special equation $y' = \lambda y$, λ constant, is usually considered sufficient, however, to give an indication of the stability of a method.

We consider first the Adams-Bashforth fourth-order method. If in (8.47) we set $f(x, y) = \lambda y$ we obtain

$$y_{n+1} - y_n - \frac{h\lambda}{24}(55y_n - 59y_{n-1} + 37y_{n-2} - 9y_{n-3}) = 0 \qquad (8.73)$$

The characteristic equation for this difference equation is

$$\beta^4 - \beta^3 - \frac{h\lambda}{24}(55\beta^3 - 59\beta^2 + 37\beta - 9) = 0$$

The roots of this equation are of course functions of $h\lambda$. It is customary to write the characteristic equation in the form

$$\rho(\beta) + h\lambda\sigma(\beta) = 0 \qquad (8.74)$$

where $\rho(\beta)$ and $\sigma(\beta)$ are polynomials defined by

$$\rho(\beta) = \beta^4 - \beta^3$$

$$\sigma(\beta) = -\frac{1}{24}(55\beta^3 - 59\beta^2 + 37\beta - 9)$$

We see that as $h \to 0$, (8.74) reduces to $\rho(\beta) = 0$, whose roots are $\beta_1 = 1$, $\beta_2 = \beta_3 = \beta_4 = 0$. For $h \neq 0$, the general solution of (8.73) will have the form

$$y_n = c_1\beta_1^n + c_2\beta_2^n + c_3\beta_3^n + c_4\beta_4^n$$

where the β_i are solutions of (8.74). It can be shown that β_1^n approaches the desired solution of $y' = \lambda y$ as $h \to 0$ while the other roots correspond to extraneous solutions. Since the roots of (8.74) are continuous functions of h, it follows that for h small enough, $|\beta_i| < 1$ for $i = 2, 3, 4$, and hence from the definition of stability that the Adams-Bashforth method is strongly stable. All multistep methods lead to a characteristic equation in the form (8.74) whose left-hand side is sometimes called the **stability polynomial**. The definition of stability can be recast in terms of the stability polynomial. A method is **strongly stable** if all the roots of $\rho(\beta) = 0$ have magnitude less than one except for the simple root $\beta = 1$.

We investigate next the stability properties of Milne's method (8.64*b*) given by

$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1}) \qquad (8.75)$$

Again setting $f(x, y) = \lambda y$ we obtain

$$y_{n+1} - y_{n-1} - \frac{h\lambda}{3}(y_{n+1} + 4y_n + y_{n-1}) = 0$$

and its characteristic equation becomes

$$\rho(\beta) + h\lambda\sigma(\beta) = 0 \qquad (8.76)$$

with

$$\rho(\beta) = \beta^2 - 1$$

$$\sigma(\beta) = \beta^2 + 4\beta + 1$$

This time $\rho(\beta) = 0$ has the roots $\beta_1 = 1$, $\beta_2 = -1$, and hence by the definition above, Milne's method is not strongly stable. To see the implications of this we compute the roots of the stability polynomial (8.76). For h small we have

$$\beta_1 = 1 + \lambda h + \mathcal{O}(h^2)$$

$$\beta_2 = -(1 - \lambda h/3) + \mathcal{O}(h^2) \qquad (8.77)$$

Hence the general solution of (8.75) is

$$y_n = c_1(1 + \lambda h + \mathcal{O}(h^2))^n + c_2(-1)^n(1 - \lambda h/3 + \mathcal{O}(h^2))^n$$

If we set $n = x_n/h$ and let $h \to 0$, this solution approaches

$$y_n = c_1 e^{\lambda x_n} + c_2(-1)^n e^{-\lambda x_n/3} \qquad (8.78)$$

In this case stability depends upon the sign of λ. If $\lambda > 0$ so that the desired solution is exponentially increasing, it is clear that the extraneous solution will be exponentially decreasing so that Milne's method will be stable. On the other hand if $\lambda < 0$, then Milne's method will be unstable since the extraneous solution will be exponentially increasing and will eventually swamp the desired solution. Methods of this type whose stability depends upon the sign of λ for the test equation $y' = \lambda y$ are said to be **weakly stable**. For the more general equation $y' = f(x, y)$ we can expect weak stability from Milne's method whenever $\partial f/\partial y < 0$ on the interval of integration.

In practice all multistep methods will exhibit some instability for some range of values of the step h. Consider, for example, the Adams-Bashforth method of order 2 defined by

$$y_{n+1} = y_n + \frac{h}{2}\{3f_n - f_{n-1}\}$$

If we apply this method to the test equation $y' = \lambda y$, we will obtain the difference equation

$$y_{n+1} - y_n - \frac{h\lambda}{2}\{3y_n - y_{n-1}\} = 0$$

and from this the stability polynomial

$$\beta^2 - \beta - \frac{h\lambda}{2}\{3\beta - 1\}$$

or the equation

$$\beta^2 - \left(1 + \frac{3h\lambda}{2}\right)\beta + \frac{h\lambda}{2} = 0$$

Case (ii) $1-4r^2\sin^2\frac{k\pi}{N}<0$.

Then

$$|\lambda|^2=\frac{1}{(2r+1)^2}\left\{\left(2r\cos\frac{k\pi}{N}\right)^2+4r^2\sin^2\frac{k\pi}{N}-1\right\}$$

$$=\frac{4r^2-1}{4r^2+4r+1}<1\quad\text{since}\quad r>0.$$

Therefore the equations are unconditionally stable for all positive r.

Brief introduction to the analytical solution of homogeneous finite-difference equations

Linear equations with constant coefficients

Consider the difference equation

$$u_{j+2}+au_{j+1}+bu_j=0,\ j=0,1,2,\cdots,\tag{3.28}$$

where a and b are real constants.

Assume that

$$u_j=Am^j$$

is a solution, where A and m are non-zero constants. Substitution into (3.28) shows that m is a root of the quadratic equation

$$m^2+am+b=0.\tag{3.29}$$

Case (i) Roots real and distinct, $m=m_1$ and $m=m_2$, say. One solution is $u_j=Am_1^j$ and another is $u_j=Bm_2^j$ where A and B are arbitrary constants. As equation (3.28) is linear in u its general solution is

$$u_j=Am_1^j+Bm_2^j.$$

Case (ii) Repeated roots, $m=m_1$ twice, say. Clearly one solution is $u_j=Am_1^j$.

Put $u_j=m_1^jf(j)$. Substitution into (3.28) and the use of $a=-2m_1$, $b=m_1^2$ leads to

$$f(j+2)-2f(j+1)+f(j)=0.$$

By inspection it is seen that $f(j)=j$ satisfies this equation. Therefore a second solution of (3.28) is $u_j=Bjm_1^j$. Hence the solution of equation (3.28) in this case is

$$u_j=(A+Bj)m_1^j.$$

Case (iii) Complex roots.

Because a and b are real the roots of (3.29) will be conjugate complex numbers, $m_1=re^{i\theta}$ and $m_2=re^{-i\theta}$, say, where $i=\sqrt{(-1)}$.

Hence $a=-r(e^{i\theta}+e^{-i\theta})=-2r\cos\theta$ and $b=r^2$. As in Case (i) the solution of (3.28) is

$$u_j=Ar^je^{ij\theta}+Br^je^{-ij\theta}=r^j\{(A+B)\cos j\theta+i(A-B)\sin j\theta\}.$$

Since A and B are arbitrary constants and $r=b^{\frac{1}{2}}$, this can be written as

$$u_j=b^{\frac{1}{2}j}(C\cos j\theta+D\sin j\theta),$$

where C and D are arbitrary constants and $\cos\theta=-a/2r=-a/2\sqrt{b}$. Methods for deriving particular integrals for non-homogeneous difference equations are given in 'Finite Difference Equations' by H. Levy and F. Lessman. (Pitman).

The eigenvalues and vectors of a common tridiagonal matrix

Let

Method to find λ and Q.

$$\mathbf{A}=\begin{bmatrix}a & b & & & \\ c & a & b & & \\ & c & a & b & \\ & & \ddots & \ddots & \\ & & & c & a\end{bmatrix}$$

be a square matrix of order N, where a, b, and c may be real or complex numbers.

Let λ represent an eigenvalue of $\mathbf{A}$ and $\mathbf{v}$ the corresponding eigenvector with components $v_1, v_2, \ldots, v_N$. Then the eigenvalue equation $\mathbf{Av}=\lambda\mathbf{v}$ gives

$$(a-\lambda)v_1+bv_2=0$$
$$cv_1+(a-\lambda)v_2+bv_3=0$$
$$\cdots$$
$$cv_{j-1}+(a-\lambda)v_j+bv_{j+1}=0$$

and

$$cv_{N-1}+(a-\lambda)v_N=0.$$

If we define $v_0 = v_{N+1} = 0$ then these N equations can be combined into the single difference equation

$$cv_{j-1} + (a-\lambda)v_j + bv_{j+1} = 0, \quad j = 1(1)N. \tag{3.30}$$

As shown, previously, the solution of (3.30) is

$$v_j = Bm_1^j + Cm_2^j, \tag{3.31}$$

where B and C are arbitrary constants and m_1, m_2 are the roots of the equation

$$c + (a-\lambda)m + bm^2 = 0. \tag{3.32}$$

(It is proved later that the roots cannot be equal.)
By equation (3.31) it follows, since $v_0 = v_{N+1} = 0$, that,

$$0 = B + C$$

and

$$0 = Bm_1^{N+1} + Cm_2^{N+1}.$$

Hence

$$\left(\frac{m_1}{m_2}\right)^{N+1} = 1 = e^{i2s\pi}, \quad s = 1(1)N,$$

where $i = \sqrt{(-1)}$. Therefore

$$\frac{m_1}{m_2} = e^{i2s\pi/(N+1)}. \tag{3.33}$$

By equation (3.32),

$$m_1 m_2 = \frac{c}{b},$$

and elimination of m_2 between (3.33) and (3.34) leads to

$$m_1 = \left(\frac{c}{b}\right)^{\frac{1}{2}} e^{is\pi/(N+1)}.$$

Similarly,

$$m_2 = \left(\frac{c}{b}\right)^{\frac{1}{2}} e^{-is\pi/(N+1)}.$$

Again by equation (3.32),

$$m_1 + m_2 = (\lambda - a)/b.$$

giving that

$$\lambda = a + b\left(\frac{c}{b}\right)^{\frac{1}{2}}\left(e^{is\pi/(N+1)} + e^{-is\pi/(N+1)}\right).$$

Hence the N eigenvalues are given by $\lambda_1 =$ $\lambda_2 = \ldots$

$$\lambda_s = a + 2b\left(\frac{c}{b}\right)^{\frac{1}{2}}\cos\frac{s\pi}{N+1}, \quad s = 1(1)N.$$

The jth component of the eigenvector is

$$v_j = Bm_1^j + Cm_2^j = B\left(\frac{c}{b}\right)^{\frac{1}{2}j}\left(e^{ijs\pi/N+1} - e^{-ijs\pi/N+1}\right)$$
$$= 2iB\left(\frac{c}{b}\right)^{\frac{1}{2}j}\sin\frac{js\pi}{N+1},$$

so the eigenvector $\mathbf{v}_s$ corresponding to λ_s can be taken as

$$\mathbf{v}_s^T = \left\{\left(\frac{c}{b}\right)^{\frac{1}{2}}\sin\frac{s\pi}{N+1}, \frac{c}{b}\sin\frac{2s\pi}{N+1}, \left(\frac{c}{b}\right)^{\frac{3}{2}} \times \sin\frac{3s\pi}{N+1}, \ldots, \left(\frac{c}{b}\right)^{\frac{N}{2}}\sin\frac{Ns\pi}{N+1}\right\}.$$

It is easily shown that the roots of equation (3.32) cannot be equal because if we assume $m_1 = m_2$ the solution of (3.32) is then

$$v_j = (B + Cj)m_1^j$$

and $v_0 = v_{N+1} = 0$ implies that $B = C = 0$, giving $\mathbf{v} = 0$, which is not possible.

An analytical solution of the classical explicit approximation to $\partial U/\partial t = \partial^2 U/\partial x^2$

Consider the equation

$$\frac{\partial U}{\partial t} = \frac{\partial^2 U}{\partial x^2}, \quad 0 < x < 1,$$

where $U = 0$ at $x = 0$ and 1, $t > 0$, and U is known when $t = 0$, $0 \leq x \leq 1$.

The classical explicit approximation to the differential equation is

$$u_{i,j+1} = ru_{i-1,j} + (1-2r)u_{i,j} + ru_{i+1,j}. \tag{3.35}$$

~~The implementation~~ *The implementation of the iteration*

In practice, the iteration defined by equation (5.57), namely,

$$(\mathbf{A}+\mathbf{N})\mathbf{u}^{(n+1)} = (\mathbf{A}+\mathbf{N})\mathbf{u}^{(n)} + (\mathbf{q}-\mathbf{A}\mathbf{u}^{(n)}),$$

where $(\mathbf{A}+\mathbf{N}) = \bar{\mathbf{L}}\bar{\mathbf{U}}$, is dealt with as follows.

Let $\mathbf{d}^{(n)} = \mathbf{u}^{(n+1)} - \mathbf{u}^{(n)}$ and $\mathbf{R}^{(n)} = \mathbf{q} - \mathbf{A}\mathbf{u}^{(n)}$.

Then by (5.57) a complete cycle of the iteration consists of the solution of

$$\bar{\mathbf{L}}\bar{\mathbf{U}}\mathbf{d}^{(n)} = \mathbf{R}^{(n)}, \tag{5.64}$$

followed by

$$\mathbf{u}^{(n+1)} = \mathbf{u}^{(n)} + \mathbf{d}^{(n)},$$

which is the iterative refinement described on page 226. Equation (5.64) would, of course, be solved by the forward and backward substitutions

$$\bar{\mathbf{L}}\mathbf{y}^{(n)} = \mathbf{R}^{(n)}$$

and

$$\bar{\mathbf{U}}\mathbf{d}^{(n)} = \mathbf{y}^{(n)}.$$

An additional acceleration parameter ω can also be introduced into the procedure by replacing (5.64) with

$$\bar{\mathbf{L}}\bar{\mathbf{U}}\mathbf{d}^{(n)} = \omega\mathbf{R}^{(n)},$$

as in reference 10. Further details concerning the calculation of α and the solution of the equations are given in Stone's paper, reference 32. His results indicate that the method is economical arithmetically in relation to older methods and that its rate of convergence is much less sensitive to the choice of iteration parameters than are the SOR and ADI methods.

Two recent direct methods

A method for 'variables separable' equations

The following method which depends upon the differential equation being 'variables separable', although that is not immediately obvious, was first proposed by Hockney, 1966, reference 17, who

considered the problem

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = g(x, y) \quad (x, y) \in D,$$

$$U = 0, \qquad (x, y) \in C,$$

where C is the boundary of the rectangular domain $D = \{(x, y): 0 < x < a,\ 0 < y < b\}$. Using Fig. 5.9 and a square mesh, the five-point difference equations approximating this problem may be written in partitioned form as

$$\begin{bmatrix} \mathbf{B} & \mathbf{I} & & & \\ \mathbf{I} & \mathbf{B} & \mathbf{I} & & \\ & \mathbf{I} & \mathbf{B} & \mathbf{I} & \\ & & \cdot & \cdot & \cdot \\ & & & \mathbf{I} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \\ \mathbf{u}_M \end{bmatrix} = \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \\ \\ \mathbf{b}_M \end{bmatrix} \tag{5.65}$$

$BU_1 + IU_2 = b_1$
$IU_1 + BU_2 + IU_3 = b_2$
. . . . . .

where $\mathbf{u}_r$ is the vector of mesh values along $y = rh$, $r = 1(1)M$, $\mathbf{b}_r$ is a known vector corresponding to $\mathbf{u}_r$ and the $N \times N$ matrix $\mathbf{B}$ is

$$\begin{bmatrix} -4 & 1 & & \\ 1 & -4 & 1 & \\ & \cdot & \cdot & \cdot \\ & & 1 & -4 \end{bmatrix},$$

From Laplacian

where N is the number of mesh points along a row parallel to Ox. By (5.65)

Reduce the matrix size. example. From 100×100 to some small matrix.

$$\begin{aligned} \mathbf{B}\mathbf{u}_1 + \mathbf{u}_2 &= \mathbf{b}_1 \\ \mathbf{u}_1 + \mathbf{B}\mathbf{u}_2 + \mathbf{u}_3 &= \mathbf{b}_2 \\ &\cdots \\ \mathbf{u}_{M-1} + \mathbf{B}\mathbf{u}_M &= \mathbf{b}_M \end{aligned} \tag{5.66}$$

Since B symmetrix we can define $Q^TQ = I \to Q^T = Q^{-1}$

Let $\mathbf{q}_r$ be an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue λ_r. Then

$$\mathbf{B}\mathbf{q}_r = \lambda_r \mathbf{q}_r, \quad r = 1(1)M,$$

and this set of equations can be written as

$$\mathbf{B}\mathbf{Q} = \mathbf{Q}\,\mathrm{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_M),$$

where $\mathbf{Q}$ is the modal matrix $[\mathbf{q}_1\ \mathbf{q}_2 \ldots \mathbf{q}_M]$. But $\mathbf{B}$ is symmetric, therefore the eigenvectors $\mathbf{q}_r$, $r = 1(1)M$, can be normalized so that $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. Hence $\mathbf{Q}^T\mathbf{B}\mathbf{Q} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_M) = \mathbf{\Lambda}$, say.

eg. $\lambda_1 < \lambda_2 < \cdots < \lambda_7$
$q_1\ q_2 \cdots q_7 \qquad Q = [q_1, q_2 \cdots q_7]$

$$\bar{\mathbf{u}}_r = \mathbf{Q}^{-1}\mathbf{u}_r \text{ and } \bar{\mathbf{b}}_r = \mathbf{Q}^{-1}\mathbf{b}_r. \tag{5.67}$$

from which it follows that

$$\mathbf{u}_r = \mathbf{Q}\bar{\mathbf{u}}_r \text{ and } \mathbf{b}_r = \mathbf{Q}\bar{\mathbf{b}}_r. \tag{5.68}$$

Substituting from (5.68) into (5.66) and premultiplying throughout with $\mathbf{Q}^{-1}$ leads to the equations

$$\begin{aligned} \Lambda\bar{\mathbf{u}}_1 + \bar{\mathbf{u}}_2 &= \bar{\mathbf{b}}_1 \\ \bar{\mathbf{u}}_1 + \Lambda\bar{\mathbf{u}}_2 + \bar{\mathbf{u}}_3 &= \bar{\mathbf{b}}_2 \\ \cdot \quad \cdot \quad &\cdot \\ \bar{\mathbf{u}}_{M-1} + \Lambda\bar{\mathbf{u}}_M &= \bar{\mathbf{b}}_M. \end{aligned} \tag{5.69}$$

Denote the ith components of $\bar{\mathbf{u}}_r$ and $\bar{\mathbf{b}}_r$ by $\bar{u}_{i,r}$ and $\bar{b}_{i,r}$ respectively and select the ith row of each of the equations (5.69). This gives the tridiagonal system of equations

$$\begin{aligned} \lambda_i\bar{u}_{i,1} + \bar{u}_{i,2} &= \bar{b}_{i,1} \\ \bar{u}_{i,1} + \lambda_i\bar{u}_{i,2} + \bar{u}_{i,3} &= \bar{b}_{i,2} \\ \bar{u}_{i,2} + \lambda_i\bar{u}_{i,3} + \bar{u}_{i,4} &= \bar{b}_{i,3} \\ \cdot \quad \cdot \quad &\cdot \\ \bar{u}_{i,M-1} + \lambda_i\bar{u}_{i,M} &= \bar{b}_{i,M}, \end{aligned} \tag{5.70}$$

for $\bar{u}_{i,r}$, $r = 1(1)M$. All the components of $\bar{\mathbf{u}}_r$, $r = 1(1)M$ can clearly be found by solving N such sets of equations for $\bar{u}_{i,r}$, $i = 1(1)N$. The procedure is therefore:

(i) Calculate the eigenvalues and eigenvectors of $\mathbf{B}$. (These are well known for the problem considered. See page 113.)

(ii) Compute $\bar{\mathbf{b}}_r = \mathbf{Q}^T\mathbf{b}_r$.

(iii) Solve equations (5.70), which is easily done.

(iv) Calculate $\mathbf{u}_r = \mathbf{Q}_r\bar{\mathbf{u}}_r$.

This method has been extended to more general self-adjoint 'variables separable' elliptic equations, to problems with derivative boundary conditions, and with irregular boundaries, see references 4 and 5. But research on the method is still relatively recent.

George's dissection method

As mentioned previously the standard Gauss elimination method for solving equations with a large band-width coefficient matrix $\mathbf{A}$ is inefficient in the sense that zero elements within the band are replaced by non-zero elements that have to be stored in the computer and used at subsequent stages of the elimination.

George, 1973, reference 15, by a combination of analysis, graph theory and intuition, formulated an ordering of the equations that gave substantial reductions in the 'fill-up', the computer storage required and in the volume of the arithmetic of the elimination process. His ordering has since been proved to be virtually optimal. For the five-point difference approximation of a Dirichlet elliptic problem defined over a rectangular region, and with a mesh giving $N\times N$ equations $\mathbf{Au} = \mathbf{b}$, the number of non-zero elements in the final upper triangular matrix $\mathbf{U}$ of $\mathbf{A} = \mathbf{LU}$ is $O(N^2 \log_2 N)$ and the volume of associated arithmetic is $O(N^3)$. The corresponding figures for the natural reading order of the mesh points is $O(N^3)$ and $O(N^4)$. For large N the savings in storage and effort are clearly considerable. A simpler but less efficient ordering has also been given by George (1972) in reference 14.

EXERCISES AND SOLUTIONS

1. The function ϕ satisfies the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + 2 = 0$$

at every point inside the square bounded by the straight lines $x = \pm 1$, $y = \pm 1$, and is zero on the boundary. Calculate a finite-difference solution using a square mesh of side $\frac{1}{2}$. (The non-dimensional form of the torsion problem for a solid elastic cylinder with a square cross-section.)

Assuming the discretization error is proportional to h^2 calculate an improved value of ϕ at the point (0, 0). (The analytical solution value is 0·589.)

Solution

Because of the symmetry there are only three unknowns: ϕ_1 at (0, 0), ϕ_2 at $(\frac{1}{2}, 0)$, ϕ_3 at $(\frac{1}{2}, \frac{1}{2})$. The equations are $8\phi_2 - 8\phi_1 + 1 = 0$, $4\phi_3 + 2\phi_1 - 8\phi_2 + 1 = 0$ and $4\phi_2 - 8\phi_3 + 1 = 0$, giving $\phi_1 = 0{\cdot}562$, $\phi_2 = 0{\cdot}438$ and $\phi_3 = 0{\cdot}344$ to 3D.

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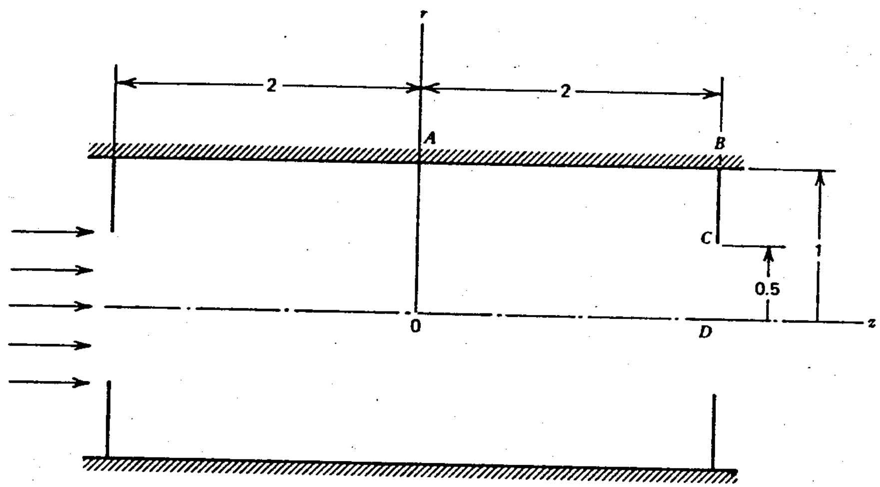

FIGURE 2.11.9 Axisymmetric flow through a tube containing repeated partitions.

metric tube containing repeated partitions (see Fig. 2.11.9). Choose a square mesh of size $h = 0.1$ for numerical computation.

Around the boundary of this region, $\psi = 0$ along OD, $\psi = 1$ along ABC, and $\partial\psi/\partial x = 0$ along both OA and CD. The derivative boundary conditions are the result of symmetry of stream function about these two sections.

2.12 Numerical Solution of Hyperbolic Partial Differential Equations

Problems concerning wave motions in fluid mechanics are governed by hyperbolic partial differential equations. One example mentioned in Section 2.9 is the supersonic flow past a thin body whose governing equation is (2.9.2). Another commonly cited example is the propagation of a one-dimensional sound wave of small amplitude, described by (see Liepmann and Roshko, 1957, p. 68)

$$\frac{\partial^2 u}{\partial t^2} = a^2 \frac{\partial^2 u}{\partial x^2} \tag{2.12.1}$$

in which t is time, x is the coordinate in the direction of wave propagation, a is the speed of sound treated as constant in the linearized analysis, and u is

the fluid speed. It can be shown that density, pressure, and temperature are all governed by equations of the same form.

In this section a numerical technique is developed for solving (2.12.1) to find u at any time $t > 0$ in the spatial domain $0 \leq x \leq L$, provided that the initial conditions of u are given at $t = 0$ and are expressed in the following form, with functions f and g to be specified for a particular problem.

$$u(x, 0) = f(x) \tag{2.12.2}$$

$$\frac{\partial u}{\partial t}(x, 0) = g(x) \tag{2.13.3}$$

Boundary conditions are to be specified at both ends of the gaseous domain, say within a channel of constant cross-sectional area. If one end of the channel is enclosed by a rigid wall, then u must be zero there at all times. On the other hand, at an end that opens to the atmosphere, the pressure there must be a constant or, alternatively, $\partial u/\partial x$ must vanish at that section.

To solve this mixed initial-boundary-value problem numerically, we divide the spatial range of the domain into small intervals of length h and the time axis into intervals of size τ. The total number of vertical grid lines is m, whereas that of horizontal grid lines can be as many as needed in a particular computation. Lines and points in the grid system are named according to Fig. 2.12.1.

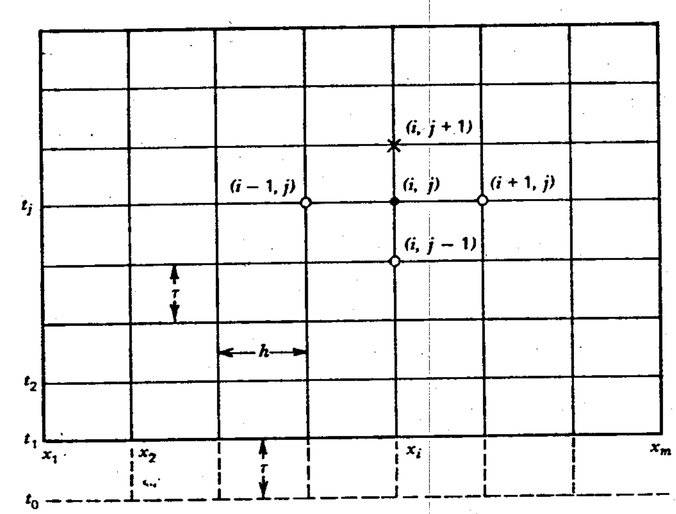

FIGURE 2.12.1 Grid system for numerical computation.

A difference equation can be derived following exactly the same procedure as that used to obtain the numerical scheme (2.10.2) for solving the Poisson equation. Using the central-difference formula to approximate the derivatives in (2.12.1), we obtain, after regrouping,

$$u_{i,j+1} = 2u_{i,j} + C^2(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) - u_{i,j-1} \tag{2.12.4}$$

where C is a dimensionless parameter called the *Courant number*, defined by

$$C \equiv \frac{a\tau}{h} \tag{2.12.5}$$

(2.12.4) computes the solution at a certain time level based on the solutions at two previous time levels.

Actually, various numerical schemes can be constructed for solving the same partial differential equation by using different finite-difference approximations. The applicability of a numerical scheme is determined by whether it is stable, that is, whether the numerical solution grows and becomes unbounded after repeatedly applying the scheme. In a way we have proved in Section 2.10 that Richardson's iterative formula is stable, and so is Liebmann's. Here we will use a different approach to find the conditions under which (2.12.4) is computationally stable. It turns out that the stability of this numerical scheme is determined by the magnitude of C.

Following *von Neumann's stability analysis*, we assume that time and space variables are separable and that the solution to (2.12.4) can be expanded in the form of a Fourier series. A representative Fourier component may be written

$$u_{i,j} = U_j\, e^{Iikh} \tag{2.12.6}$$

where U_j is the amplitude at t_j of the wave component whose wave number is k, and $I = \sqrt{-1}$. Similarly,

$$u_{i,j\pm1} = U_{j\pm1}\, e^{Iikh}, \qquad u_{i\pm1,j} = U_j\, e^{I(i\pm1)kh}$$

Substituting this into (2.12.4) gives, after canceling the common factor e^{Iikh},

$$U_{j+1} = 2U_j + C^2U_j(e^{-Ikh} + e^{Ikh} - 2) - U_{j-1}$$

By using the identity that $(e^{I\theta} + e^{-I\theta})/2 = \cos\theta$, it becomes

$$U_{j+1} = AU_j - U_{j-1} \tag{2.12.7}$$

where $A \equiv 2[1 - C^2(1 - \cos kh)]$. By introduction of an *amplification factor* λ such that

$$U_j = \lambda U_{j-1} \quad \text{and} \quad U_{j+1} = \lambda U_j = \lambda^2 U_{j-1} \tag{2.12.8}$$

(2.12.7) is reduced to

$$\lambda^2 - A\lambda + 1 = 0 \tag{2.12.9}$$

whose roots are

$$\lambda = \frac{A}{2} \pm \sqrt{\left(\frac{A}{2}\right)^2 - 1} \tag{2.12.10}$$

For $|A| \geq 2$, the roots are real, but their magnitudes are $|\lambda| \geq 1$; for $|A| < 2$ the magnitudes of the complex roots are less than 1. An inspection of (2.12.8) concludes that the amplitude grows indefinitely with increasing time unless $|\lambda| \leq 1$. Thus the inequality that $|A| \leq 2$ or, $A^2 \leq 4$, determines the condition for stability; that is, for stability,

$$[1 - C^2(1 - \cos kh)]^2 \leq 1$$

After expanding the left side and rearranging, we obtain

$$C^2 \leq \frac{2}{1 - \cos kh}$$

When $\cos kh$ varies from -1 to $+1$, the function on the right-hand side varies from 1 to infinity, of which the lowest value is chosen to insure stability. Therefore the stability criterion for the numerical scheme (2.12.4) is $C^2 \leq 1$, or

$$\frac{a\tau}{h} \leq 1 \tag{2.12.11}$$

To arrive at this expression, we have used the fact that each of the three variables on its left is positive. This relationship implies that τ and h cannot be chosen independently.

If the Courant number is chosen to be

$$\frac{a\tau}{h} = 1 \tag{2.12.12}$$

(2.12.4) takes an especially simple form.

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1} \tag{2.12.13}$$

As shown in Fig. 2.12.1, this equation states that the value of u at a grid point marked by a cross is computed from the values already computed at three circled grid points at two previous time steps. The numerical scheme (2.12.13) is commonly referred to as the *leapfrog method.* It will now be proved that this numerical method actually gives the exact solution to the differential equation (2.12.1).

It is well known that the solution to (2.12.1) satisfying initial conditions (2.12.2) and (2.12.3) is

$$u(x, t) = \tfrac{1}{2}[f(x + at) + f(x - at)] + \frac{1}{2a}\int_{x-at}^{x+at} g(v)\, dv \tag{2.12.14}$$

which can be verified by substituting it back into each of those equations. The

solution may be written in the simpler functional form

$$u(x, t) = F(x - at) + G(x + at) \tag{2.12.15}$$

where the functions F and G represent simple waves propagating without changing shape along the positive and negative x directions at constant speed a. The lines of slope $dx/dt = \pm a$ in the x-t plane, which trace the progress of the waves, are called the characteristics of the wave equation (see Liepmann and Roshko, 1957, p. 69).

When applied at a grid point (x_i, t_j), (2.12.15) becomes

$$u_{i,j} = F(x_i - at_j) + G(x_i + at_j)$$

From Fig. 2.12.1 we have

$$x_i = x_1 + (i-1)h \qquad \text{and} \qquad t_j = t_1 + (j-1)\tau$$

so that

$$u_{i,j} = F(\alpha + ih - ja\tau) + G(\beta + ih + ja\tau)$$

in which $\alpha \equiv (x_1 - h) - a(t_1 - \tau)$ and $\beta \equiv (x_1 - h) + a(t_1 - \tau)$. With $a\tau = h$, obtained from the condition that $C = 1$, it reduces to

$$u_{i,j} = F[\alpha + (i-j)h] + G[\beta + (i+j)h]$$

According to this relation, the right-hand side of (2.12.13) is rewritten

$$\begin{aligned} u_{i-1,j} + u_{i+1,j} - u_{i,j-1} &= F[\alpha + (i-j-1)h] + \cancel{G[\beta + (i+j-1)h]} \\ &\quad + \cancel{F[\alpha + (i-j+1)h]} + G[\beta + (i+j+1)h] \\ &\quad - \cancel{F[\alpha + (i-j+1)h]} - \cancel{G[\beta + (i+j-1)h]} \\ &= F[\alpha + (i-j-1)h] + G[\beta + (i+j+1)h] \end{aligned}$$

which is exactly $u_{i,j+1}$ or the left-hand side of (2.12.13). It follows that exact solution is computed for (2.12.1) by the leapfrog scheme (2.12.13).

Having introduced the concept of characteristics, we are now in a position to interpret the physical meaning of the stability criterion (2.12.11) by use of Fig. 2.12.2. An examination of (2.12.4) reveals that the solution at the grid point P is influenced by the solution at each of the grid points at previous time steps contained within two diagonals PQ and PR of slope $(dx/dt)_n = \pm h/\tau$. Thus the region $PQRP$ is the domain of dependence of point P in the numerical computation. If Pq and Pr are the backward characteristics of slope $(dx/dt)_c = \pm a$ passing through P, and if $a < h/\tau$ or, equivalently, if $|(dx/dt)_c| < |(dx/dt)_n|$, these lines will lie between PQ and PR, as shown in the figure. However, from the theory of characteristics, it is known that point P can receive signals only from the region $PqrP$, which is its physical domain of dependence. In the present case of $a\tau/h < 1$, in which the computational domain of dependence contains the physical domain of dependence, all the

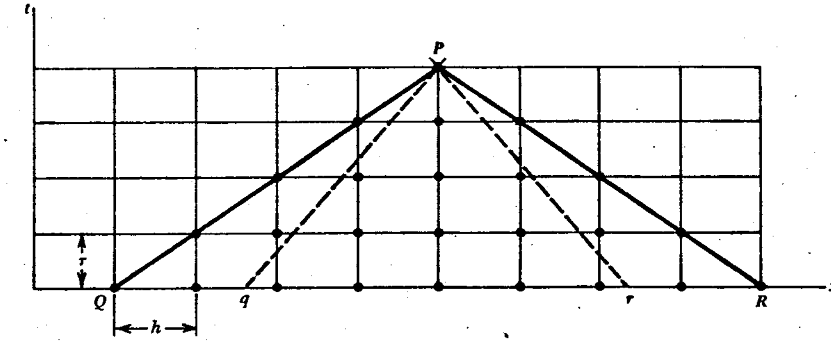

FIGURE 2.12.2 Physical interpretation of stability criterion (2.12.11).

information required to determine the condition at P is included in the computation so that the numerical scheme is stable. The result is inaccurate because of the inclusion of some unnecessary information originating from the region between PQ and Pq and the region between PR and Pr. If $a\tau/h > 1$, the characteristics Pq and Pr would be drawn outside of PQ and PR. In this case only a part of the needed information is used to determine the solution at P, and the computation is unstable. It becomes obvious that when $a\tau/h = 1$ (i.e. when the computational and physical domains of dependence coincide), the numerical solution is the exact solution.

In using the formula (2.12.13) information is needed at two previous time steps. It cannot be used directly at the initial stage to compute the solution at t_2, since conditions are specified only at the initial instant $t_1 = 0$. To help start the numerical procedure, we construct in Fig. 2.12.1 a row of fictitious grid points at $t_0 = t_1 - \tau$, and then rewrite the initial conditions (2.12.2) and (2.12.3) in index notation:

$$u_{i,1} = f_i \tag{2.12.16}$$

$$u_{i,0} = u_{i,2} - 2\tau g_i \tag{2.12.17}$$

in which f_i and g_i represent, respectively, $f(x_i)$ and $g(x_i)$. In obtaining the second expression we have approximated $\partial u/\partial t$ by the central-difference form (2.2.8). For $j = 1$ and with substitution from the preceding equations, (2.12.13) becomes

$$\begin{aligned} u_{i,2} &= u_{i-1,1} + u_{i+1,1} - u_{i,0} \\ &= f_{i-1} + f_{i+1} - u_{i,2} + 2\tau g_i \end{aligned}$$

or

$$u_{i,2} = \tfrac{1}{2}(f_{i-1} + f_{i+1}) + \tau g_i, \qquad i = 2, \cdots, m-1 \tag{2.12.18}$$

This is called the starting formula for (2.12.13).

1. A circular tube of radius R. From symmetry only the first quadrant is needed for the numerical computation. The boundary conditions at the two straight edges of the fan-shaped domain are that the variations of velocity normal to the edges are zero. To handle the curved boundary, the method of Program 2.9 may be used.

Compare the numerical result with the analytical solution for Poiseuille flow (Batchelor, 1967, p. 180) that

$$u = -\frac{1}{4\mu}\frac{dp}{dx}(R^2 - r^2)$$

$$Q = -\frac{\pi R^4}{8\mu}\frac{dp}{dx}$$

2. A triangular tube whose two slant walls make 45° angles with the third.

3. A rectangular tube containing a square inner tube.

3.6 Explicit Methods for Solving Parabolic Partial Differential Equations—Generalized Rayleigh Problem

In studying the development of a boundary layer on a body moving through an incompressible fluid, Rayleigh (1911) considered the unsteady motion of an infinitely extended fluid in response to an infinite flat plate suddenly set in motion along its own plane. If the plate is normal to the y-axis and the motion is in the x direction, the continuity equation (3.1.6) is satisfied automatically and the incompressible Navier-Stokes equation (3.1.7) is simplified to

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2} \tag{3.6.1}$$

Sometimes this equation, governing arbitary unsteady planar fluid motions, is expressed in terms of vorticity $\zeta(= -\partial u/\partial y)$ in the form

$$\frac{\partial \zeta}{\partial t} = \nu \frac{\partial^2 \zeta}{\partial y^2} \tag{3.6.2}$$

which describes the diffusion of vorticity through a one-dimensional space. According to the discussions of Section 2.9, both (3.6.1) and (3.6.2) are classified as parabolic partial differential equations. Here we will construct a numerical scheme for solving (3.6.1), examine its computational stability, and then apply it to a particular physical problem.

In numerical computations the space coordinates must be finite. Let us assume that the fluid above the plate at $y = 0$ is bounded below a finite depth that is divided into $m - 1$ equally spaced intervals of size h. If the time axis is divided into steps of size τ, a grid system is formed, as shown in Fig. 3.6.1. To approximate (3.6.1) by a finite difference equation at the grid point (i, j), the second-order spatial derivative is replaced by the central-difference formula (2.2.9) and the time derivative is replaced by the forward-difference formula (2.2.6). After rearrangement the equation has the final form

$$u_{i,j+1} = u_{i,j} + R(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \tag{3.6.3}$$

in which $R = \nu\tau/h^2$ is a dimensionless parameter. The equation states that the solution at a certain height at time interval τ later can be computed based on the present informations at the local and two neighboring stations. For given boundary conditions expressed as known functions of time, the solution at time level t_2 is computed explicitly from the initial condition at t_1 by using (3.6.3). Repeating the procedure for the successive time steps, the solution at any desired time level can be obtained. For this property the method in which (3.6.3) is applied is called an *explicit method* for solving the parabolic equation (3.6.1).

Playing a similar role as the Courant number C in (2.12.4), the parameter R in (3.6.3) cannot be arbitrarily chosen, and the limitation imposed on its magnitude is to be determined from a stability analysis of the numerical

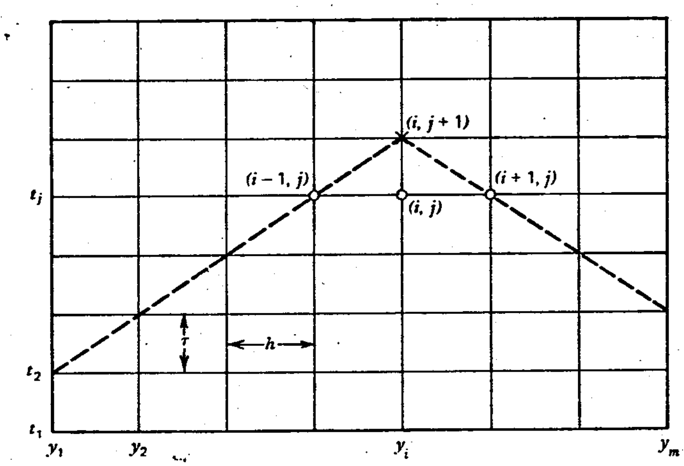

FIGURE 3.6.1 An explicit method for solving parabolic equations.

scheme. Following the technique illustrated in Section 2.12, we assume

$$u_{i,j} = U_j e^{Iikh} \tag{3.6.4}$$

and obtain, after substituting into (3.6.3),

$$U_{j+1} = [1 - 2R(1 - \cos kh)]U_j \tag{3.6.5}$$

The quantity contained within the brackets is the amplification factor λ. If $|\lambda| > 1$, $|U_{j+1}| > |U_j|$ and the amplitude of the solution becomes unbounded as $j \to \infty$. This is called an unstable situation. Thus, for stability we require $\lambda^2 \leq 1$ or, consequently, after expanding the left-hand side,

$$R \leq \frac{1}{1 - \cos kh}$$

Since the lowest value of the expression on the right-hand side is 1/2 when $\cos kh = -1$, the stability criterion derived for (3.6.3) is

$$\frac{\nu\tau}{h^2} \leq \frac{1}{2} \tag{3.6.6}$$

When the upper limiting value is used for this parameter, (3.6.3) has a particularly simple form.

$$u_{i,j+1} = \tfrac{1}{2}(u_{i-1,j} + u_{i+1,j}) \tag{3.6.7}$$

This is called the ***Bender-Schmidt recurrence equation***, which determines the solution at (y_i, t_{j+1}) as the average of the values right and left of y_i at a time t_j. However, more accurate results are obtained by using (3.6.3) for $R < 1/2$.

The differential equation (3.6.1) and its finite-difference approximation (3.6.3) apply to any unsteady planar flows bounded by two parallel infinite plates performing arbitrary parallel motions along their own planes. One of the plates may be replaced by a free surface. Furthermore, with modifications to suit cylindrical coordinates, the resulting equations apply to flows between concentric cylinders. Solving for the velocity and the related fields of these flows may be classified as the generalized Rayleigh problem.

For illustrative purposes we consider water contained between two originally stationary flat plates separated by a distance of 1 m. At an initial instant $t = 0$, the upper plate has suddenly acquired a constant speed $u_0(\doteqdot 1\ \text{m/s})$ while the lower plate is kept stationary all the time. The sudden motion of the upper plate creates a sharp velocity change there, forming a concentrated vortex sheet right below the plate. The vorticity is diffused downward, according to (3.6.2), into a region practically free of vorticity, and the velocity is redistributed accordingly. We like to find numerically the velocity distribution across the channel at different times.

In terms of the notations of Fig. 3.6.1, the initial velocity distribution is

$$u_{i,1} = 0 \quad \text{for} \quad i = 1, 2, \ldots, m-1$$
$$u_{m,1} = u_0$$

and the boundary conditions are

$$u_{i,j} = 0 \quad \text{for} \quad j > 1$$
$$u_{m,j} = u_0 \quad \text{for} \quad j > 1$$

For water $\nu = 1 \times 10^{-6}\ \text{m}^2/\text{s}$, approximately. If the space between plates is divided into 20 equal intervals, $m = 21$ and $h = 0.05$ m. Let us choose $R = 1/4$, which determines the time interval $\tau = Rh^2/\nu$, or 625 s. This time step size seems to be rather large. But it is a reasonable size for a laminar shear flow in which vorticity or velocity gradient is diffused purely by intermolecular activities characterized by a small kinematic viscosity.

In the form shown in (3.6.3), the velocity field is a two-dimensional array. This is not necessary, however, in programming for the computation. In Program 3.5 we use one-dimensional arrays UOLD(I) and UNEW(I) to denote $u_{i,j}$ and $u_{i,j+1}$, respectively. Immediately after completing the computation for one time level, we store the values of UNEW under the name UOLD, print them only at certain selected time levels, and then repeat the same computation to find UNEW at the next time level. This is an often used technique when the size of the field length of a program becomes too large to be handled by the register of a computer.

Because of the large amount of output data, the solution for u is printed every 20 time steps and plotted every 40 time steps, with only the first five curves shown in Fig. 3.6.2. The selective printing and plotting are achieved by assigning these two time-step values to variable names NPRINT and NPLOT, respectively, and doing what we did in Program 2.10—using the special property of integer division.

The output shows that a velocity discontinuity cannot exist in a viscous fluid and is smoothed out immediately by viscous diffusion. As time progresses the velocity profile approaches a linear distribution that varies from 0 at the lower plate to 1 m/s at the upper and corresponds to the solution for the Couette flow between two parallel plates in a steady shear motion.

```
C                       ***** PROGRAM 3.5 *****
C
C         EXPLICIT METHOD IS USED TO FIND THE UNSTEADY VELOCITY
C         DISTRIBUTION IN WATER CONTAINED BETWEEN A STATIONARY
C         LOWER PLATE AND AN UPPER PLATE IN AN IMPULSIVE MOTION
C
      DIMENSION UOLD(21),UNEW(21),Y(21),NSCALE(4),
     A          LABEL(3),PCHAR(5),TPLOT(5)
      REAL NU
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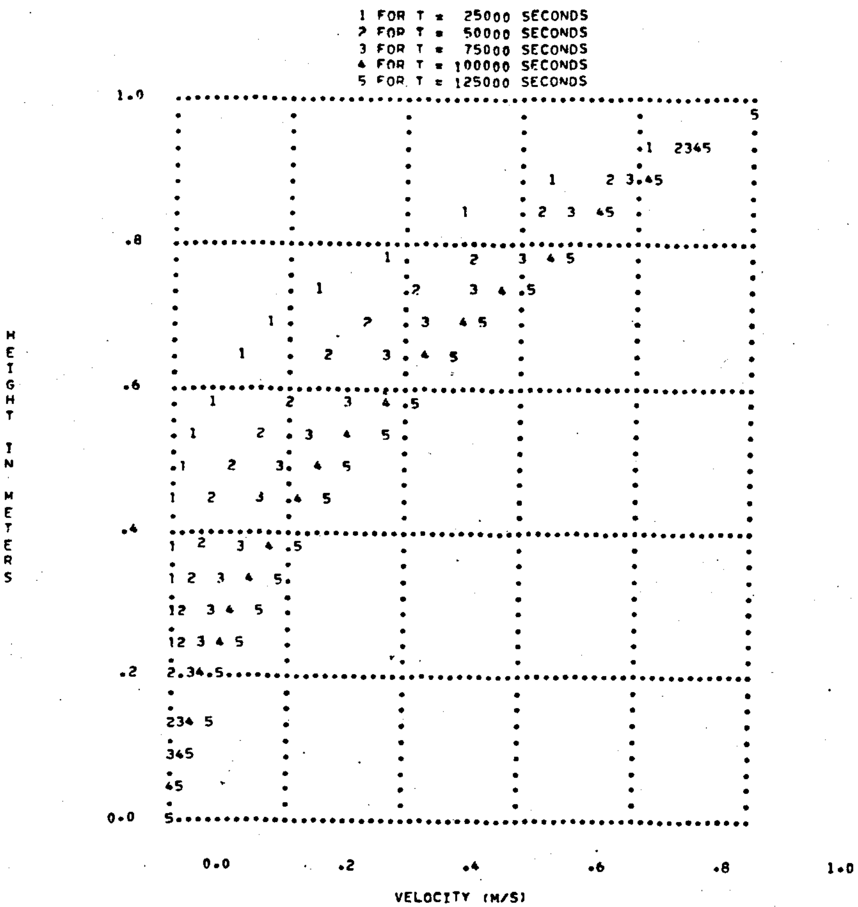


FIGURE 3.6.2 Velocity distribution at different times.

Problem 3.10 Assign a value to R that is greater that 0.5, and then run Program 3.5 to watch the growth of the solution to some unrealistic magnitudes. The result proves the validity of the stability criterion (3.6.6).

Problem 3.11 Find the velocity distribution at increasing times in the originally stationary fluid around a circular cylinder with a free surface (Fig. 3.4.1) after the cylinder is suddenly given a

rotation of tangential speed v_b at the surface. As time approaches a very large value, the solution should approach that for Problem 3.6.

Problem 3.12 Find the velocity distribution in the channel described in Program 3.5 with the upper plate replaced by one oscillating at the speed $u_0 \sin \omega t$, where $u_0 = 1$ m/s and $\omega = 1/1000\,\text{s}^{-1}$.

Problem 3.13 In approximating the differential equation (3.6.1) by the finite-difference equation (3.6.3), we used forward difference in time and central difference in space, so that the truncation error of the approximation is $O(\tau, h^2)$. An improved explicit scheme with truncation error of $O(\tau^2, h^2)$ can be constructed by also using central difference in time and by replacing the second $u_{i,j}$ on the right side of (3.6.3) by the time-average $(u_{i,j-1} + u_{i,j+1})/2$. Thus

$$\frac{u_{i,j+1} - u_{i,j-1}}{2\tau} = \nu\,\frac{u_{i-1,j} - u_{i,j-1} - u_{i,j+1} + u_{i+1,j}}{h^2}$$

$$(1+2R)u_{i,j+1} = 2R(u_{i-1,j} + u_{i+1,j}) + (1-2R)u_{i,j-1} \qquad (3.6.8)$$

This is called the ***DuFort–Frankel formula***, which involves three time levels, as does the formula (2.12.13) derived for hyperbolic differential equations.

Show that (3.6.8) is an unconditionally stable numerical scheme.

Let us take a close look at the explicit formula (3.6.3). The solution at the grid pont $(i, j+1)$ is computed, using this formula, from the solutions evaluated at three grid points $(i-1, j)$, (i, j), and $(i+1, j)$. These values, in turn, are computed from solutions in their neighborhood at the previous time step. In this way we can trace out the region of dependence of the point $(i, j+1)$, which is confined between the two dashed lines shown in Fig. 3.6.1. It means that the disturbance created at any other height in the fluid reaches the height y_i with a finite speed h/τ. This contradicts the real situation in an incompressible fluid, in which a disturbance at any point is felt immediately by all parts throughout the fluid. Thus, to improve the accuracy of (3.6.3), we may reduce the size of τ or the value of R. In so doing the dashed lines will approach the horizontal grid line passing through $(i, j+1)$ and, in the meantime, more time steps will be needed in the computation to reach the same time level. The improved accuracy in the explicit method is therefore obtained at the expense of an increased amount of computer time.

3.7 Implicit Methods for Solving Parabolic Partial Differential Equations—Starting Flow in a Channel

The deficiency associated with the explicit methods that the solution computed at one point is not affected immediately by the conditions at all other points in the fluid can be avoided by devising an alternative numerical scheme for solving the same diffusion equation (3.6.1). If we still use centered difference in space but use backward, instead of forward, difference in time, a finite-difference equation is obtained at (i, j) of the form

$$\frac{1}{\tau}(u_{i,j} - u_{i,j-1}) = \frac{\nu}{h^2}(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) \qquad (3.7.1)$$

It becomes, after regrouping,

$$Ru_{i-1,j} - (1+2R)u_{i,j} + Ru_{i+1,j} = -u_{i,j-1} \qquad (3.7.2)$$

where $R = \nu\tau/h^2$ is the same dimensionless parameter as that defined in Section 3.6.

The slight modification in approximating the time derivative causes a radical change in the procedure for obtaining a solution. Suppose the solution at $t = t_j$ is to be computed based on the solution known at the previous time step $t = t_{j-1}$; (3.7.2) shows that every three neighboring unknown values are interrelated through this linear algebraic equation. Applying (3.7.2) at all grid points interior to the boundaries at one time level gives a system of simultaneous equations that can be solved for all the unknowns at that time instant. In this way the velocities at different heights are not independent of one another; a change at one point will be felt immediately by all other points. Thus this numerical scheme is more sound than the explicit scheme on physical grounds. By using the numerical scheme the solution can no longer be computed explicitly as before, so (3.7.2) is called a formula for the *implicit method.*

The computational stability of (3.7.2) can be examined again with von Neumann's stability analysis by assuming the form already shown in (3.6.4) for the numerical solution. It can easily be verified that the resulting relationship from that analysis is

$$U_j = \frac{1}{1+2R(1-\cos kh)}U_{j-1} = \lambda U_{j-1} \qquad (3.7.3)$$

As $\cos kh$ varies from -1 to $+1$, the value of the amplification factor λ changes from $1/(1+4R)$ to 1 and can never exceed 1. Therefore this numerical scheme is stable for all positive values of R.

Although for computational stability there is no restriction on the magnitude of R as long as it is positive, a smaller value of R results in a more

accurate numerical solution. The reason for this is that after multiplying (3.7.1) through by τ, the truncated higher-order terms on the right-hand side are all multiplied by R.

We now apply the implicit method to solve a problem concerning the development of a channel flow caused by the application of a constant pressure gradient. The initially stationary incompressible fluid contained between two parallel infinite plates is set in motion by a suddenly imposed pressure gradient dp/dx along the channel. Simplified for the present geometry, the equation of motion (3.1.7) becomes

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \qquad (3.7.4)$$

If the distance between plates is $2L$ and the origin of the coordinate system is placed at the middle of the channel, the boundary and initial conditions are

$$u = 0 \quad \text{at} \quad y = \pm L \quad \text{for all } t \qquad (3.7.5)$$

$$u = 0 \quad \text{at} \quad t = 0 \quad \text{for } -L \leq y \leq L \qquad (3.7.6)$$

We know that as time increases, the velocity profile will approach its steady-state parabolic distribution

$$u_s = -\frac{1}{2\mu}\frac{dp}{dx}(L^2 - y^2) \qquad (3.7.7)$$

which is a particular solution to (3.7.4) satisfying the boundary conditions (3.7.5). By introducing the dimensionless variables

$$T = \frac{t}{L^2/\nu}, \qquad Y = \frac{y}{L}, \qquad U = u\bigg/\left(-\frac{L^2}{2\mu}\frac{dp}{dx}\right) \qquad (3.7.8)$$

and a dimensionless velocity difference

$$W = (u_s - u)\bigg/\left(-\frac{L^2}{2\mu}\frac{dp}{dx}\right) = (1 - Y^2) - U \qquad (3.7.9)$$

the governing equation (3.7.4) is simplified to

$$\frac{\partial W}{\partial T} = \frac{\partial^2 W}{\partial Y^2} \qquad (3.7.10)$$

with boundary and initial conditions

$$W = 0 \quad \text{at} \quad Y = \pm 1 \quad \text{for all } T \qquad (3.7.11)$$

$$W = 1 - Y^2 \quad \text{at} \quad T = 0 \quad \text{for } -1 \leq Y \leq 1 \qquad (3.7.12)$$

The implicit numerical scheme for solving (3.7.10) is, according to (3.7.2),

$$R\,W_{i-1,j} - (1+2R)W_{i,j} + R\,W_{i+1,j} = -W_{i,j-1} \qquad (3.7.13)$$

Then

$$\frac{\partial U}{\partial X}=\frac{\partial U}{\partial x}\frac{dx}{dX}=\frac{\partial U}{\partial x}\frac{1}{L}$$

and

$$\frac{\partial^2 U}{\partial X^2}=\frac{\partial}{\partial X}\left(\frac{\partial U}{\partial X}\right)=\frac{\partial}{\partial x}\left(\frac{1}{L}\frac{\partial U}{\partial X}\right)\frac{dx}{dX}=\frac{1}{L^2}\frac{\partial^2 U}{\partial x^2},$$

so equation (2.1) transforms to

$$\frac{\partial(uU_0)}{\partial T}=\frac{\kappa}{L^2}\frac{\partial^2(uU_0)}{\partial x^2},$$

i.e.

$$\frac{1}{\kappa L^{-2}}\frac{\partial u}{\partial T}=\frac{\partial^2 u}{\partial x^2}.$$

Writing $t=\kappa T/L^2$ and applying the function of a function rule to the left side yields

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad (2.2)$$

as the non-dimensional form of (2.1).

It should be noted that the number representing the length of the rod is 1.

An explicit method of solution

By equations (1.10) and (1.8) one finite-difference approximation to

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad (2.3)$$

is

$$\frac{u_{i,j+1}-u_{i,j}}{k}=\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2},$$

where

$$x=ih,\ (i=0,1,2,\ldots),$$

and

$$t=jk,\ (j=0,1,2,\ldots).$$

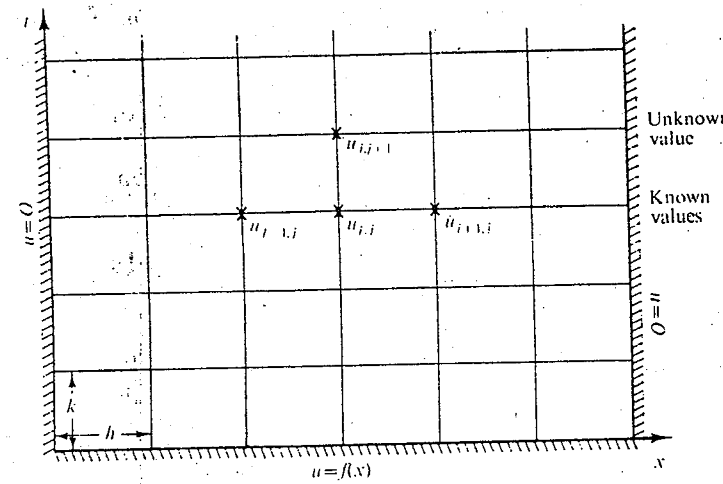


Fig. 2.1

This can be written as

$$u_{i,j+1}=u_{i,j}+r(u_{i-1,j}-2u_{i,j}+u_{i+1,j}) \qquad (2.4)$$

where $r=\delta t/(\delta x)^2=k/h^2$, and gives a formula for the unknown 'temperature' $u_{i,j+1}$ at the $(i,j+1)$th mesh point in terms of known 'temperatures' along the jth time row (Fig. 2.1). Hence we can calculate the unknown pivotal values of u along the first time row, $t=k$, in terms of known boundary and initial values along $t=0$, then the unknown pivotal values along the second time row in terms of the calculated pivotal values along the first, and so on. A formula such as this which expresses *one* unknown pivotal value directly in terms of known pivotal values is called an explicit formula.

Example 2.1

As a numerical example let us solve (2.4) given that the ends of the rod are kept in contact with blocks of melting ice and that the initial temperature distribution in non-dimensional form is

$$\begin{aligned}&(a)\ \ u=2x, && 0\leqslant x\leqslant \tfrac{1}{2},\\ &(b)\ \ u=2(1-x), && \tfrac{1}{2}\leqslant x\leqslant 1.\end{aligned} \qquad (2.5)$$

In other words we are seeking a numerical solution of $\partial u/\partial t = \partial^2 u/\partial x^2$ which satisfies

(i) $u=0$ when $x=0$ and 1 for all t. (The boundary conditions)

$$\left.\begin{array}{l}\text{(ii) } u=2x \text{ for } 0\leqslant x\leqslant\frac{1}{2},\\ \text{and } u=2(1-x) \text{ for } \frac{1}{2}\leqslant x\leqslant 1.\end{array}\right\} t=0. \text{ (The initial condition)}$$

(This initial temperature distribution could be obtained by heating the centre of the rod for a long time and keeping the ends in contact with the ice.)

For $\delta x = h = \frac{1}{10}$, the initial values and boundary values are as shown in Table 2.1. The problem is symmetric with respect to $x=\frac{1}{2}$ so we need the solution only for $0\leqslant x\leqslant\frac{1}{2}$.

TABLE 2.1

t \ x	$x=0$	0·1	0·1	0·3	0·4	0·5	0·6
$j=0$	0	0·2	0·4	0·6	0·8	1·0	0·8
$j=1$	0						
$j=2$	0						
$j=3$	0						
$j=4$	0						

Case 1

Take $\delta x = h = \frac{1}{10}$, $\delta t = k = \frac{1}{1000}$, so $r = k/h^2 = \frac{1}{10}$. Equation (2.4) then reads as

$$u_{i,j+1} = \tfrac{1}{10}(u_{i-1,j} + 8u_{i,j} + u_{i+1,j}). \qquad (2.6)$$

For pencil and paper calculations the relationship between these four function values is represented very conveniently by the 'molecule' in Fig. 2.2. The numbers in the 'atoms' are the multipliers of the function values at the corresponding mesh points.

Application of equation (2.6) to the data of Table 2.1 is shown in Table 2.2, and readers are recommended to check some of the calculations, remembering that the values of u at $x=\frac{4}{10}$ and $\frac{6}{10}$ are equal because of symmetry. (Increasing values of t, i.e. of j, are

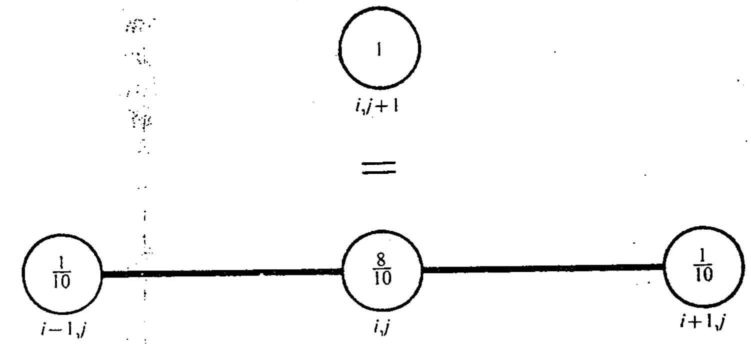


Fig. 2.2

shown moving downwards for convenience of calculation.) As examples,

$$u_{5,1} = \tfrac{1}{10}\{0{\cdot}8 + (8\times 1) + 0{\cdot}8\} = 0{\cdot}9600.$$

$$u_{4,2} = \tfrac{1}{10}\{0{\cdot}6 + (8\times 0{\cdot}8) + 0{\cdot}96\} = 0{\cdot}7960.$$

TABLE 2.2

	$i=0$, $x=0$	$i=1$, 0·1	$i=2$, 0·2	$i=3$, 0·3	$i=4$, 0·4	$i=5$, 0·5	$i=6$, 0·6
$(j=0)\ t=0{\cdot}000$	0	0·2000	0·4000	0·6000	0·8000	1·0000	0·8000
$(j=1)$ 0·001	0	0·2000	0·4000	0·6000	0·8000	0·9600	0·8000
$(j=2)$ 0·002	0	0·2000	0·4000	0·6000	0·7960	0·9280	0·7960
$(j=3)$ 0·003	0	0·2000	0·4000	0·5996	0·7896	0·9016	0·7896
$(j=4)$ 0·004	0	0·2000	0·4000	0·5986	0·7818	0·8792	0·7818
$(j=5)$ 0·005	0	0·2000	0·3999	0·5971	0·7732	0·8597	0·7732
⋮							
$(j=10)$ 0·01	0	0·1996	0·3968	0·5822	0·7281	0·7867	0·7281
⋮							
$(j=20)$ 0·02	0	0·1938	0·3781	0·5373	0·6486	0·6891	0·6486

The analytical solution of the partial differential equation satisfying these conditions is

$$u = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(\sin\tfrac{1}{2}n\pi)(\sin n\pi x)\exp(-n^2\pi^2 t).$$

Comparison of this solution with the finite-difference one at $x = 0{\cdot}3$, as given below, shows that the finite-difference solution is reasonably accurate. The percentage error is the difference of the solutions expressed as a percentage of the analytical solution of the partial differential equation.

TABLE 2.3

	Finite-difference solution ($x = 0{\cdot}3$)	*Analytical solution* ($x = 0{\cdot}3$)	*Difference*	*Percentage error*
$t = 0{\cdot}005$	0·5971	0·5966	0·0005	0·08
$t = 0{\cdot}01$	0·5822	0·5799	0·0023	0·4
$t = 0{\cdot}02$	0·5373	0·5334	0·0039	0.7
$t = 0{\cdot}10$	0·2472	0·2444	0·0028	1·1

The comparison at $x = 0{\cdot}5$ is not quite so good because of the discontinuity in the initial value of $\partial u/\partial x$, from $+2$ to -2, at this point (Equation 2.5). Inspection of Table 2.4 shows, however, that the effect of this discontinuity dies away as t increases.

TABLE 2.4

	Finite-difference solution ($x = 0{\cdot}5$)	*Analytical solution* ($x = 0{\cdot}5$)	*Difference*	*Percentage error*
$t = 0{\cdot}005$	0·8597	0·8404	0·0193	2·3
$t = 0{\cdot}01$	0·7867	0·7743	0·0124	1·6
$t = 0{\cdot}02$	0·6891	0·6809	0·0082	1·2
$t = 0{\cdot}10$	0·3056	0·3021	0·0035	1·2

It can be proved analytically that when the boundary values are constant the effect of discontinuities in initial values and initial derivatives upon the solution of a parabolic equation decreases as t increases. (See Chapter 3, exercise 20.)

An examination of Tables 2.19 and 2.21 given in exercise 1 at the end of this chapter shows that the same finite-difference solution for a problem in which the initial function and all its derivatives are continuous is very close indeed to the solution of the partial differential equation.

Richtmyer, reference 31, has shown for this particular finite-difference scheme that when the initial function and its first $(p-1)$ derivatives are continuous and the pth derivative ordinarily discontinuous (i.e., changes by finite jumps), then the difference between the solution of the partial differential equation and a convergent solution of the difference equation is of order $(\delta t)^{(p+2)/(p+4)}$, for small δt.

In this example, $p = 1$, so the difference is of order $(\delta t)^{\frac{3}{5}}$. As $(0{\cdot}001)^{\frac{3}{5}} = 0{\cdot}016$, it is seen that the finite-difference solution is actually better than the estimate indicates, a feature common to most error estimates. When all the derivatives are continuous, $p \to \infty$, and the error is of order δt.

Case 2

Take $\delta x = h = \frac{1}{10}$, $\delta t = k = \frac{5}{1000}$, so $r = k/h^2 = 0{\cdot}5$. Then equation (2.4) gives

$$u_{i,j+1} = \tfrac{1}{2}(u_{i-1,j} + u_{i+1,j}), \tag{2.7}$$

and the solution obtained by applying this finite-difference equation to the boundary and initial values is recorded in Table 2.5.

TABLE 2.5

	$i = 0$, $x = 0$	1, 0·1	2, 0·2	3, 0·3	4, 0·4	5, 0·5	6, 0·6
$T = 0{\cdot}000$	0	0·2000	0·4000	0·6000	0·8000	1·0000	0·8000
0·005	0	0·2000	0·4000	0·6000	0·8000	0·8000	0·8000
0·010	0	0·2000	0·4000	0·6000	0·7000	0·8000	0·7000
0·015	0	0·2000	0·4000	0·5500	0·7000	0·7000	0·7000
0·020	0	0·2000	0·3750	0·5500	0·6250	0·7000	0·6250
·							
·							
·							
0·100	0	0·0949	0·1717	0·2484	0·2778	0·3071	0·2778

TABLE 2.6

	Finite-difference solution ($x = 0{\cdot}3$)	*Analytical solution* ($x = 0{\cdot}3$)	*Difference*	*Percentage error*
$t = 0{\cdot}005$	0·6000	0·5966	0·0034	0·57
$t = 0{\cdot}01$	0·6000	0·5799	0·0201	3·5
$t = 0{\cdot}02$	0·5500	0·5334	0·0166	3·1
$t = 0{\cdot}1$	0·2484	0·2444	0·0040	1·6

It is seen that this finite-difference solution is not quite as good an approximation to the solution of the partial differential equation as the previous one; nevertheless it would be adequate for most technical purposes.

Case 3

Take $\delta x = \frac{1}{10}$, $\delta t = \frac{1}{100}$, so $r = \delta t/(\delta x)^2 = 1$. Then equation (2.4) gives

$$u_{i,j+1} = u_{i-1,j} - u_{i,j} + u_{i+1,j}, \tag{2.9}$$

and the solution of this finite-difference scheme is as below.

TABLE 2.7

	$i=0$	1	2	3	4	5	6
	$x=0$	0·1	0·2	0·3	0·4	0·5	0·6
$t=0{\cdot}00$	0	0·2	0·4	0·6	0·8	1·0	0·8
0·01	0	0·2	0·4	0·6	0·8	0·6	0·8
0·02	0	0·2	0·4	0·6	0·4	1·0	0·4
0·03	0	0·2	0·4	0·2	1·2	−0·2	1·2
0·04	0	0·2	0·0	1·4	−1·2	2·6	−1·2

Considered as a solution of the partial differential equation this is obviously meaningless, although it is, of course, the correct solution of equation (2.9) with respect to the initial values and boundary values given.

These three cases clearly indicate that the value of r is important and it will be proved in Chapter 3 that this explicit method is valid only when $0 < r \leq \frac{1}{2}$. (The conditions that must be satisfied for a valid expansion are dealt with both descriptively and analytically in Chapter 3 under the headings of convergence, stability and consistency. Any reader who would prefer to have an introduction to these concepts at this stage could do so by reading the descriptive treatments of these topics as they are independent of the remainder of this chapter.)

The graphs opposite compare the analytical solution of the partial differential equation (shown as continuous curves) with the finite-difference solution (shown by dots) for values of r just below and above $\frac{1}{2}$, and the same number of time-steps.

Crank–Nicolson implicit method

Although the explicit method is computationally simple it has one serious drawback. The time step $\delta t = k$ is necessarily very small

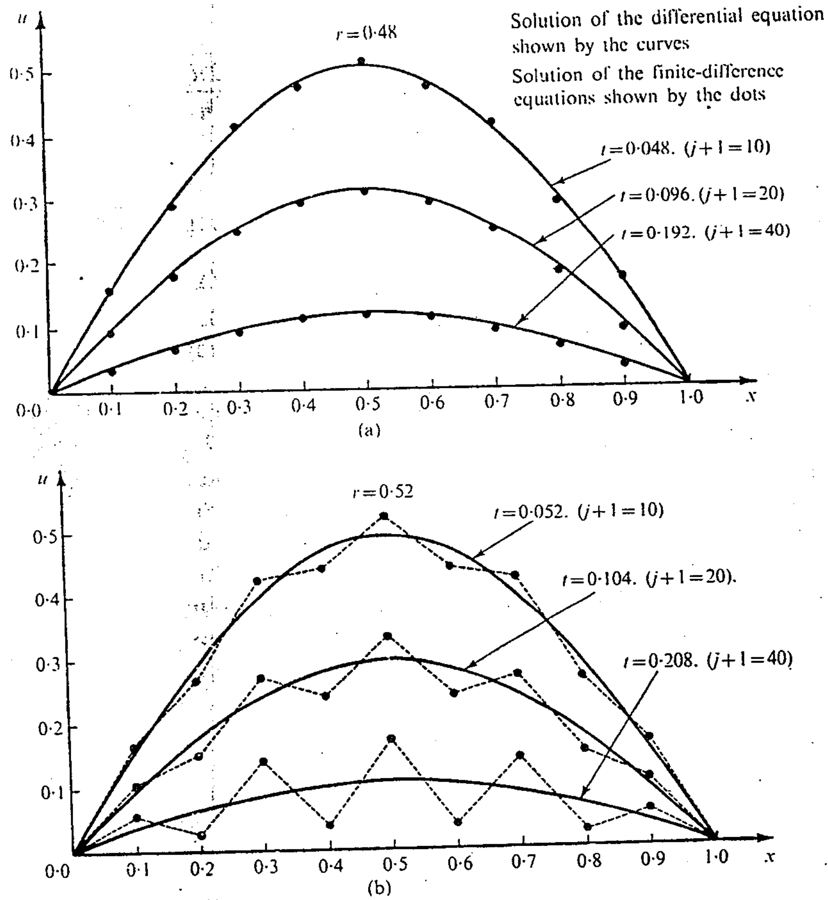


Fig. 2.3

because the process is valid only for $0 < k/h^2 \leq \frac{1}{2}$, i.e., $k \leq \frac{1}{2}h^2$, and $h = \delta x$ must be kept small in order to attain reasonable accuracy. Crank and Nicolson (1947) proposed, and used, a method that reduces the total volume of calculation and is valid (i.e., convergent and stable) for all finite values of r. They replaced $\partial^2 u/\partial x^2$ by the mean of its finite-difference representations on the $(j+1)$th and jth time rows and approximated the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

where δ_x^2 is the usual central difference. For j odd, Eqn. (2-265) becomes

$$U_{i,j+2} = U_{i,j} + 2r\,\delta_x^2 U_{i,j+1}.$$

Thus, Gordon's method is made up of a series of familiar calculations.

Another series of asymmetric approximations for the diffusion equation, $u_t = u_{xx}$, was introduced by Saul'yev [85] followed by variations due to Larkin [86], Barakat and Clark [87], and Liu [88], and related *group explicit* methods by Abdullah [89], Evans and Abdullah [90], and Evans [91]. In each case, the result is an approximation or pair of approximations that are explicit and are unconditionally stable.

The basic ideas are easy to describe. Saul'yev replaces $u_x|_{i-1/2,j}$ in

$$u_{xx}|_{i,j} = \frac{u_x|_{i+1/2,j} - u_x|_{i-1/2,j}}{h}$$

by $u_x|_{i-1/2,j+1}$, and uses the central difference approximations

$$u_x|_{i+1/2,j} = \frac{U_{i+1,j} - U_{i,j}}{h},$$

$$u_x|_{i-1/2,j+1} = \frac{U_{i,j+1} - U_{i-1,j+1}}{h}$$

for u_x and a forward difference for u_t to obtain the algorithm (Saul'yev A), which has been solved for $U_{i,j+1}$:

$$(1+r)U_{i,j+1} = U_{i,j} + r(U_{i-1,j+1} - U_{i,j} + U_{i+1,j}). \qquad (2\text{-}266)$$

The asymmetric computational molecule for Eqn. (2-266) is shown in Fig. 2-16.

An analogous form (Saul'yev B) (Problem 2-58) is

$$(1+r)U_{i,j+1} = U_{i,j} + r(U_{i+1,j+1} - U_{i,j} + U_{i-1,j}). \qquad (2\text{-}267)$$

with the computational molecule shown in Fig. 2-17.

It's important to note that Saul'yev A is explicit if the evaluation begins at the left-hand boundary ($x = 0$) and moves to the right, so that only the single value $U_{i,j+1}$ is unknown. A similar explicit nature holds for Saul'yev B if the calculation proceeds to the left from the right-hand boundary (say, $x = 1$). It is interesting to note that if $r = 1$ then Saul'yev A simplifies to

$$U_{i,j+1} = \tfrac{1}{2}(U_{i-1,j+1} + U_{i+1,j}).$$

and Saul'yev B to

$$U_{i,j+1} = \tfrac{1}{2}(U_{i+1,j+1} + U_{i-1,j})$$

—i.e., the value of $U_{i,j}$ is not needed in the computation!

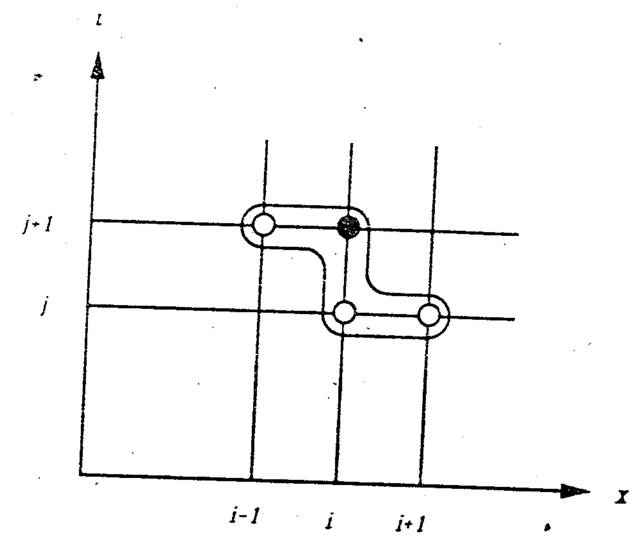


Figure 2-16 Molecule for Eqn. (2-266)

(A)

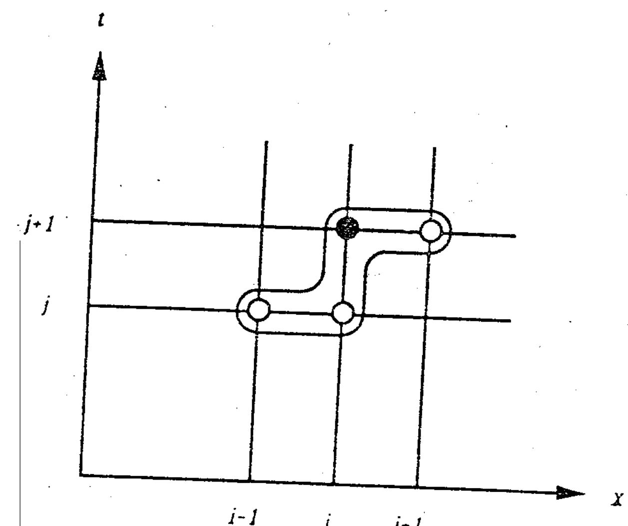


Figure 2-17 Molecule for Eqn. (2-267)

(B)

The options open to using the Saul'yev schemes are:

(i) Use Eqn. (2-266) only and proceed line by line in the t direction, but *always* from the left boundary to the right on a line.

(ii) Use Eqn. (2-267) only and proceed line by line in the t direction, but *always* from the right boundary to the left on a line.

first using Saul'yev A and then B, or the reverse. This is related to *alternating direction methods* to be discussed later.

(iv) Use Saul'yev A and Saul'yev B on the same line and average the results for the final answer (A first, and then B). This is equivalent to introducing the dummy variables $P_{i,j}$ and $Q_{i,j}$ such that

$$(1 + r)P_{i,j+1} = U_{i,j} + r(P_{i-1,j+1} - U_{i,j} + U_{i+1,j}), \quad (2\text{-}268a)$$

$$(1 + r)Q_{i,j+1} = U_{i,j} + r(Q_{i+1,j+1} - U_{i,j} + U_{i-1,j}), \quad (2\text{-}268b)$$

and

$$U_{i,j+1} = \tfrac{1}{2}(P_{i,j+1} + Q_{i,j+1}). \quad (2\text{-}269)$$

This averaging method has some computational advantage because of the possibility of truncation error cancellation.

As an alternative to Eqns. (2-268a, b) one can retain the $P_{i,j}$ and $Q_{i,j}$ from the previous step and replace $U_{i,j}$, $U_{i+1,j}$ in (2-268a) by $P_{i,j}$ and $P_{i+1,j}$, respectively, and $U_{i,j}$, $U_{i-1,j}$ in (2-268b) by $Q_{i,j}$ and $Q_{i-1,j}$.

Liu [88] used higher order approximations and obtained the algorithms (Liu A)

$$(3r + 2)U_{i,j+1} = 2(1 - 2r)U_{i,j}$$
$$+ r(U_{i-1,j} + 3U_{i+1,j} - U_{i-2,j+1} + 4U_{i-1,j+1}).$$

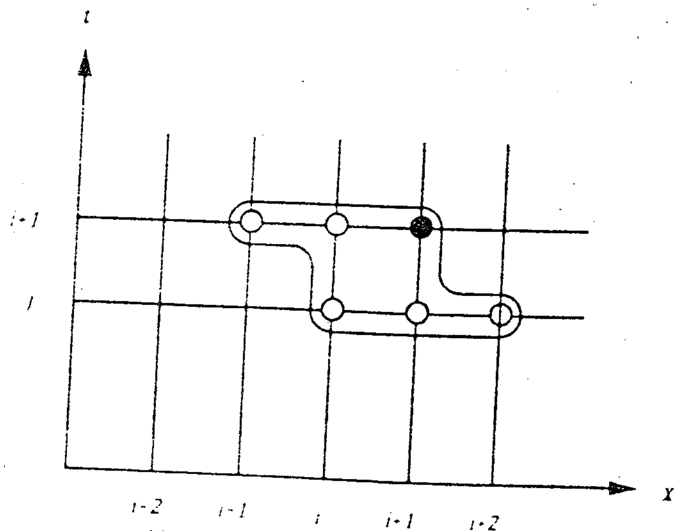


Figure 2-18(a) Molecule of Liu A: L to R

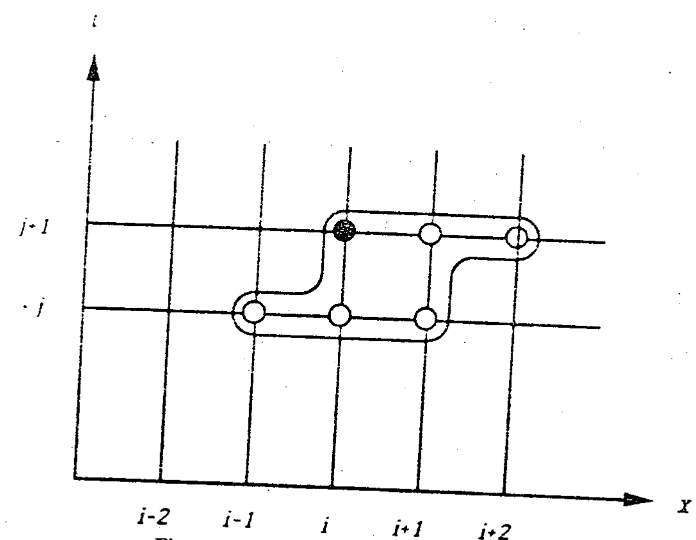


Figure 2-18(b) Molecule of Liu B: R to L

with the computational molecule shown in (Fig. 2-18a). Liu B is given by

$$(3r + 2)U_{i,j+1} = 2(1 - 2r)U_{i,j}$$
$$+ r(U_{i+1,j} + 3U_{i-1,j} - U_{i+2,j+1} + 4U_{i+1,j+1}).$$

These are analogous to the Saul'yev schemes except that the first point on any line (either from the left or the right) must be obtained by some other means. As with the Saul'yev forms, combinations can be used.

First Central-Difference Expressions Δx^2

use the modified euler's method for the first step.

non-self starting method,

$$y_i' = \frac{y_{i+1} - y_{i-1}}{2(\Delta x)}$$

$$y_i'' = \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2}$$

$$y_i''' = \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2(\Delta x)^3} \tag{5-38}$$

$$y_i'''' = \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{(\Delta x)^4}$$

Second Central-Difference Expressions Δx^4

$$y_i' = \frac{-y_{i+2} + 8y_{i+1} - 8y_{i-1} + y_{i-2}}{12(\Delta x)}$$

$$y_i'' = \frac{-y_{i+2} + 16y_{i+1} - 30y_i + 16y_{i-1} - y_{i-2}}{12(\Delta x)^2}$$

$$y_i''' = \frac{-y_{i+3} + 8y_{i+2} - 13y_{i+1} + 13y_{i-1} - 8y_{i-2} + y_{i-3}}{8(\Delta x)^3} \tag{5-39}$$

$$y_i'''' = \frac{-y_{i+3} + 12y_{i+2} - 39y_{i+1} + 56y_i - 39y_{i-1} + 12y_{i-2} - y_{i-3}}{6(\Delta x)^4}$$

First Forward-Difference Expressions

$$y_i' = \frac{y_{i+1} - y_i}{(\Delta x)}$$

$$y_i'' = \frac{y_{i+2} - 2y_{i+1} + y_i}{(\Delta x)^2}$$

$$y_i''' = \frac{y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i}{(\Delta x)^3} \tag{5-40}$$

$$y_i'''' = \frac{y_{i+4} - 4y_{i+3} + 6y_{i+2} - 4y_{i+1} + y_i}{(\Delta x)^4}$$

Second Forward-Difference Expressions

$$y_i' = \frac{-y_{i+2} + 4y_{i+1} - 3y_i}{2(\Delta x)}$$

$$y_i'' = \frac{-y_{i+3} + 4y_{i+2} - 5y_{i+1} + 2y_i}{(\Delta x)^2}$$

$$y_i''' = \frac{-3y_{i+4} + 14y_{i+3} - 24y_{i+2} + 18y_{i+1} - 5y_i}{2(\Delta x)^3} \tag{5-41}$$

$$y_i'''' = \frac{-2y_{i+5} + 11y_{i+4} - 24y_{i+3} + 26y_{i+2} - 14y_{i+1} + 3y_i}{(\Delta x)^4}$$

~~First Backward-Difference Expressions~~

$$y_i' = \frac{y_i - y_{i-1}}{(\Delta x)}$$

$$y_i'' = \frac{y_i - 2y_{i-1} + y_{i-2}}{(\Delta x)^2}$$

$$y_i''' = \frac{y_i - 3y_{i-1} + 3y_{i-2} - y_{i-3}}{(\Delta x)^3} \tag{5-42}$$

$$y_i'''' = \frac{y_i - 4y_{i-1} + 6y_{i-2} - 4y_{i-3} + y_{i-4}}{(\Delta x)^4}$$

Second Backward-Difference Expressions

$$y_i' = \frac{3y_i - 4y_{i-1} + y_{i-2}}{2(\Delta x)}$$

$O(\Delta x^2)$

$$y_i'' = \frac{2y_i - 5y_{i-1} + 4y_{i-2} - y_{i-3}}{(\Delta x)^2}$$

$$y_i''' = \frac{5y_i - 18y_{i-1} + 24y_{i-2} - 14y_{i-3} + 3y_{i-4}}{2(\Delta x)^3} \tag{5-43}$$

$$y_i'''' = \frac{3y_i - 14y_{i-1} + 26y_{i-2} - 24y_{i-3} + 11y_{i-4} - 2y_{i-5}}{(\Delta x)^4}$$

EXAMPLE 5-3

In Example 3-4 the Newton-Raphson method was used to determine the output lever angles of a crank-and-lever 4-bar linkage system for each 5° of rotation of the input crank. Now we shall determine the angular velocity and the angular acceleration of the output lever of the same type of mechanism for each 5° of rotation of the input crank, with the latter rotating at a uniform angular velocity of 100 radians/sec.

We can determine the output lever positions ϕ, corresponding to each 5° of crank rotation θ, by utilizing Freudenstein's equation and the Newton-Raphson method, as was done in Example 3-4. Such a set of values, in effect, gives us a series of points on the ϕ versus θ curve, and the ϕ values are stored in memory to provide data for the differentiation processes which follow. The slope of the ϕ-θ curve may be related to the angular velocity of the output lever $d\phi/dt$ if we realize that, with the crank rotating at a constant ω, its angular position is given by

$$\theta = \omega t$$

so that

$$\frac{d\phi}{d\theta} = \frac{1}{\omega}\frac{d\phi}{dt}$$

attach. to page 1c

Newton linearization

suppose from the differential equation we can get the diffence equations so that the diffence equations are in the form

$$f_i(u_1, u_2, \dots, u_N) = 0 \qquad i = 1, \dots, N$$

then let V_i be a known approx. to u_i for $i = 1, 2, \dots, N$ so that $f_i(V_1, V_2, \dots, V_N) \neq 0$

let $u_i = V_i + \epsilon_i$ then

$$f_i(u_1, u_2, \dots, u_N) = f_i(V_1+\epsilon_1, V_2+\epsilon_2, \dots, V_N+\epsilon_N) = f_i(V_1, V_2, V_3, \dots, V_N) + \left[\frac{\partial f_i}{\partial V_1}\Delta V_1 + \dots \frac{\partial f_i}{\partial V_N}\Delta V_N\right]$$

if $u_1, u_2, \dots, u_N$ are chosen to make $f_i(\quad) = 0$ then

$$-f_i(V_1, V_2, \dots, V_N) = \frac{\partial f_i}{\partial V_1}\Delta V_1 + \frac{\partial f_i}{\partial V_2}\Delta V_2 + \dots \frac{\partial f_i}{\partial V_N}\Delta V_N \quad \text{where } \Delta V_i = u_i - V_i$$

this leads to N equations in N unknowns $\{\Delta V_1, \dots, \Delta V_N\} = \{\epsilon_1, \epsilon_2, \dots, \epsilon_N\}$

for example $\dfrac{\partial U}{\partial t} = \alpha \dfrac{\partial^2 U^2}{\partial x^2}$ let's use CD at $i, j+\frac{1}{2}$ (CN with $O(\Delta t^2, \Delta x^2)$)

or

$$\frac{u_{ij+1} - u_{ij}}{\Delta t} = \frac{\alpha}{2\Delta x^2}\left[(u^2_{i-1j+1} - 2u^2_{ij+1} + u^2_{i+1j+1}) + (u^2_{i-1j} - 2u^2_{ij} + u^2_{i+1j})\right]$$

$\therefore$ let $u_{ij+1} = V_i$ so if $\dfrac{\Delta x^2}{\alpha \Delta t} = p$

$$U^2_{i-1} - 2(U_i^2 + pU_i) + U^2_{i+1} + \{u^2_{i-1j} - 2(u^2_{ij} - pu_{ij}) + u^2_{i+1j}\} = f_i(U_{i-1}, U_i, U_{i+1}) = 0$$

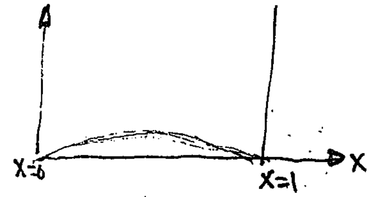
let V_i be an approximation to $u_{ij+1} = U_i$. Therefore $f_i(V_{i-1}, V_i, V_{i+1}) \neq 0$

then for $\dfrac{\partial f_i}{\partial V_{i-1}} = 2V_{i-1}$ $\quad \dfrac{\partial f_i}{\partial V_i} = -2(2V_i + p)$ $\quad \dfrac{\partial f_i}{\partial V_{i+1}} = 2V_{i+1}$

$$\therefore\ 2V_{i-1}\Delta V_{i-1} - 2(2V_i + p)\Delta V_i + 2V_{i+1}\Delta V_{i+1} = -f_i(V_1, \dots, V_N)$$
$$= -\left[V^2_{i-1} - 2(V_i^2 + pV_i) + V^2_{i+1} + \{u^2_{i-1j} - 2(u^2_{ij} - pu_{ij}) + u^2_{i+1j}\}\right]$$

so for example start $V_i = u_{ij}$; solve for ΔV_i's & check if $\|\underset{\sim}{\Delta V}\| < \epsilon$

if no, then $V_i = V_i + \Delta V_i$



let $u(x=0,t)=0$ & $u(x=1,t)=0$ & $u(x,t=0) = 4x(1-x)$

let $x=\frac{1}{6}i \quad i=0,\dots,6$

	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$	$i=5$	$i=6$	
$t=0$	0	$5/9$	$8/9$	1	$8/9$	$5/9$	0	$=u$

$$\cancel{2V_0\epsilon_0} - 2(2V_1+p)\epsilon_1 + 2V_2\epsilon_2 = -\left[\left\{\cancel{V_0^2} - 2(V_1^2+pV_1) + V_2^2\right\} + \left\{0^2 - 2\left(\tfrac{5}{9}^2 - p\tfrac{5}{9}\right) + \tfrac{8}{9}^2\right\}\right]$$

$$2V_1\epsilon_1 - 2(2V_2+p)\epsilon_2 + 2V_3\epsilon_3 = -\left[\left\{V_1^2 - 2(V_2^2+pV_2) + V_3^2\right\} + \left\{\tfrac{5}{9}^2 - 2\left(\tfrac{8}{9}^2 - p\tfrac{8}{9}\right) + 1^2\right\}\right]$$

$$2V_2\epsilon_2 - 2(2V_3+p)\epsilon_3 + 2V_4\epsilon_4 = -\left[\left\{V_2^2 - 2(V_3^2+pV_3) + V_4^2\right\} + \left\{\tfrac{8}{9}^2 - 2(1^2-p\cdot1) + \tfrac{8}{9}^2\right\}\right]$$

$$2V_3\epsilon_3 - 2(2V_4+p)\epsilon_4 + 2V_5\epsilon_5 = -\left[\left\{V_3^2 - 2(V_4^2+pV_4) + V_5^2\right\} + \left\{1^2 - 2\left(\tfrac{8}{9}^2 - p\tfrac{8}{9}\right) + \tfrac{5}{9}^2\right\}\right]$$

$$2V_4\epsilon_4 - 2(2V_5+p)\epsilon_5 + \cancel{2V_6\epsilon_6} = -\left[\left\{V_4^2 - 2(V_5^2+pV_5) + \cancel{V_6^2}\right\} + \left\{\tfrac{8}{9}^2 - 2\left(\tfrac{5}{9}^2 - p\tfrac{5}{9}\right) + 0^2\right\}\right]$$

initially take on RHS $V_1 = 5/9 \quad V_2 = 8/9 \quad V_3 = 1 \quad V_4 = 8/9 \quad V_5 = 5/9$, solve for V_i's.

in all cases V_0 & $V_6 = 0$ since $u(0,t) = u(1,t) = 0$

$p = \frac{\Delta x^2}{\alpha \Delta t}$

IF $p=1$
$$-2\left(\tfrac{10}{9}+1\right)\epsilon_1 + \tfrac{16}{9}\epsilon_2 = -\left[\left\{-2\left(\tfrac{25}{81}+1\cdot\tfrac{5}{9}\right) + \tfrac{64}{81}\right\} + \left\{0^2 - 2\left(\tfrac{25}{81} - 1\cdot\tfrac{5}{9}\right) + \tfrac{64}{81}\right\}\right]$$

$$2\left(\tfrac{5}{9}\right)\epsilon_1 - 2\left(\tfrac{16}{9}+1\right)\epsilon_2 + 2\epsilon_3 = -\left[\left\{\tfrac{25}{81} - 2\left(\tfrac{64}{81}+1\cdot\tfrac{8}{9}\right) + 1\right\} + \left\{\tfrac{25}{81} - 2\left(\tfrac{64}{81} - 1\cdot\tfrac{8}{9}\right) + 1\right\}\right]$$

$\vdots$

check to see if $\|\Delta \underline{V}\| = \|\underline{\epsilon}\| < \epsilon$

you define

$V_0 = 0$

$V_1 = V_1 + \epsilon_1 = 5/9 + \epsilon_1$

$V_2 = V_2 + \epsilon_2 = 8/9 + \epsilon_2$

$V_3 = V_3 + \epsilon_3 = 1 + \epsilon_3$

$\vdots$

V_5

V_6

Check to see if $|f_i(V_{i-1}, V_i, V_{i+1})| < \epsilon$ for $i = 1, \dots, N$

Richtmeyer's linearization used for $\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U^m}{\partial x^2}$ evaluated at x_i, t_j

uses $\frac{1}{\Delta t}(u_{i,j+1} - u_{i,j}) = \alpha \left[\theta \frac{\partial^2}{\partial x^2}(u^m_{i,j+1}) + (1-\theta)\frac{\partial^2}{\partial x^2}(u^m_{i,j})\right]$ implicit weighted average

IF $m=1$ & $\theta = \frac{1}{2}$ C-N

$$u^m_{i,j+1} = u^m_{i,j} + \Delta t \frac{\partial u^m_{i,j}}{\partial t} + \dots$$ from Taylor series

$$u^m_{i,j} + \Delta t \frac{\partial u^m_{i,j}}{\partial u_{i,j}} \underbrace{\frac{\partial u_{i,j}}{\partial t}} + \dots$$ using chain rule

$$u^m_{i,j} + \Delta t\, m\, u^{m-1}_{i,j} \frac{1}{\Delta t}(u_{i,j+1} - u_{i,j}) + \dots$$

$\therefore$

$$u^m_{i,j+1} = u^m_{i,j} + m u^{m-1}_{i,j}(u_{i,j+1} - u_{i,j})$$ replace unknown $u^m_{i,j+1}$ by approximation linear in $u_{i,j+1}$

let $w_i = u_{i,j+1} - u_{i,j}$

$$\frac{1}{\Delta t} w_i = \alpha \left[\theta \frac{\partial^2}{\partial x^2}(u^m_{i,j} + m u^{m-1}_{i,j} w_i) + (1-\theta)\frac{\partial^2}{\partial x^2}(u^m_{i,j})\right]$$

$$= \alpha \left\{\frac{\partial^2}{\partial x^2}(u^m_{i,j}) + \theta m \frac{\partial^2}{\partial x^2}(u^{m-1}_{i,j} w_i)\right\}$$

$$= \frac{\alpha}{\Delta x^2}\left[\theta m \left\{u^{m-1}_{i-1,j} w_{i-1} - 2u^{m-1}_{i,j} w_i + u^{m-1}_{i+1,j} w_{i+1}\right\} + \left\{u^m_{i-1,j} - 2u^m_{i,j} + u^m_{i+1,j}\right\}\right]$$

put this into matrix form & solved for the w_i's

then $u_{i,j+1} = u_{i,j} + w_i$

IF $m=2$

$$\begin{bmatrix} 1 + 2mC\theta u^{m-1}_{1,j} & -mC\theta u^{m-1}_{2,j} & & \\ -mC\theta u^{m-1}_{1,j} & 1 + 2mC\theta u^{m-1}_{2,j} & -mC\theta u^{m-1}_{3,j} & \\ & \ddots & \ddots & \ddots \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \end{pmatrix} = \text{RHS}$$

$$\boxed{C = \frac{\alpha \Delta t}{\Delta x^2}}$$

$\text{RHS} = C\{u^m_{i-1,j} - 2u^m_{i,j} + u^m_{i+1,j}\}$

for 1st line $-2C(u^m_{1,j}) + C u^m_{2,j}$

when $m = 1 \Rightarrow \frac{1}{\Delta t} w_i = \alpha\left[\theta \frac{\partial^2}{\partial x^2}(u_{i,j+1}) + (1-\theta)\frac{\partial^2}{\partial x^2}(u_{i,j})\right]$ scheme unconditionally stable for $\theta \geq \frac{1}{4}$

if $\theta = \frac{1}{2}$ we get crank nicholson

for $\theta < \frac{1}{4}$ $\frac{\Delta t}{\Delta x^2} > \frac{1}{2C u_{ij_{max}} (1-4\theta)}$

ELLIPTIC PDE'S

$$\left.\begin{array}{l}\nabla^2\varphi = 0 \quad (\text{LAPLACE'S EQN}) \\ \nabla^2\varphi = f(x,y)\end{array}\right\} \varphi(x,y)$$

2-D FLUID MOTION φ – velocity vector $= u\underline{i} + v\underline{j}$

φ – stream fn.

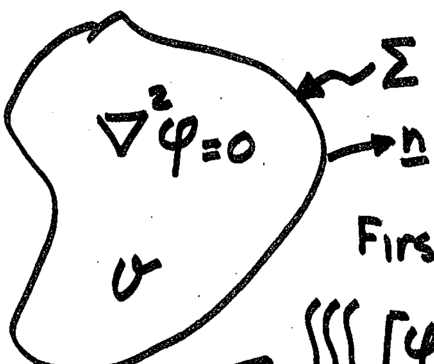
φ – Temperature

φ – torsion fn. $f = -2G\alpha$

shear modulus (G), angle of twist (α)

for solution of laplace's eqn. φ – harmonic fn.

1) If $\varphi(x,y)$ is a harmonic fn. in the region υ which is bdd by the surface Σ, then



$$\oiint_{\Sigma} \frac{\partial\varphi}{\partial n}\, d\sigma = 0$$

First Green's Theorem φ_1, φ_2 are fns of x, y, z

$$\iiint_{\upsilon} [\varphi_1 \nabla^2\varphi_2 + \nabla\varphi_1 \cdot \nabla\varphi_2]\, d\upsilon = \oiint_{\Sigma} \underline{n} \cdot \nabla\varphi_2\, d\sigma$$

$$\nabla\varphi = \underline{i}\frac{\partial\varphi}{\partial x} + \underline{j}\frac{\partial\varphi}{\partial y} + \underline{k}\frac{\partial\varphi}{\partial z} \quad ; \text{ if } \varphi_1 = 1 \quad \varphi_2 = \varphi$$

$$\iiint_{\upsilon} \underbrace{\nabla^2\varphi}_{=0}\, d\upsilon = \oiint_{\Sigma} \underline{n} \cdot \nabla\varphi\, d\sigma$$

$$0 = \oiint_{\Sigma} \underline{n} \cdot \nabla\varphi\, d\sigma = \oiint_{\Sigma} \frac{\partial\varphi}{\partial n}\, d\sigma$$

if $\nabla^2\varphi = f$ $$\iiint \nabla^2\varphi\, dV = \oint\!\!\oint_\Sigma \frac{\partial\varphi}{\partial n}\, d\sigma$$

$$\iiint f\, dV = \oint\!\!\oint_\Sigma \frac{\partial\varphi}{\partial n}\, d\sigma$$

---

if $\nabla^2\varphi = 0$ $\left.\frac{\partial\varphi}{\partial n}\right|_\Sigma = g$ (2^{nd} type of BC)

$$\iiint_V \nabla^2\varphi\, dV = \oint\!\!\oint_\Sigma \frac{\partial\varphi}{\partial n}\, d\sigma$$

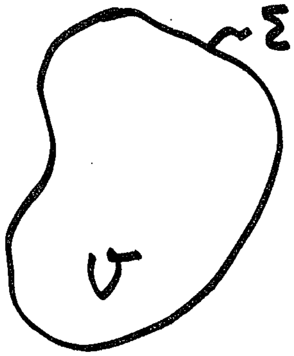
$$0 = \oint\!\!\oint_\Sigma g\, d\sigma$$

there are no source terms within the volume.

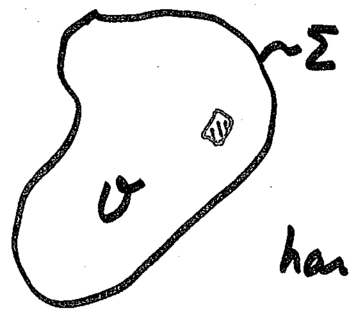
3 types of BC's
$$\left.\varphi\right|_\Sigma = g \quad \text{(Dirichlet BC - } 1^{st} \text{ type)}$$
$$\left.\frac{\partial\varphi}{\partial n}\right|_\Sigma = g \quad \text{(Neumann BC - } 2^{nd} \text{ type)}$$
nonunique unless $\oint\!\!\oint_\Sigma g\, d\sigma = 0$
$$\left.\left(a\varphi + b\frac{\partial\varphi}{\partial n}\right)\right|_\Sigma = g \quad \text{(Robin, Churchill - } 3^{rd} \text{ type)}$$

---

2) IF a fn $\varphi(x,y)$ is continuous and defined in the region $V+\Sigma$ and φ satisfies $\nabla^2\varphi = 0$ in the interior of V, then it assumes its maximum & minimum on the boundary Σ: (principle of the maximum).



3) if u & U are continuous in $V+\Sigma$ and are harmonic ($\nabla^2 u = \nabla^2 U = 0$), in V and $u \le U$ on Σ, then $u \le U$ for all points in V.



4) if u and U are continuous in $V+\Sigma$ and are harmonic in V for which $|u| \le U$ on Σ then $|u| \le U$ for all points in V.

5) If we have 3 harmonic fns. $-U, u, U$ so that

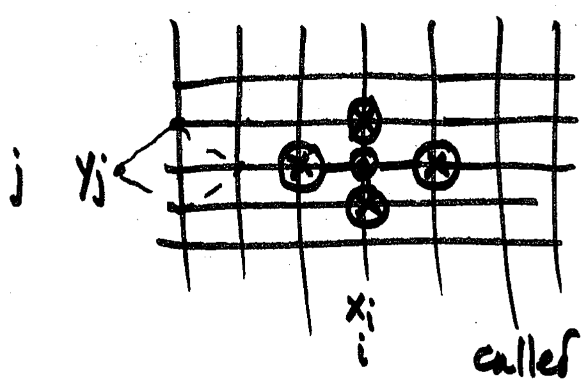
$-U \le u \le U$ on Σ, then $-U \le u \le U$ in V.

this is an out come of 4

$\nabla^2 T = f$ (S.S. heat transfer problems)

- $T(x,y) = T$
- let's write the FD equation at x_i, y_j $\quad x_i = i\Delta x,\ y_j = j\Delta y$
- CD in x & y. Look at $\Delta x = \Delta y = h$

$$T_{ij}^{(n+1)} = \frac{1}{4}\left(T_{i+1j}^{(n)} + T_{i-1j}^{(n)} + T_{ij+1}^{(n)} + T_{ij-1}^{(n)} - h^2 f_{ij}\right) \qquad f_{ij} = f(x_i, y_j)$$



To solve: first must assume some initial values on the interior

: iterate one pt. at a time

: iteration will converge

called Liebmann method if $f_{ij} = 0$

Jacobi type iteration

speed up convergence: 1) replace immediately

$$T_{ij}^{(n+1)} = \frac{1}{4}\left(T_{i-1j}^{(n+1)} + T_{i+1j}^{(n)} + T_{ij+1}^{(n)} + T_{ij-1}^{(n+1)} - h^2 f_{ij}\right)$$ Gauss-Seidel type method

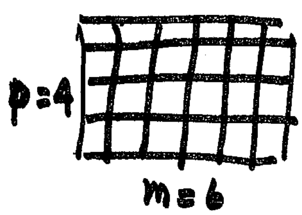
2) use successive over relaxation Southwell's method.

$$T_{ij}^{(n+1)} = T_{ij}^{(n)} + \frac{\omega}{4}\left[T_{i-1j}^{(n+1)} + T_{i+1j}^{(n)} + T_{ij-1}^{(n+1)} + T_{ij+1}^{(n)} - 4T_{ij}^{(n)} - h^2 f_{ij}\right]$$

GS eq = Southwell eqn when $\omega = 1$

$$\omega = \frac{8 - 4\sqrt{4-\alpha^2}}{\alpha^2} \qquad \alpha = \cos(\pi/m) + \cos(\pi/p)$$

$1 \le \omega \le 2$ — m & p are the # of intervals in the horiz & vert. side of a rectangular region



To ~~prove~~ <u>prove convergence</u> of the method $e_{ij}^{(n)} = T_{ij}^{(n)} - T_{ij}$

numerical — actual

for Liebmann method.

$$e_{ij}^{(n+1)} = \frac{1}{4}\left[e_{ij+1}^{(n)} + e_{ij-1}^{(n)} + e_{i+1j}^{(n)} + e_{i-1j}^{(n)}\right]$$

$$|e_{ij}^{(n+1)}| \le \frac{1}{4}\left\{|e_{ij+1}^{(n)}| + |e_{ij-1}^{(n)}| + |e_{i+1j}^{(n)}| + |e_{i-1j}^{(n)}|\right\} \le \frac{1}{4}\cdot 4\max e^{(n)}$$

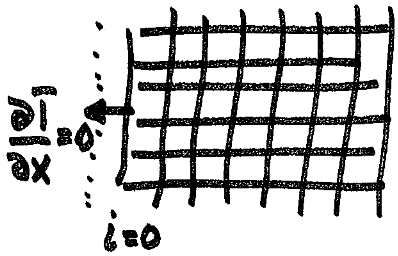
$|e_{ij}^{(n+1)}| \le e_{max}^{(n)}$ this shows error doesn't grow but is not a proof of convergence

if $e^{(n+1)} = Ke^{(n)} = K^2 e^{(n-1)} = \dots e^{(0)} K^{(n+1)}$ the for convergence

$|K| < 1$. if $K = (1 - \frac{1}{4m})$ m = # of pts in one direction

then we get convergence

If we have Neumann Type BCs



$$T_{ij} = \frac{1}{4}\left[T_{i+1j} + T_{i-1j} + T_{ij+1} + T_{ij-1} - h^2 f_{ij}\right]$$

$$\left.\frac{\partial T}{\partial x}\right|_{i=0,j} = CD \text{ in } x = \frac{T_{1j} - T_{-1j}}{2\Delta x} = 0 \qquad O(\Delta x^2, \Delta y^2)$$

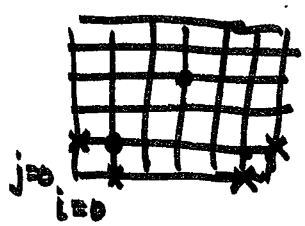
$$T_{1j} = T_{-1j} \qquad (T_{-1j} = T_{1j} - 2q\Delta x) \quad = q$$

$$T_{0j} = \frac{1}{4}\left[T_{1j} + T_{-1j} + T_{0j+1} + T_{0j-1} - h^2 f_{0j}\right]$$

$$= \frac{1}{4}\left[2T_{1j} + T_{0j+1} + T_{0j-1} - h^2 f_{0j}\right]$$

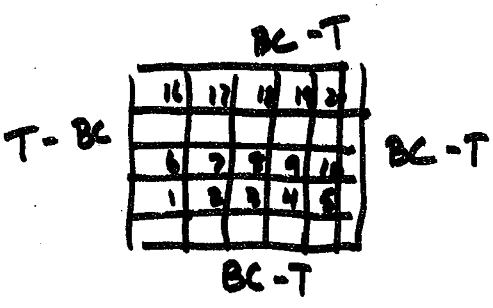
The other way to solve is by matrix methods.

$$\nabla^2 T_{ij} = 0 \qquad \left[T_{i+1j} + T_{i-1j} + T_{ij+1} + T_{ij-1} - 4T_{ij}\right] = h^2 f_{ij}$$



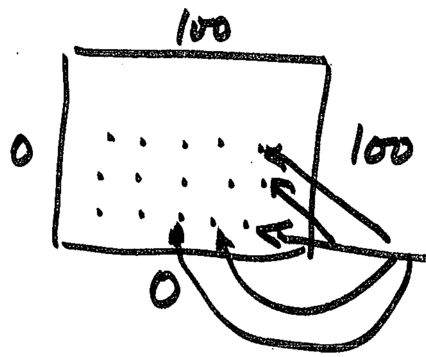
$$\left[T_{21} + \overset{BC}{T_{01}} + T_{12} + \overset{BC}{T_{10}} - 4T_{11}\right] = h^2 f_{11} \qquad i=1, j=1$$

$$\left[T_{31} + T_{11} + T_{22} + \overset{BC}{T_{20}} - 4T_{21}\right] = h^2 f_{21} \qquad i=2, j=1$$



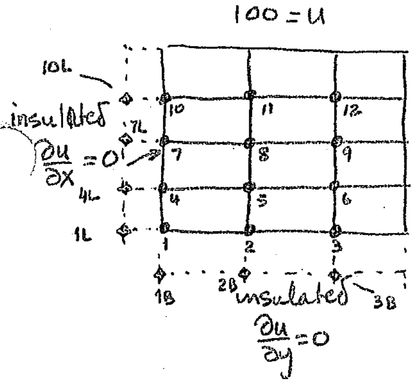
$$\begin{bmatrix} -4 & 1 & \cdots & 1 & & & \\ 1 & -4 & 1 & \cdots & 1 & & \\ & 1 & -4 & 1 & \cdots & 1 & \\ 1 & & \ddots & \ddots & \ddots & & \ddots \\ & 1 & & 1 & -4 & 1 & 1 \\ & & 1 & & \ddots & \ddots & \ddots \\ & & & 1 & \cdots & 1 & -4 \end{bmatrix} \begin{pmatrix} T_1 \\ T_2 \\ \vdots \\ T_{20} \end{pmatrix} = \begin{pmatrix} h^2 f_{11} + BC + BC \\ h^2 f_{21} + BC \\ \vdots \\ h^2 f_{51} + BC + BC \\ h^2 f_{12} + BC \\ \vdots \\ h^2 f_{54} + BC + BC \end{pmatrix}$$

how do you start the iterative process for Liebmann or G-S or Southwell



$$\Sigma(100+100+0+0) = \frac{200}{4} = 50$$

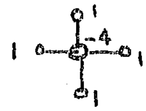
Implicit Method.



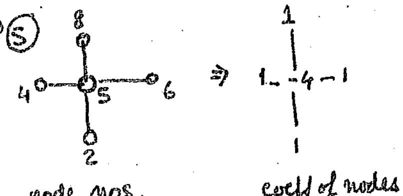
$0 = u$

$\nabla^2 u = 0 \Rightarrow u_{i+1j} + u_{i-1j} - 4u_{ij} + u_{ij+1} + u_{ij-1} = 0$ IF $\Delta x = \Delta y$

@ pt $\cancel{4}$, 5, 6, 8, 9, 11, 12

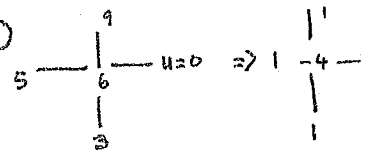


@ 5



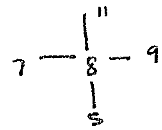
$u_4 + u_8 + u_6 + u_2 - 4u_5 = 0$

@ 6



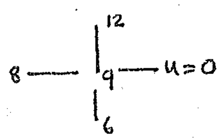
$u_5 + u_9 + 0 + u_3 - 4u_6 = 0$

@ 8



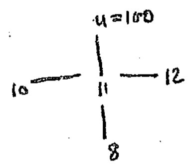
$u_7 + u_{11} + u_9 + u_5 - 4u_8 = 0$

@ 9



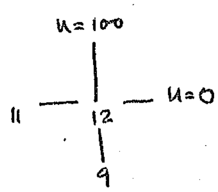
$u_8 + u_{12} + 0 + u_6 - 4u_9 = 0$

@ 11



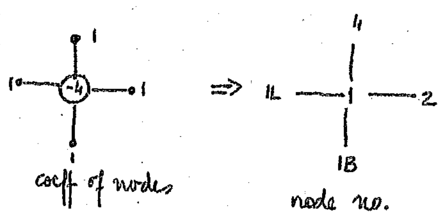
$u_{10} + 100 + u_{12} + u_8 - 4u_{11} = 0$

@ 12



$u_{11} + 100 + 0 + u_9 - 4u_{12} = 0$

@ 1

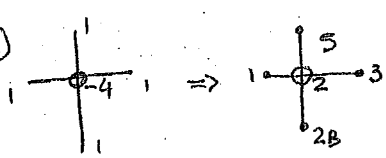


using $\left.\frac{\partial u}{\partial x}\right|_{node\,1} = 0 \Rightarrow \frac{u_2 - u_{1L}}{2\Delta x} = 0 \Rightarrow \underbrace{u_2 = u_{1L}}$

using $\left.\frac{\partial y}{\partial y}\right|_{node\,1} = 0 \Rightarrow \frac{u_4 - u_{1B}}{2\Delta y} = 0 \Rightarrow \underbrace{u_4 = u_{1B}}$

$u_{1L} + u_4 + u_2 + u_{1B} - 4u_1 = 0 \Rightarrow 2u_4 + 2u_2 - 4u_1 = 0$

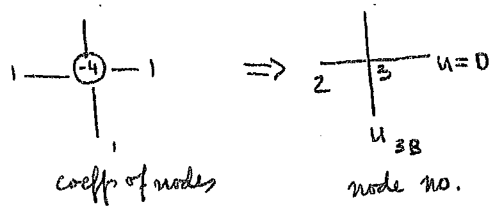
@ 2



using $\frac{\partial u}{\partial y} = \frac{u_5 - u_{2B}}{2\Delta y} = 0 \Rightarrow u_5 = u_{2B}$

$0 = u_1 + u_5 + u_3 + u_{2B} - 4u_2 \Rightarrow u_1 + 2u_5 + u_3 - 4u_2 = 0$

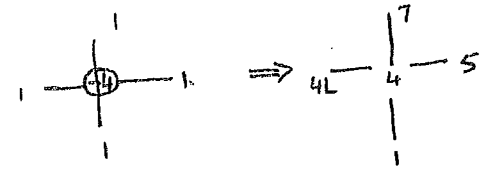
@ ③



using $\frac{\partial u}{\partial y} = \frac{u_6 - u_{3B}}{2\Delta y} = 0 \Rightarrow u_{3B} = u_6$

$u_2 + u_6 + 0 + u_{3B} - 4u_3 = 0 \Longrightarrow u_2 + 2u_6 - 4u_3 = 0$

@ ④



using $\frac{\partial u}{\partial x} = 0 \quad \frac{u_5 - u_{4L}}{2\Delta x} = 0 \Rightarrow u_{4L} = u_5$

$u_{4L} + u_7 + u_5 + u_1 - 4u_4 = 0 \Rightarrow 2u_5 + u_7 + u_1 - 4u_4 = 0$

node 7 & node 10 use same ~~method~~ to get

$2u_8 + u_{10} + u_4 - 4u_7 = 0$

& $2u_{11} + 100 + u_7 - 4u_{10} = 0$

$$\begin{bmatrix} -4 & 2 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -4 & 2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -4 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -100 \\ -100 \\ -100 \end{bmatrix} \Leftarrow M\underline{z} = \underline{t}$$

$$\begin{bmatrix} A & 2I & 0 & 0 \\ I & A & I & 0 \\ 0 & I & A & I \\ 0 & 0 & I & A \end{bmatrix} \begin{bmatrix} \underline{p} \\ \underline{q} \\ \underline{r} \\ \underline{s} \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{0} \\ \underline{b} \end{bmatrix}$$

$\underline{p} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ $\quad \underline{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\underline{q} = \begin{bmatrix} u_4 \\ u_5 \\ u_6 \end{bmatrix}$ $\quad \underline{b} = \begin{bmatrix} -100 \\ -100 \\ -100 \end{bmatrix}$

$\underline{r} = \begin{bmatrix} u_7 \\ u_8 \\ u_9 \end{bmatrix}$ $\quad A = \begin{bmatrix} -4 & 2 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 \end{bmatrix}$

$\underline{s} = \begin{bmatrix} u_{10} \\ u_{11} \\ u_{12} \end{bmatrix}$

$A\underline{p} + 2I\underline{q} = \underline{0}$ solve $A\underline{p} = -2I\underline{q}$

$\underline{p} + A\underline{q} + \underline{r} = \underline{0}$ $\quad \underline{r} = -\underline{p} - A\underline{q}$ $\quad A\underline{r} = -A\underline{p} - A^2\underline{q} = (2I - A^2)\underline{q}$

$\underline{q} + A\underline{r} + \underline{s} = \underline{0}$ $\quad \underline{s} = -\underline{q} - A\underline{r}$ $\quad \underline{s} = -\underline{q} - (2I - A^2)\underline{q} = -(3I - A^2)\underline{q}$

$\underline{r} + A\underline{s} = \underline{b}$ $\quad A\underline{r} + A^2\underline{s} = A\underline{b}$ $\quad (2I - A^2)\underline{q} - (3A^2 - A^4)\underline{q} = A\underline{b}$

$$(A^4 - 4A^2 + 2I)\underline{q} = A\underline{b}$$

solve

$A\underline{r} = (2I - A^2)\underline{q}$ ③ for $\underline{r}$

$\underline{s} = (A^2 - 3I)\underline{q}$ ② for $\underline{s}$

$(A^4 - A^2 + 2I)\underline{q} = A\underline{b}$ ① for $\underline{q}$

$A\underline{p} = -2I\underline{q}$ ④ for $\underline{p}$

In general, let

$$M = LU = \begin{bmatrix} \beta_1 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & 0 \\ 0 & 0 & \alpha_4 & \beta_4 \end{bmatrix} \begin{bmatrix} I_1 & \gamma_1 & 0 & 0 \\ 0 & I_2 & \gamma_2 & 0 \\ 0 & 0 & I_3 & \gamma_3 \\ 0 & 0 & 0 & I_4 \end{bmatrix} = \begin{bmatrix} A & 2I & 0 & 0 \\ I & A & I & 0 \\ 0 & I & A & I \\ 0 & 0 & I & A \end{bmatrix}$$

where $\beta_1 I_1 = A$ $\quad \beta_1 = A$

$\beta_1\gamma_1 = 2I$ $\quad \gamma_1 = 2\beta_1^{-1} = 2A^{-1}$

$\alpha_2 I_1 = I$ $\quad \alpha_2 = I$

$\alpha_2\gamma_1 + \beta_2 I_2 = A$ $\quad \beta_2 = A - \alpha_2\gamma_1$ or $\beta_1\beta_2 = \beta_1 A - 2I = A^2 - 2I$ $\quad \beta_2 = A - 2A^{-1}$

$\beta_2\gamma_2 = I$ $\quad \gamma_2 = \beta_2^{-1} = (A - 2A^{-1})^{-1}$

$\alpha_3 I_2 = I$ $\quad \alpha_3 = I$

$\alpha_3\gamma_2 + \beta_3 I_3 = A$ $\quad \beta_3 = A - \alpha_3\gamma_2$ $\quad \beta_2\beta_3 = \beta_2 A - I$

$\beta_3\gamma_3 = I$ $\quad \gamma_3 = \beta_3^{-1}$

$\alpha_4 I_3 = I$ $\quad \alpha_4 = I$

$\alpha_4\gamma_3 + \beta_4 I_4 = A$ $\quad \beta_4 = A - \alpha_4\gamma_3$ $\quad \beta_3\beta_4 = \beta_3 A - I$

Now let

$M\underline{z} = \underline{t}$ or $LU\underline{z} = \underline{t}$ or $U\underline{z} = \underline{y}$ & $L\underline{y} = \underline{t}$

$$L\underline{y} = \underline{t} \qquad \begin{bmatrix} \beta_1 & 0 & 0 & 0 \\ \alpha_2 & \beta_2 & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & 0 \\ 0 & 0 & \alpha_4 & \beta_4 \end{bmatrix} \begin{bmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \end{bmatrix} = \begin{bmatrix} \underline{0} \\ \underline{0} \\ \underline{0} \\ \underline{b} \end{bmatrix}$$

$\beta_1\underline{y}_1 = \underline{0}$ $\quad A\underline{y}_1 = \underline{0} \rightarrow \underline{y}_1 = \underline{0}$

$\alpha_2\underline{y}_1 + \beta_2\underline{y}_2 = \underline{0} \Rightarrow$ solve $\beta_2\underline{y}_2 = -\alpha_2\underline{y}_1$ for $\underline{y}_2$

$\alpha_3\underline{y}_2 + \beta_3\underline{y}_3 = \underline{0} \Rightarrow$ solve $\beta_3\underline{y}_3 = -\alpha_3\underline{y}_2$ for $\underline{y}_3$

$\alpha_4\underline{y}_3 + \beta_4\underline{y}_4 = \underline{b} \Rightarrow$ solve $\beta_4\underline{y}_4 = -\alpha_4\underline{y}_3 + \underline{b}$ for $\underline{y}_4$

Knowing $\begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \end{pmatrix}$ we now solve $U\underline{z} = \underline{y}$

$$\begin{bmatrix} I_1 & \gamma_1 & 0 & 0 \\ 0 & I_2 & \gamma_2 & 0 \\ 0 & 0 & I_3 & \gamma_3 \\ 0 & 0 & 0 & I_4 \end{bmatrix} \begin{pmatrix} \underline{p} \\ \underline{q} \\ \underline{r} \\ \underline{s} \end{pmatrix} = \begin{pmatrix} \underline{y}_1 \\ \underline{y}_2 \\ \underline{y}_3 \\ \underline{y}_4 \end{pmatrix}$$

Backward Subst.

$I_1\underline{p} + \gamma_1\underline{q} = \underline{y}_1$ $\quad \underline{p} = \underline{y}_1 - \gamma_1\underline{q}$ $\quad$ or $\quad \beta_1\underline{p} = \beta_1\underline{y}_1 - \underline{q}$ $\quad$ for $\underline{p}$ (3)

$I_2\underline{q} + \gamma_2\underline{r} = \underline{y}_2$ $\quad \underline{q} = \underline{y}_2 - \gamma_2\underline{r}$ $\quad$ or $\quad \beta_2\underline{q} = \beta_2\underline{y}_2 - \underline{r}$ $\quad$ for $\underline{q}$ (2)

$I_3\underline{r} + \gamma_3\underline{s} = \underline{y}_3$ $\quad \underline{r} = \underline{y}_3 - \gamma_3\underline{s}$ $\quad$ or $\quad \beta_3\underline{r} = \beta_3\underline{y}_3 - \underline{s}$ $\quad$ solve for $\underline{r}$

$I_4\underline{s} = \underline{y}_4$ $\quad \underline{s} = \underline{y}_4$ (1)

For forward subst. $\quad A\underline{y}_1 = \underline{0} \quad \underline{y}_1 = \underline{0}$

$\beta_2\underline{y}_2 = -\alpha_2\underline{y}_1 \Rightarrow \beta_1\beta_2\underline{y}_2 = -\beta_1\underline{y}_1$ or $(\beta_1 A - 2I)\underline{y}_2 = -\beta_1\underline{y}_1$

$\beta_3\underline{y}_3 = -\alpha_3\underline{y}_2 \Rightarrow \beta_2\beta_3\underline{y}_3 = -\beta_2\underline{y}_2$ or $(\beta_2 A - I)\underline{y}_3 = -\beta_2\underline{y}_2$

$\beta_4\underline{y}_4 = -\alpha_4\underline{y}_3 + \underline{b} \Rightarrow \beta_3\beta_4\underline{y}_4 = -\beta_3\underline{y}_3 + \beta_3\underline{b}$ $\quad (\beta_3 A - I)\underline{y}_4 = -\beta_3\underline{y}_3 + \beta_3\underline{b}$

α_s are never computed ; γ_s are not computed ; only βs are computed

use LU decomp. to solve for β_{i+1}
how? use LU decomp on β_i
then use $L_iU_i\beta_{i+1}$ = rhs columns

for example $\quad \beta_1\beta_2 = \beta_1 A - 2I \Rightarrow A\beta_2 = A^2 - 2I$

$(3\times3)\,(\beta_{21}\ \beta_{22}\ \beta_{23}) = (3\times3) \Rightarrow \beta_2$

cols. of β_2 $\quad$ cols of $\beta_1 A - 2I$

$\beta_2\beta_3 = \beta_2 A - I \Rightarrow \beta_3$

$\beta_3\beta_4 = \beta_3 A - I \Rightarrow \beta_4$

Knowing β_i's use them to find $\underset{\sim}{y}_i$'s

$\beta_2\underline{y}_2 = -\alpha_2\underline{y}_1 = -\underline{y}_1 \Rightarrow \underline{y}_2$

$\beta_3\underline{y}_3 = -\underline{y}_2 \Rightarrow \underline{y}_3$

$\beta_4\underline{y}_4 = -\underline{y}_3 + \underline{b} \Rightarrow \underline{y}_4$

Knowing $\underline{y}$ solve the eqs to find $\underline{p}, \underline{q}, \underline{r}, \underline{s}$

from $U\underline{z} = \underline{y}$

$I_4\underline{s} = \underline{y}_4$ $\quad \underline{s} = \underline{y}_4$

$I_3\underline{r} + \gamma_3\underline{s} = \underline{y}_3 \quad \therefore \quad \underline{r} = \underline{y}_3 - \gamma_3\underline{s} \xrightarrow{\times \text{ by } \beta_3} \beta_3\underline{r} = \beta_3\underline{y}_3 - \underline{s}$

$I_2\underline{q} + \gamma_2\underline{r} = \underline{y}_2 \quad \therefore \quad \underline{q} = \underline{y}_2 - \gamma_2\underline{r} \xrightarrow{\times \text{ by } \beta_2} \beta_2\underline{q} = \beta_2\underline{y}_2 - \underline{r}$

$I_1\underline{p} + \gamma_1\underline{q} = \underline{y}_1 \quad \therefore \quad \underline{p} = \underline{y}_1 - \gamma_1\underline{q} \xrightarrow{\times \text{ by } \beta_1} \beta_1\underline{p} = \beta_1\underline{y}_1 - \underline{q}$

change of the momentum of the object struck during the time of the stroke. Let the change in velocity of the point in the interval Δx be equal to v, where v is the initial velocity. Under the assumption that the initial velocity v is constant in Δx, then we obtain the change of momentum,

$$\rho v \Delta x = I ,$$

where ρ is the linear density of the string. Consequently, we must solve the wave equation with the initial velocity,

$$\psi_{\text{int}} = v = \frac{I}{\rho \Delta x} \qquad x, x + \Delta x ,$$

$$\psi_{\text{ext}} = 0 \qquad x, x + \Delta x ,$$

for the initial displacement.

The lengthening obtained by the action of the impulse can be described by a trapezoid whose lower base equals $(2at) + \Delta x$ and whose upper base equals $(2at) - \Delta x$, for $t > \Delta x/2a$. Obviously, the quantity $I/\Delta x = I_0$ can be interpreted as the impulse density. As $\Delta x \to 0$ the following results for the form of the displacement: the lengthening is equal to zero everywhere outside the interval $(x - at, x + at)$, and inside it is equal to $1/2a \cdot 1/\rho$. Loosely speaking, one can say that the displacement is produced by the point impulse I.

We consider now the x, t phase plane (Figure 10) and place the two characteristics through (x_0, t_0):

$$x - at = x_0 - at_0$$
$$x + at = x_0 + at_0 .$$

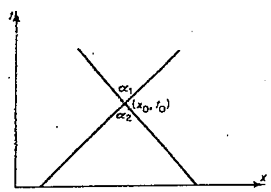


FIG. 10.

They determine two angles α_1 and α_2, the so-called upper and lower characteristic angles at the point (x_0, t_0).

The action of a point impulse at the point (x_0, t_0) produces a lengthening which in the interior of the above characteristic angles equals $1/2a \cdot 1/\rho$ and outside the interval equals zero.

Of interest to us now is the region in which the solution is uniquely defined by the initial conditions when these are prescribed in a given interval PQ of the lines $t = 0$.

Formula (2-2.9) shows that it suffices for the determination of the function u at any point $M(x, t)$ of the x, t phase plane (Figure 7) when the initial conditions in the interval PQ are known. Thus, P, Q are the points of the x axis with the coordinates $x - at$ and $x + at$. The segments MP and MQ of the characteristics passing through the point M and the segment PQ of the x axis form a triangle MPQ called the characteristic triangle of the point M.

If the initial conditions are not given on the entire line $-\infty < x < \infty$ but are given only in a fixed interval PQ, then these initial conditions define

the solution uniquely within the characteristic triangle which has the interval PQ as a base.

3. Stability of the solution

The solution of Eq. (2-2.1) is uniquely determined by the initial conditions (2-2.2). We shall prove that between this solution and the initial conditions exists a continuous dependence and, in fact, we have the theorem:

For each time interval $0 \leq t \leq t_0$ and for arbitrary ε there exists a number $\delta(\varepsilon, t_0)$ such that two solutions $u_1(x, t)$ and $u_2(x, t)$ of Eq. (2-2.1) differ from each other by an amount less than ε:

$$|u_1(x, t) - u_2(x, t)| < \varepsilon , \qquad 0 \leq t \leq t_0 ,$$

provided that the initial values

$$u_1(x, 0) = \varphi_1(x) \qquad \text{and} \qquad u_2(x, 0) = \varphi_2(x)$$
$$\frac{\partial u_1}{\partial t}(x, 0) = \psi_1(x) \qquad\qquad \frac{\partial u_2}{\partial t}(x, 0) = \psi_2(x)$$

differ from each other by an amount less than δ:

$$|\varphi_1(x) - \varphi_2(x)| < \delta , \qquad |\psi_1(x) - \psi_2(x)| < \delta . \tag{2-2.11}$$

The proof of this theorem is surprisingly simple. The functions $u_1(x, t)$ and $u_2(x, t)$ are linked † the initial values by formula (2-2.9), so that

$$|u_1(x, t) - u_2(x, t)| \leq \frac{|\varphi_1(x + at) - \varphi_2(x + at)|}{2}$$
$$+ \frac{|\varphi_1(x - at) - \varphi_2(x - at)|}{2} + \frac{1}{2a} \int_{x-at}^{x+at} |\psi_1(\alpha) - \psi_2(\alpha)| \, d\alpha ,$$

whereas on the basis of the inequality (2-2.11), there follows

$$|u_1(x, t) - u_2(x, t)| \leq \frac{\delta}{2} + \frac{\delta}{2} + \frac{1}{2a} \delta \cdot 2at \leq \delta(1 + t_0) .$$

Hence our assertion is proved if we take

$$\delta = \frac{\varepsilon}{1 + t_0} .$$

Every physically defined process must be capable of description through functions which depend continuously on those initial conditions determining the process. If the solution of a boundary-value problem depends continuously on the initial conditions, then one also says that the boundary-value problem is well set or the solution is stable.

If this continuous dependence did not exist, there could be two essentially different processes corresponding to practically the same set of initial conditions (whose difference lies within the limits of the accuracy of measurement); that is, the solution would not be stable. It cannot be asserted that such processes are determined by the initial conditions (in a physical sense). From the above theorem, it follows that the vibrations of a string are deter-

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5. Solve by Elimination

$$\begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -6 \end{bmatrix}$$

$$A \vdots b = \begin{bmatrix} 5 & -1 & 0 & 9 \\ -1 & 5 & -1 & 4 \\ 0 & -1 & 5 & -6 \end{bmatrix} \longrightarrow \begin{bmatrix} 5 & -1 & 0 & 9 \\ 0 & 24 & -5 & 29 \\ 0 & -1 & 5 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & -1 & 0 & 9 \\ 0 & 24 & -5 & 29 \\ 0 & 0 & 115 & -115 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 5 & -1 & 0 & 9 \\ 0 & 24 & -5 & 29 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & -1 & 0 & 9 \\ 0 & 24 & 0 & 24 \\ 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 5 & 0 & 0 & 10 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$\therefore \quad x_1 = 2 \quad x_2 = 1, \quad x_3 = -1$ ✓

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8. Show that the following set of equations does not have a solution.

$$\left.\begin{aligned} 3x_1 + 2x_2 - x_3 - 4x_4 &= 10 \\ x_1 - x_2 + 3x_3 - x_4 &= -4 \\ 2x_1 + x_2 - 3x_3 - \quad &= 16 \\ - x_2 + 8x_3 - 5x_4 &= 3 \end{aligned}\right\} \quad (1)$$

Solution: The coefficient matrix is

$$A = \begin{bmatrix} 3 & 2 & -1 & -4 \\ 1 & -1 & 3 & -1 \\ 2 & +1 & -3 & 0 \\ 0 & -1 & 8 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & +3 & -1 \\ 3 & 2 & -1 & -4 \\ 2 & 1 & -3 & 0 \\ 0 & -1 & 8 & -5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 3 & -1 \\ 0 & 5 & -10 & -1 \\ 0 & 3 & -9 & 2 \\ 0 & -1 & 8 & -5 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 3 & -1 \\ 0 & -1 & 8 & -5 \\ 0 & 5 & -10 & -1 \\ 0 & 3 & -9 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 3 & -1 \\ 0 & -1 & 8 & -5 \\ 0 & 0 & 30 & -26 \\ 0 & 0 & 15 & -13 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 3 & -1 \\ 0 & -1 & 8 & -5 \\ 0 & 0 & 30 & -26 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

7

$\therefore$ det(A) = 0. A is singular, the equation given doesn't have a solution

how do you know unless you also check b

1

16. (a) Solve: $$AZ = b \qquad (1)$$

A, z and b are complex. Assume the solution exists, let

$$\left.\begin{aligned} A &= B + Ci \\ z &= x + yi \\ b &= p + qi \end{aligned}\right\} \qquad (2)$$

where $i = \sqrt{-1}$. B, C, x, y, p, q are real. Substituting (2) into (1), we have

$$(Bx - Cy) + (Cx + By)i = p + qi$$

or

$$\left.\begin{aligned} Bx - Cy &= p \\ Cx + By &= q \end{aligned}\right\} \quad \checkmark \qquad (3)$$

Assume that B^{-1} and C^{-1} exist. From (3),

$$\left.\begin{aligned} (C^{-1}B + B^{-1}C)\,x &= C^{-1}p + B^{-1}q \\ (C^{-1}B + B^{-1}C)\,y &= C^{-1}q - B^{-1}p \end{aligned}\right\} \qquad (4)$$

normally do not find C^{-1} & B^{-1}

if $D = C^{-1}B + B^{-1}C$, $u = C^{-1}p + B^{-1}q$, $v = C^{-1}q - B^{-1}p$.

$$\left.\begin{aligned} Dx &= u \\ Dy &= v \end{aligned}\right\} \qquad (5)$$

Solving (5), ~~gets~~ we get u and v. Then the solution is

$$z = x + yi \qquad (6)$$

(b). It may be seen from above ~~equation~~. The complex equation is equivalent to two real equation. From complex number, we need to store the data in real and imaginary part. Hence the amount of storage is the same. X

no! A has $4n^2$ elements & $b = 2n$ elements
for real $B + Ci$ $2n^2$ elements & p & q is $2n$ elements

7 ②

Homework #2 $\frac{36}{40}$

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18. Use LU decomposition to solve the set of equations

$$\begin{bmatrix} 1 & 3 & 1 & -1 \\ 2 & 0 & 1 & 1 \\ 0 & -1 & 4 & 1 \\ 0 & 1 & 1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 16 \end{bmatrix}$$

Solution

$$A = LU = \begin{bmatrix} l_{11} & & [0] & \\ l_{21} & l_{22} & & \\ l_{31} & l_{32} & l_{33} & \\ l_{41} & l_{42} & l_{34} & l_{44} \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} & u_{14} \\ & 1 & u_{23} & u_{24} \\ & [0] & 1 & u_{34} \\ & & & u_{44} \end{bmatrix}$$

$l_{11} = a_{11} = 1, \quad l_{21} = a_{21} = 2, \; l_{31} = a_{31} = 0, \quad l_{41} = a_{41} = 0$

$u_{12} = \frac{a_{12}}{l_{11}} = \frac{3}{1} = 3, \quad u_{13} = \frac{a_{13}}{l_{11}} = \frac{1}{1} = 1. \quad u_{14} = \frac{a_{14}}{l_{11}} = -1$

$l_{22} = a_{22} - l_{21} u_{12} = 0 - 2 \cdot 3 = -6$

$l_{32} = a_{32} - l_{31} u_{12} = -1 - 0 = -1$

$l_{42} = a_{42} - l_{41} u_{12} = 1$

$u_{23} = \frac{a_{23} - l_{21} u_{13}}{l_{22}} = \frac{1 - 2 \cdot 1}{-6} = +\frac{1}{6} \qquad u_{24} = \frac{a_{24} - l_{21} u_{14}}{l_{22}} = \frac{1 - 2 \cdot (-1)}{-6} = -\frac{1}{2}$

$l_{33} = a_{33} - l_{31} u_{13} - l_{32} u_{23} = 4 - 0 - (-1) \cdot \frac{1}{6} = \frac{25}{6}$

$l_{43} = a_{43} - l_{41} u_{13} - l_{42} u_{23} = 1 - 0 - 1 \cdot \frac{1}{6} = \frac{5}{6}$

$u_{34} = \frac{a_{34} - l_{31} u_{14} - l_{32} u_{24}}{l_{33}} = \frac{1 - 0 - (-1) \cdot (-\frac{1}{2})}{25/6} = \frac{3}{25}$

$l_{44} = a_{44} - l_{41} u_{14} - l_{42} u_{24} - l_{43} u_{34} = -5 - 0 - 1 \cdot (-\frac{1}{2}) - \frac{5}{6} \cdot \frac{3}{25} = -\frac{23}{5}$

$$\therefore L = \begin{bmatrix} 1 & & [0] & \\ 2 & -6 & & \\ 0 & -1 & \frac{25}{6} & \\ 0 & 1 & \frac{5}{6} & -\frac{23}{5} \end{bmatrix} \checkmark \quad U = \begin{bmatrix} 1 & 3 & 1 & -1 \\ & 1 & \frac{1}{6} & -\frac{1}{2} \\ 0 & & 1 & \frac{3}{25} \\ & & & 1 \end{bmatrix} \checkmark$$

Applying the same transformations to the right-hand-side vector b yields

$$b_1' = \frac{b_1}{l_{11}} = 3$$

$$b_2' = \frac{b_2 - l_{21}b_1'}{l_{22}} = \frac{1-2\cdot 3}{-6} = +\frac{5}{6}$$

$$b_3' = \frac{b_3 - l_{31}b_1' - l_{32}b_2'}{l_{33}} = \frac{6-(-1)\cdot(+\frac{5}{6})}{25/6} = \frac{41}{25}$$

$$b_4' = \frac{b_4 - l_{41}b_1' - l_{42}b_2' - l_{43}b_3'}{l_{44}} = \frac{16 - 1\cdot\frac{5}{6} - \frac{5}{6}\cdot\frac{41}{25}}{-23/5} = -3$$

$\therefore$ $x_4 = b_4' = -3$

$x_3 = b_3' - u_{34}x_4 = \frac{41}{25} - \frac{3}{25}\times(-3) = +\frac{50}{25} = +2$

$x_2 = b_2' - u_{23}x_3 - u_{24}x_4 = \frac{5}{6} - \frac{1}{6}(+2) - (-\frac{1}{2})(-3) = -1$

$x_1 = b_1' - u_{12}x_2 - u_{13}x_3 - u_{14}x_4 = 3 - 3(-1) - 1\cdot(+2) - (-1)(-3) = 1$

$\therefore$ $X = (1, -1, 2, -3)$ ✓ 10

25. b)

$$H : b = \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{25}{12} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{77}{60} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{57}{60} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{319}{420} \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{25}{12} \\ 0 & 0.083 & 0.083 & 0.075 & 0.241 \\ 0 & 0.083 & 0.088 & 0.083 & 0.255 \\ 0 & 0.075 & 0.083 & 0.080 & 0.238 \end{array}\right] \rightarrow \left[\begin{array}{ccccc} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{25}{12} \\ 0 & 0.083 & 0.083 & 0.075 & 0.24 \\ 0 & 0 & 0.005 & 0.008 & 0.014 \\ 0 & 0 & 0.008 & 0.012 & 0.020 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{25}{12} \\ 0 & 0.083 & 0.083 & 0.075 & 0.241 \\ 0 & 0 & 0.005 & 0.008 & 0.014 \\ 0 & 0 & 0 & 0.000 & -0.002 \end{array}\right]$$

Hence, if chopped by only three significant digits, no solution exists. no!
[1.11 .228 1.95 .797]

c). Using only three digits, but rounding

In the same way as above, we have

$$H:b \rightarrow \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{25}{12} \\ 0 & 0.083 & 0.083 & 0.075 & 0.242 \\ 0 & 0 & 0.006 & 0.008 & 0.014 \\ 0 & 0 & 0 & 0.001 & 0.001 \end{pmatrix}$$

.988 1.42 -.428 2.10

$\therefore \; x_1 = 0.994, \quad x_2 = 1.012, \quad x_3 = 1.000 \quad x_4 = 1.000$ X X X X

d). Using five digits

d

$$H:b \rightarrow \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{25}{12} \\ 0 & 0.08333 & 0.08333 & 0.0750 & 0.24167 \\ 0 & 0 & 0.00556 & 0.00833 & 0.01389 \\ 0 & 0 & 0 & 0.00038 & 0.00031 \end{pmatrix}$$

s/b 1.000 .9995 1.0017 .9990

$x_1 = 1.00886, \quad x_2 = 0.89040, \quad x_3 = 1.27598 \quad x_4 = 0.81579$

Compare the solutions obtained. They are very sensitive to round off, i.e. H is ill-conditioned.

a). If we calculate the determinant of matrix H, the value is very small. about 10^{-5} level, i.e. the matrix is nearly singular. ✓

8

33. a). $X = [1.06, -2.15, 14.05, 0.0]$

$\|X\|_1 = \sum_{i=1}^{4} |x_i| = 17.26$

$\|X\|_2 = 14.253$

$\|X\|_\infty = 14.05$

b). $y = [1, 0, 0, 0]$

$\|y\|_1 = 1.0$ $\|y\|_2 = 1.0$, $\|y\|_\infty = 1.0$

c). $z^T = [1, 1, 1]$

$\|z^T\|_1 = 3$ $\|z^T\| = \sqrt{3}$. $\|z^T\|_\infty = 1$

d)

$$A = \begin{bmatrix} 14 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}| = 14$$

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| = 14$$

$$\|A\|_2 = \max(\lambda) = 14$$

$$\|A\|_e = \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2 \right)^{1/2} \doteq 14.457$$

e).

$$B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$\|B\|_1 = 3$ $\|B\|_\infty = 3$ $\|B\|_e = \sqrt{7}$

f).

$$B^2 = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 3 \\ 1 & 2 & 2 \end{bmatrix} \quad B^3 = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 5 & 7 \\ 3 & 4 & 5 \end{bmatrix} \quad B^4 = \begin{bmatrix} 5 & 7 & 9 \\ 9 & 12 & 16 \\ 7 & 9 & 12 \end{bmatrix}$$

$\|B^2\|_e = 4\sqrt{2}$ $\|B^3\|_e = 1$ $\|B^4\|_e = \sqrt{910} = 30.166$

g) . $AB = \begin{bmatrix} 14 & 0 & 14 \\ 3 & 3 & 3 \\ 0 & -2 & -2 \end{bmatrix}$ $\quad Az = \begin{bmatrix} 14 \\ 3 \\ -2 \end{bmatrix}$

$\|AB\|_1 = 19 \quad \|AB\|_\infty = 28. \quad \|AB\|_e = \sqrt{427} \doteq 20.664$

$\|Az\|_1 = 19 \quad \|Az\|_2 = 14.457 \quad \|Az\|_\infty = 14$

h) . $A+B = \begin{bmatrix} 15 & 0 & 1 \\ 1 & 4 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ $\quad B+B = \begin{bmatrix} 2 & 0 & 2 \\ 2 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}$ $\quad X+Y = [2.06, -2.15, 14.05, 0.0]$

$\|A+B\|_1 = 16$ $\quad \|B+B\|_1 = 6.$ $\quad \|X+y\|_1 = 18.26$

$\|A\|_1 + \|B\|_1 = 14+3 = 17 > \|A+B\|_1 = 16$

$\|B\|_1 + \|B\|_1 = 3+3 = 6 = \|B+B\|_1$

$\|X\| + \|y\|_1 = 17.26 + 1.0 = 18.26 = \|X+y\|_1$

$\therefore$ the triangle inequality holds.

8

46 .

$$\begin{bmatrix} 5 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ -6 \end{bmatrix}$$

a). Jacobi method . $x^{(0)} = (0, 0, 0)$ ~~Gauss~~

$$X_i^{(n+1)} = \frac{b_i}{a_{ii}} - \sum_{\substack{j=1 \\ j \neq i}}^{3} \frac{a_{ij}}{a_{ii}} X_j^{(n)}$$

By using the above equation, we have

$x^{(1)} = (1.800, 0.800, -1.2)$

$x^{(2)} = (1.96, 0.92 \quad -1.04)$

$x^{(3)} = (1.992, 0.984, -1.016)$

$x^{(4)} = (1.9968, 0.9952, -1.0032)$

$x^{(5)} = (1.99904, 0.99872, -1.00096)$

$x^{(6)} = (1.99744, 0.999616, -1.000256)$

5

If we take six step of iteration, the solution is

$$X = (1.999744,\ 0.999616,\ -1.000256)$$

It is very close to exact solution $X = (2, 1, -1)$

b). Gauss - Seidel method $\quad x^{(0)} = (0, 0, 0)$

$$X_i^{(n+1)} = \frac{b_i}{a_{ii}} - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} X_j^{(n+1)} - \sum_{j=i+1}^{3} \frac{a_{ij}}{a_{ii}} X_j^{(n)}$$

$$x^{(1)} = (1.8,\ 1.16,\ -0.968)$$

$$x^{(2)} = (2.032,\ 1.0128,\ -0.99744)$$

$$x^{(3)} = (2.00256,\ 1.00102,\ -0.999796)$$

$$x^{(4)} = (2.00020,\ 1.00008,\ -0.99998)$$

. . .

Compare the two method, Gauss - Seidel method converge ~~much~~ faster than Jocobi method to this problem.

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28/30

Homework Qinghua Chen

592-13-5241

57. Solve by using Newton's method

$$x^3+3y^2-21=0$$

$$x^2+2y+2=0$$

$$f(x,y)=x^3+3y^2-21=0 \qquad g(x,y)=x^2+2y+2=0$$

$$f_x'=3x^2 \qquad f_y=6y \qquad g_x=2x \qquad g_y=2$$

initial value $x_0=2.0$, $y_0=-2.2$

$$f_0=f(x_0,y_0)=1.520 \qquad g_0=g(x_0,y_0)=1.600$$

$$\Delta x_1=\begin{vmatrix}-f_0 & f_y\\ -g_0 & g_y\end{vmatrix} \Big/ \begin{vmatrix}f_x & f_y\\ g_x & g_y\end{vmatrix}=-0.3145 \qquad \Delta y_1=\begin{vmatrix}f_x & -f_0\\ g_x & -g_0\end{vmatrix} \Big/ \begin{vmatrix}f_x & f_y\\ g_x & g_y\end{vmatrix}=-0.1708$$

$$\therefore x_1=x_0+\Delta x_1=1.6854. \qquad y_1=y_0+\Delta y_1=-2.3708$$

in the same way as above, we have

$$f_1=f(x_1,y_1)=0.6496 \qquad g_1=g(x_1,y_1)=0.099$$

$$\Delta x_2=-0.0272, \qquad \Delta y_2=0.0207$$

$$x_2=1.6582 \qquad y_2=-2.3501$$

$$f_2=0.1283 \qquad g_2=0.0494$$

$$\Delta x_3=0.0151 \qquad \Delta y_3=0.0003$$

The data below are results by computer. If the tolerance permitted is 10^{-6}, then only four step of ~~ste~~ computation is enough for the given initial value $x_0=2.0$, $y_0=-2.2$

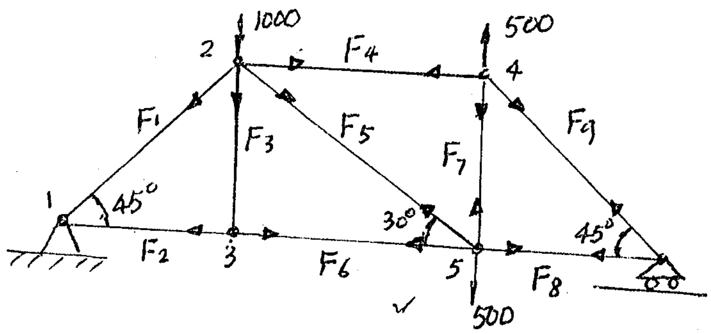
X	Y	DX	DY	FO	GO
2.0000000	-2.2000000	-0.3145834	-0.1708332	1.5200024	1.5999999
1.6854166	-2.3708332	-0.0416675	0.0207457	0.6501942	0.0989628
1.6437490	-2.3500874	-0.0007108	0.0003004	0.0099964	0.0017359
1.6430383	-2.3497870	-0.0000002	0.0000000	0.0000019	0.0000007

For initial value $x_0=-2.0$, $y_0=-3.5$, we have another set of solution

-2.0000000	-3.5000000	-0.0916667	0.3166667	7.7500000	-1.0000000
-2.0916667	-3.1833334	0.0122977	0.0215213	0.2496471	0.0084028
-2.0793691	-3.1618121	0.0000710	0.0000719	0.0004425	0.0001516
-2.0792980	-3.1617401	-0.0000001	-0.0000003	-0.0000038	0.0000000

61. The system is in equilibrium. Hence, we have

$$\Sigma F_x = 0 \qquad \Sigma F_y = 0$$



node 2: $\Sigma F_x = 0$ $\quad -F_1 \times 0.7071 + F_4 + F_5 \cdot 0.866 = 0$

$\Sigma F_y = 0$ $\quad -F_1\, 0.7071 - F_3 - F_5 \cdot 0.5 - 1000 = 0$

node 3: $\Sigma F_x = 0$ $\quad F_6 - F_2 = 0$

$\Sigma F_y = 0$ $\quad F_3 = 0$

node 4: $\Sigma F_x = 0$ $\quad F_9 \times 0.707 - F_4 = 0$

$\Sigma F_y = 0$ $\quad -F_9 \times 0.707 - F_7 + 500 = 0$

node 5: $\Sigma F_x = 0$ $\quad F_8 - F_6 - F_5 \cdot 0.866 = 0$

$\Sigma F_y = 0$ $\quad F_7 + F_5 \cdot 0.5 - 500 = 0$

node 6: $\Sigma F_x = 0$ $\quad -F_8 - F_9 \cdot 0.7071 = 0$

writing in matrix Form, we have

$$\begin{pmatrix} 0.7071 & 0 & 0 & -1 & -0.866 & 0 & 0 & 0 & 0 \\ 0.7071 & 0 & 1 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.7071 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -0.7071 \\ 0 & 0 & 0 & 0 & 0.866 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.7071 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \end{pmatrix} = \begin{pmatrix} 0 \\ -1000 \\ 0 \\ 0 \\ 500 \\ 0 \\ 0 \\ -500 \\ 0 \end{pmatrix}$$

a). Gauss Elimination: the above equation may be written as

$$\begin{bmatrix} 0.7071 & 0 & 0 & -1 & -0.866 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -0.7071 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 & -1 & 0 & 0 & -500 \\ 0 & 0 & 0 & 0 & 0.8660 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.7071 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.7071 & 0 \\ 0.7071 & 0 & 1 & 0 & 0.5 & 0 & 0 & 0 & 0 & -1000 \end{bmatrix}$$

$R_6 - R_5 \cdot \frac{0.866}{0.5}$

$-R_1 + R_3$

$$\begin{bmatrix} 0.7071 & 0 & 0 & -1 & -0.866 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -0.7071 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 & -1 & 0 & 0 & -500 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1.732 & -1 & 0 & -866 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.7071 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.7071 & 0 \\ 0 & 0 & 0 & 1 & 1.366 & 0 & 0 & 0 & 0 & -1000 \end{bmatrix}$$

$R_9 - R_4$

$+R_5 \cdot \frac{1.366}{0.5}$

$+R_7 \cdot 2.732$

$$\begin{bmatrix} 0.7071 & 0 & 0 & -1 & -0.866 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -0.7071 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 & -1 & 0 & 0 & -500 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1.732 & -1 & 0 & -866 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0.7071 & 500 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0.7071 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2.6389 & -1000 \end{bmatrix}$$

Hence, we obtain the solution

$F_1 = -1035.282$ $\quad$ $F_2 = 732.048$

$F_3 = 0.0$ $\quad$ $F_4 = -267.953$

$F_5 = -535.906$ $\quad$ $F_6 = 732.048$

$F_7 = 767.953$ $\quad$ $F_8 = 267.953$

$F_9 = -378.946$

10

(b). As can be seen from the above equation, the matrix cannot be arranged into a diagonally dominant form. If all the starting values are zero, then, the system is not convergent by Gauss-Seidel iteration (Using Program in Book. chapter 2).

65.

$$\left.\begin{aligned}
0.3\sqrt{500-P_1} &= 0.2\sqrt{P_1-P_2} + 0.2\sqrt{P_1-P_3} \\
0.2\sqrt{P_1-P_2} &= 0.1\sqrt{P_2-P_4} + 0.2\sqrt{P_2-P_3} \\
0.1\sqrt{P_1-P_3} &= 0.2\sqrt{P_2-P_3} + 0.1\sqrt{P_3-P_4} \\
0.1\sqrt{P_2-P_4} + 0.1\sqrt{P_3-P_4} &= 0.2\sqrt{P_4-0.0}
\end{aligned}\right\} \qquad (1)$$

Use relaxation method with initial values

$P_1 = 400$. $\quad P_2 = 300$. $\quad P_3 = 200$. $\quad P_4 = 100.00$

Rewrite Eq. (1) as.

$$\left.\begin{aligned}
0.3\sqrt{500-P_1} - 0.2\sqrt{P_1-P_2} - 0.2\sqrt{P_1-P_3} &= R_1 \\
0.2\sqrt{P_1-P_2} - 0.1\sqrt{P_2-P_4} - 0.2\sqrt{P_2-P_3} &= R_2 \\
0.1\sqrt{P_1-P_3} - 0.2\sqrt{P_2-P_3} - 0.1\sqrt{P_3-P_4} &= R_3 \\
0.1\sqrt{P_2-P_4} - 0.1\sqrt{P_3-P_4} - 0.2\sqrt{P_4} &= R_4
\end{aligned}\right\} \qquad (2)$$

If P_1, P_2, P_3 and P_4 are exact value, then $R_1 = R_2 = R_3 = R_4 = 0$ otherwise $R_1 \cdots R_4$ are not equal to zero. With initial value

$P_1 = 400$, $\quad P_2 = 300$, $\quad P_3 = 200$, $\quad P_4 = 100$.

1:	R_i	−1.828	−1.414	2.414	0.4142
	P_i	400.0	300.0	225.0	100.0
2:		−1.64575	−1.1462	1.9369	0.53225
		400.0	300.0	250.0	100.0
3:		−1.4495	−0.8284	1.4142	0.63896
		375.001	300.0	250.0	100.0
4:		−0.61405	−1.09636	1.30751	0.63896
		375.001	300.0	275.0	100.0
5:		−0.37798	−0.68215	0.67713	0.73709
		375.001	300.0	275.0	125.0
6:		−0.37798	−0.5908	0.77526	0.31155
		375.001	300.0	295.679	125.00
7:		−0.1592	−0.00655	−0.000072	0.39325
		375.001	300.0	295.679	148.917
8:		−0.15923	0.08716	0.09492	−0.00019
		370.629	300.0	295.679	148.917
9:		−0.00005	0.035922	0.07002	−0.000019
		370.629	300.0	296.815	148.917
10:		0.013119	0.094732	−0.000051	0.00466
		370.6290	301.4170	296.8150	148.9170
11:		0.030065	−0.000079	0.072062	0.010411
		370.6290	301.4170	298.0160	148.9170
12:		0.044101	0.060130	−0.000092	0.015339
		370.6290	302.3124	298.016	148.9170
13:		0.054899	−0.000008	0.045628	0.018959
		372.1155	302.3124	298.0160	148.9170
14:		−0.000006	0.017880	0.054306	0.018959

Hence, the approximate solution is:

$P_1 = 372.1155$, $P_2 = 302.3124$, $P_3 = 298.0160$, $P_4 = 148.9170$

The corresponding residuals are

$R_1 = -0.000006$, $R_2 = 0.01788$, $R_3 = 0.054306$, $R_4 = 0.018959$.

S/B

423.2	376.15
347.8	308.3
343.5	304.6
172.8	152.9

Method OK but original eqns. are incorrect.

8

```
C
C
C
C     THIS PROGRAM IS TO FIND THE SOLUTION OF NONLINEAR
C         SETS OF EQUATION USING NEWTON'S METHOD
C
C
      EXTERNAL F,G
      READ *,EP1,EP2
      READ *, X,Y
      DX=0.0
      DY=0.0
      PRINT 20
20    FORMAT(//,' X=','    Y=','    DX=','   DY=','   F0=','   G0=',//)
300   F0=F(X,Y)
      G0=G(X,Y)
      D11=-F0*2.0+G0*6.0*Y
      D22=-3.0*X*X*G0+2.0*X*F0
      DD=6.0*X*X-12.0*X*Y
      DX=D11/DD
      DY=D22/DD
      PRINT 100,X,Y,DX,DY,F0,G0
100   FORMAT(1X,F10.7,1X,F10.7,1X,F10.7,1X,F10.7,1X,F10.7,1X,F10.7)
      X=X+DX
      Y=Y+DY
      IF(ABS(DX).GT.EP1.OR.ABS(DY).GT.EP1) GOTO 300
      IF(ABS(F0).GT.EP2.OR.ABS(G0).GT.EP2) GOTO 300
      END
C
C
      FUNCTION F(X,Y)
      F=X**3+3.0*Y*Y-21.0
      RETURN
      END
C
C
      FUNCTION G(X,Y)
      G=X*X+2.0*Y+2.0
      RETURN
      END
```

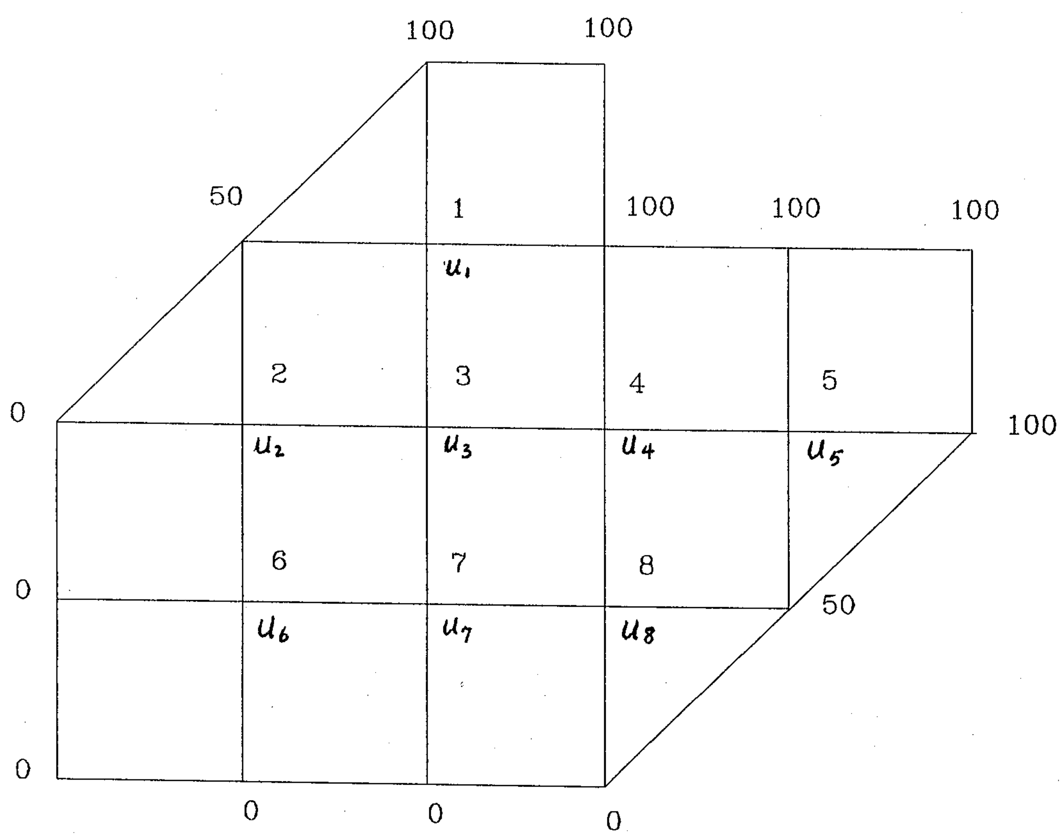
$\frac{40}{40}$

EGM 6422 Homework

Qinghua Chen

592-13-5241

(#13) The meshes are shown in the Figure. We will use the difference



equation to replace the differential equation

$$\nabla^2 u_{ij} = \frac{1}{h^2}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{ij}\right] = 0 \qquad (1)$$

$$\left.\begin{aligned}
&\frac{1}{h^2}(50+100+100+u_3-4u_1)=0\\
&\frac{1}{h^2}(0+u_3+50+u_6-4u_2)=0\\
&\frac{1}{h^2}(u_2+u_4+u_1+u_7-4u_3)=0\\
&\frac{1}{h^2}(u_3+u_5+100+u_8-4u_4)=0\\
&\frac{1}{h^2}(u_4+100+50+100-4u_5)=0\\
&\frac{1}{h^2}(0+u_7+u_2+0-4u_6)=0\\
&\frac{1}{h^2}(u_6+u_8+u_3+0-4u_7)=0\\
&\frac{1}{h^2}(u_7+50+u_4+0-4u_8)=0
\end{aligned}\right\} \qquad (2)$$

If write in matrix form, we have

$$[A][u] = [B]$$

where

$$[u] = [u_1, u_2, u_3, u_4\ u_5, u_6, u_7, u_8]^T$$

$$[B] = [250, 50, 0, 100, 250, 0, 0, 50]^T$$

$$[A] = \begin{bmatrix} 4 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4 & -1 & 0 & 0 & -1 & 0 & 0 \\ -1 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 4 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & -1 & 4 \end{bmatrix}$$

The results may be obtained by computer to be

$$[u] = [74.464, 27.688, 47.856, 65.380, 78.845, 12.895, 23.892, 34.818]^T$$

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2. Liebmann's Method

$$T_{ij}^{(n+1)} = \frac{1}{4}[T_{ij}^{(n)} + T_{i-1,j}^{(n)} + T_{i,j+1}^{(n)} + T_{i,j-1}^{(n)}]$$

$$|T_{ij}^{(n+1)} - T_{ij}^{(n)}| < \varepsilon. \quad \text{(for every point)}$$

The initial values are

$u_1 = 75.0$, $u_2 = 33.0$, $u_3 = 50.0$, $u_4 = 67.0$, $u_5 = 75.0$ $u_6 = 17.0$ $u_7 = 25.0$ $u_8 = 33.0$

$\varepsilon = 10^{-3}$

With 18 step of iteration, the results are

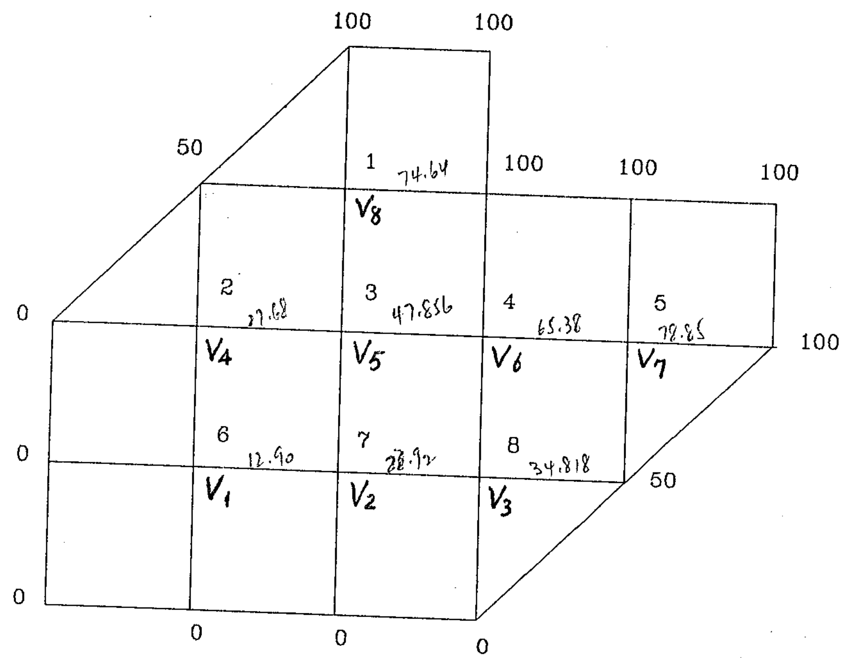
$u_1 = 74.464$, $u_2 = 27.688$, $u_3 = 47.857$. $u_4 = 65.380$, $u_5 = 78.845$. $u_6 = 12.895$. $u_7 = 23.892$

$u_8 = 34.818$.

Comparison shows that the results for two method are the same. 10

(2)

3. Use Southwell method to solve previous problem



$$U_{ij}^{(k+1)} = U_{ij}^{(k)} + \omega\left[\frac{U_{i+1,j}^{(k)} + U_{i-1,j}^{(k+1)} + U_{i,j+1}^{(k)} + U_{i,j-1}^{(k+1)} - 4U_{ij}^{(k)}}{4}\right]$$

$\therefore$

$$V_1^{(k+1)} = V_1^{(k)}(1-\omega) + \frac{\omega}{4}V_2^{(k)} + \frac{\omega}{4}V_4^{(k)}$$

$$V_2^{(k+1)} = V_2^{(k)}(1-\omega) + \frac{\omega}{4}V_1^{(k+1)} + \frac{\omega}{4}V_3^{(k)} + \frac{\omega}{4}V_5^{(k)}$$

$$V_3^{(k+1)} = V_3^{(k)}(1-\omega) + \frac{\omega}{4}V_2^{(k+1)} + \frac{\omega}{4}V_6^{(k)} + \frac{\omega}{4}\cdot 50$$

$$V_4^{(k+1)} = V_4^{k}(1-\omega) + \frac{\omega}{4}V_1^{(k+1)} + \frac{\omega}{4}V_5^{(k)} + \frac{\omega}{4}\cdot 50$$

$$V_5^{(k+1)} = V_5^{(k)}(1-\omega) + \frac{\omega}{4}V_2^{(k+1)} + \frac{\omega}{4}V_4^{(k+1)} + \frac{\omega}{4}V_6^{(k)} + \frac{\omega}{4}V_8^{(k)}$$

$$V_6^{(k+1)} = V_6^{(k)}(1-\omega) + \frac{\omega}{4}V_3^{(k+1)} + \frac{\omega}{4}V_5^{(k+1)} + \frac{\omega}{4}V_7^{(k)} + \frac{\omega}{4}\cdot 100$$

$$V_7^{(k+1)} = V_7^{(k)}(1-\omega) + \frac{\omega}{4}V_6^{(k+1)} + \frac{\omega}{4}\cdot 250$$

$$V_8^{(k+1)} = V_8^{(k)}(1-\omega) + \frac{\omega}{4}V_5^{(k+1)} + \frac{\omega}{4}\cdot 250$$

For over-relaxation factor $\omega = 1.2$, we have (for $\varepsilon = 10^{-3}$)

$u_1 = V_8 = 74.464$ $\quad$ $u_2 = V_4 = 27.688$

$u_3 = V_5 = 47.856$ $\quad$ $u_4 = V_6 = 65.380$

$u_5 = V_7 = 78.845$ $\quad$ $u_6 = V_1 = 12.895$

$u_7 = V_2 = 23.892$ $\quad$ $u_8 = V_3 = 34.818$

It needs 10 step of iteration.

For over-relaxation factor $\omega = 1.4$, we have (for $\varepsilon = 10^{-3}$)

$u_1 = V_8 = 74.464$ $\quad$ $u_2 = V_4 = 27.688$

$u_3 = V_5 = 47.856$ $\quad$ $u_4 = V_6 = 65.380$

$u_5 = V_7 = 78.845$ $\quad$ $u_6 = V_1 = 12.895$

$u_7 = V_2 = 23.892$ $\quad$ $u_8 = V_3 = 34.818$

It needs 14 step of iteration.

10

4. (#47) $\nabla^2 u = xy(x-2)(y-2)$. $0 \le x \le 2,\ 0 \le y \le 2.$

ITERATIONS WITH OVER-RELAXATION FACTOR OF 1.20

AFTER ITERATION NO. 9 MAX CHANGE IN U= 0.0005 U MATRIX IS

1.000	1.000	1.000	1.000	1.000	1.000
0.000	0.455	0.595	0.595	0.455	0.000
0.000	0.224	0.330	0.330	0.224	0.000
0.000	0.110	0.171	0.171	0.110	0.000
0.000	0.046	0.072	0.072	0.045	0.000
0.000	0.000	0.000	0.000	0.000	0.000

ITERATIONS WITH OVER-RELAXATION FACTOR OF 1.30

AFTER ITERATION NO. 9 MAX CHANGE IN U= 0.0003 U MATRIX IS

1.000	1.000	1.000	1.000	1.000	1.000
0.000	0.454	0.595	0.595	0.455	0.000
0.000	0.223	0.329	0.330	0.223	0.000
0.000	0.110	0.170	0.170	0.110	0.000
0.000	0.045	0.072	0.072	0.045	0.000
0.000	0.000	0.000	0.000	0.000	0.000

ITERATIONS WITH OVER-RELAXATION FACTOR OF 1.40

AFTER ITERATION NO. 10 MAX CHANGE IN U= 0.0003 U MATRIX IS

1.000	1.000	1.000	1.000	1.000	1.000
0.000	0.455	0.595	0.595	0.455	0.000
0.000	0.224	0.330	0.330	0.223	0.000
0.000	0.110	0.170	0.170	0.110	0.000
0.000	0.045	0.072	0.072	0.045	0.000
0.000	0.000	0.000	0.000	0.000	0.000

ITERATIONS WITH OVER-RELAXATION FACTOR OF 1.50

AFTER ITERATION NO. 11 MAX CHANGE IN U= 0.0005 U MATRIX IS

1.000	1.000	1.000	1.000	1.000	1.000
0.000	0.455	0.595	0.595	0.455	0.000
0.000	0.223	0.329	0.329	0.224	0.000
0.000	0.110	0.170	0.170	0.110	0.000
0.000	0.045	0.072	0.072	0.045	0.000
0.000	0.000	0.000	0.000	0.000	0.000

ITERATIONS WITH OVER-RELAXATION FACTOR OF 1.60

AFTER ITERATION NO. 14 MAX CHANGE IN U= 0.0007 U MATRIX IS

1.000	1.000	1.000	1.000	1.000	1.000
0.000	0.455	0.595	0.595	0.454	0.000
0.000	0.224	0.330	0.330	0.223	0.000
0.000	0.110	0.170	0.170	0.110	0.000
0.000	0.045	0.072	0.072	0.045	0.000
0.000	0.000	0.000	0.000	0.000	0.000

ITERATIONS WITH OVER-RELAXATION FACTOR OF 1.70

AFTER ITERATION NO. 19 MAX CHANGE IN U= 0.0009 U MATRIX IS

1.000	1.000	1.000	1.000	1.000	1.000
0.000	0.454	0.595	0.594	0.454	0.000
0.000	0.223	0.329	0.329	0.223	0.000
0.000	0.110	0.170	0.170	0.110	0.000
0.000	0.045	0.072	0.072	0.045	0.000
0.000	0.000	0.000	0.000	0.000	0.000

In the above calculation, we take h=0.4 and the tolerance to 0.001. Comparison of the results shows that the good relaxation factors are about 1.2 and 1.3 because in these two cases the iteration will stop after nine step. if the factor is greater than 1.4, the increase of the factor will increase the iteration step, thus the computer time.

Good!

10

```
C
C
C
C  ................................................
C   THIS PROGRAM IS USED TO SOLVE POISSON'S EQUATION
C  ................................................
C
C
      DIMENSION U(100,100)
      F(X,Y)=X*Y*(X-2.0)*(Y-2.0)
      PRINT *,'NWIDE,NHIGH,TOL,W,H'
      READ *,NWIDE,NHIGH,TOL,W,H
      NHP1=NHIGH+1
      NWP1=NWIDE+1
      DO 1,I=1,NHP1
      U(I,1)=0.0
      U(1,NWP1)=0.0
1     CONTINUE
      DO 2 I=1,NWIDE
      U(1,I)=1.0
      U(NHP1,I)=0.0
2     CONTINUE
5     SUM=0.0
      DO 10 I=1,NHP1
      SUM=SUM+U(I,1)+U(I,NWP1)
10    CONTINUE
      DO 20 I=2,NWIDE
      SUM=SUM+U(1,I)+U(NHP1,I)
20    CONTINUE
      UAVG=SUM/FLOAT(2*NWP1+2*(NHIGH-1))
      X=0.0
      Y=0.0
      DO 30 I=2,NHIGH
      DO 30 J=2,NWIDE
      U(I,J)=UAVG+H*H*F(X,Y)
30    CONTINUE
      OPEN(6,FILE='CHEN62.DAT',STATUS='NEW')
      WRITE(6,199),W
      DO 50 KNT=1,100
      CHGMAX=0.0
      DO 40 I=2,NHIGH
      DO 35 J=2,NWIDE
      RESID=W/4.0*(U(I+1,J)+U(I-1,J)+U(I,J+1)+U(I,J-1)-
     1      4.0*U(I,J)+H*H*F(X,Y))
      IF(CHGMAX.LT.ABS(RESID)) CHGMAX=ABS(RESID)
      U(I,J)=U(I,J)+RESID
35    CONTINUE
40    CONTINUE
      IF(CHGMAX.LT.TOL) GOTO 55
50    CONTINUE
55    WRITE(6,200),KNT,CHGMAX
      DO 45 I=1,NHP1
```

```
      WRITE(6,201),(U(I,J),J=1,NWP1)
45    CONTINUE
      W=W+0.1
      IF(W.LT.1.8) GOTO 5
199   FORMAT(///' ITERATIONS WITH OVER-RELAXATION FACTOR OF',F5.2)
200   FORMAT(/' AFTER ITERATION NO.',I3,' MAX CHANGE IN U=',
     1       F8.4,' U MATRIX IS'/)
201   FORMAT(1X,9F8.3)
      STOP
      END
```

Good! 30/30

Advanced Analysis of Mechanical System

Chapter 5

Qinghua Chen

592-13-5241

56-57.

THE RESULTS OF ADAMS-MOULTON'S METHOD

STEP SIZE H= 0.10000
Y... NUMERICAL SOLUTION, AY... THE ANALYTICAL SOLUTION

X=	Y=	AY=
0.000000E+00	0.000000E+00	0.000000E+00
0.100000E+00	0.100335E+00	0.100335E+00
0.200000E+00	0.202710E+00	0.202710E+00
0.300000E+00	0.309336E+00	0.309336E+00
0.400000E+00	0.422798E+00	0.422793E+00
0.500000E+00	0.546309E+00	0.546302E+00
0.600000E+00	0.684144E+00	0.684137E+00
0.700000E+00	0.842291E+00	0.842288E+00
0.800000E+00	0.102962E+01	0.102964E+01
0.900000E+00	0.126007E+01	0.126016E+01
0.100000E+01	0.155708E+01	0.155741E+01
0.110000E+01	0.196360E+01	0.196476E+01
0.120000E+01	0.256771E+01	0.257215E+01
0.130000E+01	0.358147E+01	0.360210E+01
0.140000E+01	0.565819E+01	0.579788E+01
0.150000E+01	0.118689E+02	0.141014E+02
0.160000E+01	0.555047E+02	-0.342325E+02

Compare the numerical solution and the analytical results, if $x \le 1.0$, the error is about $o(h^4)$, if $x > 1.0$, the error increases as the value of x increases. if $1.5 \le x \le 1.6$, the error is very big compared with the analytical solution.

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THE RESULTS OF R-K METHOD WITH DIFFERENT STEP SIZE

STEP SIZE H= 0.04000
Y... NUMERICAL SULUTION, AY...

X=	Y=	AY=
0.000000E+00	0.000000E+00	0.000000E+00
0.400000E-01	0.400213E-01	0.400213E-01
0.800000E-01	0.801711E-01	0.801711E-01
0.120000E+00	0.120579E+00	0.120579E+00
0.160000E+00	0.161379E+00	0.161379E+00
0.200000E+00	0.202710E+00	0.202710E+00
0.240000E+00	0.244717E+00	0.244717E+00
0.280000E+00	0.287554E+00	0.287554E+00
0.320000E+00	0.331389E+00	0.331389E+00
0.360000E+00	0.376403E+00	0.376403E+00
0.400000E+00	0.422793E+00	0.422793E+00
0.440000E+00	0.470781E+00	0.470781E+00
0.480000E+00	0.520611E+00	0.520611E+00
0.520000E+00	0.572562E+00	0.572562E+00
0.560000E+00	0.626950E+00	0.626950E+00
0.600000E+00	0.684137E+00	0.684137E+00
0.640000E+00	0.744544E+00	0.744544E+00
0.680000E+00	0.808661E+00	0.808661E+00
0.720000E+00	0.877068E+00	0.877068E+00
0.760000E+00	0.950451E+00	0.950451E+00
0.800000E+00	0.102964E+01	0.102964E+01
0.840000E+00	0.111563E+01	0.111563E+01
0.880000E+00	0.120966E+01	0.120966E+01
0.920000E+00	0.131326E+01	0.131326E+01
0.960000E+00	0.142836E+01	0.142836E+01
0.100000E+01	0.155741E+01	0.155741E+01
0.104000E+01	0.170361E+01	0.170361E+01
0.108000E+01	0.187122E+01	0.187122E+01
0.112000E+01	0.206595E+01	0.206596E+01
0.116000E+01	0.229580E+01	0.229580E+01
0.120000E+01	0.257215E+01	0.257215E+01
0.124000E+01	0.291193E+01	0.291193E+01
0.128000E+01	0.334134E+01	0.334135E+01
0.132000E+01	0.390333E+01	0.390335E+01
0.136000E+01	0.467338E+01	0.467344E+01
0.140000E+01	0.579770E+01	0.579788E+01
0.144000E+01	0.760108E+01	0.760183E+01
0.148000E+01	0.109789E+02	0.109834E+02
0.152000E+01	0.196057E+02	0.196695E+02
0.156000E+01	0.783376E+02	0.926205E+02
0.160000E+01	0.665064E+07	-0.342325E+02

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STEP SIZE H= 0.10000
y... numerical solution, ay...analytical solution

X=	Y=	ay=
0.000000E+00	0.000000E+00	0.000000E+00
0.100000E+00	0.100335E+00	0.100335E+00
0.200000E+00	0.202710E+00	0.202710E+00
0.300000E+00	0.309336E+00	0.309336E+00
0.400000E+00	0.422793E+00	0.422793E+00
0.500000E+00	0.546302E+00	0.546302E+00
0.600000E+00	0.684137E+00	0.684137E+00
0.700000E+00	0.842289E+00	0.842288E+00
0.800000E+00	0.102964E+01	0.102964E+01
0.900000E+00	0.126016E+01	0.126016E+01
0.100000E+01	0.155741E+01	0.155741E+01
0.110000E+01	0.196475E+01	0.196476E+01
0.120000E+01	0.257207E+01	0.257215E+01
0.130000E+01	0.360156E+01	0.360210E+01
0.140000E+01	0.579197E+01	0.579788E+01
0.150000E+01	0.138367E+02	0.141014E+02
0.160000E+01	0.665267E+03	-0.342325E+02

STEP SIZE H= 0.20000
Y... NUMERICAL SOLUTION, AY... THE ANALYTICAL SOLUTION

X=	Y=	AY=
0.000000E+00	0.000000E+00	0.000000E+00
0.200000E+00	0.202707E+00	0.202710E+00
0.400000E+00	0.422789E+00	0.422793E+00
0.600000E+00	0.684133E+00	0.684137E+00
0.800000E+00	0.102964E+01	0.102964E+01
0.100000E+01	0.155735E+01	0.155741E+01
0.120000E+01	0.257092E+01	0.257215E+01
0.140000E+01	0.574109E+01	0.579788E+01
0.160000E+01	0.972103E+02	-0.342325E+02

For $x < 1.0$. The errors of RK method and Adams-Moulton method are all about $O(h^4)$. For $x > 1.0$, with same step size, RK method gives better accuracy compared with analytical Results. For $x < 1.0$, if we control the error to $O(h^4)$, there is no improvement for reducing step size. but for $x > 1.0$, reducing step size will improve the accuracy.

THE RESULTS OF R-K-F METHOD FOR DIFFERENT STEP SIZE

STEP SIZE= 0.4000000E-01

X=	Y=	Y1=	E=
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.4000000E-01	0.4002135E-01	0.3801540E-01	0.3254990E-10
0.8000000E-01	0.8017110E-01	0.7814667E-01	0.3372546E-10
0.1200000E+00	0.1205793E+00	0.1185297E+00	0.3783692E-10
0.1600000E+00	0.1613795E+00	0.1592975E+00	0.4485034E-10
0.2000000E+00	0.2027100E+00	0.2005883E+00	0.5473174E-10
0.2400000E+00	0.2447167E+00	0.2425471E+00	0.6742973E-10
0.2800000E+00	0.2875543E+00	0.2853282E+00	0.8284877E-10
0.3200000E+00	0.3313894E+00	0.3290972E+00	0.1008088E-09
0.3600000E+00	0.3764029E+00	0.3740341E+00	0.1209840E-09
0.4000000E+00	0.4227932E+00	0.4203361E+00	0.1428108E-09
0.4400000E+00	0.4707805E+00	0.4682218E+00	0.1653476E-09
0.4800000E+00	0.5206108E+00	0.5179355E+00	0.1870608E-09
0.5200000E+00	0.5725618E+00	0.5697528E+00	0.2054965E-09
0.5600000E+00	0.6269495E+00	0.6239871E+00	0.2167715E-09
0.6000000E+00	0.6841368E+00	0.6809979E+00	0.2147784E-09
0.6400000E+00	0.7445438E+00	0.7412015E+00	0.1899297E-09
0.6800000E+00	0.8086614E+00	0.8050837E+00	0.1271497E-09
0.7200000E+00	0.8770679E+00	0.8732169E+00	0.2619813E-11
0.7600000E+00	0.9504515E+00	0.9462811E+00	-0.2215801E-09
0.8000000E+00	0.1029639E+01	0.1025093E+01	-0.6064474E-09
0.8400000E+00	0.1115632E+01	0.1110643E+01	-0.1251388E-08
0.8800000E+00	0.1209664E+01	0.1204144E+01	-0.2320980E-08
0.9200000E+00	0.1313264E+01	0.1307105E+01	-0.4092861E-08
0.9600000E+00	0.1428357E+01	0.1421421E+01	-0.7045787E-08
0.1000000E+01	0.1557408E+01	0.1549512E+01	-0.1202671E-07
0.1040000E+01	0.1703615E+01	0.1694517E+01	-0.2057911E-07
0.1080000E+01	0.1871217E+01	0.1860591E+01	-0.3561483E-07
0.1120000E+01	0.2065955E+01	0.2053343E+01	-0.6285354E-07
0.1160000E+01	0.2295799E+01	0.2280546E+01	-0.1140771E-06
0.1200000E+01	0.2572152E+01	0.2553286E+01	-0.2149629E-06
0.1240000E+01	0.2911930E+01	0.2887950E+01	-0.4253949E-06
0.1280000E+01	0.3341350E+01	0.3309807E+01	-0.8970710E-06
0.1320000E+01	0.3903348E+01	0.3859988E+01	-0.2055921E-05
0.1360000E+01	0.4673442E+01	0.4610226E+01	-0.5265809E-05
0.1400000E+01	0.5797886E+01	0.5697790E+01	-0.1572184E-04
0.1440000E+01	0.7601833E+01	0.7422423E+01	-0.5857534E-04
0.1480000E+01	0.1098345E+02	0.1058851E+02	-0.3064993E-03
0.1520000E+01	0.1967140E+02	0.1837542E+02	-0.2664193E-02
0.1560000E+01	0.8841963E+02	0.7649356E+02	0.7007215E+00
0.1600000E+01	0.3256551E+20	-0.1360240E+12	0.3256551E+20

STEP SIZE= 0.1000000E+00

X=	Y=	Y1=	E=
0.0000000E+00	0.0000000E+00	0.0000000E+00	0.0000000E+00
0.1000000E+00	0.1003347E+00	0.9524138E-01	0.3039591E-08
0.2000000E+00	0.2027100E+00	0.1973193E+00	0.3702521E-08
0.3000000E+00	0.3093363E+00	0.3035132E+00	0.5937592E-08
0.4000000E+00	0.4227932E+00	0.4163633E+00	0.9399885E-08
0.5000000E+00	0.5463025E+00	0.5390285E+00	0.1314363E-07
0.6000000E+00	0.6841369E+00	0.6756813E+00	0.1438579E-07
0.7000000E+00	0.8422885E+00	0.8321475E+00	0.5020880E-08
0.8000000E+00	0.1029639E+01	0.1017019E+01	-0.3884675E-07
0.9000000E+00	0.1260159E+01	0.1243730E+01	-0.1910615E-06
0.1000000E+01	0.1557409E+01	0.1534763E+01	-0.6998821E-06
0.1100000E+01	0.1964762E+01	0.1931077E+01	-0.2513833E-05
0.1200000E+01	0.2572157E+01	0.2516338E+01	-0.1000342E-04
0.1300000E+01	0.3602127E+01	0.3492782E+01	-0.4977196E-04
0.1400000E+01	0.5798128E+01	0.5510191E+01	-0.3683328E-03
0.1500000E+01	0.1409941E+02	0.1264613E+02	-0.1759596E-02
0.1600000E+01	0.8648825E+03	0.5207021E+03	0.2100224E+03

NOTE:
Y ..... CORRECTED VALUES
Y1..... PREDICTED VALUES
E ..... ERROR

For $x<1.0$, the error of RKF ~~method~~ and Adams-Moulton is about $O(h^4)$. This is because the two method all have local errors $O(h^5)$. Comparing with the analytical solutions, RKF gives better results. If we control the error to $O(h^5)$, then for $x<1.2$, there is no improvement for step size changing. But for $x>1$, reducing step size may improve~~ment~~ the accuracy; example: $x=1.4$ analytical value $y=5.79788$, $y_{h=0.1}=5.798128$. $y_{h=0.04}=5.797886$.

```
C
C
C  ***********************************************
C   THIS PROGRAM IS TO USE THE ADAMS-MOULTON
C   FORMULAR TO SOLVING FIRST ORDER DIFFERENTIAL
C   EQUATION. THE 4TH ORDER RUNGE-KUTTA PROGRAM
C   IS CALLED TO OBTAIN THE STARTING VALUES
C  ***********************************************
C
C
      SUBROUTINE RK4TH (XX0,YY0,H,YY)
C      F........ EXTERNAL FUNCTION TO EVALUATE F(XX,YY)
C      XX0,YY0.... VALUES AT THE BEGINNING OF THE INTERVAL
C      H ....... STEP SIZE
C      YY ....... YY VALUE AT THE END OF THE INTERVAL
      IMPLICIT REAL*8 (A-H,O-Z)
      F(A,B)=1.000+b*b
      XK1=H*F(XX0,YY0)
      XK2=H*F(XX0+H/2.00,YY0+XK1/2.00)
      XK3=H*F(XX0+H/2.00,YY0+XK2/2.00)
      XK4=H*F(XX0+H,YY0+XK3)
      YY=YY0+(XK1+2.00*XK2+2.00*XK3+XK4)/6.00
      RETURN
      END
C
C
C
      SUBROUTINE ADMTON(X,H,F0,F1,F2,F3,F4,Y0,Y1,PY1)
C      H ........ STEP SIZE
C      F0........ FN-3
C      F1........ FN-2
C      F2........ FN-1
C      F3........ FN
C      F4........ FN+1
C      Y0........ YN
C      Y1........ YN+1 CORRECTED VALUE
      IMPLICIT REAL*8 (A-H,O-Z)
      F(A,B)=1.0+b*b
      PY1=Y0+H*(55.0*F3-59.0*F2+37.0*F1-9.0*F0)/24.0
      F4=F(X+H,PY1)
      Y1=Y0+H*(9.0*F4+19.0*F3-5.0*F2+F1)/24.0
      RETURN
      END
C
C
C      ................................
C             MAIN PROGRAM
C      ................................
C
C

      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(30),Y(30),YP(30)
      F(A,B)=1.0+b*b
      READ *, H,X0,Y0
```

```
      OPEN (4,FILE='CHEN16.DAT',STATUS='NEW')
      X(1)=X0
      Y(1)=Y0
      F0=F(X0,Y0)
      DO 100 I=2,4
      X(I)=X(I-1)+H
      X01=X(I-1)
      Y01=Y(I-1)
      CALL RK4TH (X01,Y01,H,Y11)
      Y(I)=Y11
100   CONTINUE
      F1=F(X(2),Y(2))
      F2=F(X(3),Y(3))
      F3=F(X(4),Y(4))
      DO 200 I=5,30
      X(I)=X(I-1)+H
      X00=X(I-1)
      Y00=Y(I-1)
      CALL ADMTON(X00,H,F0,F1,F2,F3,F4,Y00,Y1,PY1)
      Y(I)=Y1
      YP(I)=PY1
      F0=F1
      F1=F2
      F2=F3
      F3=F4
200   CONTINUE
      do 400 i=1,30
400   yp(i)=tan(x(i))
      WRITE(4,300),H
300   FORMAT(//,'  STEP SIZE H=', F12.5,/,'y... numerical
     1 solution,  ay...analytical solution',/)
      WRITE(4,310)
310   FORMAT(/,'   X=          Y=          ay=',/)
      DO 320 I=1,30
      WRITE(4,330),X(I),Y(I),YP(I)
330   FORMAT(3X,E15.6,3X,E15.6,3X,E15.6)
320   CONTINUE
      END
C
C
C
```

```
C
C
C     ........................................
C         RUNGE-KUTTA-FEHLBERG METHOD
C     ........................................
C
C
      SUBROUTINE RKF(H,X0,Y0,Y,Y1,E)
      IMPLICIT REAL*8(A-H,O-Z)
      REAL*8 K1,K2,K3,K4,K5,K6
      F(A,B)=-10.0*B
      K1=H*F(X0,Y0)
      K2=H*F(X0+H/4.0,Y0+K1/4.0)
      K3=H*F(X0+3.0*H/8.0,Y0+3.0*(K1+3.0*K2)/32.0)
      K4=H*F(X0+12.0*H/13.0,Y0+(1932.0*K1-7200.0*K2+
     1 7296.0*K3)/2197.0)
      K5=H*F(X0+H,Y0+439.0*K1/216.0-8.0*K2+
     1 3680.0*K3/513.0-845.0*K4/4104.0)
      K6=H*F(X0+.5*H,Y0-8.0*K1/27.0+2.0*K2-3544.0*K3/2565.0
     1 +1859.0*K4/4104.0-11.0*K5/40.0)
      Y1=Y0+(25.0*K1/216.0+1408.0*K2/2565.0+2197.0*K4/4104.0
     1 -K5/4.0)
      Y=Y0+(16.0*K1/135.0+6656.0*K3/12825.0+
     1 28561.0*K4/56430.0-9.0*K5/50.0+2.0*K6/55.0)
      E=K1/360.0-128.0*K3/4275.0-2197.0*K4/75240.0
     1 +K5/50.0+2.0*K6/55.0
      RETURN
      END
C
C
C     ......................................
C         MAIN PROGRAM
C     ......................................
C
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION X(50),Y(50),Y1(50),E(50)
      F(A,B)=-10.0*B
      READ *,H,X0,Y0
      X(1)=X0
      Y(1)=Y0
      OPEN (6,FILE='CHEN18.DAT',STATUS='NEW')
      DO 200 I=2,30
      X(I)=X(I-1)+H
      X00=X(I-1)
      Y00=Y(I-1)
      CALL RKF(H,X00,Y00,YY,YY1,EE)
      Y(I)=YY
      Y1(I)=YY1
      E(I)=EE
200   CONTINUE
      WRITE(6,300),H
300   FORMAT(//,2X,' STEP SIZE=',E14.7,/)
      WRITE(6,310)
```

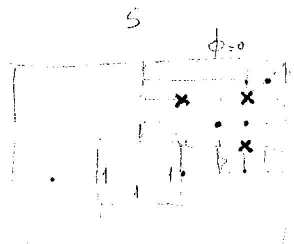
```
310  FORMAT(2X,'X=          Y=             Y1=           E=',/)
     DO 320 I=1,30
     WRITE(6,330),X(I),Y(I),Y1(I),E(I)
330  FORMAT(2X,E14.7,2X,E14.7,2X,E14.7,2X,E14.7)
320  CONTINUE
     END
```

$a=1 \quad b=1 \quad c=-1$

Do #13 pg 621

$\frac{1 \pm \sqrt{1-4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2} = \frac{dt}{dx}$

Prob. 22 in p539



1 DO. IN CHAPTER 9 PG 621 PROB. 14

- DETERMINE CHARACTERISTICS

ALSO FOR $u(x,t=0) = 2x \qquad u(0,t)=0$

$\frac{\partial u}{\partial t}(x,t=0) \qquad u(1,t)=2$

FIND $u(x,t)$ at $x=0.4$ $t=2$, @ $x=0.5$ $t=3$, at $x=0.6$ $t=5$sec

@ $x=0.3007$ $t=1$

2. DO PROBLEM ~~55~~ 32 on PG 540 in Chapter 7 ~~AB I~~

Southwell: $\omega = 1.4 - 1.5$

3. a. DEVELOP ALGORITHM FOR SAUL'YEV B (ie SHOW HOW EQN 2-267 IS FOUND)

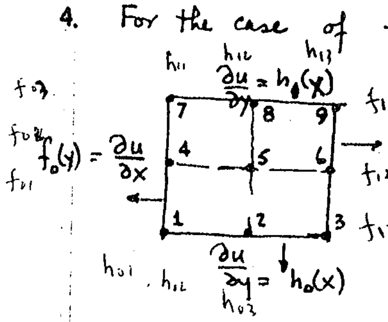
b. SHOW THAT SAUL'YEV A IS UNCONDITIONALLY STABLE BY VON NEUMANN AN.

c. FIND THE TRUNCATION ERRORS FOR SAUL'YEV A & B — first 3 terms of truncation error.

HINT: $u_{i,j+1} = u_{i,j} + \Delta t \frac{\partial u}{\partial t}(i,j) + \frac{\Delta t^2}{2}\frac{\partial^2 u}{\partial t^2}(i,j) \ldots$

$\ldots + \Delta t\, \alpha \frac{\partial^2 u}{\partial x^2}(i,j) + \frac{\Delta t^2}{2}\frac{\partial}{\partial t}\left[\alpha \frac{\partial^2 u}{\partial x^2}(i,j)\right] \ldots$

4. For the case of the problem $\nabla^2 u = 0$ in the square $0 \le x \le 1$ $\quad 0 \le y \le 1$ with conditions. let $\Delta x = \Delta y = h = \frac{1}{2}$



a. Find the equations at pts 1–9 using CD for the derivative at boundary and write them in the form $\underset{\sim}{P}\,\underset{\sim}{u} = \underset{\sim}{b}$ where $\underset{\sim}{u}^T = [u_1, u_2, \ldots, u_9]$

b. Show that $P = \begin{bmatrix} A & 2I & 0 \\ I & A & I \\ 0 & 2I & A \end{bmatrix}$ where $A, I, 0$ are 3x3 matrices

c. Show that $\det P = 0$ and no solution exists for system as discretized

counterclockwise

d. Show that $\iint \nabla^2 u\, dx\,dy = \oint \frac{\partial u}{\partial n} ds = 0$ from green's theorem (divergence theorem) and show that this gives the condition that $\int_0^{2h} h_0\, dx + \int_0^{2h} f_1\, dy + \int_{2h}^0 h_1\, dx + \int_{2h}^0 f_0\, dy = 0$

Midterm Examination

Qinghua Chen

592 - 13 - 5241

Prob #1	40
#2	50
#3	50
#4	35
	175/200

1. Show Dufort – Frankel Scheme is stable

Dufort – Frankel method

$$\frac{U_{i,j+1} - U_{i,j-1}}{2\Delta t} = \frac{\alpha}{\Delta X^2}\left[U_{i-1,j} - U_{i,j-1} - U_{i,j+1} + U_{i+1,j} \right]$$

Let $r = \frac{\alpha \Delta t}{\Delta X^2}$ Rearranging the above equation, we have

$$(1+2r)\, U_{i,j+1} = (1-2r)\, U_{i,j-1} + 2r\,(U_{i-1,j} + U_{i+1,j}) \qquad (1)$$

Assumption. $U_{i,j}$ ---- numerical value

V_{ij} ---- exact value

$e_{ij} = V_{ij} - U_{ij}$ error.

We can find the following equation for e_{ij}

$$(1+2r)\, e_{ij} = (1-2r)\, e_{ij-1} + 2r\,(e_{i-1,j} + e_{i+1,j}) \qquad (2)$$

Use von Neumann's method.

$$e_{ij} = \xi^{j} e^{I i k \Delta x} \qquad I = \sqrt{-1} \qquad (3)$$

Substituting (3) into (2), we obtain

$$(1+2r)\, \xi^{j+1} - 4r\cos k\Delta x\, \xi^{j} - (1-2r)\, \xi^{j-1} = 0$$

or
$$(1+2r)\, \xi^{2} - 4r\cos k\Delta x\, \xi - (1-2r) = 0 \qquad (4)$$

According to von Neumann's method, the necessary and sufficient conditions for stability is

$$|\xi| \leq 1 \qquad (5)$$

From (4), we have

$$\xi_{1,2} = \frac{2r\cos k\Delta x \pm \sqrt{1 - 4r^2 \sin^2 k\Delta x}}{1+2r} \qquad (6)$$

if $\cos k\Delta x = -1$ $\sin k\Delta x = 0$ $\qquad \xi_{1,2} = \frac{-2r \pm 1}{1+2r}$ or -1 & $\frac{1-2r}{1+2r} < 1$

$\cos k\Delta x = 1$ $\sin k\Delta x = 0$ $\qquad = \frac{2r \pm 1}{\,}$ or 1 & $-\left(\frac{1-2r}{\,}\right) > -1$

how can you prove this

From (6), we know that for all r, $|\xi_1| \le 1$ and $|\xi_2| \le 1$.

Hence, the DuFort -- Frankel scheme is unconditionally stable.

2 (#3) $\Delta x = 1$ in . For $k\Delta t / c\rho(\Delta x)^2 = \frac{1}{2}$

$$\Delta t = \frac{c\rho(\Delta x)^2}{2k} = \frac{0.0919 \times 0.322 \cdot 1^2}{2 \cdot 0.00517} = 2.862 \text{ sec}$$ ✓

Boundary Conditions:

$u(0,t) = \cancel{0}°C = 32\,°F \qquad U(10,t) = \cancel{0°C} = 32\,°F$

Initial conditions :

$u(x,0) = 1.8x + 32 \quad °F$

Use explicit method , For $k\Delta t / c\rho(\Delta x)^2 = \frac{1}{2}$

$$u_{i,j+1} = \frac{1}{2}(u_{i+1,j} + u_{i-1,j})$$

If expressed in cgs units, we have

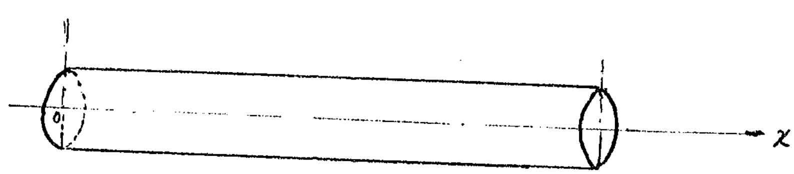
$\Delta x = 2.54$ cm . $\quad K = 0.9233$ cal/sec·cm·°C

$c = 0.0918$ cal/g·°C . $\rho = 8.921$ (g/cm³)

$$\Delta t = \frac{0.0918 \cdot 8.921 \times 2.54^2}{2 \cdot 0.9233} \doteq 2.862 \text{ sec}.$$

Boundary conditions: $\quad u(0,t) = 0°C \;,\quad u(25.4,t) = 0°C$

Initial Conditions $\quad u(x,0) = 10x$



The results are listed as follow for two unit cases.

E

CALCULATED TEMPERATUR(°C) AT x=:

t=	0.00	2.54	5.08	7.62	10.16	12.70	15.24	17.78	20.32	22.86	25.40
0.00	0.00	10.00	20.00	30.00	40.00	50.00	60.00	70.00	80.00	90.00	0.00
2.862	0.00	10.00	20.00	30.00	40.00	50.00	60.00	70.00	80.00	40.00	0.00
5.724	0.00	10.00	20.00	30.00	40.00	50.00	60.00	70.00	55.00	40.00	0.00
8.586	0.00	10.00	20.00	30.00	40.00	50.00	60.00	57.50	55.00	27.50	0.00
11.448	0.00	10.00	20.00	30.00	40.00	50.00	53.75	57.50	42.50	27.50	0.00
14.310	0.00	10.00	20.00	30.00	40.00	46.88	53.75	48.13	42.50	21.25	0.00
17.172	0.00	10.00	20.00	30.00	38.44	46.88	47.50	48.13	34.69	21.25	0.00
20.034	0.00	10.00	20.00	29.22	38.44	42.97	47.50	41.09	34.69	17.34	0.00
22.896	0.00	10.00	19.61	29.22	36.09	42.97	42.03	41.09	29.22	17.34	0.00
25.758	0.00	9.80	19.61	27.85	36.09	39.06	42.03	35.63	29.22	14.61	0.00
28.620	0.00	9.80	18.83	27.85	33.46	39.06	37.34	35.63	25.12	14.61	0.00

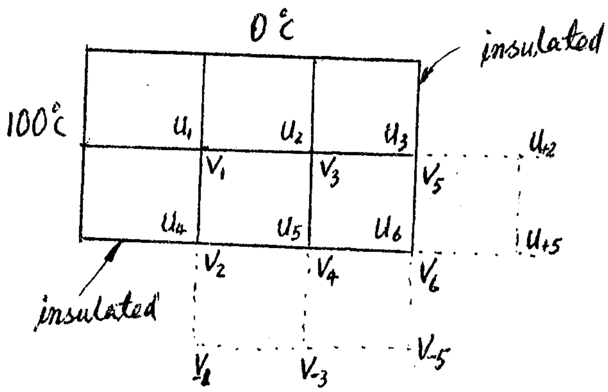
E

CALCULATED TEMPERATUR(°F) At x=:

t=	0.00	1.00	2.00	3.00	4.00	5.00	6.00	7.00	8.00	9.00	10.00
0.00	32.00	50.00	68.00	86.00	104.00	122.00	140.00	158.00	176.00	194.00	32.00
2.862	32.00	50.00	68.00	86.00	104.00	122.00	140.00	158.00	176.00	104.00	32.00
5.724	32.00	50.00	68.00	86.00	104.00	122.00	140.00	158.00	131.00	104.00	32.00
8.586	32.00	50.00	68.00	86.00	104.00	122.00	140.00	135.50	131.00	81.50	32.00
11.448	32.00	50.00	68.00	86.00	104.00	122.00	128.75	135.50	108.50	81.50	32.00
14.310	32.00	50.00	68.00	86.00	104.00	116.38	128.75	118.63	108.50	70.25	32.00
17.172	32.00	50.00	68.00	86.00	101.19	116.38	117.50	118.63	94.44	70.25	32.00
20.034	32.00	50.00	68.00	84.59	101.19	109.34	117.50	105.97	94.44	63.22	32.00
22.896	32.00	50.00	67.30	84.59	96.97	109.34	107.66	105.97	84.59	63.22	32.00
25.758	32.00	49.65	67.30	82.13	96.97	102.31	107.66	96.13	84.59	58.30	32.00
28.620	32.00	49.65	65.89	82.13	92.22	102.31	99.22	96.13	77.21	58.30	32.00

```
C
C
C ..................................................
C   THIS PROGRAM IS TO USE EXPLICIT METHOD TO SOLVE
C      PARABOLIC PARTIAL DIFFERENTIAL EQUATION
C ..................................................
C
C
      DIMENSION U(100),V(100)
      DATA DT,DX/2.862,2.54/
      FLT=0.0
      FRT=0.0
      II=0
      T=0.0
      DO 10 I=1,11
      U(I)=10.0*FLOAT(I-1)
10    CONTINUE
      OPEN (6,FILE='CHEN12.DAT',STATUS='NEW')
      WRITE(6,20)
20    FORMAT(//,'  ..........................................',
     1    /,'        CALCULATED TEMPERATUR AT:      ',
     2    /,'  ..........................................')
      WRITE(6,30)
30    FORMAT(3X,'t=',1x,'X=0',1X,'x=1.0',1x,'x=2.0',1x,'x=3.0',1x,
     1    'X=4.0',1x,'x=5.0',1x,'x=6.0',1x,'x=7.0',1x,'x=8.0',
     2    1X,'x=9.0',1x,'x=10.0')
80    WRITE(6,50) T,(U(I),I=1,11)
50    FORMAT(1X,F6.3,11(1X,F6.2))
      DO 60 I=2,10
      V(I)=U(I)
60    CONTINUE
      V(1)=FLT
      V(11)=FRT
      DO 66 I=2,10
      U(I)=(V(I+1)+V(I-1))/2.0
66    CONTINUE
      U(1)=FLT
      U(11)=FRT
      IF(II.EQ.10) GOTO 18
      T=T+DT
      II=II+1
      GOTO 80
18    AAA=0.0
      END
```

3. (#21)



$\Delta x = 2.54$ cm

$K = 0.9223$ cal/sec·cm °C

$\rho = 8.921$ g/cm²

$c = 0.0918$ cal/g·°C

$r = 2.0$

$$\Delta t = \frac{2\rho c\,\Delta x^2}{K} = 11.445 \text{ sec.}$$

Use A.D.I. Method $\Delta x = \Delta y$. $r = 2.0$

in. x-direction: $-r\,u_{i+1,j,n+1} + (1+2r)\,u_{i,j,n+1} - r\,u_{i-1,j,n+1} = r\,u_{i,j+1,n} + (1-2r)\,u_{i,j,n} + r\,u_{i,j-1,n}$

$$\left.\begin{aligned}
-2u_2 + 5u_1 - 200 &= 0 - 3V_1 + 2V_2 \\
-2u_3 + 5u_2 - 2u_1 &= 0 - 3V_3 + 2V_4 \\
-2u_{+2} + 5u_3 - 2u_2 &= 0 - 3V_5 + 2V_6 \\
-2u_5 + 5u_4 - 200 &= 2V_1 - 3V_2 + V_{-1} \\
-2u_6 + 5u_5 - 2u_4 &= 2V_3 - 3V_4 + 2V_{-3} \\
-2u_{+5} + 5u_6 - 2u_5 &= 2V_5 - 3V_6 + 2V_{-5}
\end{aligned}\right\} \qquad (1)$$

If using central difference, we have from insulated side

$$\frac{u_{+2} - u_2}{2\Delta x} = 0$$

$$\left.\begin{aligned}
u_{+2} &= u_2 \\
u_{+5} &= u_5 \\
V_{-1} &= V_1 \\
V_{-3} &= V_3 \\
V_{-5} &= V_5
\end{aligned}\right\} \qquad (2)$$

Substituting Eq.(2) into (1) yields (in matrix form)

$$[A][U] = [B][V] + [Fi] \qquad (3)$$

where

$$[U] = [u_1, u_2, u_3, u_4, u_5, u_6]^T$$

$$[V] = [v_1, v_2, v_3, v_4, v_5, v_6]^T$$

$$[Fi] = [200, 0, 0, 200, 0, 0]^T$$

$$[A] = \begin{bmatrix} 5 & -2 & 0 & 0 & 0 & 0 \\ -2 & 5 & -2 & 0 & 0 & 0 \\ 0 & -4 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & -2 & 0 \\ 0 & 0 & 0 & -2 & 5 & -2 \\ 0 & 0 & 0 & 0 & -4 & 5 \end{bmatrix} \quad [B] = \begin{bmatrix} -3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 2 \\ 4 & -3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -3 \end{bmatrix}$$

in y-direction

$$-r\,u_{i,j+1,n+2} + (1+2r)\,u_{i,j,n+2} - r\,u_{i,j-1,n+2} = r\,u_{i+1,j,n+1} + (1-2r)\,u_{i,j,n+1} + r\,u_{i-1,j,n+1}$$

$$\left.\begin{aligned} 0 + 5V_1 - 2V_2 &= 2U_2 - 3U_1 + 200 \\ -2V_1 + 5V_2 - 2V_{-1} &= 2U_5 - 3U_4 + 200 \\ 0 + 5V_3 - 2V_4 &= 2U_3 - 3U_2 + 2U_1 \\ -2V_3 + 5V_4 - 2V_{-3} &= 2U_6 - 3U_5 + 2U_4 \\ 0 + 5V_5 - 2V_6 &= 2U_{+2} - 3U_3 + 2U_2 \\ -2V_5 + 5V_6 - 2V_{-5} &= 2U_{+5} - 3U_6 + 2U_5 \end{aligned}\right\} \qquad (4)$$

Substituting Eqs (2) into (4), we have in matrix form

$$[C][V] = [D][U] + [F_2] \qquad (5)$$

where $[F_2] = [\ 200,\ 200,\ 0,\ 0,\ 0,\ 0\ ]$

$$[C] = \begin{bmatrix} 5 & -2 & 0 & 0 & 0 & 0 \\ -4 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & -2 & 0 & 0 \\ 0 & 0 & -4 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & -2 \\ 0 & 0 & 0 & 0 & -4 & 5 \end{bmatrix} \quad [D] = \begin{bmatrix} -3 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 2 & 0 \\ 2 & -3 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -3 & 2 \\ 0 & 4 & -3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & -3 \end{bmatrix}$$

With initial ~~condition~~ value. $V_1 = V_2 = V_3 = V_4 = V_5 = V_6 = 50°C$, we may solve $[U'']$ from (3). And then substituting $[U'']$ into (5), we may solve $[V^{(2)}]$ . . . . .

The results are listed as follow.

CALCULATED TEMPERATURE (C) BY A.D.I. METHOD

t, sec	u1 or v1	u2 or v3	u3 or v5	u4 or v2	u5 or v4	u6 or v6
0.0000	50.0000	50.0000	50.0000	50.0000	50.0000	50.0000
11.4450*	28.4615	-3.8462	-13.0769	76.1538	65.3846	62.3077
22.8900	43.484	21.946	15.792	55.249	33.710	27.557
34.3350*	47.8086	29.4988	25.1465	51.1960	23.8949	15.2155
45.7800	45.074	24.079	18.391	54.900	31.491	24.699
57.2250*	43.8793	22.4098	16.7731	56.6379	33.7958	26.9297
68.6700	44.780	23.640	17.964	55.360	32.061	25.250
80.1150*	44.8755	23.9994	18.5211	55.2289	31.5514	24.4623
91.5600	44.806	23.731	18.106	55.328	31.931	25.049
103.0050*	44.7053	23.6440	18.0709	55.4694	32.0540	25.0990
114.4500	44.780	23.709	18.097	55.364	31.962	25.062
125.8950*	44.7676	23.7250	18.1463	55.3814	31.9394	24.9924
137.3400	44.777	23.713	18.110	55.368	31.956	25.044

```
C
C
C .....................................................
C     THIS PROGRAM IS TO USE A.D.I. METHOD TO SOLVE
C       PARABOLIC PARTIAL DIFFERENTIAL EQUATION
C                ( FOR r=2 )
C .....................................................
C
C
      DIMENSION A(6,6),B(6,6),C(6,6),D(6,6),U(6),V(6),F1(6),G(6)
      DIMENSION F2(6)
      DATA DT,DX/11.445,2.54/
      T=0.0
      JJ=0
      DO 10 I=1,6
      F1(I)=0.0
      F2(I)=0.0
      DO 20 J=1,6
      A(I,J)=0.0
      B(I,J)=0.0
      C(I,J)=0.0
      D(I,J)=0.0
20    CONTINUE
10    CONTINUE
      F2(1)=200.0
      F2(2)=200.0
      F1(1)=200.0
      F1(4)=200.0
      A(1,1)=5.0
      A(1,2)=-2.0
      A(2,1)=-2.0
      A(2,2)=5.0
      A(2,3)=-2.0
      A(3,2)=-4.0
      A(3,3)=5.0
      A(4,4)=5.0
      A(4,5)=-2.0
      A(5,4)=-2.0
      A(5,5)=5.0
      A(5,6)=-2.0
      A(6,5)=-4.0
      A(6,6)=5.0
      B(1,1)=-3.0
      B(1,2)=2.0
      B(2,3)=-3.0
      B(2,4)=2.0
      B(3,5)=-3.0
      B(3,6)=2.0
      B(4,1)=4.0
      B(4,2)=-3.0
      B(5,3)=4.0
      B(5,4)=-3.0
```

```
      B(6,5)=4.0
      B(6,6)=-3.0
      DO 100 I=1,6
      C(I,I)=5.0
100   CONTINUE
      C(1,2)=-2.0
      C(2,1)=-4.0
      C(3,4)=-2.0
      C(4,3)=-4.0
      C(5,6)=-2.0
      C(6,5)=-4.0
      D(1,1)=-3.0
      D(1,2)=2.0
      D(2,4)=-3.0
      D(2,5)=2.0
      D(3,1)=2.0
      D(3,2)=-3.0
      D(3,3)=2.0
      D(4,4)=2.0
      D(4,5)=-3.0
      D(4,6)=2.0
      D(5,2)=4.0
      D(5,3)=-3.0
      D(6,5)=4.0
      D(6,6)=-3.0
      DO 32 I=1,6
      V(I)=50.0
32    CONTINUE
      OPEN(6,FILE='CHEN61.DAT',STATUS='NEW')
      WRITE(6,50)
50    FORMAT(//,'     CALCULATED TEMPERATURE (  C) BY A.D.I. METHOD',//
     1 't sec    u or v    u or v    u or v    u or v  u or v  u or v',/)
      WRITE(6,56),T,(V(I),I=1,6)
56    FORMAT(1X,F8.4,6(3X,F8.4))
60    DO 36 I=1,6
      SUM=0.0
      DO 38 J=1,6
      SUM=SUM+B(I,J)*V(J)
38    CONTINUE
      G(I)=SUM+F1(I)
36    CONTINUE
      T=T+DT
      CALL LSARG(6,A,6,G,1,U)
      WRITE(6,53),T,(U(I),I=1,6)
53    FORMAT(1X,F8.4,6(3X,F8.4))
      JJ=JJ+1
      DO 37 I=1,6
      SUM=0.0
      DO 39 J=1,6
      SUM=SUM+D(I,J)*U(J)
39    CONTINUE
      G(I)=SUM+F2(I)
37    CONTINUE
      T=T+DT
```

```
      CALL LSARG(6,C,6,G,1,V)
      WRITE(6,55),T,V(1),V(3),V(5),V(2),V(4),V(6)
55     FORMAT(1X,F8.4,6(3X,F8.3))
      JJ=JJ+1
      IF(JJ.LT.12) GOTO 60
      END
```

4. The formulae for A.D.I. Method are given below. (Class notes)

x:
$$-ru_{i+1,j,n+1}+(1+2r)u_{i,j,n+1}-ru_{i-1,j,n+1}=ru_{i,j+1,n}+(1-2r)u_{i,j,n}+ru_{i,j-1,n} \quad (1)$$

y:
$$-ru_{i,j+1,n+2}+(1+2r)u_{i,j,n+1}-ru_{i,j-1,n+1}=ru_{i+1,j,n}+(1-2r)u_{i,j,n}+ru_{i-1,j,n} \quad (2)$$

$$u_{i,j,n}=U^n e^{I\,ik\Delta x}\,e^{Ijk\Delta y} \quad (3)$$

not proper form $I=\sqrt{-1}$ based on separation of variables

(3) → (1).
$$[-re^{Ik\Delta x}+(1+2r)-re^{-Ik\Delta x}]U^{n+1}=[re^{Ik\Delta y}+(1-2r)+re^{-Ik\Delta y}]U^n$$

$$\therefore\quad U^{n+1}=\frac{1-2r(1-\cos k\Delta y)}{1+2r(1-\cos k\Delta x)}U^n=\frac{1-4r\sin^2\frac{k\Delta y}{2}}{1+4r^2\sin^2\frac{k\Delta x}{2}}U^n \quad (4)$$

$$\left|\frac{1-4r\sin^2\frac{k\Delta y}{2}}{1+4r^2\sin^2\frac{k\Delta x}{2}}\right|\le 1 \quad \text{for } r>0$$ Hence, the method is stable for $r>0$.

If we advance for the first step from t_n to $t_{n+\frac{1}{2}}$ and for the second step from $t_{n+1/2}$ to t_{n+1}. (See Numerical Methods for Scientific and Engineering Computation by M. K. Jain, et al).

$$-ru_{i+1,j,n+\frac{1}{2}}+(1+2r)u_{i,j,n+\frac{1}{2}}-ru_{i-1,j,n+\frac{1}{2}}=ru_{i,j+1,n}+(1-2r)u_{i,j,n}+ru_{i,j-1,n} \quad (5)$$

$$-ru_{i,j+1,n+1}+(1+2r)u_{i,j,n+1}-ru_{i,j-1,n+1}=ru_{i+1,j,n+\frac{1}{2}}+(1-2r)u_{i,j,n+\frac{1}{2}}+ru_{i,j-1,n+\frac{1}{2}} \quad (6)$$

In the same way as above, (3) → (5), (6),

x:
$$[-re^{Ik\Delta x}+(1+2r)-re^{-Ik\Delta x}]U^{n+\frac{1}{2}}=[re^{Ik\Delta y}+(1-2r)+e^{-Ik\Delta y}]U^n \quad (7)$$

y:
$$[-re^{Ik\Delta y}+(1+2r)-re^{-Ik\Delta y}]U^{n+1}=[re^{Ik\Delta x}+(1-2r)+e^{-Ik\Delta x}]U^{n+\frac{1}{2}} \quad (8)$$

Eliminating $U^{n+\frac{1}{2}}$ from (7) and (8) yields

$$U^{n+1} = U^n \frac{[1-2r(\cancel{4}1-\cos k\Delta y)][1-2r(1-\cos k\Delta x)]}{[1+2r(1-\cos k\Delta x)][1+2r(1-\cos k\Delta y)]}$$

$$= U^n \frac{\left(1-4r\sin^2\frac{k\Delta y}{2}\right)\left(1-4r\sin^2\frac{k\Delta x}{2}\right)}{\left(1+4r\sin^2\frac{k\Delta x}{2}\right)\left(1+4r\sin^2\frac{k\Delta y}{2}\right)} = U^n \xi$$

Since $0 \le \sin^2\frac{k\Delta x}{2}, \sin^2\frac{k\Delta y}{2} \le 1$ and $r > 0$, the conditions $|\xi| \le 1$ is always satisfied. Hence, the ~~comp~~ combined scheme (ADI) is unconditionally stable.

If used separately, let's discuss (7). In the same way as above, we have

$$[-re^{Ik\Delta x} + (1+2r) - re^{-Ik\Delta x}]U^{n+\frac{1}{2}} = [re^{Ik\Delta y} + (1-2r) + e^{-Ik\Delta y}]U^n$$

or
$$[1+2r(1-\cos k\Delta x)]U^{n+\frac{1}{2}} = [1-2r(1-\cos k\Delta y)]U^n$$

$$[1+2r(1-\cos k\Delta x)]^2 \overset{(U^{n+1/2})^2}{\cancel{U^{2n+1}}} = [1-2r(1-\cos k\Delta y)]^2 \cancel{U^{2n}}(U^n)^2$$

$$\xi^2 = \frac{[1-2r(1-\cos k\Delta y)]^2}{[1+2r(1-\cos k\Delta x)]^2} \le 1$$ Δx & Δy are related through r

$$[1-2r(1-\cos k\Delta y)]^2 \le [1+2r(1-\cos k\Delta x)]^2$$

$$\cancel{1} - 8r\sin^2\frac{k\Delta y}{2} + 16r^2\sin^4\frac{k\Delta y}{2} \le \cancel{1} + 8r\sin^2\frac{k\Delta x}{2} + 16r^2\sin^4\frac{k\Delta x}{2}$$

$$+r\left(\sin^2\frac{k\Delta y}{2} + \sin^2\frac{k\Delta x}{2}\right)\left[1+2r\left(\sin^2\frac{k\Delta x}{2} - \sin^2\frac{k\Delta y}{2}\right)\right] \ge 0$$

$$\therefore\ 1 + 2r\left(\sin^2\frac{k\Delta x}{2} - \sin^2\frac{k\Delta y}{2}\right) \ge 0 \qquad (9)$$

If $\sin^2\frac{k\Delta x}{2} > \sin^2\frac{k\Delta y}{2}$. the condition is always satisfied.
It's stable.

If $\sin^2\frac{k\Delta x}{2} = \sin^2\frac{k\Delta y}{2}$. stable.

If $\sin^2\frac{k\Delta x}{2} < \sin^2\frac{k\Delta y}{2}$, from (9), we have

$$\gamma \le \frac{1}{2\left(\sin^2\frac{k\Delta y}{2} - \sin^2\frac{k\Delta x}{2}\right)} \tag{10}$$

From (10), we know that the sckeme is conditionally stable. If the step size for Δx and Δy is not taking properly, it may not converge.