of the first boundary-value problem for a bounded interval. The uniqueness theorem for the first boundary-value problem for a two- or three-dimensional bounded region can be proven by a verbatim repetition of these deliberations.

Similar questions arise in the investigation of other problems, an entire