EGM 5994 EXAMINATION 27 June 1988

This examination will be 120 minutes long. This is an open book/open notes examination.

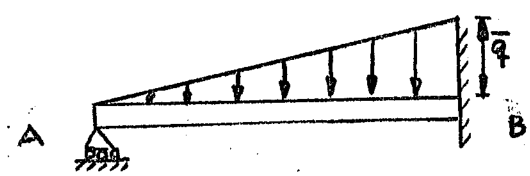
Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of this may result in failure of the exam.

PRINT NAME SIGN NAME

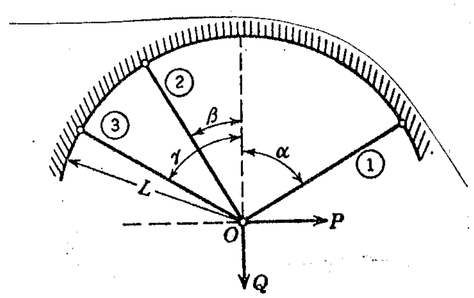
This examination consists of **three problems with several parts to one of the problems.** **Do all three problems.** Read each question carefully. Show all work!!!!!

1. For the beam under distributed load shown in the diagram below, find the reaction forces at A and B



$q = \bar{q}\frac{x}{L}$

2. If the angles of the plane truss problem illustrated in the figure are $\alpha = 60^\circ$, $\beta = 30^\circ$, $\gamma = 60^\circ$, determine the forces in its bars and the horizontal and vertical displacements u and v of joint O. Assume the bars have the same cross-sectional area A and the same elastic modulus E.



$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{bmatrix}$$

3a. Find the inverse of the matrix shown to the left using any method

$$\begin{aligned} 2x + 3y - z &= 5 \\ 4x + 4y - 3z &= 3 \\ -2x + 3y - z &= 1 \end{aligned}$$

3b. If you are given the set of simultaneous equations shown to the left, use the results of (3a) to find the values of x, y and z.

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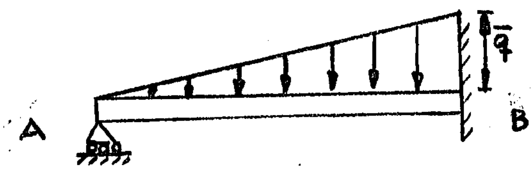

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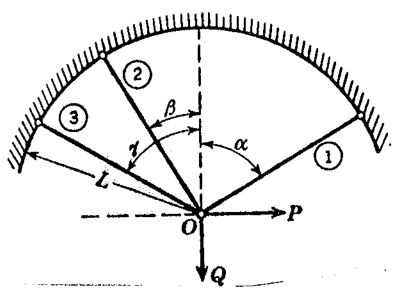

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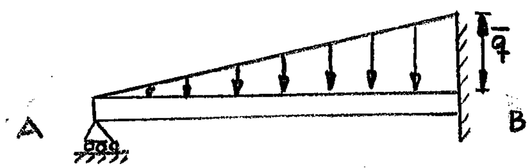

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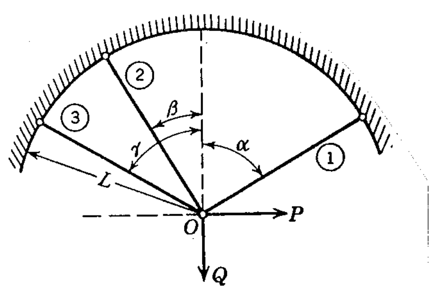

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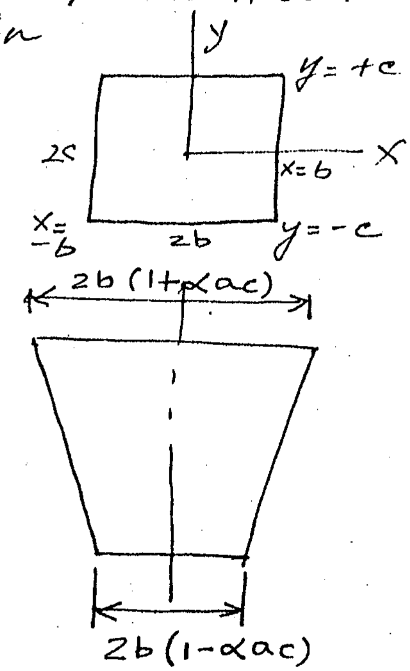
1.6. No stresses because it is not restrained
But there will be strain

$$\varepsilon_x = \varepsilon_y = \alpha T = \alpha a y$$



Elongation at Top $y = +c$ surface

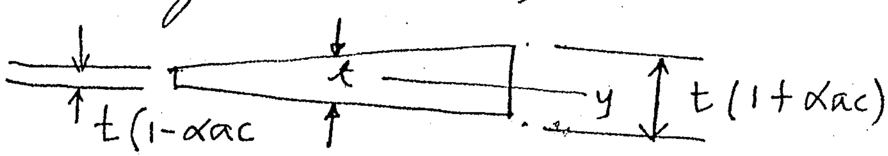
$b + b(\alpha a c)$ | $b + b(\alpha a c)$

at bottom $y = -c$ surface

$b + b(-\alpha a c)$ | $b + b(-\alpha a c)$

Comment about the shape of the top and bottom surface and give reason

Edge View (Looking down X axis)



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8 (a) Consider a cube 1 unit on each side
V = Volume = 1

Let strains $\varepsilon_x, \varepsilon_y, \varepsilon_z$ take place; then

$$V + \Delta V = (1+\varepsilon_x)(1+\varepsilon_y)(1+\varepsilon_z)$$

$$V + \Delta V = 1 + \varepsilon_x + \varepsilon_y + \varepsilon_z + \text{higher order terms}$$

$$= V + (\varepsilon_x + \varepsilon_y + \varepsilon_z)$$

$$\Delta V = \varepsilon_x + \varepsilon_y + \varepsilon_z \quad ; \text{ so } \quad \frac{\Delta V}{V} = \varepsilon_x + \varepsilon_y + \varepsilon_z$$

(b) $\sigma_x = \sigma_y = \sigma_z = -p$; so from 3D Hooke's Law

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = -\frac{1}{E}(p - \nu p - \nu p) = -\frac{(1-2\nu)p}{E}$$

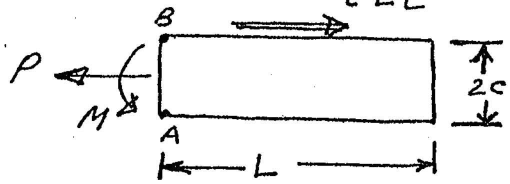
$$\therefore \frac{\Delta V}{V} = -\frac{3(1-2\nu)p}{E} \quad ; \quad B = -\frac{p}{(\Delta V/V)} = \frac{E}{3(1-2\nu)}$$

(c) $\Delta V = -p \cdot \frac{V}{B}$

if $\nu > \frac{1}{2}$, $B < 0$, $\Delta V > 0$ does not make sense

~~1.18.~~ 1.7.1 2nd ed.

$\tau L t$ = Total Shear Force



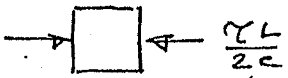
P and M are Reactions

$$\text{Axial Stress} = \frac{P}{A} = \frac{\tau L T}{2ct} = \frac{\tau L}{2c}$$

$$\text{Bending stress} = \frac{Mc}{I} = \frac{(\tau L T c)\,c}{\frac{1}{12}b(2c)^3} = \frac{3\tau L}{2c}$$

At Point A

Net Compressive Stress

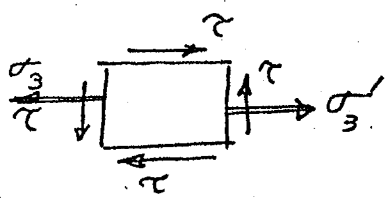


$$= \frac{3\tau L}{2c} - \frac{\tau L}{2c} = \frac{\tau L}{c}$$

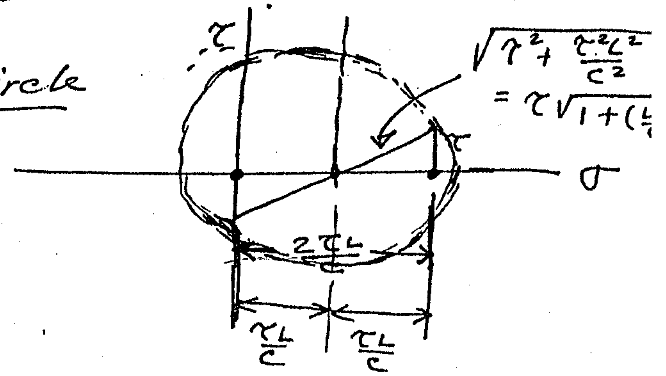
$$\tau_{MAX} = \frac{\sigma_1 - \sigma_3}{2} = \frac{1}{2}\left[0 - \left(-\frac{\tau L}{c}\right)\right] = \frac{\tau L}{2c}$$

At Point B

$$\sigma_3' \text{ (Tension)} = \frac{3\tau L}{2c} + \frac{\tau L}{2c} = \frac{4\tau L}{2c} = \frac{2\tau L}{c}$$



Mohr's Circle



Principal stresses

$$\sigma_1 = \frac{\tau L}{c} + \tau\sqrt{1 + \left(\frac{L}{c}\right)^2}$$

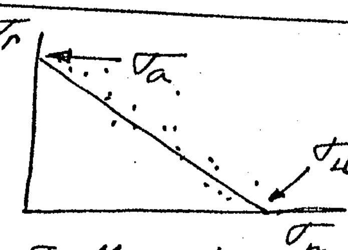
$$\sigma_2 = 0$$

$$\sigma_3 = \frac{\tau L}{c} - \tau\sqrt{1 + \left(\frac{L}{c}\right)^2}$$

$$\tau_{mx} = \frac{\sigma_1 - \sigma_3}{2} =$$

$$\text{OR} = \tau\sqrt{1 + \left(\frac{L}{c}\right)^2}$$

FATIGUE : study Fig 1.8.3 and Fig 1.8.4



$$\frac{\sigma_r}{\sigma_a / SF} + \frac{\sigma_m}{\sigma_u / SF} = 1$$

σ_u = Ult. Stress SF = SAFETY FACTOR σ_r = Alt. Stress Amplitude; σ_m = Mean Stress

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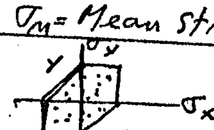
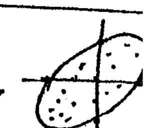
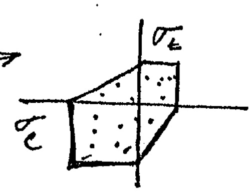
FAILURE THEORIES

Types of Failure
- Fracture
- Yielding
- Low Stiffness
- Buckling
- Creep

Failure Criterions
- × Tresca
- × Von Mises
- × Max Princ. Stress
- × Mohr Criterion

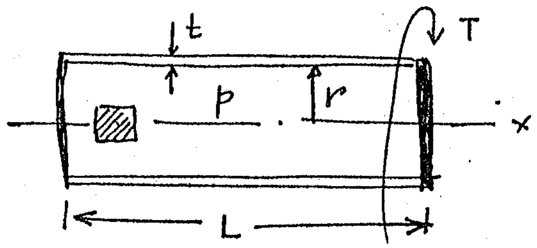


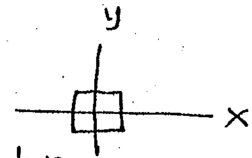




Ref: Fig 1.7.2

1.28





$$\sigma_x = \frac{pr}{2t} \qquad \left(\begin{array}{l} \text{Load} = p\pi r^2 \\ \text{Area of support} = 2\pi r t \\ \text{Stress} = p\pi r^2/2\pi r t \end{array} \right)$$

$$\sigma_y = \frac{p(2rL)}{2tL} = \frac{pr}{t}$$

$$\therefore \tau = \frac{Tr}{J} = \frac{Tr}{2\pi r^3 t}$$

So; $\sigma_x = \dfrac{2(200)}{2(4)} = 50 \text{ MPa}$; $\sigma_y = 100 \text{ MPa}$; $\tau = 29.8 \text{ MPa}$

EQ 1.4.15:

$\sigma_1 = 113.9 \text{ MPa}$; $\sigma_2 = 36.1 \text{ MPa}$; $\sigma_3 = -2 \text{ MPa}$ (inside)

Tank $\tau_{mx} = \dfrac{\sigma_1 - \sigma_3}{2} = \dfrac{113.9 + 2}{2} = 57.95 \text{ MPa}$

Test $\tau_{mx} = \dfrac{Y - 0}{2} = 100 \text{ MPa}$. $\Rightarrow$ $SF = \dfrac{100}{57.95} = 1.73$ (TRESC

EQ 1.6.16:

Tank $\sigma_e = \dfrac{1}{\sqrt{2}} \left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}$

$= \dfrac{1}{\sqrt{2}} \left[(113.9 - 36.1)^2 + (36.1 + 2)^2 + (-2 - 113.9)^2 \right]^{1/2} = 102.3 \text{ M.Pa}$.

Comment: Why Shear Stresses of EQ 1.6.16 is not used !

Test $\sigma_e = \dfrac{1}{\sqrt{2}} \left[Y^2 + Y^2 \right]^{1/2} = 200 \text{ MPa}$

$SF = \dfrac{102.3}{200} = 1.96$

1.53 a) Can use Eq. 1.7.7 with

$$\sigma_x = (SF)\frac{Mr}{\pi r^4/4} = (SF)\frac{4M}{\pi r^3}$$

$$\tau_{xy} = (SF)\frac{Tr}{\pi r^4/2} = (SF)\frac{2T}{\pi r^3}$$

$$\left(SF \frac{4M}{\pi r^3 Y} \right)^2 + 4\left(SF \frac{2T}{\pi r^3 Y} \right) = 1$$

$$r^6 = \frac{16(SF)^2}{\pi^2 Y^2}(M^2 + T^2)$$

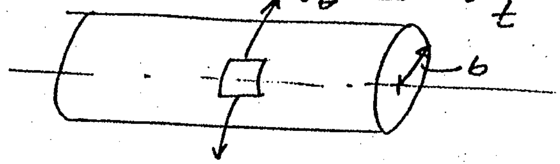
$$r = \left[\frac{4(SF)}{\pi Y} \sqrt{M^2 + T^2} \right]^{1/3}$$

b) Use Eq 1.7.8

$$r = \left[\frac{2(SF)}{\pi Y} \sqrt{4M^2 + 3T^2} \right]^{1/3}$$

Thick walled Cylinder and Spinning Disks - Ch 3
Art 3.1 thru 3.8

3.1 $\sigma_\theta = p_i \dfrac{b}{t}$



t is thickness for thin ($\frac{b}{t} \geq 10$) cyln
This means σ_θ and ε_θ are constant throughout wall thickness

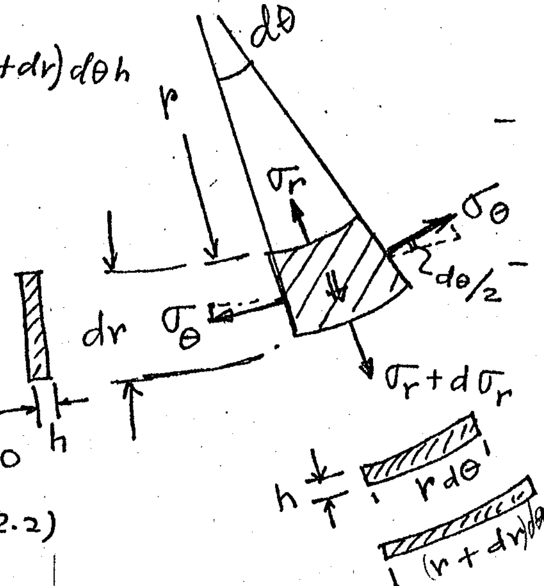
3.2. Equilibrium Equation

$\Sigma F_r = 0$ gives

$$-\sigma_r (h r\, d\theta) + (\sigma_r + d\sigma_r)(r+dr)\, d\theta\, h + (\rho\omega^2 r)(r h\, d\theta\, dr) - 2\sigma_\theta \sin\frac{d\theta}{2}(h\, dr) = 0$$

Discard products of 3 differentials and divide by $h r\, dr\, d\theta$

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} + \rho\omega^2 r = 0 \quad \cdots\cdots (EQ\ 3.2.2)$$



- Note σ_θ is constant with θ but not r
- Volume $= (h)(r\, d\theta)\, dr$

3.3. Definition of strain

$$\varepsilon_\theta = \frac{2\pi(r+u) - 2\pi r}{2\pi r} = \frac{u}{r} \quad (3.3.1)$$

$$\varepsilon_r = \frac{du}{dr} \quad \cdots\cdots (3.3.2)$$

u is Deformation

3.3 Constitutive Relation (Stress - Strain Relation) → Changes to Stress - Defor

$$\left.\begin{aligned} \sigma_r &= \frac{E}{1-\nu^2}(\varepsilon_r + \nu\varepsilon_\theta) = \frac{E}{1-\nu^2}\left(\frac{du}{dr} + \nu\frac{u}{r}\right) \\ \sigma_\theta &= \frac{E}{1-\nu^2}(\varepsilon_\theta + \nu\varepsilon_r) = \frac{E}{1-\nu^2}\left(\frac{u}{r} + \nu\frac{du}{dr}\right) \end{aligned}\right\} \cdots (3.3.3)$$

3.3 Substitution of Stress - Deformation in Equilibrium

Substitute 3.3.3 into 3.2.2. ⟹ COMPATIBILITY EQUN

$$\frac{d^2u}{dr^2} + \frac{1}{r}\frac{du}{dr} - \frac{u}{r^2} = -\frac{1-\nu^2}{E}\rho\omega^2 r \quad \cdots (3.3.4)$$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(ur)\right] = -\frac{1-\nu^2}{E}\rho\omega^2 r \quad \cdots (3.3.5)$$

SOLUTION ⟹ $$u = C_1 r + \frac{C_2}{r} - \frac{1-\nu^2}{E}\rho\omega^2\frac{r^3}{8} \quad \cdots (3.3.6)$$

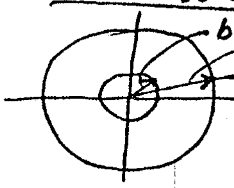
Now the deformation relation of (3.3.6) can be substituted in Stress-Deformation relation of (3.3.3) to give Stresses σ_r and σ_θ as function of r

$$\sigma_r = \frac{E}{1-\nu^2}\left[(1+\nu)c_1 - \frac{1-\nu}{r^2}c_2\right] - \frac{3+\nu}{8}\rho\omega^2 r^2 \quad \ldots(3.3.$$

$$\sigma_\theta = \frac{E}{1-\nu^2}\left[(1+\nu)c_1 + \frac{1-\nu}{r^2}c_2\right] - \frac{1+3\nu}{8}\rho\omega^2 r^2 \quad \ldots(3.3$$

Now These can be applied to specific Cases

3.4. Thick walled Cylinder $\omega = 0$; boundary condition:



$\sigma_r = -p_i$ at $r = a$

$\sigma_r = -p_o$ at $r = b$

(Make sure you understand negative sign)

Thus: $\sigma_r = \frac{E}{1-\nu^2}\left[(1+\nu)c_1 - \frac{1-\nu}{r^2}c_2\right]$

$\sigma_\theta = \frac{E}{1-\nu^2}\left[(1+\nu)c_1 + \frac{1-\nu}{r^2}c_2\right]$

Substituting boundary conditions on σ_r equation.

$$-p_i = \frac{E}{1-\nu^2}\left[(1+\nu)c_1 - \frac{1-\nu}{b^2}c_2\right] \quad \ldots\ldots \text{A}$$

$$-p_o = \frac{E}{1-\nu^2}\left[(1+\nu)c_1 + \frac{1-\nu}{a^2}c_2\right] \quad \ldots\ldots \text{B}$$

Subtracting B − A $\Rightarrow$

$$-p_o + p_i = \frac{E}{1-\nu^2}c_2\left(-\frac{1-\nu}{a^2} + \frac{1-\nu}{b^2}\right) = \frac{E(1-\nu)}{1-\nu^2}\left(\frac{a^2+b^2}{a^2b^2}\right)c_2$$

$$c_2 = \frac{(1-\nu^2)(p_i-p_o)}{E}\cdot\frac{a^2b^2}{(1-\nu)(a^2+b^2)} = \frac{(1+\nu)(1-\nu)a^2b^2(p_i-p_o)}{E(1-\nu)(a^2+b^2)}$$

$$c_2 = \frac{1+\nu}{E}\,\frac{a^2b^2(p_i-p_o)}{(a^2+b^2)}$$

Similarly by substituting c_2 into A we can get.

$$c_1 = \frac{1-\nu}{E}\,\frac{b^2p_i - a^2p_o}{a^2-b^2}$$

So c_1 and c_2 for a problem can be obtained if we know material property (E and ν), pressures (p_i and p_o) and geometry (a and b)

Finally; the stress equations may be expressed as in (3.4.3 & 4)

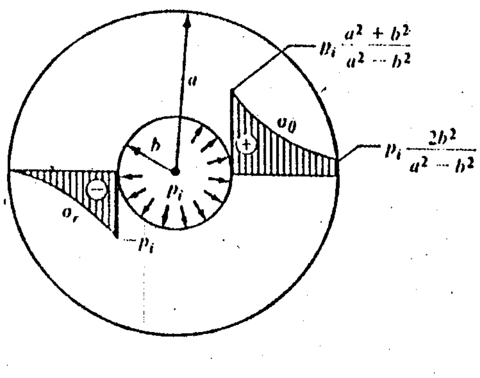
$$\sigma_r = p_i\frac{b^2}{(a^2-b^2)}\left(1-\frac{a^2}{r^2}\right) - p_o\frac{a^2}{a^2-b^2}\left(1-\frac{b^2}{r^2}\right)$$

$$\sigma_\theta = p_i\frac{b^2}{(a^2-b^2)}\left(1+\frac{a^2}{r^2}\right) - p_o\frac{a^2}{a^2-b^2}\left(1+\frac{b^2}{r^2}\right)$$

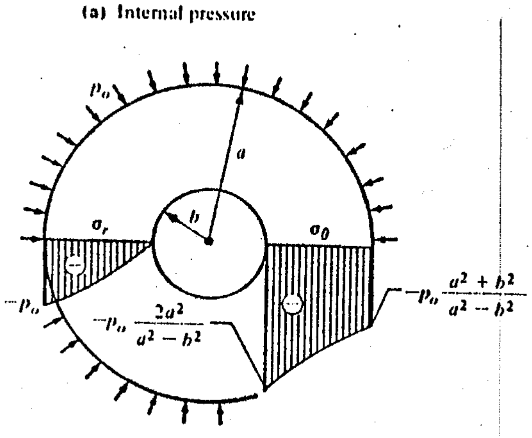
* use the ~~rest~~ rest of Art 3.4 as READING ASSIGNMENT including worked out examples.
Also pay particular attention to discussions related to Fig 3.4.2 and Fig 3.4.3

Prob. 3.2

Figure 3.4.1 is shown where effects of p_i and p_o are shown seperately



(a) Internal pressure



(b) External pressure

Fig 3.41

$$\frac{\sigma_{\theta b}}{\sigma_{\theta a}} = \frac{p_i \frac{a^2+b^2}{a^2-b^2} - p_o \frac{2a^2}{a^2-b^2}}{p_i \frac{2b^2}{a^2-b^2} - p_o \frac{a^2+b^2}{a^2-b^2}}$$

The above is obtained by substituting $r = a$ for $\sigma_{\theta a}$ and for $\sigma_{\theta b}$ $r = b$.

Non dimensionalize the above equation

$$\frac{\sigma_{\theta b}}{\sigma_{\theta a}} = \frac{p_i \frac{a^2/b^2+1}{a^2/b^2-1} - p_o \frac{2\,a^2/b^2}{a^2/b^2-1}}{p_i \frac{2}{a^2/b^2-1} - p_o \frac{a^2/b^2+1}{a^2/b^2-1}}$$

$$= \frac{p_i \frac{R^2+1}{R^2-1} - p_o \frac{2R^2}{R^2-1}}{p_i \frac{2}{R^2-1} - p_o \frac{R^2+1}{R^2-1}}$$

Where $R = a/b$; Now non dimensionalize with respect to pressure.

$$\frac{\sigma_{\theta b}}{\sigma_{\theta a}} = \frac{(p_i/p_o)\left(\frac{R^2+1}{R^2-1}\right) - \left(\frac{2R^2}{R^2-1}\right)}{(p_i/p_o)\left(\frac{2}{R^2-1}\right) - \left(\frac{R^2+1}{R^2-1}\right)}$$

$$= \frac{[1/R^2-1][(p_i/p_o)(R^2+1) - 2R^2]}{[1/R^2-1][(p_i/p_o)(2) - (R^2+1)]}$$

$$= \frac{(p_i/p_o)[R^2+1] - 2R^2}{(p_i/p_o)[2] - (R^2+1)}$$

e are looking for $\sigma_{\theta a} > \sigma_{\theta b}$ i,e; $\left|\frac{\sigma_{\theta b}}{\sigma_{\theta a}}\right| < 1$

$\longleftarrow$ -1 1 $\longrightarrow$

For $\frac{\sigma_{\theta b}}{\sigma_{\theta a}} = 1 \Rightarrow PR^2 + P - 2R^2 = 2P - R^2 - 1$ Where $P = \frac{p_i}{p_o}$

$P(R^2+1-2) = 2R^2 - R^2 - 1 \Rightarrow P(R^2-1) = (R^2-1)$

i,e; $P = 1$ (and independent of R)

i,e; $p_i = p_o$ (Hydrostatic Pressure)

For $\frac{\sigma_{\theta b}}{\sigma_{\theta a}} = -1 \Rightarrow PR^2 + P - 2R^2 = -2P + R^2 + 1 \Rightarrow P(R^2+1+2) = 2R^2 + R^2 + 1 \Rightarrow P = \frac{3R^2+1}{R^2+3}$

Since $R > 1$; for any R to satisfy $\frac{\sigma_{\theta b}}{\sigma_{\theta a}} = -1$ $P > 1$

So condition $1 < P < 3R^2+1/R^2+3$

PROB. 3.3 (This uses Ex 3.4.1 - p.94)

Ex 3.4.1 $b = 50$ mm ; $p_i = 72$ MPa ; End caps ; SF = 2 ; $Y = 480$ MPa

$E = 70$ GPa $\nu = 1/3$

USE TRESCA'S YIELD THEORY FOR $a = ?$ and $u_a =$

$\tau_{MX} = \frac{1}{2}(\sigma_\theta - \sigma_r)$. Note that at inner surface $r = b$

(Check Fig 3.4.1 p.94) $|\sigma_\theta|$ is max and + and $|\sigma_r|$ is max and − ; so shear is max at inner surface

$$\tau_{mx} = \frac{1}{2}\left[p_i \frac{a^2+b^2}{a^2-b^2} + p_i\right] = \frac{1}{2} p_i \left[\frac{a^2+b^2+a^2-b^2}{a^2-b^2}\right] = p_i \frac{a^2}{a^2-b^2} .$$

For Tresca Condition: Set $\tau_{mx} = \frac{1}{2} Y = 240$ MPa ; $p_i = SF(72) = 144$ MPa

$b = 50$ mm . (Remember Pa = N/Meter)

$a = 79.06$ mm

For u_a Find Principal stresses at $r = b$ from Eq 3.4.3, 3.4.4 and 3.4.6

$$\sigma_r = p_i \frac{b^2}{(a^2-b^2)}\left[1 - \frac{a^2}{b^2}\right] = -p_i = -72 \text{ MPa} = \sigma_3$$

$$\sigma_\theta = p_i \frac{b^2}{(a^2-b^2)}\left[1 + \frac{a^2}{b^2}\right] = 168 \text{ MPa} = \sigma_1$$

$$\sigma_z = \frac{p_i b^2}{(a^2-b^2)} = 48 \text{ MPa} = \sigma_2$$

($\sigma_1, \sigma_2, \sigma_3$ are from highest to lowest)

$$\varepsilon_{\theta b} = \frac{u_b}{b} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r - \nu\sigma_z) \text{ at } r = b$$

$$= \frac{1}{70 \times 10^9}(168 + 24 - 16)10^6 = 2.514 \times 10^{-3}$$

Therefore, the inner radius lengthens by $u_b = b\varepsilon_{\theta b}$

$= 50(2.514 \times 10^{-3}) = 0.126$ mm

at Outer Radius $r = a$ $\sigma_\theta = 96$ MPa ; $\sigma_r = 0$; $\sigma_z = 48$ MPa.

$$\varepsilon_{\theta a} = \frac{u_a}{a} = \frac{1}{70 \times 10^9}(96 + 0 - 16)10^6 = 1.143 \times 10^{-3}$$

Therefore, the outer radius lengthens by $u_a = a\,\varepsilon_{\theta a}$

$= 79.06(1.143 \times 10^{-3}) = .090$ mm.

Now For 3.3

$$\sigma_{\theta b} = p_i \frac{a^2+b^2}{a^2-b^2} = p_i \frac{1 + (b/a)^2}{1 - (b/a)^2}$$

For $a \to \infty$

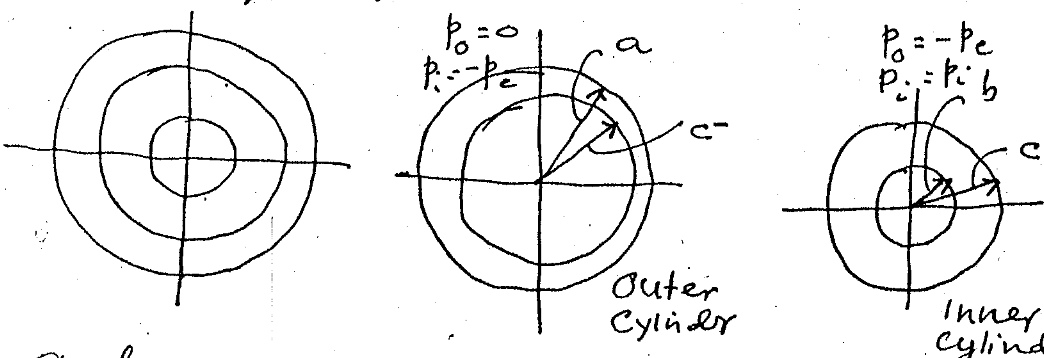
$\sigma_{\theta b} = p_i$; Then at $r = b$

$$\tau_{mx} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_{\theta b} - (-p_i)}{2} = p_i$$

By Tresca Theory; $\frac{Y}{2} = SF * \tau_{mx}$; $Y = 2(SF)p_i = 2(2)72 = 288$ MPa.

Art 3.6 COMPOUND CYLINDER

p_c = Interface pressure due to shrink fit.



goal: To make both cylinders "equal strength". meaning their τ_{mx} be same

For outer cylinder: $\tau_{mx}\big|_{r=c} = \dfrac{\sigma_\theta - \sigma_r}{2}$

$$= \frac{1}{2}\left[p_c \frac{a^2+c^2}{a^2-c^2} + \frac{p_i b^2}{a^2-b^2}\left(1+\frac{a^2}{c^2}\right) - \frac{p_i b^2}{a^2-b^2}\left(1-\frac{a^2}{c^2}\right) - (-p_c)\right] \quad \ldots (3.6.2)$$

For Inner Cylinder $\tau_{mx}\big|_{r=b} = \dfrac{\sigma_\theta - \sigma_r}{2}$

$$= \frac{1}{2}\left[-p_c \frac{2c^2}{c^2-b^2} + p_i \frac{a^2+b^2}{a^2-b^2} - (-p_i)\right] \quad \ldots (3.6.1)$$

make (3.6.1) = (3.6.2)

$$p_c\left[\frac{a^2+c^2}{a^2-c^2} + 1\right] + p_i \frac{b^2}{a^2-b^2}\left[1 + \frac{a^2}{c^2} - 1 + \frac{a^2}{c^2}\right]$$

$$= -p_c \frac{2c^2}{c^2-b^2} + p_i\left[\frac{a^2+b^2}{a^2-b^2} + 1\right]$$

$$p_c\left(\frac{a^2+c^2+a^2-c^2}{a^2-c^2}\right) + p_i \frac{b^2}{a^2-b^2}\left(\frac{2a^2}{c^2}\right) = -p_c \frac{2c^2}{c^2-b^2} + p_i\left[\frac{a^2+b^2+a^2-b^2}{a^2-b^2}\right]$$

$$p_c\left[\frac{2a^2}{a^2-c^2} + \frac{2c^2}{c^2-b^2}\right] = p_i\left[\frac{2a^2}{a^2-b^2} - \frac{2a^2b^2}{c^2(a^2-b^2)}\right]$$

$$p_c\left[\frac{2a^2}{a^2-c^2} + \frac{2c^2}{c^2-b^2}\right] = p_i \frac{2a^2c^2 - 2a^2b^2}{c^2(a^2-b^2)}$$

Finally $\Rightarrow$

$$p_c\left(\frac{a^2}{a^2-c^2} + \frac{c^2}{c^2-b^2}\right) = p_i \frac{a^2(c^2-b^2)}{c^2(a^2-b^2)} \quad \ldots (3.6.3)$$

Substitute p_c from 3.6.3 to 3.6.1

$$\tau_{mx} = \frac{p_i a^2 c^2}{c^2(a^2-c^2)+a^2(c^2-b^2)} \quad \ldots\ldots (3.6.4)$$

What is the best value of c for τ_{mx} to be minimum?

$$\frac{d\tau_{mx}}{dc} = 0$$

no need to take 2nd derivative to verify that 1st is minimum because of geometry of problem - we know max is inside.

$$c = \sqrt{ab}$$

[go to table of Differentials and differentiate 3.6.4 with respect to c]

Substituting $c=\sqrt{ab}$ in 3.6.3 and 3.6.4

$$p_c = p_i \frac{(a-b)}{2(a+b)} \qquad \tau_{mx} = \frac{p_i a}{2(a-b)}$$

INSPECT THE STRESS VARIATION IN FIG 3.6.1 - p 98

SHRINK FIT DEFORMATION: The inner diameter of the outer cylinder is 2Δ smaller than outer diameter of inner cylinder.

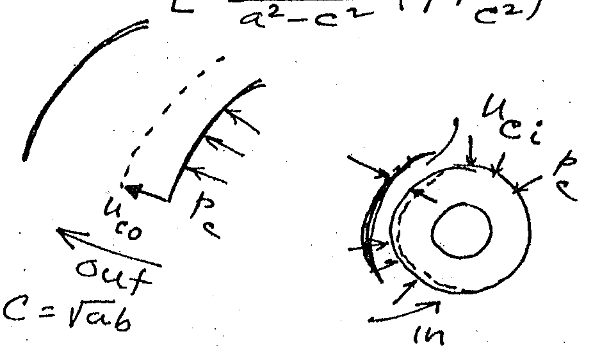
$$\varepsilon_\theta = \frac{u}{r} = \frac{1}{E}(\sigma_\theta - \nu\sigma_r)$$

at $r=c$

$$u_{co} = r\varepsilon_\theta = c\varepsilon_\theta = \frac{c}{E}\left[\frac{p_c c^2}{a^2-c^2}\left(1+\frac{a^2}{c^2}\right) - \nu(-p_c)\right] \quad \ldots (3.6.9)$$

$$u_{ci} = r\varepsilon_\theta = c\varepsilon_\theta = \frac{c}{E}\left[-\frac{p_c c^2}{a^2-c^2}\left(1+\frac{b^2}{c^2}\right) - \nu(-p_c)\right] \quad \ldots (3.6.10)$$

$$\Delta = u_{co} - u_{ci}$$
$$= |u_{co}| + |u_{ci}|$$
(actually)



Using 3.6.9 & 3.6.10 and $C=\sqrt{ab}$

$$\Delta = \frac{2c^3}{E} p_c \frac{a^2-b^2}{(a^2-c^2)(c^2-b^2)} \quad \text{or} \quad \Delta = \frac{p_i}{E}\sqrt{ab} \quad \ldots (3.6.11)$$

How MUCH T is NEEDED

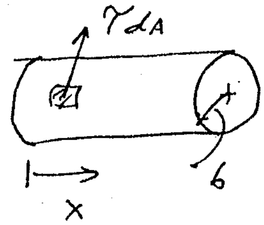
$$\varepsilon_r = \frac{\Delta}{c} = \alpha T \quad ; \text{so } \Delta = c\alpha T$$

$$T = \frac{\Delta}{c\alpha} \quad ; \text{ for } C=\sqrt{ab} \quad T = \frac{p_i}{\alpha E}$$

3.7. On the inside of Tube

$$\sigma_\theta = p_i \frac{a^2+b^2}{a^2-b^2} = 80\,(\text{MPa}) \Rightarrow p_i \frac{12^2+6^2}{12^2-6^2}$$

$= 80$; $p_i = 48$ MPa.



$$T = \int (\tau\, dA)\, b \qquad \text{and } \tau = \mu p_i = .06\,(48) = 2.88 \text{ MPa}$$

$$dA = b\, d\theta\, dx$$

$$T = \int_0^{10}\int_0^{2\pi} \tau b^2\, d\theta\, dx = 2.88\,(6)^2\, 2\pi\,(10)$$

$$= 6515 \ \text{N}\cdot\text{mm}.$$

Check the Stress of the shaft.

$$\tau_{shaft} = \frac{Tc}{J} = \frac{6515\,(6)}{\frac{\pi}{2}\, 6^4} = 19.2 \text{ MPa}.$$

(elastic)

3.8. $\Delta = u_{co} - u_{ci} = \dfrac{2c^3 p_c}{E} \dfrac{a^2-b^2}{(a^2-c^2)(c^2-b^2)}$ .. (3.6.11)

$$\Delta = \frac{2c^3 p_c}{E} \frac{1-\frac{b^2}{a^2}}{\left(1-\frac{c^2}{a^2}\right)(c^2-b^2)}$$

For $a \longrightarrow \infty$; $\Delta = \dfrac{2c^3 p_c}{E} \dfrac{1}{(c^2-b^2)}$

For $b \longrightarrow 0$; $\Delta = \dfrac{2c^3 p_c}{Ec^2} = \dfrac{2c\, p_c}{E}$

$$p_c = \frac{E\Delta}{2c} = \frac{200{,}000\ (.003)}{2\,(30)} = 10 \text{ MPa}.$$

$P = \mu\, (p_c)$ (area of contact) $= (.25)(10)(\pi . 60 . 100) = 47{,}100$ N.

Pull or Push does not matter if $\nu = 0$

If $\nu > 0$; the Bar dia decreases slightly under Tensile P so ~~less~~ less than 47,100 N is needed

For "push" dia increases due to compressive force, hence $> 47{,}100$ N is needed.

Question about the Outer Cylinder equation (3.6.2)

Due to internal pressure p_i acting at $r=b$ but at

$$\sigma_r\Big|_{r=c} = \frac{p_i\, b^2}{a^2-b^2}\left(1-\frac{a^2}{c^2}\right) \quad \text{— A}$$ Substituting $r=c$ in (3.4.3)

Due to internal pressure p_c acting at $r=c$

$$\sigma_r\Big|_{r=c} = -p_c \quad \text{— B}$$

Then $\sigma_\theta\Big|_{r=c}$ by substituting $p_i = p_c$ and $r=c$; $p_o=0$ — Rember, here $b=c$ also in eq (3.4.4)

$$\sigma_\theta\Big|_{r=c} = \frac{p_c\, c^2}{a^2-c^2}\left(1+\frac{a^2}{c^2}\right) \quad \text{— C}$$

Also for p_i acting at $r=b$ but at $\sigma_\theta\Big|_{r=c}$

$$\sigma_\theta = \frac{p_i\, b^2}{a^2-b^2}\left(1+\frac{a^2}{c^2}\right) \quad \text{— D}$$

Now if you take A, B, C, D and put it in

$$\tau_{mx}\Big|_{r=c} = \frac{\sigma_\theta - \sigma_r}{2}$$; you get

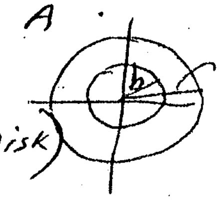
EQ (3.6.2)

PROB 3-14

Solid Flat Disk : $b=0$, $a=a+\Delta$

Outer Disk $b=a$ $a=A$

Find p_c in terms of Δ, E, ν, Hole radius $b=a$ and outer Radius $a=A$.



p_c = Contact Pressure; Let $n=\frac{b}{a}$ (For outer Disk)

using Fig 3.4.1 p.94 (For outer Disk)

$$\sigma_\theta\Big|_{r=b} = p_c\,\frac{a^2+b^2}{a^2-b^2} = \frac{1+n^2}{1-n^2}\,p_c \quad \text{and } \sigma_r = p_c$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) = \frac{p_c}{E}\left[\frac{1+n^2}{1-n^2}+\nu\right]$$

Solid Disk = Remember at $r=$ ~~outs~~ and $b=0$ Fig 3.4.1 b, $p_o = p_c$

$$\sigma_r\Big|_{r=b} = -p_c \quad ; \quad \sigma_\theta\Big|_{r=b\,(\text{outer})} = -p_o\,\frac{a^2+b^2}{a^2-b^2} = -p_c\,\frac{b^2+0}{b^2-0} = -p_c$$

$$\varepsilon_\theta = \frac{1}{E}(\sigma_\theta - \nu\sigma_r) = \frac{1}{E}\left(-p_c - \nu(-p_c)\right) = -\frac{1-\nu}{E}\,p_c$$

$$\Delta = b \times (\text{Diff in } \varepsilon_\theta)$$

$$= \frac{p_c b}{E}\left[\frac{1+n^2}{1-n^2} + \nu + (1-\nu)\right] = \frac{p_c b}{E}\,\frac{2}{1-n^2}$$

$$p_c = \frac{(1-n^2)E\Delta}{2b}$$

Alternatively, use 3.6.11

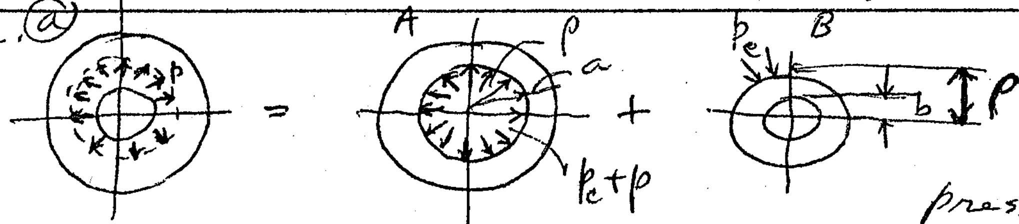
$$\Delta = \frac{2c^3 p_c}{E}\,\frac{a^2-b^2}{(a^2-c^2)(c^2-b^2)}$$

use $b = 0$
$a = a$
c = replaced by b

$$\Delta = \frac{2b^3 p_c}{E}\,\frac{a^2-0}{(a^2-b^2)(b^2-0)} = \frac{2b\,p_c}{E}\,\frac{1}{1-n^2}$$

---

22. (a)



Parts A and B must have same radial Displacement at $r = \rho$. Interface pressure p_c arises to make it so. At $r = \rho$ compute:

$\varepsilon_{\theta\rho})_A = \frac{1}{E}[\sigma_\theta - \nu\sigma_r]_A$ in EQ 3.4.3 and 3.4.4 use: $p_i = p_c + p \quad r = b = \rho;\ p_o = 0$ —— (a)

$\varepsilon_{\theta\rho})_B = \frac{1}{E}[\sigma_\theta - \nu\sigma_r]_B$; use $p_i = 0;\ p_o = p_c,\ r = a = \rho$ —— (b)

Set. $(\varepsilon_{\theta\rho})_A = (\varepsilon_{\theta\rho})_B$ and solve for
$p_c = f(p, \nu, a, b, \rho)$. To get Stress insert into eq 3.4.3 and 3.4.4 conditions (a) and (b)

b) $p = \frac{\text{Force}}{\text{AREA}} = \frac{\beta\, d(\text{vol})}{dA} = \frac{\beta(d\rho\,\rho\,d\theta\,dz)}{\rho\,d\theta\,dz} = \beta\,d\rho$

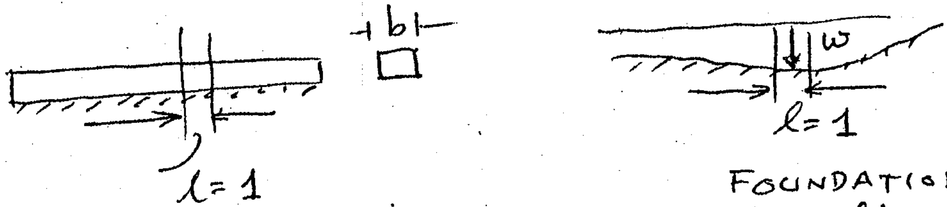
What now?

ELASTIC FOUNDATION - CH·5

Winkler Model
Linear Force - Deflection Relation
(If w is deflection at a point, then resisting <u>pressure</u> is $K_o w$ where
K_o = FOUNDATION MODULUS)

FOR BEAMS USE
$K = K_o b$ where b = width of beam

What does this mean



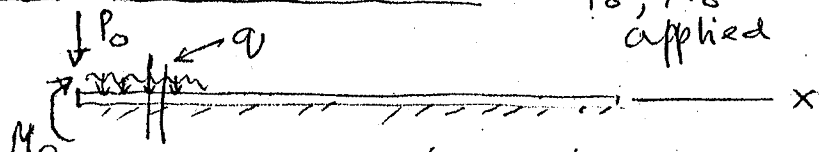
FOUNDATION AREA
$= lb = (1)(b) = b$ / unit Length

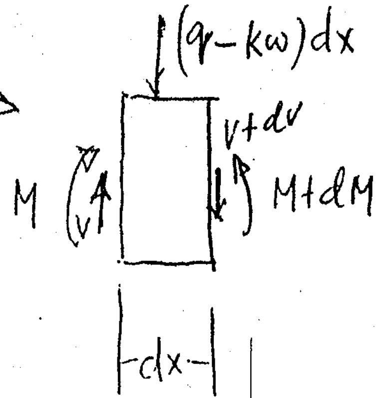
Thus RESISTING FORCE / unit Length $= w K_o (\text{AREA})$
$= w (K_o)(b) = wK$

So: $K_o = N/m^2$ per m (of Defl) ; $K = N/m$ per m (of Defl)

Thus $K_o w = N/m^2$ ← pressure $\qquad K w = N/m$ ← Force / unit Beam Length

ART 5·2 FORMULATION - P_o, M_o and q are applied positive Loads
- Support is always in contact





$$\left.\begin{aligned} \frac{dV}{dx} &= kw - q \\ \frac{dM}{dx} &= V \end{aligned}\right\} (5\cdot2\cdot1)$$

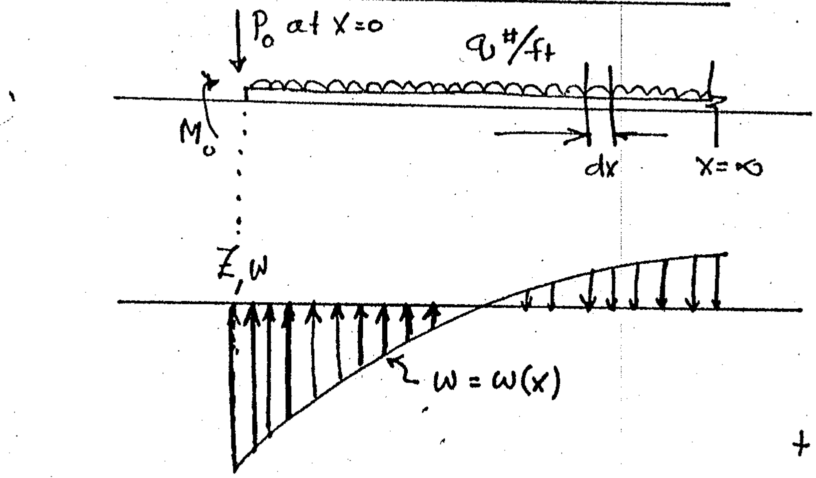
$$\frac{d^2M}{dx^2} = \frac{dV}{dx} = kw - q \qquad (5\cdot2\cdot2)$$

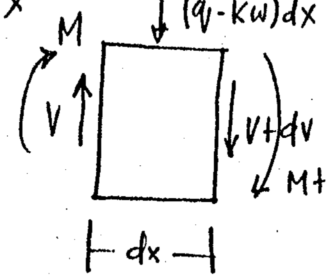
d^2w

BEAM ON ELASTIC FOUNDATION

$K_0 \rightarrow (N/m^2)/m$

$K = K_0 b = \left(\frac{N}{m^2}\cdot m\right)/m \rightarrow \left(\frac{N}{m}\right)/m$





$+\downarrow \sum F_z = 0 \quad -V + (q-kw)dx + V + dV = 0$

$$\frac{dV}{dx} = kw - q$$

$+\circlearrowright \sum M = 0 \quad M + Vdx - (q-kw)dx\cdot\frac{dx}{2} - (M+dM) = 0$

$$\frac{dM}{dx} = V$$

om Elimentary strength of Materials:

EQ (5.2.2)

$M = -EI\,w''$; So, $V = \frac{dM}{dx} = -EI\,w'''$; $kw - q = \frac{dV}{dx} = -EI\,w''''$

EQ (5.2.1)

Finally: $\boxed{EI\,w'''' + kw = q}$

SOLUTION

$$w = e^{\beta x}\left[C_1 \sin\beta x + C_2\cos\beta x\right] + e^{-\beta x}\left[C_3\sin\beta x + C_4\cos\beta x\right] + w(q)$$

where; $\beta = \left(\frac{K}{4EI}\right)^{1/4}$; C_1, C_2, C_3, C_4 are constants of integration and $w(q)$, the particular solution disappears if $q=0$

HANDY TO HAVE CONSTANTS $A_{\beta x}$, $B_{\beta x}$, $C_{\beta x}$ and $D_{\beta x}$ AS IS GIVEN IN PAGE-149 ALSO THE INFORMATIONS IN EQ 5.2.7 AND EQ 5.2.8 ARE USEFUL

$$A_{\beta x} = e^{-\beta x}(\cos\beta x + \sin\beta x) \qquad B_{\beta x} = e^{-\beta x}\sin\beta x \qquad (5.2.7)$$
$$C_{\beta x} = e^{-\beta x}(\cos\beta x - \sin\beta x) \qquad D_{\beta x} = e^{-\beta x}\cos\beta x$$

These quantities are related as follows.

$$A_{\beta x} = -\frac{1}{\beta}\frac{dD_{\beta x}}{dx} = \frac{1}{2\beta^2}\frac{d^2C_{\beta x}}{dx^2} = \frac{1}{2\beta^3}\frac{d^3B_{\beta x}}{dx^3}$$
$$B_{\beta x} = -\frac{1}{2\beta}\frac{dA_{\beta x}}{dx} = \frac{1}{2\beta^2}\frac{d^2D_{\beta x}}{dx^2} = -\frac{1}{4\beta^3}\frac{d^3C_{\beta x}}{dx^3} \qquad (5.2.8)$$
$$C_{\beta x} = \frac{1}{\beta}\frac{dB_{\beta x}}{dx} = -\frac{1}{2\beta^2}\frac{d^2A_{\beta x}}{dx^2} = \frac{1}{2\beta^3}\frac{d^3D_{\beta x}}{dx^3}$$
$$D_{\beta x} = -\frac{1}{2\beta}\frac{dC_{\beta x}}{dx} = -\frac{1}{2\beta^2}\frac{d^2B_{\beta x}}{dx^2} = \frac{1}{4\beta^3}\frac{d^3A_{\beta x}}{dx^3}$$

RELATIONSHIP AMONG CONSTANTS $A_{\beta x}$, $B_{\beta x}$, $C_{\beta x}$ and $D_{\beta x}$

$$D_{\beta x} = e^{-\beta x} \cos \beta x .$$

$$\frac{dD_{\beta x}}{dx} = -\beta e^{-\beta x} \cos \beta x - \beta e^{-\beta x} \sin \beta x = -\beta e^{-\beta x}(\cos \beta x + \sin(\beta x)) = -\beta A_{\beta x}$$

$$A_{\beta x} = e^{-\beta x}(\cos \beta x + \sin \beta x) = -\frac{1}{\beta}\frac{dD_{\beta x}}{dx}$$

$$\frac{dA_{\beta x}}{dx} = -\beta e^{-\beta x} \cos \beta x - \beta e^{-\beta x} \sin \beta x - \beta e^{-\beta x} \sin \beta x + \beta e^{-\beta x} \cos \beta x = -2\beta e^{-\beta x} \sin \beta x$$

$$B_{\beta x} = e^{-\beta x} \sin \beta x = -\frac{1}{2\beta}\frac{dA_{\beta x}}{dx}$$

$$\frac{dB_{\beta x}}{dx} = -\beta e^{-\beta x} \sin \beta x + \beta e^{-\beta x} \cos \beta x = \beta e^{-\beta x}(\cos \beta x - \sin \beta x)$$

$$C_{\beta x} = e^{-\beta x}(\cos \beta x - \sin \beta x) = \frac{1}{\beta}\frac{dB_{\beta x}}{dx}$$

Other relations shown in eq (5.2-8 - p. 148) can be similarly obtained.

The advantage of using $A_{\beta x}$, $B_{\beta x}$ ... is to avoid differentiating functions such as $e^{-\beta x}(\sin \beta x + \cos \beta x)$ ... etc.

Example: Let us assume $C_1 = C_2 = 0$ in eq 5.2.6 for certain boundary conditions as we will see later. Also $q = 0$; Therefore, $w(q) = 0$; then

$$w = e^{-\beta x}(C_3 \sin \beta x + C_4 \cos \beta x) = C_3 B_{\beta x} + C_4 D_{\beta x}$$

$$w' = C_3 \frac{dB}{dx} + C_4 \frac{dD}{dx} = C_3 (\beta C_{\beta x}) + C_4(-\beta A_{\beta x}) = \beta(C_3 C_{\beta x} - C_4 A_{\beta x}) \quad \text{.......(A)}$$

$$w'' = \beta[C_3(-2\beta D) - C_4(-2\beta B)] = 2\beta^2(C_4 B - C_3 D) \quad \text{.......(B)}$$

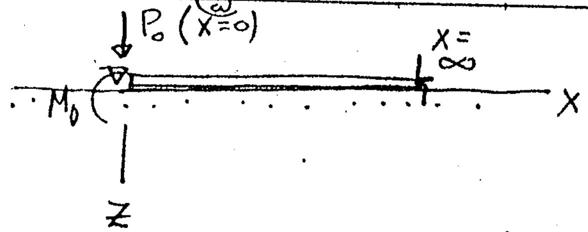
$$w''' = 2\beta^2[C_4(\beta C) - C_3(-\beta A)] = 2\beta^3(C_3 A + C_4 C) \quad \text{.......(C)}$$

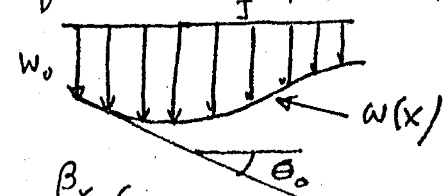
$$w'''' = 2\beta^3[C_3(-2\beta B) + C_4(-2\beta D)] = -4\beta^4[C_3 B + C_4 D] \quad \text{.......(D)}$$

$$\text{.......(E)}$$

EQNS (A) (B) (C) (D) (E) can be used with Boundary conditions.

SEMI-INFINITE BEAM WITH CONC. P_0 & M_0 (ART 5.3)





$$W = e^{\beta x}\left(C_1 \sin\beta x + C_2 \cos\beta x\right) + e^{-\beta x}\left(C_3 \sin\beta x + C_4 \cos\beta x\right)$$

B.C $\Rightarrow$ at $x=\infty$, $w(x)=0$; $e^{-\beta x}\Big|_{x=\infty} = 0$; $e^{\beta x}\Big|_{x=\infty} \neq 0$

~~W=0=~~ if $\sin\beta x = 0$, $\cos\beta x$ cannot be 0 hence for $w=0$

$C_1 = C_2 = 0$; thus:

$$W = e^{-\beta x}\left(C_3 \sin\beta x + C_4 \cos\beta x\right) = C_3 B_{\beta x} + C_4 D_{\beta x} \quad \ldots\ldots (A)$$

B.C $\rightarrow$ at $x=0$; $M = M_0$;

$$M_0 = -EI\,w''\Big|_{x=0} = -EI\,(2\beta^2)(C_4 B - C_3 D) \quad \text{FROM (C)}$$

$$= -2\beta^2 EI\,(-C_3) \quad \text{since } B_{\beta x}\Big|_{x=0} = 0 \ ; \ D_{\beta x}\Big|_{x=0} = 1$$

so; $M_0 = 2\beta^2 EI\, C_3 \Longrightarrow C_3 = \dfrac{M_0}{2\beta^2 EI}$

compare to 5.32 on page 151 SEE TABLE ON Pg-18 (HANDOUT)

It can be shown $M_0 / 2\beta^2 EI = \dfrac{2\beta^2 M_0}{K}$ as follows:

$$\beta = \left(\frac{K}{4EI}\right)^{1/4} ; \quad \beta^2 = \frac{\sqrt{K}}{2\sqrt{EI}} ; \quad 2\beta^2 = \sqrt{K/EI}$$

$$C_3 = \frac{M_0}{2\beta^2 EI} = \frac{M_0}{2\,\frac{\sqrt{K}}{2\sqrt{EI}}\,EI} = \frac{M_0}{\sqrt{K}\,\sqrt{EI}} = \frac{M_0\sqrt{K}}{\sqrt{K}\sqrt{K}\sqrt{EI}}$$

$$= \frac{M_0}{K}\left(\sqrt{\frac{K}{EI}}\right) = \frac{2\beta^2 M_0}{K}$$

B.C $\Rightarrow$ at $x=0$; $V = -P_0 = -EI\,w'''$; $A = 1$; $C = 1$ (SEE TABLE) page 16

$$P_0 = EI\,(2\beta^3)(C_3 A + C_4 C) = EI\,(2\beta^3)(C_3 + C_4)$$

$$= 2EI\beta^3 \frac{M_0}{2\beta^2 EI} + 2EI\beta^3 C_4 = \beta M_0 + 2EI\beta^3 C_4$$

$$C_4 = \frac{P_0}{2EI\beta^3} - \frac{\beta M_0}{2EI\beta^3} = \frac{P_0}{2EI\beta^3} - \frac{M_0}{2EI\beta^2}$$

It can be shown: $\boxed{C_4 = \dfrac{2\beta P_0}{K} - \dfrac{2\beta^2 M_0}{K}}$

Knowing the following interrelationship

$$w$$
$$\downarrow$$
$$\theta = \frac{dw}{dx} = w'$$
$$M = -EI\,w'' = -EI\,\theta'$$
$$V = -EI\,w''' = -EI\frac{dM}{dx} = -EI\,\theta''$$

The important relationships 5.3.3 — 5.3.6 may be Obtained

$$w = \frac{2\beta P_o}{k} D_{\beta x} - \frac{2\beta^2 M_o}{k} C_{\beta x} \qquad (5.3.3)$$

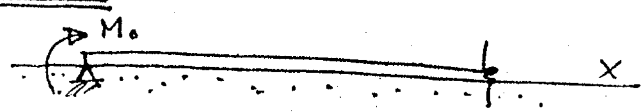
$$\theta = \frac{dw}{dx} \qquad \theta = -\frac{2\beta^2 P_o}{k} A_{\beta x} + \frac{4\beta^3 M_o}{k} D_{\beta x} \qquad (5.3.4)$$

$$M = -EI\frac{d^2w}{dx^2} \qquad M = -\frac{P_o}{\beta} B_{\beta x} + M_o A_{\beta x} \qquad (5.3.5)$$

$$V = -EI\frac{d^3w}{dx^3} \qquad V = -P_o C_{\beta x} - 2\beta M_o B_{\beta x} \qquad (5.3.6)$$

βx	$A_{\beta x}$	$B_{\beta x}$	$C_{\beta x}$	$D_{\beta x}$
0	1	0	1	1
0.02	0.9996	0.0196	0.9604	0.9800
0.04	0.9984	0.0384	0.9216	0.9600
0.10	0.9907	0.0903	0.8100	0.9003
0.20	0.9651	0.1627	0.6398	0.8024
0.30	0.9267	0.2189	0.4888	0.7077
0.40	0.8784	0.2610	0.3564	0.6174
0.50	0.8231	0.2908	0.2415	0.5323
0.60	0.7628	0.3099	0.1431	0.4530
0.70	0.6997	0.3199	0.0599	0.3798
$\pi/4$	0.6448	0.3224	0	0.3224
0.80	0.6354	0.3223	−0.0093	0.3131
0.90	0.5712	0.3185	−0.0657	0.2527
1.00	0.5083	0.3096	−0.1108	0.1988
1.10	0.4476	0.2967	−0.1457	0.1510
1.20	0.3899	0.2807	−0.1716	0.1091
1.30	0.3355	0.2626	−0.1897	0.0729
1.40	0.2849	0.2430	−0.2011	0.0419
1.50	0.2384	0.2226	−0.2068	0.0158
$\pi/2$	0.2079	0.2079	−0.2079	0
1.60	0.1959	0.2018	−0.2077	−0.0059
1.70	0.1576	0.1812	−0.2047	−0.0235
1.80	0.1234	0.1610	−0.1985	−0.0376
1.90	0.0932	0.1415	−0.1899	−0.0484
2.00	0.0667	0.1231	−0.1794	−0.0563
2.20	0.0244	0.0896	−0.1548	−0.0652
$3\pi/4$	0	0.0670	−0.1340	−0.0670
2.40	−0.0056	0.0613	−0.1282	−0.0669
2.60	−0.0254	0.0383	−0.1019	−0.0636
2.80	−0.0369	0.0204	−0.0777	−0.0573
3.00	−0.0423	0.0070	−0.0563	−0.0493
π	−0.0432	0	−0.0432	−0.0432
3.20	−0.0431	−0.0024	−0.0383	−0.0407
3.40	−0.0408	−0.0085	−0.0237	−0.0323
3.60	−0.0366	−0.0121	−0.0124	−0.0245
3.80	−0.0314	−0.0137	−0.0040	−0.0177
$5\pi/4$	−0.0279	−0.0139	0	−0.0139
4.00	−0.0258	−0.0139	0.0019	−0.0120
$3\pi/2$	−0.0090	−0.0090	0.0090	0
2π	0.0019	0	0.0019	0.0019

Prob 5.9.



In this problem $q = 0$, but $P_o \neq 0$ due to hinge support.
First Determine P_o as follows:
From (5.3.3) at $x = 0$, $w = 0$ gives

$$0 = \frac{2\beta}{k} P_o - \frac{2\beta^2}{k} M_o$$

(Note: at $x=0$, $\beta x = 0$, $C_{\beta x} = D_{\beta x} = 1$)

$$P_o = \beta M_o$$

Now; For this problem: Eq 5.3.3 becomes:

$$W = \frac{2\beta^2 M_o}{k}\left(D_{\beta x} - C_{\beta x}\right)$$

$$\Rightarrow W = \frac{2\beta^2 M_o}{k} B_{\beta x}$$

(See EQ 5.2.7 ⇒ D − C = B)

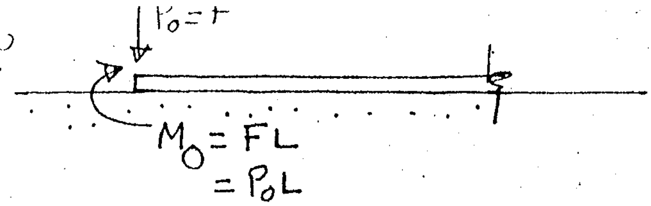
$$w' \Rightarrow \theta = \frac{2\beta^3 M_o}{k} C_{\beta x}$$

$$w'' \Rightarrow M = -EI\,w'' = M_o D_{\beta x}$$

$$w''' \Rightarrow V = \frac{dM}{dx} = -\beta M_o A_{\beta x}$$

(You should try to get these Derivatives)

5.10



(a) at $x=0$; $\theta = 0$; using (5.3.4)

$$0 = -\frac{2\beta^2 F}{k} A_{\beta x} + \frac{4\beta^3 FL}{k} D_{\beta x} \Rightarrow A_{\beta x} = D_{\beta x} = 1 \text{ @ } x=0$$

Giving: $\Rightarrow L = \frac{1}{2\beta}$

(b) Since $\sigma = \pm \frac{My}{I}$, largest stress is at max M

M is max when $\frac{dM}{dx} = 0$; but $\frac{dM}{dx} = V$

So, using (5.3.6)

$$V = \frac{dM}{dx} = 0 = -F C_{\beta x} - 2\beta(FL) B_{\beta x}$$
$$= -FC_{\beta x} - 2\beta F \frac{1}{2\beta} B_{\beta x} = -F\left(C_{\beta x} + B_{\beta x}\right)$$
$$= -F D_{\beta x}$$

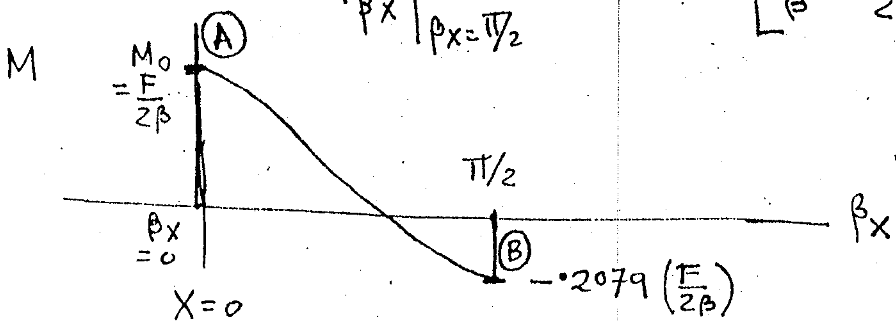
$D_{\beta x} = 0$ at $\beta x = \pi/2$; so From eq (5.3.5)

$$M\Big|_{\beta x = \pi/2} = -\frac{F}{\beta} B_{\beta x}\Big|_{\beta x = \pi/2} + M_0 A_{\beta x}\Big|_{\beta x = \pi/2}$$

$A_{\beta x}\Big|_{\beta x = \pi/2} = 0.2079$, $B_{\beta x} = 0.2079$.

NOTE $M_0 = FL = F\left(\frac{1}{2\beta}\right)$

$$M_{\beta x}\Big|_{\beta x = \pi/2} = -0.2079\left[\frac{F}{\beta} - \frac{F}{2\beta}\right] = -.2079\left(\frac{F}{2\beta}\right) = -.2079\, M_0$$



A) Max Moment $= +M_0$ at $\beta x = 0$ due to Applied moment $M_0 = \frac{F}{2\beta}$

σ_x^+ is at Bottom $\quad +M_0 \quad$ σ_x^- is at top

B) Next High moment $= -.2079 M_0$ at $\beta x = \pi/2$

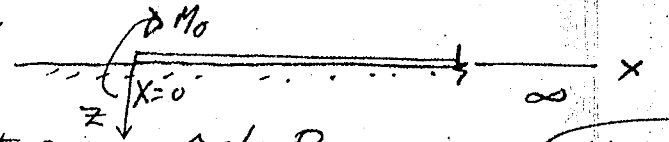
σ_x^+ is at top $\quad$ σ_x^- is at Bottom

ART 5.3 Semi Infinite Beam with P_o and M_o p-20

EQ 5.3.3 — 5.3.6 gives $w(\beta_x)$, $\theta(\beta_x)$, $M(\beta_x)$ and $V(\beta_x)$

$P_o = 0$ gives results for moment only and $M_o = 0$ give results for conc Force only.

Prob: 5.11



a) $\dfrac{|w^-|}{|w^+|} = ?$

In EQ 5.3.3, set $P_o = 0 \Rightarrow w = -\dfrac{2\beta^2 M_o}{K} C_{\beta x}$

M_o at $x=0$ is + ; so at $x=0$ where $C_{\beta x} = 1$;

$w = -\dfrac{2\beta^2}{K} M_o (1)$ (largest upward Motion at $x=0$) ; Table 5.2.1 (p.149) shows

$C_{\beta x}$ has lowest value (i,e largest − value) at $\beta_x = \frac{\pi}{2}$; $C_{\beta x} = -.2079$

So at $\beta x = \frac{\pi}{2}$; largest downward motion

$w = \dfrac{2\beta^2}{K} M_o (.2079)$

So Ratio $\rightarrow \dfrac{|w^-|}{|w^+|} = \dfrac{|1|}{|.2079|} = 4.81$

b) $\dfrac{M_{max}}{M_{min}} = ?$ Using EQ ~~33~~ 5.3.5 with $P_o = 0$

$M = M_o A_{\beta x}$

$A_{\beta x}$ (max at $\beta_{x=0}$) = 1 ; $A_{\beta x}$ (Min at $\beta x = \pi$) = −0.0432

Hence $\dfrac{M_{mx}}{M_{min}} = \dfrac{+1}{-.0432} = -23.15$

(c) F_{UP} = $\boxed{\text{Total}}$ upward force of the foundation for $w > 0 \equiv w(\text{down})^+$

F_{DOWN} = Total downward force of the foundation for $w < 0 = w(\text{up})^-$

EQ 5.3.3 $\Rightarrow w = -\dfrac{2\beta^2 M_o}{K} C_{\beta x} \Rightarrow w(\text{down})^+$ when $C_{\beta x} = -ve$; $\frac{\pi}{4} < \beta_x < \frac{5\pi}{4}$

$\Rightarrow w(\text{up})^-$ when $C_{\beta x} = +ve$; $0 < \beta_x < \frac{\pi}{4}$

also, when $w(\text{down}) \Rightarrow F_{up}$ and $w(\text{up}) \Rightarrow F_{down}$. and

$F = \int K w\, dx = -2\beta^2 M_o \displaystyle\int_{\beta x = 0}^{\beta x = \beta_x^*} C_{\beta x}\, dx \Rightarrow$ From eq (5.2.8) $C_{\beta x} = \dfrac{1}{\beta}\dfrac{dB_{\beta x}}{dx}$

$\Rightarrow \int C_{\beta x}\, dx = 1/\beta\, B_{\beta x}$

$F = -2\beta M_o B_{\beta x}\Big|_{\beta x = 0}^{\beta x = \beta_x^*} \Rightarrow F_{DOWN}\Big|_{\beta x = 0}^{\pi/4} = -2\beta M_o (0 - .3224) = 0.6448\beta M_o$

$\Rightarrow F_{UP}\Big|_{\beta x = \pi/4}^{5\pi/4} = -2\beta M_o (0.3224 - (-.0139)) = -0.6726\beta M_o \Rightarrow |F_{up}| - |F_d| = .0278\beta M_o$

PROB: 5.12

$$I = \frac{1}{12} bh^3 = \frac{1}{12}(200)(40)^3 = 106.67 \times 10^4 \ mm^4$$

$$E = 13 \times 10^9 \ Pa = \frac{13 \times 10^9}{10^6} \ N/mm^2 = 13 \times 10^3 \ N/mm^2$$

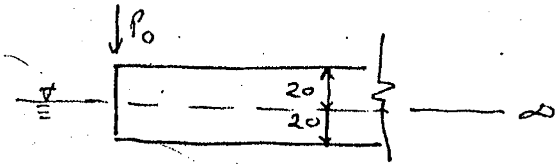
$$EI = 13 \times 106.67 \times 10^7 \ N.mm^2 = 1.387 \times 10^{10} \ N.mm^2$$

$$\gamma_{WATER} = 9810 \ N/m^3 = 9810 \times 10^{-9} \ N/mm^3$$

At depth w; pressure $p = \gamma w =$

$$K_0 = \frac{p}{w} = \gamma; \quad K = b\gamma = 200 \times 9810 \times 10^{-9} \ \frac{N}{mm^2} = 1962 \times 10^{-6} \ \frac{N}{mm^2}$$

$$\beta = \sqrt[4]{K/4EI} = \sqrt[4]{\frac{1962\,(10)^{-6}}{4 \times 1.387 \times 10^{10}}} = \sqrt[4]{\frac{1962}{4 \times 1.387 \times 10^{16}}} = 4.337 \times 10^{-4} \ \cancel{0.4337}$$



n EQ **5.3.3**, **Set** $w = 20$ mm; $M_0 = 0$; $X = 0$

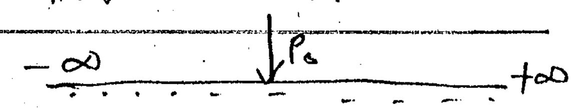
$$20 = \frac{2\beta P_0}{K} D_{\beta x} \quad ; \quad D_{\beta x} = 1 \ @ \ x = 0$$

$$20 = \frac{2\beta P_0}{K}(1) \Longrightarrow P_0 = \frac{10K}{\beta}$$

Using above values.

$$P_0 = \frac{10 \times 1962 \times 10^{-6}}{4.337 \times 10^{-4}} = \frac{19620 \times 10^{-2}}{4.337} = 45.23 \ N.$$

5.4 INFINITE BEAMS WITH CONC LOAD



KEY NOTE ⇒ The Right Half of the Beam is like a semi-Infinite Beam (ART 5.3) provided P_0 is Replaced by $P_0/2$ and M_0 is such that $\theta = 0$ at $X = 0$

Via Eq 5.3.3 ~~&~~ 5.3.4

$$\theta = -\frac{2\beta^2 (P_0/2)}{K} A_{\beta x} + \frac{4\beta^3 M_0}{K} D_{\beta x} = 0 \ \text{at } x = 0$$

gives $\Longrightarrow$ $M_0 = \dfrac{P_0}{4\beta}$

(NOTE: This is not an applied M_0. This $M_0 = \frac{P_0}{4\beta}$ acts due to application of a Load only $(P_0/2)$)

Substituting this above value of $M_0 = P_0/4\beta$ in 5.3.3

$$w = \frac{2\beta (P_0/2)}{K} D_{\beta x} - \frac{2\beta^2 (P_0/4\beta)}{K} C_{\beta x}$$

$$v = \frac{P_0 \beta}{2K}\left[2 D_{\beta x} - C_{\beta x}\right] = \frac{\beta P_0}{2K} A_{\beta x}$$

Once $W = \frac{\beta P_0}{2K} A_{\beta x}$ has been obtained according to eq (5.4.2); EQ 5.4.3 — 5.4.5 should follow by taking Derivatives

$$\theta = -\frac{\beta^2 P_0}{K} B \; ; \; M = \frac{P_0}{4\beta} C \; ; \; V = -\frac{P_0}{2} D$$

CONCENTRATED APPLIED MOMENT M_0 AT $x=0$ (p. 154)

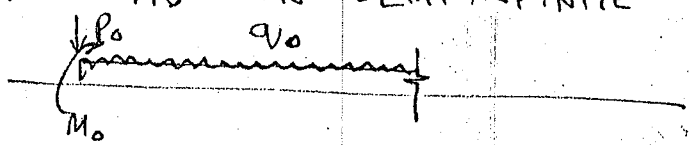
Key Note: Replace (Art 5.3) by ⟹ M_0 By $\frac{M_0}{2}$ and a condition such that P_0 produces $W=0$ at $x=0$ as per EQ 5.3.3.

$$\text{So, } 0 = \frac{2\beta P_0}{K} - \frac{2\beta^2 \left(\frac{M_0}{2}\right)}{K} \Rightarrow P_0 = \frac{M_0 \beta}{2}$$

* { Again This is not an applied P_0. This is a P_0 that resulted from application of M_0 }

EQ 5.4.8 — 5.4.11 are to be noted

DISTRIBUTED LOAD ON SEMI INFINITE BEAM (Art 5.5)

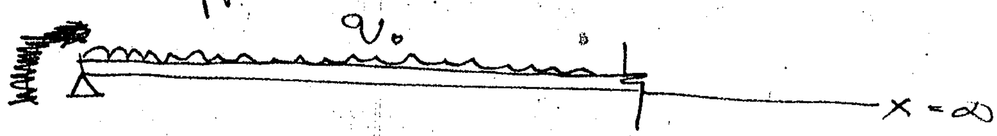


Note effect of q_0 only ($P_0 = 0$, $M_0 = 0$) will be such that $W(q) = \frac{q_0}{K}$

$$\text{So; } \quad W = C_3 B_{\beta x} + C_4 D_{\beta x} + \frac{q_0}{K}.$$

(why $C_1 = 0$ and $C_2 = 0$)

PROB: Determine support Reaction



$M_0 = 0$
$P_0 \neq 0$ even though P_0 has not been applied.
$W = 0$ @ $x = 0$

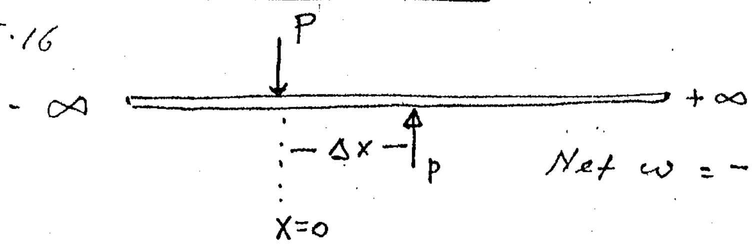
EQ 5.33.

$$W = 0 = \frac{2\beta_0 P_0}{K} \cancel{D_{\beta x}}^{=1} - \frac{2\beta^2 \cancel{M_0}^{=0}}{K} C_{\beta x} + \frac{q}{K}$$

gives $P_0 = \frac{q_0}{2\beta}$ (support Reaction)

PROBLEMS FROM CH-5

5.16



For P at $x=0$ $\quad w = \frac{\beta P}{2K} A_{\beta x}$

For P at $x = x+\Delta x$ $\quad w = \frac{\beta(-P)}{2K} A_{\beta(x+\Delta x)}$

$$\text{Net } w = -\frac{\beta P}{2K}\left(A_{\beta(x+\Delta x)} - A_{\beta x}\right)$$

$$= -\frac{\beta P \Delta x}{2K}\ \frac{\Delta A_{\beta x}}{\Delta x}$$

$$= -\frac{\beta M_0}{2K}\frac{dA_{\beta x}}{dx} = \frac{\beta^2 M_0}{K} B_{\beta x}$$

(Checks with EQ 5.4.6)

5.17. Each half of the beam behaves as a semi infinite beam with end moment M_0 and end rotation $\theta = 0.01$.

(Main Points to note: (1) Foundation is initially flat and the reference point)

By EQ 5.3.4 $\quad \theta = 0.01 = \frac{4\beta^3 M_0}{K}(1)$ giving

a)
$$M_0 = \frac{K(.01)}{4\beta^3} = \frac{0.01(.25)}{4(6.136\times10^{-4})^3} = 2.705 \text{ kN-m}$$

Substitute this value of M_0 in eq 5.3.3

$$W = -\frac{2\beta^2 M_0}{K} C_{\beta x} = -8.15 \text{ mm}$$

b) Apply P_0 to infinite beam to give $W = 8.15$ mm at $X=0$

EQ 5.4.2

$$8.15 = \frac{\beta P_0}{2K}(1) \quad ; \quad P_0 = \frac{2K}{\beta} 8.15 = 6.64 \text{ kN}$$

5.19. $\beta = \sqrt[4]{K/4EI}$ = (with proper Units)

$C = 90$ mm. $\quad K = 10 \frac{N}{mm^2}$; $E = 200\times10^9 \frac{N}{m^2}$

$I = 28.4(10)^6 \text{ mm}^4$; $P_0 = 100\text{ kN} = 10^5 N$ $\quad = 200\times\frac{10^9}{10^6}\frac{N}{mm^2}$

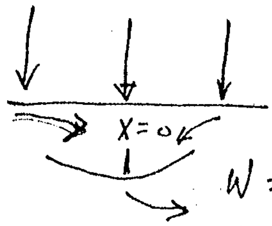
$\beta = 814.5(10^{-6})/\text{mm}$

(a) By EQ 5.4.2 at $X=0$; $\quad W = \frac{\beta P_0}{2K}(1) = \frac{814.5\times10^{-6}\times10^5}{2(10)} = 4.073$

By EQ 5.4.4 at $X=0$; $\quad M = \frac{P_0}{4\beta}(1) = 30.7(10)^6$ N.mm.

$$\sigma = \pm\frac{MC}{I} = \pm(30.7\times10^6\times90)/28.4\times10^6 = 97.3 \text{ MPa}$$
$$= 97.3 \text{ MPa}$$

(b)



$$W = \frac{\beta P_0}{2K}\left(2A_{\beta x} + A_{\beta x}\right) = \frac{814.5\times10^{-6}}{2(10)}(2\times.2924+1) = 6.45$$

For Central load $X=0$; $\beta x = 0$

For 2 side loads $X = 1700$ mm, $\beta x = 1.384$

So under Central load $\quad w = 6.45$ mm.

Similarly, from eq 5.4.4 (under central load)

$$M = \frac{P_o}{4\beta} C_{\beta x} = \frac{10^5}{4(814.5\times10^{6})}\left[2C_{\beta x} + C_{\beta x}\right]$$

(1) @ x=0

@ x=1700 mm (−0.1997)

$$= 18.43(10)^6 \text{ N·mm}$$

What about below an outer load.

$$W = \frac{\beta P_o}{2K}\left[A_0 + A_{1700\beta} + A_{3400\beta}\right]$$

$$= \frac{814.5\times10^{-6}\times10^5}{2(10)}\left[1 + .2924 + (-.0356)\right] = 5.12 \text{ mm}$$

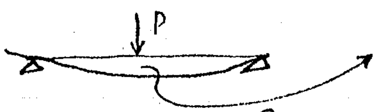
Similarly

$$M = \frac{P_o}{4\beta}\left[C_0 + C_{1700\beta} + C_{3400\beta}\right] = \frac{10^5}{4(814.5\times10^{-6})}\left(1 + (-.1997) + (-.0812)\right)$$

$$= 22.07 \text{ N·m}$$

So. Max w is beneath the Center load
max M is beneath the outer load
so σ_{mx} is ....

5.21



Simply Supported Beam: $\Delta = \frac{PL^3}{48EI}$

So For Cross Beams Spring Constant:

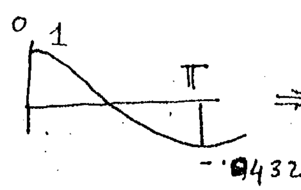
$$K = \frac{P}{\Delta} = \frac{48EI}{L^3_{}} = \frac{48(200{,}000)(25\times10^6)}{(2800)^3} = 10{,}933 \text{ N/mm}$$

$$K_{foundation} = \frac{10{,}933}{500} = 21.87 \text{ N/mm/mm} \text{ — why?}$$

by ... more than ... supports be assumed to ... foundation

$$\beta = \sqrt[4]{\frac{K}{4EI}} = 1.023(10^{-3})\text{/mm}$$

For applied P, Half wave depends on A



$\Rightarrow \beta x = \pi$

$$X = \frac{\pi}{\beta} = 3072 \text{ mm}; \quad \text{Springs/half wave} = \frac{3072}{0.5\times1000} = 6.14 > 4$$

(OK. to replace Cross Beams by Winkler formula)

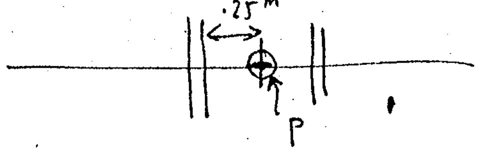
From Load to nearest end A, $X = 4250$ mm; $\beta x = 4250\times1.023\times1$
$= 4.347$

From Table (p-149) all parameters A, B, C, D are very small < 2% at $\beta x > 4.0$, hence the beam can be considered ∞. (long)

$$\text{EQ 5.4.4} \Rightarrow M = \frac{P_o}{4\beta}C = \frac{70{,}000}{4(1.023\times10^{-3})}(1) = 17.1(10)^6 \text{ N·m}$$

$$\sigma = \pm\frac{Mc}{I} = ... = 61.6 \text{ MPa}$$

Load is midway between two Cross Beams



Deflexion at cross Beam

$$w\Big|_{250\beta} = \frac{\beta P_0}{2K} A_{\beta x} = w\Big|_{\beta x = .256}$$

$A_{\beta x}$

$$= \frac{1.023 \times 10^{-3} \times 70{,}000}{2(21.870)} (.945)$$

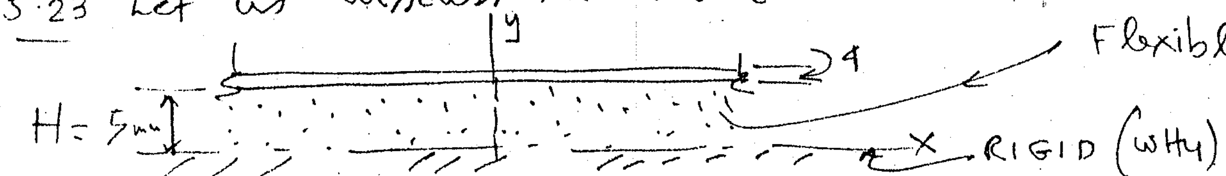
$$= 1.547 \text{ mm}$$

$$F = Kw = 10{,}933(1.547) = 16{,}915 \text{ N}$$

$$M_{Mx} = \frac{FL}{4} = \frac{(16915)(2800)}{4} = 11.84 \times 10^6 \text{ N.mm}$$

$$\sigma = \pm\frac{Mc}{I} \quad \cdots \quad = 42.6 \text{ MPa}$$

5.23 Let us discuss the model.



Estimate Foundation Modulus:
under uniform $\sigma_y \downarrow$: $\varepsilon_x = 0 = \frac{1}{E}(\sigma_x - \nu\sigma_y) \Rightarrow \sigma_x = \nu\sigma_y$

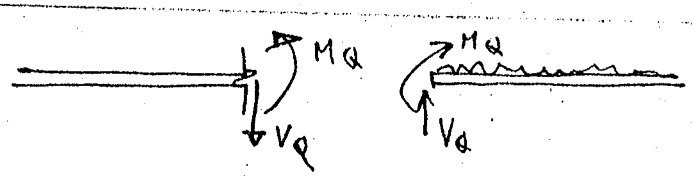
$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) = \frac{1-\nu^2}{E}\sigma_y$$

$$\Delta = H.\varepsilon_y \quad ; \quad K_0 = \frac{\sigma_y}{\Delta} \quad ; \quad K = K_0 b = \frac{\sigma_y b}{\Delta}$$

$$K = \frac{Eb}{H(1-\nu^2)} = \frac{1.6(10)^6(7)}{5(.7975)} = 2.809 \text{ N/mm}^2$$

Rest is Standard.

Hint: (5.36)



$a = 0$
$b = L$

Compute M and V at left end of span L in fig 5.5.2 by use of Eq. 5.5.4 and 5.5.5.

$$M_Q = \frac{q_0}{4\beta^2} B_{\beta L} \qquad V_Q = \frac{q_0}{4\beta}(1 - C_{\beta L})$$

Remove shear and moment at left end of loaded zone by putting these values in 5.3.3 ; 5.3.5 in opposite sense.

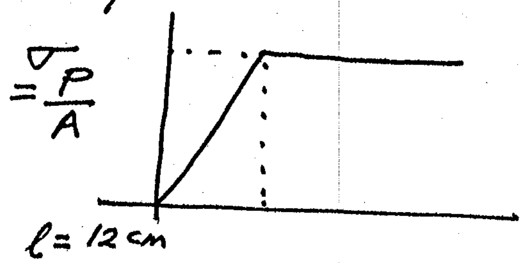
< Will discuss later >

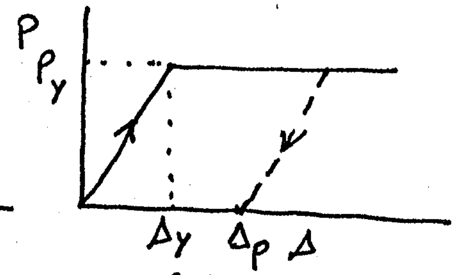
Plastic Collapse of Limit Analysis.

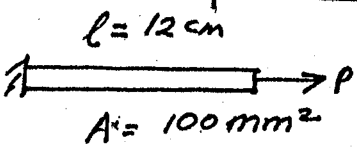
Allowable Stress Concept – Yield stress at any one point.

Limit Analysis – Load that will cause collapse of the Structure.
- Uses less material
- Does not require complete stress analysis
- Construction uncertainties and joint interactions are automatically accounted for.
- Fracture (Brittle) are not considered.

Elastic - Perfectly Plastic case (Simple and Conservative)



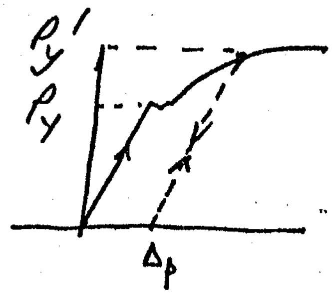




$$P_y = \sigma_y A = (240) \times 10^6 (Pa) \times 100\ (mm^2)$$
$$= 240 \times 10^6 \frac{N}{m^2} \cdot 100 \times 10^{-6} m^2$$
$$= 24\ KN$$

$$\Delta_y = \frac{P_y L}{AE} = \frac{(24 \times 10^3)(12 \times 10^{-2})}{(100 \times 10^{-6})(200 \times 10^9)} = 144\,(10)^{-6}\ m$$

In this statically determinate system, unloading after P_y has been reached will leave a ~~str~~ residual deformation Δ_p with zero stress. Further loading will again cause yielding at P_y = thus no advantage. In real life (Strain Hardening material) – This is not true.

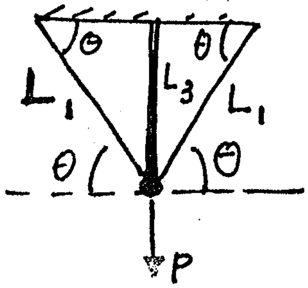


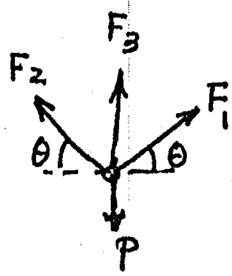
✓ Statically Indeterminate Cases offer advantage
even for idealized Elastic – Perfectly Plastic material

✓ Higher the degree of Redundancy, the greater the advantage

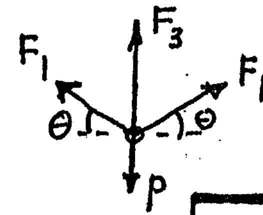
Consider the example in Page 445 - 447

Let us solve the statically indeterminate Elastic Problem first. ——— (Not in Book here)



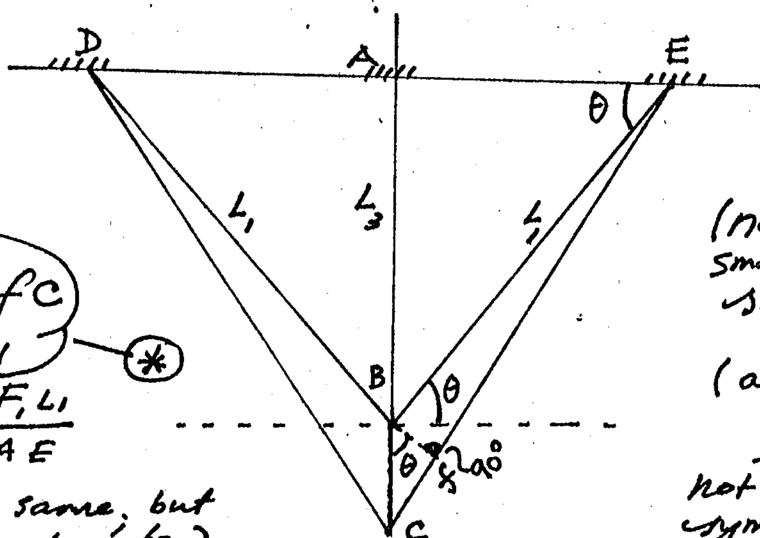


$\Sigma F_x = 0$ gives
$F_1 = F_2 \implies$



$\Sigma F_y = 0 \Rightarrow \boxed{P = F_3 + 2F_1 \sin\theta}$ —(1)

Eq 1; gives 2 unknowns; Problem of Degree of Redundancy = 1
(since number of unknowns are more than no. of Eq by 1)
Another Equation may be obtained from Deformation Condition.



$AB = L_3$
$BC = \Delta_3$
$EB = DB = L_1$
$fC = \Delta_1$

(note: $EB = Ef$ for small deformation since, $Bf \perp EC$ — Why)

(also note: AB moves vertically downward to AC. This will not be the case if symmetry is not there. But the problem still may be solved with similar Principle; but higher Redundancy)

$BC \sin\theta = fC$
$\Delta_3 \sin\theta = \Delta_1$ —(*)

$$\frac{F_3 L_3}{AE} \sin\theta = \frac{F_1 L_1}{AE}$$

(note: AE are same, but does not have to be)

Since $L_3 = L_1 \sin\theta$

$F_3 \sin^2\theta = F_1$ —(2)

Substitute (2) in (1) $\longrightarrow$ $P = F_3 + 2F_3 \sin^3\theta \longrightarrow F_3 = \dfrac{P}{(1 + 2\sin^3\theta)}$ ...(3)

Substituting (3) in (2) $\longrightarrow$ . . . . $F_1 = \dfrac{P\sin^2\theta}{(1+2\sin^3\theta)}$ ...(4)

CHECK: Using $\theta = 45°$

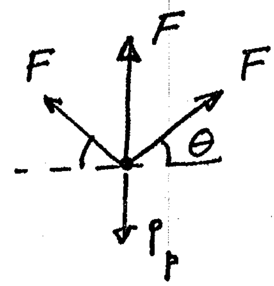
$F_3 = .5858\,P$; $F_1 = .2929\,P$ and substituting these in ...(1) the equation is checked.

Thus F_3 will yield while F_1 will still be elastic. Find P for $\frac{F_3}{A} = \sigma_y$

Let us now use the Plastic Analysis approach

TRETS

Let us assume (or we want) all three bars to yield simultaneously for "COLLAPSE". You have to set this criterion for plastic collapse first. Now find P_p for which this will happen.



F (same for all three)

$= \sigma_y A = 240\times10^6\ Pa.(100\times10^{-6})m^2$

$= 24\ KN$

$\Sigma F_y = 0 \Longrightarrow P = F + 2F\sin\theta$

For $\theta = 45^\circ \Longrightarrow P = 24 + 2(24)\left(\frac{1}{\sqrt{2}}\right)$

$P = \underline{\underline{57.94\ KN}}$

If you were to compare elastic analysis; $F_1 = .2935P$ and $F_3 = .5873P$; $F_1 = \frac{1}{2}F_3$. So when F_3 yields i.e $F_3 = 24$ KN.

$F_1 = 12$ KN making $P = 24 + 2(12)\frac{1}{\sqrt{2}} = \underline{\underline{40.92\ KN}}$

Thus the plastic collapse load P_p is $\left(\frac{57.94 - 40.92}{40.92}\right) = 41.6\%$ greater than the elastic limiting load P

DEFLECTION

Elastic case: $\Delta_{AB} = \Delta_3 = BC = \frac{F_3 L_3}{AE}$; using $F_3 = 24$ K.N at the limit (it has just reached that load)

and $L_3 = 12$ cm; $A = 100\ mm^2$, $E = 200$ GPa.

$$\Delta_{AB} = 144(10^{-6})\ m$$

Notice the other two members will have deflection compatible to a stress of 12 KN; with $L_1 = \frac{L_3}{\sin\theta}$ and $F_1 = \frac{1}{2}F_3$ one may find.

$$\Delta_{EB} = fC = \frac{F_1 L_1}{AE} = \frac{.5F_3}{AE}\left(\frac{L_3}{\sin\theta}\right) = \frac{1}{2}\frac{F_3 L_3}{AE(\frac{1}{\sqrt{2}})}$$

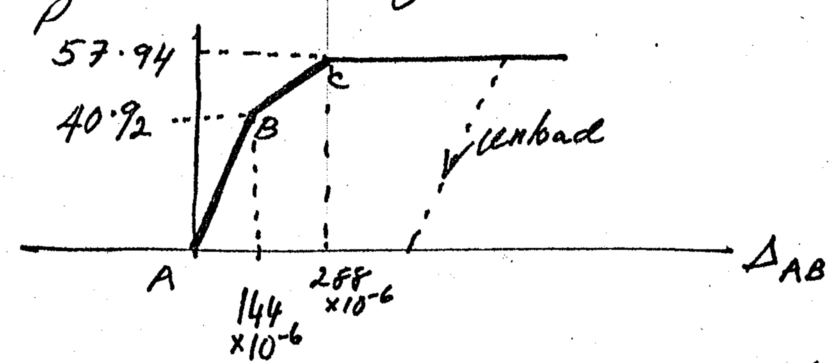
$$= \frac{F_3 L_3}{AE}\left(\frac{1}{\sqrt{2}}\right) = \Delta_3 \sin\theta$$

(This is compatible to eq * in page - 29 of Hand out)

Plastic Case: Realise that the two side members will just become plastic and the vertical member has become plastic long ago. So $\Delta_{EB} = 144(10^{-6})$; Therefore

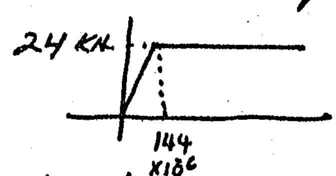
$$\Delta_{AB} = \Delta_3 = \sqrt{2}\,\Delta_{EB} = 288(10^{-6})\ m.$$

Load Deflection Behaviour of The Whole Structure



AB — all three members are elastic; at B Vertical member becomes plastic.

BC — The Vertical member keeps on deforming as per



and the two other members are still elastic.

at C two side members just become plastic.

If the unloading is done after C as shown, all three members will have residual Δ.

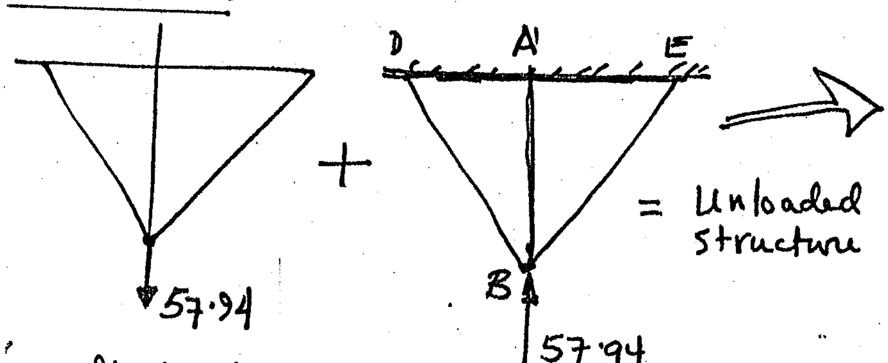
Safety Factor. For safety factor = 2

Elastic Analysis max load $= P = \frac{40.92}{2} = 20.46$ KN

Plastic Analysis max load $= P = \frac{57.94}{2} = 28.97$ KN

Note: Entire structure and each individual members are elastic in both cases, but Plastic analysis allows 41% more load.

UNLOADING



(Plastically deformed)

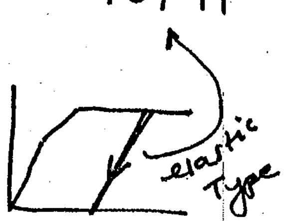
$F_1 = .2929\,(P) = .2929\,(-57.94) = -16.97$ KN

$F_3 = .5858\,(P) = .5858\,(-57.94) = -33.94$ KN

So Net residual load after unloading

$F_3 = 24 - 33.94 = -9.94$ KN

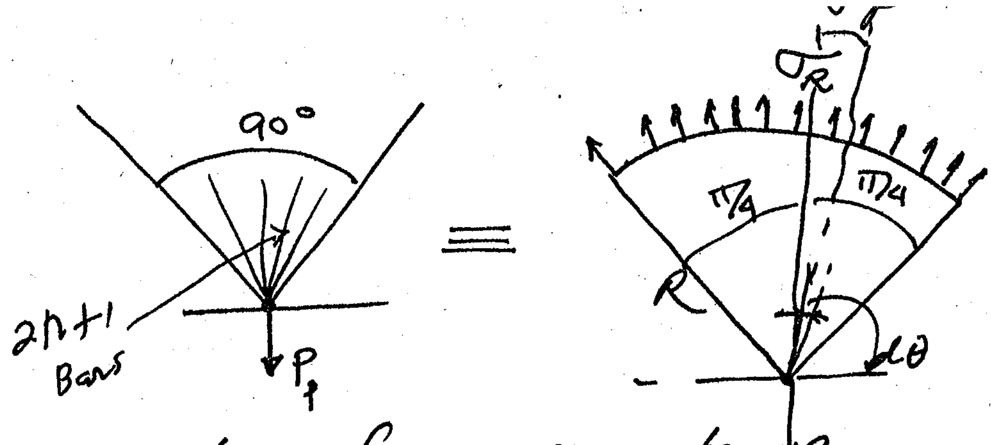
$F_1 = 24 - 16.97 = 7.03$ KN



(TREATMENT IS DIFFERENT THAN BOOK)

STUDY THE REST OF P-447 FOR CONCEPT OF RESIDUAL STRESS

Problem 11.1



Since n is large, bar forces can be smeared into a radial stress σ_R

Each bar spans an angle $d\theta = \dfrac{\pi/2}{2n} = \dfrac{\pi}{4n}$

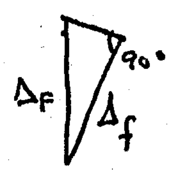
t = thickness of imaginary sheet on which σ_R acts

F = Force in one Bar

$$\sigma_R = \frac{F}{AREA} = \frac{F}{tR\,d\theta} = \frac{F}{tR\frac{\pi}{4n}} = \frac{4nF}{\pi t R}$$

$$P_p = 2\int_0^{\pi/4} \sigma_R (R t\, d\theta)\cos\theta = \frac{8nF}{\pi}\int_0^{\pi/4}\cos\theta \cdot d\theta = \frac{4\sqrt{2}\, nF}{\pi}$$

b) P_p is replaced by P_y. Here Bar force F prevails only at $\theta = 0$. Elsewhere it is less, say f. To find f in terms of F, consider Bar at $\theta = 0$



$\Delta_f = \Delta_F \cos\theta$; $\Delta_F = \dfrac{FL}{AE}$

$\Delta_f = \dfrac{F(L/\cos\theta)}{AE}$

$$f = F\cos^2\theta$$

Now apply analysis of Part (a)

$$\sigma = \frac{f}{area} = \frac{F\cos^2\theta}{tR\frac{\pi}{4n}} = \frac{4nF\cos^2\theta}{\pi t R}$$

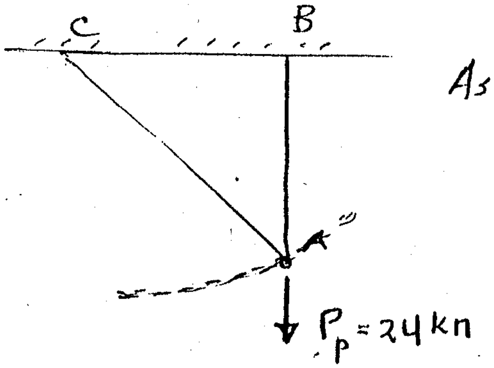
$$P_y = 2\int_0^{\pi/4} \sigma (Rt\, d\theta)\cos\theta = \frac{8nF}{\pi}\int_0^{\pi/4}\cos^3\theta\, d\theta$$

$$P_y = \frac{8nF}{\pi}\left(\sin\theta - \frac{\sin^3\theta}{3}\right)\Bigg|_0^{\pi/4} = \frac{10\sqrt{2}\, nF}{3\pi}$$

$$\frac{P_p}{P_y} = \frac{4\sqrt{2}}{\pi}\cdot\frac{3\pi}{10\sqrt{2}} = \frac{6}{5} = 1.2$$

PROB - 11.2

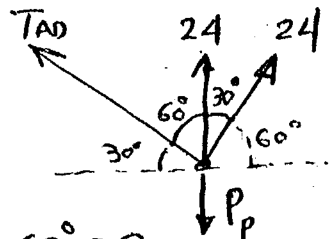
(a)



As soon as AB yields, point A swings in an arc with point C as center with essentially no change in load

— WHY

(b) Assuming Bars AC and AB Both yields; find T_{AD}



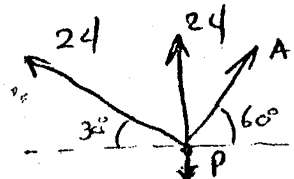
$\Sigma F_x = 0$ gives

$$-T_{AD}\cos 30^\circ + 24\cos 60^\circ = 0$$

$$T_{AD} = 13.86 \text{ KN} < 24$$

Therefore, T_{AD} will not yield.

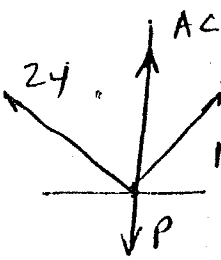
[Note: other possibilities ⟹



$$-24\cos 30^\circ + AB\cos 60^\circ = 0$$

$$AB = 24 \cdot \frac{.866}{.5} > 24$$

(not acceptable)

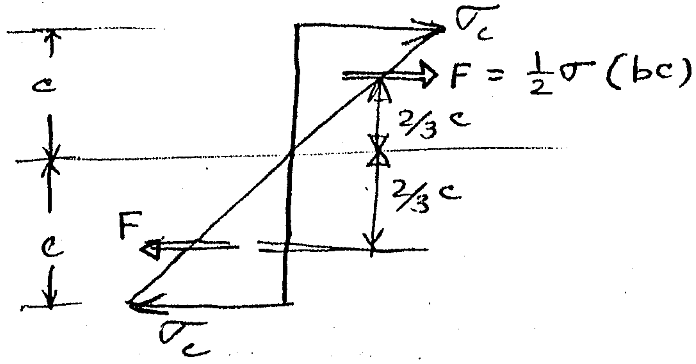


not possible since $\Sigma F_x = 0$ not satisfied.

⟹ So WHAT IS GOING TO HAPPEN

Plastic Hinge in Bending

Relation between Moment & stress



$$M = F\left(\frac{4}{3}c\right)$$

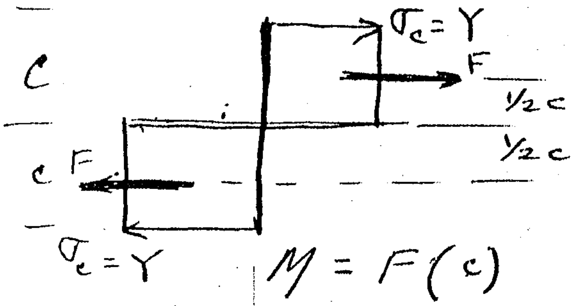
$$M = \frac{1}{2}\sigma bc\,\frac{4}{3}c$$

$$\sigma = \frac{3}{2}M/bc^2$$

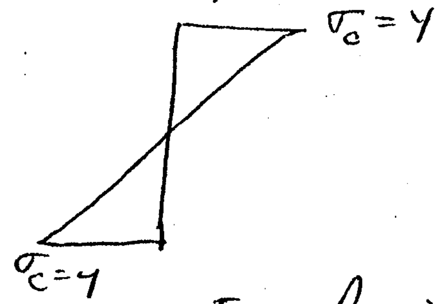
$$= \frac{Mc}{I} = \frac{Mc}{\frac{1}{12}b(2c)^3}$$

Extend the same principle.

Fully plastic



Just yielded



$$M = F(c)$$
$$= \sigma_c (bc)(c)$$
$$M_p = \sigma_c bc^2$$

if $\sigma_c = Y$ (as it is)

$$M_p = \sigma_c bc^2 = Ybc^2$$

From Previous Page

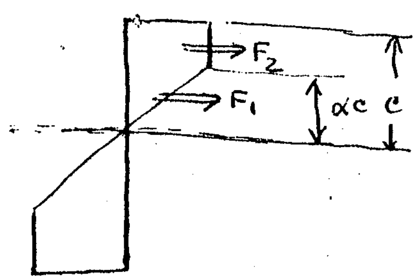
$$M_y = \frac{2}{3}\sigma_c bc^2$$

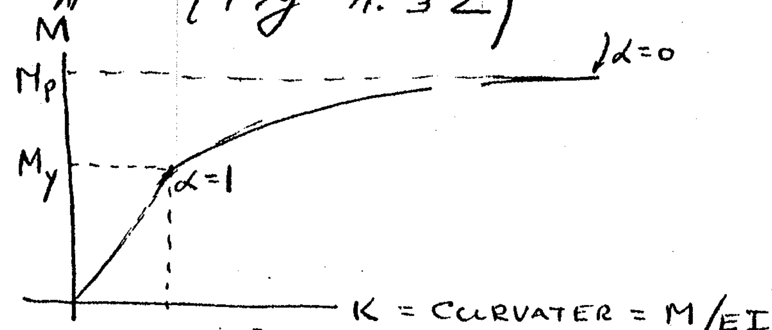
When $\sigma_c = Y$

$$M_y = \frac{2}{3} Y bc^2$$

Therefore $\dfrac{M_p}{M_y} = \dfrac{3}{2} = 1.5$ ⟸ This is for a Rectangular cross section

General Treatment (Fig 11.3.2)





$$M = \frac{4\alpha c}{3}F_1 + (1+\alpha)cF_2 = \frac{3-\alpha^2}{3}bc^2Y \quad \text{......}(11.3.1)$$

You can get $M_y = \frac{2}{3}Ybc^2$ and $M_p = Ybc^2$ by substituting $\alpha = 1$ and $\alpha = 0$ respectively on eq (11-3-1)

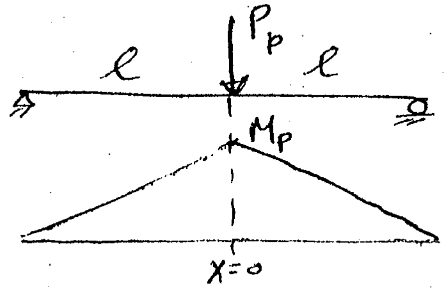
K = curvature = M/EI (only applies to the elastic core)

$$M = \frac{4\alpha c}{3}F_1 \quad ; \quad F_1 = \left(\tfrac{1}{2}Y\right)(b)(\alpha c) \quad ; \quad I = \tfrac{1}{12}b(2\alpha c)^3$$

$$K = \frac{Y}{E\alpha c} \quad \text{....}\ (11\text{-}3\text{-}4)$$

and substituting α from (11-3-4) → (11-3-1) ⟹ $M = \left[c^2 - \frac{1}{3}\left(\frac{Y}{EK}\right)\right]bY \cdots (11.3.5)$

Lateral Extent of Plastic Zone



$$M = M_p\left(\frac{\ell - x}{\ell}\right) = bc^2 Y\left(\frac{\ell - x}{\ell}\right)$$

Combining this with (11.3.1)

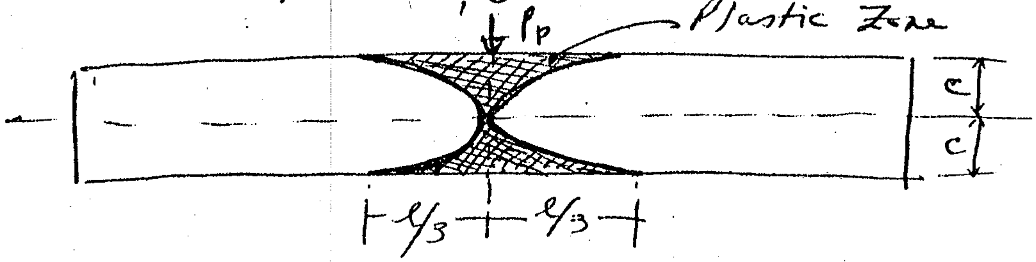
$$M = \frac{3-\alpha^2}{3}\, bc^2 Y$$

We get;

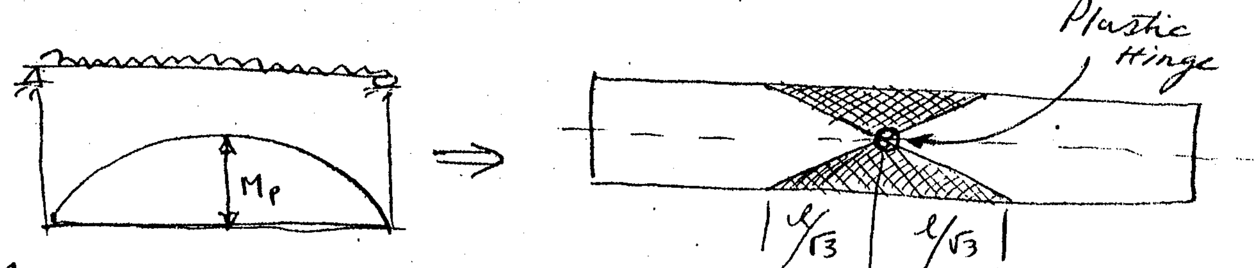
$$\frac{3-\alpha^2}{3} bc^2 Y = bc^2 Y\left(\frac{\ell - x}{\ell}\right)$$

$$\alpha = \sqrt{\frac{3x}{\ell}} \quad \text{.... (11.3.7)}$$

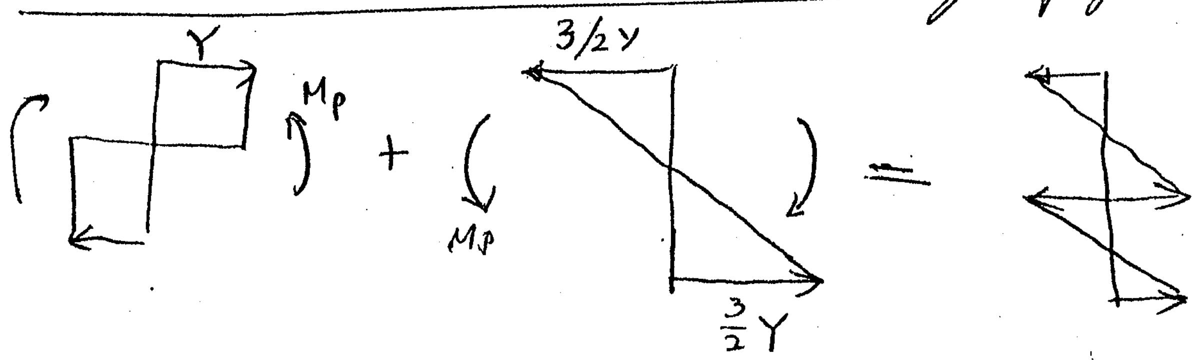
Note at $x = 0$; $\alpha = 0$ (fully plastic)
at $x = \frac{\ell}{3}$; $\alpha = 1$ (just yielded)



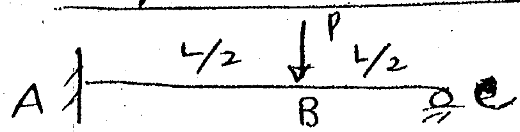
For a uniformly Distributed load case one can show that:



Residual Stress due to unloading Fig. 11.3.4



Collapse Load Calculation (11.4 – p. 451)



✓ $M_A = -\ 3PL/16$

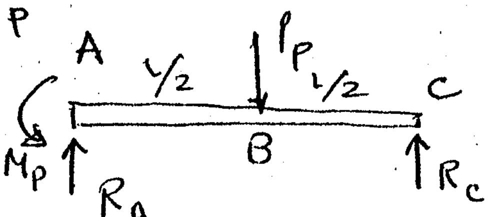
✓ $M_B = 5PL/32$

✓ So yield will begin at A

✓ It will start when $M_A = M_Y$; so corresponding $P_y = 16\, M_y/3L$

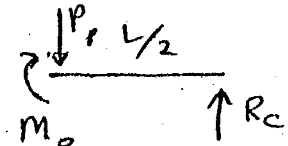
✓ Now $P = P_y$ can be gradually increased until $M_p = M_A = M_B$; Then $P = P_p$

✓ Find P_p from statics.



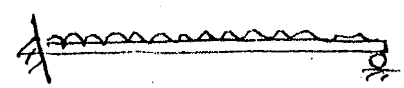
about A $\quad R_c L - P_p \frac{L}{2} + M_p = 0$

About B $\quad R_c \frac{L}{2} - M_p = 0$



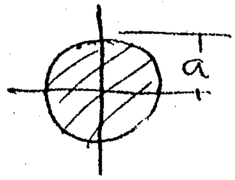
Eliminating R_c, $\quad P_p = \dfrac{6 M_p}{L}$

(No need to get involved into Virtual work now)

Similarly: for 

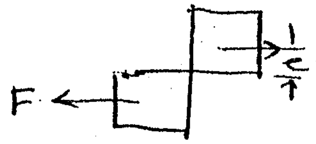
$$q_p = \frac{11.733\, M_p}{L^2}$$

PROB. 11.3 (a)



$$M_y = \frac{YI}{a} = Y \frac{\pi a^4}{a\,4} = \frac{Y a^3}{4}$$

$$F = Y \frac{\pi a^2}{2} \quad ; \quad e = \frac{4a}{3\pi}$$

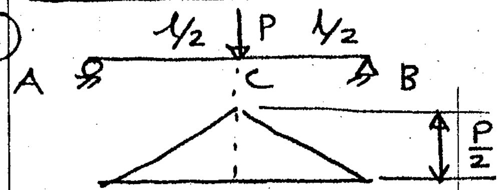


$$M_p = F(2e) = \frac{4Ya^3}{3}$$

$$\frac{M_p}{M_y} = \frac{4/3}{\pi/4} = 1.7$$

A Few Cases of End moments and Reactions

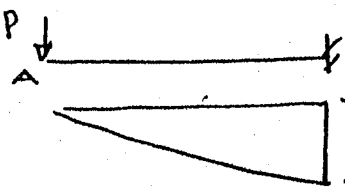
(1)



$R_A = R_B = P/2$

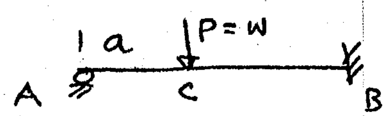
$\frac{P}{2} \cdot \frac{l}{2} = \frac{Pl}{4}$ = Moment at c

(2)



$Pl = M_B$

(3) **CASE 1c** (p. 97: Young and Roark)



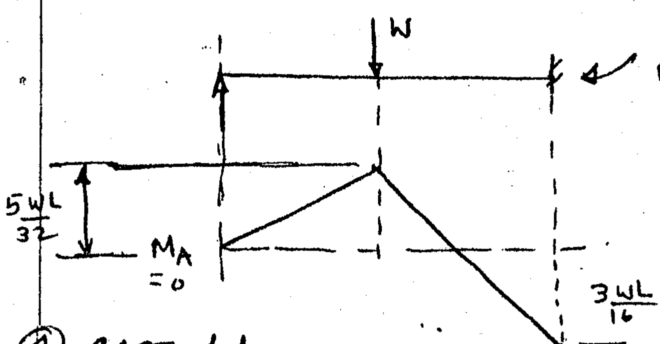
$R_A = \frac{W}{2l^3}(l-a)^2(2c+L) = \frac{W}{2L^3} \cdot \frac{l^2}{4} \cdot \frac{5}{2} l = \frac{5}{16} W$ (for $a = \frac{l}{2}$)

$R_B = \frac{11}{16} W$

$M_A = 0$

$M_c = R_A \cdot a = \frac{5}{16} Wa = \frac{5WL}{32}$ (for $a = \frac{l}{2}$)

$M_B = -\frac{Wa}{2l^2}(l^2 - a^2) = -\frac{Wl}{4l^2} \cdot \frac{3}{4} l^2 = -\frac{3WL}{16}$

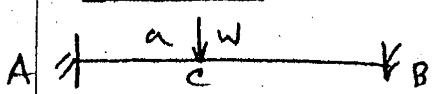


$M_B = R_A L - W\frac{l}{2}$

$= \frac{5WL}{16} - \frac{WL}{2}$

$= \frac{5WL - 8WL}{16} = -\frac{3WL}{16}$

(4) **CASE 1d**

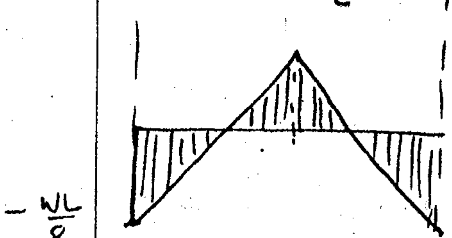


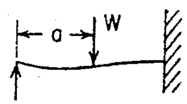
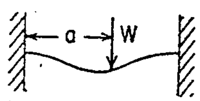
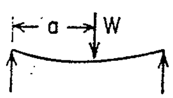
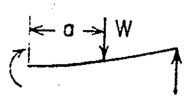
$R_A = \frac{W}{L^3}(l-a)^2(l+2a) = \frac{W}{L^3} \cdot \left(\frac{l^2}{4}\right)(2l) = \frac{W}{2}$

so $R_B = \frac{W}{2}$

$M_A = -\frac{Wa}{L^2}(l-a)^2 = -\frac{WL}{2L^2} \frac{l^2}{4} = -\frac{WL}{8}$

$M_c = -\frac{WL}{8} + \frac{W}{2} \cdot \frac{L}{2} = -\frac{WL}{8} + \frac{WL}{4} = \frac{WL}{8}$

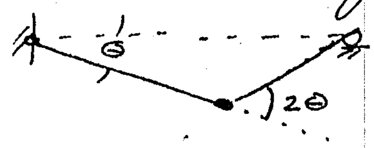


1c. Left end simply supported, right end fixed 	$R_A = \frac{W}{2l^3}(l-a)^2(2l+a) \quad M_A = 0$ $\theta_A = \frac{-Wa}{4EIl}(l-a)^2 \quad y_A = 0$ $R_B = \frac{Wa}{2l^3}(3l^2 - a^2) \quad \theta_B = 0$ $M_B = \frac{-Wa}{2l^2}(l^2 - a^2) \quad y_B = 0$	Max $+M = \frac{Wa}{2l^3}(l-a)^2(2l+a)$ at $x = a$; max possible value $= 0.174Wl$ when $a = 0.366l$ Max $-M = M_B$; max possible value $= -0.1924Wl$ when $a = 0.5773l$ Max $y = \frac{-Wa}{6EI}(l-a)^2\left(\frac{a}{2l+a}\right)^{1/2}$ at $x = l\left(\frac{a}{2l+a}\right)^{1/2}$ when $a > 0.414l$ Max $y = \frac{-Wa(l^2-a^2)^3}{3EI(3l^2-a^2)^2}$ at $x = \frac{l(l^2+a^2)}{3l^2-a^2}$ when $a < 0.414l$; max possible $y = -0.0098\frac{Wl^3}{EI}$ when $x = a = 0.414l$
1d. Left end fixed, right end fixed 	$R_A = \frac{W}{l^3}(l-a)^2(l+2a)$ $M_A = \frac{-Wa}{l^2}(l-a)^2$ $\theta_A = 0 \quad y_A = 0$ $R_B = \frac{Wa^2}{l^3}(3l - 2a)$ $M_B = \frac{-Wa^2}{l^2}(l-a)$ $\theta_B = 0 \quad y_B = 0$	Max $+M = \frac{2Wa^2}{l^3}(l-a)^2$ at $x = a$; max possible value $= \frac{Wl}{8}$ when $a = \frac{l}{2}$ Max $-M = M_A$ if $a < \frac{l}{2}$; max possible value $= -0.1481Wl$ when $a = \frac{l}{3}$ Max $y = \frac{-2W(l-a)^2a^3}{3EI(l+2a)^2}$ at $x = \frac{2al}{l+2a}$ if $a > \frac{l}{2}$; max possible value $= \frac{-Wl^3}{192EI}$ when $x = a = \frac{l}{2}$
1e. Left end simply supported, right end simply supported 	$R_A = \frac{W}{l}(l-a) \quad M_A = 0$ $\theta_A = \frac{-Wa}{6EIl}(2l-a)(l-a) \quad y_A = 0$ $R_B = \frac{Wa}{l} \quad M_B = 0$ $\theta_B = \frac{Wa}{6EIl}(l^2 - a^2) \quad y_B = 0$	Max $M = R_A a$ at $x = a$; max possible value $= \frac{Wl}{4}$ when $a = \frac{l}{2}$ Max $y = \frac{-Wa}{3EIl}\left(\frac{l^2-a^2}{3}\right)^{3/2}$ at $x = l - \left(\frac{l^2-a^2}{3}\right)^{1/2}$ when $a < \frac{l}{2}$; max possible value $= \frac{-Wl^3}{48EI}$ at $x = \frac{l}{2}$ when $a = \frac{l}{2}$ Max $\theta = \theta_A$ when $a < \frac{l}{2}$; max possible value $= -0.0642\frac{Wl^2}{EI}$ when $a = 0.423l$
1f. Left end guided, right end simply supported 	$R_A = 0 \quad M_A = W(l-a) \quad \theta_A = 0$ $y_A = \frac{-W(l-a)}{6EI}(2l^2 + 2al - a^2)$ $R_B = W \quad M_B = 0$ $\theta_B = \frac{W}{2EI}(l^2 - a^2) \quad y_B = 0$	Max $M = M_A$ for $0 < x < a$; max possible value $= Wl$ when $a = 0$ Max $\theta = \theta_B$; max possible value $= \frac{Wl^2}{2EI}$ when $a = 0$ Max $y = y_A$; max possible value $= \frac{-Wl^3}{3EI}$ when $a = 0$

Collapse load Calculation Continued

Virtual work

Work done by load $\equiv$ Work done by int moment

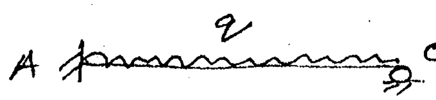


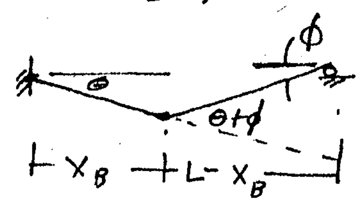
$$P_p \, \theta \frac{L}{2} = M_p(\theta) + M_p(2\theta)$$

$$P_p = \frac{6 M_p}{L}$$

Consider Fig 11.4.2 (b) and (c) for distributed load

$$q_p x_B \left(\theta \frac{x_B}{2}\right) + q_p (L - x_B)\left(\phi \frac{L - x_B}{2}\right) = M_p(\theta) + M_p(\theta + \phi)$$





Substituting $\phi = \theta x_B / (L - x_B)$

$$q_p = \frac{2 M_p}{L^2} \; \frac{2 - \frac{x_B}{L}}{\frac{x_B}{L}\left(1 - \frac{x_B}{L}\right)}$$

If we assume Hinge B is at the Mid span $x_B = L/2$ then

$$q_p = \frac{12 M_p}{L^2}$$

OR B is where max elastic moment is i.e. $x_B = \frac{5}{8} L$ then

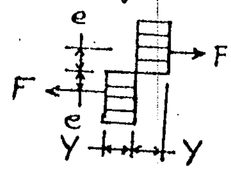
$$q_p = \frac{11.733 \, M_p}{L^2}$$

The correct hinge location is for min q_p

$$\frac{d q_p}{d x_B} = 0 \qquad \frac{x_B}{L} = 2 - \sqrt{2}$$

$$q_p = \frac{11.657 \, M_p}{L^2}$$

11.3 (c) $M_y = \frac{YI}{c} = \frac{Y}{c}\left(2\frac{bc^3}{12}\right) = \frac{Ybc^2}{6}$



$F = Y\frac{bc}{2}$, $e = \frac{c}{3}$

$M_p = F(2e) = \frac{Ybc^2}{3}$

$\frac{M_p}{M_y} = \frac{1/3}{1/6} = 2.00$

(d) $M_y = \frac{YI}{c} = \frac{Y}{7}\frac{1}{12}\left[10(14)^3 - 9.4(12)^3\right]$

$M_y = 133.3\,Y$

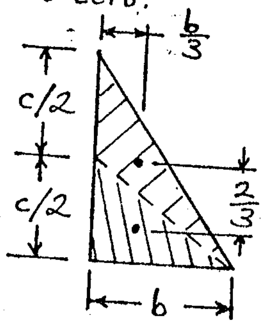
For M_p, use Eq. 11.3.3, and write

$M_p = (\text{total}) - (9.4 \times 12 \text{ rectangle})$

$M_p = 10(7)^2 Y - 9.4(6)^2 Y = 151.6\,Y$

$\frac{M_p}{M_y} = \frac{151.6}{133.3} = 1.137$

11.4. Centroids of tension and compression areas (shown by dots in the sketch below) must lie on the same vertical line, so as to form M_p moment vector perpendicular to the plane of loads. Also, the two areas must be equal so that axial forces sum to zero.



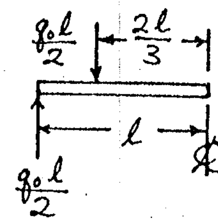
Resultant force F at centroids:

$F = Y\frac{(c/2)b}{2}$

$F = \frac{bcY}{4}$

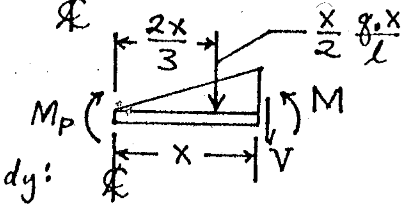
$M_p = F\frac{c}{3} = \frac{bc^2 Y}{12}$

11.7 Free body of left half of beam:



$$M_P = \frac{q_0 l}{2} l - \frac{q_0 l}{2}\frac{2l}{3} = \frac{q_0 l^2}{6}$$

Second free-body:



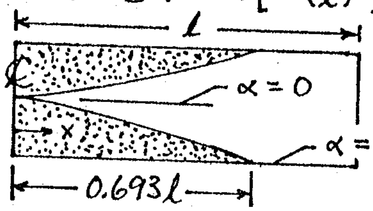
Sum moments about right end:

$$M = M_P - \frac{q_0 x^2}{2l}\frac{x}{3} = M_P - \frac{q_0 x^3}{6l}$$

$$M = M_P - \frac{q_0 l^2}{6}\frac{x^3}{l^3} = M_P\left[1-\left(\frac{x}{l}\right)^3\right]$$

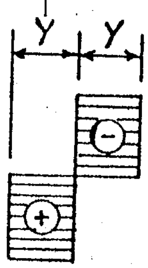
Combine with Eq. 11.3.1:

$$M_P\left(1-\frac{\alpha^2}{3}\right) = M_P\left[1-\left(\frac{x}{l}\right)^3\right], \quad \alpha = \sqrt{3}\left(\frac{x}{l}\right)^{3/2}$$



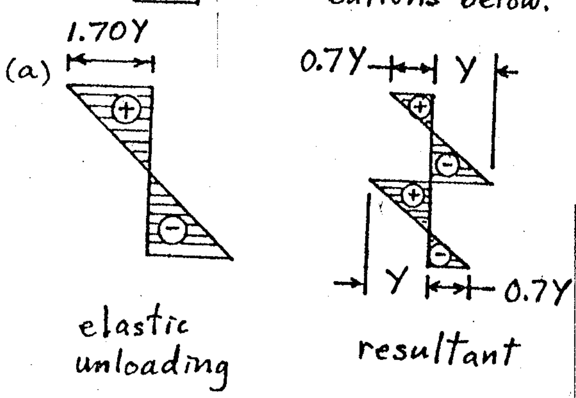
The dotted zone has yielded.

11.10

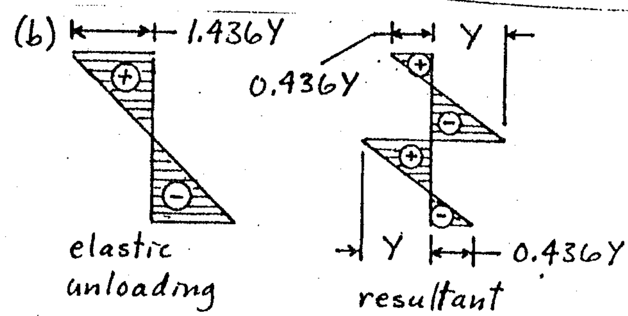


This fully plastic stress distribution is superposed on all elastic stress distributions below.

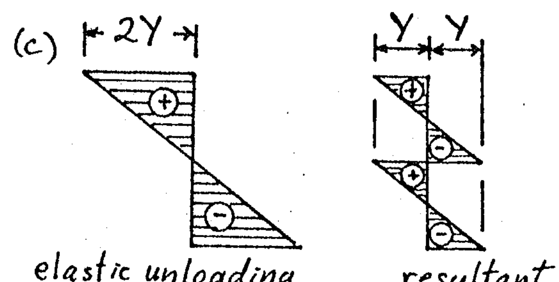
(a)



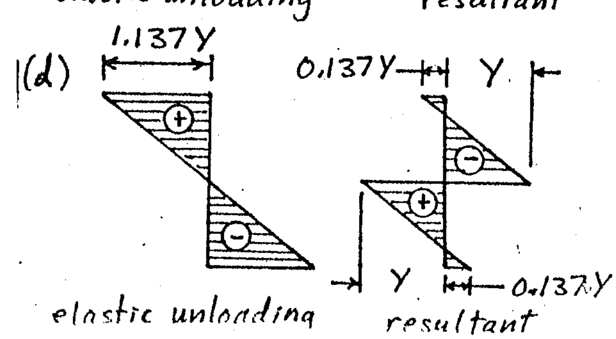
(b)



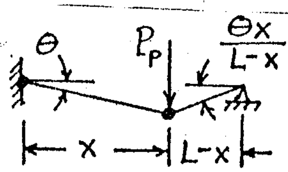
(c)



(d)



11.18



$$P_p(\theta x) = M_p\left(\theta + \theta + \frac{\theta x}{L-x}\right)$$

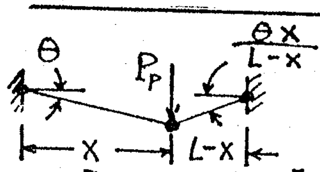
$$P_p = \frac{2L-x}{x(L-x)} M_p$$

$$\frac{dP_p}{dx} = 0 \quad \text{leads to} \quad x^2 - 4Lx + 2L^2 = 0$$

$$x = \frac{4L + \sqrt{16L^2 - 8L^2}}{2} = 0.586L$$

$$\text{Min. } P_p = 5.828\frac{M_p}{L}$$

11.19



$$P_p(\theta x) = M_p\left[2\left(\theta + \frac{\theta x}{L-x}\right)\right]$$

$$P_p = \frac{2L}{x(L-x)} M_p. \quad \frac{dP_p}{dx} = 0 \quad \text{leads to}$$

$$L - 2x = 0, \quad x = L/2 \quad (\text{as expected})$$

$$\text{Min. } P_p = 8M_p/L$$

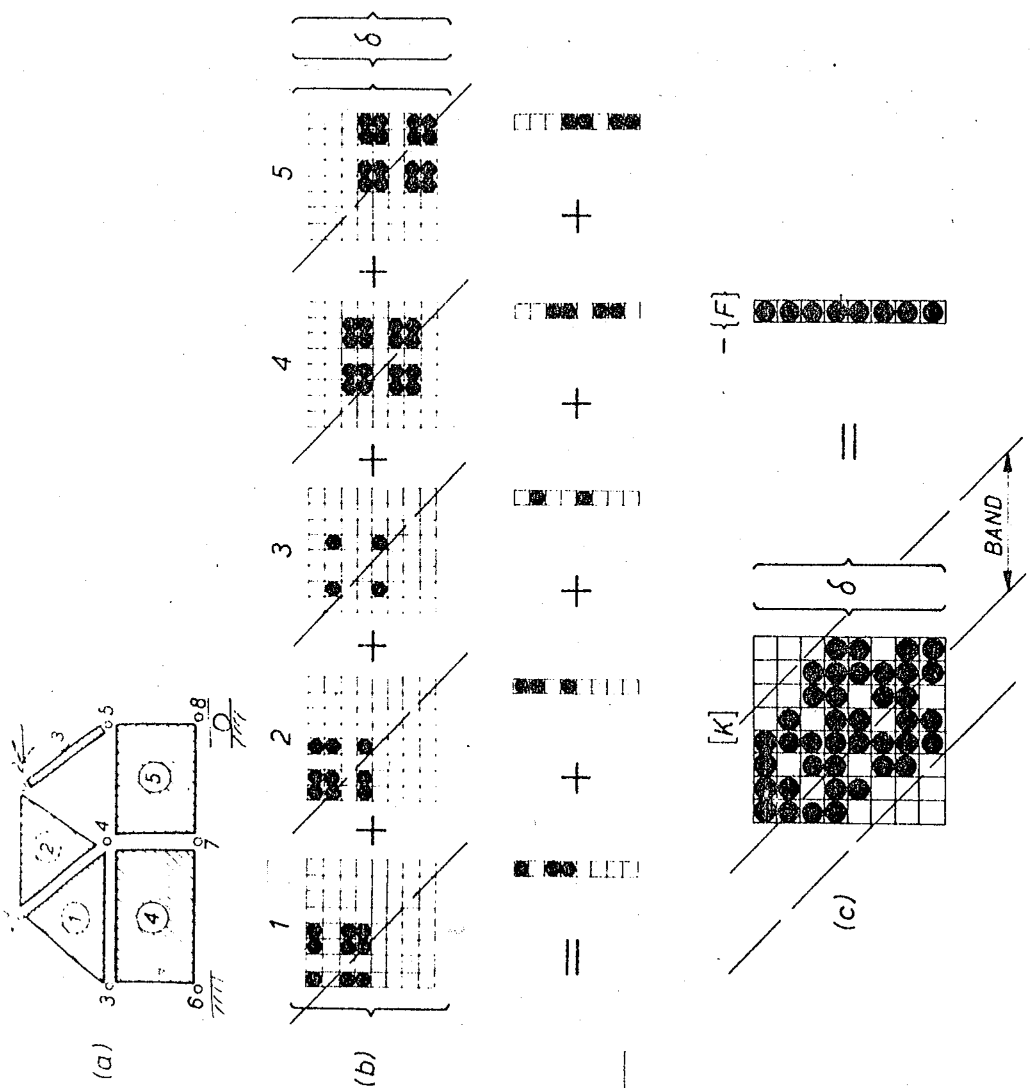


Fig. 1.4 The general pattern

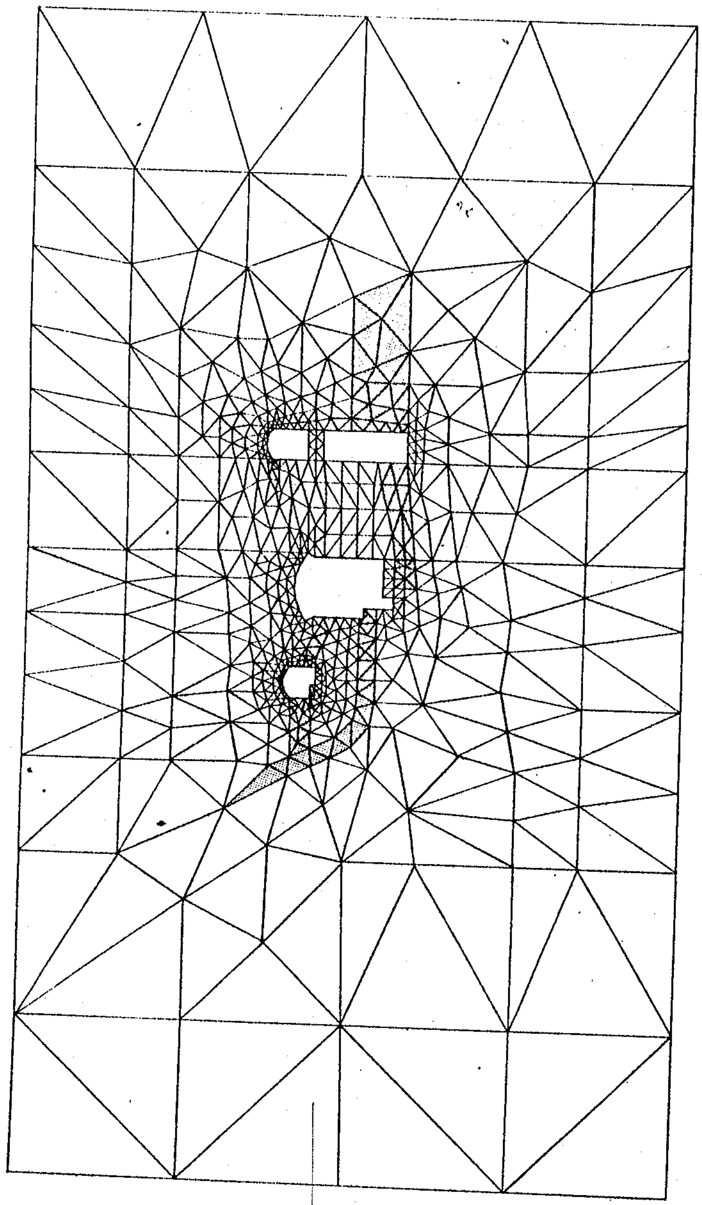


Fig. 4.14 An underground power station. Mesh used in analysis.

The comparison of these results with an analytical solution shows an exact match.

Plane Triangular-Element Systems

The simplest element for two-dimensional continuum problems is the triangular element shown in Figure 7.17. It can be used effectively to approximate practically any system configuration. Such elements were introduced by early investigators to study the torsion problem and to derive the stiffness matrix for plane stress problems.

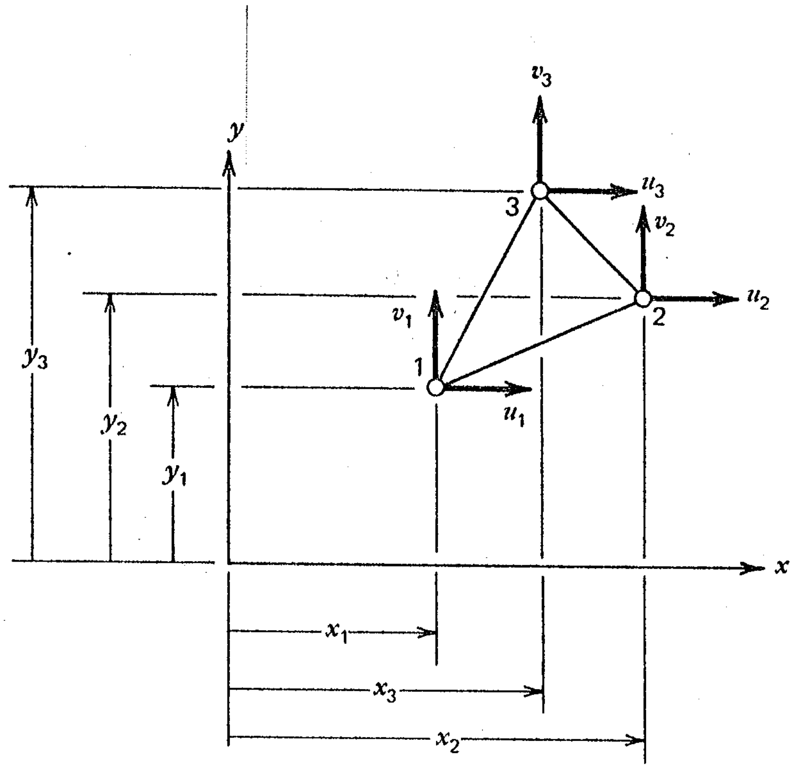
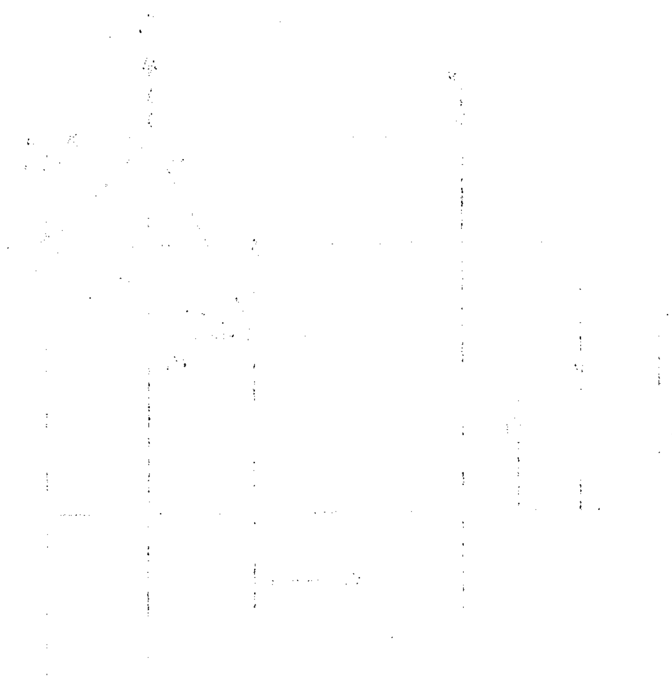


Figure 7.17 Triangular element of thickness t.

Consider the triangular element of relatively small thickness t shown in Figure 7.17. The nodes are located at the vertices and are numbered 1, 2, and 3. These are the local node numbers for the element. The corresponding global coordinates for these nodes are (x_1, y_1), (x_2, y_2), and (x_3, y_3). The displacement behavior of the element will be described with u and v values, corresponding to x and y movements respectively, at each node. Thus the displacement simulations with this element will require six degrees-of-freedom as represented by the vector relationship shown in Equation 7.26.

$$\{q\} = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3]^T \tag{7.26}$$

We make the direct assumption of linear displacements in the x-y plane so that we



can introduce the following approximating functions.

$$\begin{aligned} u_i &= a_1 + a_2 x_i + a_3 y_i \\ v_i &= b_1 + b_2 x_i + b_3 y_i \end{aligned} \tag{7.27}$$

The a's and b's can be evaluated using the matrix equations given by Equations 7.28, 7.29, and 7.30.

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [A] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}; \qquad \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = [A] \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \tag{7.28}$$

The A matrix is given by Equation 7.29.

$$[A] = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \tag{7.29}$$

Solving for the a and b vectors requires the inverse of $[A]$, as shown in Equation 7.30.

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \tag{7.30}$$

The inverse of $[A]$ is given by Equation 7.31.

$$[A]^{-1} = \frac{0.5}{\mathrm{A}} \begin{bmatrix} x_2 y_3 - x_3 y_2 & x_3 y_1 - x_1 y_3 & x_1 y_2 - x_2 y_1 \\ y_2 - y_3 & y_3 - y_1 & y_1 - y_2 \\ x_3 - x_2 & x_1 - x_3 & x_2 - x_1 \end{bmatrix} \tag{7.31}$$

In Equation 7.31, $\mathrm{A} = 0.5 \det [A]$ and is the area of the triangle, as given by Equation 7.32.

$$\mathrm{A} = 0.5\,[x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1] \tag{7.32}$$

It is left as an exercise to show that we can get Equations 7.33 and 7.34 from Equations 7.28 through 7.31.

$$u = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \tag{7.33}$$

$$v = [N_1 \quad N_2 \quad N_3] \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \tag{7.34}$$

In Equations 7.33 and 7.34, N_1, N_2, and N_3 represent the shape functions for the triangular element. A generalized form can be used to express the shape functions, as shown in Equations 7.35 and 7.36.

$$N_i = \frac{0.5}{A}\,[\alpha_i + \beta_i x + \gamma_i y] \qquad i = 1, 2, 3 \tag{7.35}$$

The α_i, β_i, and γ_i are determined from Equation 7.36.

$$\begin{array}{lll} \alpha_1 = x_2 y_3 - x_3 y_2 & \beta_1 = y_2 - y_3 & \gamma_1 = x_3 - x_2 \\ \alpha_2 = x_3 y_1 - x_1 y_3 & \beta_2 = y_3 - y_1 & \gamma_2 = x_1 - x_3 \\ \alpha_3 = x_1 y_2 - x_2 y_1 & \beta_3 = y_1 - y_2 & \gamma_3 = x_2 - x_1 \end{array} \tag{7.36}$$

Equations 7.33 and 7.34 can be written as a single matrix equation, as shown in Equation 7.37.

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \tag{7.37}$$

The two-dimensional strain displacement relations are

$$\varepsilon_x = \frac{\partial u}{\partial x}; \qquad \varepsilon_y = \frac{\partial v}{\partial y}; \qquad \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

and the corresponding stress–strain relations for a homogeneous, isotropic plane stress element are given by

$$\sigma_x = \frac{E}{1 - v^2}(\varepsilon_x + v\varepsilon_y)$$

$$\sigma_y = \frac{E}{1 - v^2}(\varepsilon_y + v\varepsilon_x)$$

$$\tau_{xy} = \frac{0.5E\gamma_{xy}}{1 + v} = 0.5E\gamma_{xy}\,\frac{1 - v}{1 - v^2}$$

The latter can be written in matrix form as shown in Equation 7.38.

$$\{\sigma\} = [C]\{\varepsilon\} \tag{7.38}$$

where

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}; \qquad \{\varepsilon\} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

and

$$[C] = \left[\frac{E}{1 - v^2}\right] \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1 - v}{2} \end{bmatrix}$$

Substituting the foregoing derivatives of the shape functions (given by Equation 7.35) gives us a general matrix equation for the strains, as shown in Equation 7.39.

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{0.5}{A} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \tag{7.39}$$

It is noted that Equation 7.39 can be written in shortened form as

$$\{\varepsilon\} = [B]\{q\}$$

To complete the derivation of the stiffness matrix, the joint force stress matrix needs to be constructed. The constant stresses relative to the triangular element are illustrated in Figure 7.18. Since the stresses are just the stress distributions, they can be converted into equivalent nodal forces. For example, the σ_x stress on the right-hand face [see Figure 7.19] yields a total force:

$$F_x^{\text{total}} = \sigma_x t(y_3 - y_1) = \sigma_x t\beta_2$$

This force produces the equivalent nodal forces f_1^x, f_2^x, and f_3^x. Part of the total load is taken by nodes 1 and 2, and part by nodes 2 and 3. That part of the force taken by nodes 1 and 2 is now assumed to be divided equally between the two nodes and is given by Equation 7.40.

$$f_1^x = f_2^x = -0.5\sigma_x t\beta_3 \tag{7.40}$$

That part of the force taken by nodes 2 and 3, which again is assumed to be divided equally, is given by Equation 7.41.

$$f_2^x = f_3^x = -0.5\sigma_x t\beta_1 \tag{7.41}$$

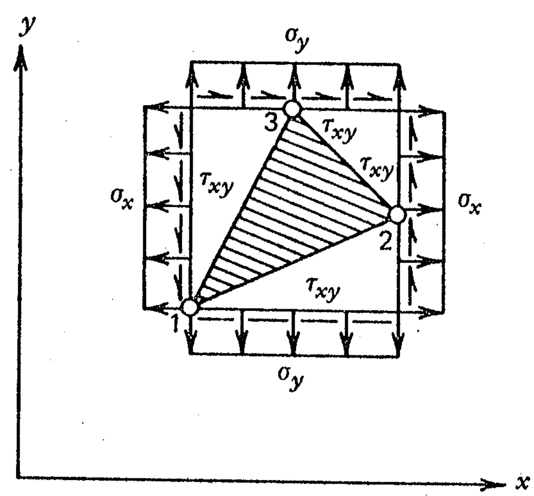
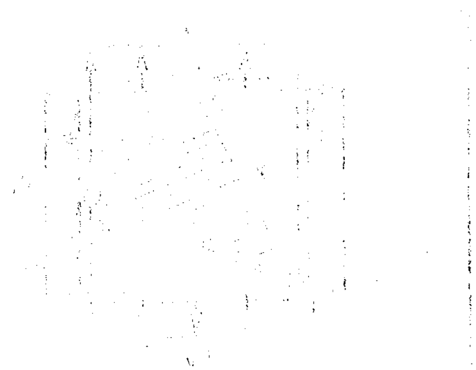


Figure 7.18 Plane stress triangular element.



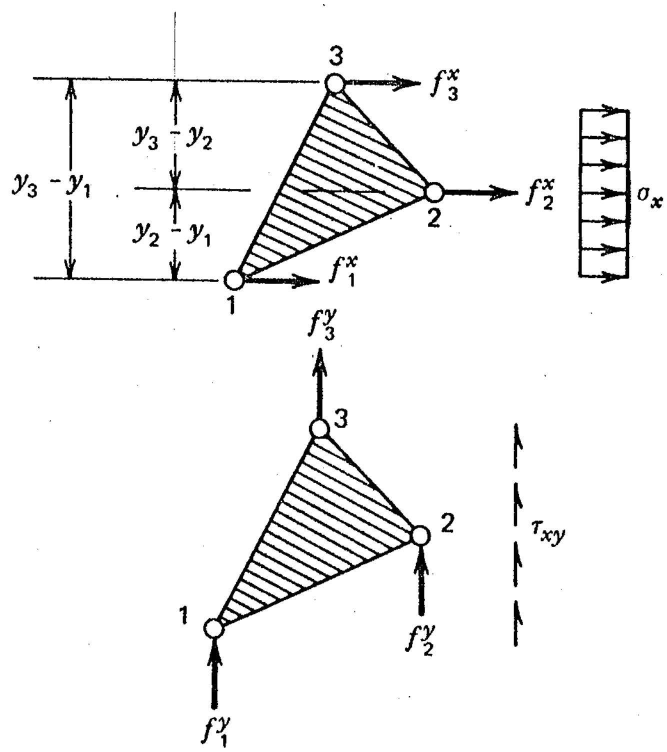


Figure 7.19 Direct stress σ_x and shear stress τ_{xy} on right-hand face with equivalent nodal forces.

Hence, for the right-hand face under the influence of σ_x, we have the equivalent nodal forces grouped under Equation 7.42. Note that $-(\beta_3+\beta_1) = -(y_1-y_3) = \beta_2$.

$$
\begin{aligned}
f_{1r}^x &= -0.5\sigma_x t\beta_3 \\
f_{2r}^x &= -0.5\sigma_x t\beta_3 - 0.5\sigma_x t\beta_1 = 0.5\sigma_x t\beta_2 \\
f_{3r}^x &= -0.5\sigma_x t\beta_1
\end{aligned}
\tag{7.42}
$$

Consulting Figure 7.19 again, we see that the shear force on the right-hand face gives us a total force of

$$F_y^{\text{total}} = \tau_{xy} t(y_3 - y_1) = \tau_{xy} t\beta_2$$

This force produces the equivalent nodal forces f_1^y, f_2^y, and f_3^y. Again part of the load is taken by nodes 1 and 2 and part by nodes 2 and 3. Assuming that it is equally divided, as before, we can express the part for nodes 1 and 2 as Equation 7.43.

$$f_1^y = f_2^y = 0.5\tau_{xy} t(y_2 - y_1) = -0.5\tau_{xy} t\beta_3 \tag{7.43}$$

Equation 7.44 gives us the part taken by nodes 2 and 3.

$$f_2^y = f_3^y = 0.5\tau_{xy} t(y_3 - y_2) = -0.5\tau_{xy} t\beta_1 \tag{7.44}$$

Thus nodal forces owing to shear forces on the right-hand face can be given by the

grouping under Equation 7.45.

$$
\begin{aligned}
f^y_{1r} &= -0.5\tau_{xy}t\beta_3 \\
f^y_{2r} &= -0.5\tau_{xy}t\beta_3 - 0.5\tau_{xy}t\beta_1 = 0.5\tau_{xy}t\beta_2 \\
f^y_{3r} &= -0.5\tau_{xy}t\beta_1
\end{aligned}
\tag{7.45}
$$

Using the same reasoning, we can develop nodal forces owing to total stress forces on the left-hand face and the top and bottom faces. From the left-hand face, we get the nodal forces shown grouped under Equation 7.46.

$$
\begin{aligned}
f^x_{1l} &= -0.5\sigma_x t\beta_2 \\
f^x_{2l} &= 0 \\
f^x_{3l} &= -0.5\sigma_x t\beta_2 \\
f^y_{1l} &= -0.5\tau_{xy}t\beta_2 \\
f^y_{2l} &= 0 \\
f^y_{3l} &= -0.5\tau_{xy}t\beta_2
\end{aligned}
\tag{7.46}
$$

From the top face, we get the nodal forces shown grouped under Equation 7.47.

$$
\begin{aligned}
f^y_{1t} &= -0.5\sigma_y t\gamma_2 \\
f^y_{2t} &= -0.5\sigma_y t\gamma_1 \\
f^y_{3t} &= 0.5\sigma_y t\gamma_3 \\
f^x_{1t} &= -0.5\tau_{xy}t\gamma_2 \\
f^x_{2t} &= -0.5\tau_{xy}t\gamma_1 \\
f^x_{3t} &= 0.5\tau_{xy}t\gamma_3
\end{aligned}
\tag{7.47}
$$

From the bottom face, we get the nodal forces shown grouped under Equation 7.48.

$$
\begin{aligned}
f^y_{1b} &= -0.5\sigma_y t\gamma_3 \\
f^y_{2b} &= -0.5\sigma_y t\gamma_3 \\
f^y_{3b} &= 0 \\
f^x_{1b} &= -0.5\tau_{xy}t\gamma_3 \\
f^x_{2b} &= -0.5\tau_{xy}t\gamma_3 \\
f^x_{3b} &= 0
\end{aligned}
\tag{7.48}
$$

The total force on any node is the sum of the nodal forces given for that node above. For example,

$$
\begin{aligned}
f^x_1 &= f^x_{1t} + f^x_{1b} + f^x_{1r} + f^x_{1l} \\
f^x_1 &= -0.5\tau_{xy}t(\gamma_2+\gamma_3) - 0.5\sigma_x t(\beta_3+\beta_2)
\end{aligned}
$$

This gives us Equation 7.49.

$$f_1^x = 0.5\sigma_x t\beta_1 + 0.5\tau_{xy} t\gamma_1 \tag{7.49}$$

In the same manner, we can get five more similar equations. However, we can express all six of these nodal force equations easier by using one matrix equation, as shown in Equation 7.50.

$$\{f\} = [\Lambda]\{\sigma\} \tag{7.50}$$

where

$$\{\sigma\} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$$[\Lambda] = 0.5t \begin{bmatrix} \beta_1 & 0 & \gamma_1 \\ 0 & \gamma_1 & \beta_1 \\ \beta_2 & 0 & \gamma_2 \\ 0 & \gamma_2 & \beta_2 \\ \beta_3 & 0 & \gamma_3 \\ 0 & \gamma_3 & \beta_3 \end{bmatrix}$$

$$\{f\} = \begin{Bmatrix} f_1^x \\ f_1^y \\ f_2^x \\ f_2^y \\ f_3^x \\ f_3^y \end{Bmatrix}$$

Substituting Equation 7.38 and the shortened form of Equation 7.39 into Equation 7.50 gives us our final element equation, Equation 7.51, relating nodal forces and displacements.

$$\{f\} = [k]\{q\} \tag{7.51}$$

The element stiffness matrix, $[k]$, consists of the product of three other matrices, as indicated in Equation 7.52.

$$[k] = [\Lambda][C][B] \tag{7.52}$$

Performing the matrix multiplication implied in Equation 7.52 gives the required stiffness matrix for the plane triangular element. Details are given as Equation 7.53.

$$[k] = \left[\frac{Et}{4A(1-\nu^2)}\right] \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ & & k_{33} & k_{34} & k_{35} & k_{36} \\ & & & k_{44} & k_{45} & k_{46} \\ & \text{symmetric} & & & k_{55} & k_{56} \\ & & & & & k_{66} \end{bmatrix} \tag{7.53}$$

where

$$
\begin{aligned}
k_{11} &= \beta_1^2 + \nu_m\gamma_1^2 & k_{34} &= \nu_p\gamma_2\beta_2 \\
k_{12} &= \nu_p\gamma_1\beta_1 & k_{35} &= \beta_2\beta_3 + \nu_m\gamma_2\gamma_3 \\
k_{13} &= \beta_1\beta_2 + \nu_m\gamma_1\gamma_2 & k_{36} &= \nu\beta_2\gamma_3 + \nu_m\gamma_2\beta_3 \\
k_{14} &= \nu\beta_1\gamma_2 + \nu_m\beta_2\gamma_1 & k_{44} &= \gamma_2^2 + \nu_m\beta_2^2 \\
k_{15} &= \beta_1\beta_3 + \nu_m\gamma_1\gamma_3 & k_{45} &= \nu\gamma_2\beta_3 + \nu_m\beta_2\gamma_3 \\
k_{16} &= \nu\beta_1\gamma_3 + \nu_m\gamma_1\beta_3 & k_{46} &= \gamma_2\gamma_3 + \nu_m\beta_2\beta_3 \\
k_{22} &= \gamma_1^2 + \nu_m\beta_1^2 & k_{55} &= \beta_3^2 + \nu_m\gamma_3^2 \\
k_{23} &= \nu\beta_2\gamma_1 + \nu_m\gamma_2\beta_1 & k_{56} &= \nu\beta_3\gamma_3 + \nu_m\beta_3\gamma_3 \\
k_{24} &= \gamma_1\gamma_2 + \nu_m\beta_1\beta_2 & k_{66} &= \gamma_3^2 + \nu_m\beta_3^2 \\
k_{25} &= \nu\gamma_1\beta_3 + \nu_m\beta_1\gamma_3 & & \text{and} \\
k_{26} &= \gamma_1\gamma_3 + \nu_m\beta_1\beta_3 & \nu_m &= \frac{1-\nu}{2} \\
k_{33} &= \beta_2^2 + \nu_m\gamma_2^2 & \nu_p &= \frac{1+\nu}{2}
\end{aligned}
$$

The stiffness matrix for a triangular element in plane strain may be obtained in the same manner as described in the foregoing except that the stress–strain relationships (Equation 7.38) must be modified. The modified stress–strain equation is given as Equation 7.54.

$$\{\sigma\} = [C']\{\varepsilon\} \tag{7.54}$$

where

$$[C'] = E[(1+\nu)(1-2\nu)]^{-1}\begin{bmatrix} 1-\nu & \nu & 0 \\ \nu & 1-\nu & 0 \\ 0 & 0 & \dfrac{1-2\nu}{2} \end{bmatrix}$$

and $\{\sigma\}$ and $\{\varepsilon\}$ are as defined before.

The required matrix multiplications are as shown in Equation 7.52, except that $[C]$ must be replaced by $[C']$. The resulting matrix is given as Equation 7.55.

$$[k'] = CK\begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} & K_{15} & K_{16} \\ & K_{22} & K_{23} & K_{24} & K_{25} & K_{26} \\ & & K_{33} & K_{34} & K_{35} & K_{36} \\ & & & K_{44} & K_{45} & K_{46} \\ & \text{symmetric} & & & K_{55} & K_{56} \\ & & & & & K_{66} \end{bmatrix} \tag{7.55}$$

where

$$
\begin{aligned}
(CK) &= Et[2(1+\nu)(1-2\nu)]^{-1} & K_{25} &= \nu\beta_3\gamma_1 + \lambda_2\beta_1\gamma_3 \\
\lambda_1 &= (1-\nu) & K_{26} &= \lambda_1\gamma_1\gamma_3 + \lambda_2\beta_1\beta_3
\end{aligned}
$$

$$\lambda_2 = \frac{1-2v}{v}$$

and

$$K_{11} = \lambda_1 \beta_1^2 + \lambda_2 \gamma_1^2$$
$$K_{12} = v\beta_1\gamma_1 + \lambda_2 \beta_1 \gamma_1$$
$$K_{13} = \lambda_1 \beta_1 \beta_2 + \lambda_2 \gamma_1 \gamma_2$$
$$K_{14} = v\beta_1\gamma_2 + \lambda_2 \gamma_1 \beta_2$$
$$K_{15} = \lambda_1 \beta_1 \beta_3 + \lambda_2 \gamma_1 \gamma_3$$
$$K_{16} = v\beta_1\gamma_3 + \lambda_2 \gamma_1 \beta_3$$
$$K_{22} = \lambda_1 \gamma_1^2 + \lambda_2 \beta_1^2$$
$$K_{23} = v\gamma_1\beta_2 + \lambda_2 \beta_1 \gamma_2$$
$$K_{24} = \lambda_1 \gamma_1 \gamma_2 + \lambda_2 \beta_1 \beta_2$$
$$K_{33} = \lambda_1 \beta_2^2 + \lambda_2 \gamma_2^2$$
$$K_{34} = v\beta_2\gamma_2 + \lambda_2 \beta_2 \gamma_2$$
$$K_{35} = \lambda_1 \beta_2 \beta_3 + \lambda_2 \gamma_2 \gamma_3$$
$$K_{36} = v\beta_2\gamma_3 + \lambda_2 \gamma_2 \beta_3$$
$$K_{44} = \lambda_1 \gamma_2^2 + \lambda_2 \beta_2^2$$
$$K_{45} = v\gamma_2\beta_3 + \lambda_2 \beta_2 \gamma_3$$
$$K_{46} = \lambda_1 \gamma_2 \gamma_3 + \lambda_2 \beta_2 \beta_3$$
$$K_{55} = \lambda_1 \beta_3^2 + \lambda_2 \gamma_3^2$$
$$K_{56} = v\beta_3\gamma_3 + \lambda_2 \gamma_3 \beta_3$$
$$K_{66} = \lambda_1 \gamma_3^2 + \lambda_2 \beta_3^2$$

Examples 7.6 and 7.7 illustrate use of the foregoing element formulations in structural plate systems.

EXAMPLE 7.6

Consider the thin flat plate system shown in Figure 7.20. Assume consistent units and take a thickness of 0.25, an elastic modulus of 30×10^6, and a Poisson ratio of 0.3. The right edge of the plate is loaded with a distributed load of 400 force units per unit area. Using a two triangular-element approximation, simulate the two-dimensional deflections and stresses.

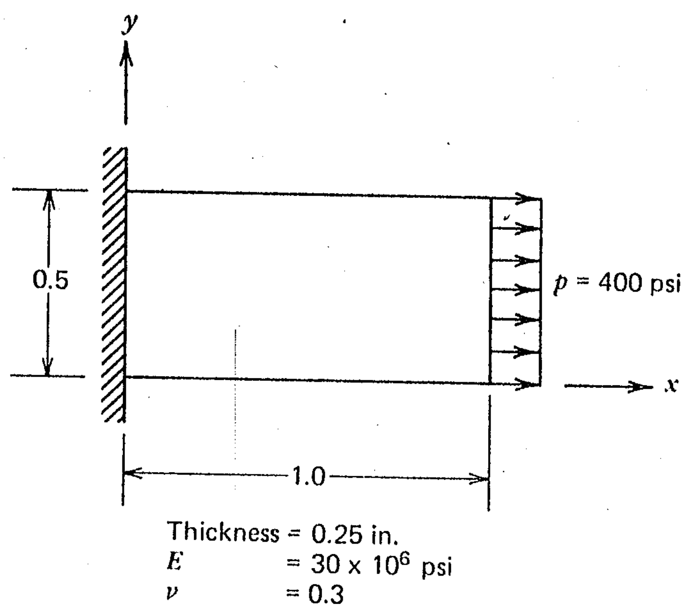


Figure 7.20 Flat plate, Example 7.6.

▶ We choose the plane stress case and visualize the finite elements and nodes as shown in Figure 7.21.

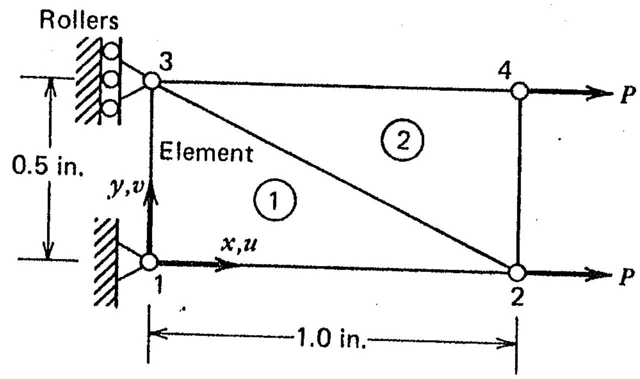


Figure 7.21 Finite-element idealization of flat plate (plane stress condition). Two elements.

For element 1 (nodes 1, 2, and 3), we have $A=0.25$, $\beta_1=y_2-y_3=-0.5$, $\beta_2=y_3-y_1=0.5$, $\beta_3=y_1-y_2=0$, $\gamma_1=x_3-x_2=-1.0$, $\gamma_2=x_1-x_3=0$, and $\gamma_3=x_2-x_1=1.0$. The stiffness matrix, using Equation 7.53, becomes

$$[k^1]=Et[4A(1-\nu^2)]^{-1}\begin{bmatrix} 0.6000 & 0.3250 & -0.2500 & -0.1750 & -0.3500 & -0.1500 \\ & 1.0875 & -0.1500 & -0.0875 & -0.0175 & -1.000 \\ & & 0.2500 & 0 & 0 & 0.1500 \\ & & & 0.0875 & 0.1750 & 0 \\ & \text{symmetric} & & & 0.3500 & 0 \\ & & & & & 1.000 \end{bmatrix}$$

For element 2 (nodes 2, 4, and 3), we have $A=0.25$, $\beta_1=0$, $\beta_2=0.5$, $\beta_3=-0.5$, $\gamma_1=-1.0$, $\gamma_2=1.0$, and $\gamma_3=0$. The stiffness matrix becomes

$$[k^2]=Et[4A(1-\nu^2)]^{-1}$$

$$\times\begin{bmatrix} 0.3500 & 0 & -0.3500 & -0.1750 & 0 & 0.1750 \\ & 1.000 & -0.1500 & -1.000 & 0.1500 & 0 \\ & & 0.6000 & 0.3250 & -0.2500 & -0.1750 \\ & & & 1.0875 & -0.1500 & -0.0875 \\ & \text{symmetric} & & & 0.2500 & 0 \\ & & & & & 0.0875 \end{bmatrix}$$

The element stiffness matrices can be combined in the direct fashion previously illustrated to form the overall global stiffness matrix. The resulting

matrix equation follows.

$$Et[4A(1-\nu^2)]^{-1}\,[KG]\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_1^x \\ F_1^y \\ F_2^x \\ F_2^y \\ F_3^x \\ F_3^y \\ F_4^x \\ F_4^y \end{Bmatrix}$$

Since the system is fixed at node 1, and restrained in v only at node 3, we get $u_1 = v_1 = u_3 = 0$. Introducing these constraints into the above equation yields the final global matrix equation to be solved. The $[KG]$ matrix is shown first for reference.

$$[KG] =$$

$$\begin{bmatrix} 0.6000 & 0.3250 & -0.2500 & -0.1750 & -0.3500 & -0.1500 & 0 & 0 \\ & 1.0875 & -0.1500 & -0.0875 & -0.0175 & -1.000 & 0 & 0 \\ & & 0.6000 & 0 & 0 & 0.3250 & -0.3500 & -0.1750 \\ & & & 1.0875 & 0.3250 & 0 & -0.1500 & -01.000 \\ & & & & 0.6000 & 0 & -0.2500 & -0.1500 \\ & & & & & 1.0875 & -0.1750 & -0.0875 \\ & \text{symmetric} & & & & & 0.6000 & 0.3250 \\ & & & & & & & 1.0875 \end{bmatrix}$$

$$Et[4A(1-\nu^2)]^{-1}\begin{bmatrix} 0.6000 & 0 & 0.3250 & -0.3500 & -0.1750 \\ & 1.0875 & 0 & -0.1500 & -1.000 \\ & & 1.0875 & -0.1750 & -0.0875 \\ & & & 0.6000 & 0.3250 \\ & \text{symmetric} & & & 1.0875 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} F_2^x \\ F_2^y \\ F_3^y \\ F_4^x \\ F_4^y \end{Bmatrix}.$$

The external loading is applied only to the right-hand face as shown in Figure 7.20. Consequently, the loading is divided equally between nodes 2 and 4.

$$F_2^x = F_4^x = 0.5P(y_4 - y_2)t$$

or

$$F_2^x = f_4^x = 0.5(400)(0.5)(0.25) = 25$$

also

$$F_2^y = F_3^y = F_4^y = 0$$

With the nodal loads specified, the global matrix equation shown above can

be solved to give the displacements.

$$\begin{Bmatrix} u_2 \\ v_2 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \left[\frac{4A(1-\nu^2)}{Et}\right] \begin{Bmatrix} 109.8901 \\ 0 \\ -16.4835 \\ 109.8901 \\ -16.4835 \end{Bmatrix} = \begin{Bmatrix} 1.3333 \\ 0 \\ -0.2000 \\ 1.3333 \\ -0.2000 \end{Bmatrix} \times 10^{-5}$$

Once the displacements have been determined, the reaction forces at the support nodes 1 and 3 can be computed. Using the appropriate equations from the original global set (e.g., for R_1^x, we use the first equation), we get the following.

$$R_1^x = (Et)[4A(1-\nu^2)]^{-1}(-0.25u_2 - 0.175v_2 - 0.15v_3)$$

$$R_1^y = (Et)[4A(1-\nu^2)]^{-1}(-0.15u_2 - 0.0875v_2 - v_3)$$

$$R_3^x = (Et)[4A(1-\nu^2)]^{-1}(0.325v_2 - 0.25u_4 - 0.15v_4)$$

The result is $R_1^x = -25$, $R_1^y = 0$, and $R_3^x = -25$, all in consistent force units.

The stresses in the elements can be obtained by using Equations 7.38 and 7.39. Thus, for plane stress, we have the following.

$$\{\sigma\} = [C]\{\varepsilon\} \qquad \text{or} \qquad \{\sigma\} = [C][B]\{q\}$$

Performing the matrix multiplications indicated in the foregoing gives the following matrix equation.

$$\{\sigma\} = E[2A(1-\nu^2)]^{-1} \begin{bmatrix} \beta_1 & \nu\gamma_1 & \beta_2 & \nu\gamma_2 & \beta_3 & \nu\gamma_3 \\ \nu\beta_1 & \gamma_1 & \nu\beta_2 & \gamma_2 & \nu\beta_3 & \gamma_3 \\ \nu_1\gamma_1 & \nu_1\beta_1 & \nu_1\gamma_2 & \nu_1\beta_2 & \nu_1\gamma_3 & \nu_1\beta_3 \end{bmatrix} \{q\}$$

where $\nu_1 = (1-\nu)/2$.

For element 1 ($E = 30 \times 10^6$, $\nu = 0.3$, $A = 0.25$, $\nu_1 = 0.30$) we have the following.

$$\{\sigma\}^1 = 6.5934 \times 10^7 \begin{bmatrix} -0.5 & -0.3 & 0.5 & 0 & 0 & 0.3 \\ -0.15 & -1.0 & 0.15 & 0 & 0 & 1.0 \\ -0.35 & -0.175 & 0 & 0.175 & 0.35 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

As expected, we get the following element stress values from the multiplication.

$$\{\sigma\}^1 = \begin{Bmatrix} 400 \\ 0 \\ 0 \end{Bmatrix}$$

In a similar manner we find the following stress values for element 2.

$$\{\sigma\}^2 = \begin{Bmatrix} 400 \\ 0 \\ 0 \end{Bmatrix}$$

We observe that, in this example, large stress gradients are not present. Consequently, we would not expect that improved accuracy could be obtained in the simulated values simply by increasing the number of elements. A BASIC computer program that can be used to simulate the displacements in this example is given as Figure 7.22.

```
100 REM EXAMPLE 7.6 AND 7.7 BASIC PROGRAM
110 CLS: CLEAR50: DEFINT I,J,L,M,N
120 PRINT"2D TRIANGULAR-ELEMENT FINITE ELEMENT EXAMPLE PROGRAM"
130 INPUT"GIVE PLATE THICKNESS";TH
140 INPUT"GIVE ELASTIC MODULUS";EM
150 INPUT"GIVE POISSON RATIO";PR
160 INPUT"GIVE NUMBER OF TRIANGULAR ELEMENTS";NT
170 INPUT"GIVE NUMBER OF NODES";NN
180 DIM N1(NT),N2(NT),N3(NT),A1(NT),A2(NT),A3(NT),B1(NT),B2(NT),B3(NT),G1(NT),G2(NT),G3(NT),X(NN),Y(NN),A(NT)
190 M=2*NN: DIM K(M,M),F(M),D(10,M),ND(10),U(NN),V(NN)
200 REM INPUT ELEMENT INFORMATION
210 FOR I=1TONT
220   PRINT"GIVE GLOBAL NODE NUMBERS (LOCAL 1,2,&3 CCW) FOR ELEMENT";I
230   INPUT N1(I),N2(I),N3(I)
240 NEXTI
250 FORI=1TONN
260   PRINT"GIVE GLOBAL COORDINATES, X , Y , FOR NODE";I
270   INPUT X(I),Y(I)
280 NEXTI
290 PRINT:PRINT">>>>> ELEMENT NODE LIST <<<<<": L=0
300 FORI=1TONT
310   PRINT"NODES 1, 2, & 3 FOR ELEMENT";I;" ARE:";N1(I);N2(I);N3(I)
320   L=L+1: IF L<14 GOTO350
330   INPUT"FOR MORE, PRESS 'ENTER'";ZZ: L=0
340   PRINT"ELEMENT NODES CONTINUED"
350 NEXTI
360 INPUT"TO CONTINUE, PRESS 'ENTER'";ZZ
370 PRINT:PRINT"////// NODE COORDINATES ///////": L=0
380 FORI=1TONN
390   PRINT" X AND Y  FOR NODE ";I;" ARE:"; X(I);Y(I)
400   L=L+1: IF L<14 GOTO430
410   INPUT"FOR MORE, PRESS 'ENTER'";ZZ: L=0
420   PRINT"NODE COORDINATES CONTINUED"
430 NEXTI
440 PRINT:INPUT"DO YOU DESIRE TO MAKE A CHANGE IN A NODE NO.?--Y/N";Q$
450   IF Q$<>"Y" GOTO490
460   INPUT"ENTER THE ELEMENT NUMBER";J
470   INPUT"GIVE THE NEW GLOBAL NODE NUMBERS ( 1, 2, 3 CCW LOCAL)";N1(J),N2(J),N3(J)
480   GOTO440
490 PRINT:INPUT"DO YOU DESIRE TO MAKE A CHANGE IN A NODE'S COORDINATES?--Y/N";Q1$
500   IF Q1$<>"Y" GOTO540
510   INPUT"GIVE THE NODE NUMBER";J
520   INPUT"GIVE THE NEW COORDINATES, X , Y";X(J),Y(J)
530   GOTO490
540 PRINT"DO YOU WISH TO SEE NODE ASSIGNMENTS & COORDINATES AGAIN?--Y/N"
550 INPUT AA$: IF AA$="Y" GOTO290
560 REM COMPUTE ELEMENT AREAS AND SHAPE FUNCTION PARAMETERS
570 FOR I=1TONT
580   A(I)=0.5*(X(N2(I))*Y(N3(I))-X(N3(I))*Y(N2(I))+X(N3(I))*Y(N1(I))-X(N1(I))*Y(N3(I))+X(N1(I))*Y(N2(I))-X(N2(I))*Y(N1(I)))
```

Figure 7.22a BASIC program, examples 7.6 and 7.7.

E-1

ess-strain:

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y) \qquad (1\text{-}24a)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x) \qquad (1\text{-}24b)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \qquad (1\text{-}27a)$$

ermal Strains

$$\varepsilon_x = \varepsilon_y = \varepsilon_z = \alpha\,\Delta T \qquad (1\text{-}28)$$

ain-Displacements

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad (1\text{-}29a)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \qquad (1\text{-}29b)$$

$$\varepsilon_z = \frac{\partial w}{\partial z} \qquad (1\text{-}29e)$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \qquad (1\text{-}29c)$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \qquad (1\text{-}29f)$$

$$\gamma_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \qquad (1\text{-}29g)$$

$$\Theta_{xy} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad (1\text{-}29d)$$

$$\Theta_{yz} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad (1\text{-}29h)$$

$$\Theta_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \qquad (1\text{-}29i)$$

E-2

1-7 SUMMARY OF IMPORTANT RELATIONSHIPS

Triaxial Stress

Strain-stress relationships:

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] \tag{1-21a}$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] \tag{1-21b}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] \tag{1-21c}$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \tag{1-27a}$$

$$\gamma_{yz} = \frac{2(1+\nu)}{E}\tau_{yz} \tag{1-27b}$$

$$\gamma_{zx} = \frac{2(1+\nu)}{E}\tau_{zx} \tag{1-27c}$$

Stress-strain relationships:

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)] \tag{1-22a}$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_y + \nu(\varepsilon_z + \varepsilon_x)] \tag{1-22b}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)] \tag{1-22c}$$

Shear relations are straightforward from Eqs. (1-27).

Biaxial Stress ($\sigma_z = \tau_{yz} = \tau_{zx} = 0$)

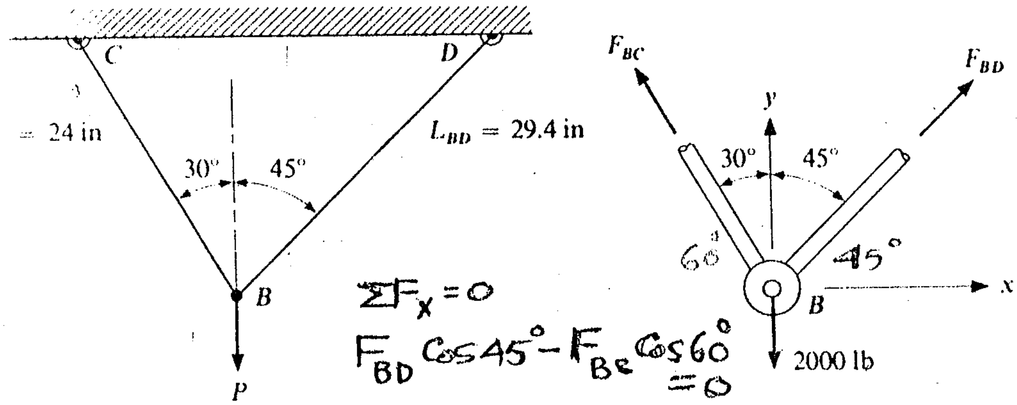
Strain-stress:

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \tag{1-23a}$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \tag{1-23b}$$

$$\varepsilon_z = -\frac{\nu}{E}(\sigma_x + \sigma_y) \tag{1-23c}$$

$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy} \tag{1-27a}$$



$\Sigma F_x = 0$

$F_{BD} \cos 45° - F_{BC} \cos 60° = 0$

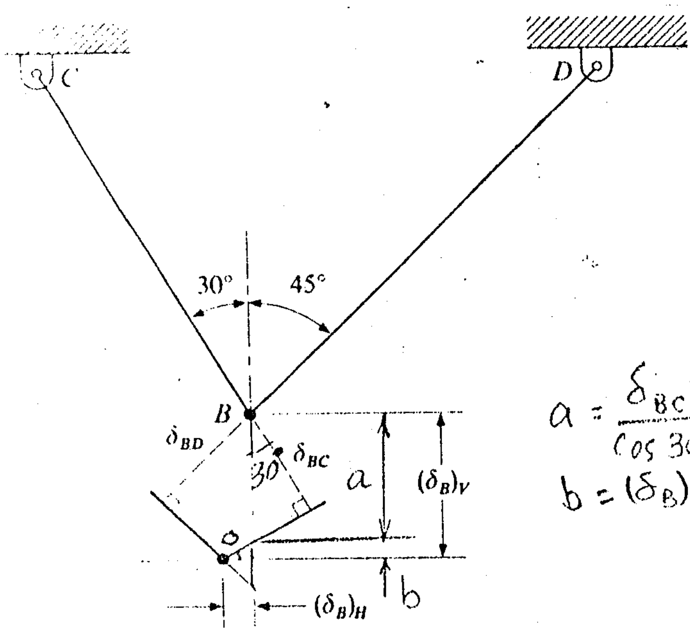
$\Sigma F_y = 0$

$F_{BD} \sin 45° + F_{BC} \sin 60° - 2000 = 0$

Example 4-1 Cables BC and BD, shown in Fig. 4-1 form angles of 30° and 45°, respectively, to the vertical. Determine the vertical and horizontal displacement of point B if a vertical force P of 2000 lb is applied at B. The cables can be considered to be of solid steel ($E = 30 \times 10^6$ lb/in²) each with a cross-sectional area of 0.2 in².

SOLUTION The first step is to determine the forces in each cable. From Fig. 4-1b, summation of forces in the x and y directions results in

$$F_{BC} = 1464 \text{ lb} \qquad F_{BD} = 1035 \text{ lb} \tag{a}$$



$a = \dfrac{\delta_{BC}}{\cos 30°}$

$b = (\delta_B)_H \tan 30°$

$$(\delta_B)_V = \frac{\delta_{BC}}{\cos 30} + (\delta_B)_H \tan 30 \qquad (b)$$

and

$$(\delta_B)_V = \frac{\delta_{BD}}{\cos 45} - (\delta_B)_H \tan 45 \qquad (c)$$

Setting the two equations equal with

$$\delta_{BC} = 5.86 \times 10^{-3} \text{ in} \qquad \delta_{BD} = 5.07 \times 10^{-3} \text{ in}$$

yields

$$(\delta_B)_H = 0.26 \times 10^{-3} \text{ in}$$

Substituting this into either of Eqs. (b) or (c) yields

$$(\delta_B)_V = 6.91 \times 10^{-3} \text{ in}$$

Figure 4-2

$$\delta_{BC} = \frac{F_{BC}L_{BC}}{AE} = \frac{(1464)(24)}{(0.2)(30 \times 10^6)} = 5.86 \times 10^{-3} \text{ in}$$

$$\delta_{BD} = \frac{F_{BD}L_{BD}}{AE} = \frac{(1035)(29.4)}{(0.2)(30 \times 10^6)} = 5.07 \times 10^{-3} \text{ in}$$

Example 4-2 Determine the vertical drop of point B in Example 4-1 by equating the total work done by the force P to the sum of the work P performs on each cable.

SOLUTION The work done by gradually applying the force P of 2000 lb is simply

$$W = \tfrac{1}{2}P(\delta_B)_V = 1000(\delta_B)_V$$

The work applied to each cable is

$$W_{BC} = \tfrac{1}{2}F_{BC}\delta_{BC} = \frac{1}{2}\frac{F_{BC}^2 L_{BC}}{AE} = 4.29 \text{ in} \cdot \text{lb}$$

$$W_{BD} = \tfrac{1}{2}F_{BD}\delta_{BD} = \frac{1}{2}\frac{F_{BD}^2 L_{BD}}{AE} = 2.62 \text{ in} \cdot \text{lb}$$

The total work is the sum of the work applied to each cable, and so

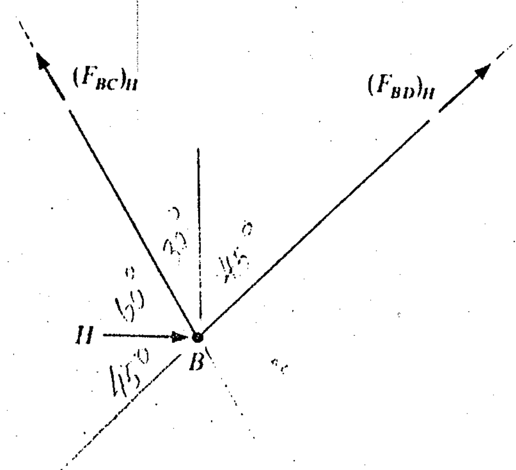
$$W = W_{BC} + W_{BD}$$

$$1000(\delta_B)_V = 4.29 + 2.62$$

$$(\delta_B)_V = 6.91 \times 10^{-3} \text{ in}$$

Thus it can be seen that a work or energy approach can simplify things considerably.†

Example 4-3 Determine the horizontal deflection of point B in Example 4-1 by the approach used in Example 4-2.



$$W_H = H(\delta_B)_H$$

The forces in cables BC and BD due to H alone are found by static analysis and are

$$(F_{BC})_H = 0.732H \qquad (F_{BD})_H = -0.897H\ddagger$$

ie additional work obtained from these forces on the cables arises from the deflections ised by the actual applied force. The additional work terms are§

$$(W_{BC})_H = (F_{BC})_H\delta_{BC} = (0.732H)(5.86 \times 10^{-3}) = 4.29H \times 10^{-3}$$

d

$$(W_{BD})_H = (F_{BD})_H\delta_{BD} = (-0.897H)(5.07 \times 10^{-3}) = -4.55H \times 10^{-3}$$

ie work due to H equals the work H performs on the elements of the system¶

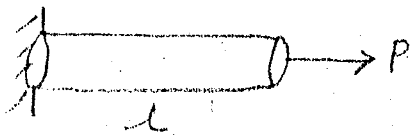
$$W_H = (W_{BC})_H + (W_{BD})_H$$

$$H(\delta_B)_H = 4.29H \times 10^{-3} - 4.55H \times 10^{-3}$$

$$(\delta_B)_H = -0.26 \times 10^{-3} \text{ in}\|$$

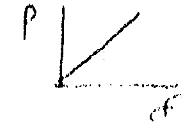
STRAIN ENERGY

u = Strain Energy / unit volume



$\sigma = P/A \quad \varepsilon = \dfrac{\Delta}{l} \quad vol = Al$

W = External Work $= \frac{1}{2} P\Delta$



$W = \frac{1}{2} P\Delta = \frac{1}{2} \frac{P}{A}(A) \frac{\Delta}{l}(l) = \frac{1}{2}\sigma\varepsilon(vol)$

$$u = w = \tfrac{1}{2}\sigma_x\varepsilon_x$$

Since the stress is uniaxial, in the elastic range $\varepsilon_x = \sigma_x/E$ and the resulting strain energy per unit volume is

$$u = \frac{1}{2E}\sigma_x^2 \tag{4-5}$$

Additional Normal Stresses

If the normal stresses σ_y and σ_z are also present,

$$u = \tfrac{1}{2}\sigma_x\varepsilon_x + \tfrac{1}{2}\sigma_y\varepsilon_y + \tfrac{1}{2}\sigma_z\varepsilon_z$$

and using the general strain-stress relationships [Eqs. (1-21)], we see that the strain energy per unit volume is

$$u = \frac{1}{2E}[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)] \tag{4-6}$$

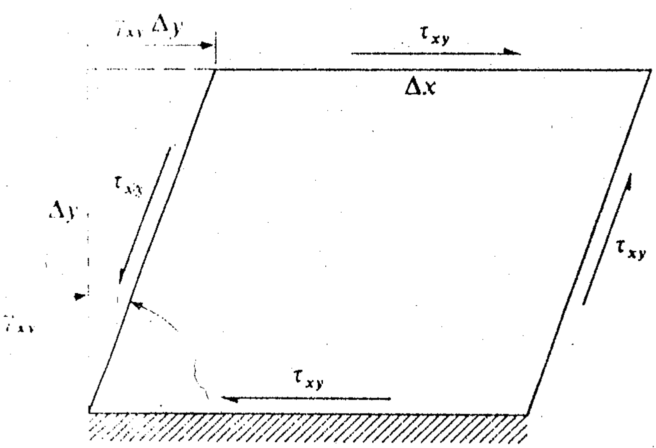


Figure 4-6

Shear Stress

The work due to τ_{xy} can be seen in Fig. 4-6. The force $\tau_{xy}\,\Delta x\,\Delta z$ moves a distance $\gamma_{xy}\,\Delta y$, which results in a work per unit volume of $\frac{1}{2}\gamma_{xy}\tau_{xy}$. When Eq. (1-27$a$) is used, the strain energy per unit volume is

$$u = \frac{1+\nu}{E}\tau_{xy}^2 \tag{4-7}$$

General State of Stress

Combining Eq. (4-6) and the strain energy per unit volume due to all the shear stresses yields, for the general case,

$$u = \frac{1}{2E}[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)$$
$$+ 2(1+\nu)(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)] \qquad (4\text{-}8)$$

Equation (4-8) can be written in terms of the principal stresses σ_1, σ_2, and σ_3, which is basically the same element transformed to eliminate the shear stresses. Thus, the strain energy per unit volume for the general case can also be expressed as

$$u = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \qquad (4\text{-}9)$$

Biaxial State of Stress

For this case (see Fig. 4-7) $\sigma_z = \tau_{yz} = \tau_{zx} = 0$, and Eq. (4-8) reduces to

$$u = \frac{1}{2E}[\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + 2(1+\nu)\tau_{xy}^2] \qquad (4\text{-}10)$$

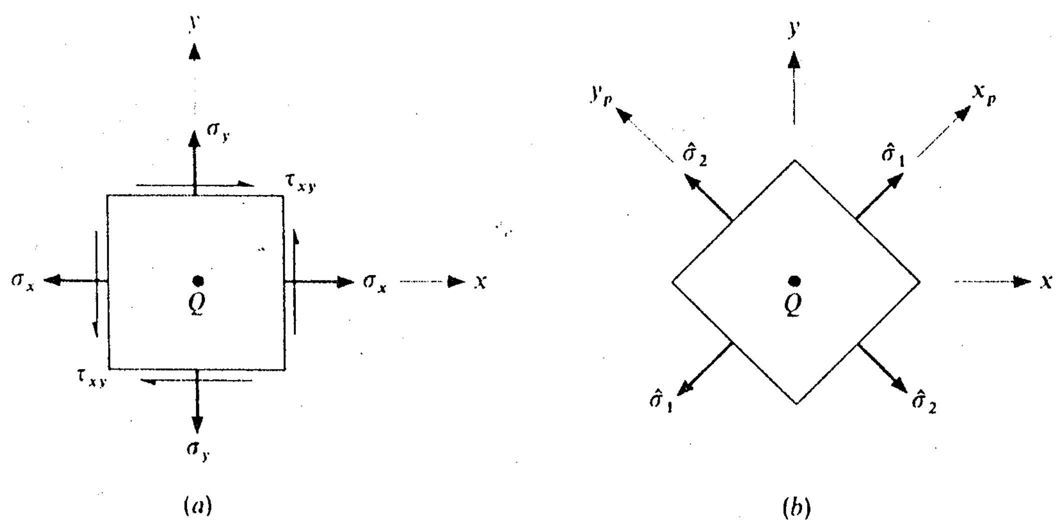


Figure 4-7

or, in terms of principal stresses (with $\hat{\sigma}_3 = 0$), from Eq. (4-9)

$$u = \frac{1}{2E}(\hat{\sigma}_1^2 + \hat{\sigma}_2^2 - 2\nu\hat{\sigma}_1\hat{\sigma}_2) \qquad (4\text{-}11)$$

In many cases of biaxial stress, $\sigma_y = 0$, and Eq. (4-10) reduces to

$$u = \frac{1}{2E}[\sigma_x^2 + 2(1+\nu)\tau_{xy}^2] \qquad (4\text{-}12)$$

4-3 TOTAL STRAIN ENERGY IN BARS WITH SIMPLE LOADING CONDITIONS

Axial Loading

The stress in a bar undergoing axial loading (see Fig. 4-8) is $\sigma_x = \pm P/A$ throughout the bar, where the plus sign indicates tension and minus compres-

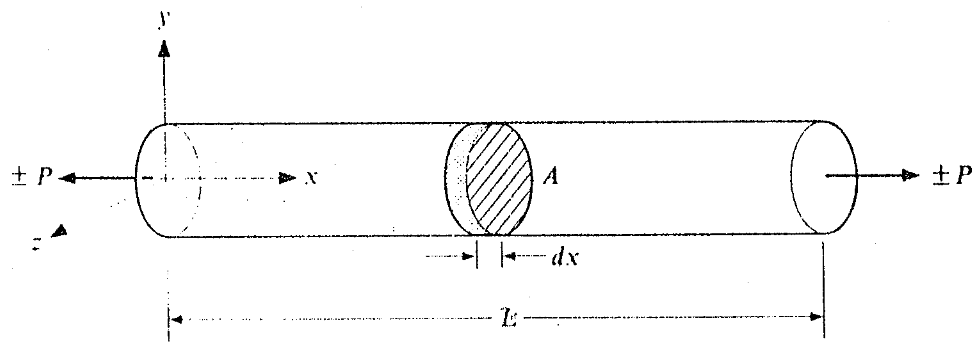


Figure 4-8

n. Thus, integrating the strain energy per unit volume [Eq. (4-5)] throughout e bar yields the total strain energy. The total strain energy is therefore

$$U_a = \frac{1}{2}\int \frac{1}{E}\left(\frac{P}{A}\right)^2 dV$$

owever, $dV = A\,dx$; thus

$$U_a = \frac{1}{2}\int_0^L \frac{P^2\,dx}{AE} \qquad (4\text{-}13)$$

f P, A, and E are constant, Eq. (4-13) reduces to

$$U_a = \frac{1}{2}\frac{P^2 L}{AE} \qquad (4\text{-}14)$$

vhich, as can be seen, is the total work performed by P since $W = \frac{1}{2}P\delta$ and $= PL/AE$.

Torsional Loading

The shear stress induced by torsion is a function of radial position, that is, $\tau = Tr/J$. Establish a dV element in which the shear stress is constant (see Fig. 4-9). Thus from Eq. (4-7)

$$U_t = \int \frac{1+\nu}{E}\left(\frac{Tr}{J}\right)^2 dx\, dA$$

In general, except for r^2 all the finite terms within the integral will be functions of x. Thus integrating with respect to dA first results in

$$U_t = \int \left[\frac{1+\nu}{E}\left(\frac{T}{J}\right)^2 \int r^2\, dA\right] dx$$

However, since $\int r^2\, dA = J$,

$$U_t = \int_0^L \frac{(1+\nu)T^2}{EJ}\, dx \tag{4-15}$$

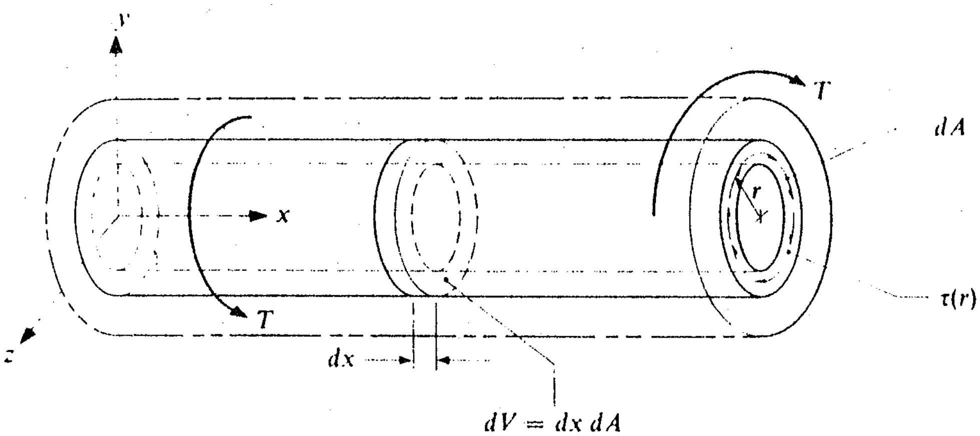


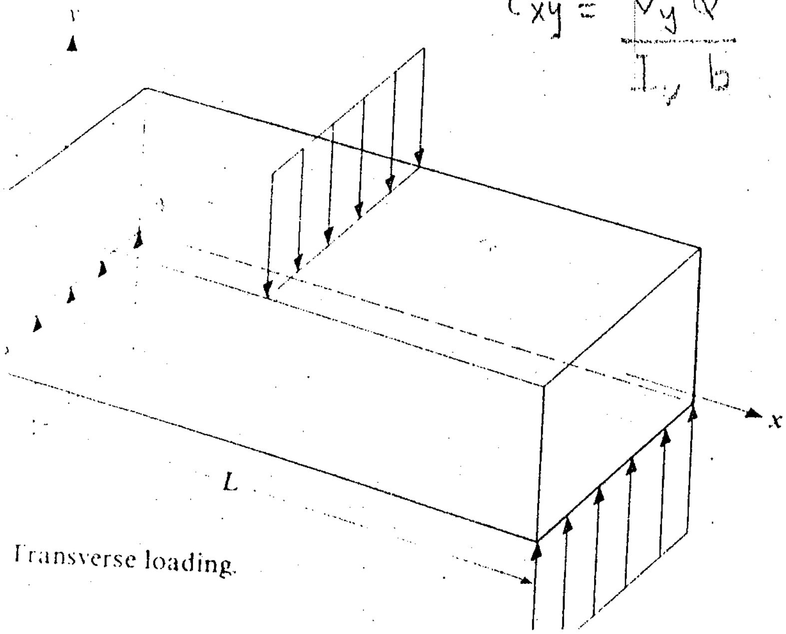
Figure 4-9

If ν, T, E, and J are not functions of x, Eq. (4-15) reduces to

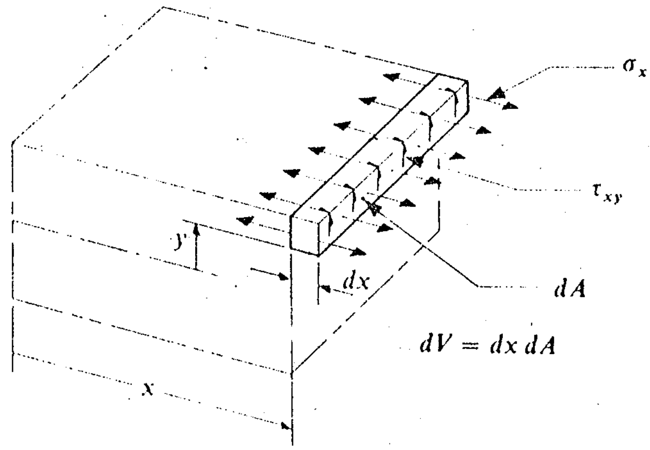
$$U_t = (1+\nu)\frac{T^2 L}{EJ} \tag{4-16}$$

$$\sigma_x = -\frac{M_z y}{I_z}$$

$$\tau_{xy} = \frac{V_y Q}{I_y b}$$



Transverse loading.



(*b*) Volume under constant stress.

from Eq. (4-12), the strain energy will be

$$U = \frac{1}{2}\int \frac{1}{E}\left[\left(\frac{M_z y}{I_z}\right)^2 + 2(1+\nu)\left(\frac{V_y Q}{I_z b}\right)^2\right] dx\, dA$$

or
$$U = \frac{1}{2}\int \frac{1}{E}\left(\frac{M_z}{I_z}\right)^2 y^2\, dx\, dA + \int \frac{1+\nu}{E}\left(\frac{V_y}{I_z}\right)^2\left(\frac{Q}{b}\right)^2 dx\, dA = U_{Bending} + U_{Shear}$$

since $\int y^2 dA = I_Z$; $\quad U_B = \frac{1}{2}\int_0^L \frac{M_Z^2}{EI_Z}\,dx$ ~~and~~ OR $U_B = \frac{1}{2EI_Z}\int_0^L M_Z^2\,dx$

The second integral in Eq. (4-17), U_s, is evaluated in a similar manner. Since V_y/I_z is a function of x alone, integration with respect to dA can be performed first as was done for U_b, but the integral $\int (Q/b)^2\, dA$ must be evaluated for the particular shape of cross section under investigation. It is common practice to assume a constant shear stress across the cross section, $\tau_{xy} = V_y/A$. This introduces a small error, and in light of the fact that for most beams $U_b \gg U_s$, this error becomes quite negligible. Therefore, if $\tau_{xy} = V_y/A$ is used instead of $\tau_{xy} = V_y Q/I_z b$, the strain energy due to the shear force V_y is

$$U_s = \int_0^L \frac{(1+\nu)V_y^2}{EA}\,dx \tag{4-22}$$

If ν, E, and A are constant,

$$U_s = \frac{1+\nu}{EA}\int_0^L V_y^2\,dx \tag{4-23}$$

E-9

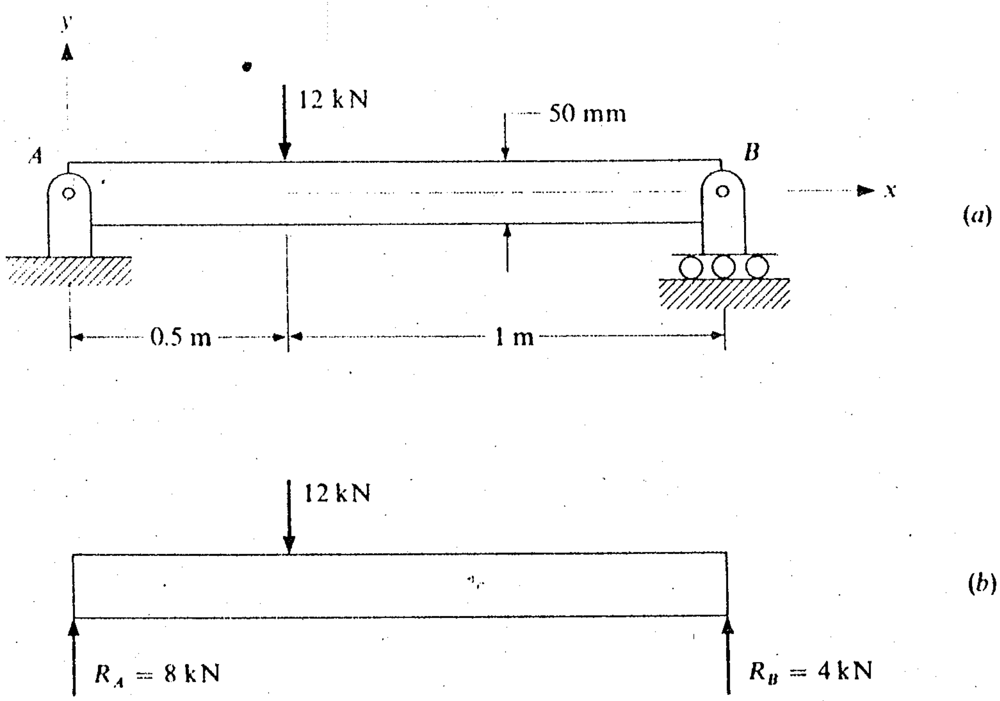


Figure 4-11

Example 4-4 Determine the total strain energy of the steel beam shown in Fig. 4-11. The cross section is rectangular, 25 mm wide, and 50 mm deep. $E_s = 20.5 \times 10^{10}\ \mathrm{N/m^2}$; $\nu = 0.3$.

SOLUTION First, solving for the reactions at A and B gives

$$R_A = 8\ \mathrm{kN} \qquad R_B = 4\ \mathrm{kN}$$

The next step is to determine the shear-force and bending-moment equations:

$$V_y = \begin{cases} -8\ \mathrm{kN} & 0 < x < 0.5 \\ 4\ \mathrm{kN} & 0.5 < x < 1.5 \end{cases}$$

$$M_z = \begin{cases} 8x\ \mathrm{kN{\cdot}m} & 0 < x < 0.5 \\ 8x - 12(x - 0.5)\ \mathrm{kN{\cdot}m} & 0.5 < x < 1.5 \end{cases}$$

When Eq. (4-19) is used, the bending energy is given by

$$U_b = \frac{1}{2EI_z}\left\{\int_0^{0.5} (8000x)^2\,dx + \int_{0.5}^{1.5} [8000x - 12{,}000(x - 0.5)]^2\,dx\right\}$$

Integration of this and substitution of $E = 20.5 \times 10^{10}\ \mathrm{N/m^2}$ and $I_z = \frac{1}{12}(0.025)(0.05)^3 = 2.604 \times 10^{-7}\ \mathrm{m^4}$ results in

$$U_b = 74.9\ \mathrm{N{\cdot}m}$$

The strain energy due to the shear force V_y is found using Eq. (4-23), which yields

$$U_s = \frac{1 + 0.3}{(20.5 \times 10^{10})(0.05)(0.025)}\left[\int_0^{0.5} (-8000)^2\,dx + \int_{0.5}^{1.5} 4000^2\,dx\right] = 0.244\ \mathrm{N{\cdot}m}$$

Thus

$$U = U_b + U_s \approx 75\ \mathrm{N{\cdot}m}$$

Note that the strain energy due to bending is considerably greater than the shear energy. This is usually the case for beams unless they are very short. Thus, in most cases, the shear energy is neglected

 E-10

4-4 THE STRAIN-ENERGY THEOREM

Consider a series of forces P_i gradually applied to a structure. At each point of load application i let δ_i be the component of the total deflection of point i along the line of action of P_i. Therefore, the total work applied is

$$W = \sum_{i=1}^{n} \int P_i \, d\delta_i$$

If energy is conserved, the total work performed on the structure increases the total strain energy of the structure. Thus $U = W$, and

$$U = \sum_{i=1}^{n} \int P_i \, d\delta_i \tag{4-24}$$

Now, fixing the points of application of all the forces except one specific point i, allow point i to deflect an infinitesimal amount $\Delta\delta_i$ in the direction of P_i by applying an infinitesimal force ΔP_i. The additional work ΔW will cause an infinitesimal change in strain energy of the system ΔU. Thus

$$\Delta W = P_i \, \Delta\delta_i + \int_0^{\Delta\delta_i} \Delta P_i \, d\delta_i = \Delta U\dagger$$

The work term in the integral is of the order of the product of two infinitesimal terms and thus can be neglected. Therefore, $\Delta U = P_i \, \Delta\delta_i$, or

$$P_i = \frac{\Delta U}{\Delta\delta_i} \tag{4-25}$$

If $\Delta\delta_i$ is now allowed to approach zero, Eq. (4-25) reduces to

$$P_i = \frac{\partial U}{\partial \delta_i} \tag{4-26}$$

Equation (4-26) actually represents n equations since $i = 1, 2, \ldots, n$.

To use Eq. (4-26), it is necessary to describe the strain energy as a function of the deflections δ_i, which may be difficult to do as the geometry of deformation is necessary. This is the disadvantage of Eq. (4-26).

Equation (4-26) is also applicable to problems where the force-deflection relation is nonlinear provided energy is conserved and rotations are small.

Example 4-5 Two cables shown in Fig. 4-12*a* are made of a nonlinear material whose stress-strain behavior is $\sigma = E\varepsilon - K\varepsilon^2$, where $E = 30 \times 10^6$ lb/in² and $K = 10 \times 10^9$ lb/in². The cross-sectional area A of each cable is 0.2 in², and both have a length L of 5 ft. Determine the vertical deflection δ_B of point B after a load P_B of 2000 lb is applied.

SOLUTION Thanks to symmetry, the strain energies in each cable are equal. Thus, it is only necessary to evaluate one cable, for example, BC. The total energy is then

$$U = 2 \int F_{BC} \, d\delta_{BC}$$

† Here P_i was considered constant through the deflection $\Delta\delta_i$. The force ΔP_i is gradually applied throughout the loading period, resulting in the integral term.

E - 11

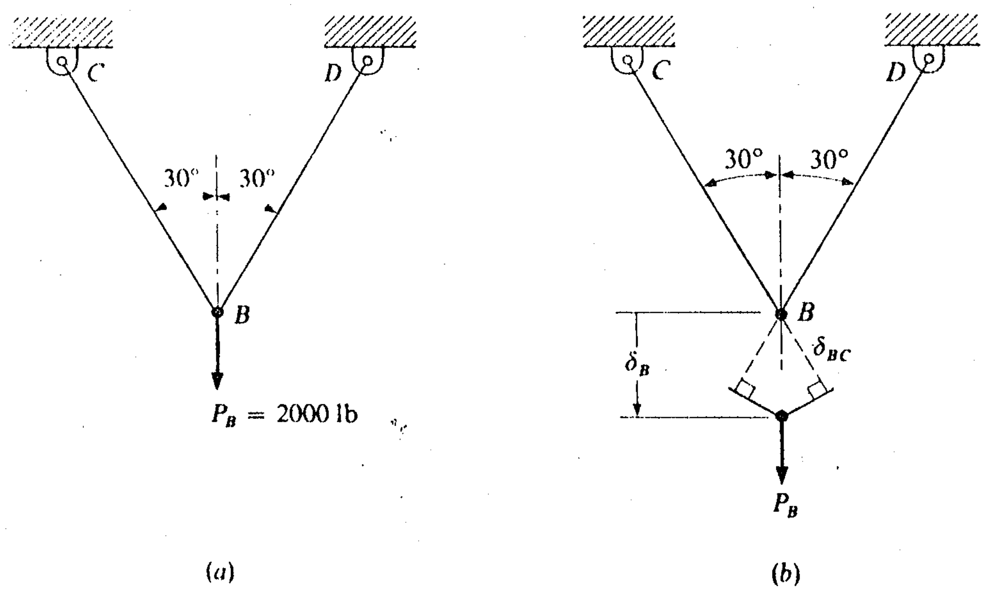


Figure 4-12

Since $F_{BC} = \sigma A = (E\varepsilon - K\varepsilon^2)_{BC}A$ and $\varepsilon_{BC} = \delta_{BC}/L$, we have

$$F_{BC} = \left[E\frac{\delta_{BC}}{L} - K\left(\frac{\delta_{BC}}{L}\right)^2\right]A \tag{a}$$

$$U = 2A\int_0^{\delta_{BC}}\left[E\frac{\delta_{BC}}{L} - K\left(\frac{\delta_{BC}}{L}\right)^2\right]d\delta_{BC} = 2AL\left[\frac{E}{2}\left(\frac{\delta_{BC}}{L}\right)^2 - \frac{K}{3}\left(\frac{\delta_{BC}}{L}\right)^3\right]$$

Now from Eq. (4-24)

$$P_B = \frac{\partial U}{\partial \delta_B} = 2A\left[E\left(\frac{\delta_{BC}}{L}\right) - K\left(\frac{\delta_{BC}}{L}\right)^2\right]\frac{\partial \delta_{BC}}{\partial \delta_B} \tag{b}$$

It is now necessary to obtain a relationship between δ_{BC} and δ_B using geometry of deformation. From Fig. 4-12*b* it can be seen that

$$\delta_{BC} = \delta_B \cos 30 \tag{c}$$

Therefore

$$\frac{\partial \delta_{BC}}{\partial \delta_B} = \cos 30 \tag{d}$$

Substituting Eqs. (*c*) and (*d*) into Eq. (*b*) yields

$$P_B = 2A\left[E\frac{\delta_B}{L} - K\cos 30\left(\frac{\delta_B}{L}\right)^2\right]\cos^2 30$$

Rearranging results in

$$\left(\frac{\delta_B}{L}\right)^2 - \frac{E}{K\cos 30}\frac{\delta_B}{L} + \frac{P_B}{2AK\cos^3 30} = 0 \tag{e}$$

Solving Eq. (*e*) yields

$$\delta_B = \frac{EL}{2K\cos 30} \pm \frac{L}{2}\left[\left(\frac{E}{K\cos 30}\right)^2 - \frac{4P_B}{2AK\cos^3 30}\right]^{1/2} \tag{f}$$

E-12

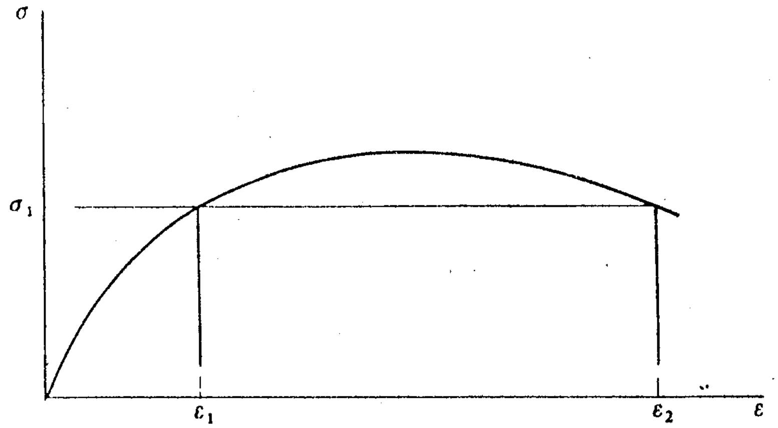


Figure 4-13

Substituting in numerical values and solving for δ_B results in

$$\delta_B = 0.0142 \text{ in} \qquad \text{or} \qquad 0.194 \text{ in}$$

Two solutions are obtained because the stress-strain relationship given is double-valued in strain, as can be seen in Fig. 4-13. Thus for a given stress in cables BC and BD, it is possible to have either strain ε_1 or ε_2. However, since the applied force is gradually increased to $P_B = 200$ lb and does not exceed this value, only the lower strain (and displacement) value is acceptable. Thus

$$\delta_B = 0.0142 \text{ in}$$

Note that in Example 4-5 the equilibrium force equations were not used in obtaining the solution. Although in this example, it would be quite easy to solve for F_{BC} and F_{BD} using the static equilibrium equations, it would not be a simple matter if the problem were statically indeterminate. When the strain-energy theorem is used, the solution of the forces comes out of the analysis without any difficulty. If δ_B is substituted into Eq. (c),

$$\delta_{BC} = 0.0142 \cos 30 = 0.0123 \text{ in}$$

Substituting δ_{BC} into Eq. (a) yields

$$F_{BC} = \left[(30 \times 10^6)\frac{0.0123}{60} - (10 \times 10^9)\left(\frac{0.0123}{60}\right)^2\right](0.2) = 1150 \text{ lb}$$

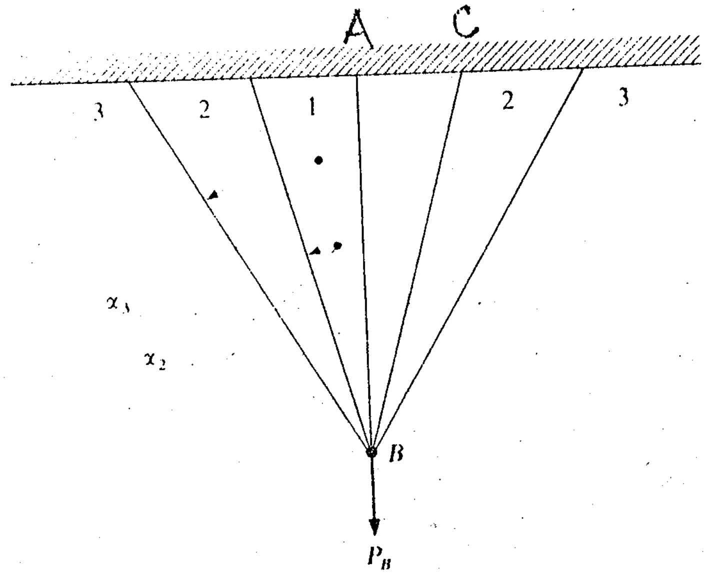
The method in evaluating forces in statically indeterminate problems is basically the same as that illustrated in Example 4-5.

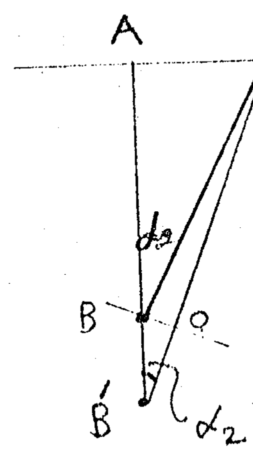
Example 4-6 Using the strain-energy method, determine the cable forces of the symmetric structure shown in Fig. 4-14. The length, area, and modulus of cable 1 are L_1, A_1, and E_1 and of cable 2 are L_2, A_2, and E_2; etc. The materials are linear.

SOLUTION The total strain energy is

$$U = \int F_1 \, d\delta_1 + 2\int F_2 \, d\delta_2 + 2\int F_3 \, d\delta_3 \qquad (a)$$

E-13





CB = CO
B'O = δ_2
BB' = δ_1

Figure 4-14

Since the force-deflection relationship for each cable is

$$F_i = \left(\frac{AE}{L}\right)_i \delta_i \tag{b}$$

Eq. (a) becomes

$$U = \left(\frac{AE}{L}\right)_1 \frac{\delta_1^2}{2} + \left(\frac{AE}{L}\right)_2 \delta_2^2 + \left(\frac{AE}{L}\right)_3 \delta_3^2 \tag{c}$$

The deflection of point B is the deflection δ_1 of cable 1, and so

$$P_B = \frac{\partial U}{\partial \delta_1} = \left(\frac{AE}{L}\right)_1 \delta_1 + 2\left(\frac{AE}{L}\right)_2 \delta_2 \frac{\partial \delta_2}{\partial \delta_1} + 2\left(\frac{AE}{L}\right)_3 \delta_3 \frac{\partial \delta_3}{\partial \delta_1} \tag{d}$$

As in Example 4-5, a geometric relationship is established:

$$\boxed{\delta_2 = \delta_1 \cos\alpha_2} \qquad \delta_3 = \delta_1 \cos\alpha_3 \tag{e}$$

and

$$\frac{\partial \delta_2}{\partial \delta_1} = \cos\alpha_2 \qquad \frac{\partial \delta_3}{\partial \delta_1} = \cos\alpha_3 \tag{f}$$

Substituting Eqs. (e) and (f) into Eq. (d) results in

$$P_B = \left[\left(\frac{AE}{L}\right)_1 + 2\left(\frac{AE}{L}\right)_2 \cos^2\alpha_2 + 2\left(\frac{AE}{L}\right)_3 \cos^2\alpha_3\right] \delta_1$$

Solving for δ_1 yields

$$\delta_1 = \frac{P_B}{K} \tag{g}$$

where

$$K = \left(\frac{AE}{L}\right)_1 + 2\left(\frac{AE}{L}\right)_2 \cos^2\alpha_2 + 2\left(\frac{AE}{L}\right)_3 \cos^2\alpha_3 \tag{h}$$

Substituting δ_1 into Eq. (b) yields the force in cable 1. Thus

$$F_1 = \left(\frac{AE}{L}\right)_1 \frac{P}{K} \tag{i}$$

Substituting δ_1 into Eqs. (e) and then substituting into Eq. (b) yields the forces in cables 2 and 3:

$$F_2 = \left(\frac{AE}{L}\right)_2 \frac{P}{K} \cos\alpha_2 \qquad F_3 = \left(\frac{AE}{L}\right)_3 \frac{P}{K} \cos\alpha_3 \tag{j}$$

E-14

As demonstrated in Example 4-7, if the load-displacement relation is *linear*, the complementary energy equals the strain energy. Thus, Eqs. (4-30) and (4-31) can be written

$$\delta_i = \frac{\partial U}{\partial P_i} \tag{4-32}$$

and

$$\theta_i = \frac{\partial U}{\partial M_i} \tag{4-33}$$

Equations (4-32) and (4-33) are statements of Castigliano's theorem as applied to problems where the force-deflection relation is linear. Since many practical engineering problems involve linear load-displacement relations and the forms of the strain energy are known for simple cases (see Sec. 4-3), the method of Castigliano provides a simple and straightforward approach to a complex collection of simple elements.

Example 4-9 For the structure shown in Fig. 4-17, determine the vertical deflection of point B. All members have a cross-sectional area A; members 1, 2, 4, and 5 are of length L, and members 3, 6, and 7 are of length $1.41L$.

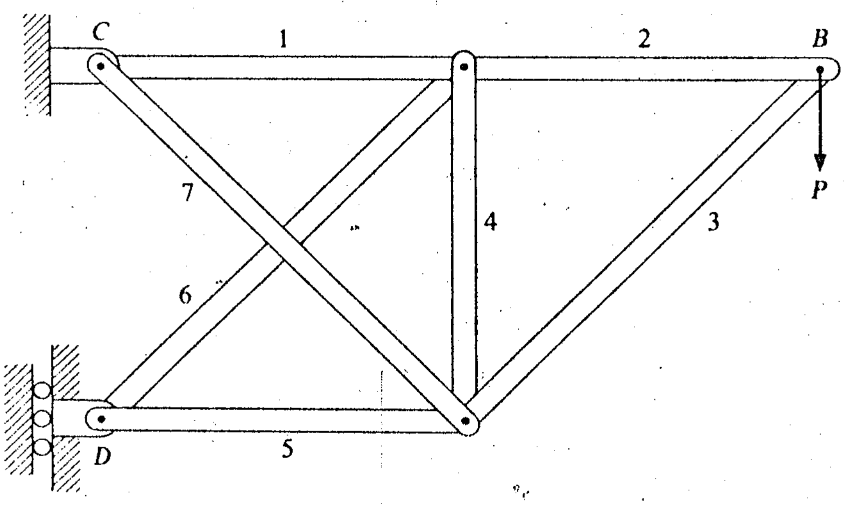


Figure 4-17

SOLUTION The total strain energy is the sum of the strain energies of all the members. Thus

$$U = U_1 + U_2 + U_3 + U_5 + U_6 + U_7$$

Since each member is loaded axially, the energy in the ith member is

$$U_i = \frac{1}{2}\frac{F_i^2 L_i}{AE} \qquad i = 1, 2, \ldots, 7$$

Thus it is necessary to determine the axial load in each member F_i using static analysis. Assuming tension positive for each member, we have

$$F_1 = P \qquad F_2 = P \qquad F_3 = -1.41P \qquad F_4 = 0$$

$$F_5 = -2P \qquad F_6 = 0 \qquad F_7 = 1.41P$$

The total energy is then given by

E-15

$$U = \frac{1}{2}\frac{P^2L}{AE} + \frac{1}{2}\frac{P^2L}{AE} + \frac{1}{2}\frac{(-1.41P)^2(1.41L)}{AE} + 0 + \frac{1}{2}\frac{(-2P)^2L}{AE} + 0 + \frac{1}{2}\frac{(1.41P)^2(1.41L)}{AE}$$

Simplifying results in

$$U = \frac{11.6}{2}\frac{P^2L}{AE}$$

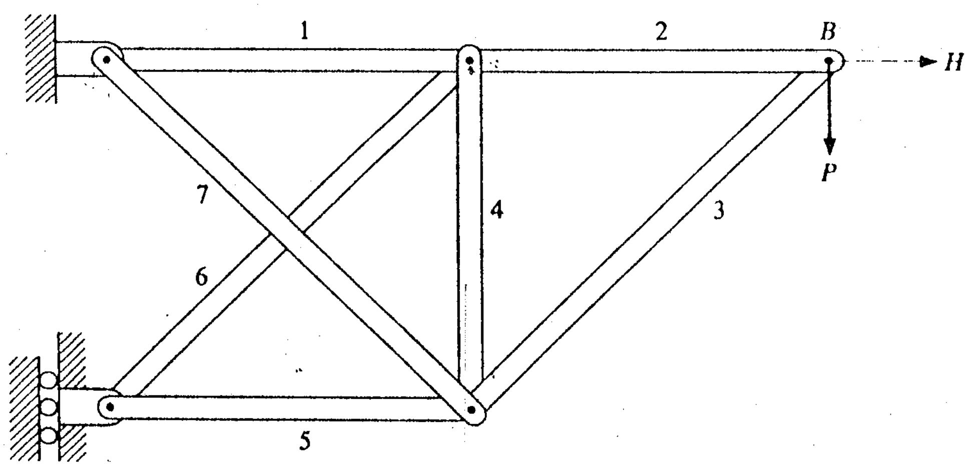
From Eq. (4-32), the vertical deflection $(\delta_B)_V$ of point B is $\partial U/\partial P$. Thus

$$(\delta_B)_V = \frac{\partial U}{\partial P} = 11.6\frac{PL}{AE}$$

In Example 4-9 in order to determine the horizontal deflection of point B using Eq. (4-32), it would be necessary to place a dummy force at B in the horizontal direction.

Example 4-10 Using Castigliano's method, determine the horizontal deflection of point B in Example 4-9.

SOLUTION Place a horizontal dummy force H at point B as shown in Fig. 4-18. A static analysis of the forces in the members caused by H alone can be added by superposition to the



forces caused by P, as determined in Example 4-9. Thus

$$F_1 = P + H \qquad F_2 = P + H \qquad F_3 = -1.41P \qquad F_4 = 0$$

$$F_5 = -2P \qquad F_6 = 0 \qquad F_7 = 1.41P$$

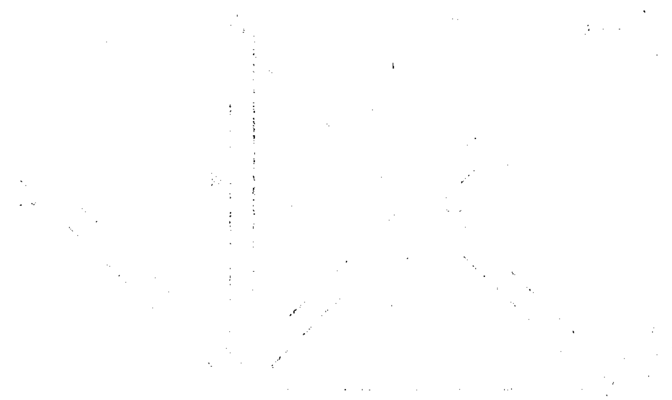
The forces can then be substituted into the strain-energy relations to obtain U_1, U_2, etc. The displacement of B in the horizontal direction is then calculated from Eq. (4-32), where,

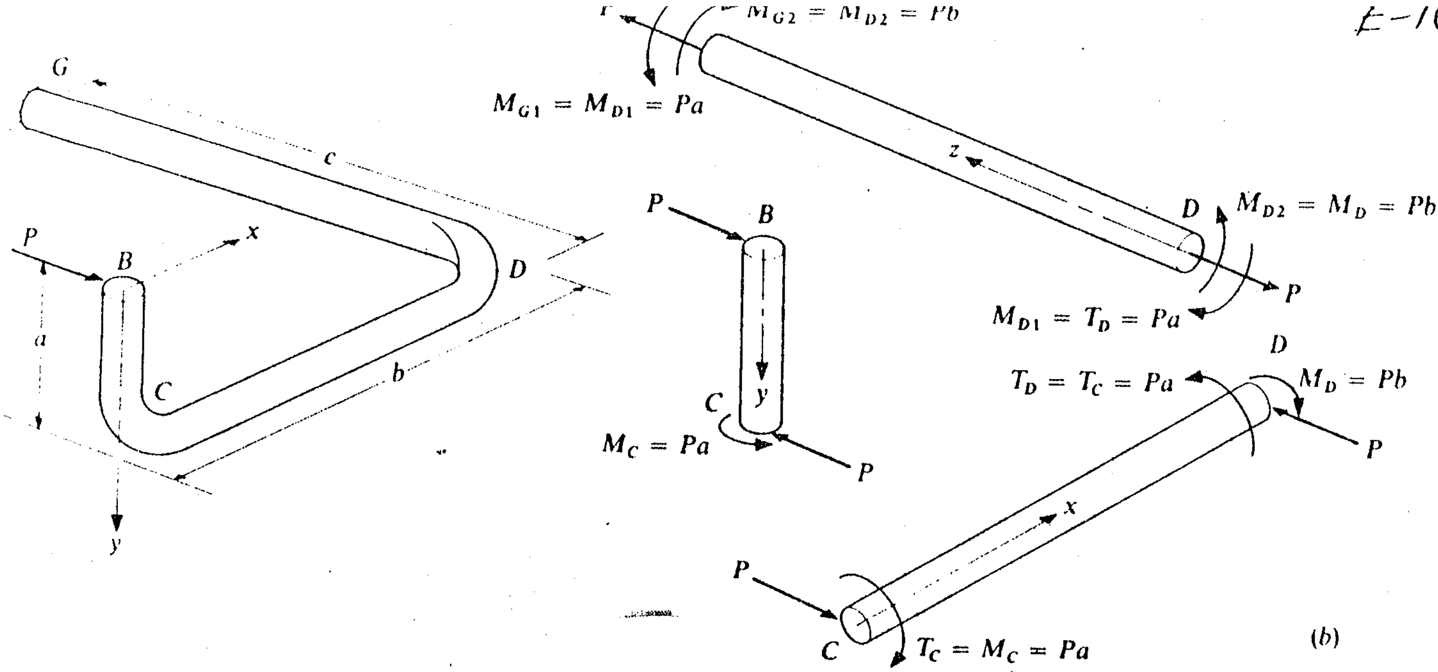
$$(\delta_B)_H = \frac{\partial U}{\partial H} = 2\frac{(P+H)L}{AE}$$

However, $H = 0$, and the final result is

$$(\delta_B)_H = 2\frac{PL}{AE}$$

The displacement of any point in the structure in any direction can be found by (1) placing a dummy force at that point in the direction desired, (2) determining the strain energy of the system due to all applied forces and the dummy force, (3) substituting the total strain energy into Eq. (4-32) and performing the partial differentiation, and (4) equating the dummy force to





Example 4-16 For the wire form shown in Fig. 4-25a determine the deflection of point B in the direction of the applied force P. (Neglect direct shear effects.) The diameter of the wire is d, the modulus of elasticity is E, and Poisson's ratio is ν.

[St]rain energy in BC is due to bending alone:

$$\frac{\partial U_{BC}}{\partial P} = \frac{1}{EI_z}\int_0^a (-Py)(-y)\,dy = \frac{Pa^3}{3EI_z} \quad (a)$$

The strain energy in CD is due to torsion and bending,†

$$\frac{\partial U_{CD}}{\partial P} = \frac{2(1+\nu)}{JE}(Pa)(a)(b) + \frac{1}{EI_z}\int_0^b (-Px)(-x)\,dx$$

$$= \frac{2(1+\nu)Pa^2b}{JE} + \frac{Pb^3}{3EI_z} \quad (b)$$

The strain energy in DG is due to an axial load and bending in two planes:‡

$$\frac{\partial U_{DG}}{\partial P} = \frac{Pc}{AE} + \frac{1}{EI_z}\int_0^c (Pb)(b)\,dz + \frac{1}{EI_z}\int_0^c (Pa)(a)\,dz$$

$$= \frac{Pc}{AE} + \frac{Pb^2c}{EI_z} + \frac{Pa^2c}{EI_z} \quad (c)$$

† Note, for torsion, $U_t = [(1+\nu)/JE]T^2L$. Therefore

$$\frac{\partial U_t}{\partial P} = \frac{2(1+\nu)}{JE}T\frac{\partial T}{\partial P}L$$

‡ Likewise, for axial loading

$$U_a = \frac{1}{2}\frac{F^2L}{AE}$$

therefore

$$\frac{\partial U_a}{\partial P} = \frac{F}{AE}\frac{\partial F}{\partial P}L$$

[Th]e deflection of point B in the direction of P is

$$(\delta_B)_P = \frac{\partial U}{\partial P} = \frac{\partial U_{BC}}{\partial P} + \frac{\partial U_{CD}}{\partial P} + \frac{\partial U_{DG}}{\partial P}$$

[Th]erefore after some rearranging and using the fact that for a circular section $J = 2I_z$, the sum [of] Eqs. (a) (b), and (c) reduces to

$$(\delta_B)_P = \frac{P}{EI_z}\left[\tfrac{1}{3}(a^3+b^3) + c(a^2+b^2) + (1+\nu)a^2b + c\frac{I_z}{A}\right]$$

Example 4-18 The curved beam shown in Fig. 4-27a has a radius of curvature R, modulus E, and a sectional moment of inertia I_z about an axis out of the page directed through the centroid of an area section. Determine the horizontal and vertical deflections of point A considering only bending due to the application of the horizontal force P.

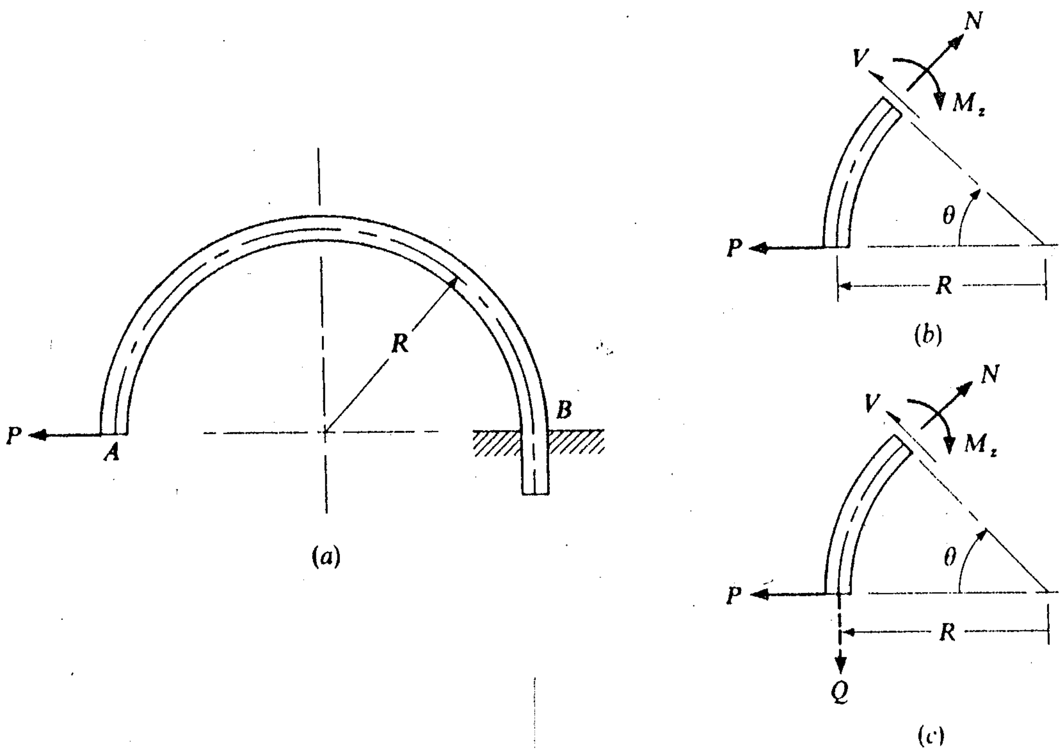


Figure 4-27

SOLUTION It is necessary to establish how the bending moment varies throughout the beam. When the variable θ is used, as shown in Fig. 4-27b, the bending moment can be determined as a function of θ. If this is done, an infinitesimal length of beam will be $R\,d\theta$. Substituting $R\,d\theta$ for dx in Eq. (4-34) gives

$$\delta_i = \frac{\partial U}{\partial P_i} = \int \frac{M_z}{EI_z}\frac{\partial M_z}{\partial P_i} R\,d\theta \tag{a}$$

Summing moments at the center of the break at θ yields

$$M_z = -PR\sin\theta \qquad 0<\theta<\pi \tag{b}$$

The horizontal deflection $(\delta_A)_H$ of point A is given by $\partial U/\partial P$; thus $\partial M_z/\partial P$ is desired. From Eq. (b)

$$\frac{\partial M_z}{\partial P} = -R\sin\theta \tag{c}$$

Substitution of Eqs. (b) and (c) into Eq. (a) results in

$$(\delta_A)_H = \frac{\partial U}{\partial P} = \int_0^{\pi} \frac{1}{EI_z}(-PR\sin\theta)(-R\sin\theta)R\,d\theta$$

Since EI_z, P, and R are constants, the integral simplifies to

$$(\delta_A)_H = \frac{PR^3}{EI_z}\int_0^{\pi}\sin^2\theta\,d\theta \tag{d}$$

After using the trigonometric identity $\sin^2\theta = (1-\cos 2\theta)/2$, integration of Eq. ($d$) reduces it to

$$(\delta_A)_H = \frac{\pi}{2}\frac{PR^3}{EI_z}$$

E-18

For the vertical deflection $(\delta_A)_V$ of point A a dummy force Q must be added, as shown in Fig. 4-27c, where

$$(\delta_A)_V = \left.\frac{\partial U}{\partial Q}\right|_{Q=0} \tag{e}$$

The bending moment as a function of θ is then given by

$$M_z = -PR \sin\theta + QR(1-\cos\theta) \qquad 0<\theta<\pi \tag{f}$$

and since $\partial M_z/\partial Q$ will be needed,

$$\frac{\partial M_z}{\partial Q} = R(1-\cos\theta) \tag{g}$$

Now that differentiation has been completed, Eqs. (f) and (g) can be substituted into Eq. (e) with $Q=0$. This results in

$$(\delta_A)_V = \int_0^\pi \frac{1}{EI_z}(-PR\sin\theta)[R(1-\cos\theta)]\,R\,d\theta \tag{h}$$

Integration of Eq. (h) gives

$$(\delta_A)_V = -2\frac{PR^3}{EI_z}$$

Example 4-18 neglected the effects of the normal and transverse forces. The effect and order of magnitude of the contribution of these forces can be seen in the following example.

SOLUTION From Fig. 4-27b the complete internal reactions are

Find horizontal defl in the previous problem for axial, transverse shear force and bending moment.

$$M_z = -PR\sin\theta \tag{a}$$

$$N = P\sin\theta \tag{b}$$

$$V = -P\cos\theta \tag{c}$$

Since N and V are varying, Eqs. (4-13) and (4-22) are used with $dx = R\,d\theta$. Differentiation with respect to P can be performed before integration, and the resulting deflection equation is

$$(\delta_A)_H = \frac{\partial U}{\partial P} = \frac{1}{EI_z}\int M_z\frac{\partial M_z}{\partial P}R\,d\theta + \frac{1}{EA}\int N\frac{\partial N}{\partial P}R\,d\theta + \frac{2(1+\nu)}{EA}\int V\frac{\partial V}{\partial P}R\,d\theta \tag{d}$$

From Eqs. (a) to (c) the derivative terms are

$$\frac{\partial M_z}{\partial P} = -R\sin\theta \tag{e}$$

$$\frac{\partial N}{\partial P} = \sin\theta \tag{f}$$

$$\frac{\partial V}{\partial P} = -\cos\theta \tag{g}$$

Substituting Eqs. (a) to (c) and (e) to (g) into Eq. (d) yields

$$(\delta_A)_H = \frac{1}{EI_z}\int_0^\pi(-PR\sin\theta)(-R\sin\theta)R\,d\theta + \frac{1}{EA}\int_0^\pi (P\sin\theta)(\sin\theta)R\,d\theta$$
$$+2\frac{1+\nu}{EA}\int_0^\pi(-P\cos\theta)(-\cos\theta)R\,d\theta$$

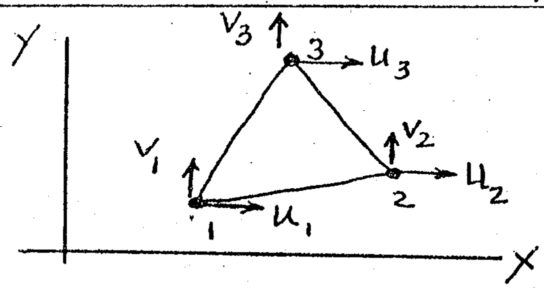
Integration and simplification result in

$$(\delta_A)_H = \frac{\pi}{2}\frac{PR^3}{EI_z} + \frac{\pi}{2}\frac{PR}{EA} + \pi(1+\nu)\frac{PR}{EA}$$

or

$$(\delta_A)_H = \frac{\pi}{2}\frac{PR^3}{EI_z}\left[1+(3+2\nu)\frac{I_z}{AR^2}\right] \tag{h}$$

LANE TRIANGULAR ELEMENT



COORDINATES	DISPLACEMENTS
(x_1, y_1)	(u_1, v_1)
(x_2, y_2)	(u_2, v_2)
(x_3, y_3)	(u_3, v_3)

NODAL QUANTITY
(This is a 6 DOF element)

$$\{q\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \cdots\cdots\cdots (26)$$

ASSUMPTION

$$u_i = a_1 + a_2 x_i + a_3 y_i$$
$$v_i = b_1 + b_2 x_i + b_3 y_i \quad \cdots\cdots\cdots (27)$$

Q: What is the difference between u_1 (NODAL DEFORMATION) and u_i (DEFORMATION ANYWHERE WITHIN THE ELEMENT)?

eq (27) we have 6 constants a's and b's

ONSTANTS IN TERMS OF NODAL DEFFORMATION

$$u_1 = a_1 + a_2 x_1 + a_3 y_1 \qquad v_1 = b_1 + b_2 x_1 + b_3 y_1$$
$$u_2 = a_1 + a_2 x_2 + a_3 y_2 \qquad v_2 = b_1 + b_2 x_2 + b_3 y_2$$
$$u_3 = a_1 + a_2 x_3 + a_3 y_3 \qquad v_3 = b_1 + b_2 x_3 + b_3 y_3$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \qquad \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = [A] \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} \; ; \; \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} = [A] \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} \quad \cdots\cdots (28)$$

NOTE: $A = .5|A| = .5(x_2 y_3 - x_3 y_2 + x_3 y_1 - x_1 y_3 + x_1 y_2 - x_2 y_1) \cdots (32)$
is the area of the triangle

From (28);

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \text{ and } \begin{Bmatrix} b_1 \\ b_2 \\ b_3 \end{Bmatrix} = [A]^{-1} \begin{Bmatrix} v_1 \\ v_2 \\ v_3 \end{Bmatrix} \cdots (30)$$

NOTE: $[A]^{-1} = \dfrac{0.5}{A} \Big[f(\text{nodal coordinates}) \Big] \quad \cdots\cdots\cdots (31)$

ENERAL DEFORMATION IN TERMS OF NODAL DEFORMATION

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \cdots\cdots\cdots (37)$$

Let us examine (37) carefully:

From (30) + (31) $\Rightarrow$ $$a_1 = \frac{0.5}{A}\left[(x_2y_3 - x_3y_2)u_1 + (x_3y_1 - x_1y_3)u_2 + (x_1y_2 - x_2y_1)u_3\right]$$

Similarly $\Rightarrow$ $$a_2 = \frac{0.5}{A}\left[(y_2-y_3)u_1 + (y_3-y_1)u_2 + (y_1-y_2)u_3\right]$$

$$a_3 = \frac{0.5}{A}\left[(x_3-x_2)u_1 + (x_1-x_3)u_2 + (x_2-x_1)u_3\right]$$

Also

$$b_1 = \frac{0.5}{A}\left[(x_2y_3 - x_3y_2)v_1 + (x_3y_1 - x_1y_3)v_2 + (x_1y_2 - x_2y_1)v_3\right]$$

$$b_2 = \frac{0.5}{A}\left[(y_2-y_3)v_1 + (y_3-y_1)v_2 + (y_1-y_2)v_3\right]$$

$$b_3 = \frac{0.5}{A}\left[(x_3-x_2)v_1 + (x_1-x_3)v_2 + (x_2-x_1)v_3\right]$$

From (27) $\Rightarrow$ $u = a_1 + a_2 x + a_3 y$

$$= u_1\left[\frac{0.5}{A}\underbrace{(x_2y_3 - x_3y_2)}_{\alpha} + \frac{.5}{A}\underbrace{(y_2-y_3)}_{\beta}x + \frac{.5}{A}\underbrace{(x_3-x_2)}_{\gamma}y\right] +$$
$$u_2\left[\frac{0.5}{A}(x_3y_1 - x_1y_3) + \frac{.5}{A}(y_3-y_1)x + \frac{.5}{A}(x_1-x_3)y\right] +$$
$$u_3\left[\frac{0.5}{A}(x_1y_2 - x_2y_1) + \frac{.5}{A}(y_1-y_2)x + \frac{.5}{A}(x_2-x_1)y\right]$$

$$= u_1\left[\frac{.5}{A}\alpha_1 + \frac{.5}{A}\beta_1 x + \frac{.5}{A}\gamma_1 y\right] +$$
$$u_2\left[\frac{.5}{A}\alpha_2 + \frac{.5}{A}\beta_2 x + \frac{.5}{A}\gamma_2 y\right] +$$
$$u_3\left[\frac{.5}{A}\alpha_3 + \frac{.5}{A}\beta_3 x + \frac{.5}{A}\gamma_3 y\right]$$

$$= N_1u_1 + N_2u_2 + N_3u_3 \quad \text{where } N_i = \frac{.5}{A}(\alpha_i + \beta_i x + \gamma_i y) \qquad (35)$$

Similarly $v = N_1v_1 + N_2v_2 + N_3v_3$

REMEMBER $N \approx f(x,y)$ also a linear function.

DERIVATIVES OF N (SHAPE FUNCTION)

$$\frac{\partial N_i}{\partial x} = \frac{.5}{A}\beta_i \approx \frac{\partial N_1}{\partial x} = \frac{.5}{A}\beta_1 = \frac{.5}{A}(Y_2 - Y_3) \text{ and so on.}$$

$$\frac{\partial N_i}{\partial y} = \frac{.5}{A}\gamma_i \approx \frac{\partial N_1}{\partial y} = \frac{.5}{A}\gamma_1 = \frac{.5}{A}(X_3 - X_2) \text{ and so on.}$$

STRAIN - DEFORMATION RELATION

$$\varepsilon_x = \frac{\partial u}{\partial x} = u_1\frac{\partial N_1}{\partial x} + u_2\frac{\partial N_2}{\partial x} + u_3\frac{\partial N_3}{\partial x} = \frac{.5}{A}\left[\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3\right]$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = v_1\frac{\partial N_1}{\partial y} + v_2\frac{\partial N_2}{\partial y} + v_3\frac{\partial N_3}{\partial y} = \frac{.5}{A}\left[\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3\right]$$

$$\gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{.5}{A}\left[\gamma_1 u_1 + \beta_1 v_1 + \gamma_2 u_2 + \beta_2 v_2 + \gamma_3 u_3 + \beta_3 v_3\right]$$

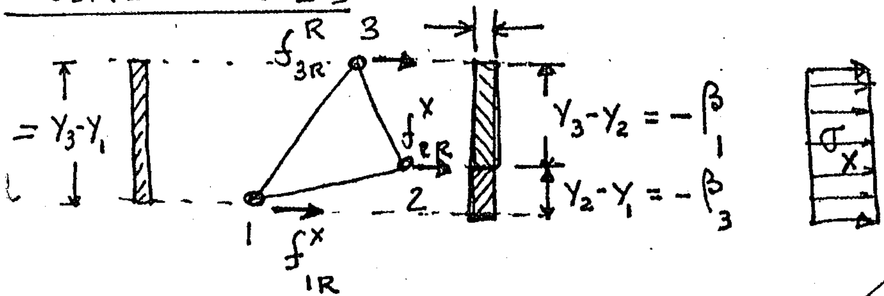
$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{.5}{A}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} \quad \ldots\ldots (39)$$

$\frac{in \cdot in}{in^2}$

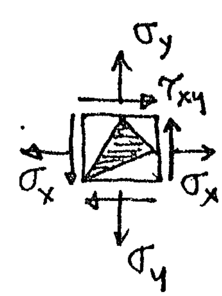
So:

$$\boxed{\begin{matrix}\{\varepsilon\} = [B]\{q\} \\ (3\times1) = (3\times6)\,(6\times1)\end{matrix}}$$

NODAL FORCES/STRESS



$$f^x_{3R} = -.5\sigma_x t\beta_1$$
$$f^x_{2R} = -.5\sigma_x t\beta_1$$
$$f^x_{2R} = -.5\sigma_x t\beta_3$$
$$f^x_{1R} = -.5\sigma_x t\beta_3$$



IGHT SIDE (σ_x)

$$\left.\begin{aligned} f^x_{1R} &= -.5\sigma_x t\beta_3 \\ f^x_{2R} &= .5\sigma_x t\beta_2 \\ f^x_{3R} &= -.5\sigma_x t\beta_1 \end{aligned}\right\} \quad \ldots\ldots\ldots (42)$$

$$B \approx \frac{in}{in^2} = \frac{1}{in}$$

RIGHT SIDE (τ_{xy})

$$\left.\begin{aligned} f_{1R}^{y} &= -.5\,\tau_{xy} t\,\beta_3 \\ f_{2R}^{y} &= .5\,\tau_{xy} t\,\beta_2 \\ f_{3R}^{y} &= -.5\,\tau_{xy} t\,\beta_1 \end{aligned}\right\} \quad \ldots\ldots\ldots\ (45)$$

LEFT SIDE (σ_x)

$$\left.\begin{aligned} f_{1L}^{x} &= -.5\,\sigma_x t\,\beta_2 \quad \text{(why - sign?)} \\ f_{2L}^{x} &= 0 \\ f_{3L}^{x} &= -.5\,\sigma_x t\,\beta_2 \end{aligned}\right\}$$

LEFT SIDE (τ_{xy})

$$\left.\begin{aligned} f_{1L}^{y} &= -.5\,\tau_{xy} t\,\beta_2 \\ f_{2L}^{y} &= 0 \\ f_{3L}^{y} &= -.5\,\tau_{xy} t\,\beta_2 \end{aligned}\right\} \quad \ldots\ldots\ (46)$$

TOP FACE (σ_y)

$$\begin{aligned} f_{1T}^{y} &= -.5\,\sigma_y t\,\gamma_2 \\ f_{2T}^{y} &= -.5\,\sigma_y t\,\gamma_1 \\ f_{3T}^{y} &= .5\,\sigma_y t\,\gamma_3 \end{aligned}$$

TOP FACE (τ_{xy})

$$\left.\begin{aligned} f_{1T}^{x} &= -.5\,\tau_{xy} t\,\gamma_2 \\ f_{2T}^{x} &= -.5\,\tau_{xy} t\,\gamma_1 \\ f_{3T}^{x} &= .5\,\tau_{xy} t\,\gamma_3 \end{aligned}\right\} \quad \ldots\ldots\ (47)$$

BOTTOM FACE (σ_y)

$$\begin{aligned} f_{1B}^{y} &= -.5\,\sigma_y t\,\gamma_3 \\ f_{2B}^{y} &= -.5\,\sigma_y t\,\gamma_3 \\ f_{3B}^{y} &= 0 \end{aligned}$$

BOTTOM FACE (τ_{xy})

$$\left.\begin{aligned} f_{1B}^{x} &= -.5\,\tau_{xy} t\,\gamma_3 \\ f_{2B}^{x} &= -.5\,\tau_{xy} t\,\gamma_3 \\ f_{3B}^{x} &= 0 \end{aligned}\right\} \quad \ldots\ldots\ (48)$$

NET $f_1 = f_{1R} + f_{1L} + f_{1T} + f_{1B} = -.5\sigma_x t\beta_3 - .5\sigma_x t\beta_2 - .5\tau_{xy} t\gamma_2 - .5\tau_{xy} t\gamma_3$

$$f_2^x = f_{2R}^x + f_{2L}^x + f_{2T}^x + f_{2B}^x = .5\sigma_x t\beta_2 + 0 - .5\tau_{xy} t\gamma_1 - .5\tau_{xy} t\gamma_3$$

$$f_3^x = f_{3R}^x + f_{3L}^x + f_{3T}^x + f_{3B}^x = -.5\sigma_x t\beta_1 - .5\sigma_x t\beta_2 + .5\tau_{xy} t\gamma_3 + 0$$

$$f_1^y = f_{1R}^y + f_{1L}^y + f_{1T}^y + f_{1B}^y = -.5\tau_{xy} t\beta_3 - .5\tau_{xy} t\beta_2 - .5\sigma_y t\gamma_2 - .5\sigma_y t\gamma_3$$

$$f_2^y = f_{2R}^y + f_{2L}^y + f_{2T}^y + f_{2B}^y = .5\tau_{xy} t\beta_2 + 0 - .5\sigma_y t\gamma_1 - .5\sigma_y t\gamma_3$$

$$f_3^y = f_{3R}^y + f_{3L}^y + f_{3T}^y + f_{3B}^y = -.5\tau_{xy} t\beta_1 - .5\tau_{xy} t\beta_2 + .5\sigma_y t\gamma_3 + 0$$

$$\begin{Bmatrix} f_1^x \\ f_1^y \\ f_2^x \\ f_2^y \\ f_3^x \\ f_3^y \end{Bmatrix} = 0.5t \begin{bmatrix} \beta_1 & 0 & \gamma_1 \\ 0 & \gamma_1 & \beta_1 \\ \beta_2 & 0 & \gamma_2 \\ 0 & \gamma_2 & \beta_2 \\ \beta_3 & 0 & \gamma_3 \\ 0 & \gamma_3 & \beta_3 \end{bmatrix} \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix}$$

$\lambda = (in)(in)$
$= in^2$

$$(6\times1) = (6\times3)(3\times1)$$

$$\{f\} = [\lambda]\{\sigma\} \quad \cdots\cdots \quad (50)$$

STRESS / STRAIN (CONSTITUTIVE) RELATION

$$\sigma_x = \frac{E}{1-\nu^2}(\varepsilon_x + \nu\varepsilon_y)$$

$$\sigma_y = \frac{E}{1-\nu^2}(\varepsilon_y + \nu\varepsilon_x)$$

$C = \frac{lb}{in^2}$

$$\tau_{xy} = \frac{1-\nu}{2E(1-\nu^2)}\gamma_{xy}$$

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \left(\frac{E}{1-\nu^2}\right) \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} \Rightarrow \{\sigma\} = [C]\{\varepsilon\} \cdots (38)$$

$$(3\times1) = (3\times3)(3\times1)$$

Consider (38) (39) and (50)

$$\underset{3\times1}{\{\sigma\}} = \underset{3\times3}{[C]}\underset{3\times1}{\{\varepsilon\}} \cdots\cdots (38) \quad ; \quad \underset{3\times1}{\{\varepsilon\}} = \underset{3\times6}{[B]}\underset{6\times1}{\{q\}} \cdots\cdots (39)$$

$$\underset{6\times1}{\{f\}} = \underset{6\times3}{[\lambda]}\underset{3\times1}{\{\sigma\}} \cdots\cdots (50)$$

FORCE — DISPLACEMENT

Substitute (38) into (50) $\Longrightarrow$
$$\underset{6\times1}{\{f\}} = \underset{6\times3}{[\lambda]}\underset{3\times3}{[C]}\underset{3\times1}{\{\varepsilon\}}$$

Substitute (39)

$$\underset{6\times1}{\{f\}} = \underset{6\times3}{[\lambda]}\underset{3\times3}{[C]}\underset{3\times6}{[B]}\underset{6\times1}{\{q\}}$$

$$\underset{6\times1}{\{f\}} = \underset{6\times6}{[K]}\underset{6\times1}{\{q\}}$$

$[K]$ IS STIFFNESS MATRIX $= [\lambda][C][B]$

β and γ

Material Property

Nodal Coordinate (Geometry)

$$K = in^2 \cdot \frac{lb}{in^2} \cdot \frac{1}{in}$$

$$K = \frac{lb}{in}$$

$$f = K\,q = \frac{lb}{in} \cdot in = lb.$$

Sliving on CAOS book

$$I_0 = 3.72(0.131)^2 = 0.0642\ \text{kg} \cdot \text{m}^2$$

$$\alpha = 36.9\ \text{rad/s}^2$$

From Example 4.2, $R_{x1} = 10.52$ N, $R_{y1} = 3.69$ N, and $M_1 = 1.34$ N · m.

1. $\Sigma F_x = ma_x$,

$$R_{x2} - R_{x1} = ma_x$$

$$R_{x2} = 10.52 + 3.72(-0.03) = 10.41\ \text{N}$$

2. $\Sigma F_y = ma_y$,

$$R_{y2} - R_{y1} - mg = ma_y$$

$$R_{y2} = 3.69 + 3.72 \times 9.8 + 3.72(-4.21) = 24.48\ \text{N}$$

3. About the center of mass of the leg, $\Sigma M = I\alpha$,

$$M_2 - M_1 - 0.169R_{x1} + 0.185R_{y1} - 0.129R_{x2} + 0.142R_{y2} = I\alpha$$

$$\begin{aligned} M_2 &= 1.34 + 0.169 \times 10.52 - 0.185 \times 3.69 + 0.129 \times 10.41 \\ &\quad - 0.142 \times 24.48 + 0.0642 \times 36.9 \\ &= 1.34 + 1.78 - 0.68 + 1.34 - 3.48 + 2.37 = 2.67\ \text{N} \cdot \text{m} \end{aligned}$$

Discussion

1. M_2 is positive. This represents a counterclockwise (extensor) moment acting at the knee. The quadricep muscles at this time are rapidly extending the swinging leg.
2. The angular acceleration of the leg is the net result of two reaction forces and one muscle moment acting at each end of the segment. Thus there may not be a single primary force causing the movement we observe. In this case each force and moment had a significant influence on the final acceleration.

4.2 FORCE TRANSDUCERS AND FORCE PLATES

In order to measure the force exerted by the body on an external body or load, we need a suitable force-measuring device. Such a device, called a *force transducer*, gives an electrical signal proportional to the applied force. There are many kinds available: strain gauge, piezoelectric, piezoresistive, capacitive, and others. All these work on the principle that the applied force causes a certain amount of strain within the transducer. For the strain gauge type a calibrated metal plate or beam within the transducer undergoes a very small change (strain) in one of its dimensions. This mechanical deflection, usually a

fraction of 1%, causes a ch
(see Section 2.2.2), resultin
force. Piezoelectric and pie
the atomic structure within
example, is a naturally foun
crystalline structure change
trical charge across approp
translated via suitable electr
Piezoresistive types exhibit
gauge, upset the balance of

4.2.1 Multidirectional For

In order to measure forces in
or tri-directional force transc
more force transducers mou
problem is to ensure that th
each of the individual transd

4.2.2 Force Plates*

The most common force act
which acts on the foot during
is three-dimensional and cons
ponents acting along the force
solved into anterior–posterior

The fourth variable needed
ground reaction vector. The fo
different pressures at each pa
under every part of the foot, w
of calculating the net effect of
Some attempts have been made
but they have been very expens
therefore necessary to use a for
a complete inverse solution.

Two common types of force
plate supported by four triaxial
the coordinates of each transdu
location of the center of pressu
seen at each of these corner tra
F_{00}, F_{X0}, F_{0Z}, and F_{XZ}, the t
+ F_{XZ}. If all four forces are equa

*Representative paper: Elftman, 1939.

fraction of 1%, causes a change in resistances connected as a bridge circuit (see Section 2.2.2), resulting in an unbalance of voltages proportional to the force. Piezoelectric and piezoresistive types require minute deformations of the atomic structure within a block of special crystalline material. Quartz, for example, is a naturally found piezoelectrical material, and deformation of its crystalline structure changes the electrical characteristics such that the electrical charge across appropriate surfaces of the block is altered and can be translated via suitable electronics to a signal proportional to the applied force. Piezoresistive types exhibit a change in resistance which, like the strain gauge, upset the balance of a bridge circuit.

4.2.1 Multidirectional Force Transducers

In order to measure forces in two or more directions it is necessary to use a bi- or tri-directional force transducer. Such a device is nothing more than two or more force transducers mounted at right angles to each other. The major problem is to ensure that the applied force acts through the central axis of each of the individual transducers.

4.2.2 Force Plates*

The most common force acting on the body is the ground reaction force, which acts on the foot during standing, walking, or running. This force vector is three-dimensional and consists of a vertical component plus two shear components acting along the force plate surface. These shear forces are usually resolved into anterior–posterior and medial–lateral directions.

The fourth variable needed is the location of the center of pressure of this ground reaction vector. The foot is supported over a varying surface area with different pressures at each part. Even if we knew the individual pressures under every part of the foot, we would be faced with the expensive problem of calculating the net effect of all these pressures as they change with time. Some attempts have been made to develop suitable pressure-measuring shoes, but they have been very expensive and are limited to vertical forces only. It is therefore necessary to use a force plate to give us all the forces necessary for a complete inverse solution.

Two common types of force plates are now explained. The first is a flat plate supported by four triaxial transducers, depicted in Figure 4.9. Consider the coordinates of each transducer to be $(0, 0)$, $(0, Z)$, $(X, 0)$, and (X, Z). The location of the center of pressure is determined by the relative vertical forces seen at each of these corner transducers. If we designate the vertical forces as F_{00}, F_{X0}, F_{0Z}, and F_{XZ}, the total vertical force is $F_Y = F_{00} + F_{X0} + F_{0Z} + F_{XZ}$. If all four forces are equal, the center of pressure is at the exact center

*Representative paper: Elftman, 1939.

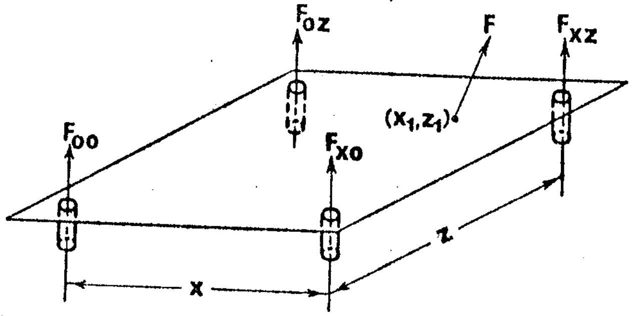


Figure 4.9 Force plate with force transducers in the four corners. Magnitude and location of ground reaction force F can be determined from the signals from the load cells in each of the support bases.

of the force plate, at ($X/2$, $Z/2$). In general,

$$x = \frac{X}{2}\left[1 + \frac{(F_{X0} + F_{XZ}) - (F_{00} + F_{0Z})}{F_y}\right] \tag{4.4}$$

$$z = \frac{Z}{2}\left[1 + \frac{(F_{0Z} + F_{XZ}) - (F_{00} + F_{X0})}{F_Y}\right] \tag{4.5}$$

A second type of force plate has one centrally instrumented pillar which supports an upper flat plate. Figure 4.10 shows the forces that act on this instrumented support. The action force of the foot F_y acts downward, and the anterior–posterior shear force can act either forward or backward. Consider a reverse shear force F_X, as shown. If we sum the moments acting about the

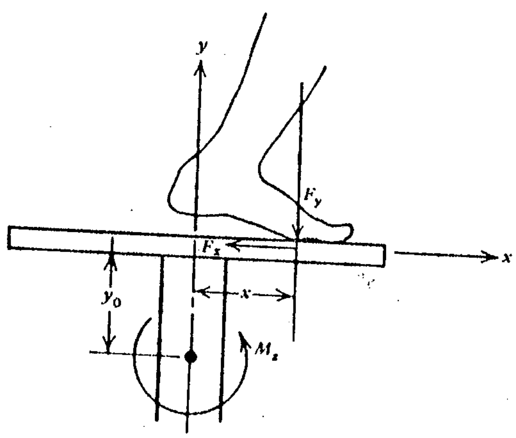


Figure 4.10 Central support type force plate, showing the location of the center of pressure of the foot and the forces and moments involved.

central axis of the support, we get

$$M_Z - F_Y \cdot x + F_X \cdot y_0 = 0$$

$$x = \frac{F_X \cdot y_0 + M_Z}{F_Y} \tag{4.6}$$

where M_Z = bending moment about axis of rotation of support
y_0 = distance from support axis to force plate surface

Since F_X, F_Y, and M_Z continuously change with time, x can be calculated to show how the center of pressure moves across the force plate. Caution is required for both types of force platforms when F_Y is very low (<2% of body weight). During this time a small error in F_Y represents a large percentage error in x and z as calculated by Equations (4.4), (4.5), and (4.6).

Typical force plate data are shown in Figure 4.11 plotted against time for a subject walking at a normal speed. The vertical reaction force F_y is very characteristic in that it shows a rapid rise at heel contact to a value in excess of body weight as full weight bearing takes place (i.e., the body mass is being ac-

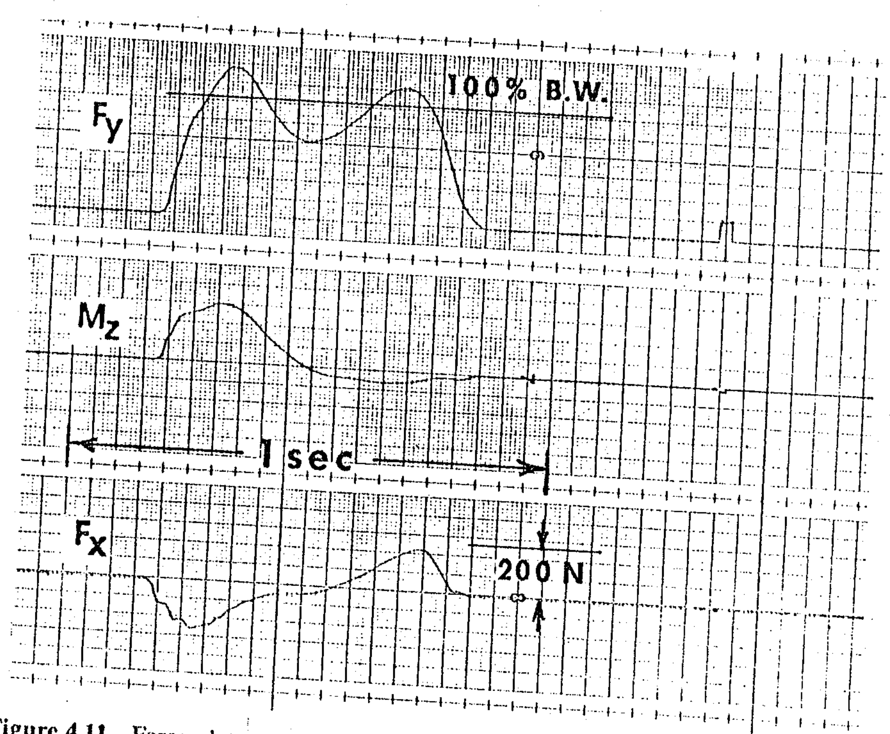


Figure 4.11 Force plate record obtained during gait, using a central support type as shown in Figure 4.10.

Chapparal

Bolt Failure on a Racing Automobile

At a recent luncheon meeting at which he was a guest speaker, Mr. Jim Hall was asked to comment on some bolt failures on his new Chaparral 2E automobiles which had been reported in the press. A number of responsible publications had reported that during some practice runs prior to the Bridgehampton Grand Prix, similar bolt failures had occurred on two of Mr. Hall's cars. According to one report[1], a bolt fell off one of the car's spoilers, into the bodywork, and onto a rear tire causing it to blow, after which the car went off the race course. This article also reported that soon afterwards, the same bolt failure occurred in a second Chaparral and that it went off the track at the same spot as the first.

[1] Competition Press, Vol. 16, No. 40, Oct. 8, 1966 (Exhibit 1)

Prepared in the Design Division of the Department of Mechanical Engineering by William J. Clemens under the direction of Professor Henry O. Fuchs with financial support from the National Science Foundation.

The Chaparral

The Chaparral is a custom racing machine designed particularly for Grand Prix style sports car races. It is revolutionary in that it is the first and only consistently successful competition automobile to make use of an automatic transmission. Perhaps its most noticeable feature is the distinguishing wing (spoiler) mounted on the rear deck of the car. Early models had the "spoiler" mounted on the automobile chassis close to the rear deck. The newer Chaparral 2E, however, has a wing mounted on struts approximately two feet above the highest point on the car. The angle of attack of the wing is controlled by a foot pedal in the driver's compartment and by changing this angle, the driver can put more force on the rear wheels of the car, increasing the car's cornering ability over that of its competitors. The wing is also used to increase aerodynamic drag of vehicle for braking.

For lateral support of the wing assembly, Mr. Hall had provided a stabilizing side link attaching the left wing strut to a rear chassis bulkhead. Because he wanted to avoid disturbing the exhaust duct on that side of the car, Mr. Hall put a bend, as shown in Exhibit 2, in the stabilizing link so that it would miss the duct.

Mr. Hall pointed out to those at the meeting that the failure the press referred to on each of his cars was not a bolt failure, but was a fracture of the rod end where the stabilizing link shown in Exhibits 2a and 2b is joined to one end of a ball joint which, in turn, is joined to the wing strut. What actually happened was that these rod ends broke and the strut fell against the rear wheel causing the reported blowouts. The stabilizing link was connected to the wing strut by a ball and socket joint, one end of the joint being rigidly attached to the link. The joint was screwed into the link and held in place by means of a lock nut. The fracture, which appeared to be a fatigue failure, was located on the threaded shaft of the joint where the lock nut adjoined the stabilizing link. Mr. Hall assumed in the initial design of the strut assembly that the main force acting on the link would not exceed that caused by a lateral acceleration of one "g". The links that broke should have withstood this force.

NEW SPOILER SPOILS CHAPARRAL 2E DEBUT

Bridgehampton, N.Y. Sept. 18 - The spoiler on the new Chaparral 2E more than lived up to its name in this second round of the Can Am series.

In practice, a bolt fell off the device on Phil Hill's car, dropped down into the bodywork and onto a tire, causing it to blow, and the car slewed off the course at turn 11 and Hill trudged back to the pits.

Hall offered the ex-world champ his own car "to get some practice in" and out went Hill again. After five tours of the track a bolt fell off the spoiler on Hall's car (Hill driving), dropped down into the bodywork and onto a tire, causing it to blow. The car slewed off the course at turn 11 and parked right next to the first car.

Then, with hill's the only car capable of proper repair, Hall withdrew his pole-sitting 2E and Hill began a classic chase of eventual winner Dan Gurney.

For 50 laps they were within fractions of a second of each other until the spoiler on the Chaparral stuck in the "brake" position and Hill dropped back to finish fourth.

Exhibit 1. From Competition Press, Vol. 16, No.40 Oct 8, 1966

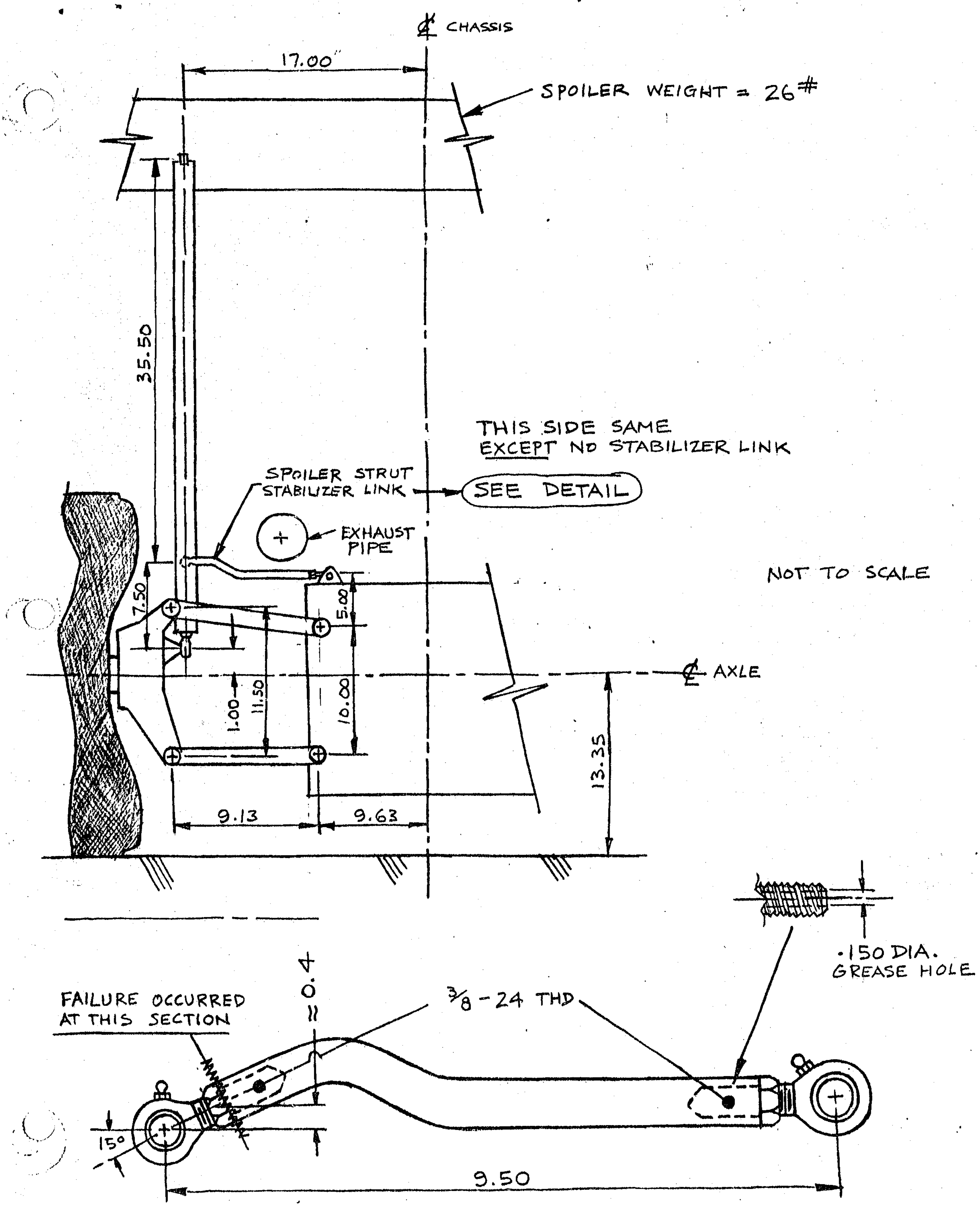
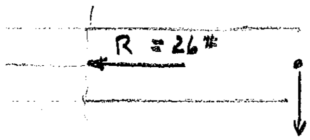


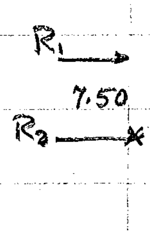
Exhibit 2a – Chaparral wing support schematic

DVN

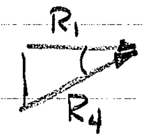
Exhibit 2b. Ball joint of the type that failed on the Chaparral stabilizing link.



35.5D   W = 26# $m = \frac{W}{g} = \frac{26^{\#}}{32.2} = .81$ slugs

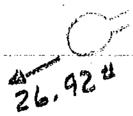


$26^{\#}(43'') - R_1(7.50) \Rightarrow R_1 = \frac{26(43)}{7.50} = 149.07^{\#}$



$R_1 = R_4 \cos\theta =$

$\frac{R_1}{R_4} = \cos(15^\circ) \quad \therefore \quad R_4 = \frac{R_1}{\cos(15^\circ)} = 26.92$ * wrong



*wrong

$A = \pi r_o^2 - \pi r_i^2 = \pi(.375^2 - .15^2)$

$= \pi(.140625 - .0225) = .3711$

$\sigma_{tens} = \frac{149.07}{.3711} = 401.7$ psi

$M = R_1 \times .4 = 59.628$ lb in

$I = \frac{\pi(d_o^4 - d_i^4)}{32} = \frac{\pi}{32}(.375^4 - .15^4) = \frac{\pi}{32}(.019775 - .000506)$

$\frac{\pi}{64}(d_o^4 - d_i^4)$

$d_o = .324$

$= .0018915$

$c = .4$ $\frac{1}{2}$ distance of bolt head. $(\frac{1}{2}d_o)$

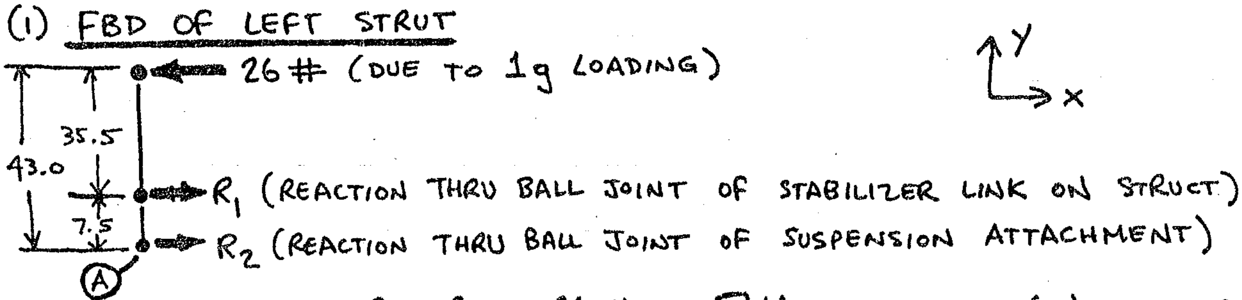
$\therefore \quad \sigma_{Bend} = \frac{Mc}{I} = \frac{(59.628)(.4)}{.0018915} = 12609.675$

$\sigma_{TOT} = \sigma_B + \sigma_T = 13011.375$ psi

ME 111 — STRESS, STRAIN AND STRENGTH — AUTUMN 1978

SOLUTION FOR HW # 1 — CHAPARRAL

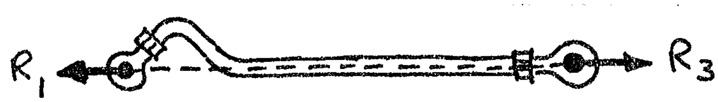
(1) FBD OF LEFT STRUT



(2 PTS.)

$$\Sigma F_x = 0 \Rightarrow R_1 + R_2 = 26\#, \quad \Sigma M_{(A)} = 0 \Rightarrow 26(43) = 7.5 R_1 \Rightarrow \underline{\underline{R_1 \cong 146\#}}$$

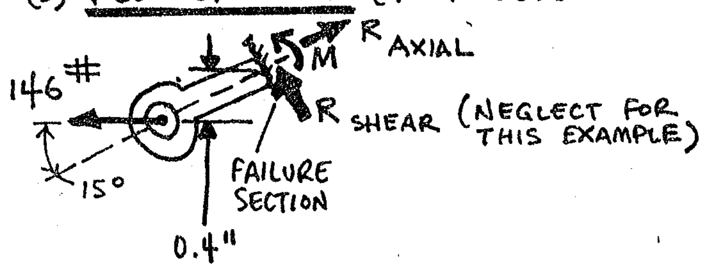
(2) FBD OF STABILIZER LINK



$$\Sigma F_x = 0 \Rightarrow R_1 = R_3 = 146\#$$

(1 PT.)

(3) FBD OF BOLT (TO FAILURE SECTION)



$$\Sigma M_{\text{FAILURE SECTION}} = 0 \Rightarrow M \cong 0.4(146) \cong \underline{\underline{58 \text{ in-}\#}}$$

$$\Sigma F_{\text{ALONG AXIS OF BOLT}} = 0 \Rightarrow R_{AXIAL} \cong 146 \cos 15° \cong \underline{\underline{141\#}}$$

(2 PTS.)

(4) CALCULATE STRESSES

MOMENT OF INERTIA OF BOLT $\equiv I = \frac{\pi}{64}(D_o^4 - D_i^4) \simeq \frac{(3.14)}{64}\left[(.324)^4 - (.150)^4\right]$ in^4

$I \simeq 5.4 \times 10^{-4}$ in^4

CROSS-SECTIONAL AREA OF BOLT $= A = \frac{\pi}{4}(D_o^2 - D_i^2) \simeq (.785)(.082) \simeq 6.44 \times 10^{-2}$ in^2

BENDING STRESS (max.) $\sigma_B = \frac{Mc}{I} = \frac{M(D_o/2)}{I} \simeq \frac{58\left(\frac{.324}{2}\right)}{(5.4 \times 10^{-4})} \simeq 17,500$ psi

AXIAL STRESS (tensile) $\sigma_A = P/A = \frac{1.41 \times 10^2}{6.44 \times 10^{-2}} \simeq 2,200$ psi

(3 PTS.)

MAX. TENSILE STRESS $= \sigma_A + \sigma_B \simeq \underline{19,700 \text{ psi}}$ (CAN ADD STRESSES SINCE IN SAME DIRECTION)

(5) DESIGN MODIFICATION

A GOOD WAY TO REDUCE THE STRESSES IS TO REMOVE THE BENDING MOMENT BY STRAIGHTENING THE LINK. THIS WAS DONE BY JIM HALL. OTHER DESIGN MODIFICATIONS ARE ALSO POSSIBLE, SUCH AS INCREASING THE BOLT SIZE.

(1 PT.)

(6) SOURCE OF 1g LOADING

THE LOADING IS DUE TO CENTRIFUGAL FORCES DEVELOPED DURING CORNERING. THE ESTIMATE OF "1g" WAS PROBABLY OBTAINED BY FIGURING SPEEDS ASSOCIATED WITH TURNS OF A GIVEN RADIUS OF CURVATURE. FOR INSTANCE, IF $V = 100$ MPH (146 FPS) AND $R = 1000'$, THEN $a = V^2/R = 21$ ft/sec^2 OF ACCELERATION (CENTRIFUGAL) OR $\simeq 2/3$ g.

(1 PT.)