MULTISTEP METHODS

$$y' = f(x,y)$$

ADAMS METHOD

PASS A QUAD. THRU $x_i, f_i = f(x_i, y_i)$

x_{i-1}, f_{i-1}

x_{i-2}, f_{i-2}

$$f(x,y) = f_i + s(f_i - f_{i-1}) + \frac{(s+1)s}{2}(f_i - 2f_{i-1} + f_{i-2})$$

$$s = \frac{x - x_i}{\Delta x}$$

$$\int_{x_i}^{x_{i+1}} \frac{dy}{dx}\,dx = \int_{x_i}^{x_{i+1}} f(x,y)\,dx$$

$$y_{i+1} = y_i + \frac{\Delta x}{12}\left[23 f_i - 16 f_{i-1} + 5 f_{i-2}\right] + O(\Delta x^4)$$

G.E. $O(\Delta x^3)$

NEWTON-GREGORY

CUBIC (x_i, f_i) (x_{i-1}, f_{i-1}) (x_{i-2}, f_{i-2})

(x_{i-3}, f_{i-3})

$$f(x,y) = f_i + s(f_i - f_{i-1}) + \frac{(s+1)s}{2}(f_i - 2f_{i-1} + f_{i-2})$$

$$+ \frac{(s+2)(s+1)s}{6}(f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3})$$

$$y_{i+1} = y_i + \frac{\Delta x}{24}\left[55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}\right]$$

$$+ \text{L.E. } O(\Delta x^5)$$

MILNE'S METHOD

$$\int_{y_{i-3}}^{y_{i+1}} \frac{dy}{dx}\,dx = \int f(\bar{x}, y)\,d\bar{x}$$

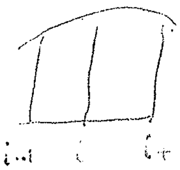
DEFINE $F(x,y)$ using $i, i-1, i-2, i-3$

use this formula & integrate over $i, i-1, i-2, i-3$

so that $y_{i+1_p} = y_{i-3} + \frac{4\Delta x}{3}(2f_i - f_{i-1} + 2f_{i-2})$

now define $f_{i+1} = f(x_{i+1}, y_{i+1p})$

$$\int_{x_{i-1}}^{x_{i+1}} \frac{dy}{dx}\,dx = \int_{x_{i-1}}^{x_{i+1}} f \; dx$$



let f be quadratic using i+1, i, i-1
SIMPSON'S RULE

$$y_{i+1_c} = y_{i-1} + \frac{\Delta x}{3}\left(f_{i+1} + 4f_i + f_{i-1}\right)$$

ADAMS MOULTON METHOD

$y' = f(x,y)$

CUBIC THRU $i, i-1, i-2, i-3$

& INTEGRATES IT OVER i TO $i+1$

$$\int_{x_i}^{x_{i+1}} y'\,dx = \int_{x_i}^{x_{i+1}} f(x,y)\,dx$$

$\Longrightarrow y_{i+1_p}$

DFN CUBIC over $i+1, i, i-1, i-2$ & INTEGRATE OVER i TO $i+1$

$$y_{i+1_p} = y_i + \frac{\Delta x}{24}\left(55f_i - 59f_{i-1} + 37f_{i-2} - 9f_{i-3}\right)$$

$$y_{i+1_c} = y_i + \frac{\Delta x}{24}\left(9f_{i+1} + 19f_i - 5f_{i-1} + f_{i-2}\right)$$

GE $O(\Delta x^4)$

Table 6.1 ADAMS FORMULAS

Order of formula	Coefficient of h	Coefficients of f_i	f_{i-1}	f_{i-2}	f_{i-3}	f_{i-4}	f_{i-5}	Local truncation error E_T
1	1	1						$\frac{1}{2}h^2y''(\xi)$
2	$\frac{1}{2}$	3	−1					$\frac{5}{12}h^3y'''(\xi)$
3	$\frac{1}{12}$	23	−16	5				$\frac{3}{8}h^4y''''(\xi)$
4	$\frac{1}{24}$	55	−59	37	−9			$\frac{251}{720}h^5y^{V}(\xi)$
5	$\frac{1}{720}$	1901	−2774	2616	−1274	251		$\frac{475}{1440}h^6y^{VI}(\xi)$
6	$\frac{1}{1440}$	4277	−7923	9982	−7298	2877	−475	$\frac{19087}{60480}h^7y^{VII}(\xi)$

represents most of the local truncation error. From the definition of the third backward difference given in Section C.2 of Appendix C we can write

$$h(\tfrac{3}{8}\nabla^3 f_i) = \tfrac{3}{8}h(f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}). \tag{6.114}$$

Backward finite-difference expressions for derivatives are given by Equations (5.123). From the third of Equations (5.123) we deduce that

$$y''''_i = \frac{y'_i - 3y'_{i-1} + 3y'_{i-2} - y'_{i-3}}{h^3} + O(h)$$

or

$$h^3y''''_i = y'_i - 3y'_{i-1} + 3y'_{i-2} - y'_{i-3} + h^3[O(h)]. \tag{6.115}$$

From Equation (6.115) we determine that

$$(f_i - 3f_{i-1} + 3f_{i-2} - f_{i-3}) = h^3y''''_i - O(h^4). \tag{6.116}$$

Combining Equations (6.116) and (6.114),

$$h(\tfrac{3}{8}\nabla^3 f_i) = \tfrac{3}{8}h^4y''''_i - O(h^5) = \tfrac{3}{8}h^4y''''(\xi)$$

where the fourth derivative of y is evaluated at some unknown x value ξ in the range of x values spanned by the one-step application of the third-order Adams formula. Thus, the local truncation error of the third-order method is

$$E_T = \tfrac{3}{8}h^4y''''(\xi) \tag{6.117}$$

as shown in Table 6.1. In similar fashion we can show that

$$h(\tfrac{1}{2}\nabla f_i) = \tfrac{1}{2}h^2y''(\xi)$$

$$h(\tfrac{5}{12}\nabla^2 f_i) = \tfrac{5}{12}h^3y'''(\xi)$$

$$\vdots$$

for the other local truncation errors shown in Table 6.1.

Carrying out the integration indicated, the general formula is

$$y_{i+1} = y_i + h(f_{i+1} - \tfrac{1}{2}\nabla f_{i+1} - \tfrac{1}{12}\nabla^2 f_{i+1} - \tfrac{1}{24}\nabla^3 f_{i+1} - \tfrac{19}{720}\nabla^4 f_{i+1} - \tfrac{27}{1440}\nabla^5 f_{i+1} - \tfrac{863}{60{,}480}\nabla^6 f_{i+1} - \cdots). \tag{6.137}$$

To obtain the third-order Adams–Moulton corrector formula from this general formula, Equation (6.137) is truncated to three terms following y_i, which yields

$$y_{i+1} = y_i + h(-f_{i+1} - \tfrac{1}{2}\nabla f_{i+1} - \tfrac{1}{12}\nabla^2 f_{i+1}).$$

Then substituting the backward differences as given in Section C.2 in Appendix C into the above, we find

$$y_{i+1} = y_i + \frac{h}{12}(-f_{i-1} + 8f_i + 5f_{i+1}) \tag{6.138}$$

which is identical with Equation (6.129) derived previously.

The fourth-order Adams–Moulton corrector formula is found by truncating Equation (6.137) to four terms following y_i, yielding

$$y_{i+1} = y_i + h(f_{i+1} - \tfrac{1}{2}\nabla f_{i+1} - \tfrac{1}{12}\nabla^2 f_{i+1} - \tfrac{1}{24}\nabla^3 f_{i+1}).$$

Substituting the backward differences from Section C.2 of Appendix C into the above gives the corrector equation

$$y_{i+1} = y_i + \frac{h}{24}(f_{i-2} - 5f_{i-1} + 19f_i + 9f_{i+1}) \tag{6.139}$$

which is identical with Equation (6.134) found previously.

Using the terms in Equation (6.137), we can obtain Adams–Moulton corrector formulas of orders 1 through 7, or of orders 1 through 6 plus a local truncation error expression for each of the six. These expressions are given in Table 6.2. In the local truncation-error expressions ξ is an unknown x value in the range of x values spanned in the one-step application of a particular Adams–Moulton formula. While explicit numerical values cannot be found for these expressions because the derivatives in them cannot be evaluated, they are useful for comparing the truncation errors of the various Adams–Moulton formulas, and for comparing the truncation errors of these formulas with those of other

Table 6.2 ADAMS–MOULTON FORMULAS AND LOCAL TRUNCATION-ERROR EXPRESSIONS

Order of formula	Coefficient of h	Coefficients of f_{i+1}	f_i	f_{i-1}	f_{i-2}	f_{i-3}	f_{i-4}	Local truncation error E_T
1	1	1						$-\frac{1}{2}h^2y''(\xi)$
2	$\frac{1}{2}$	1	1					$-\frac{1}{12}h^3y'''(\xi)$
3	$\frac{1}{12}$	5	8	−1				$-\frac{1}{24}h^4y''''(\xi)$
4	$\frac{1}{24}$	9	19	−5	1			$-\frac{19}{720}h^5y^{\mathrm{V}}(\xi)$
5	$\frac{1}{720}$	251	646	−264	106	−19		$-\frac{27}{1440}h^6y^{\mathrm{VI}}(\xi)$
6	$\frac{1}{1440}$	475	1427	−798	482	−173	27	$-\frac{863}{60480}h^7y^{\mathrm{VII}}(\xi)$

does have the disadvantage of not being self-starting, though, as we shall see later, this is not a serious disadvantage.

The development of Milne's method begins by dividing the area under a given portion of a curve $y = f(x)$ into 4 Δx-width strips, as shown in Fig. 6-27. The true area under this portion of the curve is then approximated by considering the area of these 4 strips under a second-degree parabola having 3 coordinates in common with the actual curve, as indicated by the dashed line in Fig. 6-27. The crosshatched area is the

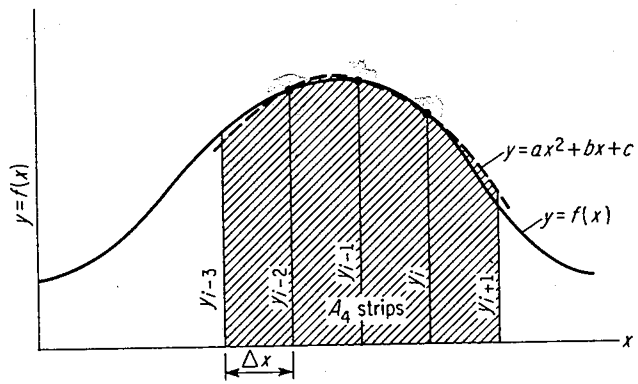


FIG. 6-27. Area in 4 strips under a curve approximated by area under a second-degree parabola.

approximate area obtained. To determine an expression for this crosshatched area in terms of Δx and the appropriate y ordinates, it is convenient to consider the 4 strips as centered on the y axis, as shown in Fig. 6-28. This arrangement does not compromise the generality of the

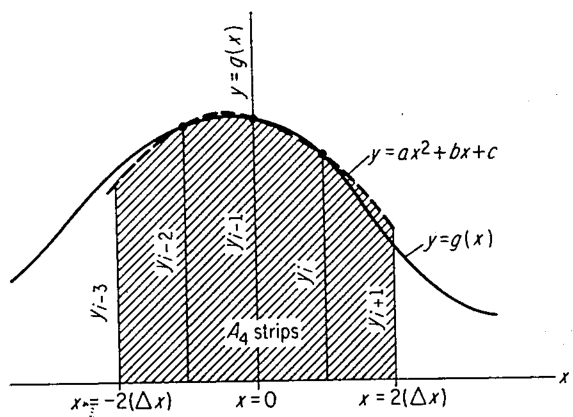


FIG. 6-28. Area of 4 strips centered on y axis.

results obtained, and it has the advantage of simplifying the intermediate expressions involved in determining the desired form of the expression for the area of 4 such strips. The crosshatched area of Fig. 6-28 is given by

$$A_{4\text{ strips}} = \int_{-2(\Delta x)}^{2(\Delta x)} (ax^2 + bx + c)dx \tag{6-83}$$

Integrating Eq. 6-83 and substituting the limits gives

$$A_{4\text{ strips}} = \frac{16}{3} a(\Delta x)^3 + 4c(\Delta x) \tag{6-84}$$

The constants a and c are determined in the manner explained on p. 286. The appropriate expressions are

$$a = \frac{y_i - 2y_{i-1} + y_{i-2}}{2(\Delta x)^2}$$
$$c = y_{i-1} \tag{6-85}$$

Substituting Eq. 6-85 into Eq. 6-84 yields, for the area of the 4 strips in terms of Δx and the y ordinates shown,

$$A_{4\text{ strips}} = \frac{4}{3}(\Delta x)[2y_i - y_{i-1} + 2y_{i-2}] \tag{6-86}$$

This expression will be used later as part of the predictor equation.

Let us consider the application of Milne's method in integrating a first-order differential equation of the form

$$y' = f(x, y) \tag{6-87}$$

where the value of y is known for $x = 0$. This technique consists, basically, of obtaining approximate values of y by the use of a *predictor* equation and then correcting these values by the iterative use of a corrector equation. Milne's predictor equation

$$P(y_{i+1}) = y_{i-3} + \frac{4}{3}(\Delta x)[2y'_i - y'_{i-1} + 2y'_{i-2}] \tag{6-88}$$

utilizes the area of 4 strips under a parabolic approximation of a curve (see Eq. 6-86) to provide a predicted value for the successive y ordinates. Milne's *corrector* equation

$$C(y_{i+1}) = y_{i-1} + \frac{\Delta x}{3}[y'_{i-1} + 4y'_i + P(y'_{i+1})] \tag{6-89}$$

provides corrected y values by using Simpson's rule for determining the area of 2 strips under a curve (see p. 286).

Assuming that the resulting y and y' curves of Eq. 6-87 have the general form of the curves shown in Fig. 6-29, the first step is to obtain a predicted value of y_4. Utilizing Eq. 6-88 with $i = 3$,

$$P(y_4) = y_0 + \frac{4}{3}(\Delta x)[2y'_3 - y'_2 + 2y'_1] \tag{6-90}$$