CONCEPT OF ELASTIC STRAIN ENERGY

- DEFORMATION OF BODY UNDER EXTERNAL LOADS – LOADS DO WORK
- ASSUME – NO KINETIC OR HEAT EXCHANGE
  - GRADUAL INCREASE IN LOAD FROM INITIAL TO FINAL STATE
  - CONSERVATION OF ENERGY $\Rightarrow$ STRAIN ENERGY IS POTENTIAL ENERGY
- FROM STATICS REMEMBER IF $\bar{F}$ CONSTANT OR NOT

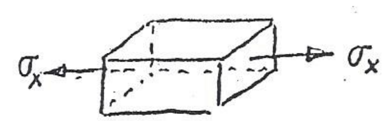
  WORK DONE ON UNIT VOLUME IS $\vec{F} \cdot d\vec{x} = \bar{\sigma} A \cdot d\bar{\epsilon}\, l = \bar{\sigma} \cdot d\bar{\epsilon}\, (Al)$
- TOTAL WORK DONE BY FORCE IS STORED AS STRAIN ENERGY GIVEN BY

$$\int_V \left[\int \bar{\sigma} \cdot d\bar{\epsilon} \right] dV = \int_V \left[\frac{\sigma^2}{2E} \right] dV = \int_V \left[\frac{E}{2} \epsilon^2 \right] dV = \int \frac{\bar{\sigma} \cdot \bar{\epsilon}}{2}\, dV$$

- REMEMBER WORK IS ADDITIVE AND DEPENDS ON FINAL & INITIAL STATES AND NOT ON PATH BETWEEN STATES
- IF WE HAVE A BODY THAT OBEYS HOOKE'S LAW AND WE APPLY A FORCE IN THE X-DIRECTION ONLY

$$\sigma_x = E\epsilon_{x_1}, \quad \epsilon_{y_1} = -\nu\epsilon_{x_1}, \quad \epsilon_{z_1} = -\nu\epsilon_{x_1} \qquad \nu - \text{POISSON RATIO}$$

- SINCE σ_x, ϵ_{x_1} ARE PARALLEL TO EACH OTHER, WORK IS DONE
- SINCE σ_x, ϵ_{y_1} OR ϵ_{z_1} ARE $\perp$ TO EACH OTHER, NO WORK DONE
- WORK DUE TO σ_x IS $\int \frac{\sigma_x \epsilon_{x_1}}{2}\, dV$



- IF WE NOW APPLY TO THIS STATE AN ADDITIONAL FORCE IN THE Y-DIRECTION KEEPING σ_x FIXED

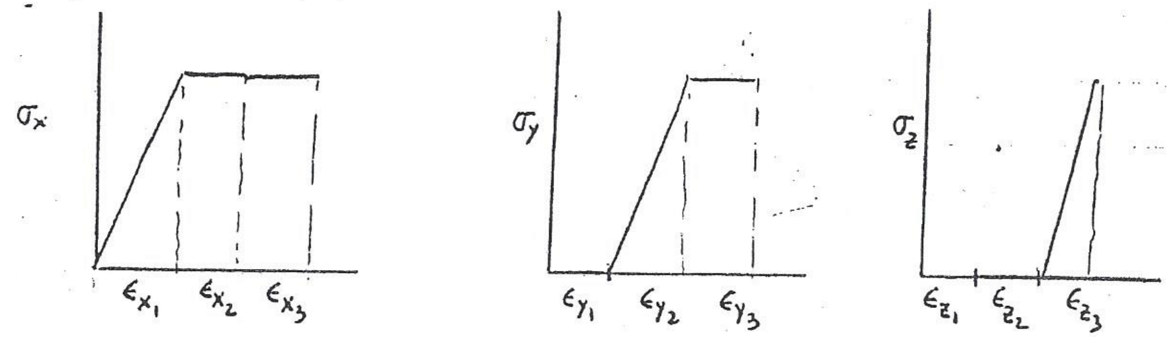
$$\sigma_y = E\epsilon_{y_2} \qquad \epsilon_{x_2} = -\nu\epsilon_{y_2} \qquad \epsilon_{z_2} = -\nu\epsilon_{y_2}$$

- SINCE σ_y, ϵ_{y_2} ARE PARALLEL, WORK IS DONE
- SINCE σ_y, ϵ_{x_2} OR ϵ_{z_2} ARE $\perp$ TO EACH OTHER, NO WORK DONE
- <u>BUT</u>

  σ_x FROM BEFORE ~~DOES~~ DO WORK DUE TO ϵ_{x_2}

• THUS ADDITIVE WORK DUE TO σ_y IS

$$\int_V \left[\frac{\sigma_y \epsilon_{y_2}}{2} + \sigma_x \epsilon_{x_2}\right] dV$$



• SIMILARLY IF WE APPLY A FORCE IN THE Z DIRECTION ONLY KEEPING σ_x, σ_y FIXED : $\sigma_z = E\epsilon_{z_3}$, $\epsilon_{x_3} = -\nu\epsilon_{z_3}$, $\epsilon_{y_3} = -\nu\epsilon_{z_3}$

• A SIMILAR ARGUMENT YIELDS THE ADDITIVE WORK DONE

$$\int_V \left[\frac{\sigma_z \epsilon_{z_3}}{2} + \sigma_x \epsilon_{x_3} + \sigma_y \epsilon_{y_3}\right] dV$$

• BY ADDING THE THREE TERMS WE GET THE TOTAL WORK DONE...

$$\int \left[\frac{\sigma_x \epsilon_{x_1}}{2} + \frac{\sigma_y \epsilon_{y_2}}{2} + \sigma_x \epsilon_{x_2} + \frac{\sigma_z \epsilon_{z_3}}{2} + \sigma_x \epsilon_{x_3} + \sigma_y \epsilon_{y_3}\right] dV$$

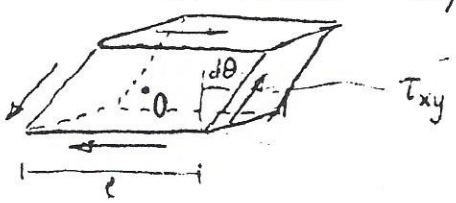
• NOTE THAT $\frac{1}{2}\sigma_x \epsilon_{x_2} = \frac{1}{2}E\epsilon_{x_1} \cdot (-\nu \epsilon_{y_2}) = \frac{1}{2}E\epsilon_{y_2} \cdot (-\nu\epsilon_{x_1}) = \sigma_y \epsilon_{y_1}/2$

• SIMILARLY $\frac{1}{2}\sigma_x \epsilon_{x_3} = \frac{1}{2}\sigma_z \epsilon_{z_1}$ & $\frac{1}{2}\sigma_y \epsilon_{y_3} = \frac{1}{2}\sigma_z \epsilon_{z_2}$

• THUS TOTAL WORK DONE IS $\frac{1}{2}\int \left[\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z\right] dV$

WHERE $\epsilon_x = \epsilon_{x_1} + \epsilon_{x_2} + \epsilon_{x_3}$, $\epsilon_y = \epsilon_{y_1} + \epsilon_{y_2} + \epsilon_{y_3}$ ...

• LOOK AT SHEAR FORCES THEY DO WORK THRU SHEAR STRAINS $d\gamma$



- CAN DO WORK BY FORCE COUPLE CAUSING BODY TO ROTATE ABOUT AN AXIS THROUGH O
- FROM STATICS WORK DONE BY MOMENT is $\bar{M}\cdot d\bar{\theta} = \bar{T}l\cdot d\bar{\theta} = \bar{\tau}Al\cdot d\bar{\theta}$

$$= \bar{\tau}\cdot d\bar{\gamma}\; Al$$

- WORK DONE $= \int_V \left[\int \bar{\tau}\cdot d\bar{\gamma}\right] dV$
- FOR A BODY OBEYING HOOKE'S LAW $\int \bar{\tau}\cdot d\bar{\gamma} = \dfrac{\tau^2}{2G} = \dfrac{G\gamma^2}{2} = \dfrac{\tau\cdot\gamma}{2}$

- BY SIMILAR MANNER WORK DONE BY SHEAR FORCES ARE

$$\int_V \left[\frac{\tau_{xy}\gamma_{xy}}{2} + \frac{\tau_{xz}\gamma_{xz}}{2} + \frac{\tau_{yz}\gamma_{yz}}{2}\right] dV$$

- TOTAL WORK DONE DUE TO ALL STRESSES IS SUM OF THE TWO

$$\int_V \tfrac{1}{2}\left[\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}\right] dV = \int_V U_{S_0}\, dV = U_S$$

- U_{S_0} IS THE STRAIN ENERGY DENSITY ; ALWAYS ≥ 0
- U_{S_0} IS ZERO ONLY WHEN ALL σ's, ϵ's, τ's AND γ's $= 0$

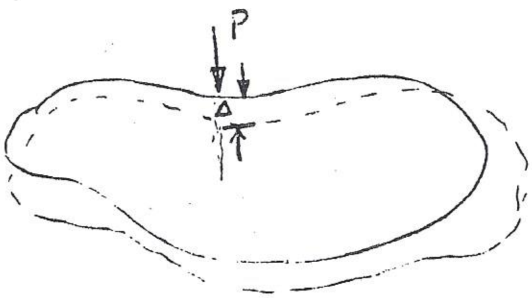
- NOTE THAT SINCE $dU_{S_0} = (\sigma_x d\epsilon_x + \sigma_y d\epsilon_y + \sigma_z d\epsilon_z + \tau_{xy} d\gamma_{xy} + \tau_{xz} d\gamma_{xz} + \tau_{yz} d\gamma_{yz})$
  AND dU_{S_0} is perfect DIFFERENTIAL, MEANING THAT

$$dU_{S_0} = \frac{\partial U_{S_0}}{\partial \epsilon_x} d\epsilon_x + \frac{\partial U_{S_0}}{\partial \epsilon_y} d\epsilon_y + \frac{\partial U_{S_0}}{\partial \epsilon_z} d\epsilon_z + \frac{\partial U_{S_0}}{\partial \gamma_{xy}} d\gamma_{xy} + \frac{\partial U_{S_0}}{\partial \gamma_{xz}} d\gamma_{xz} + \frac{\partial U_{S_0}}{\partial \gamma_{yz}}$$

$$\Rightarrow \frac{\partial U_{S_0}}{\partial \epsilon_x} = \sigma_x \quad \text{AND} \quad \frac{\partial U_{S_0}}{\partial \epsilon_y} = \sigma_y \quad \text{ETC.}$$

- SINCE σ IS RELATED TO A LOAD AND ϵ IS RELATED TO A DISPLACEMENT IN THE DIRECTION OF THAT LOAD $\Rightarrow$ WE CAN DETERMINE THE LOAD IF WE KNOW HOW THE STRAIN ENERGY VARIES WITH THE DISPLACEMENT IN THE DIRECTION OF THAT LOAD

[illegible]



$$U_s = U_s(\Delta) \Rightarrow \frac{\partial U_s}{\partial \Delta} = P$$

THIS FACT WILL BE USED LATER

- WE HAVE CONSIDERED WORK DONE = $\vec{F} \cdot d\vec{x}$

  IF $\vec{F}$ is CONSTANT $\vec{F} \cdot d\vec{x} = d(\vec{F} \cdot \vec{x})$ — MOST GENERAL EXPRESSION FOR WORK

- WHAT IF $\underset{\sim}{x}$ IS NOW CONSTANT AND $\underset{\sim}{F}$ CHANGES ie $d(\vec{F} \cdot \vec{x}) = \vec{x} \cdot d\vec{F}$
  - THIS ALSO CAUSE WORK TO BE DONE

$$d\vec{F} = d\bar{\sigma} \cdot A \qquad \vec{x} = l\bar{\epsilon} \qquad \Rightarrow \quad \vec{x} \cdot d\vec{F} = \vec{\epsilon} \cdot d\bar{\sigma} \quad (lA)$$

- JUST AS BEFORE WE CAN GO THROUGH THE PROCESS AND SHOW THAT

$$\text{TOTAL WORK DONE} = \int_V \int \left(\bar{\epsilon}_x \cdot d\bar{\sigma}_x + \bar{\epsilon}_y \cdot d\bar{\sigma}_y + \bar{\epsilon}_z \cdot d\bar{\sigma}_z + \bar{\gamma}_{xy} \cdot d\bar{\tau}_{xy} + \bar{\gamma}_{xz} \cdot d\bar{\tau}_{xz} + \bar{\gamma}_{yz} \cdot d\bar{\tau}_{yz} \right) dV$$

- THE INNER INTEGRAL IS THE COMPLEMENTARY ENERGY OF THE BODY : U_{co}
  - $dU_{co} = \left(\bar{\epsilon}_x \cdot d\bar{\sigma}_x + \bar{\epsilon}_y \cdot d\bar{\sigma}_y + \bar{\epsilon}_z \cdot d\bar{\sigma}_z + \bar{\gamma}_{xy} \cdot d\bar{\tau}_{xy} + \bar{\gamma}_{xz} \cdot d\bar{\tau}_{xz} + \bar{\gamma}_{yz} \cdot d\bar{\tau}_{yz} \right)$
  - IT IS ALSO A PERFECT DIFFERENTIAL SO THAT

$$dU_{co} = \left(\frac{\partial U_{co}}{\partial \sigma_x} d\sigma_x + \frac{\partial U_{co}}{\partial \sigma_y} d\sigma_y + \frac{\partial U_{co}}{\partial \sigma_z} d\sigma_z + \frac{\partial U_{co}}{\partial \tau_{xy}} d\tau_{xy} + \frac{\partial U_{co}}{\partial \tau_{xz}} d\tau_{xz} + \frac{\partial U_{co}}{\partial \tau_{yz}} d\tau_{yz} \right)$$

AND $\dfrac{\partial U_{co}}{\partial \sigma_x} = \epsilon_x \quad \dfrac{\partial U_{co}}{\partial \sigma_y} = \epsilon_y$ etc. $\dfrac{\partial U_{co}}{\partial \tau_{xy}} = \gamma_{xy}$

- HERE IF WE KNOW HOW THE COMPLEMENTARY ENERGY CHANGES AS LOAD CHANGES THE WE CAN FIND THE DISPLACEMENT, DUE TO THAT LOAD, IN DIRECTION OF LOAD

- REMEMBER σ_x CAN BE RELATED TO LOAD IN X-DIRECTION

  ϵ_x CAN BE RELATED TO DISPLACEMENT IN DIRECTION OF LOAD

- WHAT WE SAID ABOUT U_{S_0} & U_{C_0} IS TRUE EVEN IF BODY DOESN'T OBEY HOOKE'S LAW.
- IF BODY IS LINEALY ELASTIC : $U_{C_0} = U_{S_0}$

- NOTE : ALWAYS WRITE U_S IN TERMS OF STRAINS / DISPLACEMENTS
  U_C IN TERMS OF STRESSES / LOADS

- CLARIFY SOME PTS
  - FROM STATICS: BODY WAS RIGID - NO WORK DONE DUE TO INTERNAL LOADS
  - FOR A BODY THAT DEFORMS WORK IS DONE BY INTERNAL FORCES (U_C, U_S)
  - WHAT WE'VE JUST DISCUSSED IS WORK DONE BY INTERNAL FORCES
  - BOTH FOR NON DEFORMABLE & DEFORMABLE BODIES, THE EXTERNAL FORCES ALSO DO WORK
  - THE EXPRESSIONS FOR $U_C, U_{C_0}, U_S, U_{S_0}$ HOLD FOR NON-LINEARLY ELASTIC BODIES AS WELL
  - TOTAL WORK DONE BY BODY THAT HAS EXTERNAL LOADS APPLIED AND UNDERGOES DEFORMATION IS $W_i + W_e = \Pi$

- $$W_e = \int_S (\bar{X} u + \bar{Y} v + \bar{Z} w)\, dS$$

  S - surface of body
  u, v, w - displacements undergone by forces $\bar{X}, \bar{Y}, \bar{Z}$ applied to surface

- ASSUMPTION - BODY FORCES (LIKE WEIGHT) CAN BE ACCOUNTED FOR THROUGH W_e TERM.
- WHEN AN ELASTIC BODY IS AT REST, THE EXTERNAL FORCES + BODY FORCES + INTERNAL FORCES ARE IN A STATE OF EQUILIBRIUM

- WORK DONE BY THESE THREE SET OF FORCES IS AT A MINIMUM
- THUS $\delta\pi = \delta W_i + \delta W_e = 0$
- ALSO $\delta W_i = -\delta U_s$ REMEMBER FROM STATICS POTENTIAL ENERGY IS NEGATIVE OF WORK DONE.

- THUS FOR ANY CHANGE IN DISPLACEMENTS OF THE BODY THAT KEEPS IT IN EQUILIBRIUM

$$\delta\pi/\delta\,\text{displacements} = 0 = -\frac{\delta U_s}{\delta\,\text{displ.}} + \frac{\delta W_e}{\delta\,\text{displ.}}$$

- BUT $\dfrac{\delta U_s}{\delta\,\text{displ}} = \dfrac{\delta W_e}{\delta\,\text{displ}}$ = load, due to that displ, IN DIRECTION OF DISPLACEMENTS.

- THIS IS CASTIGLIANO'S THEOREM (FIRST)
- SIMILARLY SINCE $\delta W_i = -\delta U_s = -\delta U_c$
- FOR ANY CHANGE IN LOADS OF THE BODY THAT KEEPS IT IN EQUILIBRIUM

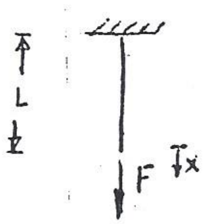
$$\delta\pi/\delta\,\text{LOADS} = 0 = -\frac{\delta U_c}{\delta\,\text{LOAD}} + \frac{\delta W_e}{\delta\,\text{LOAD}}$$

- BUT $\dfrac{\delta U_c}{\delta\,\text{LOAD}} = \dfrac{\delta W_e}{\delta\,\text{LOAD}}$ = DISPL, DUE TO THAT LOAD, IN DIR. OF LOAD

- THIS IS CASTIGLIANO'S THEOREM (SECOND)

→ USING U_c TO DETERMINE DISPLACEMENTS ←

EXAMPLE #1 EXTENSIBLE ROD



$\sigma = F/A$ $\epsilon = \dfrac{\sigma}{E} = \dfrac{F/A}{E}$ ... $= \dfrac{F}{AE} = \dfrac{x}{L}$

$U_{c_0} \cdot \text{Vol}$ $U_{s_0} \cdot \text{Vol}$

$$U_c = \frac{\sigma^2}{2E} \cdot AL = \frac{F^2 L}{2EA} \qquad U_s = \frac{E\epsilon^2}{2} \cdot AL = \frac{Ex^2}{2L} A$$

THUS $U_c = \dfrac{F^2 L}{2EA}$ $U_s = \dfrac{Ex^2}{2L} A$

$W_e = Fx$

TO FIND F, ASSUMING x IS KNOWN, DEFINE π IN TERMS OF U_s & W_e

$$W_i + W_e = \pi = -\frac{Ex^2}{2L}A + Fx = -U_s + W_e$$

$$\frac{\partial \pi}{\partial x} = -\frac{2ExA}{2L} + F = 0 \qquad F = x\frac{EA}{L} \qquad \left(\frac{\partial U_s}{\partial x} = F\right)$$

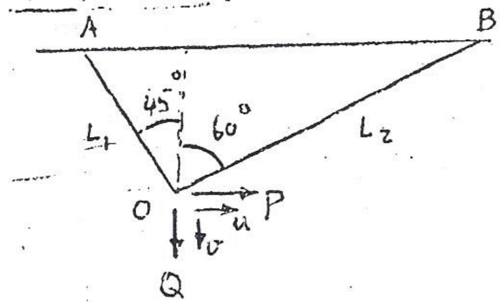
TO FIND x, ASSUMING F IS KNOWN, DEFINE π IN TERMS OF U_c & W_e

$$W_i + W_e = -U_c + W_e = \pi = -\frac{F^2 L}{2EA} + Fx$$

$$\frac{\partial \pi}{\partial F} = -\frac{2FL}{2EA} + x = 0 \qquad x = \frac{FL}{AE} \qquad \left(\frac{\partial U_c}{\partial F} = x\right)$$

- WE WILL LOOK AT TRUSSES WHERE EXTENSION/COMPRESSION IS PRIMARY LOADING

EXAMPLE #2 : TWO-ROD TRUSS

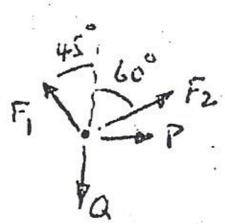


STATICALLY DETERMINATE SYSTEM

LOOK AT TWO BARS CONNECTED AT O, BARS ARE EXTENSIBLE — HAVE THE SAME CROSS SECTION, A, AND YOUNG'S MODULUS, E.

→ WANT TO FIND DISPLACEMENTS u, v GIVEN P AND Q

1) USE STATICS TO FIND FORCES IN OB & OA
2) DETERMINE U_c
3) $\frac{\partial U_c}{\partial P} = u \qquad \frac{\partial U_c}{\partial Q} = v$



$$\left.\begin{aligned} Q &= F_2 \cos 60^\circ + F_1 \cos 45^\circ \\ P &= F_1 \sin 45^\circ - F_2 \sin 60^\circ \end{aligned}\right\} \quad \begin{aligned} F_1 &= \frac{\sqrt{3}-1}{\sqrt{2}}(P + \sqrt{3}Q) \\ F_2 &= (\sqrt{3}-1)(Q-P) \end{aligned}$$

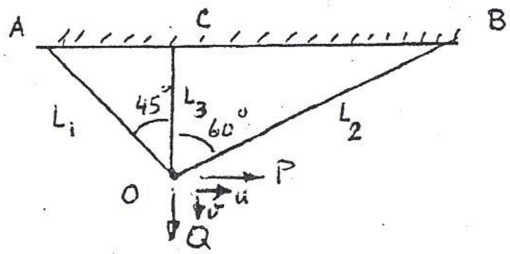
NOTE F_1 & F_2 ARE FNS OF P & Q

$$\text{Now} \quad U_c = \sum \frac{F^2 L}{2AE} = \frac{F_1^2 L_1}{2AE} + \frac{F_2^2 L_2}{2AE} = \frac{1}{2AE}\left\{\left[\frac{\sqrt{3}-1}{\sqrt{2}}(P+\sqrt{3}Q)\right]^2 L_1 + \left[(\sqrt{3}-1)(Q-P)\right]^2 L_2\right\}$$

$$u = \frac{\partial U_c}{\partial P} = \frac{2F_1 L_1}{2AE}\frac{\partial F_1}{\partial P} + \frac{2F_2 L_2}{2AE}\frac{\partial F_2}{\partial P} = \frac{\sqrt{3}-1}{\sqrt{2}}(P+\sqrt{3}Q)\frac{L_1}{AE}\cdot\frac{\sqrt{3}-1}{\sqrt{2}} + (\sqrt{3}-1)(Q-P)\frac{L_2}{AE}\left\{-(\sqrt{3}-1)\right\}$$

$$v = \frac{\partial U_c}{\partial Q} = \frac{2F_1L_1}{2AE}\frac{\partial F_1}{\partial Q} + \frac{2F_2L_2}{2AE}\frac{\partial F_2}{\partial Q} = \frac{L_1}{AE}\left\{\frac{\sqrt{3}-1}{\sqrt{2}}(P+\sqrt{3}Q)\cdot\frac{\sqrt{3}}{\sqrt{2}}(\sqrt{3}-1)\right\} + \frac{L_2}{AE}\left\{(\sqrt{3}-1)^2(Q-P)\right\}$$

EXAMPLE #3 - INDETERMINATE (STATICALLY) TRUSS



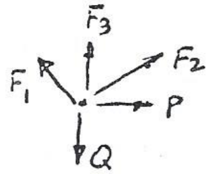
$L_3 = L \qquad L_1 = \sqrt{2}\,L \qquad L_2 = 2L$

GIVEN: A, E SAME FOR ALL THREE

WANT TO FIND DISPLACEMENTS u, v

GIVEN P & Q

STATICALLY INDETERMINATE: 3 FORCES, 2 EQS.



$$\left.\begin{aligned} Q &= F_3 + F_1\cos 45^\circ + F_2\cos 60^\circ \\ P &= F_1\sin 45^\circ - F_2\sin 60^\circ \end{aligned}\right\} \text{NOTE } P \,\&\, Q \text{ ARE FNS OF } F_1, F_2, F_3$$

• NOTE THIS WILL GIVE SAME SOLUTION FOR F_1 & F_2 IF $Q - F_3$ REPLACES Q

TO FIND F_3: 1) FIND U_c FIRST $\quad U_c = \sum \frac{F^2L}{2AE} = \frac{F_1^2L_1}{2AE} + \frac{F_2^2L_2}{2AE} + \frac{F_3^2L_3}{2AE}$

2) TAKE $\frac{\partial U_c}{\partial F_3} = 0$ THIS GIVES 3rd EQ. NEEDED

$$U_c = \frac{1}{2AE}\left\{\left[\frac{\sqrt{3}-1}{\sqrt{2}}\left(P+\sqrt{3}(Q-F_3)\right)\right]^2 L_1^2 + \left[(\sqrt{3}-1)(Q-F_3-P)\right]^2 L_2^2 + F_3^2L_3^2\right\}$$

$$\frac{\partial U_c}{\partial F_3} = \frac{F_1L_1}{AE}\frac{\partial F_1}{\partial F_3} + \frac{F_2L_2}{AE}\frac{\partial F_2}{\partial F_3} + \frac{F_3L_3}{AE} = \frac{\sqrt{3}-1}{\sqrt{2}}\left[P+\sqrt{3}(Q-F_3)\right]\frac{L_1}{AE}\left[-\sqrt{3}\frac{(\sqrt{3}-1)}{\sqrt{2}}\right] -$$

$$+ (\sqrt{3}-1)(Q-F_3-P)\frac{L_2}{AE}\left[-(\sqrt{3}-1)\right] + \frac{F_3L_3}{AE} = 0$$

3) SOLVE FOR F_3 IN TERMS OF KNOWN FORCES P & Q

$$F_3 = -0.01295\,P + 0.6883\,Q$$

4) PUT THIS INTO $F_1 = \frac{\sqrt{3}-1}{\sqrt{2}}\left(P+\sqrt{3}(Q-F_3)\right)$; $F_2 = (\sqrt{3}-1)(Q-F_3-P)$

TO FIND F_1 & F_2 IN TERMS OF P & Q

5) TAKE $\frac{\partial U_c}{\partial P}$ TO GET u & $\frac{\partial U_c}{\partial Q}$ TO GET v