solutions to difference equations

when $y' + 5y = 5$ is replaced by $y_{i+1} + 10 y_i h - y_{i-1} = 10h$

using $y' = \frac{y_{i+1} - y_{i-1}}{2h}$

this is a difference equation. As $h \to 0$ the difference equation should lead to the solution to the original equation.

In general the solution to homog. linear differ eqn is in the form.

$$y_i = c\beta^i \Rightarrow y_{i+1} + 10 y_i h - y_{i-1} = 0 \Rightarrow c\beta^{i-1}\left[\beta^2 + 10h\beta - 1\right] = 0$$

$$\beta = \frac{-10h \pm \sqrt{(10h)^2 - 4(1)(-1)}}{2} = -5h \pm \sqrt{1 + (5h)^2}$$

$$= -5h \pm \left(1 + \tfrac{1}{2}(5h^2) + \dots\right) = \begin{matrix} 1 - 5h \\ -1 - 5h \end{matrix} + O(h^2)$$

$$\therefore\ y_i = c_1 (1 - 5h)^i + c_2 (-1)^i (1 + 5h)^i \qquad \text{if } x_i = ih$$

$$= c_1 \left(1 - \frac{5x_i}{i}\right)^i + c_2 (-1)^i \left(1 + \frac{5x_i}{i}\right)^i$$

$$\lim_{i \to \infty} \left(1 + \frac{5x_i}{i}\right)^i = \lim_{i \to \infty} \left(1 + \overset{\epsilon}{\frac{5x_i}{i}}\right)^{\frac{i}{5x_i} \cdot 5x_i} = e^{5x_i} \qquad e = \lim_{\epsilon \to 0} (1+\epsilon)^{1/\epsilon}$$

$$\lim_{i \to \infty} \left(1 - \frac{5x_i}{i}\right)^i = \lim_{i \to \infty} \left(1 - \frac{5x_i}{i}\right)^{-\frac{i}{5x_i} \cdot -5x_i} = e^{-5x_i}$$

$$\therefore\ y_i = c_1 e^{-5x_i} + c_2 e^{+5x_i} (-1)^i$$

for the particular $y_{i+1} + 10 y_i h - y_{i-1} = 10h$ let $y_i^p = c$ (const)

$$c + 10hc - c = 10h \qquad \therefore\ c = 1$$

$\therefore$

$$y_i = c_1 e^{-5x_i} + c_2 e^{+5x_i} (-1)^i + 1$$

special equation $y' = \lambda y$, λ constant, is usually considered sufficient, however, to give an indication of the stability of a method.

We consider first the Adams-Bashforth fourth-order method. If in (8.47) we set $f(x, y) = \lambda y$ we obtain

$$y_{n+1} - y_n - \frac{h\lambda}{24}(55y_n - 59y_{n-1} + 37y_{n-2} - 9y_{n-3}) = 0 \tag{8.73}$$

The characteristic equation for this difference equation is

$$\beta^4 - \beta^3 - \frac{h\lambda}{24}(55\beta^3 - 59\beta^2 + 37\beta - 9) = 0$$

The roots of this equation are of course functions of $h\lambda$. It is customary to write the characteristic equation in the form

$$\rho(\beta) + h\lambda\sigma(\beta) = 0 \tag{8.74}$$

where $\rho(\beta)$ and $\sigma(\beta)$ are polynomials defined by

$$\rho(\beta) = \beta^4 - \beta^3$$

$$\sigma(\beta) = -\frac{1}{24}(55\beta^3 - 59\beta^2 + 37\beta - 9)$$

We see that as $h \to 0$, (8.74) reduces to $\rho(\beta) = 0$, whose roots are $\beta_1 = 1$, $\beta_2 = \beta_3 = \beta_4 = 0$. For $h \neq 0$, the general solution of (8.73) will have the form

$$y_n = c_1\beta_1^n + c_2\beta_2^n + c_3\beta_3^n + c_4\beta_4^n$$

where the β_i are solutions of (8.74). It can be shown that β_1^n approaches the desired solution of $y' = \lambda y$ as $h \to 0$ while the other roots correspond to extraneous solutions. Since the roots of (8.74) are continuous functions of h, it follows that for h small enough, $|\beta_i| < 1$ for $i = 2, 3, 4$, and hence from the definition of stability that the Adams-Bashforth method is strongly stable. All multistep methods lead to a characteristic equation in the form (8.74) whose left-hand side is sometimes called the **stability polynomial**. The definition of stability can be recast in terms of the stability polynomial. A method is **strongly stable** if all the roots of $\rho(\beta) = 0$ have magnitude less than one except for the simple root $\beta = 1$.

We investigate next the stability properties of Milne's method (8.64*b*) given by

$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1}) \tag{8.75}$$

Again setting $f(x, y) = \lambda y$ we obtain

$$y_{n+1} - y_{n-1} - \frac{h\lambda}{3}(y_{n+1} + 4y_n + y_{n-1}) = 0$$

and its characteristic equation becomes

$$\rho(\beta) + h\lambda\sigma(\beta) = 0 \tag{8.76}$$

with

$$\rho(\beta) = \beta^2 - 1$$

$$\sigma(\beta) = \beta^2 + 4\beta + 1$$

This time $\rho(\beta) = 0$ has the roots $\beta_1 = 1$, $\beta_2 = -1$, and hence by the definition above, Milne's method is not strongly stable. To see the implications of this we compute the roots of the stability polynomial (8.76). For h small we have

$$\beta_1 = 1 + \lambda h + \mathcal{O}(h^2)$$

$$\beta_2 = -(1 - \lambda h/3) + \mathcal{O}(h^2) \tag{8.77}$$

Hence the general solution of (8.75) is

$$y_n = c_1(1 + \lambda h + \mathcal{O}(h^2))^n + c_2(-1)^n(1 - \lambda h/3 + \mathcal{O}(h^2))^n$$

If we set $n = x_n/h$ and let $h \to 0$, this solution approaches

$$y_n = c_1 e^{\lambda x_n} + c_2(-1)^n e^{-\lambda x_n/3} \tag{8.78}$$

In this case stability depends upon the sign of λ. If $\lambda > 0$ so that the desired solution is exponentially increasing, it is clear that the extraneous solution will be exponentially decreasing so that Milne's method will be stable. On the other hand if $\lambda < 0$, then Milne's method will be unstable since the extraneous solution will be exponentially increasing and will eventually swamp the desired solution. Methods of this type whose stability depends upon the sign of λ for the test equation $y' = \lambda y$ are said to be **weakly stable**. For the more general equation $y' = f(x, y)$ we can expect weak stability from Milne's method whenever $\partial f/\partial y < 0$ on the interval of integration.

In practice all multistep methods will exhibit some instability for some range of values of the step h. Consider, for example, the Adams-Bashforth method of order 2 defined by

$$y_{n+1} = y_n + \frac{h}{2}\{3f_n - f_{n-1}\}$$

If we apply this method to the test equation $y' = \lambda y$, we will obtain the difference equation

$$y_{n+1} - y_n - \frac{h\lambda}{2}\{3y_n - y_{n-1}\} = 0$$

and from this the stability polynomial

$$\beta^2 - \beta - \frac{h\lambda}{2}\{3\beta - 1\}$$

or the equation

$$\beta^2 - \left(1 + \frac{3h\lambda}{2}\right)\beta + \frac{h\lambda}{2} = 0$$

MILNE'S

$$y_{i+1} = y_{i-1} + \frac{\Delta x}{3}\left(f_{i+1} + 4f_i + f_{i-1}\right)$$

$$f(x,y) \cong \lambda y$$

$$y_{i+1} = y_{i-1} + \frac{\lambda \Delta x}{3}\left(y_{i+1} + 4y_i + y_{i-1}\right)$$

$$y_i = \mathscr{C}\beta^i$$

$$\mathscr{C}\beta^{i-1}\left[\left(\beta^2 - 1\right) - \frac{\lambda\Delta x}{3}\left(\beta^2 + 4\beta + 1\right)\right] = 0$$

$$= \beta^2\left(1 - \frac{\lambda\Delta x}{3}\right) - \frac{4\lambda\Delta x}{3}\beta - \left(1 + \frac{\lambda\Delta x}{3}\right)$$

$$\beta = \frac{\frac{4\lambda\Delta x}{3} \pm \sqrt{\left(\frac{4\lambda\Delta x}{3}\right)^2 + 4\left(1 - \left(\frac{\lambda\Delta x}{3}\right)^2\right)}}{2\left(1 - \frac{\lambda\Delta x}{3}\right)}$$

$$\beta_1 \cong 1 + \lambda\Delta x \qquad x_i = i\Delta x$$

$$\beta_2 \cong -\left(1 - \frac{\lambda\Delta x}{3}\right)$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

$$\sqrt{1-x} \approx 1 - \frac{x}{2}$$

$$y_i = \mathscr{C}_1 e^{\lambda x_i} + \mathscr{C}_2 (-1)^i e^{-\lambda x_i/3}$$

$$y' = f(x,y) \cong \lambda y \qquad y' = \lambda y \qquad y = Ce^{sx} \quad y' = sCe^{sx} \Rightarrow \lambda = s$$

$$y = \mathscr{C}e^{\lambda x}$$

$$\left.\begin{array}{l} \text{if } \lambda > 0 \longrightarrow \text{actual soln} \\ \phantom{\text{if }} \lambda < 0 \longrightarrow \text{diverge} \end{array}\right\} \text{weakly stable}$$

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if $|\beta_i| < 1 \Longrightarrow$ strongly stable numerical techn

if $|\beta_i| > 1 \Longrightarrow$ weakly stable, possibly unstable what about $y' = f(x,y)$

$$\frac{y_{i+1} - y_i}{\Delta x} = f(x_i, y_i)$$

$$y_{i+1} = y_i + \Delta x\, f_i$$

normally we can't solve so we use $y_{i+1} = y_i + \Delta x \cdot \lambda y_i$ where $f_i \cong \lambda y_i$

for Adams - Bashforth $y_{i+1} = y_i + \frac{\Delta x \lambda}{24}\left(55 y_i - 59 y_{i-1} + 37 y_{i-2} - 9 y_{i-3}\right)$

if $y_i = \beta^i$ then $0 = \beta^{i+1} - \beta^i - \frac{\Delta x \lambda}{24}\left(55\beta^i - 59\beta^{i-1} + 37\beta^{i-2} - 9\beta^{i-3}\right)$

or $\underbrace{\beta^4 - \beta^3}_{\epsilon} - \frac{\Delta x \lambda}{24}\left(55\beta^3 - 59\beta^2 + 37\beta - 9\right) = 0$

$1 + 4\epsilon - (1 + 3\epsilon) - \frac{\Delta x}{24}\lambda\left(55 \cdot 1^{(1+\epsilon)} - 59 \cdot 1^{(1+\epsilon)} + 37 \cdot 1^{(1+\epsilon)} - 9\right) \quad -(24 + 84\epsilon)\frac{\Delta x \lambda}{24} + \epsilon \cong 0 \qquad \epsilon\left(1 - \frac{84\Delta x \lambda}{24}\right) = 24\frac{\Delta x \lambda}{24}$

$\rho(\beta) + \Delta x \lambda\, \sigma(\beta) = 0$; as $\Delta x \to 0$ this reduces to $\rho(\beta) = 0$

$\therefore \beta^3(\beta - 1) = 0$ or $\beta_{1,2,3} = 0$ & $\beta_4 = 1$ $\quad \beta$

$$\therefore y_i = c_1 \beta_1^i + c_2 i \beta_2^i + c_3 i^2 \beta_3^i + c_4 \beta_4^i$$

as $h \to 0$ $y_i \to$ solution to $y' = \lambda y$ & other solutions are extraneous

Since the original equat. is a continuous function of Δx then it follows that for small Δx $|\beta_i| < 1$ for $i = 1,2,3$ & the method is stable

What about Milne's method $y_{n+1} = y_{n-1} + \frac{h}{3}\left(f_{n+1} + 4f_n + f_{n-1}\right)$

if $f(x,y) = \lambda y$ then we get

$$(\beta^2 - 1) - \frac{\lambda h}{3}\left(\beta^2 + 4\beta + 1\right) = 0$$

$$\beta = \frac{+\frac{4\lambda h}{3} \pm \sqrt{\left(\frac{4\lambda h}{3}\right)^2 - 4\left(1 + \frac{\lambda h}{3}\right)\left(\frac{\lambda h}{3} - 1\right)}}{2\left(1 - \frac{\lambda h}{3}\right)}$$

KEEP TERMS UP TO h ONLY

$\lim_{\epsilon \to 0}(1+\epsilon)^{1/\epsilon} = e$

$\beta_1 = 1 + \lambda h$

$\beta_2 = -(1 - \lambda h/3)$

and the solution is $y_n \to c_1 e^{\lambda x_n} + c_2 (-1)^n e^{-\lambda x_n/3}$

$(1 + \lambda h)^{\frac{1}{\lambda h}\lambda x_i} \Rightarrow e^{\lambda x_i}$