EGM 5615 Synthesis of Engineering Mechanics

Review of elementary formulae

Here we critically re-examine some basic formulae in connection with underlying assumptions.

1. Simple tension of a bar of cross section area A, length L under load P. E is Young's modulus.

Stress, $\sigma = \frac{P}{A}$. Deflection, $\Delta = \frac{PL}{AE}$.

2. Twisting of a bar by torque T. r = radial coordinate, J = polar moment of inertia, L = length, G = shear modulus.

Stress $\tau = \frac{Tr}{J}$, twist angle, $\theta = \frac{TL}{JG}$.

3. Beam bending. Moment M is applied to beam of area moment I. y is distance, perpendicular to the long axis of the beam, from neutral axis. x is coordinate along beam. v = displacement.

Stress $\sigma = \frac{My}{I}$, curvature, $\frac{d^2v}{dx^2} = \frac{M}{EI}$.

Geometrical assumptions

Axial load in tension must be centered, otherwise there is superposed bending.

In torsion, cross section must be circular. There can be a hole, but it must be on center. Plane sections perpendicular to the rod axis were assumed to rotate but remain plane. For non-circular sections, there is warp of cross sections.

In bending, moment vector must be along a principal axis of inertia. For the deflection equation to be valid, deflections must be small enough that the second derivative is a good approximation to the curvature.

In bending, depth of cross section should exceed width. Otherwise the beam tends to act as a plate, and structural stiffness is perhaps 10% greater for many structural materials. Why? The Poisson effect is restrained in cylindrical plate bending, while it is free to occur in beam bending.

Material assumptions

Demonstration: stretch or bend viscoelastic putty. Observe time dependent behavior.

We have assumed *elastic* material behavior, specifically linearly elastic.

Demonstration: bend copper wire. It stays bent. Deflection depends not only on applied moment, but also exhibits a threshold effect and hysteresis.

We have assumed *elastic* material behavior, specifically linearly elastic.

Demonstration: Bend off-axis honeycomb, observe twist.

We have assumed *isotropic* elastic behavior.

Demonstration: Bend stack of paper, observe slip between sheets. Stack is much easier to bend than to stretch in comparison with a block of wood of similar thickness.

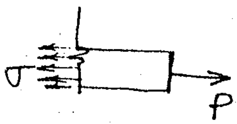
We have assumed *homogeneity*.

All materials are in fact heterogeneous, if only due to their atomic structure. Real materials such as steel or aluminum, have larger scale heterogeneities such as dislocations, grain boundaries, and inclusions. Often, we can get away with an assumption of homogeneity if the heterogeneities are much smaller than any size scale of interest in the deformation field.

ADVANCED MECHANICS OF MATERIALS — PROF. LEVY

1.1 (a)

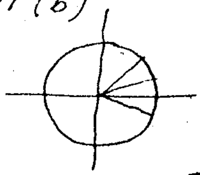
1. Plane Section remain Plain and translate with respect to one another in the load direction
2. Axial strain ϵ constant over cross section and lengthwise.
3. Uniaxial stress $\sigma = E\epsilon$ - constitutive relation



4. $P - \sigma A = 0 \;;\; \sigma = \frac{P}{A}$

$$\frac{P}{A} = E\,\frac{\Delta}{L} \;;\quad \Delta = \frac{PL}{AE}$$

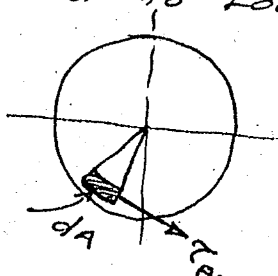
1.1 (b)



1. Establish geometry of deformation
   - Plane section remain Plane and radial straight lines remain straight.
2. Strain distribution.
   - Shear strain proportional to radial Coordinate

$$\gamma_{\theta z} = K\rho \quad (K = \text{constant})$$

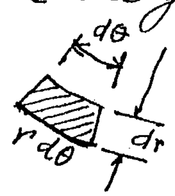
3. Constitutive relation.
   - $\tau_{\theta z} = G\gamma_{\theta z} = GK\rho$
4. Stress to Load relation and Free Body Diagram.



$$dA = (\rho\, d\theta)\, dr$$

$$T = \int \rho\,(\tau_{\theta z}\, dA)$$

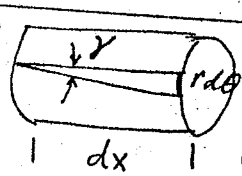
$$= GK\int_0^{2\pi}\!\int_0^R \rho^3\, dr\, d\theta = \frac{GK\pi}{2}R^4 = GKJ$$



So $GK = \frac{T}{J}$; applying to step 3

$$\tau_{\theta z} = \frac{T\rho}{J} \qquad \gamma_{\theta z} = \frac{\tau_{\theta z}}{G} = \frac{Tr}{JG} = Kr \quad K = \frac{T}{JG}$$

1 (c)

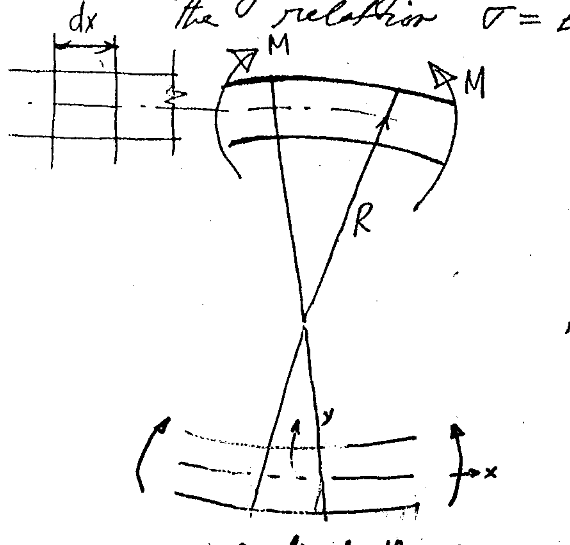


$$\frac{r\,d\theta}{dx} = \gamma = \frac{\tau}{G}$$

$$\theta = \int^L \frac{d\theta}{dx}dx = \int_0^L \frac{\gamma\, dx}{\rho} = \int_0^L \frac{\tau}{\rho G}\, dx = \int_0^L \frac{Tr}{\rho G J}\, dx$$

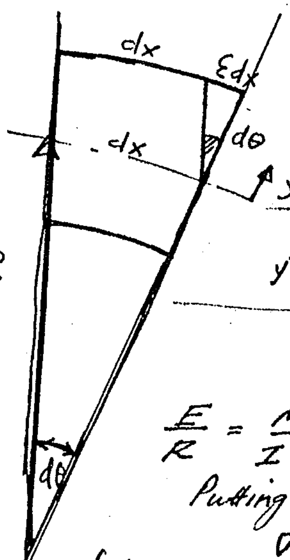
for Constant values $\theta = \frac{TL}{GJ}$

1) Bar must be straight and of constant cross section and have a plane of symmetry that contains the axis. M is the moment vector perpendicular to the plane of symmetry. The material is homogeneous and displays the relation $\sigma = E\epsilon$



$\epsilon = \frac{l_{bot} - l}{l} = \frac{(l + y d\theta) - l}{l} = \frac{-y d\theta}{R d\theta}$

$\sigma = E\epsilon = -Ey/R$

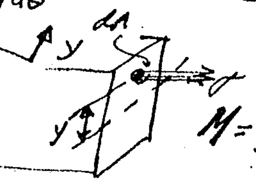


$-M = \int \sigma dA \cdot y = 0$

$y\,d\theta = \epsilon\,dx = \frac{dx}{R}$

$\epsilon = \frac{y}{R}$

$\sigma = E\epsilon = \frac{Ey}{R}$ ..... (A)



$M = \int_A (\sigma dA) y$

$= \int_A \frac{E}{R} y^2 dA = \frac{E}{R} \int_A y^2 dA = \frac{EI}{R}$

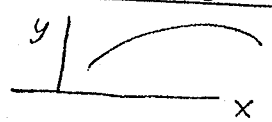
$\frac{E}{R} = \frac{M}{I}$ .. (B)

Putting (B) in (A)

$\sigma = \frac{My}{I}$

$M = +\frac{E}{R} I$

$-\frac{1}{R} = \frac{-\left(\frac{d^2y}{dx^2}\right)}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{3/2}}$



for any curve $y = f(x)$ from calculus

Small deflection (slope $= \frac{dy}{dx}$) assume $\left|\frac{dy}{dx}\right| \ll 1$

$\frac{1}{R} = -\frac{d^2y}{dx^2}$ ... (c)

Between (B) in 1.1(d) and (c)

$\frac{E}{R} = \frac{M}{I} \Rightarrow \frac{1}{R} = \frac{M}{EI}$

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

$T = \int (\tau dA) r = \int_0^{2\pi} \int_0^a \tau (r dr d\theta) r$

$= 2\pi \int_0^a \tau r^2 dr = 2\pi c \int_0^a \gamma\, r^2 dr$

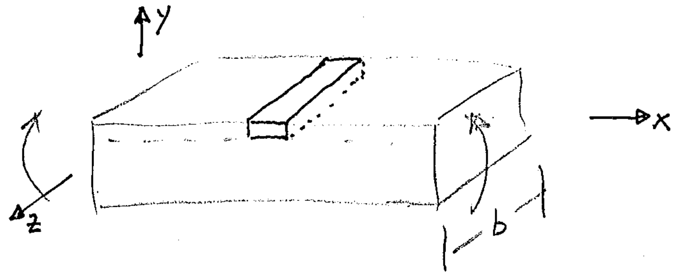
$= 2\pi cK \int_0^a r^{1/2} r^2 dr$ (assuming $\gamma = K r^{1/2}$)

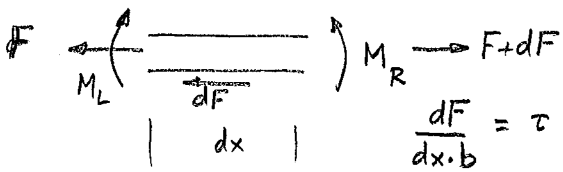
$T = 2\pi cK \frac{2}{7} a^{7/2}$

But $\tau = c\gamma = cK r^{1/2}$

$\therefore cK = \frac{\tau}{r^{1/2}}$

$\tau = \frac{7 T r^{1/2}}{4\pi a^{7/2}}$





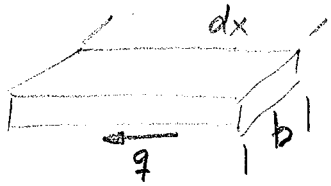
$$\frac{dF}{dx \cdot b} = \tau$$

$$F = \int \sigma_L dA = \int -\frac{M_L y}{I_z} dy \cdot b = -\frac{M_L}{I_z}\int y\,dA = -\frac{M_L Q}{I_z}$$

$$F+dF = \int \sigma_R dA = \int -\frac{M_R}{I_z} y\,dA = -\frac{M_R Q}{I_z} \qquad M_R = M_L + dM$$

$$dF = (F+dF) - F = -(M_R - M_L)\frac{Q}{I} = -dM\frac{Q}{I}$$

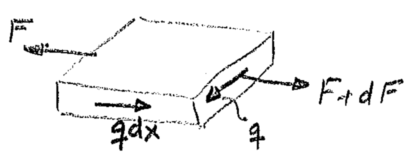
$$dM = -V dx \Rightarrow \frac{VQ}{I} dx = dF \text{ or } \frac{dF}{dx} = \frac{VQ}{I} = \text{shear flow } q$$



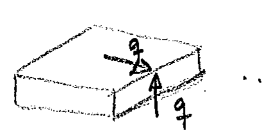
but $q = \tau b$ for thin sections = const

$\therefore \quad \tau = \frac{VQ}{Ib}$

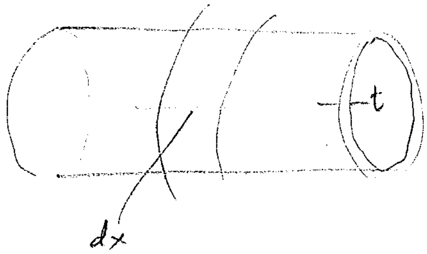
problem is that we assume that τ is constant throughout thickness

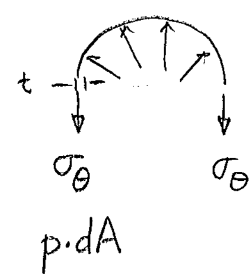


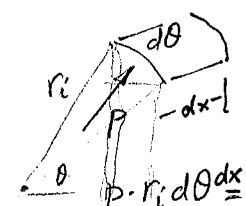
also you can look at



LESSON #2



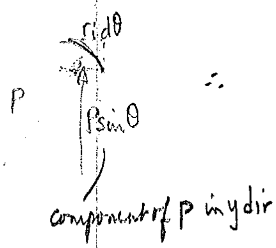




$p \cdot r_i d\theta\, dx = \underset{\sim}{dF}$

$dF \sin\theta\, dx = dF_y$

$dF \cos\theta\, dx = dF_x$



$$\therefore \quad F_y = \int_0^{\pi} dF \sin\theta = \int_0^{\pi} p r_i d\theta\, dx \sin\theta = -p r_i\, dx \cos\theta \Big|_0^{\pi} = 2 p r_i\, dx$$

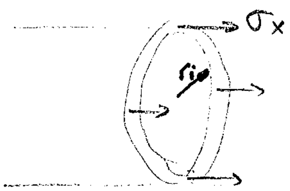
$$F_y - 2\sigma_\theta \cdot dx \cdot t = 0 \qquad \sigma_\theta = \frac{F_y}{2dxt} = \frac{2pr_i dx}{2dx \cdot t} = \frac{pr_i}{t}$$

cylindrical

assume σ_θ = const along thickness

$$F_x = \int_0^{\pi} dF \cos\theta = \int_0^{\pi} p r_i d\theta\, dx \cos\theta = p r_i\, dx \sin\theta \Big|_0^{\pi} = 0$$

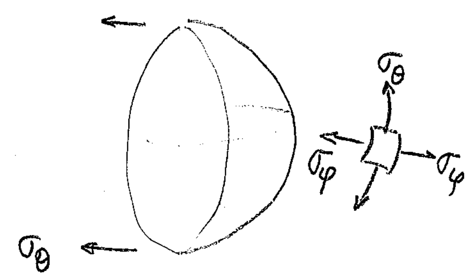
on proj of area on which it acts



$$\sigma_x \cdot 2\pi r_i t = p \cdot \pi r_i^2$$

$$\sigma_x = \frac{pr_i}{2t}$$

GOOD ONLY IF STRAINS ASSUMED CONST. THROUGH THICKNESS



$$\sigma_\theta \cdot 2\pi r_i t = p \cdot \pi r_i^2$$

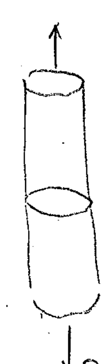
$$\sigma_\theta = \frac{pr_i}{2t}$$

$$\sigma_\varphi = \frac{pr_i}{2t}$$ by symmetry

Axial loading in one dimension

Assume plane cross section remain plane when deformed

$$\epsilon = \frac{\sigma}{E}$$ only if not heated or cooled

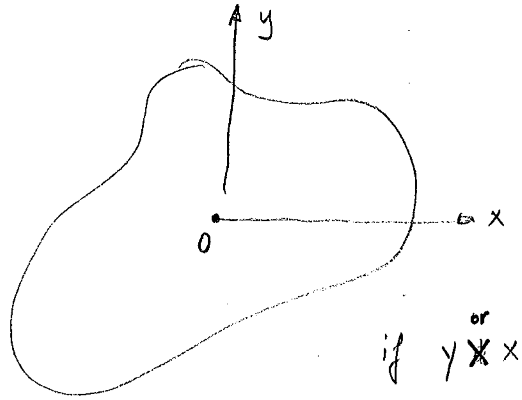


if yes then $\epsilon = \frac{\sigma}{E} + \alpha \Delta T$

$$u = \int \epsilon\, dx = \int \left(\frac{\sigma}{E} + \alpha \Delta T \right) dx = \int \left(\frac{P}{AE} + \alpha \Delta T \right) dx$$

LESSON # 3

Properties of plane areas



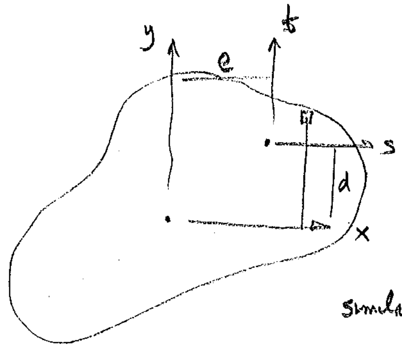
$$I_{yy} = \int x^2 dA$$

$$I_{xx} = \int y^2 dA$$

$$I_{xy} = \int yx\, dA$$

if y or x are axes of symmetry then $I_{yx} = 0$

Parallel axes theorem



$$I_{xx} = \int y^2 dA = \int (t+d)^2 dA$$

$$= \int (t^2 + 2td + d^2)\, dA$$

$$= I_{ss} + 2d \int t\, dA + \int d^2 dA$$

$$I_{xx} = I_{ss} + d^2 A$$

(second term = 0 if centroidal)

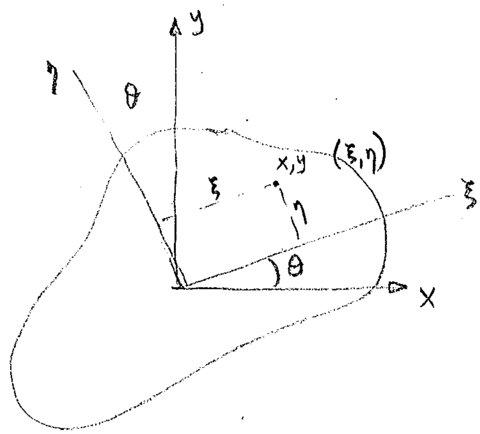
similarly $\quad I_{yy} = I_{tt} + e^2 A$

$$I_{xy} = \int yx\, dA = \int (t+d)(s+e)\, dA$$

$$= \int ts\, dA + d\int s\, dA + e\int t\, dA + de\int dA$$

(middle two terms = 0 if centroidal)

$$I_{xy} = I_{ts} + deA$$



$$\xi = x\cos\theta + y\sin\theta$$

$$\eta = y\cos\theta - x\sin\theta$$

$$I_{\eta\eta} = \int \xi^2 dA = \int (y^2\sin^2\theta + 2xy\sin\theta\cos\theta + x^2\cos^2\theta)\, dA$$

$$= I_{xx}\sin^2\theta + I_{xy}\sin 2\theta + I_{yy}\cos^2\theta$$

$$= I_{xx}\left(\frac{1-\cos 2\theta}{2}\right) + I_{xy}\sin 2\theta + I_{yy}\left(\frac{1+\cos 2\theta}{2}\right)$$

$$I_{\eta\eta} = \frac{I_{xx}+I_{yy}}{2} + \frac{1}{2}(I_{yy}-I_{xx})\cos 2\theta + I_{xy}\sin 2\theta$$

replace θ by $\theta - 90$ $\quad I_{\xi\xi} = \frac{I_{xx}+I_{yy}}{2} - \frac{1}{2}(I_{yy}-I_{xx})\cos 2\theta - I_{xy}\sin 2\theta$

$$I_{\eta\eta} + I_{\xi\xi} = I_{xx} + I_{yy}$$

$$I_{\xi\eta} = \int \xi\eta\, dA = I_{xy}\cos^2\theta + \left(\frac{I_{xx}-I_{yy}}{2}\right)\sin 2\theta - I_{xy}\sin^2\theta$$

$$= I_{xy}\cos 2\theta + \quad ''$$

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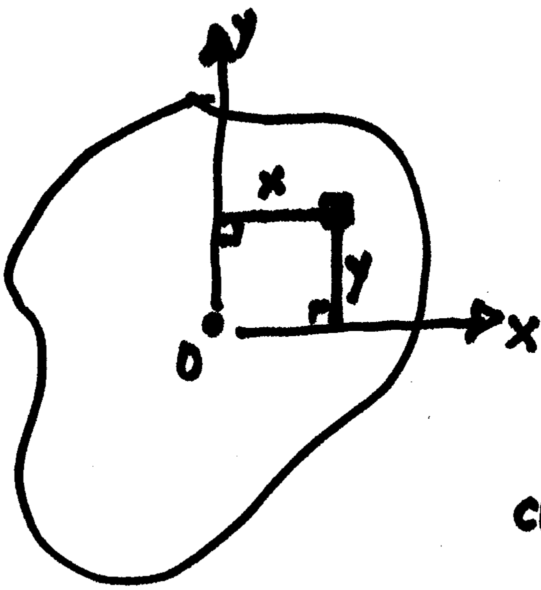


$$I_{yy} = \int_A x^2 dA$$

$$I_{xx} = \int_A y^2 dA$$

cross-mom of Inertia $I_{yx} = \int_A xy\,dA$

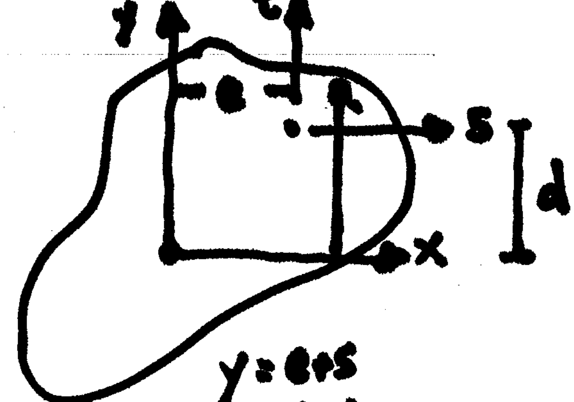
if x or y are axes of symmetry $I_{xy} = 0$

I_{xx}, I_{yy} are principal moments of inertia

x, y are principal axes

$$I_0 \ddot{\theta} = \sum T_0$$

Parallel Axis theorem



y = e+s
x = d+t

s, t are princ. axes $I_{st} = 0$

I_{ss}, I_{tt} known

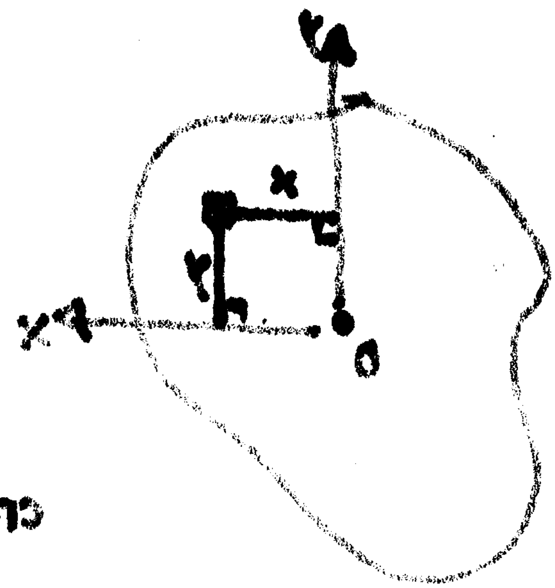
$$I_{xx} \stackrel{?}{=} \int_A y^2 dA = \int_A [s+d]^2 dA$$

$$= \int_A (s^2 + 2sd + d^2) dA$$

$$= \int_A s^2 dA + \int_A 2sd \cdot dA + \int_A d^2 \cdot dA$$

$$I_{yy} = \int_A x^2 dA$$

$$I_{xx} = \int_A y^2 dA$$



cross-mom of Inertia $I_{yx} = \int_A xy\,dA$

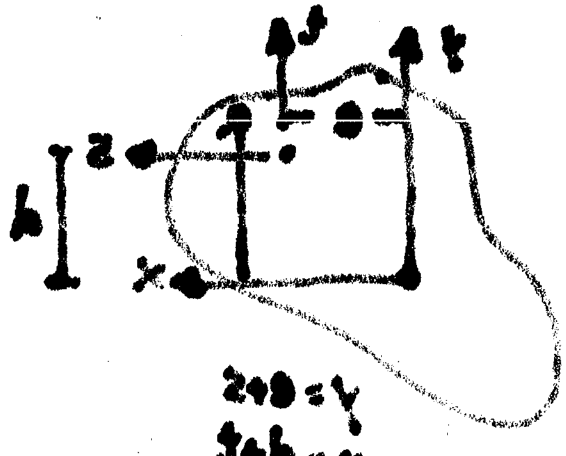
if x or y are axes of symmetry $I_{xy} = 0$

I_{xx}, I_{yy} are principal moments of inertia

x,y are principal axes

$\Sigma T_o = I_o \ddot{\theta}$

Parallel Axis theorem



s,t are princ. axes $I_{st} = 0$

I_{ss}, I_{tt} known

$$I_{xx} = \int_A y^2 dA = \int_A [s+d]^2 dA$$

$$= \int_A (s^2 + 2sd + d^2) dA$$

$$= \int_A s^2 dA + \int_A 2sd\,dA + \int_A d^2 dA$$

$$I_{xx} = I_{tt} + 2d\int_A s\,dA + d^2A \quad \text{(crossed out, } = 0\text{)}$$

$$\bar{s} = \frac{\int s\,dA}{\int dA}$$

$$\underline{I_{xx} = I_{tt} + d^2A}$$

$$I_{tt} = \int_A s^2\,dA$$

$$\underline{I_{yy} = \int x^2\,dA = \int [e+t]^2\,dA = I_{ss} + e^2A}$$

$$I_{ss} = \int t^2\,dA$$

$$I_{st} = \int_A ts\,dA$$

$$I_{xy} = \int (s+d)(e+t)\,dA$$

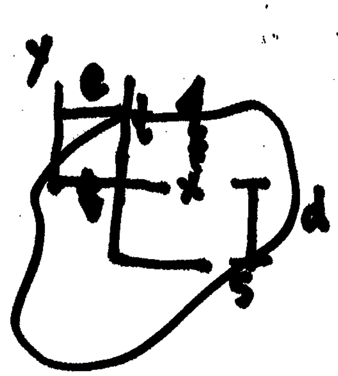
$$= \int [s\cdot e + s\cdot t + d\cdot e + d\cdot t]\,dA$$

$$I_{xy} = e\int s\,dA + \int st\,dA + d\cdot e\int dA + d\int t\,dA$$

IF s & t are princ axes

$$\bar{t} = \frac{\int t\,dA}{\int dA}$$

$$\underline{I_{xy} = d\cdot e\cdot A}$$



$\dfrac{\dot{A}bz\}}{Ab\}} = \bar{z}$ $\qquad A^s b + \dot{A}bz\rfloor bS + {}_{yy}I = {}_{xx}I$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 0 \quad A$

$Ab\dot{z}\}_A = {}_{yy}I \qquad \underline{A^s h + {}_{yy}I = {}_{xx}I}$

$Ab^s f[= {}_{zz}I \qquad \underline{A^s g + {}_{zz}I = Ab^s[f+g]\} = Ab^s x\} = {}_{yy}I}$

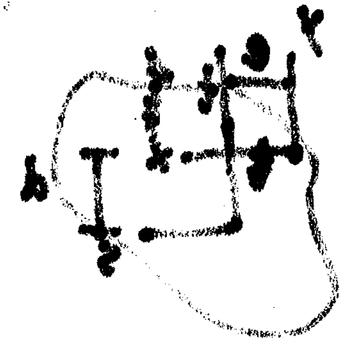
$Ab(f+g)(h+z)\} = {}_{yx}I \qquad Abzf\}_A = {}_{fz}I$

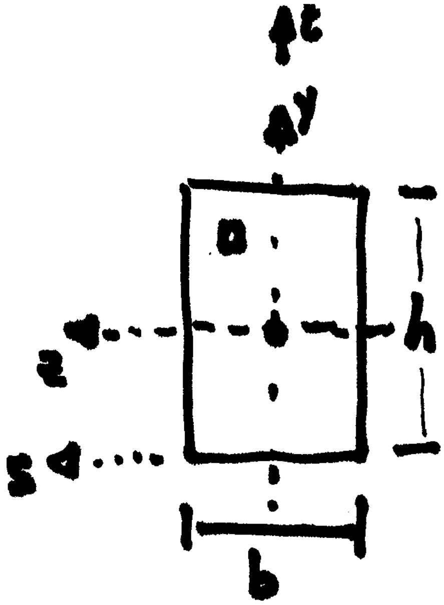
$Ab[f\cdot h + g\cdot h + f\cdot z + gz]\} =$

$\dot{A}bf\}h + Ab\}g\cdot h + \dot{A}bfz\} + \dot{A}bz\}g = {}_{yx}I$

$\dfrac{Abf\}}{Ab\}} = \bar{f}$ $\quad$ ssnd sniηq nsa ʇ bz ₮,

$\underline{A\cdot g\cdot h = {}_{yx}I}$





$$I_{zz} = \int y^2 \, dA = \int_{-h/2}^{h/2}\int_{-b/2}^{b/2} y^2 \, dz \cdot dy$$

$$= \frac{y^3}{3}\Big|_{-h/2}^{h/2} \; z\Big|_{-b/2}^{b/2} = \frac{2}{3}\left(\frac{h}{2}\right)^3 \cdot b$$

$$I_{zz} = \frac{bh^3}{12}$$

$$I_{zy} = \int y \cdot z \, dA = \int_{-h/2}^{h/2}\int_{-b/2}^{b/2} y \cdot z \cdot dz \cdot dy$$

$$= \frac{y^2}{2}\Big|_{-h/2}^{h/2} \; \frac{z^2}{2}\Big|_{-b/2}^{b/2} = 0$$

$d = \frac{h}{2}$

$$I_{ss} = I_{zz} + d^2 A$$

$$= \frac{bh^3}{12} + \left(\frac{h}{2}\right)^2 \cdot bh$$

$$I_{ss} = \frac{bh^3}{3}$$

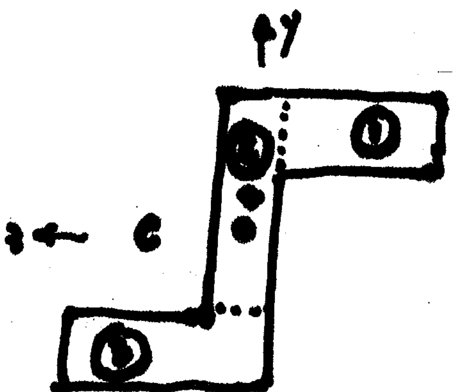
	$\bar{z}$	$\bar{y}$	A	$\bar{z}A$	$\bar{y}A$
①			·	·	·
②			·	·	·
③			—	—	—

$\bar{y} = \frac{\Sigma \bar{y}A}{\Sigma A}$

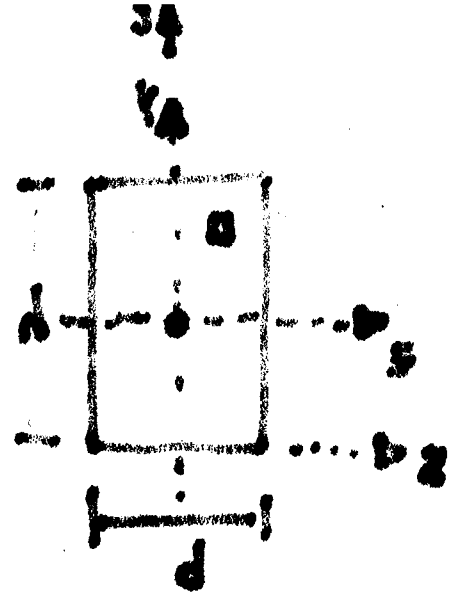
$\bar{z} = \frac{\Sigma \bar{z}A}{\Sigma A}$

- FIND CENTROID OF COMPOSITE BODY C
- FIND THE MOM OF INERTIA OF ① ABOUT C
- " " " " " OF ② ABOUT C
- " " " " " OF ③ " C

MOM OF INERTIA OF COMPOSITE = Σ MOM OF INERTIA



I_{zz}, I_{yy}, I_{yz}



$I_{zz} = \int y^2\, dA = \int_{-h/2}^{h/2}\int_{-b/2}^{b/2} y^2\, dz \cdot dy$

$= \frac{y^3}{3}\Big|_{-h/2}^{h/2} \cdot z\Big|_{-b/2}^{b/2} = \frac{2}{3}\left(\frac{h}{2}\right)^3 \cdot b$

$I_{zz} = \frac{bh^3}{12}$

$I_{yz} = \int y \cdot z\, dA = \int_{-h/2}^{h/2}\int_{-b/2}^{b/2} y \cdot z \cdot dz \cdot dy$

$= \frac{y^2}{2}\Big|_{-h/2}^{h/2} \frac{z^2}{2}\Big|_{-b/2}^{b/2} = 0$

$I_{\bar z} = I_{zz} + d^2 A \qquad d = \frac{h}{2}$

$= \frac{bh^3}{12} + \left(\frac{h}{2}\right)^2 \cdot bh$

$I_{\bar z} = \frac{bh^3}{3}$

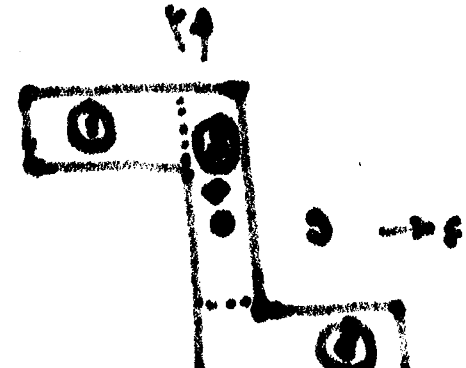
$\bar y = \frac{\Sigma A\bar y}{\Sigma A}$

$\bar z = \frac{\Sigma A\bar z}{\Sigma A}$

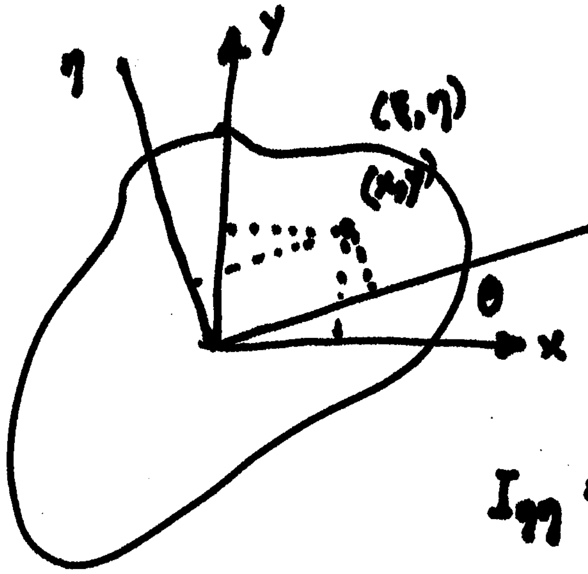
	$\bar z$	$\bar y$	A	$A\bar z$	$A\bar y$
①					
②					
③					

- FIND CENTROID OF COMPOSITE BODY C
- FIND THE MOM OF INERTIA OF ① ABOUT C
- " " " " " OF ③ ABOUT C
- " " " " " OF ② " C

MOM OF INERTIA OF COMPOSITE = Σ MOM OF INERTIA



I_{zz}, I_{yy}, I_{yz}



I_{xx}, I_{yy}, I_{xy} are given

$I_{\xi\xi}, I_{\eta\eta}, I_{\xi\eta}$?

$\xi = x\cos\theta + y\sin\theta$

$\eta = y\cos\theta - x\sin\theta$

$$I_{\eta\eta} = \int \xi^2 dA = \int (x\cos\theta + y\sin\theta)^2 dA$$

$$= \int (x^2\cos^2\theta + 2xy\cos\theta\sin\theta + y^2\sin^2\theta) dA$$

$$I_{\eta\eta} = \cos^2\theta\, I_{yy} + \sin 2\theta\, I_{xy} + \sin^2\theta\, I_{xx}$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$1 = \sin^2\theta + \cos^2\theta$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

- $$I_{\eta\eta} = \frac{I_{xx}+I_{yy}}{2} + \frac{1}{2}(I_{yy}-I_{xx})\cos 2\theta + I_{xy}\sin 2\theta$$

- $$I_{\xi\xi} = \int \eta^2 dA = \int (y\cos\theta - x\sin\theta)^2 dA$$
$$= \frac{I_{xx}+I_{yy}}{2} - \frac{1}{2}(I_{yy}-I_{xx})\cos 2\theta - I_{xy}\sin 2\theta$$

$I_{xx} + I_{yy} = I_{\xi\xi} + I_{\eta\eta}$

FIRST INVARIANT OF MOM. OF INERTIA

- $$I_{\xi\eta} = \int \eta \cdot \xi\, dA = I_{xy}\cos 2\theta + \frac{I_{xx}-I_{yy}}{2}\sin 2\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

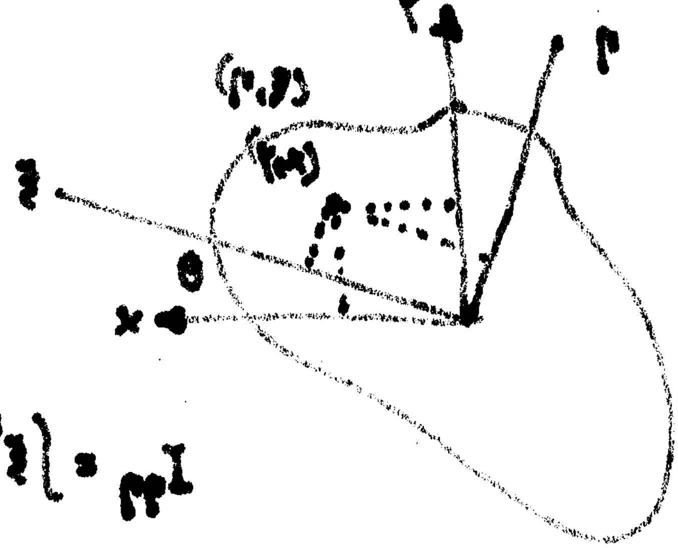
ξ, η are principal if $I_{\xi\eta} = 0$ $\quad \tan 2\theta_p = \dfrac{I_{xy}}{\frac{I_{yy}-I_{xx}}{2}}$

$\theta_p,\ \theta_p + 90$

S	A
T	C

I_{xx}, I_{yy}, I_{xy} are given

$I_{\xi\xi}, I_{\eta\eta}, I_{\xi\eta}$?



$\xi = x\cos\theta + y\sin\theta$

$\eta = y\cos\theta - x\sin\theta$

$$I_{\eta\eta} = \int \xi^2 dA = \int (x\cos\theta + y\sin\theta)^2 dA$$

$$= \int(x^2\cos^2\theta + 2xy\sin\theta\cos\theta + y^2\sin^2\theta)\,dA$$

$$I_{\eta\eta} = \cos^2\theta\, I_{yy} + \sin 2\theta\, I_{xy} + \sin^2\theta\, I_{xx}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$I_{\eta\eta} = \frac{I_{xx}+I_{yy}}{2} + \frac{1}{2}(I_{yy}-I_{xx})\cos 2\theta + I_{xy}\sin 2\theta \quad \bullet$$

$$I_{\xi\xi} = \int \eta^2 dA = \int (y\cos\theta - x\sin\theta)^2 dA \quad \bullet$$

$$= \frac{I_{xx}+I_{yy}}{2} - \frac{1}{2}(I_{yy}-I_{xx})\cos 2\theta - I_{xy}\sin 2\theta$$

$I_{xx}+I_{yy} = I_{\xi\xi}+I_{\eta\eta}$

FIRST INVARIANT OF MOM. OF INERTIA

$$I_{\xi\eta} = \int \xi\cdot\eta\, dA = I_{xy}\cos 2\theta + \frac{I_{xx}-I_{yy}}{2}\sin 2\theta \quad \bullet$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

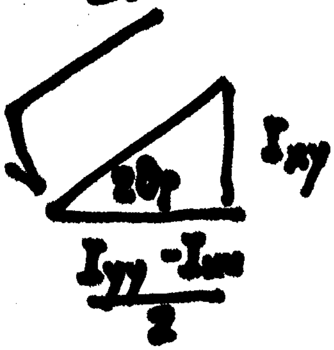
ξ, η are principal if $I_{\xi\eta}=0$ $\quad \tan 2\theta_p = \dfrac{I_{xy}}{\frac{I_{yy}-I_{xx}}{2}}$

$\theta_p,\ \theta_p + 90$

S	A
T	C

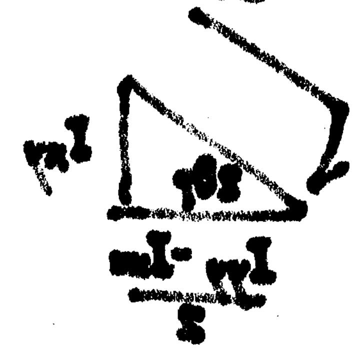
$$I_{max,min} = \frac{I_{xx}+I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}$$

$$\frac{dI_{\eta\eta}}{d\theta} = 0 \Rightarrow I_{\xi\eta} = 0$$



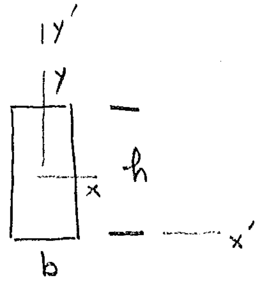
$$I_{max,\,min} = \frac{I_{xx}+I_{yy}}{2} \pm \sqrt{\left(\frac{I_{xx}-I_{yy}}{2}\right)^2 + I_{xy}^2}$$

$$\frac{\partial I_{xx}}{\partial \theta} = 0 \Rightarrow I_{xy} = 0$$



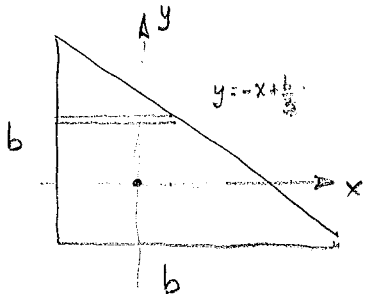
$\sigma = \frac{My}{I}$ if I is max σ is lowered

I is min σ is increased



$$I_{xx} = \frac{bh^3}{12} \qquad I_{yy} = \frac{hb^3}{12} \qquad I_{xy} = 0$$

$$I_{x'x'} = \frac{bh^3}{12} + \left(\frac{h}{2}\right)^2 \cdot hb = \frac{bh^3}{3}$$



$$I_{xx} = I_{yy} = \frac{b^4}{36} \qquad I_{xy} = -\frac{b^4}{72}$$

~~$I_{xx} \int_{-b/3}^{2b/3} \left(-x + \frac{b}{3}\right)^2 dx\,dy$~~

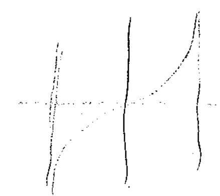
since $I_{\eta\eta}$, $I_{\xi\xi}$, $I_{\xi\eta}$ are fns of θ

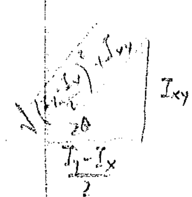
define principal moments of inertia so that $\frac{dI_{\xi\xi}}{d\theta} = 0 \iff I_{\xi\eta} = 0$

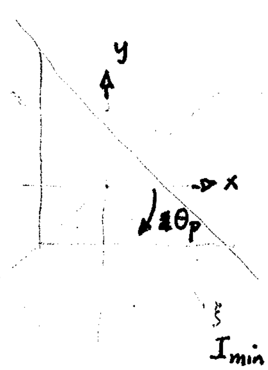
$$\therefore \frac{dI_{\xi\xi}}{d\theta} = 0 - \tfrac{1}{2}(I_{yy} - I_{xx})(-\sin 2\theta)\cdot 2 + I_{xy}\cos 2\theta \cdot 2 = 0$$

$$\therefore 0 = I_{xy}\cos 2\theta + \frac{I_x - I_y}{2}\sin 2\theta \quad \text{or} \quad \tan 2\theta_p = \frac{2I_{xy}}{I_y - I_x}$$

Principal axes here are at $\tan 2\theta_p = \dfrac{2\left(-\frac{b^4}{72}\right)}{\frac{b^4}{36} - \frac{b^4}{36}} = -\infty$







$$2\theta_p = -90^\circ \qquad \theta = -45^\circ$$

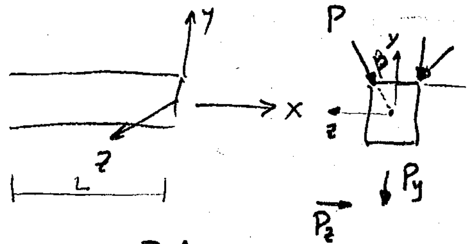
$$I_{\xi\xi} = \frac{I_x + I_y}{2} - \frac{1}{2}(I_y - I_x)\frac{\left(\frac{I_y - I_x}{2}\right)}{\sqrt{\quad}} - \frac{I_{xy}\cdot I_{xy}}{\sqrt{\quad}}$$

$$I_{min} = \frac{I_x + I_y}{2} - \sqrt{\quad} = \frac{b^4}{72}$$

$$I_{\eta\eta} = \frac{I_x + I_y}{2} + \frac{1}{2}(I_y - I_x)\frac{\left(\frac{I_y - I_x}{2}\right)}{\sqrt{\quad}} + I_{xy}\cdot\frac{I_{xy}}{\sqrt{\quad}}$$

$$I_{max} = \frac{I_x + I_y}{2} + \sqrt{\quad} = \frac{b^4}{24}$$

$I_{\xi\eta} = 0$ since line $\eta\eta$ = axis of symmetry

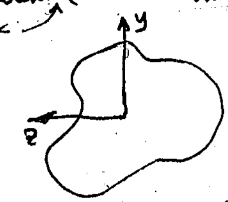


$$\sigma_x = -\frac{P_y \cdot l\, y}{I_{zz}} = -\frac{M_z y}{I_{zz}} \qquad P_y = P\cos\beta \quad P_z = P\sin\beta$$

$$\sigma_x = \frac{P_z l z}{I_{yy}} = \frac{M_y z}{I_{yy}} \qquad M_z = -P_y l \quad M_y = P_z l$$

$$\sigma_x = -\frac{M_z y}{I_{zz}} + \frac{M_y z}{I_{yy}}$$ if y & z are prin

what if they aren't (but are centroidal)
prinic



$$\sigma_x = \frac{-\left(M_z I_y + M_y I_{yz}\right)y + \left(M_y I_z + M_z I_{yz}\right)z}{I_y I_z - I_{yz}^2}$$

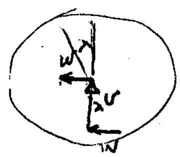
where is neutral axis (where $\sigma_x = 0$)

Peery

beam deflection in unsymmetric bending

$$\tan\lambda = \frac{z}{y} = \left(\frac{M_y I_z + M_z I_{yz}}{M_z I_y + M_y I_{yz}}\right)$$

orientation of neutral axis



$$\frac{w}{v} = \tan\lambda$$

$$\Delta = \sqrt{w^2 + v^2} = \sec\lambda \cdot v$$

$$\Delta = \frac{v}{\cos\lambda}$$

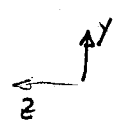
but if $I_{yz} = 0$ $\quad \frac{d^2\Delta}{dx^2} = \frac{1}{\cos\lambda}\frac{d^2 v}{dx^2} = \frac{M}{EI\cos\lambda}$

$$\frac{d^2\Delta}{dx^2} = \frac{M_y}{E\left(I_y - I_{yz}\tan\lambda\right)}$$

$$q = \frac{\left(I_y Q_z - I_{yz} Q_y\right)V_y + \left(Q_y I_z - I_{yz} Q_z\right)V_z}{I_y I_z - I_{yz}^2}$$

$$Q_z = \int y\,dA = \bar{y} A_s$$

$$Q_y = \int z\,dA = \bar{z} A_s$$



$$V_z I_z Q_y - V_y I_{yz} Q_y + V_y I_y Q_z - V_z I_{yz} Q_z$$

$$q = \frac{V_y\left(I_y Q_z - I_{yz} Q_y\right) + V_z\left(I_z Q_y - Q_z I_{yz}\right)}{I_y I_z - I_{yz}^2}$$

LESSON # 4

$$\epsilon_{x}' = l_1^2 \epsilon_x + m_1^2 \epsilon_y + n_1^2 \epsilon_z + l_1 m_1 \gamma_{xy} + m_1 n_1 \gamma_{yz} + n_1 l_1 \gamma_{zx}$$

$\vdots \quad = \; \ldots$

$\epsilon_{z}' = \; \ldots$

$$\gamma_{x'y'} = -2\, l_1 m_1 (\epsilon_x - \epsilon_y) + (l_1 m_2 + m_1 l_2)\, \gamma_{xy}$$

$\vdots \quad = \; \ldots$

$\gamma_{z'x'} = \; \ldots$

ϵ_1, ϵ_2 2-D max, min $\gamma_{12} = 0$

$\epsilon_1, \epsilon_2, \epsilon_3$ 3-D max, min, other $\gamma_{12} = \gamma_{23} = \gamma_{31} = 0$

9-24.1

2-D:

$\epsilon_x + \epsilon_y = I_{1\epsilon}$ invariant

$\epsilon_x \epsilon_y - \frac{1}{4} \gamma_{xy}^2 = I_{2\epsilon}$ invariant

3-D:

$\epsilon_x + \epsilon_y + \epsilon_z = I_{1\epsilon}$ invariant

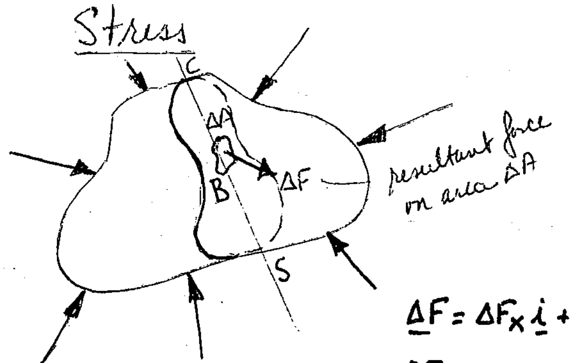
$\epsilon_x \epsilon_y + \epsilon_y \epsilon_z + \epsilon_z \epsilon_x - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) = I_{2\epsilon}$

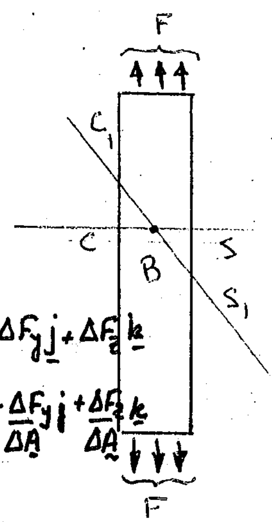
$\epsilon_x \epsilon_y \epsilon_z + \frac{1}{4}(\gamma_{xy}\gamma_{yz}\gamma_{zx}) - \frac{1}{4}(\epsilon_x \gamma_{yz}^2 + \epsilon_y \gamma_{zx}^2 + \epsilon_z \gamma_{xy}^2) = I_{3\epsilon}$

1.139

Stress

FIG 1   9-24



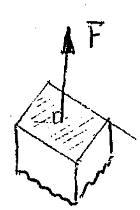


$\underline{\Delta F} = \Delta F_x \underline{i} + \Delta F_y \underline{j} + \Delta F_z \underline{k}$

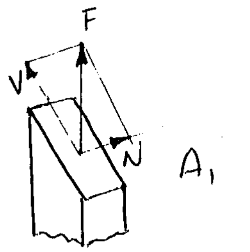
$\underset{\sim}{t} = \frac{\Delta \underline{F}}{\Delta \underline{A}} = \frac{\Delta F_x}{\Delta \underline{A}} \underline{i} + \frac{\Delta F_y}{\Delta \underline{A}} \underline{j} + \frac{\Delta F_z}{\Delta \underline{A}} \underline{k}$

$\Delta \underline{A} = \Delta A \underline{n}$ $\underline{n} \perp \Delta A$

if ΔA in y-z plane, $\underline{n} = \underline{i}$



$\frac{F}{A}$ = Intensity of internal forces in the axial direction.



F/A_1 = Intensity of load in axial direction

$F/A_1 < F/A$

$\sigma = \frac{N}{A_1}$ normal stress

σ & τ are uniformly distributed $\tau = \frac{V}{A_1}$ shear stress

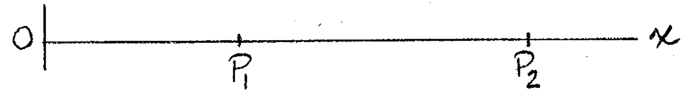
stress at $B = \left(\frac{\Delta F}{\Delta A}\right)_{\Delta A \to 0}$ we define at vector $\bar{n} \perp$ to the plane

↓ need 2 vectors

those quantities needing 2 vectors to define it are <u>tensors</u> ∴ stress is a <u>tensor</u>.

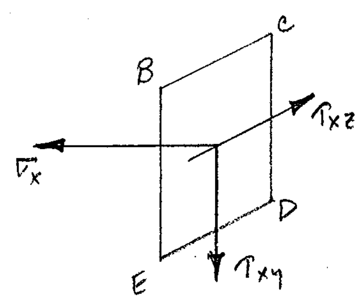
9-29-70

FIG 3 9-29



Positive face of the parallelopiped is defined as the one with the larger coordinate.

FIG. 2a 9-29



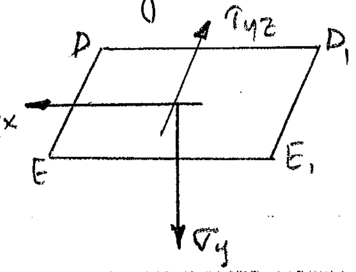
σ_x is defined positive

τ_{xy} " " "

$\underline{t}_x = \sigma_x \underline{i} + \sigma_{xy} \underline{j} + \sigma_{xz} \underline{k}$ $\Delta \underline{A} = \Delta A \underline{i}$

Sign convention:

FIG. 2b 9-29



σ → tensile stress +

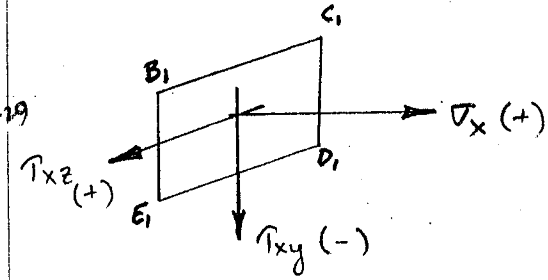
→ compressive " −

τ → pos. direct for positive face : +

→ neg. " " neg " : +

$\underline{t}_y = \sigma_y \underline{j} + \tau_{yx} \underline{i} + \tau_{yz} \underline{k}$ $\Delta \underline{A} = \Delta A \underline{j}$

FIG. 2c
9-29



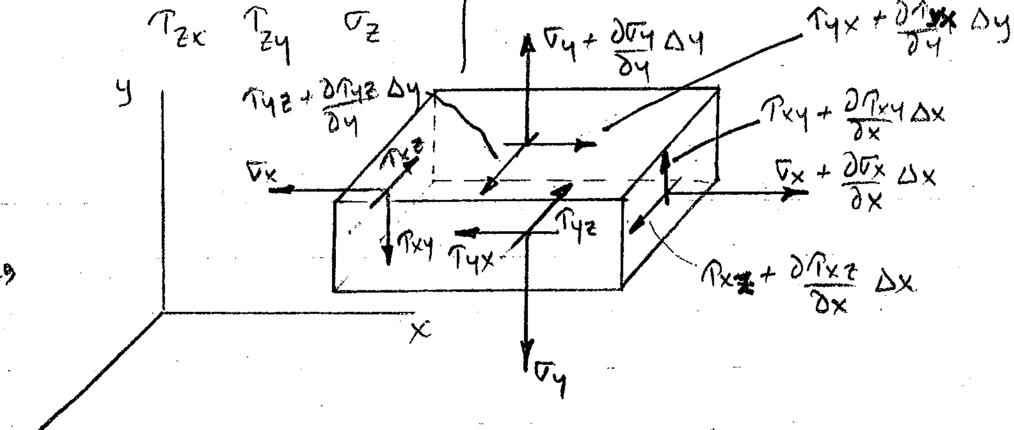
$$\underline{t}_x = \sigma_x \underline{i} - \tau_{xy} \underline{j} + \tau_{xz} \underline{k}$$

$$\Delta \underline{A} = \Delta A \underline{i}$$

3 planes exist at a pt with 3 stress components each
⇒ 9 stress component all together at a point.

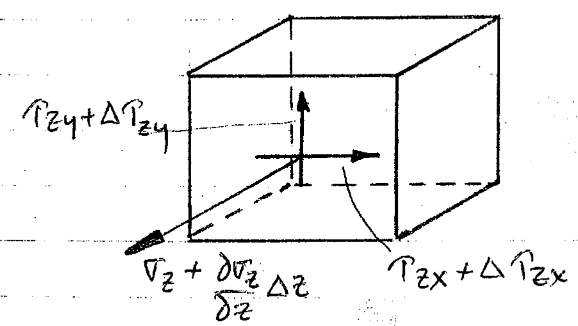
$$\left.\begin{matrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{matrix}\right\} \text{Stress tensor}$$

FIG. 1
9-29



Stresses are continuous functions at $P(x,y,z)$
they change from pt to pt.

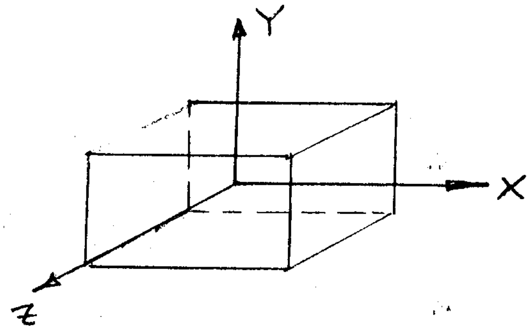
FIG. 4a
9-29



$$\Delta \tau_{zy} = \frac{\partial \tau_{zy}}{\partial z} \Delta z$$

$$\Delta \tau_{zx} = \frac{\partial \tau_{zx}}{\partial z} \Delta z$$

FIG 4b 9-29



Body Forces

$$\sum F_x = \left(\sigma_x + \frac{\partial \sigma_x}{\partial x}\Delta x\right)\Delta y \Delta z - \sigma_x \Delta y \Delta z + \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right)\Delta$$

$$- \tau_{yx}\Delta x \Delta z + \left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\Delta z\right)\Delta x \Delta y - \tau_{zx}\Delta x \Delta y$$

$$+ X \Delta x \Delta y \Delta z = 0$$

$$\left[\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X\right]\Delta x \Delta y \Delta z = 0$$

9-29.1

$$\left.\begin{aligned}
&\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X = 0 \\
&\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + Y = 0 \\
&\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + Z = 0
\end{aligned}\right\} \text{Equilibrium}$$

look at moment about z-axis

$$\sum M_{cz} = \left(\tau_{xy} + \frac{\partial \tau_{xy}}{\partial x}\Delta x\right)\Delta y \Delta z \frac{\Delta x}{2} + \tau_{xy}\Delta y \Delta z \frac{\Delta x}{2}$$

$$- \left(\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y}\Delta y\right)\Delta x \Delta z \frac{\Delta y}{2} - \tau_{yx}\Delta x \Delta z \frac{\Delta y}{2} = 0$$

$$\cong (\tau_{xy} - \tau_{yx})\Delta x \Delta y \Delta z = 0 \longrightarrow \boxed{\tau_{xy} = \tau_{yx}}$$

9-29.2

$$\left.\begin{aligned}
\tau_{xy} &= \tau_{yx} \\
\tau_{yz} &= \tau_{zy} \\
\tau_{zx} &= \tau_{xz}
\end{aligned}\right\} \text{from moment eq.}$$

Complementary Shear St

6 unknowns, 3 equations makes proper problem statically *indeterminate*.

10-5-70

FIG. 1 10-5

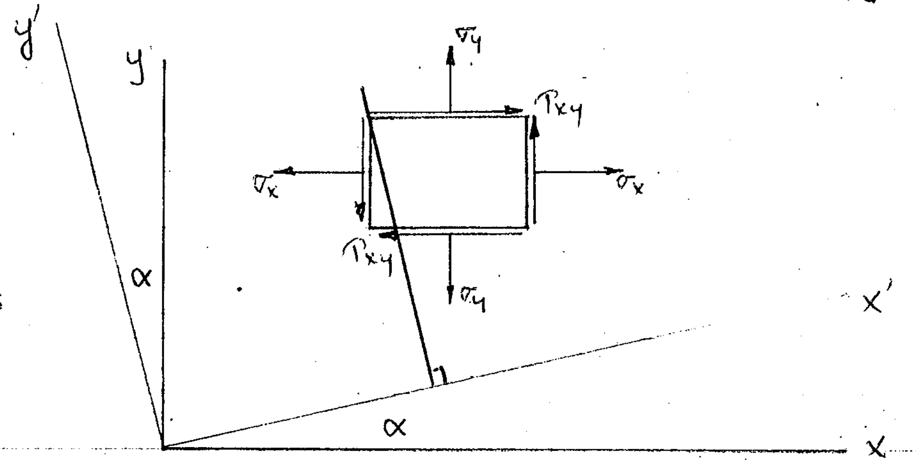
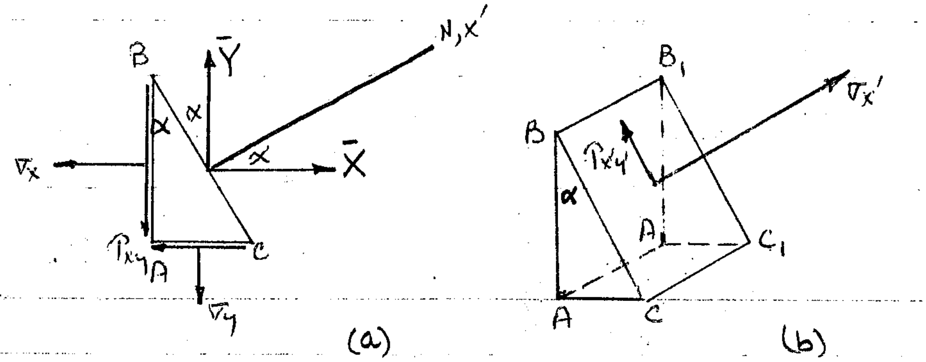


FIG. 2 10-5



$\overline{AB} = \overline{BC}\cos\alpha$

$\overline{AC} = \overline{BC}\sin\alpha$

$ABB_1A_1A = A_x = A_n\cos\alpha$

$ACC_1A_1A = A_y = A_n\sin\alpha$

$BB_1C_1CB = A_n$

If $A_n = 1$ then

$$\bar{X} = \sigma_x A_x + \tau_{xy} A_y = \sigma_x A_n\cos\alpha + \tau_{xy} A_n\sin\alpha = (\sigma_x\cos\alpha + \tau_{xy}\sin\alpha)A_n$$

$$\bar{Y} = \tau_{xy} A_x + \sigma_y A_y = \tau_{xy} A_n \cos\alpha + \sigma_y A_n \sin\alpha = (\tau_{xy}\cos\alpha + \sigma_y \sin\alpha) A_n$$

$$F_n = A_n \sigma_x' = (\bar{X}\cos\alpha + \bar{Y}\sin\alpha)\ ; \left(\text{thus } \sigma_x' = \frac{\bar{X}\cos\alpha + \bar{Y}\sin\alpha}{A_n}\right)$$
$$= \sigma_x \cos^2\alpha + 2\tau_{xy}\sin\alpha\cos\alpha + \sigma_y \sin^2\alpha$$

10-5,1
$$\boxed{\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\alpha + \tau_{xy}\sin 2\alpha}$$

$$A_n \tau_{x'y'} = \bar{Y}\cos\alpha - \bar{X}\sin\alpha\ ; \left(\text{thus } \tau_{x'y'} = \frac{\bar{Y}\cos\alpha - \bar{X}\sin\alpha}{A_n}\right)$$
$$= \tau_{xy}\cos^2\alpha + (\sigma_y - \sigma_x)\sin\alpha\cos\alpha - \tau_{xy}\sin^2\alpha$$
$$= \tau_{xy}(\cos^2\alpha - \sin^2\alpha) - \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\alpha$$

10-5,2
$$\boxed{\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\alpha + \tau_{xy}\cos 2\alpha}$$

for σ_y' use $\alpha = \alpha + 90^\circ$ and get
$$\sigma_y' = \frac{\sigma_x + \sigma_y}{2} - \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\alpha + \tau_{xy}\sin 2\alpha\right]$$

	x	y
x'	l	m
y'	$l_2 = -m$	$m_2 = l$

$l_2 = \cos(90+\alpha) = -\sin\alpha = -m$
$m_2 = \sin(90+\alpha) = \cos\alpha = l$

$\cos\alpha = l$ $\quad \bar{X} = (l\sigma_x + m\tau_{xy})A_n$
$\sin\alpha = m$ $\quad \bar{Y} = (l\tau_{xy} + m\sigma_x)A_n$

$l^2 + m^2 = 1$
$$\sigma_x' = (l\bar{X} + m\bar{Y})/A_n$$
$$= l^2\sigma_x + 2lm\tau_{xy} + m^2\sigma_y$$

$$\frac{\partial \sigma_x'}{\partial l} = 2\left[l\sigma_x + m\tau_{xy}\right] + 2\left[l\tau_{xy} + m\sigma_y\right]\frac{\partial m}{\partial l} = 0$$

if $l^2 + m^2 = 1$ $\quad 2l\,dl + 2m\,dm = 0$ $\quad \frac{dm}{dl} = -l/m$

$$\underset{①}{\frac{l\sigma_x + m\tau_{xy}}{l}} = \underset{②}{\frac{l\tau_{xy} + m\sigma_y}{m}} = \underset{③}{\sigma}$$

$$\bar{X}/lA_n = \bar{Y}/mA_n = \sigma = \frac{\underset{\sim}{t_n}}{A_n}$$

this is force $\frac{\bar{X}}{\text{area} \perp \text{to } \bar{X}} = \frac{\text{force } \bar{Y}}{\text{area} \perp \text{to } \bar{Y}}$

what if $\underset{\sim}{t_n}$ is a traction on principal plane
$\therefore t_n = \sigma$

LESSON # 5

from ① & ③ $(\sigma_x - \sigma) l + m \tau_{xy} = 0$ either $l = m = 0$ impossible or det $= 0$

from ② & ③ $(\sigma_y - \sigma) m + l \tau_{xy} = 0$

therefore,

$$\begin{vmatrix} \sigma_x - \sigma & \tau_{xy} \\ \tau_{xy} & \sigma_y - \sigma \end{vmatrix} = 0$$

or, $\sigma^2 - (\sigma_x + \sigma_y)\sigma + \left[(\sigma_x \sigma_y) - \tau_{xy}^2\right] = 0$ $\quad \sigma^2 - I_1 \sigma + I_2 = 0$

10-5,3

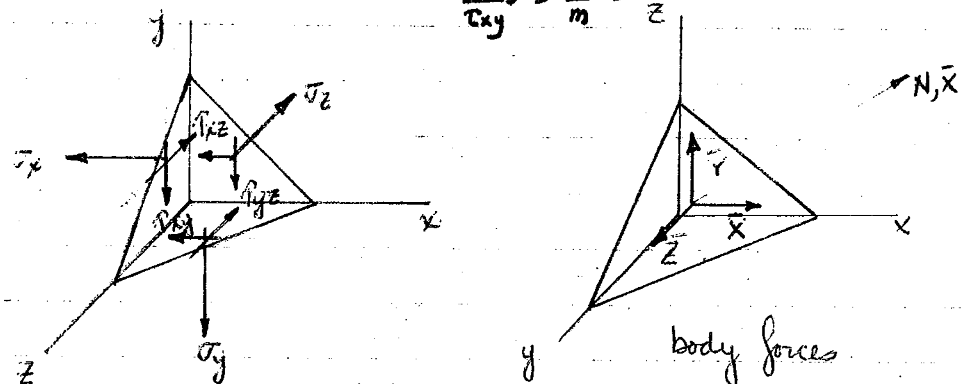
$$\sigma = \frac{\sigma_x + \sigma_y}{2} \pm \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$$

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

$$\frac{-\sigma_x + \sigma}{\tau_{xy}} = \frac{m}{l} = \frac{\sin\theta}{\cos\theta} = \tan\theta$$

$$\frac{\sigma - \sigma_y}{\tau_{xy}} = \frac{l}{m} = \cot\theta$$

FIG. 3 10-5



(a) (b)

$A_x = l_1 A_n$

$A_y = m_1 A_n$

$A_z = n_1 A_n$

$\bar{X} = (l_1 \sigma_x + m_1 \tau_{xy} + n_1 \tau_{zx}) A_n$

$\bar{Y} = (l_1 \tau_{xy} + m_1 \sigma_y + n_1 \tau_{zy}) A_n$

$\bar{Z} = (l_1 \tau_{xz} + m_1 \tau_{yz} + n_1 \sigma_z) A_n$

$$\bar{X}/l_1 A_n = \bar{Y}/m_1 A_n = \frac{\bar{Z}}{n_1 A_n} = \sigma$$

	X	Y	Z
X'	l_1	m_1	n_1
Y'			
Z'			

from FIG. $1_{10\text{-}5}$ we can obtain

$$(\sigma_x')_{\substack{max \\ min}} = \left(\frac{\sigma_x+\sigma_y}{2}\right) \pm \left[\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$$

$$\tan 2\alpha = \frac{2\tau_{xy}}{\sigma_x-\sigma_y}$$

In 2-D the invariants are ($\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$ in 3D)

$$\sigma^2 - I_1\sigma + I_2 = 0$$

10-6.1 $I_{1\sigma} = \sigma_x + \sigma_y$; $I_{2\sigma} = \sigma_x\sigma_y - \tau_{xy}^2$

In 3-D the invariants become

$$I_{1\sigma} = \sigma_x + \sigma_y + \sigma_z = \sigma_1+\sigma_2+\sigma_3 \quad ; \quad I_{2\sigma} = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - (\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2) = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

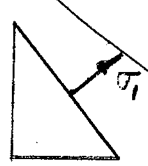
$$I_{3\sigma} = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{zx}\tau_{yz} - (\sigma_x\tau_{yz}^2 + \sigma_y\tau_{zx}^2 + \sigma_z\tau_{xy}^2) = \sigma_1\sigma_2\sigma_3$$

from $\frac{\partial\sigma_x'}{\partial l}$ we had obtained $\frac{l\sigma_x + m\tau_{xy}}{l} = \frac{l\tau_{xy}+m\sigma_y}{m} = \frac{\bar{X}}{l} = \frac{\bar{Y}}{m}$

or $m/l = \bar{Y}/\bar{X} = \tan\alpha$

if ratio holds then we have only σ_1 and no shear

FIG. $1_{10\text{-}6}$



From Strain displacements we have 6 equations

$$\epsilon_x = \frac{\partial u}{\partial x} \quad ; \quad \epsilon_y = \frac{\partial v}{\partial y} \quad ; \quad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \quad ; \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad ; \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

From Stress Equilibrium Equations we got

$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} + \bar{X} = 0$$

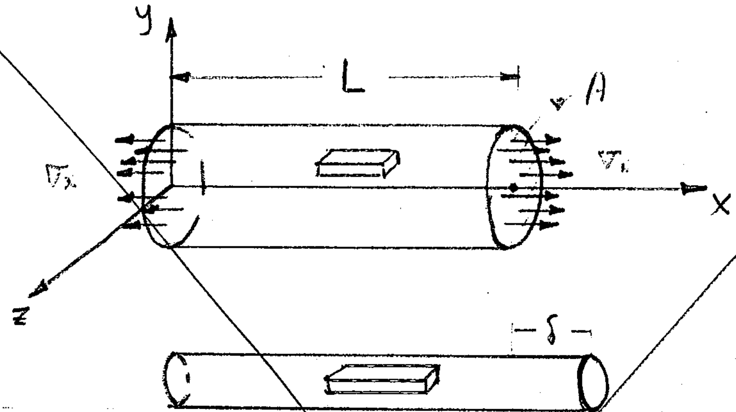
$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \bar{Y} = 0$$

$$\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + \bar{Z} = 0$$

Total :- 9 equations 15 unknowns

Stress - Strain Relationships

FIG. 2 10-6



cross-section must remain the same - isotropic properties

$\delta = \frac{FL}{AE}$ deformation of bar

$\frac{\delta}{L} = \epsilon_x = \frac{F}{AE} = \frac{\sigma_x}{E}$; E - modulus of elasticity

$\epsilon_y = \epsilon_z = -\nu\epsilon_x$; ν - Poisson's ratio

if $\epsilon_y = \frac{\sigma_y}{E}$ then the effect of this is $\epsilon_x = \epsilon_z = -\nu\epsilon_y$

" $\epsilon_z = \frac{\sigma_z}{E}$ " " " " " " $\epsilon_x = \epsilon_y = -\nu\epsilon_z$

true if we are below Elastic or Proportional limit (FIG. 1(c) 9-15)

Within elastic limit we can use the principle of superposition

$$\therefore \boxed{\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)}$$

2. The components of stress at a point in a body referred to a rectangular Cartesian system of coordinates are given by

σx= 5 Mpa	τxy= 5 MPa	τxz = 8 MPa
τyx = 5 Mpa	σy = 0 MPa	τyz = -7.5 MPa
τzx = 8 MPa	τzy = -7.5 Mpa	σz = -3 MPa

a) Find the principal stresses $\sigma 1$, $\sigma 2$, and $\sigma 3$ and
b) the directions that accompany these principal stresses;

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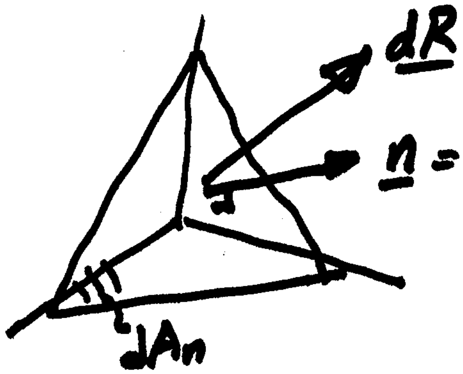
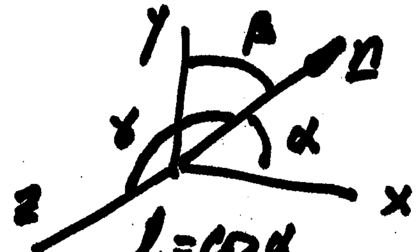


$\underline{n} = l\underline{i} + m\underline{j} + n\underline{k}$

l, m, n - directional cosines



$l = \cos\alpha$

$m = \cos\beta$ $\quad l^2 + m^2 + n^2 = 1$

$n = \cos\gamma$

$$\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

$$\underline{t}_n = \underline{n} \cdot \sigma = (l\sigma_x + m\tau_{yx} + n\tau_{zx})\underline{i} + (l\tau_{xy} + m\sigma_y + n\tau_{zy})\underline{j} + (l\tau_{xz} + m\tau_{yz} + n\sigma_z)\underline{k}$$

$$t_n dA_n = F_x\underline{i} + F_y\underline{j} + F_z\underline{k}$$

ex: $\quad F_x = (l\sigma_x + m\tau_{yx} + n\tau_{zx})dA_n$

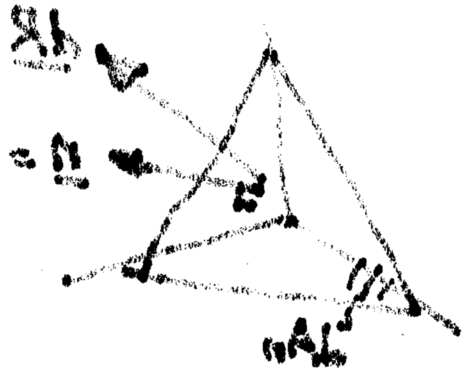
IF plane was a plane on which principal stress exists

$$\mathbb{p} = \underline{n}p\underline{n} \qquad \underline{t} = \underline{n} \cdot \mathbb{p} = p\underline{n} = p(l\underline{i} + m\underline{j} + n\underline{k})$$

$$\underline{F} = p\underline{n}dA_n = p(l\underline{i} + m\underline{j} + n\underline{k})dA_n$$

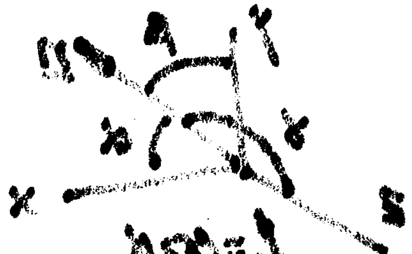
$$\Rightarrow \quad \begin{aligned} l\sigma_x + m\tau_{yx} + n\tau_{zx} &= pl \\ l\tau_{xy} + m\sigma_y + n\tau_{zy} &= pm \\ l\tau_{xz} + m\tau_{yz} + n\sigma_z &= pn \end{aligned} \quad \Rightarrow \quad \begin{bmatrix} \sigma_x - p & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - p & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - p \end{bmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$$\det(\quad) = 0$$



$\underline{n} = l\underline{i} + m\underline{j} + n\underline{k}$

l, m, n – directional cosines



$l = \cos\alpha$
$m = \cos\beta$
$n = \cos\gamma$

$(l^2 + m^2 + n^2) = 1$

$$D = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

$$\underline{t}_n = \underline{n} \cdot D = \underline{i}(l\sigma_x + m\tau_{yx} + n\tau_{zx}) + \underline{j}(l\tau_{xy} + m\sigma_y + n\tau_{zy}) + \underline{k}(l\tau_{xz} + m\tau_{yz} + n\sigma_z)$$

$$\underline{t}_n dA_n = F_x\underline{i} + F_y\underline{j} + F_z\underline{k}$$

ex: $F_x = (l\sigma_x + m\tau_{yx} + n\tau_{zx}) dA_n$

if plane was a plane on which principal stress acting

$$\underline{t} = \underline{n} \cdot \sigma = \sigma\underline{n} = \sigma(l\underline{i} + m\underline{j} + n\underline{k}) \qquad \underline{n}\,\sigma\,\underline{n} = \sigma$$

$$\underline{F} = \sigma\underline{n}\,dA_n = \sigma(l\underline{i} + m\underline{j} + n\underline{k})\,dA_n$$

$$\begin{aligned} l\sigma &= l\sigma_x + m\tau_{yx} + n\tau_{zx} \\ m\sigma &= l\tau_{xy} + m\sigma_y + n\tau_{zy} \\ n\sigma &= l\tau_{xz} + m\tau_{yz} + n\sigma_z \end{aligned} \Rightarrow \begin{bmatrix} \sigma_x - \sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{bmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$\det(\quad) = 0$

$$p^3 - I_{1\sigma} p^2 + I_{2\sigma} p - I_{3\sigma} = 0$$

$$I_{1\sigma} = \sigma_x + \sigma_y + \sigma_z$$

$$I_{2\sigma} = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - (\tau_{xy}\tau_{yx} + \tau_{xz}\tau_{zx} + \tau_{yz}\tau_{zy})$$

$$I_{3\sigma} = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - (\sigma_x\tau_{yz}^2 + \sigma_y\tau_{xz}^2 + \sigma_z\tau_{xy}^2)$$

$p = \sigma_1 > \sigma_2 > \sigma_3$ principal stresses

$$I_{1\sigma} = \sum \sigma_i = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_{2\sigma} = \sum_{i \neq j} \sigma_i\sigma_j = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_{3\sigma} = \sum_{i \neq j \neq k} \sigma_i\sigma_j\sigma_k = \sigma_1\sigma_2\sigma_3$$

$$\sigma = \begin{pmatrix} 5 & 5 & 8 \\ 5 & 0 & -7.5 \\ 8 & -7.5 & -3 \end{pmatrix} \qquad \sigma - pI = \begin{pmatrix} 5-p & 5 & 8 \\ 5 & 0-p & -7.5 \\ 8 & -7.5 & -3-p \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_{1\sigma} = 5 + 0 - 3 = 2$$

$$I_{3\sigma} = -806.25$$

$$I_{2\sigma} = -15 - (25 + 64 + 56.25) = -160.25$$

$$f(p) = p^3 - 2p^2 - 160p + 806.25 = 0$$

$$f'(p) = 3p^2 - 4p - 160$$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

$\sigma_1 = 9.94$ MPa $\quad \sigma_3 = -13.82$ MPa $\quad \sigma_2 = 5.88$ MPa

$$p^3 - I_{1\sigma} p^2 + I_{2\sigma} p - I_{3\sigma} = 0$$

$$I_{1\sigma} = \sigma_x + \sigma_y + \sigma_z$$

$$I_{2\sigma} = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - (\tau_{xy}\tau_{yx} + \tau_{yz}\tau_{zy} + \tau_{zx}\tau_{xz})$$

$$I_{3\sigma} = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - (\sigma_x\tau_{yz}^2 + \sigma_y\tau_{zx}^2 + \sigma_z\tau_{xy}^2)$$

$p = \sigma_1 > \sigma_2 > \sigma_3$ principal stresses

$$I_{1\sigma} = \sum \sigma_i = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_{2\sigma} = \sum_{i \neq j} \sigma_i\sigma_j = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_{3\sigma} = \sum_{i \neq j \neq k} \sigma_i\sigma_j\sigma_k = \sigma_1\sigma_2\sigma_3$$

$$\sigma = \begin{pmatrix} 5 & 5 & 8 \\ 5 & 0 & -7.5 \\ 8 & -7.5 & -3 \end{pmatrix} \qquad \sigma - pI = \begin{pmatrix} 5-p & 5 & 8 \\ 5 & 0-p & -7.5 \\ 8 & -7.5 & -3-p \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_{1\sigma} = 5 + 0 - 3 = 2$$

$$I_{3\sigma} = -806.25$$

$$I_{2\sigma} = -15 - (25 + 64 + 56.25) = -160.25$$

$$f(p) = p^3 - 2p^2 - 160p + 806.25 = 0$$

$$f'(p) = 3p^2 - 4p - 160$$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

$\sigma_1 = 9.94$ MPa $\qquad \sigma_2 = -13.85$ MPa $\qquad \sigma_3 = 5.88$ MPa

$$\begin{pmatrix} 5-9.94 & 5 & 8 \\ 5 & 0-9.94 & -7.5 \\ 8 & -7.5 & -3-9.94 \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4.94 & 5 & 8 \\ 5 & -9.94 & -7.5 \\ 8 & -7.5 & -12.94 \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Choose $l = 1$

$$-4.94\,l + 5m + 8n = 0$$
$$5l - 9.94m - 7.5n = 0$$

$$5m + 8n = +4.94$$
$$-9.94m - 7.5n = -5$$

Solve for M, N

$$M^2 + N^2 + l^2 = M^2 + N^2 + 1 = d > 1$$

$$\frac{M^2 + N^2 + l^2}{d} = 1 \qquad m = \sqrt{\frac{M^2}{d}} \cdot \text{sign}(M)$$

$$n = \sqrt{\frac{N^2}{d}} \cdot \text{sign}(N)$$

$$l = \sqrt{\frac{1}{d}}$$

$$\begin{pmatrix} -.844 & 5 & 8 \\ 5 & -.444 & -.72 \\ 8 & -.72 & -.12.44 \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Assume $l = 1$

$$-.444\,l + 5m + 8n = 0$$

$$5\,l - .444\,m - .72\,n = 0$$

$$5m + 8n = .444$$

$$-.444\,m - .72\,n = -5$$

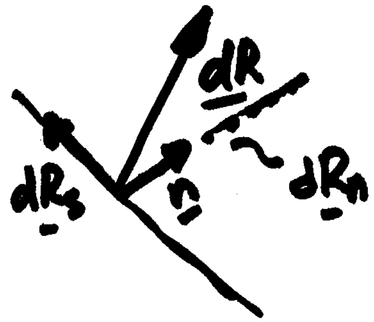
Solve for M, K

$$M^2 + K^2 + l^2 = M^2 + K^2 + 1 = d > 1$$

$$m = \frac{M}{\sqrt{d}} \qquad 1 = \frac{M^2 + K^2 + 1}{d}$$

$$n = \frac{K}{\sqrt{d}}$$

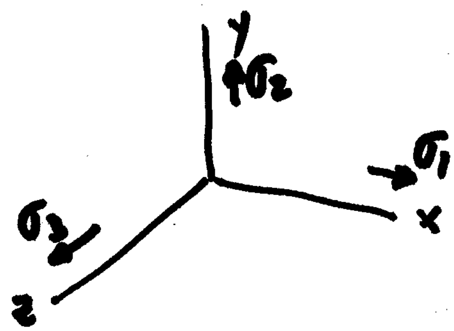
$$l = \frac{1}{\sqrt{d}}$$



$$\frac{1}{A_n}\sqrt{dR^2 - dR_n^2} = \frac{dR_s}{A_n} = \tau_s$$

$$\sigma_p = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix} \qquad \underline{n} \cdot \sigma_p = \underline{t}_n \qquad \underline{n} = l\underline{i} + m\underline{j} + n\underline{k}$$

$$\sigma_p = \underline{i}\,\sigma_1\,\underline{i} + \underline{j}\,\sigma_2\,\underline{j} + \underline{k}\,\sigma_3\,\underline{k}$$



$$\underline{t}_n = \underline{n} \cdot \sigma_p = l\sigma_1\underline{i} + m\sigma_2\underline{j} + n\sigma_3\underline{k}$$

force due to $t_n = (l\sigma_1\underline{i} + m\sigma_2\underline{j} + n\sigma_3\underline{k})A_n$

$\underline{t}_n \cdot \underline{n}$ = normal stress

$$= l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3 = \sigma_n$$

$$d\underline{R}_n = (l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3)A_n$$

$$d\underline{R} = (l\sigma_1\underline{i} + m\sigma_2\underline{j} + n\sigma_3\underline{k})A_n$$

$$d\underline{R} \cdot d\underline{R} = (l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2)A_n^2$$

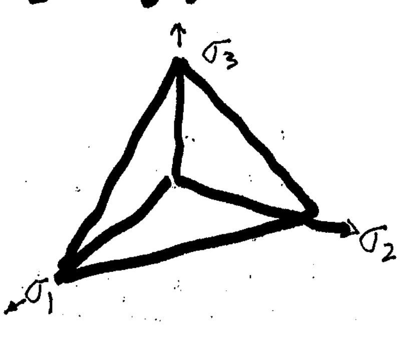
$$dR_n^2 = (l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3)^2 A_n^2$$

$$\tau_s = \frac{1}{A_n}\sqrt{(l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2)A_n^2 - (l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3)^2 A_n^2}$$

Octahedral Stress

$l = m = n$

$l^2 + m^2 + n^2 = 1 \Rightarrow \frac{1}{\sqrt{3}}$



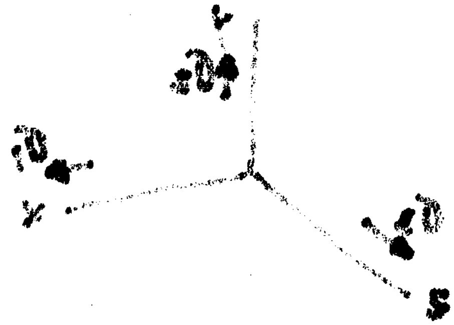
$t^n = \frac{\Delta F}{\Delta A} = \lim_{\Delta A_n \to 0} \frac{\Delta F_n - \Delta F_n}{\Delta A_n}$

$n = li + mj + nk \qquad \sigma_p \cdot n = t^n$

$\sigma_p = i\sigma_x i + j\sigma_y j + k\sigma_z k$

$$\sigma_p = \begin{pmatrix} \sigma_x & 0 & 0 \\ 0 & \sigma_y & 0 \\ 0 & 0 & \sigma_z \end{pmatrix}$$

$t^n = n \cdot \sigma_p = l\sigma_x i + m\sigma_y j + n\sigma_z k$



traction vector $t^n = (l\sigma_x i + m\sigma_y j + n\sigma_z k)\Delta A_n$

normal stress $= t^n \cdot n$

$\sigma_n = l^2\sigma_x + m^2\sigma_y + n^2\sigma_z =$

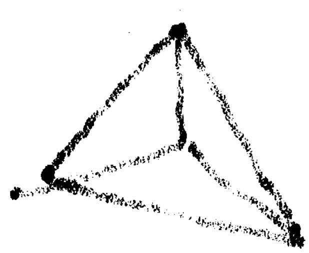
$\Delta F_n = (l^2\sigma_x + m^2\sigma_y + n^2\sigma_z)\Delta A_n$

$\Delta F = (l\sigma_x i + m\sigma_y j + n\sigma_z k)\Delta A_n$

$\Delta F \cdot \Delta F = (l^2\sigma_x^2 + m^2\sigma_y^2 + n^2\sigma_z^2)\Delta A_n^2$

$\Delta F_n^2 = (l^2\sigma_x + m^2\sigma_y + n^2\sigma_z)^2 \Delta A_n^2$

$\tau^2 = \frac{1}{\Delta A_n^2}\left[(l^2\sigma_x^2 + m^2\sigma_y^2 + n^2\sigma_z^2)\Delta A_n^2 - (l^2\sigma_x + m^2\sigma_y + n^2\sigma_z)^2\Delta A_n^2\right]$

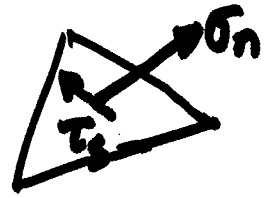


Octahedral stress

$l = m = n$

$l^2 + m^2 + n^2 = 1 \Rightarrow l = \frac{1}{\sqrt{3}}$

$$\frac{dR_n}{A_n} = \sigma_n = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3} I_{1\sigma}$$



$$dR^2 = \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) A_n^2$$

$$dR_n^2 = \left[\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)\right]^2 A_n^2$$

$$\tau_s = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

$$\sigma_n = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

$$\tau_s = \frac{1}{3}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]^{1/2}$$

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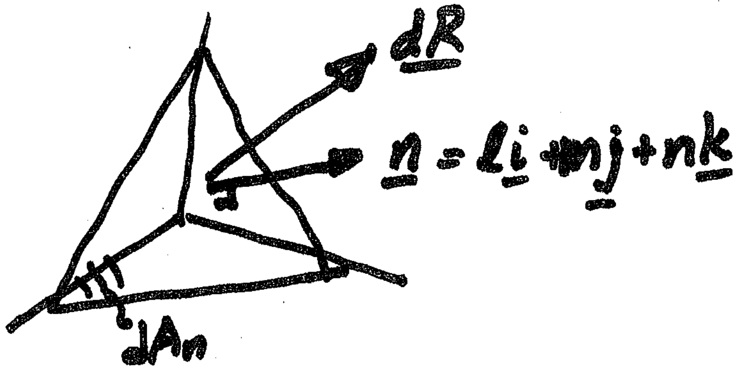
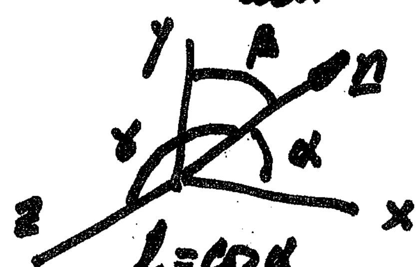


l, m, n - directional cosines



$l = \cos\alpha$

$m = \cos\beta \qquad l^2 + m^2 + n^2 = 1$

$n = \cos\gamma$

$$\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

$$\underline{t}_n = \underline{n}\cdot\sigma = (l\sigma_x + m\tau_{yx} + n\tau_{zx})\underline{i} + (l\tau_{xy} + m\sigma_y + n\tau_{zy})\underline{j} + (l\tau_{xz} + m\tau_{yz} + n\sigma_z)\underline{k}$$

$$t_n dA_n = F_x\underline{i} + F_y\underline{j} + F_z\underline{k}$$

ex: $F_x = (l\sigma_x + m\tau_{yx} + n\tau_{zx})dA_n$

IF plane was a plane on which principal stress exists

$$\mathbb{P} = \underline{n}p\underline{n} \qquad \underline{t} = \underline{n}\cdot\mathbb{P} = p\underline{n} = p(l\underline{i} + m\underline{j} + n\underline{k})$$

$$\underline{F} = p\underline{n}dA_n = p(l\underline{i} + m\underline{j} + n\underline{k})dA_n$$

$$\Rightarrow \begin{aligned} l\sigma_x + m\tau_{yx} + n\tau_{zx} &= pl \\ l\tau_{xy} + m\sigma_y + n\tau_{zy} &= pm \\ l\tau_{xz} + m\tau_{yz} + n\sigma_z &= pn \end{aligned} \Rightarrow \begin{bmatrix} \sigma_x - p & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - p & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - p \end{bmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$$

$$\det(\quad) = 0$$

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$$p^3 - I_{1\sigma} p^2 + I_{2\sigma} p - I_{3\sigma} = 0$$

$$I_{1\sigma} = \sigma_x + \sigma_y + \sigma_z$$

$$I_{2\sigma} = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - (\tau_{xy}\tau_{yx} + \tau_{xz}\tau_{zx} + \tau_{yz}\tau_{zy})$$

$$I_{3\sigma} = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - (\sigma_x\tau_{yz}^2 + \sigma_y\tau_{xz}^2 + \sigma_z\tau_{xy}^2)$$

p: $\sigma_1 > \sigma_2 > \sigma_3$ principal stresses

$$I_{1\sigma} = \sum \sigma_i = \sigma_1 + \sigma_2 + \sigma_3$$

$$I_{2\sigma} = \sum_{i \neq j} \sigma_i\sigma_j = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$$

$$I_{3\sigma} = \sum_{i \neq j \neq k} \sigma_i\sigma_j\sigma_k = \sigma_1\sigma_2\sigma_3$$

$$\sigma = \begin{pmatrix} 5 & 5 & 8 \\ 5 & 0 & -7.5 \\ 8 & -7.5 & -3 \end{pmatrix} \qquad \sigma - pI = \begin{pmatrix} 5-p & 5 & 8 \\ 5 & 0-p & -7.5 \\ 8 & -7.5 & -3-p \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$I_{1\sigma} = 5 + 0 - 3 = 2$$

$$I_{3\sigma} = -806.25$$

$$I_{2\sigma} = -15 - (25 + 64 + 56.25) = -160.25$$

$$f(p) = p^3 - 2p^2 - 160p + 806.25 = 0$$

$$f'(p) = 3p^2 - 4p - 160$$

$$p_{n+1} = p_n - \frac{f(p_n)}{f'(p_n)}$$

$\sigma_1 = 9.94$ MPa $\quad \sigma_3 = -13.82$ MPa $\quad \sigma_2 = 5.88$ MPa

$$\begin{pmatrix} 5-9.94 & 5 & 8 \\ 5 & 0-9.94 & -7.5 \\ 8 & -7.5 & -3-9.94 \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -4.94 & 5 & 8 \\ 5 & -9.94 & -7.5 \\ 8 & -7.5 & -12.94 \end{pmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Choose $l = 1$

$$-4.94\,l + 5m + 8n = 0$$
$$5l - 9.94m - 7.5n = 0$$

$$5m + 8n = +4.94$$
$$-9.94m - 7.5n = -5$$

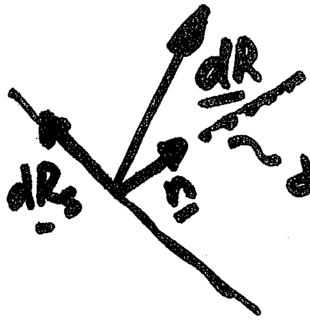
Solve for M, N

$$M^2 + N^2 + l^2 = M^2 + N^2 + 1 = d > 1$$

$$\frac{M^2 + N^2 + l^2}{d} = 1 \qquad m = \sqrt{\frac{M^2}{d}}$$

$$n = \sqrt{\frac{N^2}{d}}$$

$$l = \sqrt{\frac{1}{d}}$$

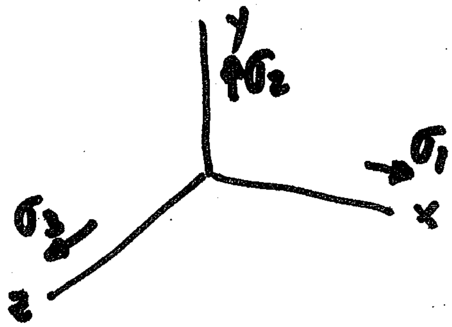


$$\frac{1}{A_n}\sqrt{dR^2 - dR_n^2} = \frac{dR_s}{A_n} = \tau_s$$

$$\sigma_p = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$$

$\underline{n}\cdot\sigma_p = \underline{t}_n \qquad \underline{n} = l\underline{i} + m\underline{j} + n\underline{k}$

$\sigma_p = \underline{i}\sigma_1\underline{i} + \underline{j}\sigma_2\underline{j} + \underline{k}\sigma_3\underline{k}$



$\underline{t}_n = \underline{n}\cdot\sigma_p = l\sigma_1\underline{i} + m\sigma_2\underline{j} + n\sigma_3\underline{k}$

force due to $t_n = (l\sigma_1\underline{i} + m\sigma_2\underline{j} + n\sigma_3\underline{k})A_n$

$\underline{t}_n\cdot\underline{n}$ = normal stress

$= l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3 = \sigma_n$

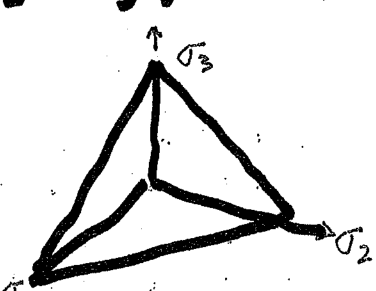
$d\underline{R}_n = (l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3)A_n$

$d\underline{R} = (l\sigma_1\underline{i} + m\sigma_2\underline{j} + n\sigma_3\underline{k})A_n$

$d\underline{R}\cdot d\underline{R} = (l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2)A_n^2$

$dR_n^2 = (l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3)^2 A_n^2$

$$\tau_s = \frac{1}{A_n}\sqrt{(l^2\sigma_1^2 + m^2\sigma_2^2 + n^2\sigma_3^2)A_n^2 - (l^2\sigma_1 + m^2\sigma_2 + n^2\sigma_3)^2 A_n^2}$$

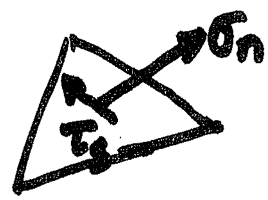


Octahedral Stress

$l = m = n$

$l^2 + m^2 + n^2 = 1 \Rightarrow \frac{1}{\sqrt{3}}$

$\frac{dR_{n}}{A_n}$ $\quad \sigma_n = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) = \frac{1}{3} I_{1\sigma}$



$$dR^2 = \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) A_n^2$$

$$dR_n^2 = \left[\frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)\right]^2 A_n^2$$

$$\tau_s = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2}$$

$$\sigma_n = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z)$$

$$\tau_s = \frac{1}{3}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]^{1/2}$$

Maximum shear stress

Suppose the principal stresses are $\sigma_1 > \sigma_2 > \sigma_3$.

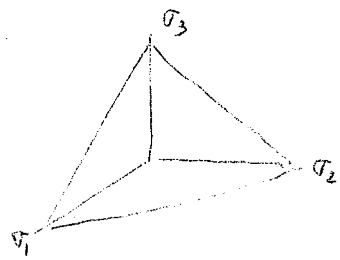
Then the maximum shear stress is

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

and it acts on planes 45 degrees from the principal stresses. Proof: Sokolnikoff, Mathematical Theory of Elasticity, page 50-53.

Octahedral shear stress

An octahedral plane makes equal angles with the principal stress directions. It is given that name because there are eight such planes forming an octahedron about the origin. Octahedral shear stress is of interest in the context of failure criteria.



Consider *force* $d\mathbf{R}$ on an arbitrary plane of area dA,

$$d\mathbf{R} = \sigma_1 l \; dA \, \underset{\sim}{i} + \sigma_2 m \; dA \, \underset{\sim}{j} + \sigma_3 n \; dA \, \underset{\sim}{k} .$$

The *traction vector* T has dimensions of stress:

$$T = \frac{dR}{dA}$$

The stress normal to this plane is

$$\sigma_n = d\mathbf{R} \cdot \mathbf{n}/dA. \;\; = \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2$$

The shear stress is, by the Pythagorean theorem,

$$\tau_s = d\mathbf{R}_s/dA = (1/dA)\sqrt{dR^2 - dR_n^2} \;,$$

but $\mathbf{n} = l \; \underset{\sim}{i} + m \, \underset{\sim}{j} \; + n \, \underset{\sim}{\mathbf{k}}$, with l, m, n as direction cosines.

For the octahedral plane, $l = m = n = 1/\sqrt{3}$, since it makes equal angles with the axes and since the sum of the squares of the direction cosines equals 1.

$$d\mathbf{R}^2/dA^2 = (d\mathbf{R} \cdot d\mathbf{R})/dA^2 = \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2 = \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

The octahedral normal stress is, by σ_n, $= \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2$

$$\sigma_{oct,\,n} = (1/3)(\sigma_1 + \sigma_2 + \sigma_3)$$

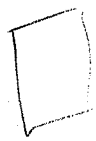
The octahedral shear stress is, by τ_s, dR, and $\mathbf{n}$,

$$\tau_{oct} = \{\sigma_n^2 - \sigma_{oct,\,n}^2\}^{1/2} .$$

gives traction "||

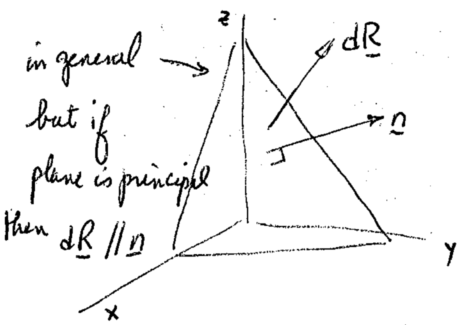
$$\underset{\sim}{n}\cdot\sigma = (l\underset{\sim}{i} + m\underset{\sim}{j} + n\underset{\sim}{k}) \cdot (i\sigma_{xx}\underset{\sim}{i} + i\sigma_{xy}j + i\sigma_{xz}\underset{\sim}{k} + \dots$$

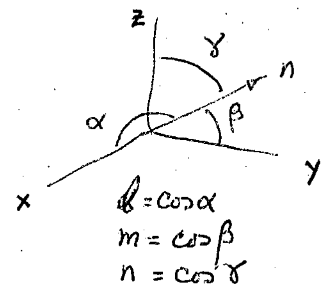
$$= \underset{\sim}{t_n} = l\sigma_{xx} i + l\tau_{xy} j + l\tau_{xz} k + \dots$$
$$+ m\tau_{yx} i + m\sigma_y j + m\tau_{yz} k + \dots$$
$$+ n\tau_{zx} i + n\tau_{zy} j + n\sigma_z k$$



now

$$\underset{\sim}{t_n}\, dA_n = \text{Force on plane } A = F_x\underset{\sim}{i} + F_y\underset{\sim}{j} + F_z\underset{\sim}{k}$$
$$F_x = (l\sigma_{xx} + m\tau_{yx} + n\tau_{zx})dA_n$$
$$F_y = (l\tau_{xy} + m\sigma_y + n\tau_{zy})dA_n$$
$$F_z = (l\tau_{xz} i + n\tau_{zy} j + n\sigma_z)dA_n$$





$$d\underset{\sim}{R} = dR_x\underset{\sim}{i} + dR_y\underset{\sim}{j} + dR_z\underset{\sim}{k} = p(l\underset{\sim}{i} + m\underset{\sim}{j} + n\underset{\sim}{k})dA_n$$
$$\hookrightarrow = \sigma\underset{\sim}{n}\, dA_n = \sigma dA_n(l\underset{\sim}{i} + m\underset{\sim}{j} + n\underset{\sim}{k}) \Rightarrow p = \sigma dA_n$$

& $\sigma dA_n \cdot l = dR_x$ & $\sigma dA_n m = dR_y$ & $\sigma dA_n \cdot n = dR_z$

for equilib

$$F_x - dR_x = 0 \qquad F_y - dR_y = 0 \qquad F_z - dR_z = 0$$

or

$$\begin{bmatrix} \sigma_x - \sigma & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_y - \sigma & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_z - \sigma \end{bmatrix} \begin{pmatrix} l \\ m \\ n \end{pmatrix} \begin{matrix} = 0 \\ = 0 \\ = 0 \end{matrix}$$

either $l, m, n = 0$ not possible since $l^2 + m^2 + n^2 = 1$

or $\det(\sigma - \sigma I) = 0 \Rightarrow \sigma^3 - I_{1\sigma}\sigma^2 + I_{2\sigma}\sigma - I_{3\sigma} = 0$

$$I_{1\sigma} = \sigma_x + \sigma_y + \sigma_z = \sum_{i=1}^{3}\sigma_i$$
$$I_{2\sigma} = \sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - (\tau_{xy}\tau_{yx} + \tau_{xz}\tau_{zx} + \tau_{yz}\tau_{zy}) = \prod_{\substack{i,j=1 \\ i\neq j}}^{3}\sigma_i\sigma_j$$
$$I_{3\sigma} = \sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - (\sigma_x\tau_{yz}^2 + \sigma_y\tau_{xz}^2 + \sigma_z\tau_{yx}^2) = \prod_{\substack{i,j,k=1 \\ i\neq j\neq k}}^{3}\sigma_i\sigma_j\sigma_k$$

σ solves the cubic & gives $\sigma_1 \geq \sigma_2 \geq \sigma_3$

use any value of σ_i to get the l_i, m_i, n_i that go with it

$\sigma_x = 20 \qquad \sigma_y = 30 \qquad \sigma_z = -10$

$\tau_{xy} = 40 \qquad \tau_{yz} = 25 \qquad \tau_{zx} = -30$

$I_{1\sigma} = \Sigma\, \sigma_x + \sigma_y + \sigma_z = 40$

$I_{2\sigma} = 20 \cdot 30 + 30(-10) + (-10)20 - \left[(40)^2 + (25)^2 + (-30)^2\right] = -3025$

$I_{3\sigma} = 20 \cdot 30 \cdot (-10) + 2(40)(25)(-30) - \left[20 \cdot 25^2 + 30 \cdot (-30)^2 + (-10)40^2\right] = -89500$

$\sigma^3 - 40\sigma^2 - 3025\sigma + 89500 = 0$

if $\sigma^3 + b\sigma^2 + c\sigma + d = 0$ then $\sigma = y - b/3$ and $\sigma = y + \frac{40}{3}$

$\Rightarrow y^3 + py + q = 0$

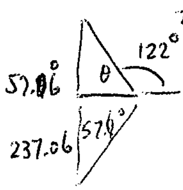
$p = c - \frac{1}{3}b^2 = -3025 - \frac{1}{3}(-40)^2 = -3558.3\overline{3}$

$q = d - \frac{1}{3}bc + (2/27)b^3 = 44425.9256$

$A = -\frac{1}{2}q + \sqrt{(1/27)p^3 + \frac{1}{4}q^2}$

$B = -\frac{1}{2}q - \sqrt{\quad}$

$W = \frac{-1 + i\sqrt{3}}{2} \qquad W^2 = \frac{-1 - i\sqrt{3}}{2}$



here $\frac{1}{27}p^3 + \frac{1}{4}q^2 = -1175276042$, & $\sqrt{\quad} = 34282.3i$

$A = -22212.96 + 34282.3i = 40849.624\, e^{0.683\pi i} = re^{i\theta}$

$B = -22212.96 - 34282.3i = 40849.624\, e^{1.317\pi i}$

$y_1 = A^{1/3} + B^{1/3} = 34.44\, e^{.227683\pi i} + 34.44\, e^{.439\pi i} = 2r^{1/3}\,\mathrm{Re}(e^{i\theta/3}) = 2r^{1/3}\cos\theta/3$

$y_1 = 2 \cdot 34.44 \cdot \cos(0.227683\pi) = 68.88(.7549) = 51.997$

and $\sigma_1 = y_1 + \frac{40}{3} = 51.997 + 13.33 = 65.33$ MPa

$y_2 = 2r^{1/3}\cos(\theta/3 + 120°) = 2 \cdot 34.44 \cos(160.98°) = -65.12$

$\sigma_2 = y_2 + \frac{40}{3} = -65.12 + 13.33 = -51.79$ MPa

$y_3 = 2r^{1/3}\cos(\theta/3 + 240) = 2 \cdot 34.44 \cos(280.98) = 13.12$

$\sigma_3 = y_3 + \frac{40}{3} = 13.12 + 13.33 = 26.45$ MPa

so ordered, $\sigma_1 \geq \sigma_3 \geq \sigma_2$

The cubic formula is the closed-form solution for a cubic equation, i.e., the roots of a cubic polynomial. A general cubic equation is of the form

$$z^3 + a_2 z^2 + a_1 z + a_0 = 0 \tag{1}$$

To solve the general cubic (□), it is reasonable to begin by attempting to eliminate the a_2 term by making a substitution of the form

$$z \equiv x - \lambda. \tag{2}$$

Then

$$(x-\lambda)^3 + a_2 (x-\lambda)^2 + a_1 (x-\lambda) + a_0 = 0 \tag{3}$$
$$(x^3 - 3\lambda x^2 + 3\lambda^2 x - \lambda^3) + a_2 (x^2 - 2\lambda x + \lambda^2) + a_1 (x-\lambda) + a_0 = 0 \tag{4}$$
$$x^3 + (a_2 - 3\lambda) x^2 + (a_1 - 2 a_2 \lambda + 3\lambda^2) x + (a_0 - a_1 \lambda + a_2 \lambda^2 - \lambda^3) = 0. \tag{5}$$

The x^2 is eliminated by letting $\lambda = a_2/3$, so

$$z \equiv x - \tfrac{1}{3} a_2. \tag{6}$$

Then

$$z^3 = \left(x - \tfrac{1}{3} a_2\right)^3 = x^3 - a_2 x^2 + \tfrac{1}{3} a_2^2 x - \tfrac{1}{27} a_2^3 \tag{7}$$
$$a_2 z^2 = a_2 \left(x - \tfrac{1}{3} a_2\right)^2 = a_2 x^2 - \tfrac{2}{3} a_2^2 x + \tfrac{1}{9} a_2^3 \tag{8}$$
$$a_1 z = a_1 \left(x - \tfrac{1}{3} a_2\right) = a_1 x - \tfrac{1}{3} a_1 a_2, \tag{9}$$

so equation (□) becomes

$$x^3 + (-a_2 + a_2) x^2 + \left(\tfrac{1}{3} a_2^2 - \tfrac{2}{3} a_2^2 + a_1\right) x - \left(\tfrac{1}{27} a_2^3 - \tfrac{1}{9} a_2^3 + \tfrac{1}{3} a_1 a_2 - a_0\right) = 0 \tag{10}$$
$$x^3 + \left(a_1 - \tfrac{1}{3} a_2^2\right) x - \left(\tfrac{1}{3} a_1 a_2 - \tfrac{2}{27} a_2^3 - a_0\right) = 0 \tag{11}$$
$$x^3 + 3 \cdot \frac{3 a_1 - a_2^2}{9} x - 2 \cdot \frac{9 a_1 a_2 - 27 a_0 - 2 a_2^3}{54} = 0. \tag{12}$$

Defining

$$p \equiv \frac{3 a_1 - a_2^2}{3} \tag{13}$$
$$q \equiv \frac{9 a_1 a_2 - 27 a_0 - 2 a_2^3}{27} \tag{14}$$

then allows (□) to be written in the standard form

$$x^3 + p x = q. \tag{15}$$

The simplest way to proceed is to make Vieta's substitution

$$x = w - \frac{p}{3w}, \tag{16}$$

which reduces the cubic to the equation

$$w^3 - \frac{p^3}{27 w^3} - q = 0, \tag{17}$$

which is easily turned into a quadratic equation in w^3 by multiplying through by w^3 to obtain

$$(w^3)^2 - q(w^3) - \frac{1}{27} p^3 = 0 \tag{18}$$

(Birkhoff and Mac Lane 1996, p. 106). The result from the quadratic formula is

$$w^3 = \frac{1}{2}\left(q \pm \sqrt{q^2 + \frac{4}{27} p^3}\right) = \frac{1}{2} q \pm \sqrt{\frac{1}{4} q^2 + \frac{1}{27} p^3} \tag{19}$$

$$= R \pm \sqrt{R^2 + Q^3}, \tag{20}$$

where Q and R are sometimes more useful to deal with than are p and q. There are therefore six solutions for w (two corresponding to each sign for each root of w^3). Plugging w back in to (⁄) gives three pairs of solutions, but each pair is equal, so there are three solutions to the cubic equation.

Equation (⁄) may also be explicitly factored by attempting to pull out a term of the form $(x - B)$ from the cubic equation, leaving behind a quadratic equation which can then be factored using the quadratic formula. This process is equivalent to making Vieta's substitution, but does a slightly better job of motivating Vieta's "magic" substitution, and also at producing the explicit formulas for the solutions. First, define the intermediate variables

$$Q \equiv \frac{3a_1 - a_2^2}{9} \tag{21}$$

$$R \equiv \frac{9a_2 a_1 - 27a_0 - 2a_2^3}{54} \tag{22}$$

(which are identical to p and q up to a constant factor). The general cubic equation (⁄) then becomes

$$x^3 + 3Qx - 2R = 0. \tag{23}$$

$$x^3 + px = q, \tag{56}$$

in the variable x, then $a_2 = 0$, $a_1 = p$, and $a_0 = -q$, and the intermediate variables have the simple form (cf. Beyer 1987)

$$Q = \frac{1}{3} p \quad (57)$$

$$R = \frac{1}{2} q \quad (58)$$

$$D \equiv Q^3 + R^2 = \left(\frac{p}{3}\right)^3 + \left(\frac{q}{2}\right)^2. \quad (59)$$

The solutions satisfy Vieta's formulas

$$z_1 + z_2 + z_3 = -a_2 \quad (60)$$

$$z_1 z_2 + z_2 z_3 + z_1 z_3 = a_1 \quad (61)$$

$$z_1 z_2 z_3 = -a_0 . \quad (62)$$

In standard form (□), $a_2 = 0$, $a_1 = p$, and $a_0 = -q$, so eliminating q gives

$$p = -(z_i^2 + z_i z_j + z_j^2) \quad (63)$$

for $i \neq j$, and eliminating p gives

$$q = -z_i z_j (z_i + z_j) \quad (64)$$

for $i \neq j$. In addition, the properties of the symmetric polynomials appearing in Vieta's formulas give

$$z_1^2 + z_2^2 + z_3^2 = -2 p \quad (65)$$

$$z_1^3 + z_2^3 + z_3^3 = 3 q \quad (66)$$

$$z_1^4 + z_2^4 + z_3^4 = 2 p^2 \quad (67)$$

$$z_1^5 + z_2^5 + z_3^5 = -5 p q. \quad (68)$$

The equation for z_1 in Cardano's formula does not have an i appearing in it explicitly while z_2 and z_3 do, but this does not say anything about the number of real and complex roots (since S and T are themselves, in general, complex). However, determining which roots are real and which are complex can be accomplished by noting that if the polynomial discriminant $D > 0$, one root is real and two are complex conjugates; if $D = 0$, all roots are real and at least two are equal; and if $D < 0$, all roots are real and unequal. If $D < 0$, define like our problem

$$\theta \equiv \cos^{-1}\left(\frac{R}{\sqrt{-Q^3}}\right). \quad (69)$$

Then the real solutions are of the form

$$z_1 = 2\sqrt{-Q} \cos\left(\frac{\theta}{3}\right) - \frac{1}{3} a_2 \quad (70)$$

$$z_2 = 2\sqrt{-Q}\cos\left(\frac{\theta + 2\pi}{3}\right) - \tfrac{1}{3}a_2 \tag{71}$$

$$z_3 = 2\sqrt{-Q}\cos\left(\frac{\theta + 4\pi}{3}\right) - \tfrac{1}{3}a_2. \tag{72}$$

This procedure can be generalized to find the real roots for any equation in the standard form (1) by using the identity

$$\sin^3\theta - \tfrac{3}{4}\sin\theta + \tfrac{1}{4}\sin(3\theta) = 0 \tag{73}$$

(Dickson 1914) and setting

$$x \equiv \sqrt{\frac{4|p|}{3}}\, y \tag{74}$$

(Birkhoff and Mac Lane 1996, pp. 90-91), then

$$\left(\frac{4|p|}{3}\right)^{3/2} y^3 + p\sqrt{\frac{4|p|}{3}}\, y = q \tag{75}$$

$$y^3 + \tfrac{3}{4}\frac{p}{|p|}\, y = \left(\frac{3}{4|p|}\right)^{3/2} q \tag{76}$$

$$4y^3 + 3\,\mathrm{sgn}(p)\, y = \tfrac{1}{2} q \left(\frac{3}{|p|}\right)^{3/2} \equiv C. \tag{77}$$

If $p > 0$, then use

$$\sinh(3\theta) = 4\sinh^3\theta + 3\sinh\theta \tag{78}$$

to obtain

$$y = \sinh\left(\tfrac{1}{3}\sinh^{-1} C\right). \tag{79}$$

If $p < 0$ and $|C| \geq 1$, use

$$\cosh(3\theta) = 4\cosh^3\theta - 3\cosh\theta, \tag{80}$$

and if $p < 0$ and $|C| \leq 1$, use

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta, \tag{81}$$

to obtain

$$y = \begin{cases} \cosh\left(\frac{1}{3}\cosh^{-1} C\right) & \text{for } C \geq 1 \\ -\cosh\left(\frac{1}{3}\cosh^{-1} |C|\right) & \text{for } C \leq -1 \\ \cos\left(\frac{1}{3}\cos^{-1} C\right) \text{ [three solutions]} & \text{for } |C| < 1. \end{cases} \quad (82)$$

The solutions to the original equation are then

$$x_i = 2\sqrt{\frac{|p|}{3}}\, y_i - \tfrac{1}{3} a_2. \quad (83)$$

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Maximum shear stress

Suppose the principal stresses are $\sigma_1 > \sigma_2 > \sigma_3$.

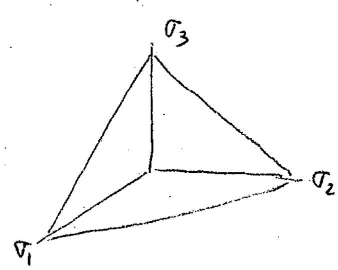
Then the maximum shear stress is

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

and it acts on planes 45 degrees from the principal stresses. Proof: Sokolnikoff, Mathematical Theory of Elasticity, page 50-53.

Octahedral shear stress

An octahedral plane makes equal angles with the principal stress directions. It is given that name because there are eight such planes forming an octahedron about the origin. Octahedral shear stress is of interest in the context of failure criteria.



Consider *force* **dR** on an arbitrary plane of area dA,

$$\mathbf{dR} = \sigma_1 l \; dA \, \underset{\sim}{i} + \sigma_2 m \, dA \, \underset{\sim}{j} + \sigma_3 n \, dA \, \underset{\sim}{k}.$$

The *traction vector* T has dimensions of stress:

$$T = \frac{dR}{dA}$$

The stress normal to this plane is

$$\sigma_n = \mathbf{dR} \cdot \mathbf{n}/dA. = \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2$$

The shear stress is, by the Pythagorean theorem,

$$\tau_s = d\mathbf{R}_s/dA = (1/dA)\sqrt{dR^2 - dR_n^2},$$

but $\mathbf{n} = l \; \underset{\sim}{\mathbf{i}} + m \underset{\sim}{j} + n \underset{\sim}{\mathbf{k}}$, with l, m, n as direction cosines.

For the octahedral plane, $l = m = n = 1/\sqrt{3}$, since it makes equal angles with the axes and since the sum of the squares of the direction cosines equals 1.

$$d\mathbf{R}^2/dA^2 = (d\mathbf{R} \cdot d\mathbf{R})/dA^2 = \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2 = \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

The octahedral normal stress is, by σ_n, $= \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2$

$$\sigma_{oct,\, n} = (1/3)(\sigma_1 + \sigma_2 + \sigma_3)$$

The octahedral shear stress is, by τ_s, dR, and **n**,

$$\tau_{oct} = \{\sigma_n^2 - \sigma_{oct,\, n}^2\}^{1/2}.$$

$$\tau_{oct} = (1/3)\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \frac{1}{3}\left[(\sigma_x-\sigma_y)^2 + (\sigma_y-\sigma_z)^2 + (\sigma_z-\sigma_x)^2 + 6\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right)\right]^{1/2}$$

Strain energy

Consider a spring. F = kx. The work done in compressing the spring is

$W = \int_0^x F\,dx = \frac{1}{2}kx^2$. Since the spring is elastic, this is the strain energy U.

Now consider the strain energy in compressing a cubic block of side L and Young's modulus E.

F = kx

$\varnothing$ $\sigma A = \sigma L^2 = kx = kxL/L = k\,\varepsilon\,L$, so

$\sigma = \frac{k\varepsilon}{L}$. So E = k/L. So $U = \frac{1}{2}kx^2 = \frac{1}{2}(EL)(\varepsilon L)^2 = \frac{1}{2}E\varepsilon^2 L^3 = \frac{1}{2E}\sigma_x^2 L^3$.

So $\frac{1}{2}E\varepsilon^2$ represents strain energy per unit **volume**.

Three-dimensional forms for the strain energy density are obtained by superposition. Poisson related strains enter in the energy expression because stresses in an orthogonal direction do work as the material deforms due to Poisson effects.

$$U_0 = \frac{1}{2E}\{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z)\} + \frac{1}{2G}\{\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\}.$$

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In the study of **yield** criteria it is expedient to separate the effects of normal and shear stress.

Mean normal stress: $\sigma_a = \frac{1}{3}[\sigma_x + \sigma_y + \sigma_z] = \frac{1}{3}[\sigma_1 + \sigma_2 + \sigma_3]$, since the trace is invariant.

This is also the normal stress on an octahedral plane.

Deviatoric stress: The trace of this is zero by construction. This produces change of shape only, no change of volume. By contrast, a hydrostatic stress produces change of volume only, no change of shape *in an isotropic material*. A general state of stress can be written as a sum of hydrostatic and deviatoric stresses.

$$\mathbf{s} = \begin{pmatrix} \sigma_x - \sigma_a & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y - \sigma_a & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z - \sigma_a \end{pmatrix}.$$

Distortional energy, $U_{0d} = \frac{3}{4G}\tau_{oct}^2 = \frac{1}{6G}\sigma_{eff}^2$.

$$\sigma_{eff} = \frac{1}{\sqrt{2}}\{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\{\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\}\}^{1/2}.$$

Uses of concept of strain energy density.

Ø Strain energy is used in criteria for **yield**, which will soon be discussed.

Ø Consider some examples on the magnitudes of strain energy. Consider conversion of strain energy as potential energy into kinetic energy associated with velocity v.

$$\frac{1}{2}\,E\varepsilon^2 L^3 = \frac{1}{2}\,m\,v^2, \text{ so } v^2 = E\varepsilon^2 \frac{L^3}{m} = E\varepsilon^2 \frac{1}{\rho}\,, \text{ so}$$

$$v = \varepsilon\sqrt{\frac{E}{\rho}}\,.$$

Steel, E = 200 GPa, $\rho = 7000$ kg/m^3, $\varepsilon = 0.001$, so v = 5.35 m/s = 12 mph.
Bone, E = 20 GPa, $\rho = 2000$ kg/m^3, $\varepsilon = 0.01$, so v = 32 m/s = 72 mph.
Rubber, E = 1 MPa, $\rho = 1200$ kg/m^3, $\varepsilon = 1$, so v = 29 m/s = 65 mph.

We could imagine high strength steel at $\varepsilon = 0.01$, so v = 120 mph.

Rubber at $\varepsilon = 5$ (we surely have nonlinearity at that point), v = 324 mph.

Defect free glass, for strains approaching the theoretical limit,
E = 70 GPa, $\rho = 2000$ kg/m^3, $\varepsilon = 0.07$, so v = 414 m/s = 926 mph. When this breaks, it shatters into powder.

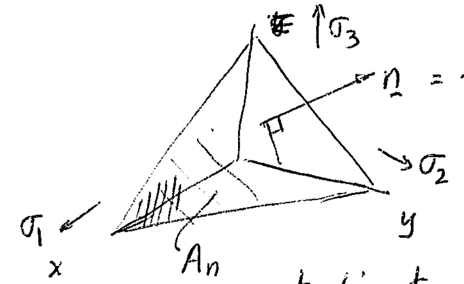
Applications: catapults, slingshots, shattering in brittle failure; kangaroos.

Energy is absorbed in creating fracture surfaces, so during fracture it does not all get converted into kinetic energy.

Discuss table of strain energy per mass data, after Gordon.

LESSON #6

Octahedral plane = plane that makes equal angles with principal stress directions



since any traction $t_n = n \cdot \sigma = n \cdot \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{pmatrix}$

$$= n \cdot (i\sigma_1 i + j\sigma_2 j + k\sigma_3 k)$$

$$t_n = l\sigma_1 i + m\sigma_2 j + n\sigma_3 k$$

and force due to $t_n = t_n (A_n) = dR$

then to find normal stress to plane

$$\sigma_n = \frac{t_n A_n \cdot n}{A_n} = \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2 = \frac{dR \cdot n}{A_n} = \frac{dR_n}{A_n}$$

$$\tau_s = \frac{dR_s}{dA_n} = \frac{1}{dA_n}\sqrt{dR^2 - dR_n^2}$$

$$dR \cdot dR = dR^2 = l^2\sigma_1^2 A_n^2 + m^2\sigma_2^2 A_n^2 + n^2\sigma_3^2 A_n^2 = A_n^2 \cdot \tfrac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) = A_n^2 \cdot \tfrac{1}{3}(\Sigma \sigma_i^2)$$

$$dR_n^2 = [(\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2)A_n]^2 = \tfrac{1}{9}A_n^2(\sigma_1 + \sigma_2 + \sigma_3)^2 = A_n^2/9\,[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$

$$\Longrightarrow \begin{cases} \tau_s = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} & \text{when } n = l = m = \frac{1}{\sqrt{3}} \\ \sigma_n = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \end{cases}$$

then
$$dR^2 - dR_n^2 = \tfrac{2}{9}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) - \tfrac{2}{9}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) = \tfrac{2}{9}(\sigma_1 + \sigma_2 + \sigma_3)^2 - \tfrac{6}{9}(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)$$
$$= \frac{2I_1^2 - 6I_2}{9}$$

If princ. stresses not known.

$$\tau_s = \frac{1}{3}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{xz}^2 + \tau_{yz}^2)\right]^{1/2}$$

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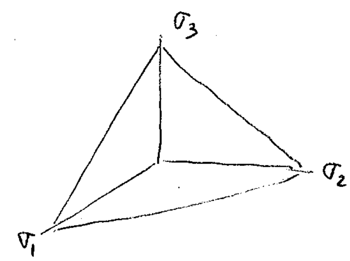

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∅ Strain energy is used in criteria for **yield**, which will soon be discussed.

∅ Consider some examples on the magnitudes of strain energy. Consider conversion of strain energy as potential energy into kinetic energy associated with velocity v.

$\frac{1}{2}\ E\varepsilon^2 L^3 = \frac{1}{2}\ m\, v^2$, so $v^2 = E\varepsilon^2 \frac{L^3}{m} = E\varepsilon^2 \frac{1}{\rho}$, so

$v = \varepsilon \sqrt{\frac{E}{\rho}}$.

Steel, E = 200 GPa, ρ = 7000 kg/m^3, ε = 0.001, so v = 5.35 m/s = 12 mph.
Bone, E = 20 GPa, ρ = 2000 kg/m^3, ε = 0.01, so v = 32 m/s = 72 mph.
Rubber, E = 1 MPa, ρ = 1200 kg/m^3, ε = 1, so v = 29 m/s = 65 mph.

We could imagine high strength steel at ε = 0.01, so v = 120 mph.

Rubber at ε = 5 (we surely have nonlinearity at that point), v = 324 mph.

Defect free glass, for strains approaching the theoretical limit,
E = 70 GPa, ρ = 2000 kg/m^3, ε = 0.07, so v = 414 m/s = 926 mph. When this breaks, it shatters into powder.

Applications: catapults, slingshots, shattering in brittle failure; kangaroos.

Energy is absorbed in creating fracture surfaces, so during fracture it does not all get converted into kinetic energy.

Discuss table of strain energy per mass data, after Gordon.

NORMAL AND SHEAR STRESSES ON OBLIQUE PLANE

- Normal and shear stresses on general oblique plane if we have given principal or coordinate stresses

Then,

NORMAL : $\sigma = \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2$

OR $\sigma = \sigma_x l^2 + \sigma_y m^2 + \sigma_z n^2 + 2(\tau_{xy} lm + \tau_{yz} mn + \tau_{xz} ln)$

SHEAR : $\tau = \left[(\sigma_1 - \sigma_2)^2 l^2 m^2 + (\sigma_2 - \sigma_3) m^2 n^2 + (\sigma_3 - \sigma_1)^2 n^2 l^2\right]^{\frac{1}{2}}$

OR

$$\tau = \Big[\underbrace{(\sigma_x l + \tau_{xy} m + \tau_{xz} n)^2}_{p_x^2} + (\tau_{xy} l + \sigma_y m + \tau_{yz} n)^2 + \dots$$

$$+ (\tau_{xz} l + \tau_{yz} m + \sigma_z n)^2 - \sigma^2\Big]^{\frac{1}{2}} \qquad p^2 = \sigma^2 + \tau^2$$

$$p_x^2 + p_y^2 + p_z^2 = \sigma^2 + \tau^2$$

- There is a unique plane (oblique) that has $l = m = n = \frac{1}{\sqrt{3}}$ relative to principal directions

There are 8 such planes.

Substituting $l = m = n = \frac{1}{\sqrt{3}}$ into transformation equ's gives:

$\sigma_{oct} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$ = mean stress (hydrostatic)

$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{\frac{1}{2}}$

(appl. in failure theory)

3D- MOHR'S CIRCLE (USAGE)

$$l^2 + m^2 + n^2 = 1 \qquad (3)$$

- The following is mathematical treatise to prove that possible stress states are in shaded area:

$$\sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2 = \sigma$$
$$\sigma_1^2 l^2 + \sigma_2^2 l^2 + \sigma_3^2 n^2 = \tau^2 + \sigma^2$$
$$l^2 + m^2 + n^2 = 1$$

Solving for l, m, n, we can express these three eqs. in form of other three eqs. that are expressed in terms of each single cosine (l, m or n).

Using Cramer's rule express l^2 (and the same for rest)

$$l^2 = \frac{\begin{vmatrix} \sigma & \sigma_2 & \sigma_3 \\ \tau^2+\sigma^2 & \sigma_2^2 & \sigma_3^2 \\ 1 & 1 & 1 \end{vmatrix}}{\begin{vmatrix} \sigma_1 & \sigma_2 & \sigma_3 \\ \sigma_1^2 & \sigma_2^2 & \sigma_3^2 \\ 1 & 1 & 1 \end{vmatrix}}$$

Solving for l^2: (some algebra)

$$l^2 = \frac{(\sigma-\sigma_2)(\sigma-\sigma_3)+\tau^2}{(\sigma_1-\sigma_2)(\sigma_1-\sigma_3)} \geq 0$$

numerator
denominator

then, $$m^2 = \frac{(\sigma-\sigma_1)(\sigma-\sigma_3)+\tau^2}{(\sigma_2-\sigma_3)(\sigma_2-\sigma_1)} \geq 0$$

$$n^2 = \frac{(\sigma-\sigma_1)(\sigma-\sigma_2)+\tau^2}{(\sigma_3-\sigma_1)(\sigma_3-\sigma_2)} \geq 0$$

From inequalities follows that:

$$(\sigma-\sigma_2)(\sigma-\sigma_3)+\tau^2 \geq 0$$
$$(\sigma-\sigma_1)(\sigma-\sigma_3)+\tau^2 \leq 0$$
$$(\sigma-\sigma_1)(\sigma-\sigma_2)+\tau^2 \geq 0$$

$(\sigma-\sigma_2)(\sigma-\sigma_3) = \sigma^2-(\sigma_2+\sigma_3)\sigma+\sigma_2\sigma_3$

complete square

$\sigma^2 - 2\frac{(\sigma_2+\sigma_3)}{2}\sigma + \sigma_2\sigma_3 + \left(\frac{\sigma_2+\sigma_3}{2}\right)^2 - \left(\frac{\sigma_2+\sigma_3}{2}\right)^2$

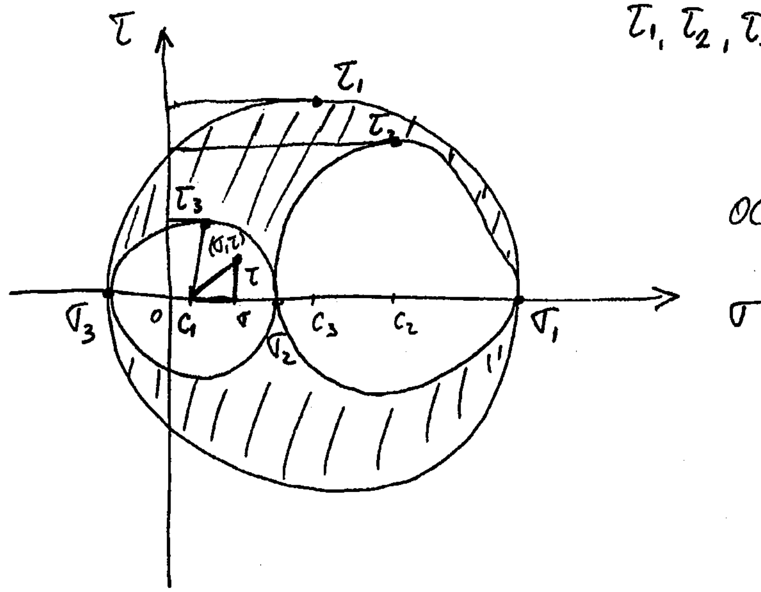
$= \left[\sigma - \left(\frac{\sigma_2+\sigma_3}{2}\right)\right]^2 - \left(\frac{\sigma_2+\sigma_3}{2}\right)^2 + \sigma_2\sigma_3$

$= [\quad]^2 - \left(\frac{\sigma_2-\sigma_3}{2}\right)^2$

Those equations can be rewritten in the form:

$$\begin{cases} \tau^2 + \left(\sigma - \frac{\sigma_2+\sigma_3}{2}\right)^2 \geq \left(\frac{\sigma_2-\sigma_3}{2}\right)^2 = \tau_3^2 \\ \tau^2 + \left(\sigma - \frac{\sigma_3+\sigma_1}{2}\right)^2 \leq \left(\frac{\sigma_3-\sigma_1}{2}\right)^2 = \tau_1^2 \\ \tau^2 + \left(\sigma - \frac{\sigma_1+\sigma_2}{2}\right)^2 \geq \left(\frac{\sigma_1-\sigma_2}{2}\right)^2 = \tau_2^2 \end{cases}$$

equations represent region between circles (shaded)



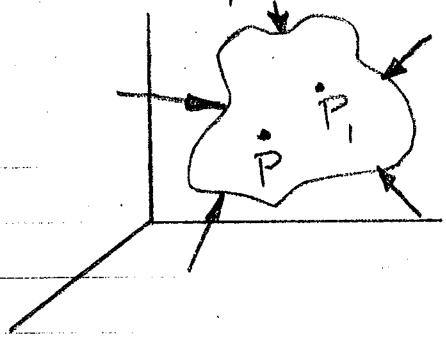
τ_1, τ_2, τ_3 - radii of the Mohr's circle

$OC_1 = \frac{1}{2}(\sigma_3+\sigma_2)$

B is continuous material
homogeneous
elastic
isotropic - behavior is independent of direction.

The material's crystal may not be isotropic but since crystal orientation may be very random in material the average behavior of the material may be isotropic

FIG. 3 9-15



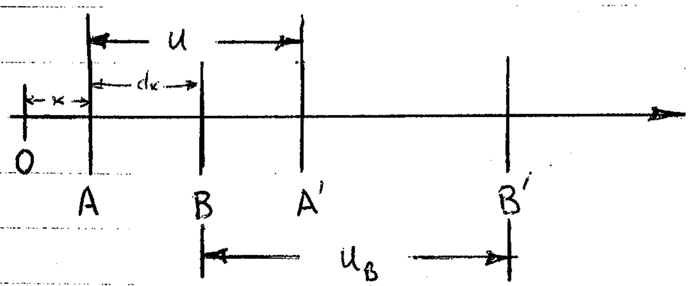
$P(x, y, z)$
$P_1(x+u, y+v, z+w)$

$u = u(x, y, z)$
$v = v(x, y, z)$
$w = w(x, y, z)$

u, v, w are continuous functions whose derivatives are also continuous.

1-D Case

FIG. 4 9-15



AB are displaced to A'B' along x

$u_B = u + du = u + \frac{du}{dx} dx$

$A' = A + u$
$B' = x + dx + u_B$

$$\epsilon_x = \frac{A'B' - AB}{AB} = \frac{du}{dx} \qquad 9\text{-}15.1$$

If $u(x,y,z)$ we write derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$

Stress is defined as the change in force / change in unit area as the change in unit area goes to zero, or.

$$\text{stress} = \lim_{\Delta A \to 0} \frac{\Delta F}{\Delta A}$$

9-17-70

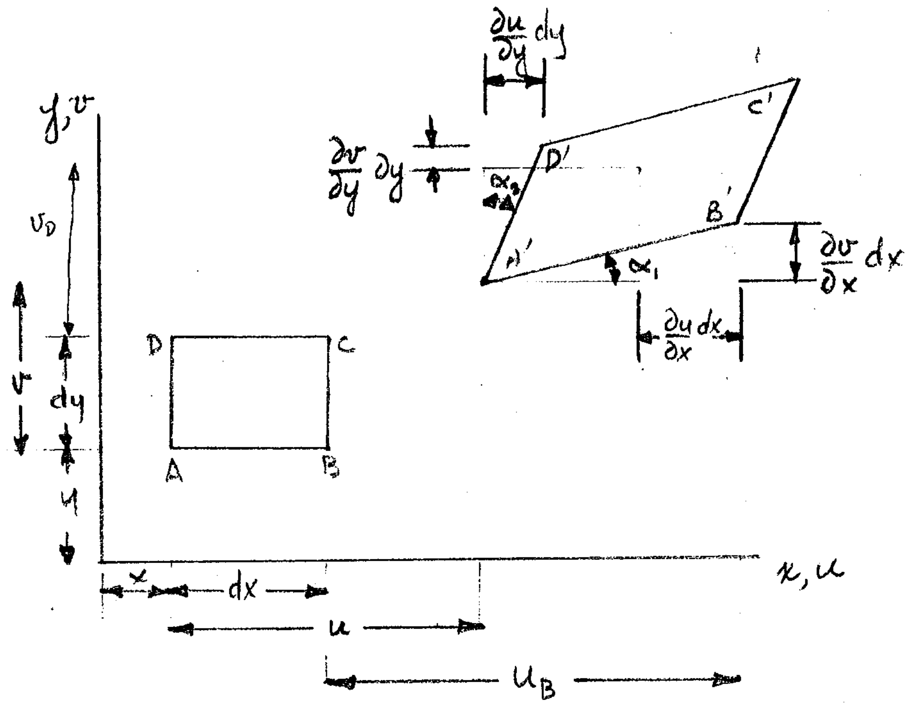


FIG 1 9-17

$$u_B = u + \frac{\partial u}{\partial x} dx$$

$$v_D = v + \frac{\partial v}{\partial y} dy$$

$\epsilon = du/dx$ in 1-D case.

longitudinal strain

$$\epsilon_x = \frac{A'B' - AB}{AB}$$

$$A'B' = (1+\epsilon_x) AB = (1+\epsilon_x) dx$$

$$\overline{A'B'}^2 = \left(dx + \frac{\partial u}{\partial x} dx\right)^2 + \left(\frac{\partial v}{\partial x} dx\right)^2$$

$$(1 + 2\epsilon_x + \epsilon_x^2) dx^2 = dx^2 + 2\frac{\partial u}{\partial x} dx^2 + \left(\frac{\partial u}{\partial x}\right)^2 dx^2 + \left(\frac{\partial v}{\partial x}\right)^2 dx^2$$

thus $1 + 2\epsilon_x + \epsilon_x^2 = 1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

however $\epsilon_x^2 \ll \epsilon_x$ and $\left(\frac{\partial u}{\partial x}\right)^2, \left(\frac{\partial v}{\partial x}\right)^2 \ll \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ resp.

we can then neglect them. if so we obtain

9-17.1

$$\boxed{\begin{aligned}\epsilon_x &= \frac{\partial u}{\partial x}\\ \epsilon_y &= \frac{\partial v}{\partial y}\end{aligned}}$$

and in the n-direct $\boxed{\epsilon_n = \frac{\partial u_n}{\partial n}}$ can be gotten similarly.

Strain, then, along any direction = partial of the displacement along that direction, with respect to that direction (as shown above).

rigid body translation produces no longitudinal strains, $\epsilon_n = 0$.

$$\alpha_1 \approx \tan\alpha_1 = \frac{(\partial v/\partial x)\,dx}{\left(1+\frac{\partial u}{\partial x}\right)dx} = \partial v/\partial x$$

because $\left(\frac{\partial u}{\partial x}\right)^2 << \frac{\partial u}{\partial x} << 1$

$\partial u/\partial x$ can be neglected therefore $\alpha_1 \approx \tan\alpha_1 \approx \frac{\partial v}{\partial x}$

$$\alpha_2 \approx \tan\alpha_2 \approx \frac{(\partial u/\partial y)\,dy}{\left(1+\frac{\partial v}{\partial y}\right)dy} \approx \frac{\partial u}{\partial y}$$

$\alpha_1 + \alpha_2 = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$ and $\gamma_{xy} = \alpha_1 + \alpha_2$

9-17.2

$$\boxed{\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}}$$

Shear strain is by definition an increase in γ or a decrease in the right $\measuredangle$.

Compatibility of Strains.

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0$$ interchange x, y, z for the other two

Proof: $$\frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} - \left(\frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} \right) \equiv 0$$

also $$2\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} \right)$$ interchange x, y, z

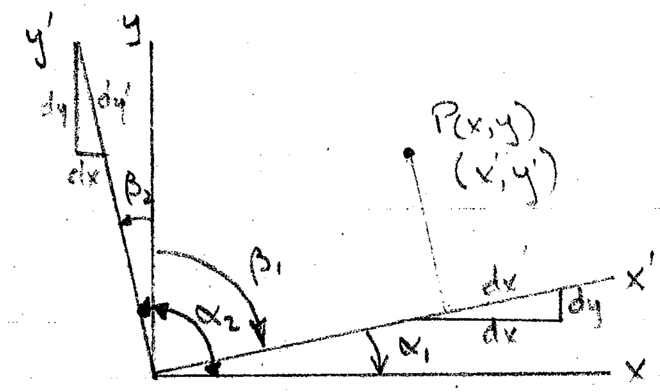
9-17.3
$$\epsilon_x = \frac{\partial u}{\partial x}, \quad \epsilon_y = \frac{\partial v}{\partial y}, \quad \epsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

The assumptions we have made are:

1. Material continuous
2. homogeneous
3. isotropic
4. elastic
5. continuous functions & derivatives.

FIG. 2
9-17



The surface given by Eq. (119) being drawn with the point O (Fig. 128) as center, we may identify δx, δy, δz in Eqs. (*b*) with x, y, z in Eqs. (*e*).

We consider now the special case when ω_x, ω_y, ω_z are zero. Then the right-hand sides of Eqs. (*e*) are the same as the right-hand sides of Eqs. (*b*) but for a factor 2. Consequently the displacement given by Eqs. (*b*) is normal to the surface given by Eq. (119). Considering the point O_1 (Fig. 128) as a point on the surface, this means that the displacement of O_1 is normal to the surface at O_1. Hence if OO_1 is one of the principal axes of strain, that is, one of the principal axes of the surface, the displacement of O_1 is in the direction of OO_1, and therefore OO_1 does not rotate. The displacement in question will correspond to step 1.

In order to complete the displacement, we must restore to Eqs. (*b*) the terms in ω_x, ω_y, ω_z. But these terms correspond to a small rigid-body rotation having components ω_x, ω_y, ω_z about the x, y, z axes. Consequently these quantities, given by (122), express the rotation of step 3—that is, the rotation of the principal axes of strain at the point O. They are called simply the *components of rotation*.

PROBLEMS

1. What is the equation, of the type $f(x,y,z) = 0$, of the surface with center at O which becomes a sphere $x'^2 + y'^2 + z'^2 = r^2$ *after* the homogeneous deformation of Art. 80? What kind of surface is it?
2. Show that if the rotation is zero throughout the body (irrotational deformation), the displacement vector is the gradient of a scalar potential function.

   Indicate one or more examples of such irrotational deformation from the problems treated in the text.

Timoshenko & Goodier
Theory of Elasticity

chapter | 8

General Theorems

stresses & strains in 3-D

84 | Differential Equations of Equilibrium

In the discussion of Art. 74 we considered the stress at a point of an elastic body. Let us consider now the variation of the stress as we change the position of the point. For this purpose the conditions of equilibrium of a small rectangular parallelepiped with the sides δx, δy, δz (Fig. 129) must be studied. The components of stresses acting on the sides of this small element and their positive directions are indicated in the figure. Here we take into account the small changes of the components of stress due to the small increases δx, δy, δz of the coordinates. Thus designating the mid-points of the sides of the element by 1, 2, 3, 4, 5, 6 as in Fig. 129, we distinguish between the value of σ_x at point 1, and its value at point 2, writing these $(\sigma_x)_1$ and $(\sigma_x)_2$, respectively. The symbol σ_x itself denotes, of course, the value of this stress component at the point x, y, z. In calculating the *forces* acting on the element we consider the sides as very small, and the force is obtained by multiplying the stress at the centroid of a side by the area of this side.

It should be noted that the body force acting on the element, which was neglected as a small quantity of higher order in discussing the equilibrium of a tetrahedron (Fig. 126), must now be taken into account, because it is of the same order of magnitude as the terms due to variations of the stress components, which we are now considering. If we let X, Y, Z denote the components of this force per unit volume of the element, then the equation of equilibrium obtained by summing all the forces acting on the element in the x direction is

$$[(\sigma_x)_1 - (\sigma_x)_2]\,\delta y\,\delta z + [(\tau_{xy})_3 - (\tau_{xy})_4]\,\delta x\,\delta z\,[(\tau_{xz})_5 - (\tau_{xz})_6]\,\delta x\,\delta y + X\,\delta x\,\delta y\,\delta z = 0$$

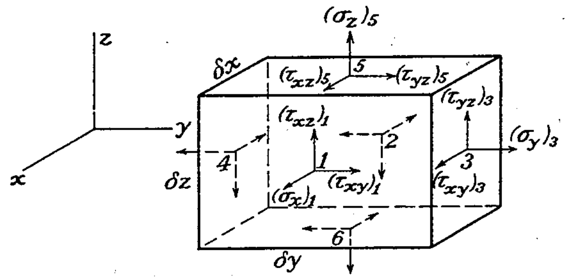


Fig. 129

The two other equations of equilibrium are obtained in the same manner. After dividing by $\delta x\ \delta y\ \delta z$ and proceeding to the limit by shrinking the element down to the point x, y, z, we find

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y &= 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z &= 0 \end{aligned} \tag{123}$$

Equations (123) must be satisfied at all points throughout the volume of the body. The stresses vary over the volume of the body, and when we arrive at the surface they must be such as to be in equilibrium with the external forces on the surface of the body. These conditions of equilibrium at the surface can be obtained from Eqs. (108). Taking a tetrahedron $OBCD$ (Fig. 126) so that the side BCD coincides with the surface of the body, and denoting by $\bar{X}, \bar{Y}, \bar{Z}$ the components of the surface forces per unit area at this point, Eqs. (108) become

$$\begin{aligned} \bar{X} &= \sigma_x l + \tau_{xy} m + \tau_{xz} n \\ \bar{Y} &= \sigma_y m + \tau_{yz} n + \tau_{xy} l \\ \bar{Z} &= \sigma_z n + \tau_{xz} l + \tau_{yz} m \end{aligned} \tag{124}$$

in which l, m, n are the direction cosines of the external normal to the surface of the body at the point under consideration.

If the problem is to determine the state of stress in a body submitted to the action of given forces, it is necessary to solve Eqs. (123), and the solution must be such as to satisfy the boundary conditions (124). These equations, containing six components of stress, $\sigma_x, \ldots, \tau_{yz}$, are not sufficient for the determination of these components. The problem is a statically indeterminate one, and in order to obtain the solution we must proceed as in the case of two-dimensional problems, i.e., the elastic deformations of the body must also be considered.

85 | Conditions of Compatibility

It should be noted that the six components of strain at each point are completely determined by the three functions u, v, w, representing the components of displacement. Hence, the components of strain cannot be taken arbitrarily as functions of x, y, z but are subject to relations that follow from Eqs. (2).

Thus, from Eqs. (2),

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x\, \partial y^2} \qquad \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2\, \partial y} \qquad \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} = \frac{\partial^3 u}{\partial x\, \partial y^2} + \frac{\partial^3 v}{\partial x^2\, \partial y}$$

from which

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} \tag{a}$$

Two more relations of the same kind can be obtained by cyclical interchange of the letters x, y, z.

From the derivatives

$$\frac{\partial^2 \epsilon_x}{\partial y\, \partial z} = \frac{\partial^3 u}{\partial x\, \partial y\, \partial z} \qquad \frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 v}{\partial x\, \partial z} + \frac{\partial^2 w}{\partial x\, \partial y}$$

$$\frac{\partial \gamma_{xz}}{\partial y} = \frac{\partial^2 u}{\partial y\, \partial z} + \frac{\partial^2 w}{\partial x\, \partial y} \qquad \frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u}{\partial y\, \partial z} + \frac{\partial^2 v}{\partial x\, \partial z}$$

we find that

$$2\frac{\partial^2 \epsilon_x}{\partial y\, \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \tag{b}$$

Two more relations of the kind (b) can be obtained by interchange of the letters x, y, z. We thus arrive at the following six differential relations between the components of strain, which must be satisfied by virtue of Eqs. (2):

$$\begin{aligned} \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} & 2\frac{\partial^2 \epsilon_x}{\partial y\, \partial z} &= \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\ \frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z} & 2\frac{\partial^2 \epsilon_y}{\partial x\, \partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\ \frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} &= \frac{\partial^2 \gamma_{xz}}{\partial x\, \partial z} & 2\frac{\partial^2 \epsilon_z}{\partial x\, \partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) \end{aligned} \tag{125}$$

These differential relations[1] are called the *conditions of compatibility*.

[1] Proofs that these six equations are sufficient to ensure the existence of a displacement corresponding to a given set of functions $\epsilon_x, \ldots, \gamma_{xy} \ldots$, may be found in

By using Hooke's law [Eqs. (3)] conditions (125) can be transformed into relations between the components of stress. Take, for instance, the condition

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y\,\partial z} \tag{c}$$

From Eqs. (3) and (4), using the notation (7), we find

$$\epsilon_y = \frac{1}{E}[(1+\nu)\sigma_y - \nu\Theta]$$

$$\epsilon_z = \frac{1}{E}[(1+\nu)\sigma_z - \nu\Theta]$$

$$\gamma_{yz} = \frac{2(1+\nu)\tau_{yz}}{E}$$

Substituting these expressions in (c), we obtain

$$(1+\nu)\left(\frac{\partial^2 \sigma_y}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial y^2}\right) - \nu\left(\frac{\partial^2 \Theta}{\partial z^2} + \frac{\partial^2 \Theta}{\partial y^2}\right) = 2(1+\nu)\frac{\partial^2 \tau_{yz}}{\partial y\,\partial z} \tag{d}$$

The right side of this equation can be transformed by using the equations of equilibrium (123). From these equations we find

$$\frac{\partial \tau_{yz}}{\partial y} = -\frac{\partial \sigma_z}{\partial z} - \frac{\partial \tau_{xz}}{\partial x} - Z$$

$$\frac{\partial \tau_{yz}}{\partial z} = -\frac{\partial \sigma_y}{\partial y} - \frac{\partial \tau_{xy}}{\partial x} - Y$$

Differentiating the first of these equations with respect to z and the second with respect to y, and adding them together, we find

$$2\frac{\partial^2 \tau_{yz}}{\partial y\,\partial z} = -\frac{\partial^2 \sigma_z}{\partial z^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial}{\partial x}\left(\frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial y}\right) - \frac{\partial Z}{\partial z} - \frac{\partial Y}{\partial y}$$

or, by using the first of Eqs. (123),

$$2\frac{\partial^2 \tau_{yz}}{\partial y\,\partial z} = \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_z}{\partial z^2} + \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}$$

Substituting this in Eq. (d) and using, to simplify the writing, the symbol

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., p. 49, Cambridge University Press, New York, 1927, and I. S. Sokolnikoff, "Mathematical Theory of Elasticity," p. 25, 1956. The equations themselves were given by B. de Saint-Venant in his edition of the book by C. L. M. H. Navier, "Résumé des Leçons sur l'Application de la Mécanique," app. 3, Carilian-Goeury, Paris, 1864.

we find

$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_x - \frac{\partial^2\Theta}{\partial x^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2\Theta}{\partial x^2}\right) = (1+\nu)\left(\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right) \tag{e}$$

Two analogous equations can be obtained from the two other conditions of compatibility of the type (c).

Adding together all three equations of the type (e) we find

$$(1-\nu)\nabla^2\Theta = -(1+\nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) \tag{f}$$

Substituting this expression for $\nabla^2\Theta$ in Eq. (e),

$$\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial x^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial X}{\partial x} \tag{g}$$

We can obtain three equations of this kind, corresponding to the first three of Eqs. (125). In the same manner the remaining three conditions (125) can be transformed into equations of the following kind:

$$\nabla^2\tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2\Theta}{\partial y\,\partial z} = -\left(\frac{\partial Z}{\partial y} + \frac{\partial Y}{\partial z}\right) \tag{h}$$

If there are no body forces or if the body forces are constant, Eqs. (g) and (h) become

$$\begin{aligned}
(1+\nu)\nabla^2\sigma_x + \frac{\partial^2\Theta}{\partial x^2} &= 0 \qquad & (1+\nu)\nabla^2\tau_{yz} + \frac{\partial^2\Theta}{\partial y\,\partial z} &= 0\\
(1+\nu)\nabla^2\sigma_y + \frac{\partial^2\Theta}{\partial y^2} &= 0 \qquad & (1+\nu)\nabla^2\tau_{xz} + \frac{\partial^2\Theta}{\partial x\,\partial z} &= 0\\
(1+\nu)\nabla^2\sigma_z + \frac{\partial^2\Theta}{\partial z^2} &= 0 \qquad & (1+\nu)\nabla^2\tau_{xy} + \frac{\partial^2\Theta}{\partial x\,\partial y} &= 0
\end{aligned} \tag{126}$$

We see that in addition to the equations of equilibrium (123) and the boundary conditions (124) the stress components in an isotropic body must satisfy the six conditions of compatibility (g) and (h) or the six conditions (126). This system of equations is generally sufficient for determining the stress components without ambiguity (see Art. 96).

The conditions of compatibility contain only second derivatives of the stress components. Hence, if the external forces are such that the equations of equilibrium (123) together with the boundary conditions (124) can be satisfied by taking the stress components either as constants or as linear functions of the coordinates, the equations of compatibility are satisfied identically and this stress system is the correct solution of the problem. Several examples of such problems will be considered in Chap. 9.

86 | Determination of Displacements

When the components of stress are found from the previous equations, the components of strain can be calculated by using Hooke's law [Eqs. (3) and (6)]. Then Eqs. (2) are used for the determination of the displacements u, v, w. Differentiating Eqs. (2) with respect to x, y, z we can obtain 18 equations containing 18 second derivatives of u, v, w, from which all these derivatives can be determined. For u, for instance, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial \epsilon_x}{\partial x} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial \epsilon_y}{\partial x} \qquad \frac{\partial^2 u}{\partial z^2} = \frac{\partial \gamma_{xz}}{\partial z} - \frac{\partial \epsilon_z}{\partial x}$$
$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial \epsilon_x}{\partial y} \qquad \frac{\partial^2 u}{\partial x\, \partial z} = \frac{\partial \epsilon_x}{\partial z} \qquad \frac{\partial^2 u}{\partial y\, \partial z} = \frac{1}{2}\left(\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}\right) \tag{a}$$

The second derivatives for the two other components of displacement v and w can be obtained by cyclical interchange in Eqs. (a) of the letters x, y, z.

Now u, v, w can be obtained by double integration of these second derivatives. The introduction of arbitrary constants of integration will result in adding to the values of u, v, w linear functions in x, y, z, as it is evident that such functions can be added to u, v, w without affecting such equations as (a). To have the strain components (2) unchanged by such an addition, the additional linear functions must have the form

$$\begin{aligned} u' &= a + by - cz \\ v' &= d - bx + ez \\ w' &= f + cx - ey \end{aligned} \tag{b}$$

This means that the displacements are not entirely determined by the stresses and strains. On the displacements found from the differential Eqs. (123), (124), and (126) a displacement like that of a rigid body can be superposed. The constants a, d, f in Eqs. (b) represent a translatory motion of the body, and the constants b, c, e are the three rotations of the rigid body around the coordinate axes. When there are sufficient constraints to prevent motion as a rigid body, the six constants in Eqs. (b) can easily be calculated so as to satisfy the conditions of constraint. Several examples of such calculations will be shown later.

87 | Equations of Equilibrium in Terms of Displacements

One method of solution of the problems of elasticity is to eliminate the stress components from Eqs. (123) and (124) by using Hooke's law and to express the strain components in terms of displacements by using Eqs. (2). In this manner we arrive at three equations of equilibrium containing only the three unknown functions u, v, w. Substituting in the first of Eqs. (123) from (11),

$$\sigma_x = \lambda e + 2G\frac{\partial u}{\partial x} \qquad e = \epsilon_x + \epsilon_y + \epsilon_z \tag{a}$$

and from (6),

$$\begin{aligned} \tau_{xy} &= G\gamma_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \tau_{xz} &= G\gamma_{xz} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \end{aligned} \tag{b}$$

we find

$$(\lambda + G)\frac{\partial e}{\partial x} + G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + X = 0$$

The two other equations can be transformed in the same manner. Then, using the symbol ∇^2 (see page 238), the equations of equilibrium (123) become

$$\begin{aligned} (\lambda + G)\frac{\partial e}{\partial x} + G\,\nabla^2 u + X &= 0 \\ (\lambda + G)\frac{\partial e}{\partial y} + G\,\nabla^2 v + Y &= 0 \\ (\lambda + G)\frac{\partial e}{\partial z} + G\,\nabla^2 w + Z &= 0 \end{aligned} \tag{127}$$

and, when there are no body forces,

$$\begin{aligned} (\lambda + G)\frac{\partial e}{\partial x} + G\,\nabla^2 u &= 0 \\ (\lambda + G)\frac{\partial e}{\partial y} + G\,\nabla^2 v &= 0 \\ (\lambda + G)\frac{\partial e}{\partial z} + G\,\nabla^2 w &= 0 \end{aligned} \tag{128}$$

Differentiating these equations, the first with respect to x, the second with respect to y, and the third with respect to z, and adding them together, we find

$$(\lambda + 2G)\,\nabla^2 e = 0$$

i.e., the volume expansion e satisfies the differential equation

$$\frac{\partial^2 e}{\partial x^2} + \frac{\partial^2 e}{\partial y^2} + \frac{\partial^2 e}{\partial z^2} = 0 \tag{129}$$

The same conclusion holds also when body forces are constant throughout the volume of the body.

Substituting from such equations as (*a*) and (*b*) into the boundary conditions (124) we find

$$\bar{X} = \lambda el + G\left(\frac{\partial u}{\partial x} l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n\right) + G\left(\frac{\partial u}{\partial x} l + \frac{\partial v}{\partial x} m + \frac{\partial w}{\partial x} n\right) \quad (130)$$

. . . .

Equations (127) together with the boundary conditions (130) define completely the three functions u, v, w. From these the components of strain are obtained from Eqs. (2) and the components of stress from Eqs. (9) and (6). Applications of these equations will be shown in Chap. 14.

88 | General Solution for the Displacements

It is easily verified by substitution that the differential equations (128) of equilibrium in terms of displacement are satisfied by[1]

$$u = \phi_1 - \alpha \frac{\partial}{\partial x}(\phi_0 + x\phi_1 + y\phi_2 + z\phi_3)$$

$$v = \phi_2 - \alpha \frac{\partial}{\partial y}(\phi_0 + x\phi_1 + y\phi_2 + z\phi_3)$$

$$w = \phi_3 - \alpha \frac{\partial}{\partial z}(\phi_0 + x\phi_1 + y\phi_2 + z\phi_3)$$

where $4\alpha = 1/(1 - \nu)$ and the four functions ϕ_0, ϕ_1, ϕ_2, ϕ_3 are harmonic, i.e.,

$$\nabla^2\phi_0 = 0 \qquad \nabla^2\phi_1 = 0 \qquad \nabla^2\phi_2 = 0 \qquad \nabla^2\phi_3 = 0$$

It can be shown that this solution is general, even when ϕ_0 is omitted.[2]

This form of solution has been adapted to curvilinear coordinates by Neuber and applied by him in the solution of problems of solids of revolution[3] generated by hyperbolas (the hyperbolic groove on a cylinder) and ellipses (cavity in the form of an ellipsoid of revolution) transmitting tension, bending, torsion, or shear force transverse to the axis with accompanying bending.

[1] This solution was given independently by P. F. Papkovitch, *Compt. Rend.*, vol. 195, pp. 513 and 754, 1932, and by H. Neuber, *Z. Angew. Math. Mech.*, vol. 14, p. 203, 1934. Other general solutions were given by B. Galerkin, *Compt. Rend.*, vol. 190, p. 1047, 1930, and by Boussinesq and Kelvin—see Todhunter and Pearson, "History of Elasticity," vol. 2, pt. 2, p. 268. See also R. D. Mindlin, *Bull. Am. Math. Soc.*, 1936, p. 373.

[2] For discussion of the number of functions needed for completeness, see P. M. Naghdi and C. S. Hsu, *J. Math. Mech.*, vol. 10, pp. 233–246, 1961, and references given there.

[3] H. Neuber, "Kerbspannungslehre," 2d ed., Springer-Verlag OHG, Berlin, 1958. This book also contains solutions of two-dimensional problems. See Chap. 6 above.

89 | The Principle of Superposition

The solution of a problem of a given elastic solid with given surface and body forces requires us to determine stress components, or displacements, that satisfy the differential equations and the boundary conditions. If we choose to work with stress components, we have to satisfy: (1) the equations of equilibrium (123); (2) the compatibility conditions (125); (3) the boundary conditions (124). Let σ_x . . . , τ_{xy} . . . , be the stress components so determined, and due to surface forces $\bar{X}$, $\bar{Y}$, $\bar{Z}$ and body forces X, Y, Z.

Let σ_x', . . . , τ_{xy}' . . . be the stress components in the same elastic solid due to surface forces $\bar{X}'$, $\bar{Y}'$, $\bar{Z}'$ and body forces X', Y', Z'. Then the stress components $\sigma_x + \sigma_x'$, . . . , $\tau_{xy} + \tau_{xy}'$, . . . will represent the stress due to the surface forces $\bar{X} + \bar{X}'$, . . . and the body forces $X + X'$, ... This holds because all the differential equations and boundary conditions are *linear*. Thus, adding the first of Eqs. (123) to the corresponding equation

$$\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}'}{\partial y} + \frac{\partial \tau_{xz}'}{\partial z} + X' = 0$$

we find

$$\frac{\partial}{\partial x}(\sigma_x + \sigma_x') + \frac{\partial}{\partial y}(\tau_{xy} + \tau_{xy}') + \frac{\partial}{\partial z}(\tau_{xz} + \tau_{xz}') + X + X' = 0$$

and similarly from the first of (124) and its counterpart we have by addition

$$\bar{X} + \bar{X}' = (\sigma_x + \sigma_x')l + (\tau_{xy} + \tau_{xy}')m + (\tau_{xz} + \tau_{xz}')n$$

The compatibility conditions can be combined in the same manner. The complete set of equations shows that $\sigma_x + \sigma_x'$, . . . , $\tau_{xy} + \tau_{xy}'$ . . . satisfy all the equations and conditions determining the stress due to forces $\bar{X} + \bar{X}'$, . . . , $X + X'$, ... This is an instance of the principle of superposition. It is readily extended to other types of boundary conditions such as given displacements.

In deriving our equations of equilibrium (123) and boundary conditions (124), we made no distinction between the position and form of the element before loading, and its position and form after loading. As a consequence, our equations and the conclusions drawn from them are valid only so long as the small displacements in the deformation do not affect substantially the action of the external forces. There are cases, however, in which the deformation must be taken into account. Then the justification of the principle of superposition given above fails. The beam under simultaneous thrust and lateral load affords an example of this kind,

$$x' = x\cos\alpha_1 + y\sin\alpha_1$$
$$y' = y\cos\alpha_1 - x\sin\alpha_1$$

if $\cos\alpha_1 = l_1$ $\quad \cos\alpha_2 = l_2$

$\cos\beta_1 = m_1$ $\quad \cos\beta_2 = m_2$

since $\alpha_1 = -\beta_2$ $\quad \beta_2 + \beta_1 + \alpha_1 = -\alpha_2$

$\cos\alpha_1 = \cos(-\beta_2) = \cos\beta_2$ $\quad \beta_2 = \alpha_1$ $\quad \underline{\underline{l_1 = m_2}}$

since $\beta_2 + \beta_1 + \alpha_1 = 90^\circ + \alpha_1$ then

$\cos(\alpha_2) = \cos(\alpha_2) = \cos(90^\circ + \alpha_1) = -\sin\alpha_1 = -m_1$

$\underline{\underline{l_2 = -m_1}}$

$\frac{dx}{dx'} = \cos\alpha_1 = l_1$ $\quad \frac{dy}{dx'} = \sin\alpha_1 = m_1$

$\frac{dx}{dy'} = +\sin\beta_2 = +l_2 = -m_1$ $\quad \frac{dy}{dy'} = l_1$

$l_1^2 + m_1^2 = 1$ $\quad ((\cos\alpha_1)^2 + (\sin\alpha_1)^2 = 1)$

$l_2^2 + m_2^2 = 1$ $\quad ((\cos\alpha_2)^2 + (\sin\alpha_2)^2 = 1)$

$l_1^2 + l_2^2 = 1$

$m_2^2 + m_1^2 = 1$

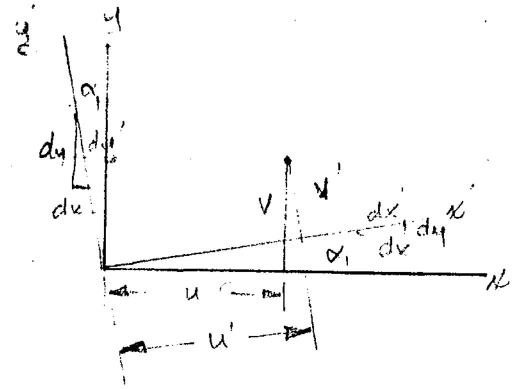
$l_1 m_1 + l_2 m_2 = 0$

$l_1 l_2 + m_1 m_2 = 0$

9-22-70

	x	y
x'	l_1	m_1
y'	$l_2 = -m_1$	$m_2 = l_1$

FIG 1 9-22



$$u' = l_1 u + m_1 v \qquad \frac{dx}{dx'} = \frac{dy}{dy'} = l_1$$

$$v' = -m_1 u + l_1 v$$

$$\cos\alpha_1 = l_1 \;;\; \sin\alpha_1 = m_1 \qquad \frac{dy}{dx'} = -\frac{dx}{dy'} = m_1$$

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\epsilon_x' = \frac{\partial u'}{\partial x'} = \frac{\partial}{\partial x'}\left(l_1 u + m_1 v\right)$$

$$\frac{\partial}{\partial x'}(\quad) = \frac{\partial}{\partial x}(\quad)\frac{dx}{dx'} + \frac{\partial}{\partial y}(\quad)\frac{dy}{dx'}$$

$$\epsilon_x' = l_1\left(\frac{\partial u}{\partial x}\underbrace{\frac{dx}{dx'}}_{l_1} + \frac{\partial u}{\partial y}\underbrace{\frac{dy}{dx'}}_{m_1}\right) + m_1\left(\frac{\partial v}{\partial x}\underbrace{\frac{dx}{dx'}}_{l_1} + \frac{\partial v}{\partial y}\underbrace{\frac{dy}{dx'}}_{m_1}\right)$$

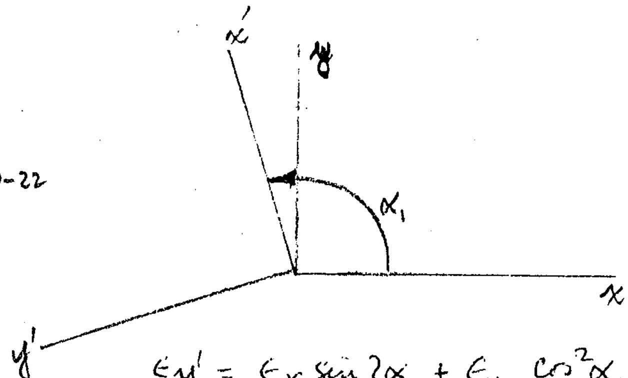
$$= l_1^2\frac{\partial u}{\partial x} + m_1^2\frac{\partial v}{\partial y} + l_1 m_1\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}\right)$$

$$\epsilon_x' = l_1^2\epsilon_x + m_1^2\epsilon_y + l_1 m_1 \gamma_{xy}$$

$$\epsilon_x' = \epsilon_x\cos^2\alpha_1 + \epsilon_y\sin^2\alpha_1 + \gamma_{xy}\sin\alpha_1\cos\alpha_1$$

9-22.1 $$\boxed{\epsilon_x' = \frac{\epsilon_x+\epsilon_y}{2} + \frac{\epsilon_x-\epsilon_y}{2}\cos 2\alpha_1 + \frac{\gamma_{xy}}{2}\sin 2\alpha_1}$$

FIG 2
9-22



$$\epsilon_{y'} = \epsilon_x \sin^2\alpha_1 + \epsilon_y \cos^2\alpha_1 - \gamma_{xy}\sin\alpha_1\cos\alpha_1$$

9-22.2

$$\boxed{\epsilon_{y'} = \frac{\epsilon_x+\epsilon_y}{2} - \frac{\epsilon_x-\epsilon_y}{2}\cos 2\alpha_1 - \frac{\gamma_{xy}}{2}\sin 2\alpha_1}$$

$$\gamma_{x'y'} = \frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}$$

$$= \frac{\partial u'}{\partial x}\frac{dx}{dy'} + \frac{\partial u'}{\partial y}\frac{dy}{dy'} + \frac{\partial v'}{\partial x}\frac{dx}{dx'} + \frac{\partial v'}{\partial y}\frac{dy}{dx'}$$

$$= -\left(l_1\frac{\partial u}{\partial x} + m_1\frac{\partial v}{\partial x}\right)m_1 + \left(-m_1\frac{\partial u}{\partial x} + l_1\frac{\partial v}{\partial x}\right)l_1$$

$$+\left(l_1\frac{\partial u}{\partial y} + m_1\frac{\partial v}{\partial y}\right)l_1 + \left(-m_1\frac{\partial u}{\partial y} + l_1\frac{\partial v}{\partial y}\right)m_1$$

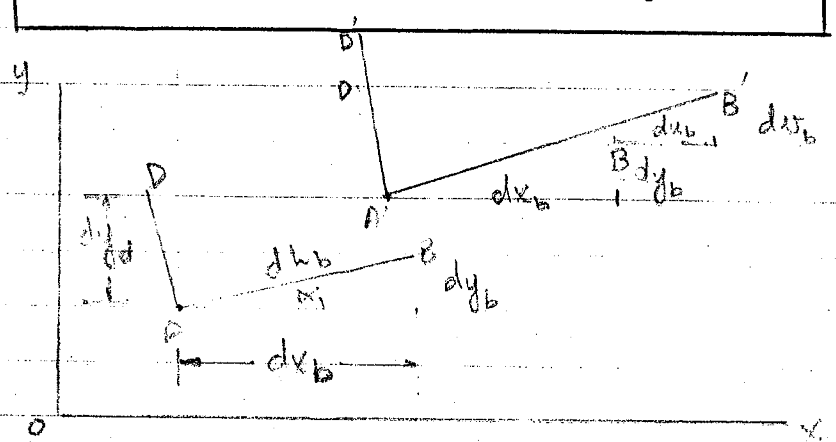
$$= -2l_1m_1\frac{\partial u}{\partial x} + 2l_1m_1\frac{\partial v}{\partial y} + l_1^2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) - m_1^2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)$$

$$\gamma_{x'y'} = -2(\epsilon_x-\epsilon_y)\sin\alpha_1\cos\alpha_1 + (\cos^2\alpha_1 - \sin^2\alpha_1)\gamma_{xy}$$

9-22.3

$$\boxed{\gamma_{x'y'} = -(\epsilon_x-\epsilon_y)\sin 2\alpha_1 + \gamma_{xy}\cos 2\alpha_1}$$

FIG. 3
9-22



$$\epsilon_x' = \frac{A'B' - AB}{AB}$$

$$(1+\epsilon_x') = A'B'/AB \qquad \overline{A'B'}^2 = (dx_b + du_b)^2 + (dy_b + dv_b)^2$$

$$= dx_b^2 + 2dx_b\, du_b + du_b^2 + dy_b^2 + 2dy_b\, dv_b + dv_b^2$$

$$u_B = u + \frac{\partial u}{\partial x} dx_b + \frac{\partial u}{\partial y} dy_b$$

$$(1 + 2\epsilon_x' + \epsilon_x'^2)\, dL_b^2 = dL_b^2 + 2\left(\frac{\partial u}{\partial x} dx_b^2 + \frac{\partial u}{\partial y} dx_b\, dy_b\right) + 2\left(\frac{\partial v}{\partial x} dx_b\, dy_b + \frac{\partial v}{\partial y} dy_b^2\right) + \left(\frac{\partial u}{\partial x} dx_b + \frac{\partial u}{\partial y} dy_b\right)^2 + \left(\frac{\partial v}{\partial x} dx_b + \frac{\partial v}{\partial y} dy_b\right)$$

neglect these terms since they are small

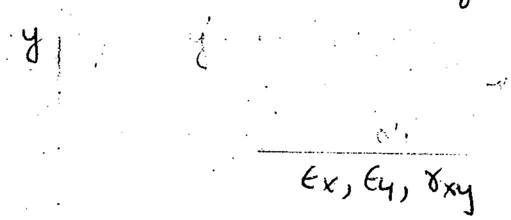
$$\frac{dx_b}{dL_b} = l_1 \qquad \frac{dy_b}{dL_b} = m_1$$

$$\epsilon_x' = l_1^2 \frac{\partial u}{\partial x} + m_1^2 \frac{\partial v}{\partial y} + l_1 m_1 \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\epsilon_x' = l_1^2 \epsilon_x + m_1^2 \epsilon_y + \gamma_{xy}(l_1 m_1)$$

FIG. 4

9-22



$$\frac{\partial \epsilon_x'}{\partial(\alpha_1)} = \frac{\epsilon_x - \epsilon_y}{2}(-2\sin 2\alpha_1) + \frac{\gamma_{xy}}{2}(2\cos 2\alpha_1) = 0$$

9-22.4 $$\boxed{\tan 2\alpha_1 = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y} \equiv \tan 2\alpha = \frac{\gamma_{xy}}{\epsilon_x - \epsilon_y}}$$

$$\tan 2\alpha = \tan(\pi + 2\alpha); \quad \alpha = \frac{\pi}{2} + \alpha$$

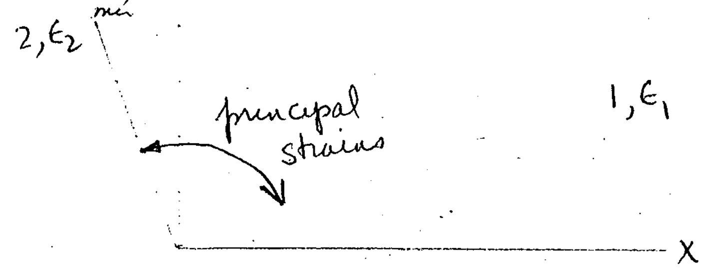
$$\cos 2\alpha = \pm (\epsilon_x - \epsilon_y) / \left[(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2\right]^{1/2}$$

$$\sin 2\alpha = \pm \gamma_{xy} / [(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2]^{1/2}$$

$$(\epsilon_{x'})_{\substack{max,\ min \\ 1,\ 2}} = \frac{\epsilon_x + \epsilon_y}{2} \pm \frac{1}{2}\left[(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2\right]^{1/2}$$

$$(\gamma_{x'y'})_{\epsilon_{x'}\ max,\ min} = \cos 2\alpha_1 \left[-(\epsilon_x - \epsilon_y)\tan 2\alpha_1 + \gamma_{xy}\right] = 0$$

FIG. 5
9-22



9-22.5

$$\epsilon_{1,2} = \frac{\epsilon_x + \epsilon_y}{2} \pm \frac{1}{2}\left[(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2\right]^{1/2}$$

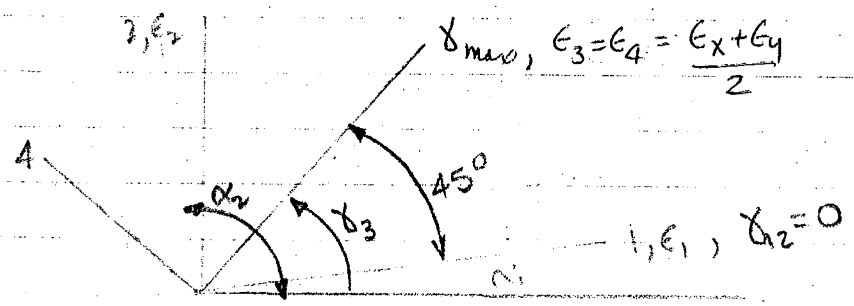
$$\gamma_{1,2} = 0$$

$$\frac{\partial \gamma_{x'y'}}{\partial \alpha_1} = -2(\epsilon_x - \epsilon_y)\cos 2\alpha_1 - 2\gamma_{xy}\sin 2\alpha_1 = 0$$

9-22.6

$$\tan 2\alpha_1 = -\frac{\epsilon_x - \epsilon_y}{\gamma_{xy}}$$

FIG. 6
9-22



$$\epsilon_{x'} = \frac{\epsilon_x + \epsilon_y}{2} + \frac{\cos 2\alpha_1}{2}\left[(\epsilon_x - \epsilon_y) + \gamma_{xy}\tan 2\alpha_1\right]$$

(0 written above; terms crossed out)

9-22.7

$$\gamma_{max} = \pm\left[(\epsilon_x - \epsilon_y)^2 + (\gamma_{xy})^2\right]^{1/2}$$

9-22.8

$$\epsilon_1 = \frac{\epsilon_x + \epsilon_y}{2} + \frac{1}{2}\left[(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2\right]^{1/2}$$

$\epsilon_2 =$ " $-$ "

$\gamma_{1,2} = 0.$ $\tan 2\alpha = \dfrac{\gamma_{xy}}{\epsilon_x - \epsilon_y}$

$$\gamma_{max} = \left[(\epsilon_x - \epsilon_y)^2 + \gamma_{xy}^2\right]^{1/2}$$

$\epsilon_x' = \epsilon_y' = \dfrac{\epsilon_x + \epsilon_y}{2}$ $\tan 2\alpha = -\dfrac{\epsilon_x - \epsilon_y}{\gamma_{xy}}$

$\boxed{\epsilon_x + \epsilon_y = \epsilon_1 + \epsilon_2}$ unchanged no matter what α_1 is
Invariant at a point.

$\boxed{\epsilon_x \epsilon_y - \frac{1}{4}\gamma_{xy}^2 = \epsilon_x'\epsilon_y' - \frac{1}{4}\gamma_{x'y'}^2}$ Invariant

9-24-70

2-D $\begin{cases} \epsilon_x, \epsilon_y \\ \gamma_{xy} \end{cases}$ function of u, v

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} - \frac{\partial^2 \gamma_{xy}}{\partial x \partial y} = 0$$

3 differential eq.
2 unknowns
1 Compatability Eq.

	x	y
x'	l_1	m_1
y'	l_2	m_2

3-D $\begin{cases} \epsilon_x, \epsilon_y, \epsilon_z \\ \gamma_{xy}, \gamma_{yz}, \gamma_{zx} \end{cases}$ function (u, v, w)

6 differential eq.
3 unknowns
6 Compatability Eq (1.48)

	X	Y	Z
X'	l_1	m_1	n_1
Y'	l_2	m_2	n_2
Z'	l_3	m_3	n_3

also

$$2\frac{\partial^2 \epsilon_x}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right)$$

$$2\frac{\partial^2 \epsilon_y}{\partial x \partial z} = \frac{\partial}{\partial y}\left(+ \text{ " } - \text{ " } + \text{ " }\right)$$

$$2\frac{\partial^2 \epsilon_z}{\partial x \partial y} = \frac{\partial}{\partial z}\left(+ \text{ " } + \text{ " } - \text{ " }\right)$$

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

	X_1	X_2	X_3
X_1'	l_{11}	l_{12}	l_{13}
X_2'	l_{21}	l_{22}	l_{23}
X_3'	l_{31}	l_{32}	l_{33}

$$\epsilon'_{\alpha\beta} = l_{\alpha i} l_{\beta j} \epsilon_{ij}$$

$$\epsilon_x' = l_1^2 \epsilon_x + m_1^2 \epsilon_y + n_1^2 \epsilon_z + l_1 m_1 \gamma_{xy} + m_1 n_1 \gamma_{yz} + n_1 l_1 \gamma_{zx}$$

$$\vdots \quad = \dots$$

$$\epsilon_z' = \dots$$

$$\gamma_{x'y'} = 2(\epsilon_x l_1 l_2 + \epsilon_y m_1 m_2 + \epsilon_z n_1 n_2) + \gamma_{xy}(l_1 m_2 + l_1 m_2) + \gamma_{yz}(m_1 n_2 + n_1 m_2) + \gamma_{xz}(n_1 l_2 + l_1 n_2)$$

~~$2 l_1 m_1 (\epsilon_x - \epsilon_y) + (l_1 m_2 + m_1 l_2)\gamma_{xy}$~~

$$\vdots \quad = \dots$$

$$\gamma_{z'x'} = \dots$$

ϵ_1, ϵ_2 2-D max, min $\quad \gamma_{12} = 0$

$\epsilon_1, \epsilon_2, \epsilon_3$ 3-D max, min, other $\quad \gamma_{12} = \gamma_{23} = \gamma_{31} = 0$

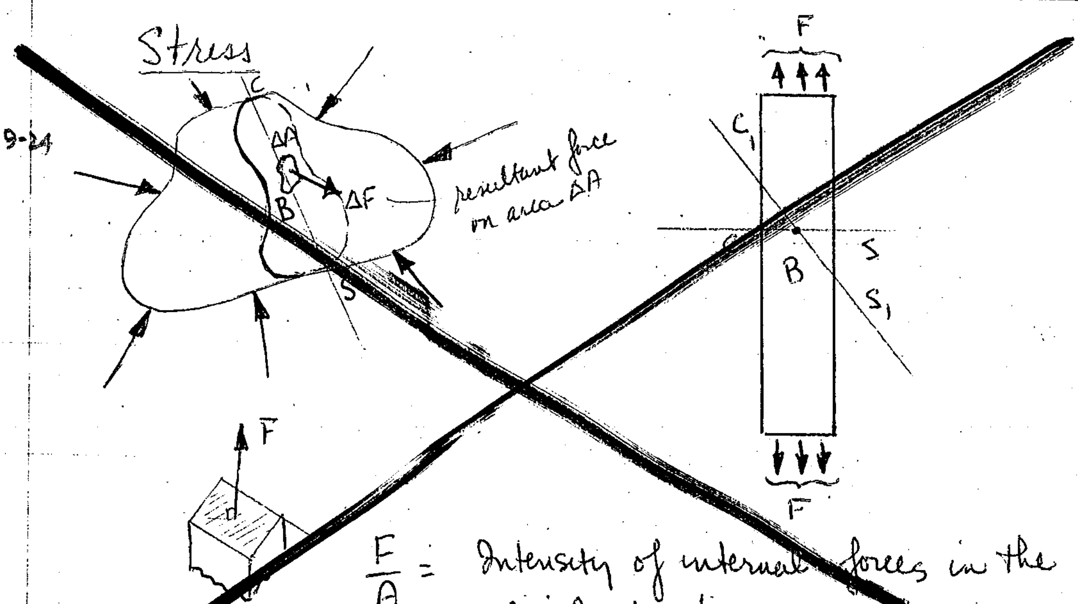
9-24.1

2-D
$$\epsilon_x + \epsilon_y = I_{1\epsilon} \quad \text{invariant}$$
$$\epsilon_x \epsilon_y - \frac{1}{4}\gamma_{xy}^2 = I_{2\epsilon} \quad \text{invariant}$$

3-D
$$\epsilon_x + \epsilon_y + \epsilon_z = I_{1\epsilon} \quad \text{invariant}$$
$$\epsilon_x \epsilon_y + \epsilon_y \epsilon_z + \epsilon_z \epsilon_x - \frac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) = I_{2\epsilon}$$
$$\epsilon_x \epsilon_y \epsilon_z + \frac{1}{4}(\gamma_{xy}\gamma_{yz}\gamma_{zx}) - \frac{1}{4}(\epsilon_x \gamma_{yz}^2 + \epsilon_y \gamma_{zx}^2 + \epsilon_z \gamma_{xy}^2) = I_{3\epsilon}$$

1.139

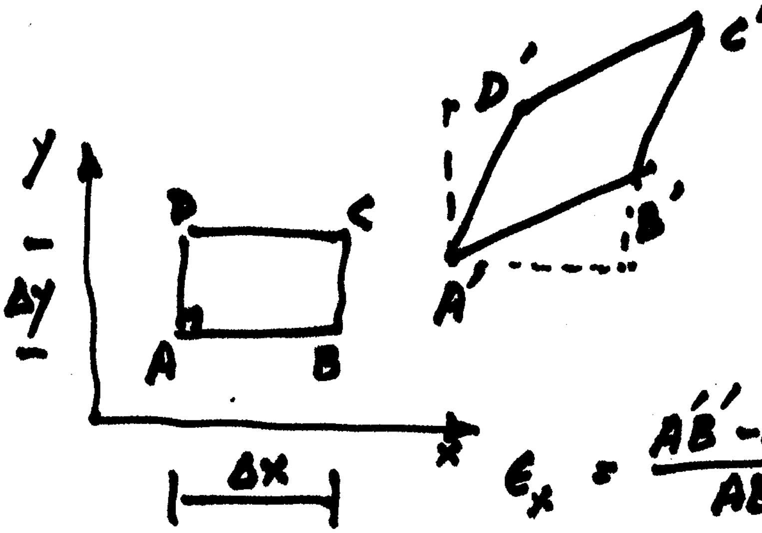
FIG 1 9-24



$\frac{F}{A}$ = Intensity of internal forces in the axial direction.

EGM 5615

9/29/2010



$$\epsilon_x = \frac{A'B' - AB}{AB} = \frac{\frac{\partial u}{\partial x}\Delta x}{\Delta x} \cong \frac{\partial u}{\partial x}$$

$$\epsilon_y = \frac{A'D' - AD}{AD} = \frac{\partial v/\partial y \, \Delta y}{\Delta y} \cong \frac{\partial v}{\partial y}$$

$$\gamma_{yx} = \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$u(x,y,z)$ – disp in x dir
$v(x,y,z)$ – " " y dir

$$\epsilon_z = \frac{\partial w}{\partial z}$$

w – disp in z dir $w(x,y,z)$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

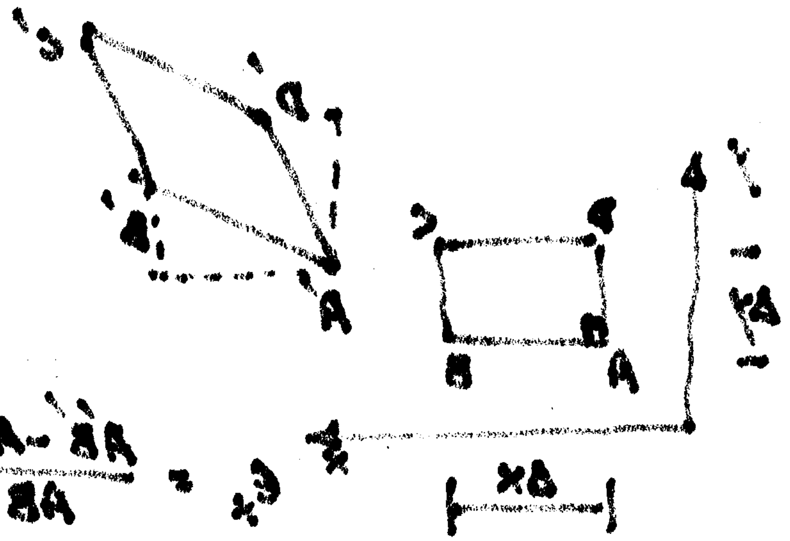
γ_{zx} (= γ_{xz}), γ_{zy} (= γ_{yz})

STRAIN TENSOR

$$\epsilon = \begin{pmatrix} \epsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_y & \gamma_{yz}/2 \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \epsilon_z \end{pmatrix} \qquad \sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

$$\sigma = E\epsilon \iff \sigma = C\epsilon$$

$$\tau_{xy} = \tau_{yx} \qquad \gamma_{xy} = \gamma_{yx}$$



$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{\frac{\partial u}{\partial x}\Delta x}{\Delta x} = \frac{\partial u}{\partial x}$$

$$\varepsilon_y = \frac{A'D' - AD}{AD} = \frac{\frac{\partial v}{\partial y}\Delta y}{\Delta y} = \frac{\partial v}{\partial y}$$

$u(x,y,z)$ – disp in x dir
$v(x,y,z)$ – " " y dir

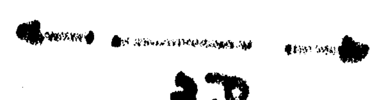
$$\gamma_{xy} = \gamma_{yx} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$w(x,y,z)$ w – disp in z dir $\varepsilon_z = \frac{\partial w}{\partial z}$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \gamma_{zy}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \gamma_{zx}$$

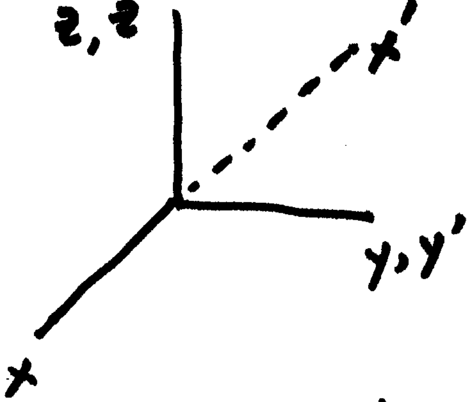
$$\sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix} \qquad \varepsilon = \begin{pmatrix} \varepsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \varepsilon_y & \frac{\gamma_{yz}}{2} \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \varepsilon_z \end{pmatrix}$$

STRAIN TENSOR



$\sigma = E\varepsilon \iff \sigma = C\varepsilon$

$\tau_{xy} = \tau_{yx}$ $\gamma_{xy} = \gamma_{yx}$



$$\sigma'_{mn} = \sigma_{ij}\, l_{mi}\, l_{nj} \qquad l_{\alpha\beta} = \cos(x_\alpha, x_\beta)$$

	x	y	z
x'	-1	0	0
y'	0	1	0
z'	0	0	1

$j, i = 1 \Rightarrow x$
$= 2 \quad y$
$3 \quad z$

$$\sigma'_1 = \sigma_1 \qquad \sigma'_4 = \sigma_4$$
$$\sigma'_2 = \sigma_2 \qquad \sigma'_5 = -\sigma_5 \qquad \sigma'_5 = \tau_{y'z'} \quad \sigma_5 = \tau_{yz}$$
$$\sigma'_3 = \sigma_3 \qquad \sigma'_6 = -\sigma_6$$

$$\varepsilon'_1 = \varepsilon_1 \qquad \varepsilon'_4 = \varepsilon_4$$
$$\varepsilon'_2 = \varepsilon_2 \qquad \varepsilon'_5 = -\varepsilon_5$$
$$\varepsilon'_3 = \varepsilon_3 \qquad \varepsilon'_6 = -\varepsilon_6$$

$$\sigma'_1 = C_{11}\varepsilon'_1 + C_{12}\varepsilon'_2 + \dots + C_{15}\varepsilon'_5 + C_{16}\varepsilon'_6$$
$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + \dots + C_{15}\varepsilon_5 + C_{16}\varepsilon_6$$
$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + \dots - C_{15}\varepsilon_5 - C_{16}\varepsilon_6$$
$$\Rightarrow C_{16}, C_{15} = 0$$

for $\sigma_2, \sigma'_2 \Rightarrow C_{25} = C_{26} = 0$

$\sigma_3, \sigma'_3 \Rightarrow C_{35} = C_{36} = 0$

$\sigma_4, \sigma'_4 \Rightarrow C_{45} = C_{46} = 0$

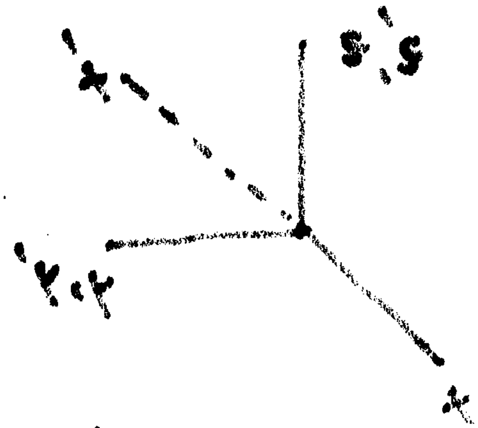
$\sigma_5, \sigma'_5 \Rightarrow C_{55}, C_{56} \neq 0$

$\sigma_6, \sigma'_6 \Rightarrow C_{56}, C_{66} \neq 0$

$\ell_{np} = \cos(x_n, x'_p)$ $\qquad \sigma'_{mn} = \sigma_{ij}\, \ell_{mi}\, \ell_{nj}$

$x \Rightarrow 1 = i, j$
$y \quad 2$
$z \quad 3$

	x	y	z
x'	-1	0	0
y'	0	1	0
z'	0	0	1



$\sigma'_{yz} = \sigma_{yz} \qquad \sigma'_{xz} = -\sigma_{xz} \qquad \sigma'_{xy} = -\sigma_{xy}$

$\sigma'_1 = \sigma_1 \qquad \sigma'_4 = \sigma_4$
$\sigma'_2 = \sigma_2 \qquad \sigma'_5 = -\sigma_5$
$\sigma'_3 = \sigma_3 \qquad \sigma'_6 = -\sigma_6$

$\varepsilon'_1 = \varepsilon_1 \qquad \varepsilon'_4 = \varepsilon_4$
$\varepsilon'_2 = \varepsilon_2 \qquad \varepsilon'_5 = -\varepsilon_5$
$\varepsilon'_3 = \varepsilon_3 \qquad \varepsilon'_6 = -\varepsilon_6$

$\sigma'_1 = c_{11}\varepsilon'_1 + c_{12}\varepsilon'_2 + \dots + c_{15}\varepsilon'_5 + c_{16}\varepsilon'_6$

$\sigma_1 = c_{11}\varepsilon_1 + c_{12}\varepsilon_2 + \dots - c_{15}\varepsilon_5 - c_{16}\varepsilon_6$ $\Big] \Rightarrow c_{15}, c_{16} = 0$

$\sigma_1 = c_{11}\varepsilon_1 + c_{12}\varepsilon_2 + \dots + c_{15}\varepsilon_5 + c_{16}\varepsilon_6$

for $\sigma_2, \sigma'_2 \Rightarrow c_{25} = c_{26} = 0$

$\sigma_3, \sigma'_3 \Rightarrow c_{35} = c_{36} = 0$

$\sigma_4, \sigma'_4 \Rightarrow c_{45} = c_{46} = 0$

$\sigma_5, \sigma'_5 \Rightarrow c_{55}, c_{56} \neq 0$

$\sigma_6, \sigma'_6 \Rightarrow c_{65}, c_{66} \neq 0$

Stress Strain Relations and Elastic Symmetry

References: Sokolnikoff, *Mathematical Theory of Elasticity* pp. 56-71
Malvern, *Introduction to the mechanics of a Deformable Medium*, pp 273-294

Triclinic Crystal (Most General Anisotropic material)

constants.

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46}\\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56}\\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}\end{bmatrix}}_{C\ \text{(Stiffness) Matrix}} \begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{bmatrix}$$

Monoclinic Crystal

one plane of elastic symmetry. e.g. YZ plane. Reflection of X axis leaves constants unchanged

Direction Cosines of Transformation:

$$\begin{array}{c|ccc} & x & y & z\\ \hline x' & -1 & 0 & 0\\ y' & 0 & 1 & 0\\ z' & 0 & 0 & 1\end{array}$$

"C" matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0\\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0\\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0\\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & C_{56}\\ 0 & 0 & 0 & 0 & C_{56} & C_{66}\end{bmatrix}$$

Orthorhombic (orthotropic) Material

3 mutually orthogonal planes of symmetry. Reflection of x, y, and z leaves constants unchanged

Direction cosines (reflect y axis)

$$\begin{array}{c|ccc} & x & y & z\\ \hline x' & 1 & 0 & 0\\ y' & 0 & -1 & 0\\ z' & 0 & 0 & 1\end{array}$$

"C" Matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66}\end{bmatrix}$$

Cubic Material

Interchange of 2 axes (i.e. rotate 90° then reflect) leaves constants unchanged.

Direction cosines (interchange Y & Z)

$$\begin{array}{c|ccc} & x & y & z\\ \hline x' & 1 & 0 & 0\\ y' & 0 & 0 & 1\\ z' & 0 & 1 & 0\end{array}$$

"C" matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & C_{44}\end{bmatrix}$$

Isotropic Material:

Any coordinate transformation leaves Elastic Constants unchanged

"C" matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{44}^{*} & & \\ & & & & C_{44}^{*} & \\ & & & & & C_{44}^{*}\end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & 2\mu & & \\ & & & & 2\mu & \\ & & & & & 2\mu\end{bmatrix}$$

$$^{*}C_{44} = \frac{C_{11} - C_{12}}{2}$$

λ, μ, are the Lamé constants $\qquad \mu = G \qquad \lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$

$\sigma = C\,\epsilon$ may be written $\qquad \sigma_{ij} = \lambda\,\epsilon_{kk}\,\delta_{ij} + 2\mu\,\epsilon_{ij} \qquad G = \dfrac{E}{2(1+\nu)}$

where ϵ_{kk} is the dilatation

$\sigma_1 = \sigma_x \quad \sigma_2 = \sigma_y \quad \sigma_3 = \sigma_z \quad \tau_{xy} = \sigma_4 \quad \tau_{yz} = \sigma_5 \quad \tau_{zx} = \sigma_6$

$\epsilon_1 = \epsilon_x \quad \epsilon_2 = \epsilon_y \quad \epsilon_3 = \epsilon_z \quad \epsilon_4 = \gamma_{xy}/2 \quad \dfrac{\gamma_{yz}}{2} = \epsilon_5 \quad \dfrac{\gamma_{zx}}{2} = \epsilon_6$

$\epsilon_4 = 2\epsilon_{xy}$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{\gamma_{12}}{2}$$

if σ_x is gradually applied, work $= \frac{1}{2}\sigma_x \epsilon_x = \frac{1}{2} C_{11}\sigma_x^2$

Now apply σ_y while σ_x is present. Causes work $\frac{1}{2}\sigma_y \epsilon_y + \sigma_x \epsilon_x' = \frac{1}{2} C_{22}\sigma_y^2 + C_{12}\sigma_x\sigma_y$ since $\epsilon_x' = C_{12}\sigma_y$

total work $= \frac{1}{2} C_{11}\sigma_x^2 + \frac{1}{2} C_{22}\sigma_y^2 + C_{12}\sigma_x\sigma_y$

now change order & apply σ_y first. work $= \frac{1}{2}\sigma_y \epsilon_y = \frac{1}{2} C_{22}\sigma_y^2$

Now apply σ_x while σ_y is present. work $= \frac{1}{2}\sigma_x \epsilon_x + \sigma_y \epsilon_y' = \frac{1}{2} C_{11}\sigma_x^2 + C_{21}\sigma_y\sigma_x$ since $\epsilon_y' = C_{21}\sigma_x$

total work $= \frac{1}{2} C_{22}\sigma_y^2 + \frac{1}{2}\sigma_x^2 C_{11} + C_{21}\sigma_x\sigma_y$

since works must be equal $\Rightarrow C_{12} = C_{21}$

Do same for other directions

if σ_x is gradually applied, work $= \frac{1}{2}\sigma_x \epsilon_x = \frac{1}{2} C_{11}\sigma_x^2$

Now apply σ_y while σ_x is present. Causes work $\frac{1}{2}\sigma_y \epsilon_y + \sigma_x \epsilon_x' = \frac{1}{2} C_{22}\sigma_y^2 + C_{12}\sigma_x\sigma_y$ since $\epsilon_x' = C_{12}\sigma_y$

total work $= \frac{1}{2} C_{11}\sigma_x^2 + \frac{1}{2} C_{22}\sigma_y^2 + C_{12}\sigma_x\sigma_y$

now change order & apply σ_y first. work $= \frac{1}{2}\sigma_y \epsilon_y = \frac{1}{2} C_{22}\sigma_y^2$

Now apply σ_x while σ_y is present. work $= \frac{1}{2}\sigma_x\epsilon_x + \sigma_y \epsilon_y' = \frac{1}{2} C_{11}\sigma_x^2 + C_{21}\sigma_y\sigma_x$ since $\epsilon_y' = C_{21}\sigma_x$

total work $= \frac{1}{2} C_{22}\sigma_y^2 + \frac{1}{2}\sigma_x^2 C_{11} + C_{21}\sigma_x\sigma_y$

since work must be equal $\Rightarrow C_{12} = C_{21}$

Do same for other directions

$C_{12} = (C_{12} + C_{21})\,\varepsilon_1\varepsilon_2$

$$U = \tfrac{1}{2}C_{11}\varepsilon_1^2 + C_{12}\varepsilon_1\varepsilon_2 + C_{13}\varepsilon_1\varepsilon_3 + \dots + C_{16}\varepsilon_1\varepsilon_6$$
$$+\tfrac{1}{2}C_{22}\varepsilon_2^2 + C_{23}\varepsilon_2\varepsilon_3 + \dots + C_{26}\varepsilon_2\varepsilon_6$$
$$\vdots$$
$$+\tfrac{1}{2}C_{66}\varepsilon_6^2$$

Now we note that

$\sigma_1 = \dfrac{\partial U}{\partial \varepsilon_1} = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + \dots C_{16}\varepsilon_6$ and this is true for any σ_r $(r = 1, \dots, 6)$

Now $\sigma_r = C_{rp}\varepsilon_p \quad (r, p = 1 \cdots 6) = \dfrac{\partial U}{\partial \varepsilon_p}$

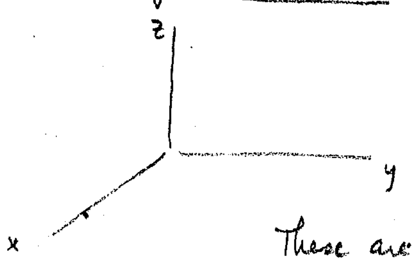
Most general stress-strain relation of a linearly elastic solid involves 21 independent constants C_{pr}

Material scientists & physicists provided theory of crystals in order to determine (

Crystal theory

Triclinic	involves 21 constants	
Monoclinic	13	1 symmetry
Orthotropic/Orthorhombic	9	2 symmetries
Tetragonal / Trigonal	7, 6	
Hexagonal	5	
Cubic	3	
Isotropic	2	material properties in all directions are exactly the same.

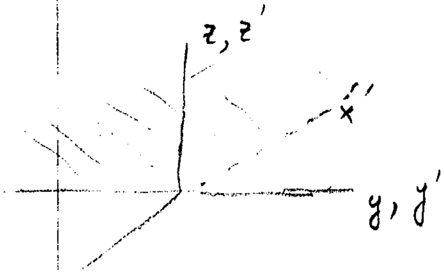
To reduce from <u>21 to 2</u> we introduce cartesian plane & use symmetries of the crystal



These are original relations involving σ_i, c_{ij} & ε_j

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 + \dots + C_{16}\varepsilon_6$$
$$\sigma_2 = C_{12}\varepsilon_1 + \dots + C_{26}\varepsilon_6$$
$$\vdots$$
$$\sigma_6 = C_{16}\varepsilon_6 + \dots + C_{66}\varepsilon_6$$

Now let the yz plane be the plane of symmetry ie (x, x')



	x	y	z
x'	-1	0	0
y'	0	1	0
z'	0	0	1

$l_{\alpha\beta} = \cos(x_\alpha, x_\beta)$

remember $\sigma_{mn}' = \sigma_{ij} l_{mi} l_{nj}$ $\qquad \varepsilon_{mn}' = \varepsilon_{ij} l_{mi} l_{nj}$

$\therefore \quad \sigma_1' = \sigma_1 \quad \sigma_4' = \sigma_4 \qquad \varepsilon_1' = \varepsilon_1 \quad \varepsilon_4' = \varepsilon_4$

$\sigma_2' = \sigma_2 \quad \sigma_5' = -\sigma_5 \qquad \varepsilon_2' = \varepsilon_2 \quad \varepsilon_5' = -\varepsilon_5$

$\sigma_3' = \sigma_3 \quad \sigma_6' = -\sigma_6 \qquad \varepsilon_3' = \varepsilon_3 \quad \varepsilon_6' = -\varepsilon_6$

now $\sigma_1' = C_{11}\varepsilon_1' + \dots + C_{16}\varepsilon_6'$ ok. C_{ij} is invariant under transformation.

$\sigma_1 = C_{11}\varepsilon_1 + \dots - C_{15}\varepsilon_5 - C_{16}\varepsilon_6$ by substituting for $\sigma_1', \varepsilon_1', \dots, \varepsilon_6'$

but originally $\sigma_1 = C_{11}\varepsilon_1 + \dots + C_{15}\varepsilon_5 + C_{16}\varepsilon_6$ must also be true $\therefore C_{15} = C_{16} = 0$

$\sigma_2 = \sigma_2'$ from 2nd relation $\Rightarrow C_{25} = C_{26} = 0$

$\sigma_3 = \sigma_3'$ from 3rd relation $\Rightarrow C_{35} = C_{36} = 0$

$\sigma_4 = \sigma_4'$ from 4th relation $\Rightarrow C_{45} = C_{46} = 0$

$\sigma_5' = -\sigma_5$ no change $\Rightarrow -\sigma_5 = -C_{55}\varepsilon_5 - C_{56}\varepsilon_6$

$\sigma_6' = -\sigma_6$ no change $\quad -\sigma_6 = -C_{65}\varepsilon_5 - C_{66}\varepsilon_6$

reduced const C_{ij} by 8

These are the coeff ($\neq 0$) for <u>monoclinic</u> $C_{56}, C_{66}, C_{11}, C_{12}, C_{13}, C_{14}, C_{55}$
$C_{22}, C_{23}, C_{24}, C_{33}, C_{34}, C_{44}$

Now we look at symmetry in the (y, y')

then for orthorhombic $C_{14} = C_{24} = C_{34} = C_{56} = 0$ Non zero coeff $C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}$
C_{66}, C_{55}, C_{44}

	x	y	z
x'	1	0	0
y'	0	-1	0
z'	0	0	1

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & & \\ C_{12} & C_{22} & C_{23} & C_{24} & & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & & \\ C_{14} & C_{24} & C_{34} & C_{44} & & \\ & 0 & & & C_{55} & C_{56} \\ & & & & C_{56} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \sigma_6 \end{pmatrix}$$

monoclinic

$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3$

$\sigma_2 = C_{12}\varepsilon_2 + C_{22}\varepsilon_2 + C_{23}\varepsilon_3 \qquad 0$

$\sigma_3 = C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{33}\varepsilon_3$

$\sigma_4 = \qquad C_{44}\varepsilon_4 \quad 0 \quad 0$

$\sigma_5 = \quad 0 \qquad 0 \quad C_{55}\varepsilon_5 \quad 0$

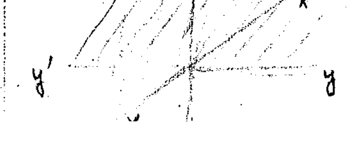
$\sigma_6 = \qquad 0 \quad 0 \quad C_{66}\varepsilon_6$

orthorhombic

$\sigma_1' = \sigma_1 \quad \sigma_4' = -\sigma_4 \qquad \varepsilon_1' = \varepsilon_1 \quad \varepsilon_4' = -\varepsilon_4$

$\sigma_2' = \sigma_2 \quad \sigma_5' = \sigma_5 \qquad \varepsilon_2' = \varepsilon_2 \quad \varepsilon_5' = \varepsilon_5$

$\sigma_3' = \sigma_3 \quad \sigma_6' = -\sigma_6 \qquad \varepsilon_3' = \varepsilon_3 \quad \varepsilon_6' = -\varepsilon_6$



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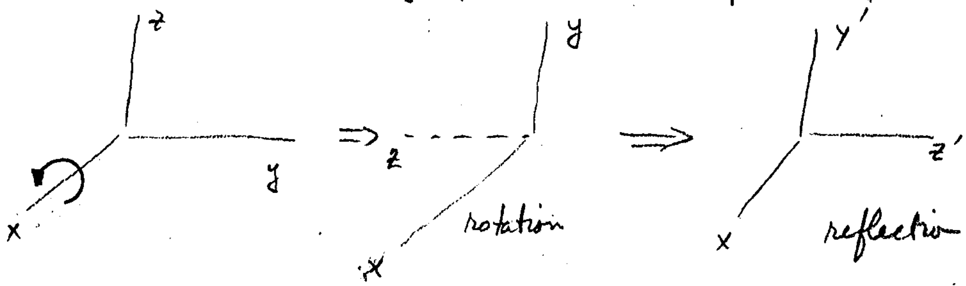
cubic material – interchange of 2 axes rotate by 90° & reflect.



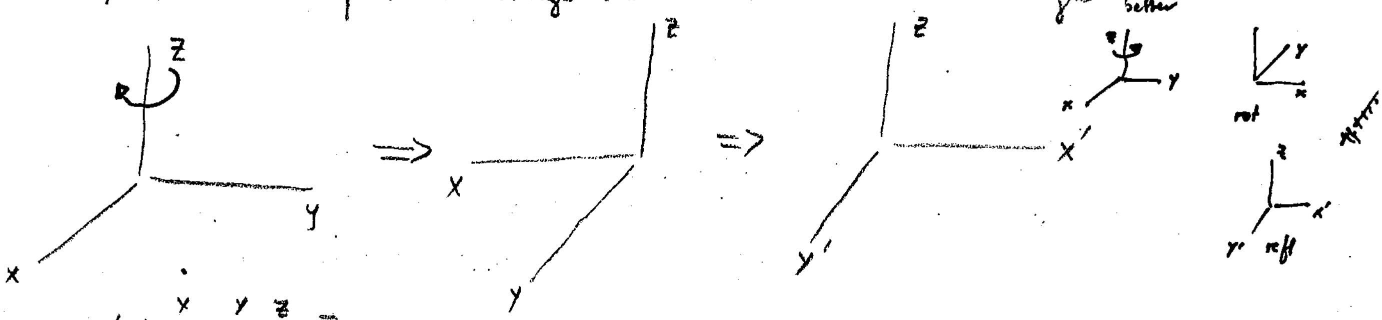
$$\begin{array}{c} \\ x' \\ y' \\ z' \end{array}\begin{array}{c} x \quad y \quad z \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\end{array}$$

$$\left\{\begin{array}{ll} \sigma_1' = \sigma_1 & \sigma_4' = \sigma_4 \\ \sigma_2' = \sigma_3 & \sigma_5' = \sigma_6 \\ \sigma_3' = \sigma_2 & \sigma_6' = \sigma_6 \end{array}\right\} \quad \begin{array}{ll} \epsilon_1' = \epsilon_1 & \epsilon_4' = \epsilon_4 \\ \epsilon_2' = \epsilon_3 & \epsilon_5' = \epsilon_6 \\ \epsilon_3' = \epsilon_2 & \epsilon_6' = \epsilon_6 \end{array}$$

$$\begin{pmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_3' \\ \sigma_4' \\ \sigma_5' \\ \sigma_6' \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_3 \\ \sigma_2 \\ \sigma_4 \\ \sigma_6 \\ \sigma_5 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{13} & C_{12} & & & \\ C_{12} & C_{23} & C_{22} & & & \\ C_{13} & C_{33} & C_{23} & & & \\ & & & C_{44} & & \\ & & & & & C_{55} \\ & & & & C_{66} & \end{pmatrix}\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66} \end{pmatrix}$$

$$\Rightarrow \quad C_{13} = C_{12}, \quad C_{22} = C_{33}, \quad C_{55} = C_{66}$$

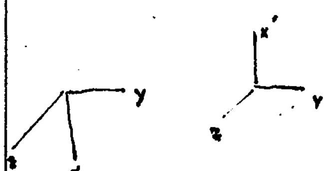
now since this is only 1 interchange we can do another interchange better



$$\begin{array}{c} \\ x' \\ y' \\ z' \end{array}\begin{array}{c} x \quad y \quad z \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{array}$$

using this we will get $C_{11} = C_{22}$, $C_{12} = C_{23}$, $C_{44} = C_{55}$ such that

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & C_{44} \end{pmatrix}$$



to go to isotropic just rotate about the x axis

$$\begin{array}{c|ccc} & x & y & z \\ x' & 1 & 0 & 0 \\ y' & 0 & \cos\theta & \sin\theta \\ z' & 0 & -\sin\theta & \cos\theta \end{array}$$

$\sigma_4' = \frac{(\sigma_3 - \sigma_2)}{2} \sin 2\theta + \sigma_4 (\cos 2\theta)$ but $\sigma_4' = C_{44}\varepsilon_4'$

and $\varepsilon_4' = (\varepsilon_3 - \varepsilon_2) \sin 2\theta + \varepsilon_4 \cos 2\theta$

from the cubic $\sigma_3 - \sigma_2 = (C_{11} - C_{12})(\varepsilon_3 - \varepsilon_2)$ and $\sigma_4 = C_{44}\varepsilon_4$ then we obtain $\sigma_4' = \frac{C_{11} - C_{12}}{2} (\varepsilon_3 - \varepsilon_2) \sin 2\theta + C_{44}\varepsilon_4 \cos 2\theta = C_{44}(\varepsilon_3 - \varepsilon_2)\sin 2\theta + C_{44}\varepsilon_4 \cos 2\theta = C_{44}\varepsilon_4'$

since θ is arbitrary $\Rightarrow$ $C_{44} = \frac{C_{11} - C_{12}}{2}$

if we define $C_{44} = \mu$, $C_{12} = \lambda \Rightarrow C_{11} = \lambda + 2\mu$ λ, μ are Lamé constants.

and $\boxed{\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu \varepsilon_{ij}}$ Generalized Hooke's Law

ε_{kk} is the dilatation change in volume.

We can go directly to isotropic from the general strain energy density
U being a scalar it is invariant under any transformation for an isotropic material

Normally $U = f'(\varepsilon_{ij}') = \frac{1}{2} c_{ijkl}\varepsilon_{ij}\varepsilon_{kl}$ for anisotropic
but $U = f(\varepsilon_{ij})$ for isotropic and since it is invariant under any transformation we can pick U as a fn of the strain invariants or

$U = f(I, II, III)$; since III involves cubic terms of ε_{ij} we can drop it to the first approx $\therefore$

$U = f(I, II) \cong \bar{A} I^2 + B\, II$

where $I = \varepsilon_{qq}$

$II = \varepsilon_{rs}\varepsilon_{rs} - \varepsilon_{qq}\varepsilon_{rr}$

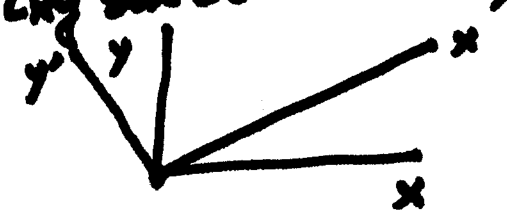
$U \cong \bar{A}\varepsilon_{qq}\varepsilon_{rr} + B\varepsilon_{rs}\varepsilon_{rs} - B\varepsilon_{qq}\varepsilon_{rr}$

$= A\varepsilon_{qq}\varepsilon_{rr} + B\varepsilon_{rs}\varepsilon_{rs}$

$$\sigma_x' = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_y' = (\quad) - (\quad)\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\sigma_x' = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\sin 2\theta$$



	x	y	z
x'	$\cos\theta$	$\sin\theta$	0
y'	$-\sin\theta$	$\cos\theta$	0
z'	0	0	1

$l_{\alpha\beta}$ $\quad$ $\sigma_x', \sigma_y', \tau_{xy}'$

$$\sigma_{mn}' = \sigma_{ij}\, l_{mi}\, l_{nj} \qquad j\ 1,2,3 \quad i\ 1,2,3$$

λ, μ LAME CONST.

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

$$\sigma_x' = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_y' = (\quad) - (\quad)\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\sigma_x' = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\sin 2\theta$$

	x	y	z
x'	$\cos\theta$	$\sin\theta$	0
y'	$-\sin\theta$	$\cos\theta$	0
z'	0	0	1

find $\sigma_x', \sigma_y', \tau_{x'y'}$

$\sigma_{mn} = \sigma_{ij}\, l_{mi}\, l_{nj}$ $\quad$ $i = 1,2,3$; $j = 1,2,3$

λ, μ LAME CONST.

$$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$$

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

Concept of superposition

The effect of a compound cause, say a loading configuration, is the sum of the effects of the individual causes.

Motivation: solve problems involving complex load configurations based on simpler solutions.

Caveat: Superposition depends on linearity, both material and geometrical. Geometrical nonlinear situations include large deformation of a bar in bending, and contact between two spheres.

Safety factor

$$\text{Safety factor SF} = \frac{\text{failure load}}{\text{working load}} \;.$$

Failure does not necessarily mean fracture. It may mean excessive deformation, damage, or any effect which causes the structure or structural element to no longer function as intended.

Example: What if airplane wings could be made of an infinitely strong but not infinitely stiff material?

§2.1 Plane elasticity

Method of solution.

Recall in the mechanics of materials method, we began with an assumption about the deformation field. We only checked the stresses to make sure they agreed with the applied loads in terms of *resultants* .

In the elasticity method, one must simultaneously satisfy:
1. equilibrium conditions in a continuum sense at each point,
2. continuity of the displacement field,
3. boundary conditions at the surface.

If they are all satisfied exactly, we have an exact solution.

The theory of elasticity permits one to deal with problems which are not necessarily geometrically simple.

Few new elasticity solutions are now being discovered. Even so, study of elasticity aids in the development of physical insight.

Stress-strain relations

Elementary form of **Hooke's law** for a linear, isotropic, elastic solid:

$$\varepsilon_{xx} = \frac{1}{E}\ \{\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\}$$

$$\varepsilon_{yy} = \frac{1}{E}\ \{\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}\},$$

$$\varepsilon_{zz} = \frac{1}{E}\ \{\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}\}.$$

These three are complete, but sometimes a shear relation is also presented,

$$\gamma_{xy} = \frac{1}{G}\ \tau_{xy} \;.$$

Plane stress in xy plane means σ_{zz}, σ_{yz}, and σ_{zx} are zero. Then,

$$\varepsilon_{xx} = \frac{1}{E}\ \{\sigma_{xx} - \nu\sigma_{yy}\}$$

$$\varepsilon_{yy} = \frac{1}{E}\ \{\sigma_{yy} - \nu\sigma_{xx}\}$$

$$\gamma_{xy} = \frac{1}{G}\ \tau_{xy}\ .$$

Although they are simpler in the compliance formulation, one can solve for stress and present them in the modulus formulation, for plane stress.

$$\sigma_{xx} = \frac{E}{1 - \nu^2}\ \{\varepsilon_{xx} + \nu\varepsilon_{yy}\}$$

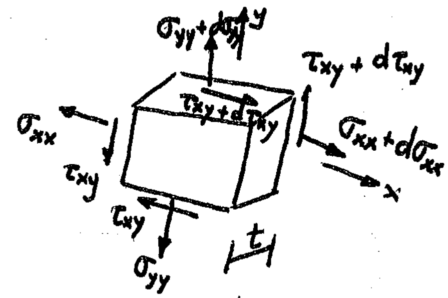
$$\sigma_{yy} = \frac{E}{1 - \nu^2}\ \{\varepsilon_{yy} + \nu\varepsilon_{xx}\}$$

$$\tau_{xy} = G\ \gamma_{xy}\ .$$

§2.2 Equilibrium equations, boundary conditions, Saint Venant's principle

We are familiar with the application of Newton's first law of equilibrium to macroscopic objects. In solving problems on a continuum scale, a differential form of the equations of **equilibrium** is needed.

Consider a free-body diagram of a differential element of thickness t of material.



From sum of forces in the x direction,

$$-t\,\sigma_{xx}\,dy - \tau_{xy}\,t\,dx + \{\sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}dx\}t\,dy + \{\tau_{xy} + \frac{\partial\tau_{xy}}{\partial y}dy\}\,t\,dx = 0$$

From sum of forces in the y direction,

$$-t\,\tau_{xy}\,dy - \sigma_{yy}\,t\,dx + \{\tau_{xy} + \frac{\partial\tau_{xy}}{\partial x}dx\}t\,dy + \{\sigma_{yy} + \frac{\partial\sigma_{yy}}{\partial y}dy\}\,t\,dx = 0$$

Simplifying, the equilibrium equations are:

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0 \text{ in x direction}$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0 \text{ in y direction.}$$

The equilibrium relations in the index notation are as follows for force and moment respectively: The Einstein summation convention assumed in which repeated indices are summed over. The comma represents differentiation with respect to the spatial coordinate corresponding to the index after the comma.

σ_{ji} is stress
G_i is a body force, or force per unit volume.
e_{ijk} is the permutation symbol
m_{ji} is a moment per unit area or couple stress. It is neglected in classical elasticity.
C_i is a body moment, or couple per unit volume.

$$\sigma_{ji,j} + G_i = 0 \qquad (1)$$

$$e_{ijk}\,\sigma_{jk} + m_{ji,j} + C_i = 0 \qquad (2)$$

Body forces arise due to gravitation.
Body moments arise due to electromagnetic interactions in magnetic materials.

Couple stresses represent a distributed average of moments upon fibers, ribs, layers, or other structural elements in composite materials.

In classical elasticity, in the absence of body couples or surface couples, Eq. 2 reduces to
$\sigma_{jk} = \sigma_{kj}$,

that is, the stress is symmetric.
If body couples or surface couples are permitted, the stress can become asymmetric.

Boundary conditions.

Boundary conditions entail prescription of stress or displacement upon the surface of the object in question. In many problems, the surface tractions (stresses at the surface) are zero over much of the surface.

Saint Venant's principle

Saint-Venant's principle is important in the application of elasticity solutions in many practical situations in which boundary conditions are satisfied in the sense of **resultants** rather than pointwise. For example, a bending moment may be applied to a beam via a complex array of bolted joints, which generate a locally complex stress pattern. In view of Saint-Venant's principle, one expects to observe bending type stresses far from the ends.

Saint-Venant's principle states that a localized **self-equilibrated** load system produces stresses which decay with distance more rapidly than stresses due to forces and moments. It is applicable in many situations of interest in engineering.

Demonstration: stress fields for concentrated loads which give rise to compression or bending, as seen with a photoelastic demonstrator.

There are some counter-examples. Consider a sandwich panel with rigid face sheets and an elastic material of Poisson's ratio ν sandwiched between them. For Poisson's ratios in the vicinity of 0.5, stresses applied to the end will decay with distance z as $\sigma(z) \propto e^{-\gamma z}$. The decay rate is

$$\gamma \propto \sqrt{\frac{3(1-2\nu)}{3-4\nu}} \; .$$

The distance $1/\gamma$, over which there is significant stress, diverges as Poisson's ratio approaches $\frac{1}{2}$.

In some thin-walled structures, localized self-equilibrated loads may propagate a significant distance. Saint-Venant's principle is inapplicable for such structures.

Constitutive relations

We mostly deal with linear isotropic elastic materials in this class. Many other possibilities exist.

Anisotropic: Dependent upon direction, referring to the material properties of composites, aggregates, single crystals, and oriented polycrystalline materials.

Creep: Time dependent strain in response to step stress; a manifestation of viscoelastic behavior.

Cubic: A type of anisotropic symmetry in which the unit cells are cube shaped. There are three independent elastic constants. Material is invariant to 90 degree rotations.

Elastic: Stress-strain path for loading is identical to the path for unloading, with immediate recovery to zero upon unloading.

Elastic-perfectly plastic: Elastic up to yield point after which strain increases with no increase in stress.

Elastic-plastic with work hardening: Beyond yield point, stress increases with strain.

Hexagonal: A type of anisotropic symmetry in which the unit cells are hexagonally shaped. Material is invariant to 60 degree rotations about an axis. There are five independent elastic constants. **Transverse isotropy** is mechanically equivalent to hexagonal although the structure may be random in the transverse direction.

Homogeneous: Material properties are identical at every point in the body. Concept of symmetry is expressed here as translational symmetry: material is invariant to translations. Homogeneous materials may be isotropic or anisotropic. At the atomic scale all materials are heterogeneous, but for many engineering applications we may view them as continuous media.

Isotropic: Independent of direction, referring to material properties. There are two independent elastic constants for a linearly elastic material. Engineering constants are E, G, B, ν, but they are interrelated.

Linear: Stress is proportional to strain, assuming all other variables upon which stress or strain might depend are held constant.

Orthotropic: A type of anisotropic symmetry in which the unit cells are shaped like rectangular parallelepipeds. In crystallography, this is called orthorhombic. There are nine independent elastic constants. Principal directions are mutually orthogonal. Material is invariant to reflections in two or three orthogonal planes.

Piezoelectric: In some crystalline or polycrystalline materials which lack a center of symmetry, there is coupling in which both stress and electric field contribute to the strain.

Thermoelastic: In all materials with a nonzero coefficient of thermal expansion, there is thermoelastic coupling in which both stress and temperature changes contribute to the strain.

Triclinic: A type of anisotropic symmetry in which the unit cells are oblique parallelepipeds with unequal sides and angles. There are 21 independent elastic constants.

Viscoelastic: Relation between stress and strain depends upon time or upon frequency.

Principal stress

Principal stresses are normal stresses which act on mutually perpendicular planes. They include the absolute largest and smallest normal stresses at a given point.

Recall Mohr's circle is a tool for 2-D transformations of stress and special 3-D transformations. It can be used to determine principal stresses under those special circumstances. It is not applicable to general 3-D transformations.

Consider a free body diagram of a cut corner of a unit cube. This is called a Cauchy tetrahedron.

CONCEPT OF ELASTIC STRAIN ENERGY

- DEFORMATION OF BODY UNDER EXTERNAL LOADS - LOADS DO WORK
- ASSUME - NO KINETIC OR HEAT EXCHANGE
  - GRADUAL INCREASE IN LOAD FROM INITIAL TO FINAL STATE
  - CONSERVATION OF ENERGY ⇒ STRAIN ENERGY IS POTENTIAL ENERGY
- FROM STATICS REMEMBER IF $\vec{F}$ CONSTANT OR NOT

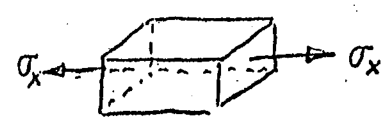
  WORK DONE ON UNIT VOLUME IS $\vec{F}\cdot d\vec{x} = \bar{\sigma}A \cdot d\bar{\epsilon}\, l = \bar{\sigma}\cdot d\bar{\epsilon}\,(Al)$
- TOTAL WORK DONE BY FORCE IS STORED AS STRAIN ENERGY GIVEN BY

$$\int_V \left[\int \bar{\sigma}\cdot d\bar{\epsilon}\right] dV = \int_V \left[\frac{\sigma^2}{2E}\right] dV = \int_V \left[\frac{E}{2}\epsilon^2\right] dV = \int \frac{\bar{\sigma}\cdot\bar{\epsilon}}{2}\, dV$$

- REMEMBER WORK IS ADDITIVE AND DEPENDS ON FINAL & INITIAL STATES AND NOT ON PATH BETWEEN STATES
- IF WE HAVE A BODY THAT OBEYS HOOKE'S LAW AND WE APPLY A FORCE IN THE X-DIRECTION ONLY

$$\sigma_x = E\epsilon_{x_1}, \quad \epsilon_{y_1} = -\nu\epsilon_{x_1}, \quad \epsilon_{z_1} = -\nu\epsilon_{x_1} \qquad \nu - \text{POISSON RATIO}$$

- SINCE σ_x, ϵ_{x_1} ARE PARALLEL TO EACH OTHER, WORK IS DONE
- SINCE σ_x, ϵ_{y_1} OR ϵ_{z_1} ARE ⊥ TO EACH OTHER, NO WORK DONE
- WORK DUE TO σ_x IS $\int \frac{\sigma_x \epsilon_{x_1}}{2}\, dV$



- IF WE NOW APPLY TO THIS STATE AN ADDITIONAL FORCE IN THE Y-DIRECTION KEEPING σ_x FIXED

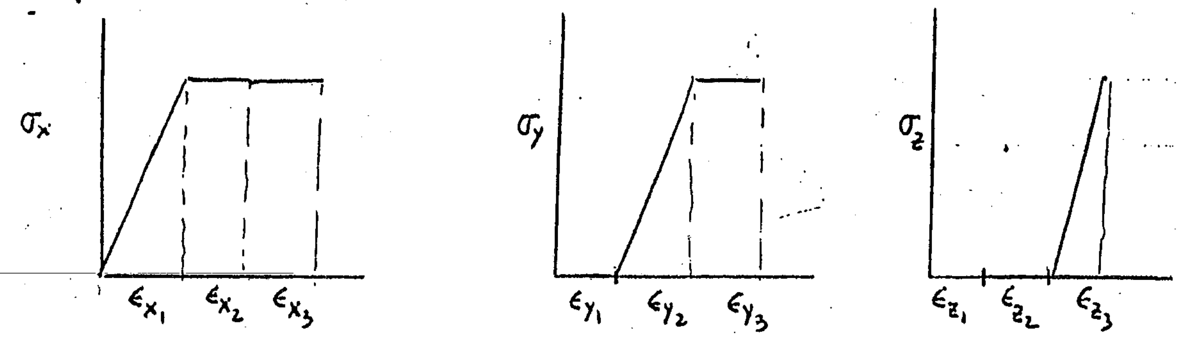
$$\sigma_y = E\epsilon_{y_2} \qquad \epsilon_{x_2} = -\nu\epsilon_{y_2} \qquad \epsilon_{z_2} = -\nu\epsilon_{y_2}$$

- SINCE σ_y, ϵ_{y_2} ARE PARALLEL, WORK IS DONE
- SINCE σ_y, ϵ_{x_2} OR ϵ_{z_2} ARE ⊥ TO EACH OTHER, NO WORK DONE
- BUT

  σ_x FROM BEFORE DOES DO WORK DUE TO ϵ_{x_2}

- THUS ADDITIVE WORK DUE TO σ_y IS

$$\int_V \left[\frac{\sigma_y \epsilon_{y_2}}{2} + \sigma_x \epsilon_{x_2}\right] dV$$



- SIMILARLY IF WE APPLY A FORCE IN THE Z DIRECTION ONLY KEEPING σ_x, σ_y FIXED : $\sigma_z = E\epsilon_{z_3}$, $\epsilon_{x_3} = -\nu\epsilon_{z_3}$, $\epsilon_{y_3} = -\nu\epsilon_{z_3}$
- A SIMILAR ARGUMENT YIELDS THE ADDITIVE WORK DONE

$$\int_V \left[\frac{\sigma_z \epsilon_{z_3}}{2} + \sigma_x \epsilon_{x_3} + \sigma_y \epsilon_{y_3}\right] dV$$

- BY ADDING THE THREE TERMS WE GET THE TOTAL WORK DONE...

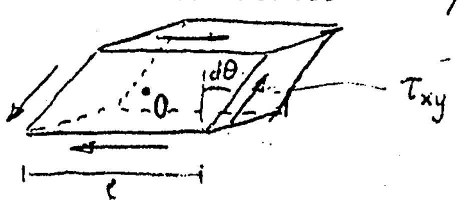
$$\int \left[\frac{\sigma_x \epsilon_{x_1}}{2} + \frac{\sigma_y \epsilon_{y_2}}{2} + \sigma_x \epsilon_{x_2} + \frac{\sigma_z \epsilon_{z_3}}{2} + \sigma_x \epsilon_{x_3} + \sigma_y \epsilon_{y_3}\right] dV$$

  - NOTE THAT $\frac{1}{2}\sigma_x \epsilon_{x_2} = \frac{1}{2}E\epsilon_{x_1} \cdot (-\nu \epsilon_{y_2}) = \frac{1}{2}E\epsilon_{y_2} \cdot (-\nu\epsilon_{x_1}) = \sigma_y \epsilon_{y_1}/2$
  - SIMILARLY $\frac{1}{2}\sigma_x \epsilon_{x_3} = \frac{1}{2}\sigma_z \epsilon_{z_1}$ & $\frac{1}{2}\sigma_y \epsilon_{y_3} = \frac{1}{2}\sigma_z \epsilon_{z_2}$

- THUS TOTAL WORK DONE IS $\frac{1}{2}\int \left[\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z\right] dV$

  WHERE $\epsilon_x = \epsilon_{x_1} + \epsilon_{x_2} + \epsilon_{x_3}$, $\epsilon_y = \epsilon_{y_1} + \epsilon_{y_2} + \epsilon_{y_3}$ ...

- LOOK AT SHEAR FORCES THEY DO WORK THRU SHEAR STRAINS $d\gamma$



- CAN DO WORK BY FORCE COUPLE CAUSING BODY TO ROTATE ABOUT AN AXIS THROUGH 0
- FROM STATICS WORK DONE BY MOMENT IS $\vec{M}\cdot d\vec{\theta} = \bar{T}\ell \cdot d\bar{\theta} = \bar{\tau} A\ell \cdot d\bar{\theta}$
$= \bar{\tau}\cdot d\bar{\gamma}\ A\ell$
- WORK DONE $= \int_V \left[\int \bar{\tau}\cdot d\bar{\gamma}\,\right] dV$
- FOR A BODY OBEYING HOOKE'S LAW $\int \bar{\tau}\cdot d\bar{\gamma} = \frac{\tau^2}{2G} = \frac{G\gamma^2}{2} = \frac{\tau\cdot\gamma}{2}$

- BY SIMILAR MANNER WORK DONE BY SHEAR FORCES ARE

$$\int_V \left[\frac{\tau_{xy}\gamma_{xy}}{2} + \frac{\tau_{xz}\gamma_{xz}}{2} + \frac{\tau_{yz}\gamma_{yz}}{2} \right] dV$$

- TOTAL WORK DONE DUE TO ALL STRESSES IS SUM OF THE TWO

$$\int_V \frac{1}{2}\left[\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}\right] dV = \int_V U_{s_0}\, dV = U_s$$

- U_{s_0} IS THE STRAIN ENERGY DENSITY ; ALWAYS ≥ 0
- U_{s_0} IS ZERO ONLY WHEN ALL σ's, ϵ's, τ's AND γ's $= 0$

remember $\sigma_{ij} = \bar{c}_{ijkl}\,\epsilon_{kl}$ or $\sigma_i = c_{ij}\epsilon_j$ & $c_{ij} = c_{ji}$

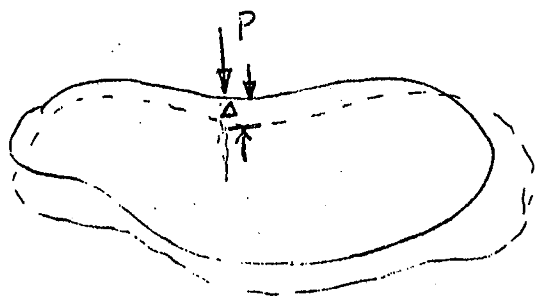
- NOTE THAT SINCE $dU_{s_0} = \frac{1}{2}\left(\sigma_x\, d\epsilon_x + \sigma_y\, d\epsilon_y + \sigma_z\, d\epsilon_z + \tau_{xy}\, d\gamma_{xy} + \tau_{xz}\, d\gamma_{xz} + \tau_{yz}\, d\gamma_{yz}\right)^{(*)}$
AND dU_{s_0} is perfect DIFFERENTIAL, MEANING THAT

$$dU_{s_0} = \frac{\partial U_{s_0}}{\partial \epsilon_x} d\epsilon_x + \frac{\partial U_{s_0}}{\partial \epsilon_y} d\epsilon_y + \frac{\partial U_{s_0}}{\partial \epsilon_z} d\epsilon_z + \frac{\partial U_{s_0}}{\partial \gamma_{xy}} d\gamma_{xy} + \frac{\partial U_{s_0}}{\partial \gamma_{xz}} d\gamma_{xz} + \frac{\partial U_{s_0}}{\partial \gamma_{yz}}$$

by putting dfn of $\sigma_i = c_{ij}\epsilon_j$ into (*) and differentiating with each of the ϵ_j

$\Rightarrow \frac{\partial U_{s_0}}{\partial \epsilon_x} = \sigma_x$ AND $\frac{\partial U_{s_0}}{\partial \epsilon_y} = \sigma_y$ ETC.

- SINCE σ IS RELATED TO A LOAD AND ϵ IS RELATED TO A DISPLACEMENT IN THE DIRECTION OF THAT LOAD $\Rightarrow$ WE CAN DETERMINE THE LOAD IF WE KNOW HOW THE STRAIN ENERGY VARIES WITH THE DISPLACEMENT IN THE DIRECTION OF THAT LOAD.
- REMEMBER $U = V_c \cdot V_0 \ell$



$$U_s = U_s(\Delta) \Rightarrow \frac{\partial U_s}{\partial \Delta} = P$$

THIS FACT WILL BE USED LATER

- WE HAVE CONSIDERED WORK DONE $= \vec{F} \cdot d\vec{x}$

  IF $\vec{F}$ is CONSTANT $\vec{F} \cdot d\vec{x} = d(\vec{F} \cdot \vec{x})$ — MOST GENERAL EXPRESSION FOR WORK

- WHAT IF $\underset{\sim}{X}$ IS NOW CONSTANT AND $\underset{\sim}{F}$ CHANGES ie $d(\vec{F} \cdot \vec{x}) = \vec{x} \cdot d\vec{F}$

  - THIS ALSO CAUSE WORK TO BE DONE

$$d\vec{F} = d\vec{\sigma} \cdot A \qquad \vec{X} = l\vec{\epsilon} \qquad \Rightarrow \quad \vec{x} \cdot d\vec{F} = \vec{\epsilon} \cdot d\vec{\sigma} \quad (lA)$$

- JUST AS BEFORE WE CAN GO THROUGH THE PROCESS AND SHOW THAT

$$\text{TOTAL WORK DONE} = \int_V \frac{1}{2} \int (\bar{\epsilon}_x \cdot d\bar{\sigma}_x + \bar{\epsilon}_y \cdot d\bar{\sigma}_y + \bar{\epsilon}_z \cdot d\bar{\sigma}_z + \bar{\gamma}_{xy} \cdot d\bar{\tau}_{xy} + \bar{\gamma}_{xz} \cdot d\bar{\tau}_{xz} + \bar{\gamma}_{yz} \cdot d\bar{\tau}_{yz}) dV^{(**)}$$

just as $\sigma_{ij} = C_{ijkl}\epsilon_{kl}$ or $\sigma_i = C_{ij}\epsilon_j$ we can write $\epsilon_j = a_{ij}\sigma_j$ & $a_{ij} = a_{ji}$

- THE INNER INTEGRAL IS THE COMPLEMENTARY ENERGY OF THE BODY : U_{co}
- $dU_{co} = \frac{1}{2}(\bar{\epsilon}_x \cdot d\bar{\sigma}_x + \bar{\epsilon}_y \cdot d\bar{\sigma}_y + \bar{\epsilon}_z \cdot d\bar{\sigma}_z + \bar{\gamma}_{xy} \cdot d\bar{\tau}_{xy} + \bar{\gamma}_{xz} \cdot d\bar{\tau}_{xz} + \bar{\gamma}_{yz} \cdot d\bar{\tau}_{yz})$
  - IT IS ALSO A PERFECT DIFFERENTIAL SO THAT

$$dU_{co} = \left(\frac{\partial U_{co}}{\partial \sigma_x} d\sigma_x + \frac{\partial U_{co}}{\partial \sigma_y} d\sigma_y + \frac{\partial U_{co}}{\partial \sigma_z} d\sigma_z + \frac{\partial U_{co}}{\partial \tau_{xy}} d\tau_{xy} + \frac{\partial U_{so}}{\partial \tau_{xz}} d\tau_{xz} + \frac{\partial U_{so}}{\partial \tau_{yz}} d\tau_{yz} \right)$$

now by putting $\epsilon_i = a_{ij}\sigma_j$ into (**) and differentiating wrt ϵ_i we get

AND $\frac{\partial U_{co}}{\partial \sigma_x} = \epsilon_x \quad \frac{\partial U_{co}}{\partial \sigma_y} = \epsilon_y$ etc. $\frac{\partial U_{co}}{\partial \tau_{xy}} = \gamma_{xy}$

- HERE IF WE KNOW HOW THE COMPLEMENTARY ENERGY CHANGES AS LOAD CHANGES THE WE CAN FIND THE DISPLACEMENT, DUE TO THAT LOAD, IN DIRECTION OF LOAD

- REMEMBER σ_x CAN BE RELATED TO LOAD IN X-DIRECTION

  ϵ_x CAN BE RELATED TO DISPLACEMENT IN DIRECTION OF LOAD

- WHAT WE SAID ABOUT U_{s_0} & U_{c_0} IS TRUE EVEN IF BODY DOESN'T OBEY HOOKE'S LAW.
- IF BODY IS LINEALY ELASTIC : $U_{c_0} = U_{s_0}$
- NOTE : ALWAYS WRITE U_s IN TERMS OF STRAINS / DISPLACEMENTS
  U_c IN TERMS OF STRESSES / LOADS
- CLARIFY SOME PTS
  - FROM STATICS: BODY WAS RIGID - NO WORK DONE DUE TO INTERNAL LOADS
  - FOR A BODY THAT DEFORMS WORK IS DONE BY INTERNAL FORCES (U_c , U_s)
  - WHAT WE'VE JUST DISCUSSED IS WORK DONE BY INTERNAL FORCES
  - BOTH FOR NON DEFORMABLE & DEFORMABLE BODIES, THE EXTERNAL FORCES ALSO DO WORK
  - THE EXPRESSIONS FOR U_c, U_{c_0}, U_s, U_{s_0} HOLD FOR NON-LINEARLY ELASTIC BODIES AS WELL
  - TOTAL WORK DONE BY BODY THAT HAS EXTERNAL LOADS APPLIED AND UNDERGOES DEFORMATION IS $W_i + W_e = \Pi$
- $$W_e = \int_S (X u + Y v + Z w)\, dS$$
  S - surface of body
  u, v, w - displacements undergone by forces X, Y, Z applied to surface
- ASSUMPTION - BODY FORCES (LIKE WEIGHT) CAN BE ACCOUNTED FOR THROUGH W_e TERM.
- WHEN AN ELASTIC BODY IS AT REST, THE EXTERNAL FORCES + BODY FORCES + INTERNAL FORCES ARE IN A STATE OF EQUILIBRIUM

- WORK DONE BY THESE THREE SET OF FORCES IS AT A MINIMUM
- THUS $\delta\pi = \delta W_i + \delta W_e = 0$
- ALSO $\delta W_i = -\delta U_s$ REMEMBER FROM STATICS POTENTIAL ENERGY IS NEGATIVE OF WORK DONE.

- THUS FOR ANY CHANGE IN DISPLACEMENTS OF THE BODY THAT KEEPS IT IN EQUILIBRIUM

$$\delta\pi / \delta\,\text{displacements} = 0 = -\frac{\delta U_s}{\delta \text{displ.}} + \frac{\delta W_e}{\delta \text{displ.}}$$

- BUT $\frac{\delta U_s}{\delta \text{displ}} = \frac{\delta W_e}{\delta \text{displ}}$ = load, due to that displ, IN DIRECTION OF DISPLACEMENTS.

- THIS IS CASTIGLIANO'S THEOREM (FIRST)
- SIMILARLY SINCE $\delta W_i = -\delta U_s = -\delta U_c$
- FOR ANY CHANGE IN LOADS OF THE BODY THAT KEEPS IT IN EQUILIBRIUM

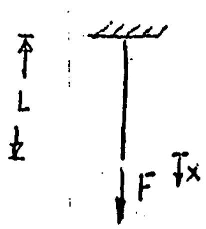
$$\delta\pi / \delta\,\text{LOADS} = 0 = -\frac{\delta U_c}{\delta \text{LOAD}} + \frac{\delta W_e}{\delta \text{LOAD}}$$

- BUT $\frac{\delta U_c}{\delta \text{LOAD}} = \frac{\delta W_e}{\delta \text{LOAD}}$ = DISPL, DUE TO THAT LOAD, IN DIR. OF LOAD

- THIS IS CASTIGLIANO'S THEOREM (SECOND)

→ USING U_c TO DETERMINE DISPLACEMENTS ←

EXAMPLE #1 EXTENSIBLE ROD



$\sigma = F/A \qquad \epsilon = \frac{\sigma}{E} = \frac{F}{AE} = \frac{x}{L}$

$U_c = \frac{\sigma^2}{2E} \cdot AL = \frac{F^2 L}{2EA}$ ($U_{c_0} \cdot \text{Vol}$) $\qquad U_s = \frac{E\epsilon^2}{2} \cdot AL = \frac{Ex^2}{2L} A$ ($U_{s_0} \cdot \text{Vol}$)

THUS $U_c = \frac{F^2 L}{2EA} \qquad U_s = \frac{Ex^2}{2L} A$

$W_e = Fx$

TO FIND F, ASSUMING x IS KNOWN, DEFINE Π IN TERMS OF U_s & W_e

$$W_i + W_e = \Pi = -\frac{Ex^2}{2L}A + Fx = -U_s + W_e$$

$$\frac{\partial \Pi}{\partial x} = -\frac{2ExA}{2L} + F = 0 \qquad F = x\frac{EA}{L} \qquad \left(\frac{\partial U_s}{\partial x} = F\right)$$

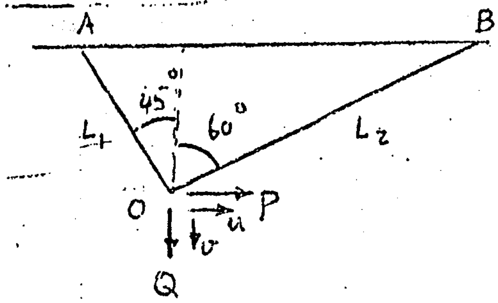
TO FIND x, ASSUMING F IS KNOWN, DEFINE Π IN TERMS OF U_c & W_e

$$W_i + W_e = -U_c + W_e = \Pi = -\frac{F^2L}{2EA} + Fx$$

$$\frac{\partial \Pi}{\partial F} = -\frac{2FL}{2EA} + x = 0 \qquad x = \frac{FL}{AE} \qquad \left(\frac{\partial U_c}{\partial F} = x\right)$$

- WE WILL LOOK AT TRUSSES WHERE EXTENSION/COMPRESSION IS <u>PRIMARY LOADING</u>

<u>EXAMPLE #2 : TWO-ROD TRUSS</u>

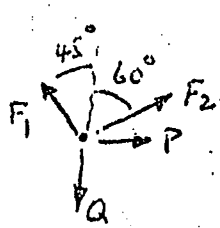


STATICALLY DETERMINATE SYSTEM

LOOK AT TWO BARS CONNECTED AT O, BARS ARE EXTENSIBLE — HAVE THE SAME CROSS SECTION, A, AND YOUNG'S MODULUS, E.

→ WANT TO FIND DISPLACEMENTS u, v GIVEN P AND Q

1) USE STATICS TO FIND FORCES IN OB & OA
2) DETERMINE U_c
3) $\frac{\partial U_c}{\partial P} = u \qquad \frac{\partial U_c}{\partial Q} = v$



$$\left.\begin{aligned} Q &= F_2\cos 60° + F_1 \cos 45° \\ P &= F_1 \sin 45° - F_2 \sin 60° \end{aligned}\right\} \quad \begin{aligned} F_1 &= \frac{\sqrt{3}-1}{\sqrt{2}}(P+\sqrt{3}Q) \\ F_2 &= (\sqrt{3}-1)(Q-P) \end{aligned}$$

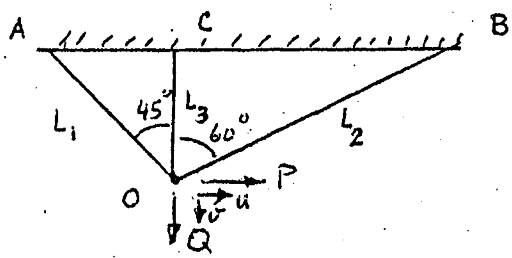
<u>NOTE</u> F_1 & F_2 ARE FNS OF P & Q

$$\text{Now} \quad U_c = \sum \frac{F^2L}{2AE} = \frac{F_1^2 L_1}{2AE} + \frac{F_2^2 L_2}{2AE} = \frac{1}{2AE}\left\{\left[\frac{\sqrt{3}-1}{\sqrt{2}}(P+\sqrt{3}Q)\right]^2 L_1 + \left[(\sqrt{3}-1)(Q-P)\right]^2 L_2\right\}$$

$$u = \frac{\partial U_c}{\partial P} = \frac{2F_1L_1}{2AE}\frac{\partial F_1}{\partial P} + \frac{2F_2L_2}{2AE}\frac{\partial F_2}{\partial P} = \frac{\sqrt{3}-1}{\sqrt{2}}(P+\sqrt{3}Q)\frac{L_1}{AE}\cdot\frac{\sqrt{3}-1}{\sqrt{2}} + (\sqrt{3}-1)(Q-P)\frac{L_2}{AE}\left\{-(\sqrt{3}-1)\right\}$$

$$v = \frac{\partial U_c}{\partial Q} = \frac{2F_1L_1}{2AE}\frac{\partial F_1}{\partial Q} + \frac{2F_2L_2}{2AE}\frac{\partial F_2}{\partial Q} = \frac{L_1}{AE}\left\{\frac{\sqrt{3}-1}{\sqrt{2}}(P+\sqrt{3}Q)\cdot\frac{\sqrt{3}}{\sqrt{2}}(\sqrt{3}-1)\right\} + \frac{L_2}{AE}\left\{(\sqrt{3}-1)^2(Q-P)\right\}$$

EXAMPLE #3 - INDETERMINATE (STATICALLY) TRUSS



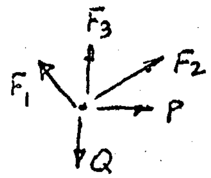
$L_3 = L \qquad L_1 = \sqrt{2}\,L \qquad L_2 = 2L$

GIVEN: A, E SAME FOR ALL THREE

WANT TO FIND DISPLACEMENTS u, v

GIVEN P & Q

STATICALLY INDETERMINATE: 3 FORCES, 2 EQS.



$$Q = F_3 + F_1\cos 45° + F_2\cos 60°$$
$$P = F_1\sin 45° - F_2\sin 60°$$

} NOTE P & Q ARE FNS OF F_1, F_2, F_3

• NOTE THIS WILL GIVE SAME SOLUTION FOR F_1 & F_2 IF $Q - F_3$ REPLACES Q

TO FIND F_3 : 1) FIND U_c FIRST $\quad U_c = \sum \frac{F^2L}{2AE} = \frac{F_1^2L_1}{2AE} + \frac{F_2^2L_2}{2AE} + \frac{F_3^2L_3}{2AE}$

2) TAKE $\frac{\partial U_c}{\partial F_3} = 0$ THIS GIVES 3rd EQ. NEEDED

$$U_c = \frac{1}{2AE}\left\{\left[\frac{\sqrt{3}-1}{\sqrt{2}}\left(P+\sqrt{3}(Q-F_3)\right)\right]^2 L_1^* + \left[(\sqrt{3}-1)(Q-F_3-P)\right]^2 L_2^* + F_3^2L_3^*\right\}$$

$$\frac{\partial U_c}{\partial F_3} = \frac{F_1L_1}{AE}\frac{\partial F_1}{\partial F_3} + \frac{F_2L_2}{AE}\frac{\partial F_2}{\partial F_3} + \frac{F_3L_3}{AE} = \frac{\sqrt{3}-1}{\sqrt{2}}\left[P+\sqrt{3}(Q-F_3)\right]\frac{L_1}{AE}\left[-\sqrt{3}\frac{(\sqrt{3}-1)}{\sqrt{2}}\right] -$$

$$+ (\sqrt{3}-1)(Q-F_3-P)\frac{L_2}{AE}\left[-(\sqrt{3}-1)\right] + \frac{F_3L_3}{AE} = 0$$

remember $L_3 = L$, $L_1 = \sqrt{2}L$, $L_3 = 2L$

3) SOLVE FOR F_3 IN TERMS OF KNOWN FORCES P & Q. Since $\frac{L}{AE} \neq 0$, divide out

$$F_3 = -0.01295P + 0.6883Q$$

4) PUT THIS INTO $F_1 = \frac{\sqrt{3}-1}{\sqrt{2}}\left(P+\sqrt{3}(Q-F_3)\right)$; $F_2 = (\sqrt{3}-1)(Q-F_3-P)$

TO FIND F_1 & F_2 IN TERMS OF P & Q namely $F_1 = .6337P + .2794Q$
$F_2 = -.6373P + .2282Q$

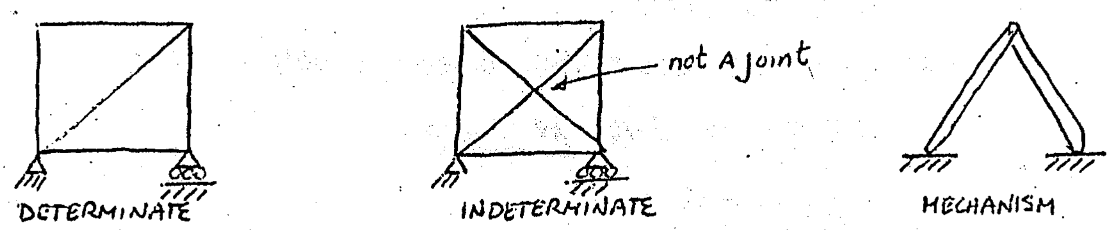
5) TAKE $\frac{\partial U_c}{\partial P}$ TO GET u & $\frac{\partial U_c}{\partial Q}$ TO GET v since $U_c = U_c$

$u = \frac{L}{AE}(1.3969P - .1295Q) \qquad v = \frac{L}{AE}(-.1295P + .6883Q)$

ASIDE

- WHEN IS TRUSS STATICALLY DETERMINATE
  - MUST CHECK BOTH EXTERNAL & INTERNAL CONDITIONS
  - FOR A TRUSS HAVING j JOINTS & n BARS

EQNS OF EQUILIB.
- IF $2j - 3 = n$ STATICALLY DETERMINATE INTERNALLY
- IF $< n$ STATICALLY INDETERMINATE INTERNALLY
- IF $> n$ IT IS A MECHANISM AND IT IS STATICALLY DETERMINATE



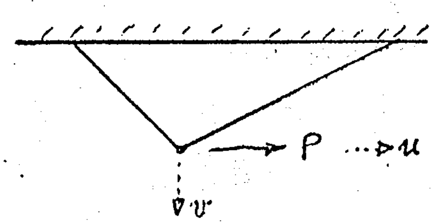
- FOR A SPACE TRUSS $3j - 6 = n$ (3-D TRUSS)

- EXTERNAL EQNS OF EQUILIB. $\Sigma F_x = 0 \quad \Sigma F_y = 0 \quad \Sigma M = 0$
  - IF NO. OF UNKNOWNS $>$ NO. OF EQNS STATICALLY INDETERMINATE
  - IF " " " $\leq$ " " " DETERMINATE

- FOR A SPACE TRUSS EQS OF EQUILIB. $\Sigma F_x, \Sigma F_y, \Sigma F_z, \Sigma M_x, \Sigma M_y, \Sigma M_z = 0$

- WE SEE WE CAN FIND DISPLACEMENT IN DIRECTION OF FORCE USING CASTIGLIANO'S THEOREM $\left(\frac{\partial U_c}{\partial P} = u \right)$

- WHAT IF WE WANT DISPLACEMENT OF A POINT WHERE THERE IS NO FORCE APPLIED?

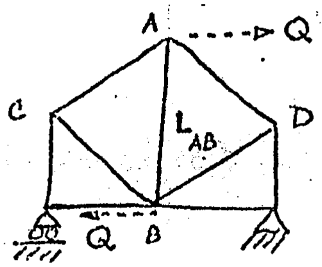


$$\frac{\partial U_c}{\partial P} = u$$

HOW DO WE GET v?

- PUT A FICTITIOUS FORCE Q IN DIRECTION OF v; FIND U_c; TAKE $\lim_{Q \to 0} \frac{\partial U_c}{\partial Q} = v$

- CAN WE USE CASTIGLIANO'S THEOREM TO DETERMINE ROTATIONS OF A BAR IN A TRUSS? YES
- HOW? 1) APPLY A <u>COUPLE</u> WHOSE FORCES ARE ⊥ TO BAR
  2) FIND U_c DUE TO THAT COUPLE, AFTER FINDING FORCES IN BARS OF TRUSS USING EQUILIB
  3) TAKE $\frac{\partial U_c}{\partial (\text{FORCE OF COUPLE})}$; THEN TAKE limit AS THAT FORCE GOES TO ZERO
  4) TAKE THE RESULT AND DIVIDE BY LENGTH OF BAR. THIS GIVES ROTATION OF BAR <u>IN RADIANS</u>



FIND $U_c = \sum \frac{F_i^2 L_i}{2A_i E_i}$

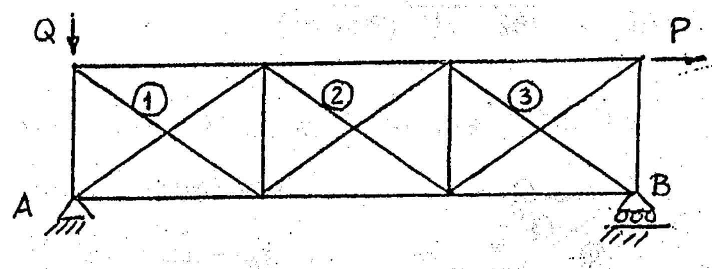
- HERE THE F_i's WOULD BE FUNCTIONS OF Q.
- NOW TAKE $\frac{\partial U_c}{\partial Q}$; TAKE $\lim_{Q\to 0} \frac{\partial U_c}{\partial Q} = L_{AB}\theta_{AB}$
- NOW TAKE $\frac{1}{L_{AB}} \cdot (L_{AB}\theta_{AB}) = \theta_{AB}$ IN RADIANS

- ALSO NOTE THAT $L_{AB} Q$ REPRESENTS THE MOMENT OF THE COUPLE
- WHAT IF <u>AT A</u> A FORCE P EXISTED ALREADY AND YOU WANTED THE ROTATION OF BAR AB? <u>MUST ADD THE COUPLE IN ADDITION TO THE EXISTING FORCE SYSTEM</u>.

- THE ABOVE SYSTEM IS DETERMINATE. WHAT IF A BAR WERE PLACED ACROSS CD MAKING THE SYSTEM INDETERMINATE. HOW WOULD YOU PROCEED? 1) WRITE EQUILIBRIUM EQS AND FIND FORCES IN TERMS OF THE INDETERMINATE FORCE F_{CD} (AS WE DID BEFORE)
  2) FIND U_c ; TAKE $\frac{\partial U_c}{\partial F_{CD}} = 0$ TO GET F_{CD}
  3) SUBSTITUTE TH RESULT INTO THE FORCES FOUND FROM EQUILIBRIUM; FORM U_c ; TAKE $\frac{\partial U_c}{\partial Q}\Big|$ $= L_{AB}\theta_{AB}$ ETC.

- THE DEGREE OF INDETERMINACY IS THE DIFFERENCE BETWEEN n AND $2j-3$ FOR A PLANE TRUSS OR n AND $3j-6$ FOR A SPACE TRUSS. THIS IS THE NUMBER OF EQUATIONS NEEDED VIA CASTIGLIANO'S THEOREM

EXAMPLE



8 JOINTS

$2j-3=13$

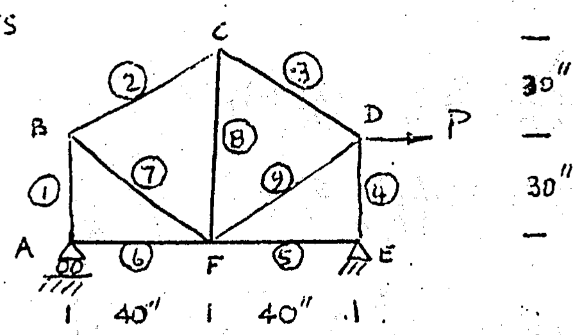
$n=16$

$n-(2j-3)=3$

- THIS IS DEGREE OF INDETERMINACY 3
- THUS AFTER YOU FIND U_c THEN $\frac{\partial U_c}{\partial F_1}=0$; $\frac{\partial U_c}{\partial F_2}=0$; $\frac{\partial U_c}{\partial F_3}=0$
- THESE GIVE THE REQUIRED EQ'S TO SOLVE FOR F_1, F_2, F_3

- TO FIND THE REACTION FORCES AT A & B: USE TRUSS METHODS LEARNED IN STATICS OR USE CASTIGLIANO'S THEOREM WITH $u_A = v_A = 0$ AND $v_B = 0$

EXAMPLES



$P = 4000$ lb

$A = .1 \text{ in}^2$

$E = 30\times10^6$ psi

$CF = 60''$ $BC, BF, FD, CD = 50''$

$BA = DE = 30'' = L$

THIS IS STATICALLY DETERMINATE: EXTERNAL/INTERNAL

USE EXTERNAL EQUILIBRIUM TO FIND AT A $V_A = \frac{3P}{8}\downarrow$

E $H_E = P \leftarrow$ $V_E = \frac{3P}{8}\uparrow$

USE JOINT METHOD OF STATICS TO FIND

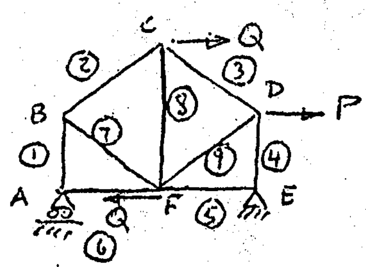
$F_1 = -F_4 = \frac{3P}{8} = -\frac{F}{8}$; $F_2 = F_3 = -F_7 = \frac{5P}{16}$; $F_6 = 0$; $F_5 = -P$; $F_9 = \frac{15P}{16}$

$$U_c = \sum \frac{F_i^2 L_i}{2A_i E_i} = \frac{L}{2AE}\left\{\left(\frac{3P}{8}\right)^2\cdot 1 + \left(\frac{5P}{16}\right)^2\cdot \frac{5}{3} + \left(\frac{5P}{16}\right)^2\cdot\frac{5}{3} + \left(-\frac{3P}{8}\right)^2\cdot 1 + (-P)^2\cdot\frac{4}{3}\right.$$

$$\left. + (0)^2\cdot \frac{4}{3} + \left(-\frac{5P}{16}\right)^2\cdot\frac{5}{3} + \left(-\frac{3P}{8}\right)^2\cdot 2 + \left(\frac{15P}{16}\right)^2\cdot\frac{5}{3}\right\} = \frac{739}{384}\frac{P^2L}{AE}$$

$$u_d = \frac{\partial U_c}{\partial P} = \frac{739}{192}\frac{PL}{AE} = \frac{739\,(4000)(30'')}{192\ \ .1\ \ (30\times 10^6)} = .154\,in \qquad \text{DISPL OF D DUE TO P}$$

TO FIND ROTATION OF CF, ASSUME LOADS Q AT C & F



USE EQUILIB. EQNS TO FIND

$V_E = \frac{3P}{8} + \frac{3Q}{4}$ ↑ $\quad V_A = \frac{3P}{8} + \frac{3Q}{4}$ ↓

$H_E = P$ ←

USE JOINT EQUIL. TO FIND : $F_1 = -F_4 = \frac{3P}{8} + \frac{3Q}{4}$; $F_2 = -F_7 = \frac{5P}{16} + \frac{5Q}{8}$; $F_3 = \frac{5P}{16} - \frac{5Q}{8}$

$F_5 = -P + 0$; $F_6 = 0 + 0$; $F_8 = -\frac{3P}{8} + 0$; $F_9 = \frac{15P}{16} + \frac{5Q}{8}$

$$U_c = \sum \frac{F_i^2 L_i}{2A_i E_i} = \frac{L}{2AE}\left\{\left(\frac{3P}{8}+\frac{3Q}{4}\right)^2\cdot 1 + \left(\frac{5P}{16}+\frac{5Q}{8}\right)^2\cdot\frac{5}{3} + \left(\frac{5P}{16}-\frac{5Q}{8}\right)^2\cdot\frac{5}{3} + \left(-\frac{3P}{8}-\frac{3Q}{4}\right)^2\cdot 1\right.$$

$$+ (-P+0)^2\cdot\frac{4}{3} + (0+0)^2\cdot\frac{4}{3} + \left(-\frac{5P}{16}-\frac{5Q}{8}\right)^2\cdot\frac{5}{3} + \left(-\frac{3P}{8}+0\right)^2\cdot 2$$

$$\left. + \left(\frac{15P}{16}+\frac{5Q}{8}\right)^2\cdot\frac{5}{3}\right\}$$

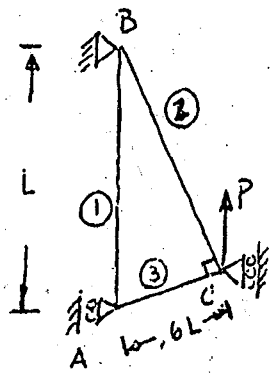
$$\left.\frac{\partial U_c}{\partial Q}\right|_{Q=0} = \frac{L}{AE}\left\{\left(\frac{3P}{8}\right)\cdot 1\cdot\frac{3}{4} + \left(\frac{5P}{16}\right)\cdot\frac{5}{3}\cdot\frac{5}{8} + \left(\frac{5P}{16}\right)\cdot\frac{5}{3}\cdot -\frac{5}{8} + \left(-\frac{3P}{8}\right)\cdot 1\cdot -\frac{3}{4} + 0 + 0 +\right.$$

$$\left.\left(-\frac{5P}{16}\right)\cdot\frac{5}{3}\cdot -\frac{5}{8} + 0 + \left(\frac{15P}{16}\right)\cdot\frac{5}{3}\cdot\frac{5}{8}\right\} = \frac{179}{96}\frac{PL}{AE}$$

Now

$$\theta_{CF} = \frac{1}{L_{CF}}\left.\frac{\partial U_c}{\partial Q}\right|_{Q=0} = 0.00124 \text{ radians} \quad \text{or } 0.0712^\circ$$

- REMEMBER: TRUSSES ASSUME LOAD AT JOINTS, WEIGHTLESS AND ONLY EXTEND BUT **DO** <u>NOT</u> BEND
- WHAT ABOUT IF WE WANT TO FIND LOADS GIVEN THE DISPLACEMENTS



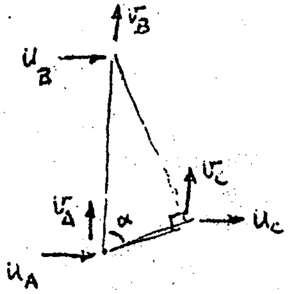
$P = 2500$ LB $\quad A = .1\ in^2 \quad E = 30 \times 10^6$ psi

$L_1 = L, \quad L_2 = .8L, \quad L_3 = .6L$

DFN: ELONGATION - CHANGE IN LENGTH OF BAR ALONG ITS LINE OF ACTION DUE TO LOAD IN BAR

EXAMPLE: $e_1 = \epsilon_1 L_1 = \frac{\sigma_1}{E_1} L_1 = \frac{F_1 L_1}{A_1 E_1}$

- CONSIDER ELONGATION POSITIVE IN DIRECTION OF POSITIVE (TENSILE FORCE)



$u_B, v_B; u_C, u_A = 0$ BY BOUNDARY CONDITIONS

$$e_1 = -v_A$$
$$e_2 = -v_c \sin\alpha = -.8v_c$$
$$e_3 = v_c \cos\alpha - v_A \cos\alpha = (v_c - v_A).6$$

MUST WRITE ELONGATIONS INTERMS OF DISPLACEMENTS

- HERE WE ASSUME POSITIVE DISPLACEMENT IN POSITIVE X & Y DIRECTION

$$U_s = \sum \frac{A_i E_i}{2} \frac{e_i^2}{L_i} = \sum \frac{AE}{2} \left\{ \frac{(-v_A)^2}{L} + \frac{(-.8v_c)^2}{.8L} + \frac{(.6[v_c - v_A])^2}{.6L} \right\}$$

- BUT AT A THERE IS NO VERTICAL LOAD ∴ $\frac{\partial U_s}{\partial v_A} = 0 = \frac{AE}{L}\left[(-v_A)(-1) + .6[v_c - v_A](-1)\right]$

$$\boxed{0 = \frac{AE}{L}(1.6v_A - .6v_c)}$$

BUT AT C $\frac{\partial U_s}{\partial v_c} = P$ → $P = AE\left[\left(\frac{-.8v_c}{.8L}\right)(-.8) + .6\frac{(v_c - v_A)}{.6L}(.6)\right]$

$$\boxed{P = \frac{AE}{L}\left[1.4v_c - .6v_A\right]}$$

SOLUTION OF THESE GIVE

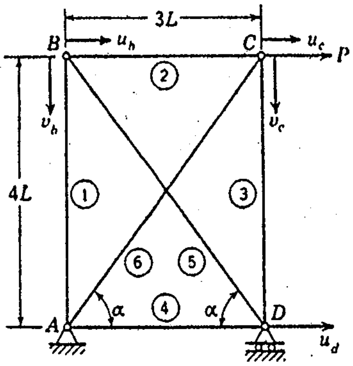
$$v_A = \frac{15PL}{47AE} \qquad v_c = \frac{40}{47}\frac{PL}{AE}$$

NOW: $e_1 = -v_A = -\frac{15PL}{47AE} \qquad e_2 = -.8v_c = \frac{-32PL}{47AE} \qquad e_3 = .6(v_c - v_A) = \frac{15PL}{47AE}$

THUS $F_1 = \frac{AEe_1}{L_1} = \frac{-15P}{47}; \quad F_2 = \frac{AEe_2}{L_2} = \frac{-32PL}{47(.8L)} = -\frac{40P}{47}; \quad F_3 = \frac{AEe_3}{L_3} = \frac{15PL}{47(.6L)} = \frac{25P}{47}$

Example 15.4 For the six-bar truss supported and loaded in its own plane as shown in Fig. 15.6*a*, determine the forces in the bars and the displacement components of the joints. All the bars of the truss have the same cross-sectional area A and the same elastic modulus E.

STATICALLY INDETERMINATE TRUSS $2j-3<n$



(a)

$$\frac{4LF_1}{AE} = e_1 = -v_b \qquad \frac{3LF_4}{AE} = e_4 = u_d$$

$$\frac{3LF_2}{AE} = e_2 = u_c - u_b \qquad \frac{5LF_5}{AE} = e_5 = (u_d - u_b)\cos\alpha - v_b \sin\alpha$$

$$\frac{4LF_3}{AE} = e_3 = -v_c \qquad \frac{5LF_6}{AE} = e_6 = u_c\cos\alpha - v_c \sin\alpha$$

NOTE: $u_A, v_A, v_d = 0$ BOUNDARY CONDITIONS

$$U_s = \sum \frac{A_i E_i e_i^2}{2L_i}\ ; \quad \frac{\partial U_s}{\partial u_c} = P\ ; \quad \frac{\partial U_s}{\partial u_b} = \frac{\partial U_s}{\partial v_b} = \frac{\partial U_s}{\partial u_d} = 0 = \frac{\partial U_s}{\partial v_c}$$

$$U_s = \frac{AE}{2}\left\{ \frac{v_b^2}{4L} + \frac{(u_c-u_b)^2}{3L} + \frac{v_c^2}{4L} + \frac{(u_d)^2}{3L} + \frac{[(u_d-u_b)\cos\alpha - v_b\sin\alpha]^2}{5L} + \frac{[u_c\cos\alpha - v_c\sin\alpha]^2}{5L} \right\}$$

$$\frac{\partial U_s}{\partial u_c} = P = \frac{AE}{L}\left\{ \frac{(u_c-u_b)}{3}\cdot 1 + \frac{(u_c\cos\alpha - v_c\sin\alpha)\cos\alpha}{5} \right\} = \frac{-AE}{375L}(125u_b - 152u_c + v_c\cdot 36)$$

$$\frac{\partial U_s}{\partial u_d} = 0 = \frac{AE}{L}\left\{ \frac{u_d}{3}\cdot 1 + \frac{[(u_d-u_b)\cos\alpha - v_b\sin\alpha]\cos\alpha}{5} \right\} = \frac{-AE}{375L}(27u_b + 36v_b - 152u_d)$$

$$\frac{\partial U_s}{\partial u_b} = 0 = \frac{AE}{L}\left\{ \frac{(u_c-u_b)(-1)}{3} + \frac{[(u_d-u_b)\cos\alpha - v_b\sin\alpha](-\cos\alpha)}{5} \right\} = \frac{-AE}{375L}(27u_d + 125u_c - 36v_b - 152u_b)$$

$$\frac{\partial U_s}{\partial v_b} = 0 = \frac{AE}{L}\left\{ \frac{v_b}{4}\cdot 1 + \frac{[(u_d-u_b)\cos\alpha - v_b\sin\alpha](-\sin\alpha)}{5} \right\} = \frac{-AE}{375L}\left(-36u_b + \frac{567}{4}v_b + 36u_d\right)$$

$$\frac{\partial U_s}{\partial v_c} = 0 = \frac{AE}{L}\left\{ \frac{v_c}{4}\cdot 1 + \frac{[u_c\cos\alpha - v_c\sin\alpha](-\sin\alpha)}{5} \right\} = \frac{-AE}{375L}\left(\frac{567}{4}v_c - 36u_c\right)$$

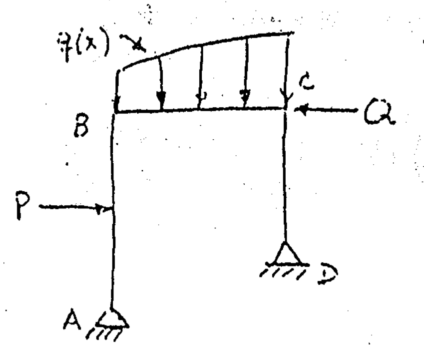
FROM THESE WE GET $u_b = \dfrac{21PL}{2AE}$ $\quad u_c = \dfrac{189}{16}\dfrac{PL}{AE}$ $\quad u_d = \dfrac{21PL}{16AE}$ $\quad v_b = \dfrac{-7PL}{3AE}$

$v_c = +\dfrac{3PL}{AE}$;

AND $F_1 = \dfrac{7P}{12}$, $F_2 = \dfrac{7P}{16}$, $F_3 = -3P$, $F_4 = \dfrac{7P}{16}$, $F_5 = \dfrac{15P}{16}$; $F_6 = -\dfrac{35P}{48}$

GO TO NEXT PAGE

- WHAT IF STRUCTURE BENDS? BEAM UNDER END LOADING
  FRAME UNDER TRANSVERSE LOADING
- HERE BENDING IS PRIMARY MODE OF LOADING.



ACTUALLY TOTAL SOLUTION WILL INVOLVE EXTENSIONAL, BENDING AND SHEARING EFFECTS

REMEMBER $U_{s_e} = \int_V \frac{E\epsilon^2}{2} dV$ $\qquad U_{c_e} = \int \frac{\sigma^2}{2E} dV$

DUE TO TENSION

7.3 PRINCIPLE OF STATIONARY POTENTIAL ENERGY

TABLE 7.2.1 Factors k_y and k_z for use in Eq. 7.2.3, where y and z are centroidal principal axes of the cross section.

Cross-Sectional Type	k_y	k_z
Rectangle	1.20	1.20
Solid circle	≈1.11	≈1.11
Thin-walled cylinder	2.00	2.00
I section, web parallel to z axis	≈1.20[a]	≈1.00[b]
Closed thin-walled box section	≈1.00[b]	≈1.00[b]

[a]For area A in this calculation use the combined cross-sectional areas of the flanges.
[b]For area A in this calculation use the cross-sectional area of the web (or webs in the box section).

• SIMILARLY $U_{s_s} = \int_V \frac{G\gamma^2}{2} dV$ and $U_{c_s} = \int_V \frac{\tau^2}{2G} dV$ DUE TO SHEAR

• FOR BENDING ENERGY $\sigma = \frac{-My}{I}$ AND $\epsilon = -Ky = +\frac{d^2v}{dx^2} y$ ... K IS CURVATURE

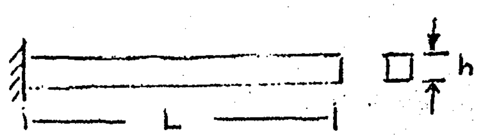
FOR A BEAM

$$U_{s_b} = \int_V \frac{E}{2}\epsilon^2 dV = \int_V \frac{E}{2}\left(+\frac{d^2v}{dx^2}y\right)^2 dV = \int_0^L \frac{E}{2}(v'')^2\left(\int_A y^2 dA\right)dx$$

$$= \int_0^L \frac{EI}{2}(v'')^2 dx$$

$$U_{c_b} = \int_V \frac{\sigma^2}{2E} dV = \int_V \frac{M^2y^2}{2EI^2} dV = \int_0^L \frac{M^2}{2EI^2}\left(\int_A y^2 dA\right)dx = \int_0^L \frac{M^2}{2EI}dx$$

• LOOK AT SHEAR EFFECTS – IF BEAM IS <u>NOT LONG</u> (SHORT BEAM), MUST ACCOUNT FOR THEM



IF $h/L \sim 1$ MUST ACCOUNT FOR SHEAR
OTHER WISE – NO.

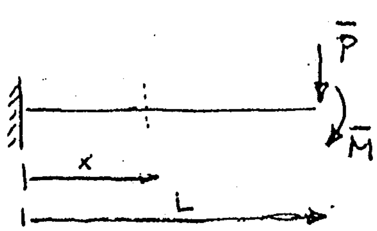
• IF NECESSARY $$\int_V \frac{\tau^2}{2G} dV = \int_0^L \int_A \frac{\tau^2 dA}{2G} dx = \int_0^L k\frac{V^2}{2GA} dx$$

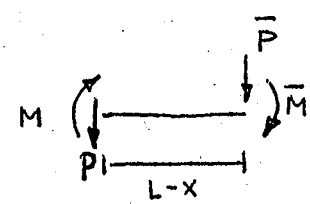
k IS REPORTED IN TABLE ~~7.2.1 Pg 231~~ 4.4 pg 87 2nd ed.; V IS SHEAR FORCE FOUND FROM

$$-\frac{dM}{dx} = V$$

A IS CROSS-SECTIONAL AREA

• LOOK AT SEVERAL CASES





$M + \bar{P}(L-x) + \bar{M} = 0$

$M = -[\bar{M} + \bar{P}(L-x)]$

AT POSITION X

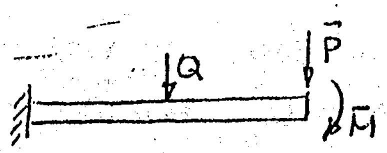
BY $U_{c_b} = \int_0^L \frac{M^2}{2EI} dx = \int_0^L \frac{(-[\bar{M} + \bar{P}(L-x)])^2}{2EI} dx$

$\frac{\partial U_{c_b}}{\partial \bar{P}}$ = DISPL (VERTICAL) WHERE $\bar{P}$ IS APPLIED = $\int_0^L \frac{1}{EI} \left(-[\bar{M} + \bar{P}(L-x)]\right)\left(-(L-x)\right) dx$, where $(-(L-x)) = \frac{\partial M}{\partial \bar{P}}$

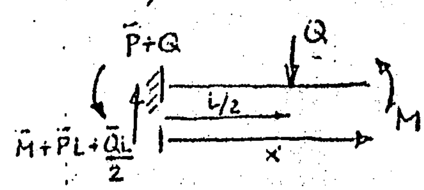
$\frac{\partial U_{c_b}}{\partial \bar{M}}$ = ROTATION WHERE $\bar{M}$ IS APPLIED = $\int_0^L \frac{1}{EI} \left(-[\bar{M} + \bar{P}(L-x)]\right)(-1)\, dx$, where $(-1) = \frac{\partial M}{\partial \bar{M}}$

- THIS IS A STATICALLY DETERMINATE PROBLEM. NOTE THAT LOAD & MOMENT GIVE THE TWO EQUATIONS OF EQUILIBRIUM

- WHAT IF WE WANTED TO FIND DISPLACEMENT AT $x = L/2$?
- PUT LOAD Q THERE, FIND U_{c_b} ; THEN TAKE $\left.\frac{\partial U_{c_b}}{\partial Q}\right|_{Q=0}$

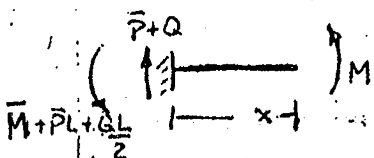


DEFINE M $\quad 0 \le x \le L/2 \quad$ & $\quad L/2 \le x \le L$



$+M + (Q)(x - L/2) - (\bar{P}+Q)x + (\bar{M} + \bar{P}L + \frac{QL}{2}) = M - \bar{P}(x-L) + \bar{M} = 0$

$M = +\bar{P}(x-L) - \bar{M} \qquad$ FOR $\quad L/2 \le x \le L$



$+M - (\bar{P}+Q)x + (\bar{M} + \bar{P}L + \frac{QL}{2}) = M + \bar{P}(L-x) + Q(L/2 - x) + \bar{M} = 0$

$M = -[\bar{M} + \bar{P}(L-x) + Q(L/2 - x)] \qquad$ FOR $\quad 0 \le x \le L/2$

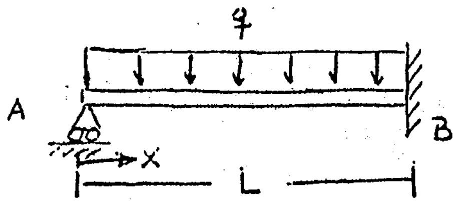
$$U_{c_b} = \int_0^L \frac{M^2}{2EI} dx = \int_0^{L/2} \frac{(-[\bar{M} + \bar{P}(L-x) + Q(L/2-x)])^2}{2EI} dx + \int_{L/2}^{L} \frac{(-[\bar{M} + \bar{P}(L-x)])^2}{2EI} dx$$

$$\frac{\partial U_{c_b}}{\partial Q} = \int_0^{L/2} \frac{2\left(-[\bar{M} + \bar{P}(L-x) + Q(L/2-x)]\right)}{2EI}\left(-[L/2 - x]\right) dx$$

where $(-[L/2 - x]) = \partial M / \partial Q$

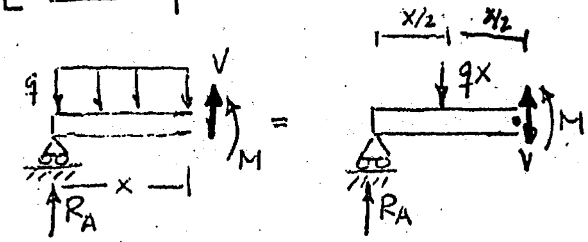
$$v\Big|_{x=L/2} = \left.\frac{\partial U_{c_b}}{\partial Q}\right|_{Q=0} = \int_0^{L/2} \frac{2\left(-[\bar{M} + \bar{P}(L-x)]\right)}{2EI}\left(-[L/2 - x]\right) dx$$

STATICALLY INDETERMINATE BEAMS



AT A REACTION UPWARD
B MOMENT + REACTION } 4 UNKNOWNS, 3 EQNS

LOOK AT BEAM AT DISTANCE x FROM LEFT END



$V = +qx + R_A$

$M + q\frac{x^2}{2} - R_A x = 0$

$M = R_A x - q\frac{x^2}{2} \qquad 0 \le x \le L$

$$U_{c_b} = \int_0^L \frac{M^2}{2EI}\,dx = \int_0^L \frac{\left(R_A x - q\frac{x^2}{2}\right)^2}{2EI}\,dx$$

• WHAT IS $\frac{\partial U_{c_b}}{\partial R_A} = U_A = 0 = \int_0^L \frac{\left(R_A x - q\frac{x^2}{2}\right)}{EI}\,x\,dx \implies R_A = \frac{3qL}{8}$ (where $x = \frac{\partial M}{\partial R_A}$)

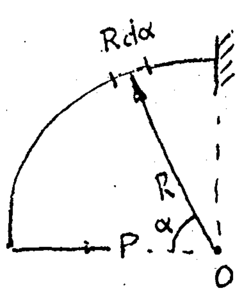
• WHAT IF WE WANTED TO FIND THE ROTATION OF THE BEAM AT $x = L/4$?

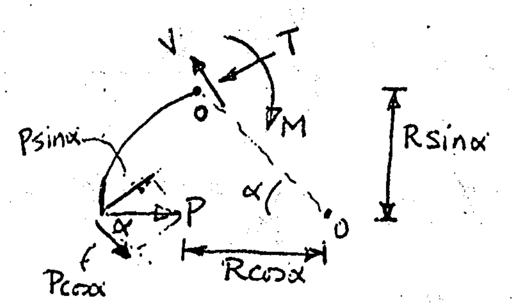
SUPERPOSE A MOMENT M_1 AT $x = L/4$, FIND U_{c_b}, TAKE $\left.\frac{\partial U_{c_b}}{\partial M_1}\right|_{M_1=0} = \theta\Big|_{L/4}$

• WHAT IF THERE ARE TWO LOADS CALLED P? WHAT DOES $\frac{\partial U_{c_b}}{\partial P}$ MEAN?

• HOW WOULD I FIND THE DISPLACEMENT UNDER ONE OF THESE LOADS?

LET'S LOOK AT A CIRCULAR CANTILEVERED BEAM UNDER HORIZONTAL LOAD





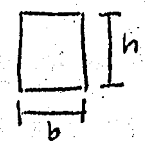
$T = P\sin\alpha$

$V = P\cos\alpha$

$M = PR\sin\alpha$

FROM EQUILIBRIUM -

IF THE CROSS-SECTION IS RECTANGULAR



$A = bh \quad I = \frac{bh^3}{12}$

$$\sigma = T/A = \frac{P\sin\alpha}{A} \qquad \tau = \frac{3V(h^2-4y^2)}{2bh^3} = \frac{3P\cos\alpha\,(h^2-4y^2)}{2bh^3}$$

$$U_{c_t} = U_{c_e} + U_{c_s} + U_{c_b} \qquad U_{c_e} = \int_V \frac{\sigma^2}{2E}\,dV = \int_0^{\pi/2} \frac{P^2\sin^2\alpha}{2A^2E}\,A\cdot R\,d\alpha$$

$$= \int_0^{\pi/2} \frac{P^2R}{2AE}\sin^2\alpha\,d\alpha = \frac{\pi P^2 R}{8AE}$$

$$U_{c_s} = \int_V \frac{\tau^2}{2G}\,dV = \int_0^{\pi/2} \frac{R\,d\alpha}{2G} \int_{-h/2}^{h/2} \frac{9V^2}{(2bh^3)^2}\,b(h^2-4y^2)^2\,dy \qquad dV = dx\,dy\,dz = (R\,d\alpha)(dy)b$$

$$= \frac{3\pi P^2 R}{20GA} \quad \text{and multiply by } k = 1.2$$

$$U_{c_b} = \int_V \frac{\sigma^2}{2E}\,dV = \int_0^{\pi/2} \frac{M^2}{2EI}\,R\,d\alpha = \int_0^{\pi/2} \frac{P^2R^3}{2EI}\sin^2\alpha\,d\alpha = \frac{\pi P^2R^3}{8EI}$$

$$U_{c_t} = \frac{\pi P^2 R^3}{8EI}\left[1 + \frac{1}{12}\left(\frac{h}{R}\right)^2 + \frac{E}{10G}\left(\frac{h}{R}\right)^2\right]$$

BENDING EXTENSION SHEAR

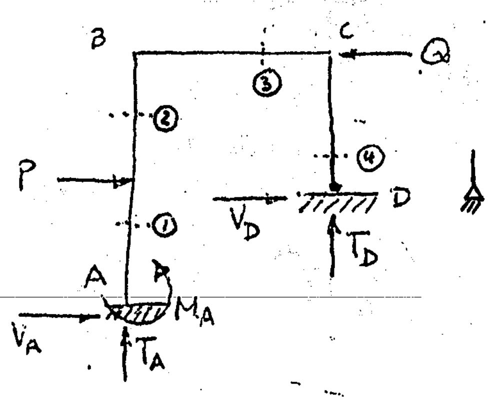
AND $\frac{\partial U_{c_t}}{\partial P}$ = DISPL IN DIRECTION OF $P = \frac{\pi P R^3}{4EI}\left[1 + \frac{1}{12}\left(\frac{h}{R}\right)^2 + \frac{E}{10G}\left(\frac{h}{R}\right)^2\right]$

IF $\frac{E}{G} = 2(1+\nu)$ AND $\nu = .3 \quad \frac{E}{10G} = .26$

IF $h/R = \frac{1}{10} \quad \frac{1}{12}\left(\frac{h}{R}\right)^2 = .00083 \quad \frac{E}{10G}\left(\frac{h}{R}\right)^2 = .0026$

<u>NOTE</u> EXTENSION & SHEAR CONTRIB. MUCH SMALLER

WHAT ABOUT A FRAME?



1) CONSIDER FRAME AS FREE BODY
2) THE REACTION COMPONENTS ARE V_A, T_A, M_A AT A AND V_D, T_D (PINNED END) AT D
3) SINCE 3 EQNS OF EQUILIB ONLY
$\Rightarrow$ 2 OF THESE UNKNOWNS CAN BE WRITTEN IN TERMS OF OTHER 3

e.g. $V_A = V_A(V_D, T_A, T_D) \quad M_A = M_A(V_D, T_A, T_D)$

4) NOW FIND M IN TERMS OF THESE QUANTITIES OVER EACH RANGE

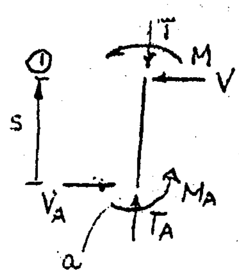
FROM A TO P   P TO B   B TO C   C TO D

5) USE $$U_{c_b} = \sum \int \frac{M^2}{2EI}\,ds = \int_A^P \frac{M^2}{2EI}\,ds + \int_P^B \frac{M^2}{2EI}\,ds + \int_B^C \frac{M^2\,ds}{2EI} + \int_C^D \frac{M^2\,ds}{2EI}$$

6) THEN TAKE $\dfrac{\partial U_{c_b}}{\partial M_A} = 0$ $\dfrac{\partial U_{c_b}}{\partial V_A} = 0$ SINCE THESE ARE THE INDETERMINATE QUANTITIES

7) THESE TWO EQUATIONS AND THE EQUILIBRIUM EQUATIONS GIVE 5 EQNS & 5 UNKN

EXAMPLE : BETWEEN A - P



$\Sigma F_H = 0 \Rightarrow V = V_A$

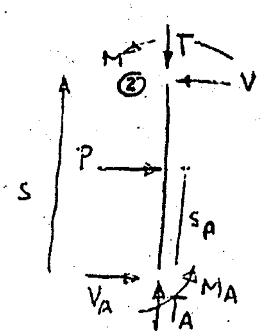
$\Sigma F_V = 0 \Rightarrow T = T_A$

$\Sigma M_a = 0 = M + Vs + M_A = 0$

$M = -(V_A s + M_A)$

$$\therefore \int_A^P \frac{M^2\,ds}{2EI} = \int_0^{s_P} \frac{[-(V_A s + M_A)]^2\,ds}{2EI}$$

BETWEEN P - B



$V = P + V_A$

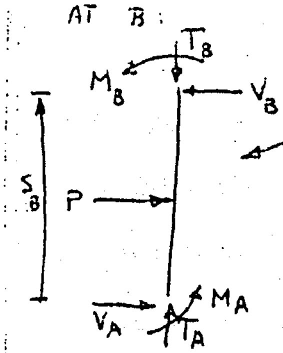
$T = T_A$

$M = -P(s - s_P) - V_A s - M_A$

$$\int_P^B \frac{M^2\,ds}{2EI} = \int_{s_P}^{s} \frac{[-(P\{s - s_P\} + V_A s + M_A)]^2\,ds}{2EI}$$

BETWEEN B - C

AT B :

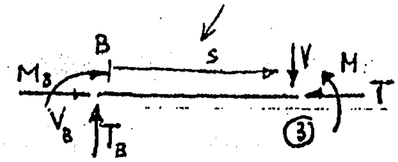


$V_B = P + V_A$

$T_B = T_A$

$M_B = -P(s_B - s_P) - V_A s_B - M_A$

SECTION B - C



$T = V_B = P + V_A$

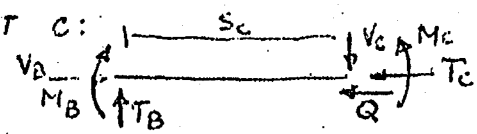
$V = T_B = T_A$

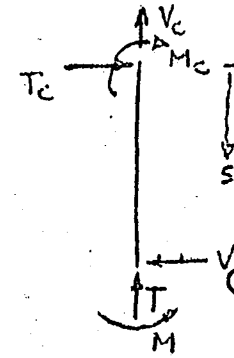
$M = T_B s + M_B = T_A s - [P(s_B - s_P) + V_A s_B + M_A]$

$$\int_B^C \frac{M^2\,ds}{2EI} = \int_0^s \frac{(T_A s + M_B)^2\,ds}{2EI}$$

BETWEEN C – D

AT C:





$T_c = V_B - Q = P + V_A - Q$

$V_c = T_B = T_A$

$M_c = T_B s_c + M_B = T_A s_c - [P(s_B - s_p) + V_A s_B + M_A]$

$T = -V_c = -T_A$

$V = T_c = P + V_A - Q$

$M = M_c + T_c s = M_c$ [illegible] $+ (P + V_A - Q)s$

$$\int_c^D \frac{M^2 ds}{2EI} = \int_0^{s_D} \frac{(M_c + T_c s)^2}{2EI} ds$$

NOW $U_{c_b} = \sum \int \frac{M^2}{2EI} ds$ AND TAKE $\frac{\partial U_{c_b}}{\partial M_A} = 0$ & $\frac{\partial U_{c_b}}{\partial V_A} = 0$

FROM EQUILIB.

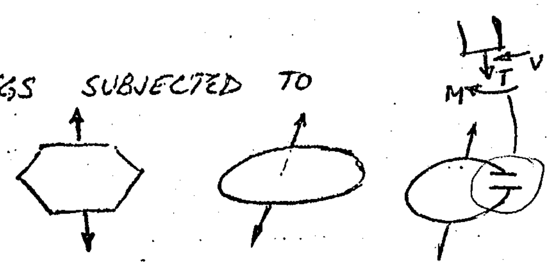
$\Sigma F_H = 0 \quad V_A + P + Q + V_D = 0$

$\Sigma F_V = 0 \quad T_A + T_D = 0$

$\Sigma M_A = 0 \quad M_A - P s_p + Q s_B - V_D (s_B - s_p) + T_D s_c = 0$

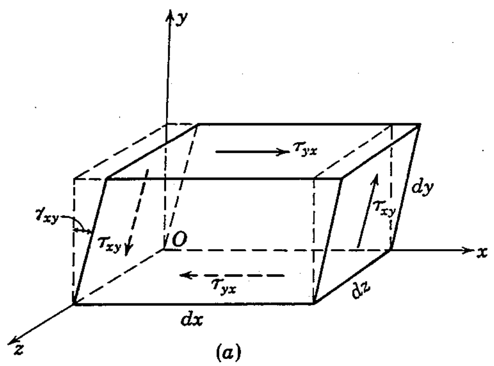
GIVE 5 EQNS
5 UNKNOWNS V_A, V_D, T_A, M_A

- WHAT ABOUT CLOSED PLANE FRAMES & RINGS SUBJECTED TO LOADS THAT ARE SELF EQUILIBRATING?



- YOU DO SAME THING AS WITH A FRAME. THERE WILL BE 3 STATICALLY INDETERMINATE REACTIONS THEY ARE INTERNAL REACTIONS M, T, V. THESE REACTIONS MUST MINIMIZE $\Pi = W_i = -U_{c_b} = -U_{c_s}$
- FORM $U_{c_b} = \sum \int \frac{M^2}{2EI} ds$ AROUND THE CLOSED PERIMETER
- THEN TAKE $\frac{\partial U_{c_b}}{\partial M} = 0 \quad \frac{\partial U_{c_b}}{\partial V} = 0 \quad \frac{\partial U_{c_b}}{\partial T} = 0$

Structural Mechanics w/ Intro to Elasticity & Plasticity by Patel & Venkatraman



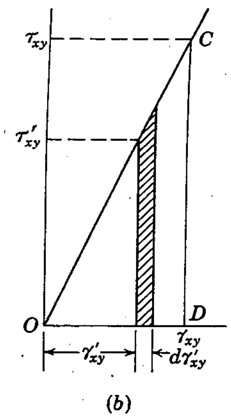


Fig. 4.2

Substitution for τ'_{xy} on the right side of this equation from Eq. (3.8) and integration yield the relations

$$dU_{1\tau} = \frac{G\gamma_{xy}^2}{2}\,dV = \frac{\tau_{xy}^2}{2G}\,dV = \frac{\tau_{xy}\gamma_{xy}}{2}\,dV \tag{4.11}$$

Here, the terms $G\gamma_{xy}^2/2 = \tau_{xy}^2/2G = \tau_{xy}\gamma_{xy}/2$ represent the area of the triangle OCD in Fig. 4.2*b*.

By considering the gradual application of the shear stresses τ_{yz} and τ_{zx} on the appropriate faces of the parallelepiped in a manner analogous to that of τ_{xy}, the corresponding shear-strain energies stored in its volume are easily seen to be $dU_{2\tau} = (\tau_{yz}\gamma_{yz}/2)\,dx\,dy\,dz$ and $dU_{3\tau} = (\tau_{zx}\gamma_{zx}/2)\,dx\,dy\,dz$. Therefore, the total shear-strain energy dU_τ accumulated in the parallelepiped in its deformed shape because of the action τ_{xy}, τ_{yz}, and τ_{zx} is

$$dU_\tau = dU_{1\tau} + dU_{2\tau} + dU_{3\tau} = \tfrac{1}{2}(\tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})\,dV \tag{4.12}$$

In the general three-dimensional state of stress, the total strain energy stored in an elementary volume dV is the sum

$$dU_s = dU_\sigma + dU_\tau = U_{s0}\,dV$$
$$= \tfrac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})\,dV \tag{4.13}$$

Here, by implicit definition, U_{s0} is the total strain energy stored in a unit volume of the material. It is known as the *strain-energy density*. Integration of this density over the volume V yields the total strain energy stored in the body as

$$U_s = \int_V U_{s0}\,dV \tag{4.14}$$

Now, with the stress-strain relations in Eqs. (3.5*a*) and (3.8) and the strain and stress invariants defined by Eqs. (1.38), (2.6), and (2.7), the strain-energy density U_{s0} can be expressed, after some algebraic manipulations, in the following alternate forms:

$$U_{s0} = \frac{E}{2(1+\nu)(1-2\nu)}\{(1-\nu)(\epsilon_x+\epsilon_y+\epsilon_z)^2 - 2(1-2\nu)$$
$$\times\,[\epsilon_x\epsilon_y + \epsilon_y\epsilon_z + \epsilon_z\epsilon_x - \tfrac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)]\}$$
$$= \frac{1}{2}\left[\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}I_{1\epsilon}^2 - 4GI_{2\epsilon}\right] \tag{4.15a}$$

$$U_{s0} = \frac{1}{2E}[(\sigma_x+\sigma_y+\sigma_z)^2$$
$$-\,2(1+\nu)(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)]$$
$$= \frac{1}{2}\left(\frac{I_{1\sigma}^2}{E} - \frac{I_{2\sigma}}{G}\right) \tag{4.15b}$$

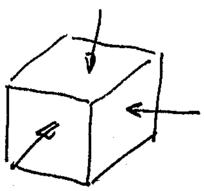
mention this !!

These expressions show that the strain-energy density is invariant under orthogonal rotations of coordinates. Therefore, the function U_{s0} can be expressed solely in terms of the principal strains, or the principal stresses. Thus, by denoting the principal directions by 1, 2, and 3, Eq. (4.15) reduces to

$$U_{s0} = \frac{E}{2(1+\nu)(1-2\nu)}[(1-\nu)(\epsilon_1+\epsilon_2+\epsilon_3)^2$$
$$-\,2(1-2\nu)(\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1)]$$
$$= \frac{E}{2(1+\nu)(1-2\nu)}[(1-2\nu)(\epsilon_1^2+\epsilon_2^2+\epsilon_3^2) + \nu(\epsilon_1+\epsilon_2+\epsilon_3)^2] \tag{4.16a}$$

$$U_{s0} = \frac{1}{2E}[(\sigma_1+\sigma_2+\sigma_3)^2 - 2(1+\nu)(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]$$
$$= \frac{1}{2E}\{(1-2\nu)(\sigma_1^2+\sigma_2^2+\sigma_3^2)$$
$$+\,\nu[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]\} \tag{4.16b}$$

For values of Poisson's ratio in the range of $-1 \le \nu \le 0.5$, these equations show that the strain-energy density U_{s0} is always positive except in the single instance when all stresses and strains vanish simultaneously, in which case its value is zero. While additional properties of the density function U_{s0} could be discussed here on the basis of Eqs. (4.15) and (4.16), it is more instructive to deal with these in the context of the less restrictive discussion of strain energy and its relation to the generalized Hooke's law [Eq. (3.14)] presented in the next section.



$p = \sigma_x + \sigma_y + \sigma_3$

$\sigma_x = \sigma_y = \sigma_z = \frac{p}{3}$

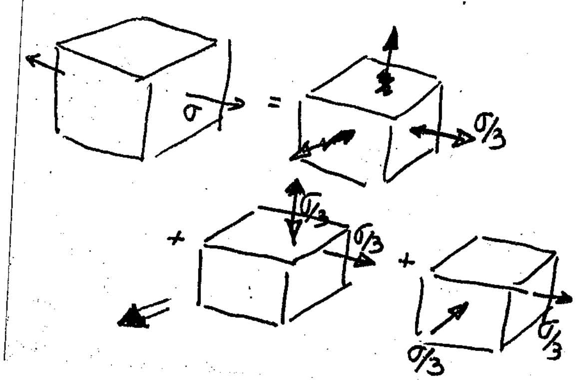
$\epsilon_x = \frac{1-2\nu}{E}\frac{p}{3} = \epsilon_y = \epsilon_z$

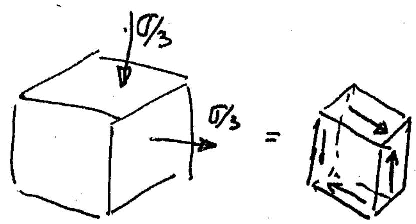
$\therefore \; e = \frac{\Delta V}{V} = 3\left(\frac{1-2\nu}{E}\right)\frac{p}{3}$

So hydrostatic produces volume changes or dilatational change

$\epsilon_x \frac{\sigma_x}{2} = \frac{1-2\nu}{2E}\frac{p^2}{9} \quad \text{or} \quad \frac{1-2\nu}{2E}\frac{p^2}{3} = U_{s_{dilat}}$

$U_{s_0} = \frac{1}{2E}\left(\sigma_x^2 + \sigma_y^2 + \sigma_z^2\right) - \frac{\nu}{E}\left(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x\right) + \frac{1}{2G}\left(\tau_{xy}^2 + \dots\right)$





$\tau = \frac{\sigma_1 - \sigma_3}{2} = \sigma/3$

Here $U = U_o$ because the structure has unit volume and is uniformly stressed. Similarly, for a state of pure shear, Fig. 2.6-1b,

$$U_o = \frac{G\gamma_{xy}^2}{2} \quad \text{or} \quad U_o = \frac{\tau_{xy}^2}{2G} \tag{2.6-3}$$

Multiaxial States of Stress. Consider first an isotropic and linearly elastic body in a state of plane stress. Nonzero stresses are σ_x, σ_y, and τ_{xy}. Let these stresses be applied one after another. If σ_x is applied first, with σ_y and τ_{xy} both zero, U_o is given by Eq. 2.6-2. If σ_y is now added, it produces the strains

$$\epsilon_y = \frac{\sigma_y}{E} \quad \text{and} \quad \epsilon_x = -\nu\epsilon_y = -\nu\frac{\sigma_y}{E} \tag{2.6-4}$$

and the following contribution to U_o.

$$\int_0^{\epsilon_y} \sigma_y\, d\epsilon_y + \sigma_x \epsilon_x = \frac{\sigma_y^2}{2E} - \nu\frac{\sigma_x\sigma_y}{E} \tag{2.6-5}$$

No integration is needed to obtain $\sigma_x \epsilon_x$ in Eq. 2.6-5 because σ_x remains constant as x-direction strain is produced by σ_y. Another contribution to U_o comes from Eq. 2.6-3. The final result for a state of plane stress, from Eqs. 2.6-2, 2.6-3, and 2.6-5, is

$$U_o = \frac{1}{2E}[\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y] + \frac{\tau_{xy}^2}{2G} \tag{2.6-6}$$

The foregoing argument can be extended to the fully three-dimensional case by adding the stresses σ_z, τ_{yz}, and τ_{zx}. The result is

$$U_o = \frac{1}{2E}[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)] + \frac{1}{2G}[\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2] \tag{2.6-7}$$

Strain Energy of Distortion. In an arbitrary state of stress, the average normal stress is

$$\sigma_a = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \tag{2.6-8}$$

which, incidentally, is the normal stress on an octahedral plane. *Deviatoric stresses* are given the symbol s and are defined as follows.

$$s_x = \sigma_x - \sigma_a \quad s_y = \sigma_y - \sigma_a \quad s_z = \sigma_z - \sigma_a \quad s_{xy} = \tau_{xy} \quad s_{yz} = \tau_{yz} \quad s_{zx} = \tau_{zx} \tag{2.6-9}$$

An arbitrary state of stress can be represented as the sum of two states: (1) a hydrostatic state in which principal stresses are $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_a$, and (2) a state in which all ... No change of shape is produced by the hydrostatic state. No

change in volume due to deviatoric

Strain energy of distortion, associate
predict the onset of yielding. An expression
volume, U_{od}, is obtained from Eq. 2.6-7 by
toric stresses. Also, we eliminate E by using

$$U_{od} = \frac{1}{12G}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\ldots$$

Alternative expressions for U_{od} are

$$U_{od} = \frac{3}{4G}\tau_{oct}^2 \quad \text{a}$$

where τ_{oct} is given by Eq. 2.4-3, 2.4-4, or 2.4

$$\sigma_e = \frac{1}{\sqrt{2}}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma\ldots$$

From Eqs. 2.4-5 and 2.6-12, $\tau_{oct} = \sqrt{2}\,\sigma_e/$
is a uniaxial state of stress. If the state of
sible that $\sigma_e > \sigma_1$: for example, if princi
$\sigma_e = \sqrt{3}\sigma_1$. The von Mises failure criterion

2.7 STRESS CONCENTRATION

Stress in a solid is rarely uniform. It rises
geneity or abrupt changes in geometry.
boundaries in metal, small inclusions of fc
stituents of concrete, and the cell struc
changes in geometry include tool marks a
geometry are common, such as threads on
way in a shaft. Peak stress associated with
the geometry is accurately known and th
been tabulated.

Stress Concentration Factors. Conside
sion, Fig. 2.7-1a. It is obvious that stress m
section containing the hole, because there
a reduced area. However, the mechanics
mula for σ_{max} because we have no reliable
tion. The problem can be solved (with d
Most stress concentration problems are to
been solved experimentally. Results have
elastic material, in the form of *stress conc*
the problem in Fig. 2.7-1a,

$$\sigma_{max} = K_t\,\sigma_{nom} \quad \text{where}$$

ıe structure has unit volume and is uniformly stressed. ·e shear, Fig. 2.6-1b,

$$U_o = \frac{G\gamma_{xy}^2}{2} \quad \text{or} \quad U_o = \frac{\tau_{xy}^2}{2G} \tag{2.6-3}$$

ss. Consider first an isotropic and linearly elastic body in a ero stresses are σ_x, σ_y, and τ_{xy}. Let these stresses be applied ıpplied first, with σ_y and τ_{xy} both zero, U_o is given by Eq. 2.6- ɔduces the strains

$$= \frac{\sigma_y}{E} \quad \text{and} \quad \epsilon_x = -\nu\epsilon_y = -\nu\frac{\sigma_y}{E} \tag{2.6-4}$$

ıtion to U_o.

$$\int_0^{\epsilon_y} \sigma_y \, d\epsilon_y + \sigma_x \epsilon_x = \frac{\sigma_y^2}{2E} - \nu\frac{\sigma_x\sigma_y}{E} \tag{2.6-5}$$

:o obtain $\sigma_x \epsilon_x$ in Eq. 2.6-5 because σ_x remains constant as x- ɜd by σ_y. Another contribution to U_o comes from Eq. 2.6-3. of plane stress, from Eqs. 2.6-2, 2.6-3, and 2.6-5, is

$$_o = \frac{1}{2E}[\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y] + \frac{\tau_{xy}^2}{2G} \tag{2.6-6}$$

an be extended to the fully three-dimensional case by adding . The result is

$$[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)]$$
$$\frac{1}{G}[\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2] \tag{2.6-7}$$

rtion. In an arbitrary state of stress, the average normal

$$\frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \tag{2.6-8}$$

normal stress on an octahedral plane. *Deviatoric stresses* are ɜ defined as follows.

$$- \sigma_a \qquad s_y = \sigma_y - \sigma_a \qquad s_z = \sigma_z - \sigma_a$$
$$s_{yz} = \tau_{yz} \qquad s_{zx} = \tau_{zx} \tag{2.6-9}$$

ɜs can be represented as the sum of two states: (1) a hydrosta- al stresses are $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_a$, and (2) a state in which all o change of shape is produced by the hydrostatic state. No

Strain energy of distortion, associated with the deviatoric state, can be used to predict the onset of yielding. An expression for the strain energy of distortion per unit volume, U_{od}, is obtained from Eq. 2.6-7 by replacing the written stresses by the deviatoric stresses. Also, we eliminate E by using Eq. 2.5-4. Thus

$$U_{od} = \frac{1}{12G}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)] \tag{2.6-10}$$

Alternative expressions for U_{od} are

$$U_{od} = \frac{3}{4G}\tau_{oct}^2 \quad \text{and} \quad U_{od} = \frac{1}{6G}\sigma_e^2 \tag{2.6-11}$$

where τ_{oct} is given by Eq. 2.4-3, 2.4-4, or 2.4-5, and σ_e is an "effective" stress defined as

$$\sigma_e = \frac{1}{\sqrt{2}}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)]^{1/2} \tag{2.6-12}$$

From Eqs. 2.4-5 and 2.6-12, $\tau_{oct} = \sqrt{2}\,\sigma_e/3$. Conveniently, σ_e reduces to $\sigma_e = \sigma_x$ if σ_x is a uniaxial state of stress. If the state of stress is hydrostatic, then $\sigma_e = 0$. It is possible that $\sigma_e > \sigma_1$: for example, if principal stresses are $\sigma_1 = -\sigma_3$ and $\sigma_2 = 0$, then $\sigma_e = \sqrt{3}\sigma_1$. The von Mises failure criterion is often stated in terms of σ_e.

2.7 STRESS CONCENTRATION

Stress in a solid is rarely uniform. It rises to local peaks because of material inhomogeneity or abrupt changes in geometry. Material inhomogeneities include crystal boundaries in metal, small inclusions of foreign material, small voids, the various constituents of concrete, and the cell structure of wood. Unintentional and random changes in geometry include tool marks and surface scratches. *Intentional* changes in geometry are common, such as threads on a bolt, teeth on a gear, an oil hole, and a keyway in a shaft. Peak stress associated with a change in geometry is easy to calculate if the geometry is accurately known and the associated stress concentration factor has been tabulated.

Stress Concentration Factors. Consider a central circular hole in a plate under tension, Fig. 2.7-1a. It is obvious that stress must be greater than σ_o somewhere on a cross section containing the hole, because there the axial force $P = \sigma_o Dt$ must be carried by a reduced area. However, the mechanics of materials method cannot provide a formula for σ_{max} because we have no reliable way of predicting the geometry of deformation. The problem can be solved (with difficulty) by the theory of elasticity method. Most stress concentration problems are too complicated for either method. Many have been solved experimentally. Results have been tabulated for an isotropic and linearly elastic material, in the form of *stress concentration factors* K_t. Using them is simple. For the problem in Fig. 2.7-1a,

$$\sigma_{max} = K_t\,\sigma_{nom} \quad \text{where} \quad \sigma_{nom} = \frac{P}{A_{net}} = \frac{P}{(D - 2r)t} \tag{2.7-1}$$

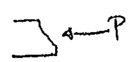
תיאוריות החוזק

ידוע שמבנה שלא מתוכנן כהוגן מתמוטט. יש כמה אפשרויות שגורם להתמוטטות:

1. ההתמוטטות קורה בגלל שמופיע סדקים או שסדקים שנמצאים מתחילים להאריך את גודלם

- יכולים להשבר מתחת עומס מתמיד ומעייפות.
- לשבר פריך (BRITTLE) אין תגובה פלסטית (BRITTLE FRACTURE – NO PLASTICITY)
- לפריכות במתח השטח-החתך של השבר הוא אנכי לעומס.

לפריכות בלחץ השטח-החתך של השבר הוא משופע לעומס.

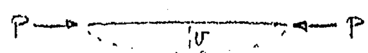


LARGE DEFORMATION FOR LOAD STRESS

2. ההתמוטטות יכולה לקרות בגלל כניעת העיבוד זאת אומרת שהגוף הולך ומאריך את המידות שלו יותר מאה שמקובל. לשינוי קטן במאמץ יש שינוי גדול לעיבור (YIELD)

LARGE DEFORMATION WHEN

3. ההתמוטטות יכולה לקרות בגלל קריסה. המוט יכול להציג בתזוזה גדולה בכיוון האנכי לעומס הצירי. (BUCKLING)



CREEP DUE TO TEMP. ↑

4. בגלל טמפרטורות גבוהות יכול להיות זחילה (CREEP). המידות יכולות להשתנות עד שהגוף מפריע לגוף קרוב לו כמו במנוע (בוכנה, טבעות בוכנה) PISTON

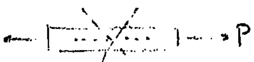
• לכן צריך לדעת וגם לתאר מתי גוף נכשל

נדון בשיטות שונות על שם TRESCA, VON MISES, קריטריון מאמץ ראשי מקסימלי, קריטריון עיבור ראשי מקסימלי, קריטריון MOHR, קריטריון NORRIS. אבל יש הרבה תיאוריות בשוק.

MAX SHEAR THEORY

1. קריטריון TRESCA: גם ידוע כי התיאוריה של המאמץ הגזירה המקסימלי, ותיאורית COULOMB

התמוטטות מתחילה כאשר שהמאמץ הגזירה המקסימלי מגיע לערך מסויים



התיאוריה הזאת באה מפני שיש חומרים מתמתחים (גמישים) SOFT MAT שכשלונם הוא דרך מישור שהוא 45 מעלות לציר של עומס מתיחה. מישורים לגזירה מקסימליות הם 45 מעלות למישור העומס (FAILURE ON 45° PLANES TO TENSILE LOAD)

DUE TO SOFT MATERIALS FAILURE AT 45° TO LOAD DIR.

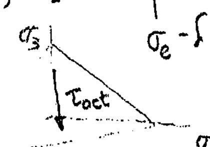
לניסיון חד-צירי $\sigma_y = \sigma_1$ בהתחלת התמוטטות

$$\tau_{max} = \frac{\sigma_y}{2} = \frac{\sigma_1 - \sigma_2}{2}$$

2. קריטריון VON MISES: גם ידוע כי התיאוריה של MAXWELL. מבוסס ברעיון אנרגיה.

התמוטטות מתחילה כאשר המאמץ האפקטיבי (EFFECTIVE) מגיע לערך מסויים, σ_e
אבל יש יחס לגזירה אוקטהדרלי (OCTOHEDRAL SHEAR) ל-σ_e
הגזירה הנמצאת על מישור שעושה זוויות של 45° לכל הצירים הראשיים של המאמצים.

RELATED TO CHANGE IN SHAPE & ENERGY DUE TO DEVIATORIC STRESSES



$\frac{2\sigma_y^2}{12G} = \frac{1}{12G}[\ ]^2$ אנרגית התעוותות

$$\sigma_e = \frac{3}{\sqrt{2}}\tau_{oct} = \frac{1}{\sqrt{2}}\left((\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)\right)^{1/2} = \sigma_{yp}$$

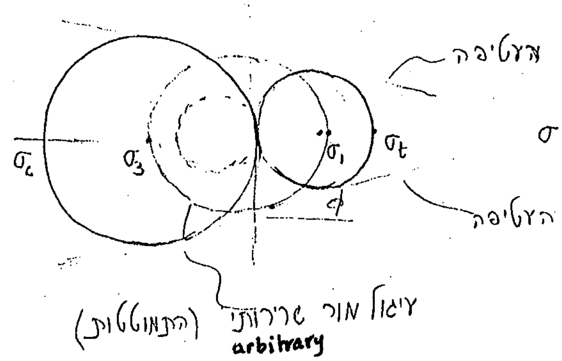
אע"פ ש- $\tau_{max} > \tau_{oct}$, הוא מתארת את המצב הכללי של המאמצים כאשר הכניעה מתחילה.

3. קריטריון של המאמץ הראשי המקסימלי: כאשר המאמץ הראשי σ_1 מגיע לערך מסויים, מסכרתי במתיחות. רואים זה בחומרים שבירים (פריכים) BRITTLE שהמישור הפריק הוא מישור שניצב ל- σ_1

MAX PRINC STRESS
$\sigma_1 > \sigma_t^{ULT}$
→ BRITTLE MATERIAL
FRACTURE SURFACE
$\perp$ to σ_1

4. קריטריון מור: התמוטטות מתחילה כאשר עיגול מור הגדול ביותר ($\sigma_1 - \sigma_3$) סמוך לעטיפת התמוטטות, מור עובד טוב בחומר פריך (LARGEST MOHR'S CIRCLE ($\sigma_1 - \sigma_3$) TOUCHES A FAILURE ENVELOPE).
לשרטט את העטיפה ההתמוטטות - לעשות ניסיונות - אחד בעומס צירי מתוח והשני בעומס צירי לחוץ

FAILURE - WHEN LARGEST MOHR CIRCLE TOUCHES FAILURE ENVELOPE



התמוטטות מתחילה כ- $\frac{\sigma_1}{\sigma_t} - \frac{\sigma_3}{|\sigma_c|} \geq 1$. כש- הלחץ הוא דומיננטי (DOMINANT), ההתמוטטות קורת דרך הגזירה במישור משופע בזווית של $45^\circ + \frac{1}{2}\phi$ לכיוון של σ_3.

FRACTURE OR BEGINNING OF YIELD OCCURS WHEN ONE OF THE EQUATIONS ARE SATISFIED

5. בחומר אורתוטרופי (ORTHOTROPIC MATERIAL), פריכות או התחלת כניעה קורת כאשר אחת מהמשוואות הללו מסופקת. זה הקריטריון של NORRIS

$$\frac{\sigma_x^2}{S_x} - \frac{\sigma_x \sigma_y}{S_x S_y} + \frac{\sigma_y^2}{S_y^2} + \frac{\tau_{xy}^2}{S_{xy}^2} \geq 1$$

$$\frac{\sigma_y^2}{S_y^2} - \frac{\sigma_y \sigma_z}{S_y S_z} + \frac{\sigma_z^2}{S_z^2} + \frac{\tau_{yz}^2}{S_{yz}^2} \geq 1$$

$$\frac{\sigma_z^2}{S_z^2} - \frac{\sigma_z \sigma_x}{S_z S_x} + \frac{\sigma_x^2}{S_x^2} + \frac{\tau_{zx}^2}{S_{zx}^2} \geq 1$$

כאן x, y, z הם הצירים הראשיים של החומר.
כאן $S_x, S_y, S_{xy}, S_{yz}, S_{zx}, S_z$ הם הערכים שמקבלים מניסיונות מיוחדים.

S_x is DESIGNED FROM TENSILE TEST IF σ_x IS TENSILE
S_y " " COMPRESSION TEST IF σ_y IS COMPRESSIVE

THEORY APPLICABLE TO SOFT MATERIALS DON'T APPLY TO BRITTLE ONES (apply)

6. תיאוריה שמיישמים לחומרים מתמתחים (SOFT) בדרך כלל לא עובדת בחומרים פריכים

7. היך מיישמים לתיאוריות האלו כאשר יש מצב מאמץ דו-צירי, כי התיאוריות האלו עובדים, בדרך כלל, במצב מאמץ חד-צירי.

HOW TO USE FOR BIAXIAL WHEN MOST THEORIES ONLY WORK FOR UNIAXIAL STATES

8. צריך לקבל את הערכים של σ_1, או τ_{max} או σ_e לכול המצבים ולבדוק מהווה המצב ההכי גרוע, אם לא היו תיאוריות התמוטטות

MOHR CIRCLE

$$\frac{\sigma_1}{\sigma_t} - \frac{\sigma_3}{|\sigma_c|} \geq 1$$

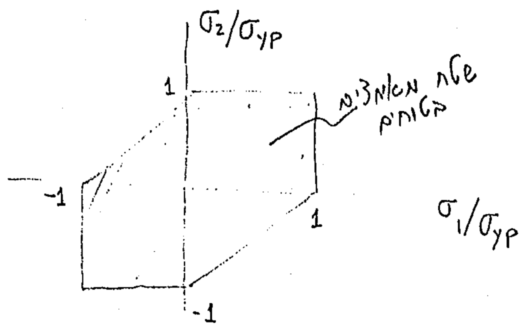
$\sigma_1 = 12.9$ $\sigma_3 = -30.9$

$\sigma_t = 14$ $\sigma_c = 120$

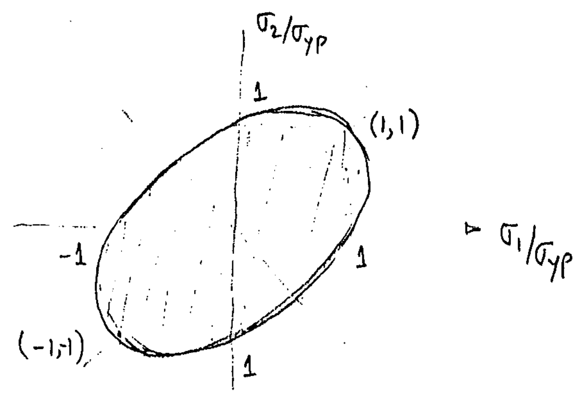
$$\frac{12.9}{14} + \frac{30.9}{120} \geq 1$$

$$0.92 + 0.25 > 1$$

מכאן רואים ששלוש מהתיאוריות נותנות לנו שיש התמוטטות. ולכן צריכים לתקן את הבדיקה ולתכנן (DESIGN) את התיכנון בחזרה.



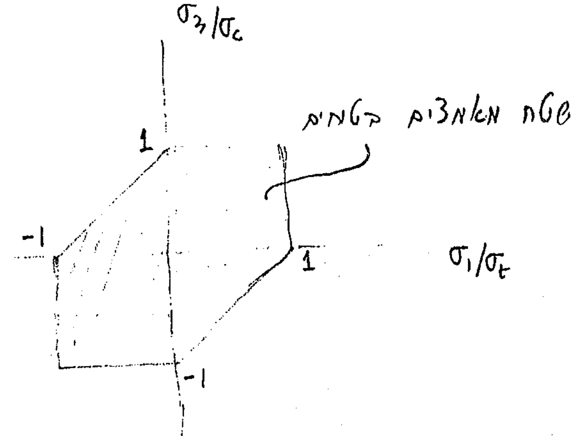
לפי TRESCA



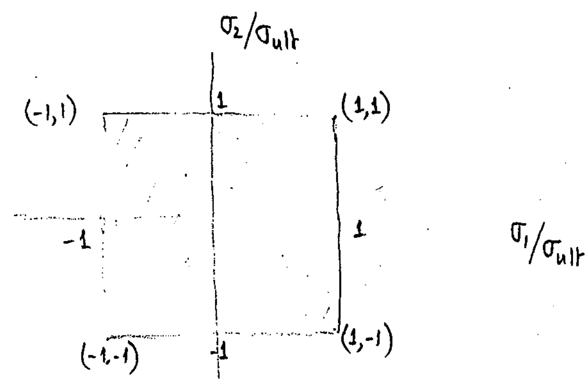
לפי VON MISES

נ.ב.: מאמץ הכניעה בלחץ ~~ומתיחה~~ ומתיחה אותו ערך

WHEN $\sigma_{yp_c} = \sigma_{yp_t}$



לפי MOHR



σ_{ult} - המאמץ המרבי

לפי המאמץ הראשי המקסימלי

VON MISES IS MORE FORGIVING & CONTAINS TRESCA

TEST RESULTS FOLLOW VON MISES

קריטריון TRESCA כלול בשל VON MISES. תוצאות ניסיוניות מסכימות יותר עם VON MISES.

בתיאוריות פלסטיות משתמשים עם שני התיאוריות האלו - VON MISES או TRESCA

Experimental data follow von Mises. IN THEORY OF PLASTICITY, WE WILL CONCENTRATE ON THESE 2.

נתרכז בשניהם

כי זה יקח הרבה כסף, והרבה זמן, אי אפשר לבדוק את כל המצבים המאמציים

לכן משתמשים בתיאוריות ועושים ניסיונות במתח חד-צירי

ומקבלים על עצמינו ש-התמוטטות בכל שיטה אחרת מוצג

בהתמוטטות במצב מתוח חד-צירי

CANT CHECK ALL STRESS STATE
USE THEORY W/ UNIAXIAL TENSION TEST. ASSUME THAT FAILURE IN TENSION INDICATES FAILURE IN ALL OTHERS

דוגמא - COOK & YOUNG

- TENSION & COMPRESSION TESTS OF A BRITTLE MATERIAL GIVE $\sigma_t = 14$ MPa & $\sigma_c = 120$ MPa
- UNDER LOADING CONDITION, A CERTAIN STATE OF STRESS EXISTS $\sigma_x = 0$, $\sigma_y = -18$ MPa, $\tau_{xy} = 20$ MPa

IS THIS STATE OF STRESS SAFE? CONSIDER PLANE STRESS

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2}$$

$$= \frac{0 - 18}{2} + \sqrt{\frac{(18)^2}{4} + (20)^2} = -9.0 + 21.9 = 12.9 \text{ MPa}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\frac{(\sigma_x - \sigma_y)^2}{4} + \tau_{xy}^2} = -9.0 - 21.9 = -30.9 \text{ MPa}$$

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(0+18)^2 + (-18-0)^2 + (0-0)^2 + 6\left(20^2 + 0^2 + 0^2\right)\right]^{1/2} = 39.04 \text{ MPa}$$

קריטריון המאמץ הראשי המקסימאלי

MAX STRESS SHOWS SAFE

מפני ש- $(\sigma_1 < \sigma_t)$ $12.9 < 14$ החומר במצב טוב בקריטריון הזה.

קריטריון TRESCA

$$12.9 > 0 > -30.9 \Leftarrow \sigma_1 > \sigma_2 > \sigma_3$$

TRESCA & VON MISES SHOW UNSAFE, BUT NORMALLY DON'T WORK FOR BRITTLE

$$\tau_{max} = \frac{12.9 + 30.9}{2} = 21.9 \text{ MPa} = \frac{\sigma_1 - \sigma_2}{2}$$

מפני ש- בניסיון מאמץ מתוח חד-צירי הוא $14\text{MPa} = \sigma_t$

ו- τ_{max} שווה ל- $\sigma_{yp}/2 = \sigma_t/2 = 7\text{MPa}$

לפי TRESCA $\tau_{max} > \sigma_{yp}/2$ לכן החומר במצב לא טוב

והתחיל להתמוטט.

קריטריון VON MISES

מפני ש- $\sigma_e = 39.04$ ובניסיון מאמץ מתוח חד-צירי הוא $\sigma_{yp} = \sigma_t$

ו- $(\sigma_e > \sigma_{yp})$. לכן לפי VON MISES החומר מתחיל להתמוטט.

קריטריון MOHR

במצב ללא התמוטטות $\frac{\sigma_1}{\sigma_t} - \frac{\sigma_3}{|\sigma_c|} \le 1 \Leftarrow \frac{12.9}{14} - \frac{(-30.9)}{120} = 1.18$

לכן יש התמוטטות

Failure criteria

Failure criteria deal with failure of a material, in contrast with failure of a structure.

Brittle materials

Maximum normal stress criterion

Failure if $\sigma_1 \geq \sigma_t^{ult}$, that is , if the largest tensile principal stress exceeds the ultimate strength. The other principal stresses are ignored. Similarly in compression, the applied stress is compared with the ultimate strength in compression. This criterion is simplistic since if all the principal stresses are compressive, most materials are much stronger than would be expected based on uniaxial tests.

Mohr criterion

This takes into account ultimate tensile, compressive and shear stresses. Represent each state by a Mohr circle. Draw an 'envelope' tangent to the circles. Mohr suggested that, provided an arbitrary state of stress was represented by Mohr circle within that envelope, failure would not occur. Sometimes a simplified form is taken in which the shear test is ignored. Then the envelope is a straight line and failure is predicted if $\sigma_1/\sigma_t^{ult} - \sigma_3/\sigma_c^{ult} \geq 1$. This also is not ery realistic for hydrostatic compression.

Ductile materials: yield criteria

Maximum shear stress criterion (Tresca criterion)

Yield when $\tau_{max} \geq \tau_Y$.

In principal stress space this looks like a hexagon.

Tension test to yield.

Recall that the maximum shear stress is $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$, but the minimum principal stress is zero so for tension, $\tau_Y = 0.5\, \sigma_Y$.

Von Mises

Yield when $\sigma_{eff} \geq \tau_Y$.

Recall $\sigma_{eff} = \frac{1}{\sqrt{2}}\{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\{\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\}\}^{1/2}$.

Distortional energy, $U_{0d} = \frac{3}{4G}\ \tau_{oct}^2 = \frac{1}{6G}\ \sigma_{eff}^2$.

In principal stress space, the Von Mises criterion looks like an ellipse.

Fatigue

A material loaded through multiple cycles will break at a stress considerably less than the ultimate strength for a single application of load. Fatigue is quantified by the S-N curve, in which the number N of cycles is plotted logarithmically.

The effect of cyclic stresses is to initiate microcracks at centers of stress concentration within the material or on the surface resulting in the growth and propagation of cracks leading to failure.

As for fatigue testing, the rate of crack growth can be plotted in a log-log scale versus time. Testing the fatigue properties to generate an S-N curve entails monitoring the number of cycles to failure at various stress levels. This test requires a large number of specimens compared with the crack propagation test.

The *endurance limit* is the stress below which the material will not fail in fatigue no matter how many cycles are applied. Not all materials exhibit an endurance limit. (a practical limit is often chosen as 10^7 cycles).

The presence of a saline environment exacerbates fatigue.
Surface roughness exacerbates fatigue. A polished surface is better.
Rubbing or 'fretting' exacerbates fatigue. Re-design the part or use lubricants.
Heat treatment to introduce residual surface compression can be helpful.

Stress concentration

Ratio of local maximum stress to applied stress in the absence of the heterogeneity is called the stress concentration factor or SCF.

Stress concentration factors are determined from

- Elasticity theory
- Experiment
- Finite elements.

Stress concentration factors arise from

- Holes
- Notches
- Grooves
- Heterogeneities in loading
- Heterogeneities in material

Some particular values for holes and inclusions

Circular hole in plane uniaxial tension, SCF = K = 3.0

Elliptic hole, with a as major axis, b as minor axis, ρ as radius of curvature

$$SCF = K = (1 + 2\frac{a}{b}) = 1 + 2\sqrt{\frac{a}{\rho}}$$

Example, consider glass with theoretical strength of $\sigma^{ult} = 14$ GPa, with cracks 2 μm long with radius of curvature 1 A (0.1 nm). Then the strength of glass with these cracks is $\sigma = 14$ GPa/[SCF] = 70 MPa. This is about the strength of common glass.

Spherical cavity in uniaxial tension

$$SCF|_{polar} = -\frac{3+15\nu}{14-10\nu} SCF|_{eq\theta\theta} = \frac{27-15\nu}{14-10\nu} SCF|_{eq\psi\psi} = -\frac{3-15\nu}{14-10\nu}$$

Spherical cavity in biaxial tension

Spherical cavity in pure shear

$$SCF = \frac{15(1-\nu)}{7-5\nu}$$

Rigid cylindrical inclusion in uniaxial tension

$$SCF|_{polar} = \frac{1}{2}\left(3-2\nu+\frac{1}{3-4\nu}\right) \qquad SCF|_{eq} = \frac{1}{2}\left(1+2\nu-\frac{3}{3-4\nu}\right)$$

Rigid spherical inclusion in uniaxial tension

$$SCF|_{polar} = \frac{2}{1+\nu} + \frac{1}{4-5\nu} \qquad SCF|_{eq} = \frac{\nu}{1+\nu} - \frac{5\nu}{8-10\nu}$$

Rigid spherical inclusion in hydrostatic tension

contact region radius: $a = 0.880\ (FR/E)^{1/3}$. It increases slowly with force.

peak compressive stress: $p_0 = 0.616\ (FE^2/R^2)^{1/3}$. It increases slowly with force.

Stress vs radial position in region, $\sigma_z = -\, p_0 \dfrac{\sqrt{a^2 - r^2}}{a}$, a parabolic distribution.

Overall 3-D pattern of stress is complex and multiaxial. Cracks may develop below the surface in ball bearings.

$$SCF|_{radial} = 3\,\frac{1-\nu}{1+\nu}$$

Reference: Goodier, J. N., "Concentration of stress around spherical and cylindrical inclusions and flaws", *Trans. ASME* Vol. 55, 1933, 39-44. (later called *J. Applied Mech.*, Vol. 1)

Observe that for the three dimensional cases, the stress concentration factor depends on the Poisson's ratio of the material in question.

A heterogeneous load distribution

Consider a **rigid circular cylindrical indenter** of radius R pressed with load F on a semi infinite solid substrate. This could represent a building erected upon compliant earth, or an industrial press operation. A solution for an elastic solid of Young's modulus E and Poisson's ratio ν is available. The indenter displacement is (Timoshenko, S. P. and Goodier, J. N., *Theory of Elasticity*, McGraw Hill, 1982.)

$$w = \frac{F(1 - \nu^2)}{2RE}\,.$$

The pressure distribution q(r) as a function of radial coordinate r is

$$q(r) = \frac{F}{2\pi R\sqrt{R^2 - r^2}}\,.$$

Observe that the pressure becomes **singular** at the edge. The indenter is idealized as perfectly rigid (much stiffer than the elastic substrate), and with a perfectly sharp edge.

Uses of concept of stress concentration.

- Find stress distribution (nominal) in the absence of holes.
- Multiply nominal stresses by the appropriate stress concentration factors. Many of these may be obtained from a handbook.
- The largest stress will cause failure. It is not necessarily the largest nominal stress.
- In the design process it is sensible to ameliorate stress concentrations by avoiding sharp re-entrant corners, and rounding them off when they are an unavoidable part of a structure.

Demonstrations, by photoelasticity. Circular hole at center of a compressed bar. Circular hole in bar subjected to pure bending. Circular notches in bar subjected to pure bending.

Contact stress

From the theory of elasticity, we have several interesting solutions for spheres and cylinders in contact.

For spheres of radius R of Young's modulus E, Poisson's ratio ν, under force F,

Fracture mechanics

As elliptic hole becomes progressively narrower, the ellipse approaches a crack shape and SCF = K $\rightarrow\infty$. Actual observed stress concentration factors for cracks are not infinite.

Therefore a material with one perfectly sharp crack will have **zero** strength, since the stress concentration factor becomes infinite. Experimentally, even for brittle materials, strength is reduced by cracks but not infinitely.

A criterion based on energy balance rather than on stress has therefore been adopted. The approach is due to Griffith. The energy relations are visualized as follows. The energy required to extend a crack is linear in crack length because the energy is expended in creating new free surfaces. By contrast the strain energy available comes from a roughly semicircular region around the crack and is therefore quadratic in crack length. As the crack propagates, the material in that region is unloaded and its energy is made available to drive the crack. When the crack is long enough that an increment in crack length gives rise to an energy release equal to energy expended, catastrophic crack growth impends.

Griffith proposed an **energy** approach to fracture. The elastic energy stored in a test specimen of unit thickness, in a circular region around a crack of length a, is:

$$2\pi a^2 \frac{1}{2E} \sigma^2 \qquad \text{(F1)}$$

Recall that $\frac{1}{2}\ E\varepsilon^2 = \frac{1}{2E}\ \sigma^2$ represents a strain energy per unit volume.

The elastic energy for a brittle material is twice the area under the stress strain curve. The elastic energy is used to create two new surfaces as the crack propagates. The surface energy, 4 γa (γ is the surface energy; it is an energy per unit area.) should be smaller than the elastic energy for the crack to grow. Thus, the incremental changes of both energies for the crack to grow can be written,

$$\frac{d}{da}\left(\frac{\pi(a\sigma)^2}{E}\right) = \frac{d}{da}(4\gamma a) \qquad \text{(F2)}$$

Hence,

$$\sigma = \sigma_f = \sqrt{\frac{2\gamma E}{\pi a}} \qquad \text{(F3)}$$

Since for a given material E and γ are constants,

$$\sigma_f = \frac{K}{\sqrt{\pi a}} \qquad \text{(F4)}$$

In this case K has the units of psi $\sqrt{in}$ or MPa $\sqrt{m}$ and is proportional to the energy required for fracture.
K is a measure of *fracture toughness,* called the stress intensity factor. Cracks and stress concentrations also occur in ductile materials, but their effect is usually not as serious as in brittle ones since local yielding which occurs in the region of peak stress will effectively blunt the crack and alleviate the stress concentration.

The **stress intensity factor** K is the criterion for fracture in cracked objects. For a small Mode I crack of length a,

$K_1 = \sigma \sqrt{\pi a}$ f(a/c).

Here f(a/c) is a dimensionless function of loading geometry; it expresses the effect of crack length in relation to block size. σ is the stress required for fracture in the absence of a crack. The units for K are MPa$\sqrt{m}$, in contrast to the stress concentration factor which is dimensionless. Observed that there is no characteristic length scale in the classical theory of elasticity. The length scale must come from other considerations.

Fracture occurs when K_1 exceeds a critical value, K_{1c} determined from experiment. This is the fracture toughness based on a static test. The fracture toughness for a dynamic situation is NOT the same as for a static situation

Formulas for K are valid over a range of geometrical parameters, specifically, thickness $t \geq 2.5\ (K_{1c}/\sigma_y)^2$, and crack length $a \geq 2.5\ (K_{1c}/\sigma_y)^2$.

In a thick block, the stress field around the tip of the crack is triaxial, since the Poisson contraction in the highly stressed region near the crack is restrained by the surrounding material, which is not so highly stressed. This triaxial stress causes brittle behavior in seemingly ductile materials, since shear deformation is suppressed.

If the block is thinner than the above limit, toughness depends on thickness. If the crack length is less than the above limit, then the material may undergo yield before any fracture occurs from the crack.

Be aware that K_{1c} depends on temperature, and often drops precipitously at low temperature.

Example

Estimate the size of the surface flaw in a glass whose modulus of elasticity and surface energy are 70 GPa and 800 erg/cm^2 respectively. Assume that the glass breaks at a tensile stress of 100 MPa.

Answer
From equation (F4), and keeping in mind the transformation from cgs to SI units,

$$a = \frac{2\gamma E}{\pi \sigma_f^2}$$

$$= \frac{2 \times 800 \text{ dyne/cm} \times 70 \text{ GPa}}{\pi\,(100 \text{ MPa})^2}$$

$$= \underline{3.565\ \mu m}$$

To two significant figures, a = 3.6 μm.
[Note that if the crack is on the surface its length is a, if it is inside the specimen it is 2a. Remember 1 erg = 1 dyne cm]

Example (adapted from example on page 62-63)

Consider a 1 cm crack in a large plate of steel. Assume the plate is large enough that f(a/c) ≅1.

What stress gives rise to fracture for a weaker or 'mild' steel (σ_y = 500 MPa, K_{1c} = 175 MPa√m) and a high strength steel (σ_y = 1410 MPa, K_{1c} = 50 MPa√m).

Solution: Use $K_1 = \sigma\, f(a/c) \sqrt{\pi a}$, so with f(a/c) ≅ 1, $\sigma = K_{1c}/\sqrt{\pi a}$.

Weaker steel A, σ = 987 MPa, which exceeds the yield strength, so it undergoes yield at 500 MPa rather than fracture from the crack.

Stronger steel, σ = 282 MPa, which is less than the yield strength 1410 MPa, so it fractures catastrophically.

Consequently, in the presence of 1 cm cracks, the stronger steel actually is weaker than the 'mild' weaker steel. Moreover, the consequence of an overload is more severe for the stronger steel since it catastrophically fractures rather than yields in bulk.

Example (adapted from Gordon, *Structures*)

Suppose we have a large structure such as a ship or a bridge and wish to tolerate a 1 meter long crack without catastrophic failure. Consider 'mild' steel (σ_y = 500 MPa, K_{1c} = 175 MPa√m) .

Solution-

With f(a/c) ≅ 1, $\sigma = K_{1c}/\sqrt{\pi a}$ = 90 MPa or 14,000 psi.

In **foam**, Gibson and Ashby [*Cellular solids*] predict toughness K_{1c} proportional to [√(cell size)](density)$^{3/2}$.

Stress concentrations: appendix

Experimental stress concentrations in composite materials are consistently less than the theoretical ones. The non-classical fracture behavior has been dealt with using point stress and average stress criteria, however that approach cannot account for non-classical strain distributions in objects under small load. Such differences may be accounted for via a generalized continuum approach. Reduced stress concentration factors for small holes are known experimentally in fibrous composite materials. The fracture strength of graphite epoxy plates with holes depends on the size of the hole [1]. Moreover the strain around small holes and notches in fibrous composites well below the yield point is smaller than expected classically [2,3], while for large holes, the strain field follows classical predictions [4].

1. R.F. Karlak, "Hole effects in a related series of symmetrical laminates", in *Proceedings of failure modes in composites, IV*, The metallurgical society of AIME, Chicago, 106-117, (1977)
2. J.M. Whitney, and R.J. Nuismer, "Stress fracture criteria for laminated composites containing stress concentrations", *J. Composite Materials*, **8**, (1974) 253-275.
3. M. Daniel, "Strain and failure analysis of graphite-epoxy plates with cracks", *Experimental Mechanics,* **18** (1978) 246-252, .
4. R. E. Rowlands, I. M. Daniel, and J. B. Whiteside "Stress and failure analysis of a glass-epoxy plate with a circular hole", *Experimental Mechanics*, **13**, (1973) 31-37

Stress concentration

Ratio of local maximum stress to applied stress in the absence of the heterogeneity is called the stress concentration factor or SCF.

Stress concentration factors are determined from

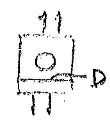
- Elasticity theory
- Experiment
- Finite elements.

Stress concentration factors arise from

- Holes
- Notches
- Grooves
- Heterogeneities in loading
- Heterogeneities in material

Some particular values for holes and inclusions

Circular hole in plane uniaxial tension, SCF = K = 3.0

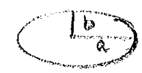


$3.0 - f\left(\frac{r}{D}\right)$

Elliptic hole, with a as major axis, b as minor axis, ρ as radius of curvature

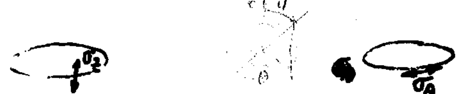
$$SCF = K = \left(1 + 2\frac{a}{b}\right) = 1 + 2\sqrt{\frac{a}{\rho}}$$

@ semi major $\rho = \frac{b^2}{a}$; @ semi minor $\rho = a^2/b$



Example, consider glass with theoretical strength of σ^{ult} = 14 GPa, with cracks 2 μm long with radius of curvature 1 A (0.1 nm). Then the strength of glass with these cracks is σ = 14 GPa/[SCF] = 70 MPa. This is about the strength of common glass.

Spherical cavity in uniaxial tension



σ_θ, σ_r

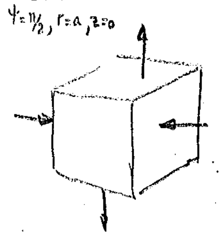
$$SCF|_{polar} = -\frac{3+15\nu}{14-10\nu} \quad SCF|_{eq\theta\theta} = \frac{27-15\nu}{14-10\nu} \quad SCF|_{eq\psi\psi} = -\frac{3-15\nu}{14-10\nu}$$

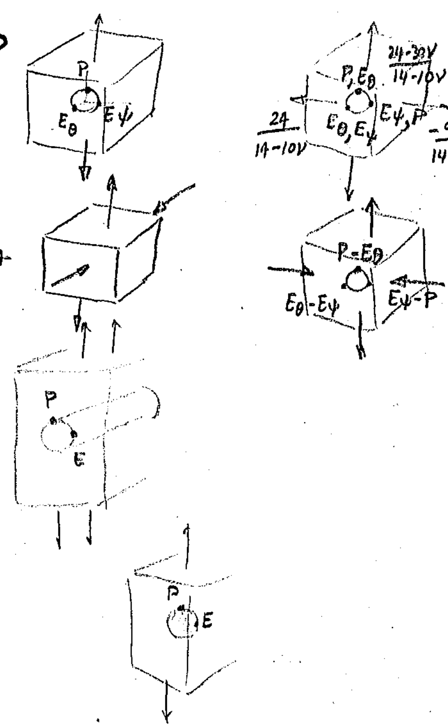
($\psi = 0, \pi$, $r = a$; $\psi = \pi/2$, $r = a$; $\psi = \pi/2$, $r = a$, $z = 0$)

Spherical cavity in biaxial tension

Spherical cavity in pure shear

$$SCF = \frac{15(1-\nu)}{7-5\nu}$$





Rigid cylindrical inclusion in uniaxial tension

$$SCF|_{polar} = \frac{1}{2}\left(3-2\nu + \frac{1}{3-4\nu}\right) \qquad SCF|_{eq} = \frac{1}{2}\left(1+2\nu - \frac{3}{3-4\nu}\right)$$

Rigid spherical inclusion in uniaxial tension

$$SCF|_{polar} = \frac{2}{1+\nu} + \frac{1}{4-5\nu} \qquad SCF|_{eq} = \frac{\nu}{1+\nu} - \frac{5\nu}{8-10\nu}$$

Rigid spherical inclusion in hydrostatic tension

$$SCF|_{radial} = 3\,\frac{1-\nu}{1+\nu}$$

Reference: Goodier, J. N., "Concentration of stress around spherical and cylindrical inclusions and flaws", *Trans. ASME* Vol. 55, 1933, 39-44. (later called *J. Applied Mech.*, Vol. 1)

Observe that for the three dimensional cases, the stress concentration factor depends on the Poisson's ratio of the material in question.

A heterogeneous load distribution

Consider a **rigid circular cylindrical indenter** of radius R pressed with load F on a semi infinite solid substrate. This could represent a building erected upon compliant earth, or an industrial press operation. A solution for an elastic solid of Young's modulus E and Poisson's ratio ν is available. The indenter displacement is (Timoshenko, S. P. and Goodier, J. N., *Theory of Elasticity*, McGraw Hill, 1982.)

$$w = \frac{F(1 - \nu^2)}{2RE}.$$

The pressure distribution q(r) as a function of radial coordinate r is

$$q(r) = \frac{F}{2\pi R\sqrt{R^2 - r^2}}.$$

unit force $w = \frac{P(1-\nu^2)}{\pi E R}$

for constant load q radius of load is R $w = \frac{2(1-\nu^2)qR}{\pi E}$

Observe that the pressure becomes **singular** at the edge. The indenter is idealized as perfectly rigid (much stiffer than the elastic substrate), and with a perfectly sharp edge.

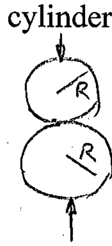
Uses of concept of stress concentration.

- ∅ Find stress distribution (nominal) in the absence of holes.
- ∅ Multiply nominal stresses by the appropriate stress concentration factors. Many of these may be obtained from a handbook.
- ∅ The largest stress will cause failure. It is not necessarily the largest nominal stress.
- ∅ In the design process it is sensible to ameliorate stress concentrations by avoiding sharp re-entrant corners, and rounding them off when they are an unavoidable part of a structure.

Demonstrations, by photoelasticity. Circular hole at center of a compressed bar. Circular hole in bar subjected to pure bending. Circular notches in bar subjected to pure bending.

Contact stress (Hertz Contact Stresses)

From the theory of elasticity, we have several interesting solutions for spheres and cylinders in contact.



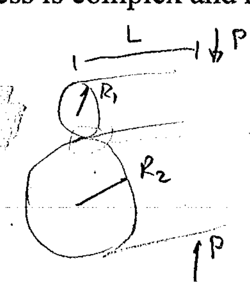
For spheres of radius R of Young's modulus E, Poisson's ratio ν, under force F, contact region radius: $a = 0.880\ (FR/E)^{1/3}$. It increases slowly with force.

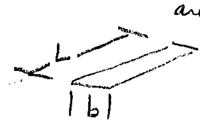
peak compressive stress: $p_0 = 0.616\ (FE^2/R^2)^{1/3}$. It increases slowly with force.

Stress vs radial position in region, $\sigma_z = -\, p_0 \frac{\sqrt{a^2 - r^2}}{a}$, a parabolic distribution.

Overall 3-D pattern of stress is complex and multiaxial. Cracks may develop below the surface in ball bearings.

For parallel cylinders





area of contact $b = 3.04\sqrt{\frac{P}{LE}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}}$

$E_{1,2} = E \quad \nu_{1,2} = .3$

$p_0 = 0.418\sqrt{\frac{PE}{L}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$

$L >> 5b$

Most failure criteria for composite materials address failure of an individual ply. Failure of the laminated structure is more complicated because of unresolved questions about interlaminar stresses, how degraded plies unload, and how failure of some plies influences the remaining intact plies. At present, no failure criterion intended for a laminated structure is reliable enough to be used without experimental confirmation.

3.5 INTRODUCTION TO FRACTURE MECHANICS

Cracks and Brittle Fracture. One expects that materials such as glass and rock will fail in a brittle manner. A normally ductile material such as structural steel may also fail in a brittle manner if it contains a crack in a region of tensile stress. Typically a crack begins at a stress raiser and grows gradually, due to cyclic loading or due to corrosion under steady loading. When a crack reaches a "critical length" it suddenly propagates as a brittle fracture, and the part or structure breaks, perhaps completely in two. Complete separation may be prevented by progagation of the crack into a "crack arrester" such as an existing hole, or by deformations that happen to relieve the mechanism that causes the crack to propagate. Crack propagation speeds may exceed 1000 m/s.

The Liberty cargo ships of World War II are classic examples of this kind of failure. Of some 2700 built, more than 100 broke in two. Part of the trouble was welded construction, in which edges of adjacent plates were welded together. (Previously, ships were constructed of overlapping plates connected by rivets, thus incorporating "crack arresters" in the structure.) Also, the material itself was made more susceptible to brittle fracture by heat of the welding process and by cold conditions in which these ships often operated.

The state of stress at a crack tip causes material there to lose ductility. Consider, for example, a flat plate with a crack oriented perpendicular to the direction of load (Fig. 3.5-1a). Near the crack tip, normal stresses in the plane of the plate are tensile and very large. Consequently, due to the Poisson effect, material around the crack tip tries to contract in the thickness direction (normal to the plate surface). However, the

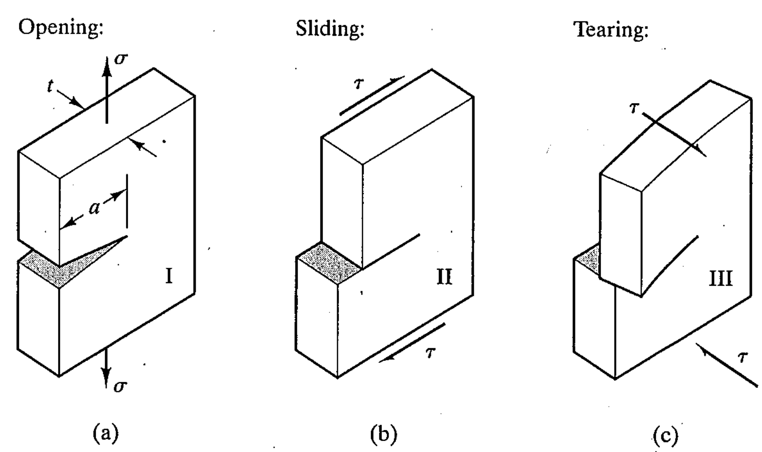
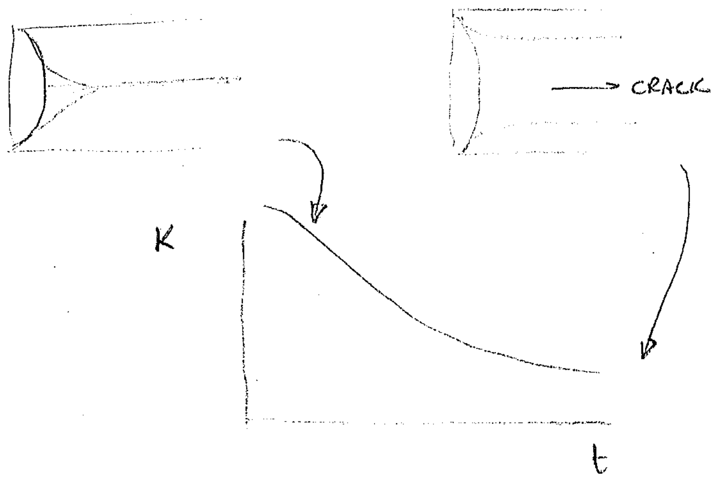
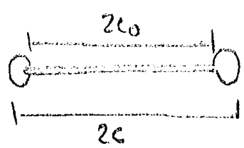


FIGURE 3.5-1 The three crack modes, commonly named I, II, and III.

IF TOO THIN SHEAR LIPS FORM BY TESTS $B \geq 2.5 \left(\frac{K_{Ic}}{\sigma_{yp}} \right)^2$



Crack length requirement ALSO CONSIDERATION OF PLASTIC ZONE IN FRONT OF CRACK & THAT FRACTURE OCCURS BEFORE YIELD



$2c \doteq 2c_0 + R$

$R = \frac{1}{6} \cdot \frac{1}{\pi} \left(\frac{K_I}{\sigma_{yp}} \right)^2$ in plane strain

IF $2c_0 > \frac{1}{2} \left(\frac{K_{Ic}}{\sigma_{yp}} \right)^2$ FRACTURE

$2c_0 < \frac{1}{2} \left(\frac{K_{Ic}}{\sigma_{yp}} \right)^2$ YIELD

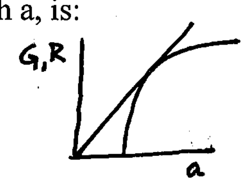
SRAWLEY & BROWN $c_0 > 2.5 \left(\frac{K_{Ic}}{\sigma_{yp}} \right)^2$

Fracture mechanics

As elliptic hole becomes progressively narrower, the ellipse approaches a crack shape and SCF = K $\rightarrow\infty$. Actual observed stress concentration factors for cracks are not infinite.

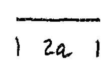
Therefore a material with one perfectly sharp crack will have **zero** strength, since the stress concentration factor becomes infinite. Experimentally, even for brittle materials, strength is reduced by cracks but not infinitely.
A criterion based on energy balance rather than on stress has therefore been adopted. The approach is due to Griffith. The energy relations are visualized as follows. The energy required to extend a crack is linear in crack length because the energy is expended in creating new free surfaces. By contrast the strain energy available comes from a roughly semicircular region around the crack and is therefore quadratic in crack length. As the crack propagates, the material in that region is unloaded and its energy is made available to drive the crack. When the crack is long enough that an increment in crack length gives rise to an energy release equal to energy expended, catastrophic crack growth impends.

Griffith proposed an **energy** approach to fracture. The elastic energy stored in a test specimen of unit thickness, in a circular region around a crack of length a, is:



$$2\pi a^2 \frac{1}{2E}\sigma^2 = 2\cdot \text{strain energy density} \cdot \pi a^2 \cdot 1 \cdot \qquad \text{(F1)}$$

Recall that $\frac{1}{2}E\varepsilon^2 = \frac{1}{2E}\sigma^2$ represents a strain energy per unit volume.
The elastic energy for a brittle material is twice the area under the stress strain curve. The elastic energy is used to create two new surfaces as the crack propagates. The surface energy, 4 γa (γ is the surface energy; it is an energy per unit area.) should be smaller than the elastic energy for the crack to grow. Thus, the incremental changes of both energies for the crack to grow can be written,

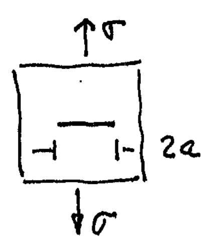


2 surfaces ∴ energy = $2\gamma\cdot 2a$
surface energy

$$\frac{d}{da}\left(\frac{\pi(a\sigma)^2}{E}\right) = \frac{d}{da}(4\gamma a) \qquad \text{(F2)}$$

$$\frac{2\pi a\sigma^2}{E} = 4\gamma$$

Hence,



$$\sigma = \sigma_f = \sqrt{\frac{2\gamma E}{\pi a}} \qquad \text{(F3)}$$

Since for a given material E and γ are constants,

$$\sigma_f = \frac{K}{\sqrt{\pi a}} \qquad \text{(F4)}$$

In this case K has the units of psi $\sqrt{\text{in}}$ or MPa $\sqrt{\text{m}}$ and is proportional to the energy required for fracture.
K is a measure of *fracture toughness,* called the stress intensity factor. Cracks and stress concentrations also occur in ductile materials, but their effect is usually not as serious as in brittle ones since local yielding which occurs in the region of peak stress will effectively blunt the crack and alleviate the stress concentration.

The **stress intensity factor** K is the criterion for fracture in cracked objects. For a small Mode I crack of length a,

$K_1 = \sigma \sqrt{\pi a}\ f(a/c)$.

Here f(a/c) is a dimensionless function of loading geometry; it expresses the effect of crack length in relation to block size. σ is the stress required for fracture in the absence of a crack. The units for K are MPa√m, in contrast to the stress concentration factor which is dimensionless. Observed that there is no characteristic length scale in the classical theory of elasticity. The length scale must come from other considerations.

Fracture occurs when K_1 exceeds a critical value, K_{1c} determined from experiment. This is the fracture toughness based on a static test. The fracture toughness for a dynamic situation is NOT the same as for a static situation.

Formulas for K are valid over a range of geometrical parameters, specifically, thickness $t \geq 2.5\ (K_{1c}/\sigma_y)^2$, and crack length $a \geq 2.5\ (K_{1c}/\sigma_y)^2$.

In a thick block, the stress field around the tip of the crack is triaxial, since the Poisson contraction in the highly stressed region near the crack is restrained by the surrounding material, which is not so highly stressed. This triaxial stress causes brittle behavior in seemingly ductile materials, since shear deformation is suppressed.

If the block is thinner than the above limit, toughness depends on thickness. If the crack length is less than the above limit, then the material may undergo yield before any fracture occurs from the crack.

Be aware that K_{1c} depends on temperature, and often drops precipitously at low temperature.

Example

Estimate the size of the surface flaw in a glass whose modulus of elasticity and surface energy are 70 GPa and 800 erg/cm^2 respectively. Assume that the glass breaks at a tensile stress of 100 MPa.

Answer

From equation (F4), and keeping in mind the transformation from cgs to SI units,

$$a = \frac{2\gamma E}{\pi \sigma_f^2}$$

$$= \frac{2 \times 800\ \text{dyne/cm} \times 70\ \text{GPa}}{\pi\ (100\ \text{MPa})^2}$$

$$= \underline{3.565\ \mu\text{m}}$$

To two significant figures, a = 3.6 μm.
[Note that if the crack is on the surface its length is a, if it is inside the specimen it is 2a.
Remember 1 erg = 1 dyne cm]

I. ASTM model for K_{Ic} testing

A. designed to produce valid K_{Ic} results - How?

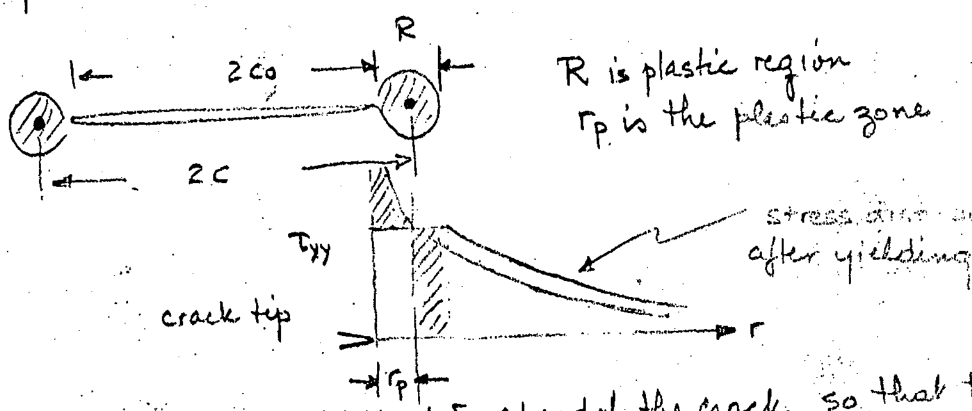
1. must meet $c_0 \geq 2.5 \left(K_{Ic}/\sigma_y\right)^2$
2. " " $B \geq 2.5 \left(K_{Ic}/\sigma_y\right)^2$; $W/B \cong 2$
3. starting crack length must be 0.45 - 0.55 W (width of specimen)
4. crack must be sharp and must be introduced via a fatigue crack starting from a V-notch
5. The fatigue crack must be introduced by <u>low</u> type cycling
6. A displacement gage will be used to accurately measure the <u>relative</u> displacement of two precisely located gages positions
7. Temperature and load rate requirements

B. Why these requirements -

1. $c_0 \geq 2.5 \left(K_{Ic}/\sigma_y\right)^2$. this is a requirement that is necessary and sufficient in order for LEFM to hold

<u>Proof</u>:

Consider a plate loaded in tension



- We assume that the stresses are redistributed ahead of the crack so that the load bearing capacity in front of the crack is unchanged when yielding occurs. We assume that the shaded areas under the graph are the same.
- Thus $2c = 2c_0 + 2r_p = 2c_0 + R$ is the <u>effective</u> length of the crack
- In plane strain mode I $R = \frac{1}{6\pi}\left(K_I/\sigma_y\right)^2$

and $c = c_0 + \frac{1}{12\pi}\left(K_I/\sigma_y\right)^2$

if the stress σ ↑ K_I ↑ also K_I ↑ due to the plastic zone correction.

Thus $$K_I = \sigma\sqrt{\pi c_0}\left\{1 - \frac{1}{12}\left(\sigma/\sigma_y\right)^2\right\}^{-1/2} \quad (1)$$

- In a test as $\sigma \to \sigma_y$, $K_I \to K_{Ic}$

- If σ reaches σ_y before $K_I = K_{Ic}$ we get yielding and by our elastic-plastic model r_p (and R) $\to \infty$. Hence we violate the LEFM assumption of small scale yielding

- We want $K_I = K_{Ic}$ before $\sigma = \sigma_y$. Thus let $K_I = K_{Ic}$ in (1) and solve for the crack length $2c_0$

$$2c_0 = \frac{2}{\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2\left\{\left(\sigma_y/\sigma\right)^2 - \frac{1}{12}\right\} \qquad K_I = K_{Ic}$$

This will cause unstable crack growth

- The crack length that produces yielding is when $\sigma_y = \sigma$

or $$2c_0 = \frac{11}{12}\cdot\frac{2}{\pi}\left(K_{Ic}/\sigma_y\right)^2 \sim \frac{1}{2}\left(\frac{K_{Ic}}{\sigma_y}\right)^2$$

if $\sigma > \sigma_y$ then $2c_0 < \frac{1}{2}\left(K_{Ic}/\sigma_y\right)^2$ unacceptable

$\sigma < \sigma_y$ then $2c_0 > \frac{1}{2}\left(K_{Ic}/\sigma_y\right)^2$ or $c_0 > \frac{1}{4}\left(K_{Ic}/\sigma_y\right)^2$

- Because we want to make adequate measurements of K_{Ic}

we want $c_0 >> \frac{1}{4}\left(K_{Ic}/\sigma_y\right)^2$

Srawley and Brown suggested that $c_0 \geqslant 2.5\left(K_{Ic}/\sigma_y\right)^2$ and this is accepted as the standard.

2. $B \geq 2.5 \left(K_{Ic}/\sigma_y \right)^2$: This requirement arises from the consideration that we want only MODE I type fracture

Proof:

- As has been shown to you in class in order for cracks to propagate for perfectly brittle materials, the crack extension force $G_I = 2\gamma_s$, where γ_s is the surface energy. However for materials that deform plastically, then crack extension will only occur when $G_I = 2\gamma_s + p$ where p is the plastic work of crack extension. p is not a constant and depends on the size of the plastic zone, σ_y, the work hardening rate, etc AND they all in turn depend on the crack length.

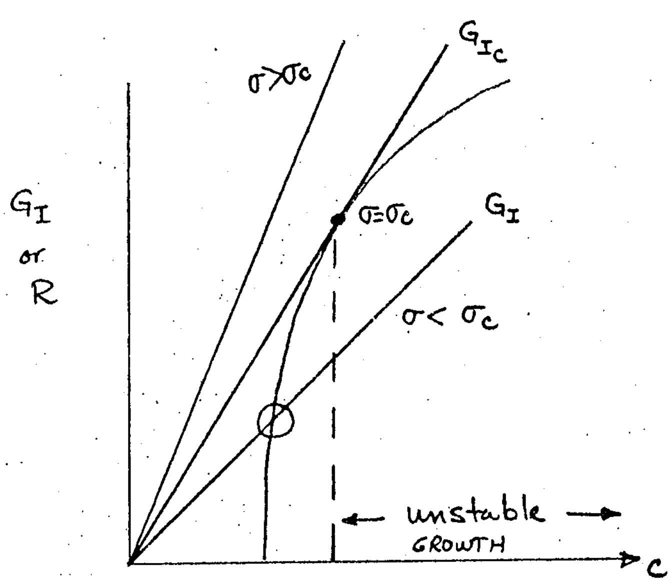
- If we define $R \equiv$ crack extension resistance $= 2\gamma_s + p$, then for unstable growth we must have that

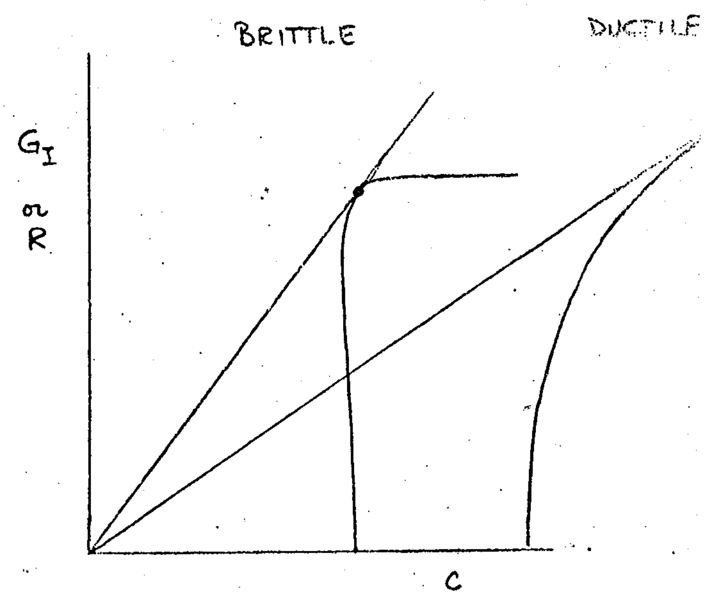
$$G_I \geq R$$

and also

$$\partial G_I / \partial c \geq \frac{\partial R}{\partial c}$$

Thus if we remember that $G_I = \frac{\sigma^2 \pi c}{2\mu}(1-\nu)$ and look at a typical G_I versus c curve,





Note that the brittle material shows little plastic deformation and has a well defined $G_I = R$ point of intersection and occurs below the

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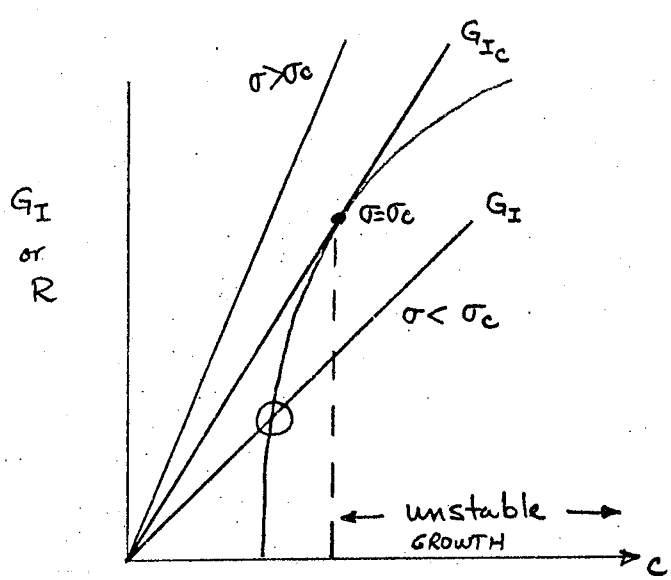
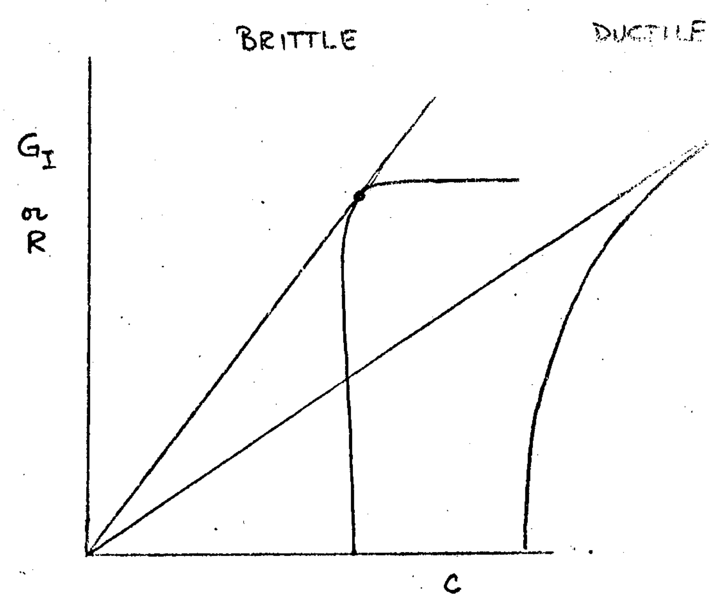
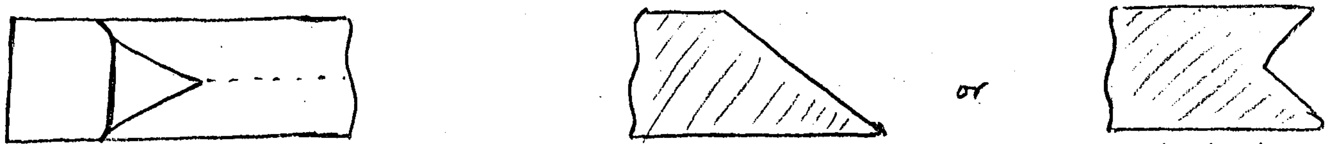




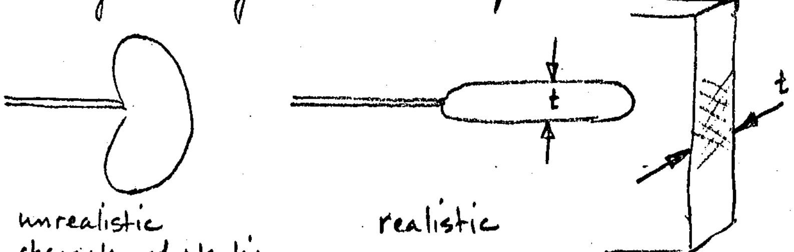
Note that the brittle material shows little plastic deformation and has a well defined $G_I = R$ point of intersection and occurs below the

what has been found is that as the plate is made thinner the R curve will vary and will no longer have a distinct intersection point. The reason for this is the growth of "shear lips" from the free surface and the thinness of the plate (plane stress conditions).



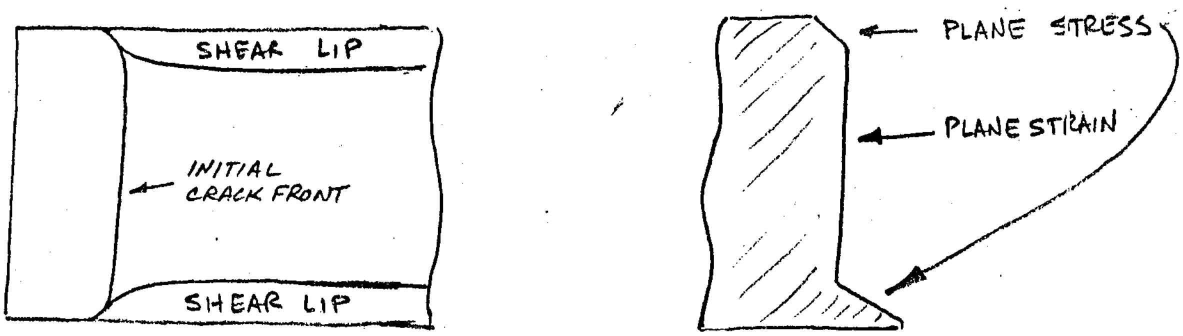
DIR. OF crack propagation → SLANT V SLANT

The growth of the shear lips is due to the plastic zone being constrained in the thickness direction. So it will spread in front of the crack tip. The mechanism that will cause crack extension will be due to failure in <u>shear</u> (mode III); hence we see the slant formation.

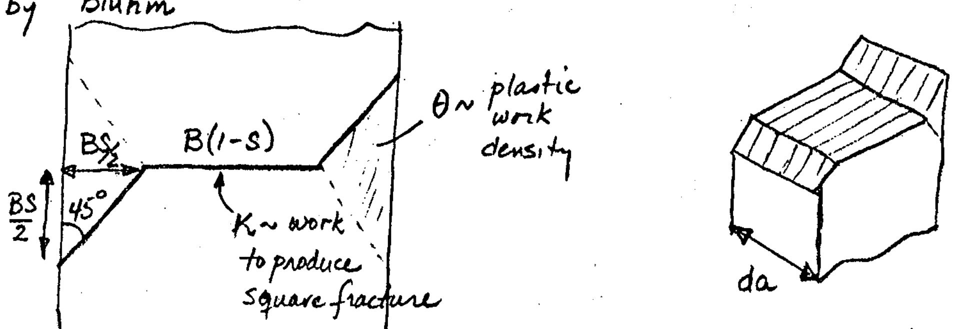


unrealistic character of plastic zone. realistic

As the plate width is increased, the formation of the shear lips is reduced due to the plane strain effect and the cross-section will look like this



Many have proposed models to describe what occurs here. One such model is that of Krafft, Sullivan and Boyle (1961) modified by Bluhm

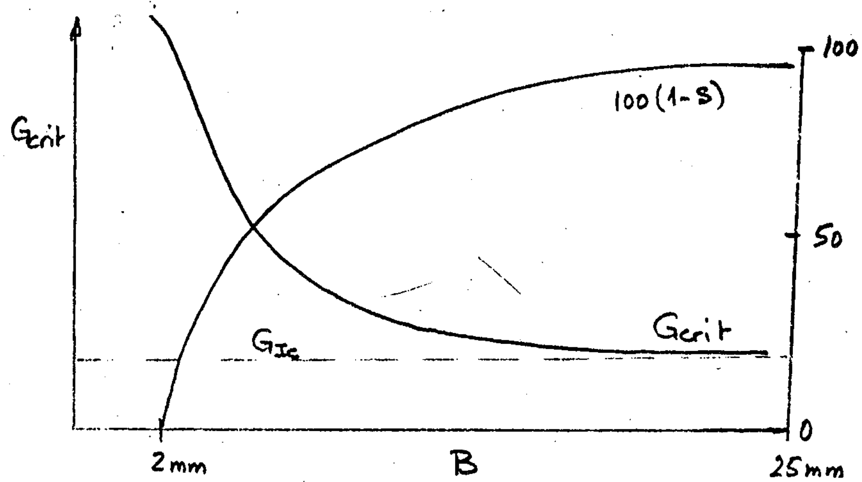


1. square fracture $\neq f(c_0)$
2. shear lips are assumed to occur at 45°
3. flat fracture is a surface phenomenon
4. Shear lip is volumetric

The work done to create the crack surface da is:

Now $G_I = \frac{1}{B}\frac{dW}{da} = \kappa(1-S) + \frac{BS^2\theta}{2}$. Note that S is picked so that BS = constant as crack length ↑

Thus as $B \to \infty$ $S \to 0$ and $G_I \to \kappa$.

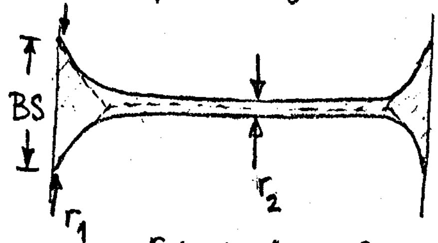


From data for Aluminum 7075-T6

$K \sim 200\ \mathrm{kJ/m^2}$

$\theta \sim 20\ \mathrm{kJ/m^2}$

Look at the plastic zone and superpose the model of Krafft:



$BS \gg r_{\text{plane stress}} \sim r_p = \frac{1}{2\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2$ (*)

since $r_1 > r_2$ (plane strain). If (*) is true then plain strain conditions will extend over most of the cross-section and we will have essentially mode I fracture.

$r_p/B \ll 1$ or $\frac{r_p}{B} < S \ll 1$

hence $B \gtrsim \left(\frac{K_{Ic}}{\sigma_y}\right)^2$. To determine the real equation, tests were done on many types of metals and here are some of the results.

Example: Maraging Steel $\sigma_y = 259\ KSI$

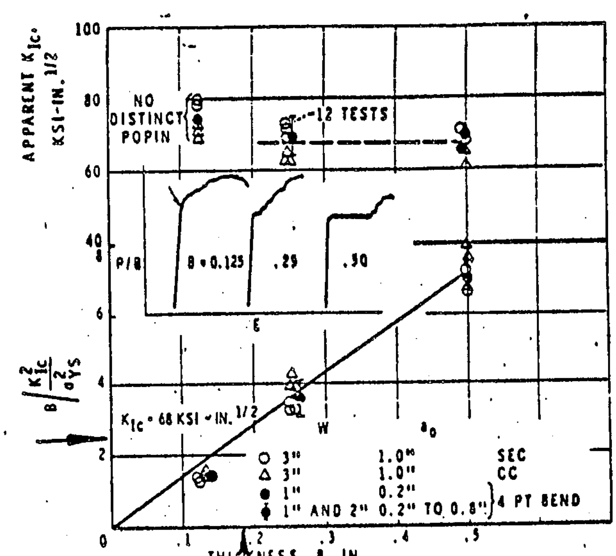


FIG. 14—*Effect of thickness on popin behavior and apparent K_{Ic} for 259 ksi*

conclusion:

$B \geqslant 2.5\left(\frac{K_{Ic}}{\sigma_y}\right)^2$

Specimen Size Requirements

We have argued that to limit yielding we must make large samples with long cracks. Thus $K_I \to K_{I_c}$ before $\sigma \to \sigma_y$. From our analysis we expect

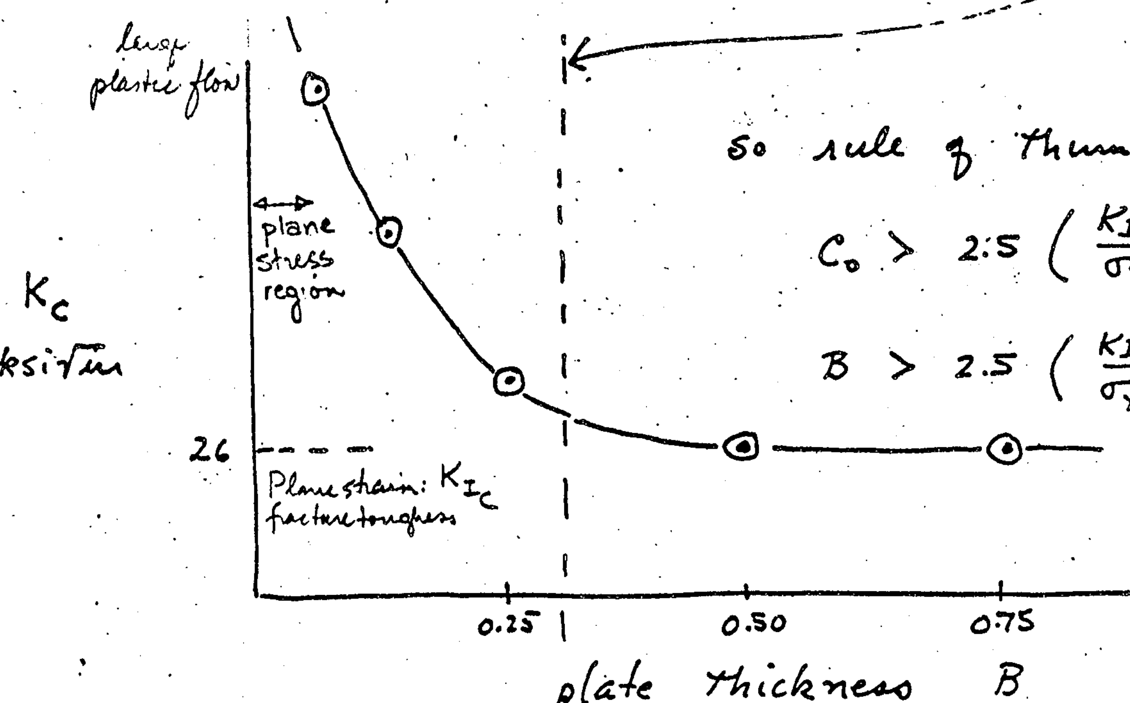
$$C_o \geq 2.5 \left(\frac{K_{I_c}}{\sigma_y} \right)^2$$

this needs to be checked. Also, How thick must sample be for plane strain conditions?

Consider 7075-T6 (MS&E 202C experiment).

$\sigma_y = 75$ ksi

$K_{I_c} = 26$ ksi $\sqrt{in}$ $\qquad 2.5 \left(\frac{K_{I_c}}{\sigma_y} \right)^2 = 0.3$ in



so rule of thumb:

$$C_o > 2.5 \left(\frac{K_{I_c}}{\sigma_y} \right)^2$$

$$B > 2.5 \left(\frac{K_{I_c}}{\sigma_y} \right)^2$$

← usually more difficult to achieve due to how the specimen is machi[ned]

; much lower stresses. Thickness-direction tension at the
il just ahead of the crack tip is therefore in a state of *triax-*
ure more than plastic flow.

agate at great speed after reaching a critical length? The
rgy considerations, which are summarized as follows.
ometry of Fig. 3.5-1a, which is shown again in Fig. 3.5-2a.
crack an amount da is independent of crack length a, so
luce the crack varies linearly with a (Fig. 3.5-2b). As the
uced in material alongside the crack, and stored strain
rgy released varies approximately quadratically with a.
gine that a crack of length a nullifies the uniaxial state of
of radius a, shown dashed in Fig. 3.5-2a. Strain energy
ortional to the volume of this disc, $V = \pi a^2 t/2$.) When a
ment of strain energy released to equal the increment
the crack, sudden fracture impends. Thus $dU_e = dU_s$ in
al crack length. Energy needed to drive the crack is sup-
d material. There is no need (and insufficient time) for
ork of external forces acting through a distance. The fail-
ould be if the crack were absent because the crack pro-
bonds to be broken sequentially rather than all at once.
d as shown in Fig. 3.5-1. In practice, Mode I is most com-
le. For any mode, one can calculate a *stress intensity fac-*
are it with an allowable value to determine whether
nsity factor is *not* a stress concentration factor! Indeed,
is it is not necessary to use stress concentration data:
d not be calculated.

ider only isotropic materials, and only Mode I cracks
ress intensity factor for a Mode I crack is denoted by K_I

$$K_I = \beta\sigma\sqrt{\pi a} \qquad (3.5\text{-}1)$$

at would exist if the crack were absent. Thus, stress σ is
Multiplier β is dimensionless and depends on geometry
3.5-1). Dimension a is defined as either the full crack
n geometry. Units of K_I are MPa$\sqrt{\text{m}}$. Fracture impends
K_{Ic} known as *fracture toughness.* K_{Ic} can be considered a
of specimen thickness t, if the specimen is sufficiently
nsion at the crack tip to develop fully. Also, the crack
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$$\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 \quad \text{and} \quad a \geq 2.5\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 \qquad (3.5\text{-}2)$$

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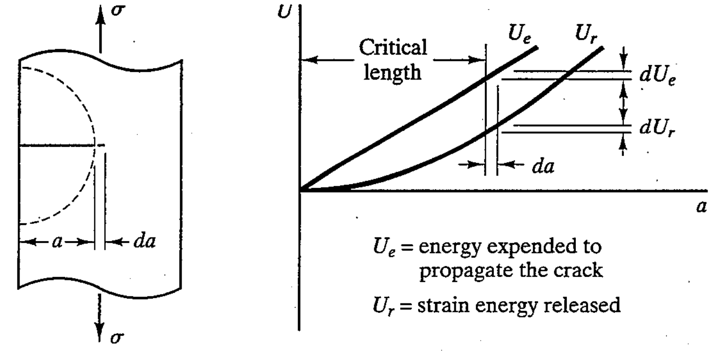
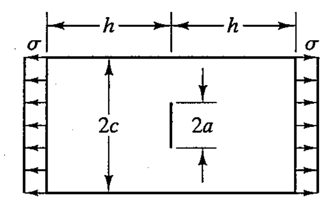
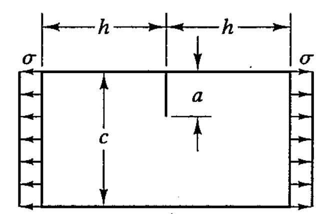
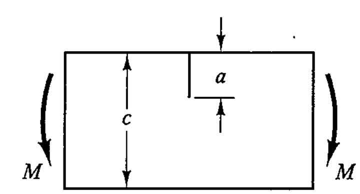
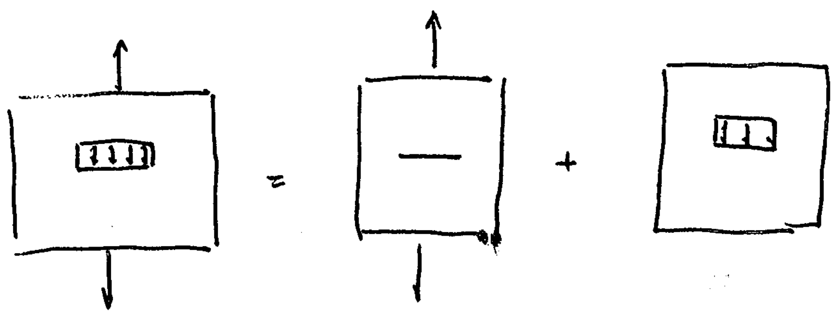


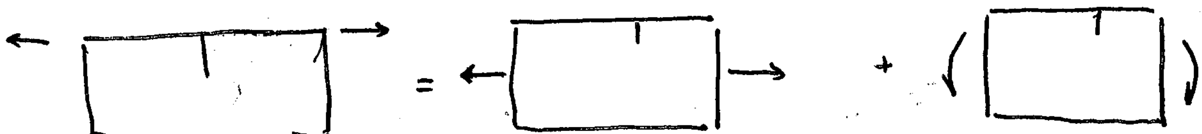
FIGURE 3.5-2 (a) Plate with an edge crack of length a. (b) Energy relations for crack extension.

TABLE 3.5-1 Stress intensity data for flat plates, of isotropic material and uniform thickness, with in-plane loading [3.6].

Tension, central crack of length $2a$ 	$K_I = \beta\sigma\sqrt{\pi a}$ $\beta = \dfrac{1 - 0.5(a/c) + 0.326(a/c)^2}{\sqrt{1-(a/c)}}$ Accurate to within 1% for all a/c, provided h/c is "large"
Tension, edge crack of length a 	$K_I = \beta\sigma\sqrt{\pi a}$ $\beta = [1.12 - 0.23(a/c) + 10.6(a/c)^2 - 21.7(a/c)^3 + 30.4(a/c)^4]$ Accurate to within 1% for $a/c \leq 0.6$, provided $h/c > 1$ and sides are free to rotate
Pure bending, edge crack of length a 	$K_I = \beta\sigma\sqrt{\pi a}$ $\sigma = \dfrac{M(c/2)}{I} = \dfrac{6M}{tc^2}$ $\beta = [1.12 - 1.39(a/c) + 7.32(a/c)^2 - 13.1(a/c)^3 + 14.0(a/c)^4]$ Accurate to within 1% for $a/c \leq 0.6$



$$K_I = K_{I_{(1)}} + K_{I_{(2)}}$$



$$K_{I_{TOT}} = \sum K_I^{(i)}$$

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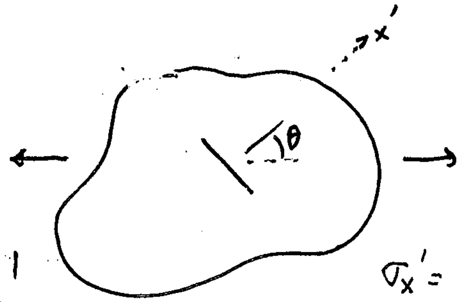

σ_x

find $\sigma_{x'}$ & $\tau_{x'y'}$

$K_I = \sigma_{x'}\sqrt{\pi a}$

$K_{II} = \tau_{x'y'}\sqrt{\pi a}$

$$\sigma_{x'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = \tau_{xy}\cos 2\theta - \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta \; ?$$

$K_I = \sigma_x'\sqrt{\pi a}$

$K_{II} = \tau_{x'y'}\sqrt{\pi a}$

$K_{I_c} = \sigma_y\sqrt{\pi a}$

$K_{II_c} = \tau\sqrt{\pi a}$

$$\left(\frac{K_I}{K_{I_c}}\right)^2 + \left(\frac{K_{II}}{K_{II_c}}\right)^2 \geq 1$$

$$\left(\frac{\sigma_x}{\sigma_{yp}}\right)^2 + \left(\frac{\tau_{xy}}{\tau_{max}}\right)^2 \geq 1$$

Example (adapted from example on page 62-63)

Consider a 1 cm crack in a large plate of steel. Assume the plate is large enough that $f(a/c) \cong 1$.

What stress gives rise to fracture for a weaker or 'mild' steel ($\sigma_y = 500$ MPa, $K_{1c} = 175$ MPa$\sqrt{m}$) and a high strength steel ($\sigma_y = 1410$ MPa, $K_{1c} = 50$ MPa$\sqrt{m}$).

Solution: Use $K_1 = \sigma f(a/c) \sqrt{\pi a}$, so with $f(a/c) \cong 1$, $\sigma = K_{1c}/\sqrt{\pi a}$.

Weaker steel A, $\sigma = 987$ MPa, which exceeds the yield strength, so it undergoes yield at 500 MPa rather than fracture from the crack.

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Consequently, in the presence of 1 cm cracks, the stronger steel actually is weaker than the 'mild' weaker steel. Moreover, the consequence of an overload is more severe for the stronger steel since it catastrophically fractures rather than yields in bulk.

Example (adapted from Gordon, *Structures*)

Suppose we have a large structure such as a ship or a bridge and wish to tolerate a 1 meter long crack without catastrophic failure. Consider 'mild' steel ($\sigma_y = 500$ MPa, $K_{1c} = 175$ MPa$\sqrt{m}$).
Solution-

With $f(a/c) \cong 1$, $\sigma = K_{1c}/\sqrt{\pi a} = 90$ MPa or 14,000 psi.

In **foam**, Gibson and Ashby [*Cellular solids*] predict toughness K_{1c} proportional to [√(cell size)](density)$^{3/2}$.

Stress concentrations: appendix

Experimental stress concentrations in composite materials are consistently less than the theoretical ones. The non-classical fracture behavior has been dealt with using point stress and average stress criteria, however that approach cannot account for non-classical strain distributions in objects under small load. Such differences may be accounted for via a generalized continuum approach. Reduced stress concentration factors for small holes are known experimentally in fibrous composite materials. The fracture strength of graphite epoxy plates with holes depends on the size of the hole [1]. Moreover the strain around small holes and notches in fibrous composites well below the yield point is smaller than expected classically [2,3], while for large holes, the strain field follows classical predictions [4].

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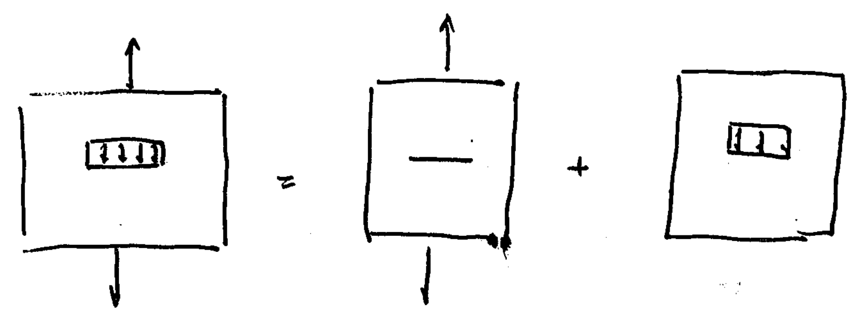
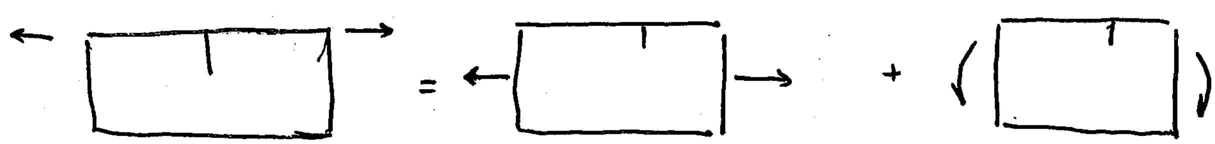


$$K_I = K_{I_{(1)}} + K_{I_{(2)}}$$



$$K_{I_{TOT}} = \sum K_I^{(i)}$$

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σ_x

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$$K_{I_c} = \sigma_y \sqrt{\pi a}$$

$$K_{II_c} = \tau_{max} \sqrt{\pi a}$$

$$\left(\frac{K_I}{K_{I_c}}\right)^2 + \left(\frac{K_{II}}{K_{II_c}}\right)^2 \geq 1$$

$$\left(\frac{\sigma_x}{\sigma_{yp}}\right)^2 + \left(\frac{\tau_{xy}}{\tau_{max}}\right)^2 \geq 1$$

$$\sigma_x' = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = \tau_{xy} \cos 2\theta - \left(\frac{\sigma_x - \sigma_y}{2}\right)^{?} \sin 2\theta$$

Fatigue

A material loaded through multiple cycles will break at a stress considerably less than the ultimate strength for a single application of load. Fatigue is quantified by the S-N curve, in which the number N of cycles is plotted logarithmically.

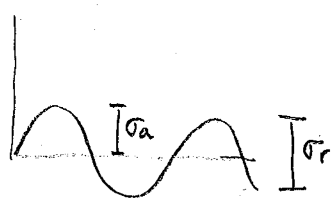
The effect of cyclic stresses is to initiate microcracks at centers of stress concentration within the material or on the surface resulting in the growth and propagation of cracks leading to failure.

As for fatigue testing, the rate of crack growth can be plotted in a log-log scale versus time. Testing the fatigue properties to generate an S-N curve entails monitoring the number of cycles to failure at various stress levels. This test requires a large number of specimens compared with the crack propagation test.

The *endurance limit* is the stress below which the material will not fail in fatigue no matter how many cycles are applied. Not all materials exhibit an endurance limit. (a practical limit is often chosen as 10^7 cycles).

The presence of a saline environment exacerbates fatigue.
Surface roughness exacerbates fatigue. A polished surface is better.
Rubbing or 'fretting' exacerbates fatigue. Re-design the part or use lubricants.
Heat treatment to introduce residual surface compression can be helpful.

FATIGUE DEPENDS ON $R = \dfrac{\sigma_{min}}{\sigma_{max}}$

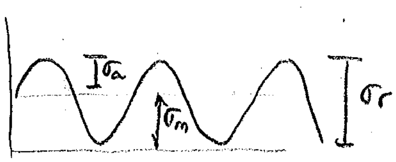


$\sigma_m = 0$
$R = -1$ fully reversed

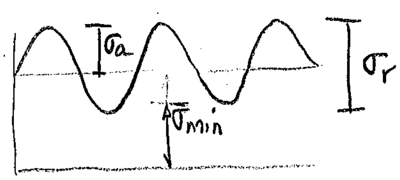
σ_a = stress amplitude $\dfrac{\sigma_{max} - \sigma_{min}}{2}$

σ_r = stress range $= 2\sigma_a$

$\sigma_m = \dfrac{\sigma_{max} + \sigma_{min}}{2}$



$\sigma_{min} = 0$
$R = 0$



$\sigma_{min} \neq 0$
$R > 0$

NOTCH GEOMETRY EFFECTS as $\rho \uparrow$ (FOR SAME $\Delta\sigma$) NO OF CYCLES $\uparrow$ UNTIL CRACK STARTS

as ρ = const if $\Delta\sigma \uparrow$ NO OF CYCLES $\downarrow$

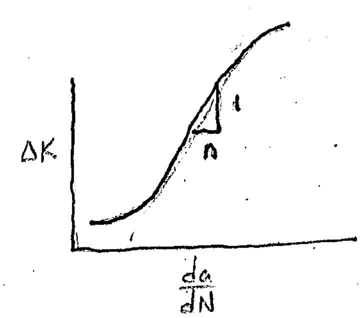
ALSO $\dfrac{da}{dN}$ has been found to vary as $\dfrac{(\Delta K)^n}{(1-R)K_c - \Delta K}$

$\Delta K = K_{max} - K_{min}$

and $K_c = \max K$ when $\dfrac{da}{dN} \to \infty$

$K_{max} = \sigma_{max}\sqrt{\pi a}$
$K_{min} = \sigma_{min}\sqrt{\pi a}$

$$\int_{a_0}^{a} da = \int_0^N \frac{(\Delta K)^n}{(1-R)K_c - \Delta K}\, dN$$



EGM 5615

11/8/2010

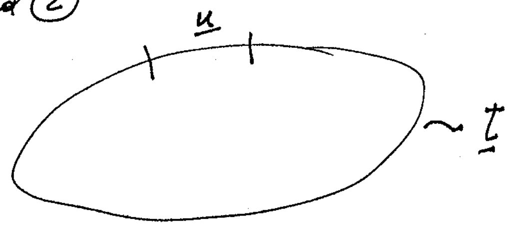
INTRO TO ELASTICITY

UNIQUENESS OF SOLUTIONS

1) IF WE SPECIFY A TRACTION VECTOR $\sigma \cdot \underline{n} = \underline{t}_n = \sigma_{in}$ at all points on the boundary then the displacement vector $\underline{u}$ (u, v, w) is unique up to a rigid body motion (Neumann-type problem)

2) IF $\underline{u}$ IS defined everywhere on the boundary, then this gives a unique solution to the problem (Dirichlet-type problem)

3) Mix ① and ②



t_x, t_y, u_z

u_x, u_y, t_z

Cannot do $\underline{u_x, t_x}, u_y$

EGM 5615

11/8/2010

Fracture mechanics

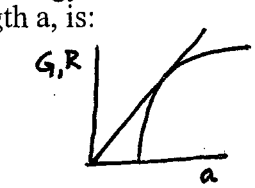
As elliptic hole becomes progressively narrower, the ellipse approaches a crack shape and SCF = K $\rightarrow\infty$. Actual observed stress concentration factors for cracks are not infinite.

Therefore a material with one perfectly sharp crack will have **zero** strength, since the stress concentration factor becomes infinite. Experimentally, even for brittle materials, strength is reduced by cracks but not infinitely.

A criterion based on energy balance rather than on stress has therefore been adopted. The approach is due to Griffith. The energy relations are visualized as follows. The energy required to extend a crack is linear in crack length because the energy is expended in creating new free surfaces. By contrast the strain energy available comes from a roughly semicircular region around the crack and is therefore quadratic in crack length. As the crack propagates, the material in that region is unloaded and its energy is made available to drive the crack. When the crack is long enough that an increment in crack length gives rise to an energy release equal to energy expended, catastrophic crack growth impends.

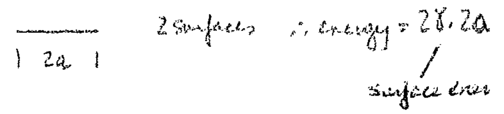
Griffith proposed an **energy** approach to fracture. The elastic energy stored in a test specimen of unit thickness, in a circular region around a crack of length a, is:

$$2\pi a^2 \frac{1}{2E}\sigma^2 = 2 \cdot \text{strain energy density} \cdot \pi a^2 \cdot 1 \qquad \text{(F1)}$$



Recall that $\frac{1}{2}E\varepsilon^2 = \frac{1}{2E}\sigma^2$ represents a strain energy per unit volume.

The elastic energy for a brittle material is twice the area under the stress strain curve. The elastic energy is used to create two new surfaces as the crack propagates. The surface energy, 4 γa (γ is the surface energy; it is an energy per unit area.) should be smaller than the elastic energy for the crack to grow. Thus, the incremental changes of both energies for the crack to grow can be written,

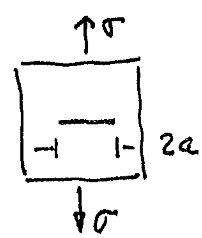


$$\frac{d}{da}\left(\frac{\pi(a\sigma)^2}{E}\right) = \frac{d}{da}(4\gamma a) \qquad \text{(F2)}$$

$$\frac{2\pi a\sigma^2}{E} = 4\gamma$$

Hence,

$$\sigma = \sigma_f = \sqrt{\frac{2\gamma E}{\pi a}} \qquad \text{(F3)}$$



Since for a given material E and γ are constants,

$$\sigma_f = \frac{K}{\sqrt{\pi a}} \qquad \text{(F4)}$$

In this case K has the units of psi $\sqrt{in}$ or MPa $\sqrt{m}$ and is proportional to the energy required for fracture.

K is a measure of *fracture toughness,* called the stress intensity factor. Cracks and stress concentrations also occur in ductile materials, but their effect is usually not as serious as in brittle ones since local yielding which occurs in the region of peak stress will effectively blunt the crack and alleviate the stress concentration.

Example (adapted from example on page 62-63)

Consider a 1 cm crack in a large plate of steel. Assume the plate is large enough that $f(a/c) \cong 1$.

What stress gives rise to fracture for a weaker or 'mild' steel ($\sigma_y = 500$ MPa, $K_{1c} = 175$ MPa$\sqrt{m}$) and a high strength steel ($\sigma_y = 1410$ MPa, $K_{1c} = 50$ MPa$\sqrt{m}$).

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much lower stresses. Thickness-direction tension at the just ahead of the crack tip is therefore in a state of *triax-* re more than plastic flow.

;ate at great speed after reaching a critical length? The ;y considerations, which are summarized as follows. metry of Fig. 3.5-1a, which is shown again in Fig. 3.5-2a. rack an amount da is independent of crack length a, so ice the crack varies linearly with a (Fig. 3.5-2b). As the ced in material alongside the crack, and stored strain gy released varies approximately quadratically with a. ne that a crack of length a nullifies the uniaxial state of f radius a, shown dashed in Fig. 3.5-2a. Strain energy rtional to the volume of this disc, $V = \pi a^2 t/2$.) When a ient of strain energy released to equal the increment ie crack, sudden fracture impends. Thus $dU_e = dU_s$ in l crack length. Energy needed to drive the crack is sup- material. There is no need (and insufficient time) for ırk of external forces acting through a distance. The fail- ›uld be if the crack were absent because the crack pro- bonds to be broken sequentially rather than all at once. l as shown in Fig. 3.5-1. In practice, Mode I is most com- :. For any mode, one can calculate a *stress intensity fac-* re it with an allowable value to determine whether ısity factor is *not* a stress concentration factor! Indeed, ; it is not necessary to use stress concentration data: l not be calculated.

ler only isotropic materials, and only Mode I cracks :ss intensity factor for a Mode I crack is denoted by K_I

$$K_I = \beta\sigma\sqrt{\pi a} \qquad (3.5\text{-}1)$$

ıt would exist if the crack were absent. Thus, stress σ is ultiplier β is dimensionless and depends on geometry 3.5-1). Dimension a is defined as either the full crack ı geometry. Units of K_I are MPa$\sqrt{\text{m}}$. Fracture impends $_{Ic}$ known as *fracture toughness.* K_{Ic} can be considered a of specimen thickness t, if the specimen is sufficiently ısion at the crack tip to develop fully. Also, the crack ıinimum. Recommended minimum dimensions are

$$\frac{}{}\!\!\left(\frac{{}_{Ic}}{{}_{Y}}\right)^2 \quad \text{and} \quad a \geq 2.5\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 \qquad (3.5\text{-}2)$$

; determined by a tension test of the material, and t is thickness is less than the value described by Eq. 3.5-2,

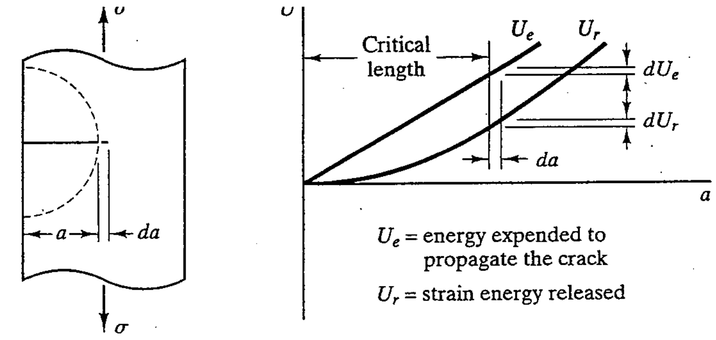

FIGURE 3.5-2 (a) Plate with an edge crack of length a. (b) Energy relations for crack extension.

TABLE 3.5-1 Stress intensity data for flat plates, of isotropic material and uniform thickness, with in-plane loading [3.6].

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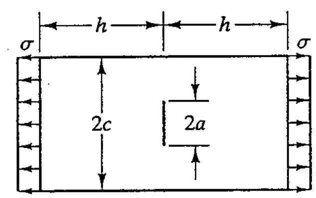
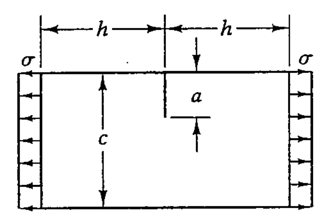


$K_I = \beta\sigma\sqrt{\pi a}$

$\beta = \dfrac{1 - 0.5(a/c) + 0.326\,(a/c)^2}{\sqrt{1-(a/c)}}$

Accurate to within 1% for all a/c, provided h/c is "large"

Tension, edge crack of length a

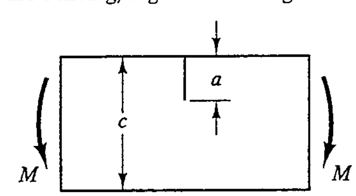


$K_I = \beta\sigma\sqrt{\pi a}$

$\beta = [1.12 - 0.23\,(a/c) + 10.6\,(a/c)^2 - 21.7\,(a/c)^3 + 30.4\,(a/c)^4]$

Accurate to within 1% for $a/c \leqslant 0.6$, provided $h/c > 1$ and sides are free to rotate

Pure bending, edge crack of length a

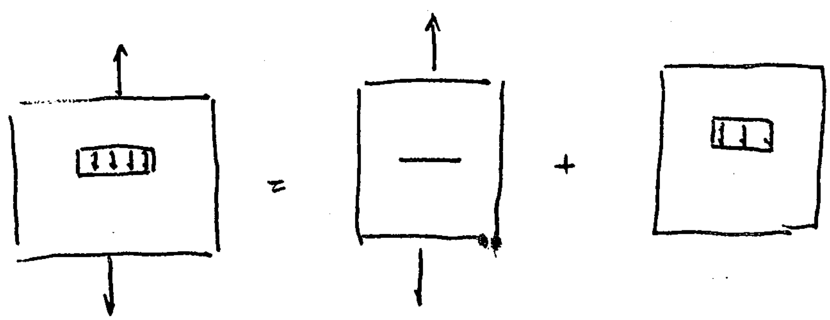


$K_I = \beta\sigma\sqrt{\pi a}$

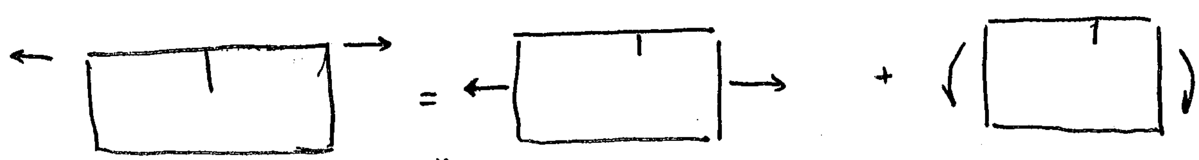
$\sigma = \dfrac{M(c/2)}{I} = \dfrac{6M}{tc^2}$

$\beta = [1.12 - 1.39\,(a/c) + 7.32\,(a/c)^2 - 13.1\,(a/c)^3 + 14.0\,(a/c)^4]$

Accurate to within 1% for $a/c \leqslant 0.6$



$K_I = K_{I_{(1)}} + K_{I_{(2)}}$



$K_{I_{TOT}} = \sum K_I^{(i)}$

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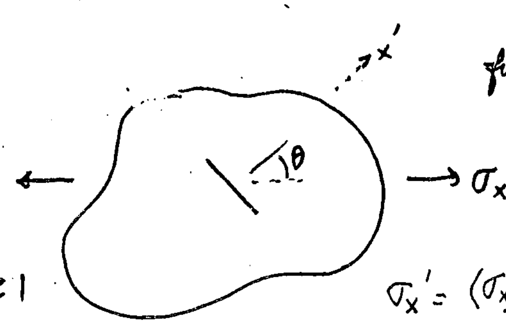

find $\sigma_{x'}$ & $\tau_{x'y'}$

$K_I = \sigma_{x'} \sqrt{\pi a}$

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$K_I = \sigma_x' \sqrt{\pi a}$
$K_{II} = \tau_{x'y'} \sqrt{\pi a}$
$K_{I_c} = \sigma_y \sqrt{\pi a}$
$K_{II_c} = \tau_{[illegible]} \sqrt{\pi a}$

$$\left(\frac{K_I}{K_{I_c}}\right)^2 + \left(\frac{K_{II}}{K_{II_c}}\right)^2 \geq 1$$

$$\left(\frac{\sigma_x}{\sigma_{yp}}\right)^2 + \left(\frac{\tau_{xy}}{\tau_{max}}\right)^2 \geq 1$$

$$\sigma_x' = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = \tau_{xy}\cos 2\theta - \left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta \ ?$$

Fatigue

A material loaded through multiple cycles will break at a stress considerably less than the ultimate strength for a single application of load. Fatigue is quantified by the S-N curve, in which the number N of cycles is plotted logarithmically.

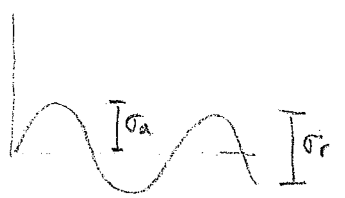
The effect of cyclic stresses is to initiate microcracks at centers of stress concentration within the material or on the surface resulting in the growth and propagation of cracks leading to failure.

As for fatigue testing, the rate of crack growth can be plotted in a log-log scale versus time. Testing the fatigue properties to generate an S-N curve entails monitoring the number of cycles to failure at various stress levels. This test requires a large number of specimens compared with the crack propagation test.

The *endurance limit* is the stress below which the material will not fail in fatigue no matter how many cycles are applied. Not all materials exhibit an endurance limit. (a practical limit is often chosen as 10^7 cycles).

The presence of a saline environment exacerbates fatigue.
Surface roughness exacerbates fatigue. A polished surface is better.
Rubbing or 'fretting' exacerbates fatigue. Re-design the part or use lubricants.
Heat treatment to introduce residual surface compression can be helpful.

FATIGUE DEPENDS ON $R = \frac{\sigma_{min}}{\sigma_{max}}$



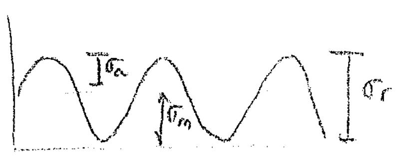
σ_a = stress amplitude $\frac{\sigma_{max} - \sigma_{min}}{2}$

σ_r = stress range $= 2\sigma_a$

$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}$

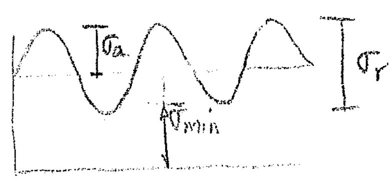
$\sigma_m = 0$

$R = -1$ fully reversed



$\sigma_{min} = 0$

$R = 0$



$\sigma_{min} \neq 0$

$R > 0$

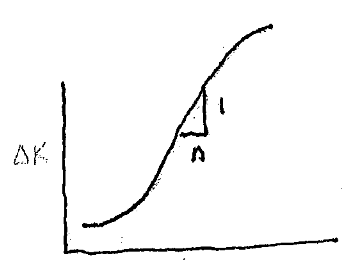
NOTCH GEOMETRY EFFECTS as $\rho\uparrow$ (FOR SAME $\Delta\sigma$) NO OF CYCLES $\uparrow$ UNTIL CRACK STARTS

as ρ = const if $\Delta\sigma\uparrow$ NO OF CYCLES $\downarrow$

ALSO $\frac{da}{dN}$ has been found to vary as $\frac{(\Delta K)^n}{(1-R)K_c - \Delta K}$

$\Delta K = K_{max} - K_{min}$

and K_c = max K when $\frac{da}{dN} \to \infty$



$K_{max} = \sigma_{max}\sqrt{\pi a}$

$K_{min} = \sigma_{min}\sqrt{\pi a}$

$$\int_{a_0}^{a} da = \int_0^N \frac{(\Delta K)^n}{(1-R)K_c - \Delta K}\, dN$$

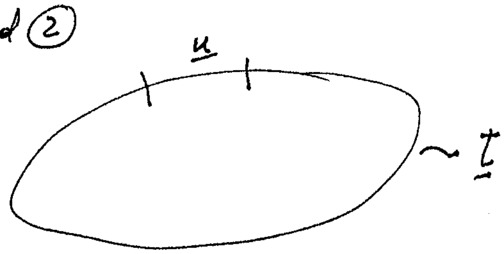
INTRO TO ELASTICITY

UNIQUENESS OF SOLUTIONS

1) IF WE SPECIFY A TRACTION VECTOR $\underline{\underline{\sigma}} \cdot \underline{n} = \underline{t}_n = \sigma_{in}$ at all points on the boundary then the displacement vector $\underline{u}\ (u, v, w)$ is unique up to a rigid body motion (Neumann-type problem)

2) IF $\underline{u}$ IS defined everywhere on the boundary, then this gives a unique solution to the problem (Dirichlet-type problem)

3) Mix ① and ②



$\underline{t}_x, \underline{t}_y, u_z$

u_x, u_y, t_z

Cannot do $\underline{\underline{u_x, t_x}}, u_y$

ME 238B : Theory of Elasticity

Barnett Rm 550K

Homework – handed out on Monday, Due on Monday 1/3 grade

TA Rich King 264 Durand. TH 2-4 F 12-2

Mid-Term: Take home 1/3 grade

Final : In class $\frac{1}{3}$ grade

Rough Outline

a. 2-D Problems

Fourier Series, Integral transforms

Complex variable methods

Cracks, Inclusions

b. 3-D Problems

Green's functions methods.

Boundary Integral Eqn. Method.

Will try to take a physical notion and derive p.d.e. for it.

a. $C_{ijkl} \frac{\partial^2 u_k}{\partial x_i \partial x_l} = 0$ (static, no. boundary forces)

or $\frac{\partial \sigma_{ij}}{\partial x_i} = 0 \Rightarrow \sigma_{ij} = C_{ijkl}\, e_{kl}$; $e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$

uses the stress function approach ; requires use of compat eqn

In 2-D we will have 2 P.D.E. and 2 unknowns → 2 B.C.

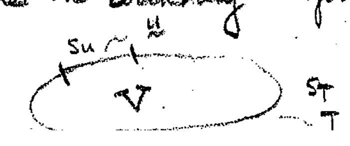
3-D " " " 3 " " 3 " → 3 B.C.

Uniqueness Thm (a review)

(1) We specify traction vector $\underset{\sim}{T}$ at all points on the boundary (T_x, T_y, T_z) This is a "dead loading" problem. $\underset{\sim}{u}$ is unique only to a rigid body motion.

(2) Specify $\underset{\sim}{u}$ everywhere on boundary – gives a unique solution

(3) Mix (1) and (2)



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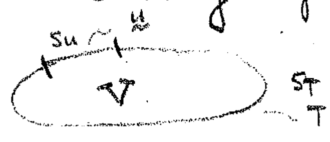


a. T_x, u_y, u_z

b. T_x, T_y, u_z

cannot specify T_x, u_y, u_x (cannot specify u_x & T_x)

c. for elastic foundation $T_x + k u_x = 0$

The uniqueness proof assumes that the volume is finite and "No elastic singularities".

i.e. (1) $\underline{u}$ discontinuous (shrink fit, dislocations, cracks)

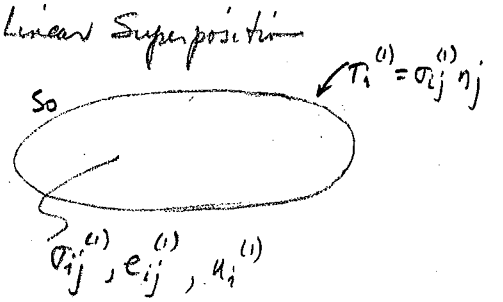
(2) $|\sigma_{ij}| \to \infty$

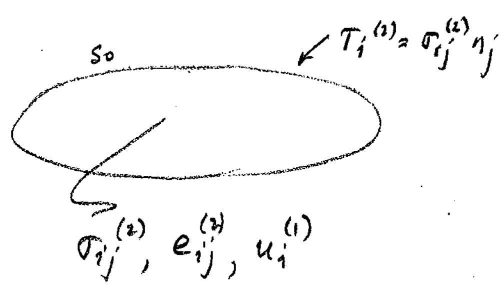
<u>Semi Inverse Method</u>

Obtain by any fashion a solution to the governing PDE (equil eqns) w/out regard for Boundary conditions. Then try to figure out what problem you have solved.

If solution is not what you need, generate another. Then linearly superpose these problems to get different solutions.

<u>Linear Superposition</u>





Equil. $\sigma_{ij,i}^{(1)} = 0$

$e_{ij}^{(1)} = \frac{1}{2}\left(u_{i,j}^{(1)} + u_{j,i}^{(1)}\right)$

$\sigma_{ij}^{(1)} = C_{ijkl}\, e_{kl}^{(2)}$

$T_i^{(1)} = \sigma_{ij}^{(1)} n_j$ on S_o

$\sigma_{ij,i}^{(2)} = 0$

$e_{ij}^{(2)} = \frac{1}{2}\left(u_{i,j}^{(2)} + u_{j,i}^{(2)}\right)$

$\sigma_{ij}^{(2)} = C_{ijkl}\, e_{kl}^{(2)}$

$T_i^{(2)} = \sigma_{ij}^{(2)} n_j$ on S_o

if $\Sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \Rightarrow \Sigma_{ij,i} = 0$ ⇐ this works no matter what

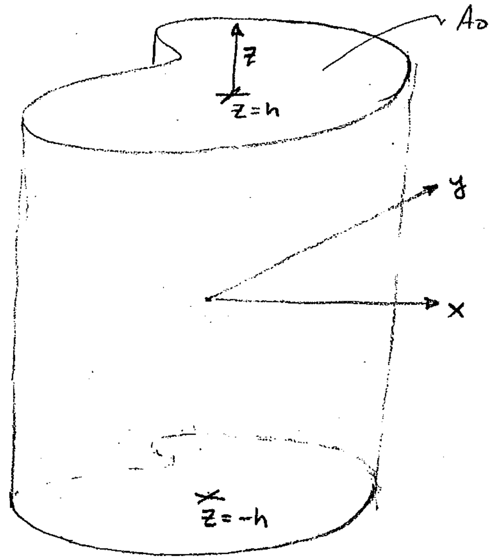
if $E_{ij} = e_{ij}^{(1)} + e_{ij}^{(2)}$ and $U_i = u_i^{(1)} + u_i^{(2)} \Rightarrow E_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$

this works only for infinitesimal strains

⇓

$\Sigma_{ij} = C_{ijkl} E_{ij}$ and $T_i = T_i^{(1)} + T_i^{(2)} = \Sigma_{ij} n_j$ on S_0

Simple extension problem



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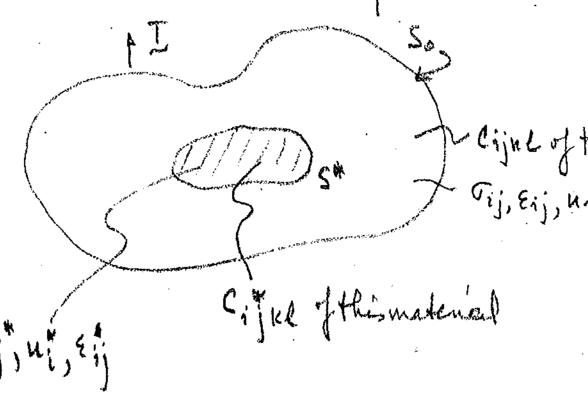
One problem not mentioned in class last time

Inclusion Boundary conditions



S_0 : $\sigma_{ij} n_j = T_i$

S^* : $u_i^* = u_i$ perfectly bonded

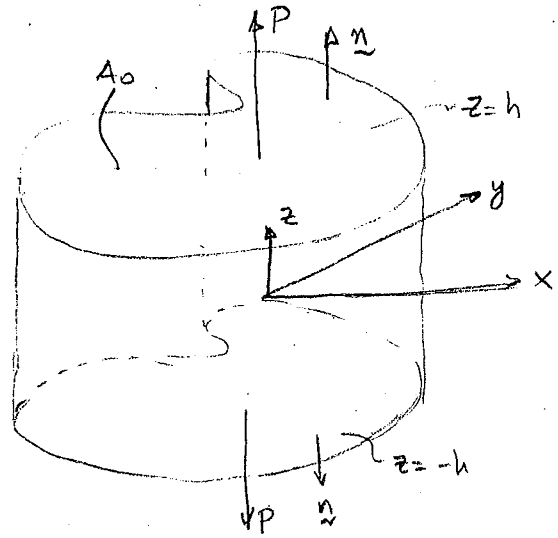
$n_j \sigma_{ij}^* = \sigma_{ij} n_j$

Back to the problem we started last time

Simple Extension

(Any constant state of stress satisfies equil eqns)

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0$$



Traction BC.

@ $z=h$ $\quad T_z = \frac{P}{A_0} = \sigma_{zj} n_j \qquad n_x = n_y = 0; \; n_z = +1$

$$\boxed{T_z = \frac{P}{A_0} = \sigma_{zz} \text{ @ } z=h} \qquad \boxed{T_x = \sigma_{xj} n_j \Rightarrow \sigma_{xz} = 0 \text{ @ } z=h}$$

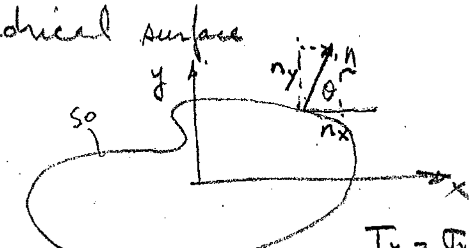
$$T_y = 0 \Rightarrow \boxed{\sigma_{yz} = 0 \text{ @ } z=h}$$

@ $z=-h$ $\qquad n_x = n_y = 0 \; ; \; n_z = -1$

$$T_z = \frac{-P}{A_0} = \sigma_{zj} n_j = \sigma_{zz}(-1) \Rightarrow \boxed{\sigma_{zz} = \frac{P}{A_0} \text{ on } z=-h}$$

$$T_x, T_y = 0 \qquad \boxed{\sigma_{xz} = \sigma_{yz} = 0 \text{ on } z=-h}$$

on the cylindrical surface



$$T_x = T_y = T_z = 0$$

$$n_x = \cos\theta \quad n_y = \sin\theta \quad n_z = 0$$

$$T_x = \sigma_{xx}\cos\theta + \sigma_{xy}\sin\theta = 0$$

$$T_y = \sigma_{yx}\cos\theta + \sigma_{yy}\sin\theta = 0$$

$$T_z = \sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta = 0$$

if $T_x, T_y, T_z = 0$ for any θ $\Rightarrow$

pick $\sigma_{xx} = \sigma_{xy} = \sigma_{yy} = \sigma_{zx} = \sigma_{zy} = 0$ on S_o

Hence if we pick $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{zx} = \sigma_{zy} = 0$ everywhere and $\sigma_{zz} = \frac{P}{A}$ on $z = th$

Assume for this problem $\sigma_{zz} = \frac{P}{A_o}$; all others $\sigma_{ij} = 0$

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right) = \frac{-\nu}{E}\frac{P}{A_o} = \frac{\partial u_x}{\partial x}$$

$$e_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right) = \frac{-\nu P}{E A_o} = \frac{\partial u_y}{\partial y}$$

$$e_{zz} = \frac{\sigma_{zz}}{E} = \frac{P}{EA_o} = \frac{\partial u_z}{\partial z}$$

$$\sigma_{xz} = \sigma_{yz} = \sigma_{xy} = 0 \Rightarrow e_{xz} = e_{yz} = e_{xy} = 0$$

$$u_x = -\frac{\nu P}{A_o E} x + f(y,z)$$

$$u_y = -\frac{\nu P}{E A_o} y + g(x,z)$$

$$u_z = \frac{P}{EA_o} z + h(y,x)$$

$$e_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \Rightarrow \frac{\partial f}{\partial z} + \frac{\partial h}{\partial x} = 0$$

must use compatibility to show $\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 g}{\partial x \partial z} = \frac{\partial^2 h}{\partial y \partial x} = 0$

~~$\Rightarrow \frac{\partial f}{\partial z} = -\frac{\partial h}{\partial x} = k(y)$~~

~~$\therefore f = k_1(y) z + k_2(y)$~~

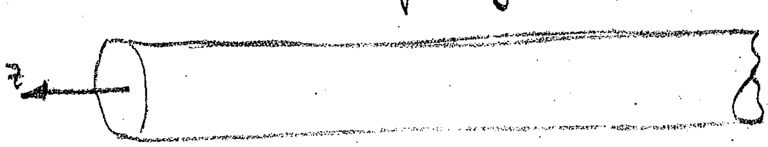
~~$h = -k_1(y) x + k_3(y)$~~

→ HW #1 complete and solve showing solution is/includes a rigid body rotation/translation

2-D elastostatic problems (isotropic materials)

<u>Plain strain</u>

Elastic solid very long in 1 direction



$$e_{zz} = 0$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2\mu}\sigma_{yz} = 0$$

$$e_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{2\mu}\sigma_{xz} = 0$$

$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq 0$ and also not fns of z

$\therefore$ $u_x = u_x(x,y)$; $u_y = u_y(x,y)$; $u_z = 0$
$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq 0$; $\neq$ fns of z

Since $e_{zz} = 0 = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right) \Rightarrow \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \neq$ fn of z

each cross section has same thing happening as any other cross section. Plane strain normally simulates the effects at center of a very thick plate

The Equil Eqns reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \qquad \text{since } \sigma_{zx} = 0 \qquad (1)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \qquad \text{since } \sigma_{zy} = 0 \qquad (2)$$

since $\sigma_{zz} \neq$ fn of z $\Longrightarrow$ $\frac{\partial \sigma_{zz}}{\partial z} \equiv 0$ in third eq

Solution by Airy Stress fn

Define a fn ϕ $\ni$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \; ; \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \; ; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

put into equil (1) $\therefore$ $\frac{\partial^3 \phi}{\partial x \partial y^2} - \frac{\partial^3 \phi}{\partial x \partial y^2} = 0$

also same for (2) $\frac{\partial^3 \phi}{\partial x^2 \partial y} - \frac{\partial^3 \phi}{\partial x^2 \partial y} = 0$

But what does ϕ satisfy? Look at Hooke's law

$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$ assume a displ field exists $\Rightarrow$ $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

now $\sigma_{ij,i} = \lambda \delta_{ij} e_{kk,i} + 2\mu e_{ij,i} = \lambda e_{kk,j} + 2\mu e_{ij,i}$

subst. the displ gradient relationship $= \lambda (u_{k,kj}) + \mu (u_{j,ii} + u_{i,ji})$

$= (\lambda+\mu) u_{k,kj} + \mu u_{j,ii}$

now differentiate once

$(\lambda+\mu)(u_{k,kjj}) + \mu (u_{j,iij}) = 0$

or $(\lambda + 2\mu)(u_{k,kjj}) = 0$

or $(\lambda+2\mu) \nabla^2 e_{kk} = 0 \Rightarrow \nabla^2 \sigma_{kk} = 0 \Rightarrow \nabla^4 \phi = 0$

now $\sigma_{ii} = \overset{3}{\delta_{ii}} \lambda e_{kk} + 2\mu e_{ii}$ (no summation) $\Rightarrow \sigma_{kk} = (3\lambda + 2\mu) e_{kk}$ or $\nabla^2 \sigma_{kk} = (3\lambda+2\mu)\nabla^2 e_{kk} = 0$. Next time will prove ↑

1/10/79

<u>Plain Strain</u> From last time

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0 \end{aligned}\right\} \quad \begin{aligned} \sigma_{xx} &= \phi_{,yy} \\ \sigma_{xy} &= -\phi_{,xy} \\ \sigma_{yy} &= \phi_{,xx} \end{aligned}$$ where ϕ is the airy stress fn

using $\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$ material relation w/

Equil $\sigma_{ij,i} = 0$ and $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$(\lambda+\mu) u_{i,ij} + \mu u_{j,ii} = 0$

now take $\frac{\partial}{\partial x_j}$ $(\lambda+\mu) u_{i,ijj} + \mu u_{j,iij} = 0$ since dummy indices rep $i \to j$, $j \to i$ in 2nd relation but $i,jji = i,ijj$

hence $(\lambda+2\mu)(u_{i,i})_{,jj} = (\lambda+2\mu)(e_{ii})_{,jj} = 0$

$\therefore \nabla^2 e_{ii} = 0$ where $\nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i} = (\ )_{,ii}$

now $\sigma_{ii} = \lambda \overset{3}{\delta_{ii}} e_{kk} + 2\mu e_{ii} = (3\lambda+2\mu) e_{ii}$

$\therefore \nabla^2 e_{ii} \Rightarrow \nabla^2 \sigma_{ii} = 0$

<u>Now in plane strain</u> $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ from $\epsilon_{zz} = 0$

$\nabla^2 \sigma_{ii} = \nabla^2 (1+\nu)(\sigma_{xx} + \sigma_{yy}) = (1+\nu) \nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0$

using Airy stress fns

$\nabla^2(\sigma_{xx} + \sigma_{yy}) = \nabla^2(\phi_{,yy} + \phi_{,xx}) = \nabla^2(\nabla^2 \phi) = \nabla^4 \phi = 0$

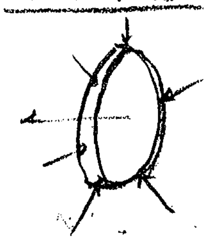
in Plane stress $\sigma_{zz}=0 \Rightarrow$ $\epsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx}+\sigma_{yy})$

$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx}-\nu\sigma_{yy})$ $\quad \epsilon_{kk} = (\sigma_{xx}+\sigma_{yy})\frac{1-2\nu}{E}$

$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy}-\nu\sigma_{xx})$

<u>Plane Stress</u> Plane $\Rightarrow \sigma_{zx}=\sigma_{zy}=0$ Plane Stress also $\Rightarrow \sigma_{zz}=0$

If it is small thickness & hence must vary from 0 to 0 over a small thickness: assume 0 everywhere



$0 = \begin{pmatrix}\sigma_{zz}\\ \sigma_{zx}\\ \sigma_{zy}\end{pmatrix}$ on flat faces

$\sigma_{zz}=0 \Rightarrow e_{zz} = -\frac{\nu}{E}(\sigma_{xx}+\sigma_{yy})$

$\sigma_{zx}=0 \Rightarrow e_{zx}=0$

$\sigma_{zy}=0 \Rightarrow e_{zy}=0$

Tentatively Assume $\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq f(z)$ or $u_x = u_x(x,y)$, $u_y = u_y(x,y)$

Timoshenko & Goodier Rd 274-277 | HW #1b prove that this assumption is inconsistent |

however we can see that u_x, u_y have z^2 component & that for $z \ll 1$ then we can assume the above w/o loss in accuracy

hence, we can define generalized displ for disc w/ thickness h

$$U_x(x,y) = \frac{1}{h}\int_0^h u_x(x,y,z)\,dz$$

We will now prove that $\nabla^4\phi = 0$ is DE for plane strain & plane stress for certain conditions

<u>Plane strain</u>

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy}+\sigma_{zz})\right) \quad \sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$$

$$= \frac{1}{E}\left\{\sigma_{xx}(1-\nu^2) - \sigma_{yy}\,\nu(1+\nu)\right\}$$

$$= \frac{1+\nu}{E}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$$

now $\mu = \frac{E}{2(1+\nu)}$ $\quad \therefore e_{xx} = \frac{1}{2\mu}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$

$$e_{xy} = \frac{\sigma_{xy}}{2\mu}$$

<u>Plane stress</u>

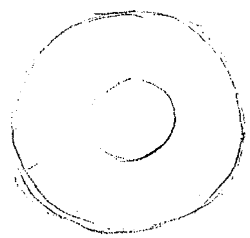
$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy}+\sigma_{zz}^{\,0})\right) = \frac{1}{E}(\sigma_{xx}-\nu\sigma_{yy}) = \frac{1}{2\mu}\left\{\frac{\sigma_{xx}}{1+\nu} - \frac{\nu}{1+\nu}\sigma_{yy}\right\}$$

$e_{xy} = \frac{1}{2\mu}\sigma_{xy}$ $\qquad e_{ii} = \frac{1}{2\mu}\frac{(1-\nu)}{1+\nu}(\sigma_{xx}+\sigma_{yy})$

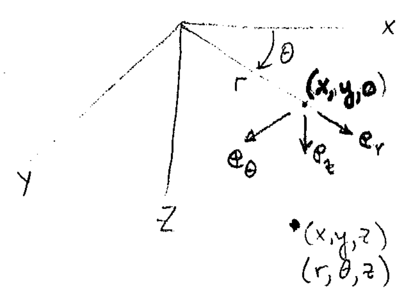
Do this first before starting on cylindrical

12/4/78

Orthogonal Curvilinear Coordinates will be discussed in order to do torsional problem of a hollow surface.



Cylindrical Coordinate systems.



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

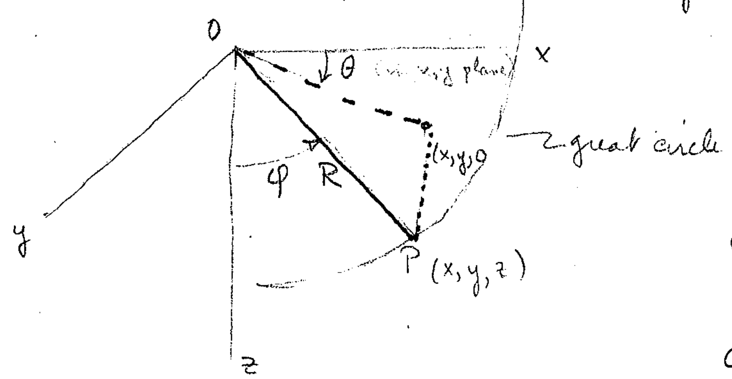
invertible

$r = \sqrt{x^2+y^2}$

$\theta = \arctan(y/x)$

$z = z$

Spherical Coordinate System.



$x = R\sin\varphi\cos\theta$

$y = R\sin\varphi\sin\theta$

$z = R\cos\varphi$

$R = \sqrt{x^2+y^2+z^2}$

$\theta = \tan^{-1}(y/x)$

$\varphi = \arctan\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

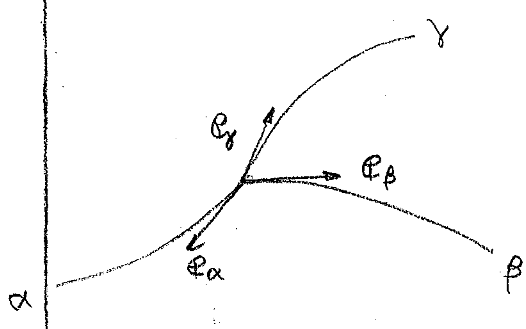
General orthogonal curvilinear coordinates (α, β, γ)

$\alpha = \alpha(x, y, z)$ look at α = const this defines a surface

$\beta = \beta(x, y, z)$ look at β = " " " " "

$\gamma = \gamma(x, y, z)$ " " γ = " " " " "

the intersection of these 3 surfaces define a point p.



$e_{\alpha_i} \cdot e_{\alpha_j} = \delta_{ij}$, $e_{\alpha_i} \times e_{\alpha_j} = e_{\alpha_k}\, e_{ijk}$

in r, θ, z coord.

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta r} + \frac{\partial}{\partial z}\sigma_{zr} + \frac{(\sigma_{rr} + \sigma_{\theta\theta})}{r} + f_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2}{r}\sigma_{r\theta} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + f_\theta = 0$$

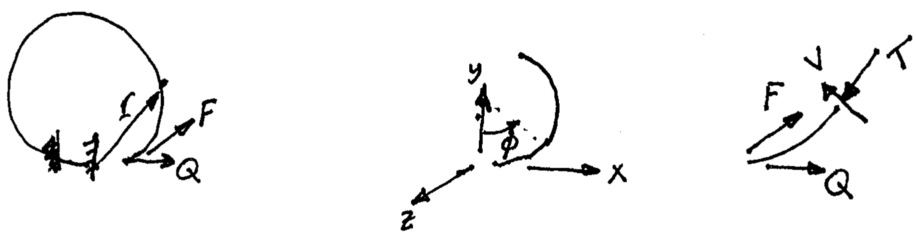
$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = 0$$

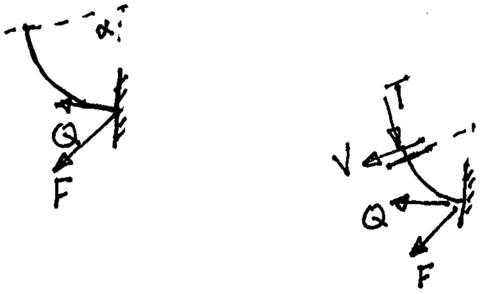
$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

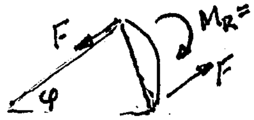
$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right)$$

$$\varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$





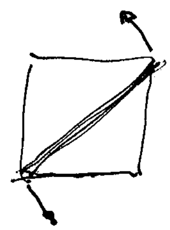


$M_R = F\sqrt{2}R\sqrt{1-\cos\varphi}$

$F_z = F$
$F_\varphi = 0$
$F_R = 0$

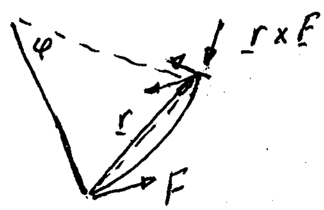
$2R^2 - 2R^2 \cos\varphi$

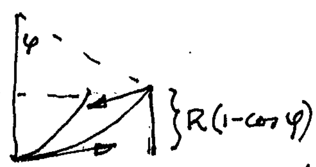
$R\sqrt{2}\sqrt{1-\cos\varphi}$



$(\sqrt{2}R)^2 + R^2 - 2\sqrt{2}R^2 \cos 45$

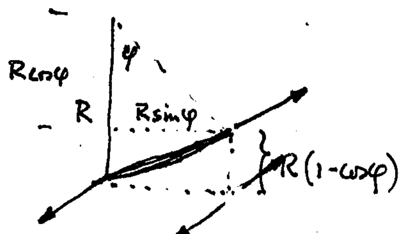
$2R^2 + R^2 - 2\sqrt{2}R^2 \cdot \frac{\sqrt{2}}{2} = R^2$





$M_z = QR(1-\cos\varphi)$

$\int_0^{2\pi} \frac{M_z^2}{2EI_z} R\,d\varphi$



$M_z = 0$

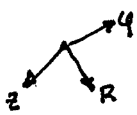
$M_y = FR\sin\varphi$

$M_x = FR(1-\cos\varphi)$ twisting mom

$R^2(1-\cos\varphi)^2 + R^2\sin\varphi$

$R^2 + R^2 - 2R^2\cos\varphi$

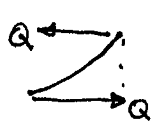
$M_R = 2R^2(1-\cos\varphi)$



$(R\sin\varphi\,\underline{i} + R(1-\cos\varphi)\underline{j}) \times F_z \underline{k}$

$-F_z R\sin\varphi\,\underline{j} + F_z R(1-\cos\varphi)\,\underline{i}$

$M_y \qquad\qquad M_x$

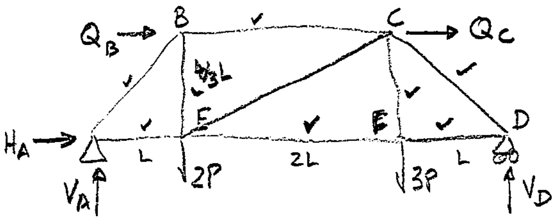


$QR(1-\cos\varphi) = M_z$

$$U_{c_n} = \int_0^{2\pi} \frac{T^2}{2GJ} R\,d\varphi + \int_0^{2\pi} \frac{M_y^2}{2EI_y} R\,d\varphi$$

$$\frac{\partial U_c}{\partial F} = \int_0^{2\pi} \frac{2T}{2GJ} \frac{\partial T}{\partial F} R\,d\varphi + \int_0^{2\pi} \frac{2M_y}{2EI} \frac{\partial M_y}{\partial F}$$

$J = I_x + I_y$



$\Sigma F_x = 0 \quad H_A + Q_B + Q_C = 0$

$\Sigma F_y = 0 \quad V_A + V_D - 5P = 0$

$+\circlearrowright \Sigma M_A = 0 \quad 2P \cdot L + 3P(3L) - V_D \cdot 4L + Q_B \cdot \frac{4L}{3} + Q_C \cdot \frac{4L}{3} = 0$

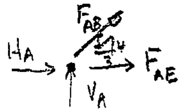
$V_D = \frac{Q_B + Q_C}{3} + \frac{11P}{4}$

$V_A = \frac{-Q_B - Q_C}{3} + \frac{9P}{4}$

$H_A = -Q_B - Q_C$

For (a) let $Q_C = 0$

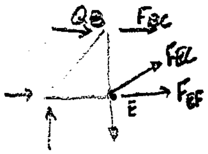
(b) let $Q_B = 0$



$V_A + F_{AB} \cdot \frac{4}{5} = 0 \quad F_{AB} = -V_A \cdot \frac{5}{4} = \frac{Q_B + Q_C}{12} 5 - \frac{45P}{16}$

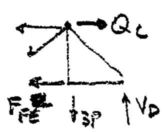
$F_{AE} + F_{AB} \cdot \frac{3}{5} + H_A = 0$

$F_{AE} = -F_{AB} \cdot \frac{3}{5} - H_A = \frac{3(Q_B + Q_C)}{4} + \frac{27P}{16}$

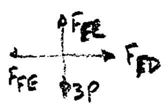


$+\circlearrowright \Sigma M_E = Q_B \cdot \frac{4L}{3} + F_{BC} \cdot \frac{4L}{3} + V_A \cdot L = 0$

$F_{BC} = -Q_B - V_A \cdot \frac{3}{4} = -Q_B + \frac{Q_B + Q_C}{4} - \frac{27P}{16} = -\frac{3}{4} Q_B + \frac{1}{4} Q_C - \frac{27P}{16}$

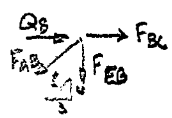


$+\circlearrowright \Sigma M_C = F_{FE} \cdot \frac{4L}{3} - V_D \cdot L = 0 \quad F_{FE} = V_D \cdot \frac{3}{4} = \frac{Q_B + Q_C}{4} + \frac{33P}{16}$



$F_{FD} = F_{FE} = \frac{Q_B + Q_C}{4} + \frac{33P}{16}$

$F_{FE} = 3P$ ✓



$F_{AB} \cdot \frac{4}{5} + F_{EB} = 0$

$-F_{AB} \cdot \frac{4}{5} = F_{EB} = -\frac{(Q_B + Q_C)}{3} + \frac{9}{4} P$

$F_{CD} \cdot \frac{4}{5} + V_D = 0 \quad F_{CD} = -V_D \cdot \frac{5}{4} = -\frac{5}{12}(Q_C + Q_B) - \frac{55}{16} P$

~~$Q_C - F_{BC} + F_{BC} \cdot \frac{3}{5} - F_{EC} \cdot \frac{3}{\sqrt{13}} = 0$~~

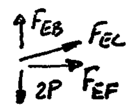
~~$Q_C + \frac{2}{3} Q_B - \frac{1}{3} Q_C + \frac{27P}{16} - \frac{1}{4}(Q_C + Q_B) - \frac{33}{16} P - F_{FC} \cdot \frac{3}{\sqrt{13}} = 0$~~

~~$\frac{\sqrt{13}}{3}\left(\frac{5}{12} Q_C + \frac{5}{12} Q_B - \frac{6P}{16}\right) = F_{FC}$~~

~~$F_{CE} + F_{DC} \frac{4}{5} + F_{FC} \cdot \frac{2}{\sqrt{13}} = 0$~~

~~$3P + \frac{1}{3}(Q_C + Q_B) - \frac{11}{4} P + F_{FC} \cdot \frac{2}{\sqrt{13}}$~~

~~$+ \frac{P}{4}$~~



$F_{EC} \cdot \frac{2}{\sqrt{13}} + F_{EB} - 2P = 0$

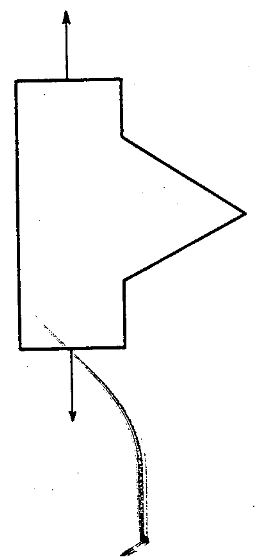
$F_{EB} = 2P - F_{EC} \frac{2}{\sqrt{13}}$

$F_{EC} = (2P - F_{EB}) \cdot \frac{\sqrt{13}}{2}$

$= P\sqrt{13} + \frac{Q_B + Q_C}{6} \sqrt{13} - \frac{9\sqrt{13} P}{8}$

$F_{EC} = -\frac{\sqrt{13}}{8} P + \frac{Q_B}{6}\sqrt{13} + \frac{Q_C \sqrt{13}}{6}$

8. The figure represents a "tooth" on a plate in a state of plane stress in the plane of the paper. The faces of the tooth (the two straight lines) are free from force. On the supposition that the stress components are all finite and continuous throughout the region, prove that there is no stress at all at the apex of the tooth.



chapter | 3

Two-dimensional Problems in Rectangular Coordinates

18 | Solution by Polynomials

It has been shown that the solution of two-dimensional problems, when body forces are absent or are constant, is reduced to the integration of the differential equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \qquad (a)$$

having regard to boundary conditions (20). In the case of long rectangular strips, solutions of Eq. (*a*) in the form of polynomials are of interest. By taking polynomials of various degrees, and suitably adjusting their coefficients, a number of practically important problems can be solved.[1]

Beginning with a polynomial of the second degree

$$\phi_2 = \frac{a_2}{2}x^2 + b_2xy + \frac{c_2}{2}y^2 \qquad (b)$$

which evidently satisfies Eq. (*a*), we find from Eqs. (29), putting $\rho g = 0$,

$$\sigma_x = \frac{\partial^2 \phi_2}{\partial y^2} = c_2 \qquad \sigma_y = \frac{\partial^2 \phi_2}{\partial x^2} = a_2 \qquad \tau_{xy} = -\frac{\partial^2 \phi_2}{\partial x \partial y} = -b_2$$

All three stress components are constant throughout the body, i.e., the stress function (*b*) represents a combination of uniform tensions or compressions[2] in two perpendicular directions and a uniform shear. The

[1] A. Mesnager, *Compt. Rend.*, vol. 132, p. 1475, 1901. See also A. Timpe, *Z. Math. Physik*, vol. 52, p. 348, 1905.

[2] The arrows in Fig. 21 are all drawn in the standard sense, as defined in Art. 3. The numbers a_2, $-b_2$, c_2 attached to them may be positive or negative. Thus all possibilities can be covered without changing the directions of the arrows. In Fig. 22, however, the arrows show directly the intended directions of the applied forces.

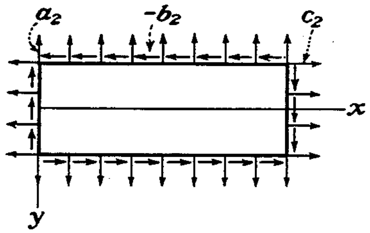


Fig. 21

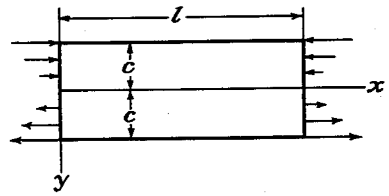


Fig. 22

forces on the boundaries must equal the stresses at these points as discussed on page 28; in the case of a rectangular plate with sides parallel to the coordinate axes, these forces are shown in Fig. 21.

Let us consider now a stress function in the form of a polynomial of the third degree:

$$\phi_3 = \frac{a_3}{3(2)} x^3 + \frac{b_3}{2} x^2 y + \frac{c_3}{2} xy^2 + \frac{d_3}{3(2)} y^3 \qquad (c)$$

This also satisfies Eq. (a). Using Eqs. (29) and putting $\rho g = 0$, we find

$$\sigma_x = \frac{\partial^2 \phi_3}{\partial y^2} = c_3 x + d_3 y$$

$$\sigma_y = \frac{\partial^2 \phi_3}{\partial x^2} = a_3 x + b_3 y$$

$$\tau_{xy} = -\frac{\partial^2 \phi_3}{\partial x \, \partial y} = -b_3 x - c_3 y$$

For a rectangular plate, taken as in Fig. 22, assuming all coefficients except d_3 equal to zero, we obtain pure bending. If only coefficient a_3 is different from zero, we obtain pure bending by normal stresses applied to the sides $y = \pm c$ of the plate. If coefficient b_3 or c_3 is taken different from zero, we obtain not only normal but also shearing stresses acting on the sides of the plate. Figure 23 represents, for instance, the case in

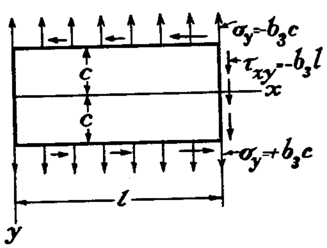


Fig. 23

$d_3 \neq 0$ pure bending
$a_3 \neq 0$
b_3 or $c_3 \neq 0$ normal on $\pm y = c$
& τ_{xy} on $\pm x = l$

which all coefficients except b_3 in function (c) are equal to zero. Along the sides $y = \pm c$ we have uniformly distributed tensile and compressive stresses, respectively, and shearing stresses proportional to x. On the side $x = l$ we have only the constant shearing stress $-b_3 l$, and there are no stresses acting on the side $x = 0$. An analogous stress distribution is obtained if coefficient c_3 is taken different from zero.

In taking the stress function in the form of polynomials of the second and third degrees we are completely free in choosing the magnitudes of the coefficients, since Eq. (a) is satisfied whatever values they may have. In the case of polynomials of higher degrees Eq. (a) is satisfied only if certain relations between the coefficients are satisfied. Taking, for instance, the stress function in the form of a polynomial of the fourth degree,

$$\phi_4 = \frac{a_4}{4(3)} x^4 + \frac{b_4}{3(2)} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{3(2)} xy^3 + \frac{e_4}{4(3)} y^4 \qquad (d)$$

and substituting it into Eq. (a), we find that the equation is satisfied only if

$$e_4 = -(2c_4 + a_4)$$

The stress components in this case are

$$\sigma_x = \frac{\partial^2 \phi_4}{\partial y^2} = c_4 x^2 + d_4 xy - (2c_4 + a_4) y^2$$

$$\sigma_y = \frac{\partial^2 \phi_4}{\partial x^2} = a_4 x^2 + b_4 xy + c_4 y^2$$

$$\tau_{xy} = -\frac{\partial^2 \phi_4}{\partial x \, \partial y} = -\frac{b_4}{2} x^2 - 2c_4 xy - \frac{d_4}{2} y^2$$

Coefficients $a_4, \ldots, d_4$ in these expressions are arbitrary, and by suitably adjusting them we obtain various conditions of loading of a rectangular plate. For instance, taking all coefficients except d_4 equal to zero, we find

$$\sigma_x = d_4 xy \qquad \sigma_y = 0 \qquad \tau_{xy} = -\frac{d_4}{2} y^2 \qquad (e)$$

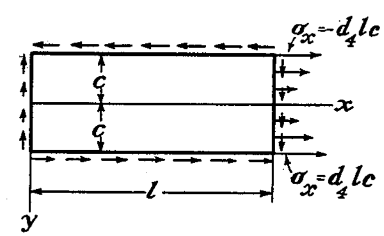


Fig. 24

Assuming d_4 positive, the forces acting on the rectangular plate shown in Fig. 24 and producing the stresses (e) are as given. On the longitudinal sides, $y = \pm c$ are uniformly distributed shearing forces; on the ends, shearing forces are distributed according to a parabolic law. The shearing forces acting on the boundary of the plate reduce to the couple[1]

$$M = \frac{d_4c^2l}{2} 2c - \frac{1}{3}\frac{d_4c^2}{2} 2cl = \frac{2}{3} d_4c^3l$$

This couple balances the couple produced by the normal forces along the side $x = l$ of the plate.

Let us consider a stress function in the form of a polynomial of the fifth degree.

$$\phi_5 = \frac{a_5}{5(4)} x^5 + \frac{b_5}{4(3)} x^4y + \frac{c_5}{3(2)} x^3y^2 + \frac{d_5}{3(2)} x^2y^3 + \frac{e_5}{4(3)} xy^4 + \frac{f_5}{5(4)} y^5 \qquad (f)$$

Substituting in Eq. (a) we find that this equation is satisfied if

$$e_5 = -(2c_5 + 3a_5)$$
$$f_5 = -\tfrac{1}{3}(b_5 + 2d_5)$$

The corresponding stress components are:

$$\sigma_x = \frac{\partial^2\phi_5}{\partial y^2} = \frac{c_5}{3} x^3 + d_5x^2y - (2c_5 + 3a_5)xy^2 - \frac{1}{3}(b_5 + 2d_5)y^3$$

$$\sigma_y = \frac{\partial^2\phi_5}{\partial x^2} = a_5x^3 + b_5x^2y + c_5xy^2 + \frac{d_5}{3} y^3$$

$$\tau_{xy} = -\frac{\partial^2\phi_5}{\partial x\, \partial y} = -\frac{1}{3} b_5x^3 - c_5x^2y - d_5xy^2 + \frac{1}{3}(2c_5 + 3a_5)y^3$$

Again coefficients $a_5, \ldots, d_5$ are arbitrary, and in adjusting them we obtain solutions for various loading conditions of a plate. Taking, for instance, all coefficients, except d_5, equal to zero we find

$$\sigma_x = d_5(x^2y - \tfrac{2}{3}y^3)$$
$$\sigma_y = \tfrac{1}{3}d_5y^3 \qquad (g)$$
$$\tau_{xy} = -d_5xy^2$$

The normal forces are uniformly distributed along the longitudinal sides of the plate (Fig. 25a). Along the side $x = l$, the normal forces consist of two parts, one following a linear law and the other following the law of a

[1] The thickness of the plate is taken equal to unity.

σ_y = const on $\pm y = c$
τ_{xy} linear on $\pm y = c$
parab on $+x = l$
σ_x = f(cubic parabola

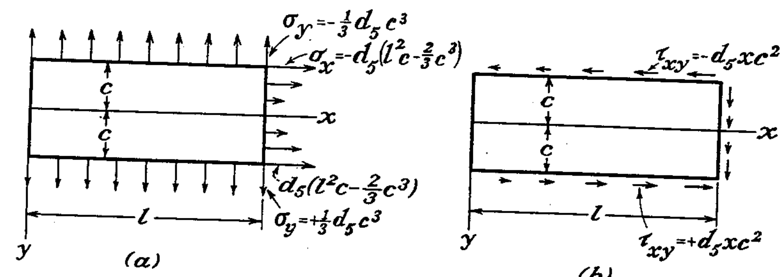


Fig. 25

cubic parabola. The shearing forces are proportional to x on the longitudinal sides of the plate and follow a parabolic law along the side $x = l$. The distribution of these stresses is shown in Fig. 25b.

Since Eq. (a) is a linear differential equation, a sum of several solutions of this equation is also a solution. We can superpose the elementary solutions considered in this article to arrive at new solutions of practical interest. Several examples of the application of this method of superposition will be considered.

19 | End Effects. Saint-Venant's Principle

In the previous article several solutions for rectangular plates were obtained from very simple forms of the stress function ϕ. In each case the boundary forces must be distributed exactly as the solution itself requires. In the case of pure bending, for instance (Fig. 22), the loading on the ends must consist of normal traction (σ_x, at $x = 0$ or $x = l$) proportional to y. If the couples on the ends are applied in any other manner, the solution given in Art. 18 is no longer correct. Another solution must be found if the changed boundary conditions on the ends are to be exactly satisfied. Many such solutions have been obtained (some are referred to later) not only for rectangular regions but for prismatic, cylindrical, and tapered shapes. These show that a change in the distribution of the load on an end, without change of the resultant, alters the stress significantly only near the end. In such cases then, simple solutions such as those of the present chapter can give sufficiently accurate results except near the ends.

The change of distribution of the load is equivalent to the superposition of a system of forces statically equivalent to zero force and zero couple. The expectation that such a system, applied to a small part of the surface of the body, would give rise to localized stress and strain only, was enunciated by Saint-Venant[1] in 1855 and came to be

[1] B. de Saint-Venant, "Mémoires des Savants Etrangers," vol. 14, 1855.

known as *Saint-Venant's principle.* It accords with common experience in a variety of circumstances not confined to small strains in elastic materials obeying Hooke's law—for instance, the application of a small clamp to a length of thick rubber tube causes appreciable strain only in the immediate neighborhood of the clamp.

For bodies extended in two or three dimensions, such as disks, spheres, or the semi-infinite solid, the stress or strain due to loading on a small part of the body may be expected to diminish with distance on account of "geometrical divergence," whether or not the resultant is zero. It has been shown[1] that vanishing of the resultant is not an adequate criterion for the degree of localization.

20 | Determination of Displacements

When the components of stress are found from the previous equations, the components of strain can be obtained by using Hooke's law, Eqs. (3) and (6). Then the displacements u and v can be obtained from the equations

$$\frac{\partial u}{\partial x} = \epsilon_x \qquad \frac{\partial v}{\partial y} = \epsilon_y \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy} \tag{a}$$

The integration of these equations in each particular case does not present any difficulty, and we shall have several examples of their application. It may be seen at once that the strain components (a) remain unchanged if we add to u and v the linear functions

$$u_1 = a + by \qquad v_1 = c - bx \tag{b}$$

in which a, b, and c are constants. This means that the displacements are not entirely determined by the stresses and strains. A displacement like that of a rigid body can be superposed on the displacements due to the internal strains. The constants a and c in Eqs. (b) represent a translatory motion of the body and the constant b is a small angle of rotation of the rigid body about the z axis.

It has been shown (see page 31) that in the case of constant body forces the stress distribution is the same for plane stress distribution or plane strain. The displacements are different for these two problems, however, since in the case of plane stress distribution the components of strain, entering into Eqs. (a), are given by equations

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

[1] R. von Mises, *Bull. Am. Math. Soc.*, vol. 51, p. 555, 1945; E. Sternberg, *Quart. Appl. Math.*, vol. 11, p. 393, 1954; E. Sternberg and W. T. Koiter, *J. Appl. Mech.*, vol. 25, pp. 575–581, 1958.

and in the case of plane strain the strain components are:

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] = \frac{1}{E}[(1 - \nu^2)\sigma_x - \nu(1 + \nu)\sigma_y]$$

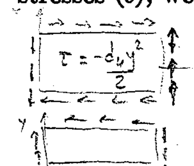
$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] = \frac{1}{E}[(1 - \nu^2)\sigma_y - \nu(1 + \nu)\sigma_x]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

It is easily verified that these equations can be obtained from the preceding set for plane stress by replacing E in the latter by $E/(1 - \nu^2)$, and ν by $\nu/(1 - \nu)$. These substitutions leave G, which is $E/2(1 + \nu)$, unchanged. The integration of Eqs. (a) will be shown later in discussing particular problems.

21 | Bending of a Cantilever Loaded at the End

Consider a cantilever having a narrow rectangular cross section of unit width bent by a force P applied at the end (Fig. 26). The upper and lower edges are free from load, and shearing forces, having a resultant P, are distributed along the end $x = 0$. These conditions can be satisfied by a proper combination of pure shear with the stresses (e) of Art. 18 represented in Fig. 24. Superposing the pure shear $\tau_{xy} = -b_2$ on the stresses (e), we find



$$\sigma_x = d_4xy \qquad \sigma_y = 0 \qquad \tau_{xy} = -b_2 - \frac{d_4}{2}y^2 \tag{a}$$

To have the longitudinal sides $y = \pm c$ free from forces we must have

$$(\tau_{xy})_{y=\pm c} = -b_2 - \frac{d_4}{2}c^2 = 0$$

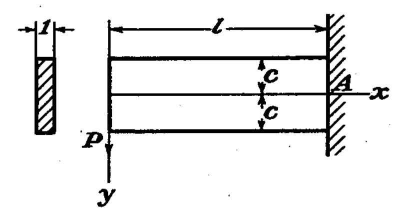


Fig. 26

from which

$$d_4 = -\frac{2b_2}{c^2}$$

To satisfy the condition on the loaded end the sum of the shearing forces distributed over this end must be equal to P. Hence[1]

$$-\int_{-c}^{c} \tau_{xy}\, dy = \int_{-c}^{c} \left(b_2 - \frac{b_2}{c^2} y^2\right) dy = P$$

from which

$$b_2 = \frac{3}{4}\frac{P}{c}$$

Substituting these values of d_4 and b_2 in Eqs. (a) we find

$$\sigma_x = -\frac{3}{2}\frac{P}{c^3}xy \qquad \sigma_y = 0$$

$$\tau_{xy} = -\frac{3P}{4c}\left(1 - \frac{y^2}{c^2}\right)$$

Noting that $\frac{2}{3}c^3$ is the moment of inertia I of the cross section of the cantilever, we have

$$\sigma_x = -\frac{Pxy}{I} \qquad \sigma_y = 0 \qquad \qquad (b)$$

$$\tau_{xy} = -\frac{P}{I}\frac{1}{2}(c^2 - y^2)$$

This coincides completely with the elementary solution as given in books on the strength of materials. It should be noted that this solution represents an exact solution only if the shearing forces on the ends are distributed according to the same parabolic law as the shearing stress τ_{xy} and the intensity of the normal forces at the built-in end is proportional to y. If the forces at the ends are distributed in any other manner, the stress distribution (b) is not a correct solution for the ends of the cantilever, but, by virtue of Saint-Venant's principle, it can be considered satisfactory for cross sections at a considerable distance from the ends.

Let us consider now the displacement corresponding to the stresses (b). Applying Hooke's law we find

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E} = -\frac{Pxy}{EI} \qquad \epsilon_y = \frac{\partial v}{\partial y} = -\frac{\nu\sigma_x}{E} = \frac{\nu Pxy}{EI} \qquad (c)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G} = -\frac{P}{2IG}(c^2 - y^2) \qquad (d)$$

[1] The minus sign before the integral follows from the rule for the sign of shearing stresses. Stress τ_{xy} on the end $x = 0$ is positive if it is upward (see p. 4).

The procedure for obtaining the components u and v of the displacement consists in integrating Eqs. (c) and (d). By integration of Eqs. (c) we find

$$u = -\frac{Px^2y}{2EI} + f(y) \qquad v = \frac{\nu Pxy^2}{2EI} + f_1(x)$$

in which $f(y)$ and $f_1(x)$ are as yet unknown functions of y only and x only. Substituting these values of u and v in Eq. (d) we find

$$-\frac{Px^2}{2EI} + \frac{df(y)}{dy} + \frac{\nu Py^2}{2EI} + \frac{df_1(x)}{dx} = -\frac{P}{2IG}(c^2 - y^2)$$

In this equation some terms are functions of x only, some are functions of y only, and one is independent of both x and y. Denoting these groups by $F(x)$, $G(y)$, K, we have

$$F(x) = -\frac{Px^2}{2EI} + \frac{df_1(x)}{dx} \qquad G(y) = \frac{df(y)}{dy} + \frac{\nu Py^2}{2EI} - \frac{Py^2}{2IG}$$

$$K = -\frac{Pc^2}{2IG}$$

and the equation may be written

$$F(x) + G(y) = K$$

Such an equation means that $F(x)$ must be some constant d and $G(y)$ some constant e. Otherwise $F(x)$ and $G(y)$ would vary with x and y, respectively, and by varying x alone, or y alone, he equality would be violated. Thus

$$e + d = -\frac{Pc^2}{2IG} \qquad (e)$$

and $$\frac{df_1(x)}{dx} = \frac{Px^2}{2EI} + d \qquad \frac{df(y)}{dy} = -\frac{\nu Py^2}{2EI} + \frac{Py^2}{2IG} + e$$

Functions $f(y)$ and $f_1(x)$ are then

$$f(y) = -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$

$$f_1(x) = \frac{Px^3}{6EI} + dx + h$$

Substituting in the expressions for u and v we find

$$u = -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} + dx + h \qquad (g)$$

The constants d, e, g, h may now be determined from Eq. (e) and from the three conditions of constraint that are necessary to prevent the beam from moving as a rigid body in the xy plane. Assume that the point A, the centroid of the end cross section, is fixed. Then u and v are zero for $x = l$, $y = 0$, and we find from Eqs. (g)

$$g = 0 \qquad h = -\frac{Pl^3}{6EI} - dl$$

The deflection curve is obtained by substituting $y = 0$ into the second of Eqs. (g). Then

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^3}{6EI} - d(l - x) \tag{h}$$

For determining the constant d in this equation, we must use the third condition of constraint, eliminating the possibility of rotation of the beam in the xy plane about the fixed point A. This constraint can be realized in various ways. Let us consider two cases: (1) When an element of the axis of the beam is fixed at the end A. Then the condition of constraint is

$$\left(\frac{\partial v}{\partial x}\right)_{\substack{x=l\\y=0}} = 0 \tag{k}$$

(2) When a vertical element of the cross section at the point A is fixed. Then the condition of constraint is

$$\left(\frac{\partial u}{\partial y}\right)_{\substack{x=l\\y=0}} = 0 \tag{l}$$

In the first case we obtain from Eq. (h)

$$d = -\frac{Pl^2}{2EI}$$

and from Eq. (e) we find

$$e = \frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}$$

Substituting all the constants in Eqs. (g), we find

$$\begin{aligned} u &= -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + \left(\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}\right)y \\ v &= \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} \end{aligned} \tag{m}$$

The equation of the deflection curve is

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} \tag{n}$$

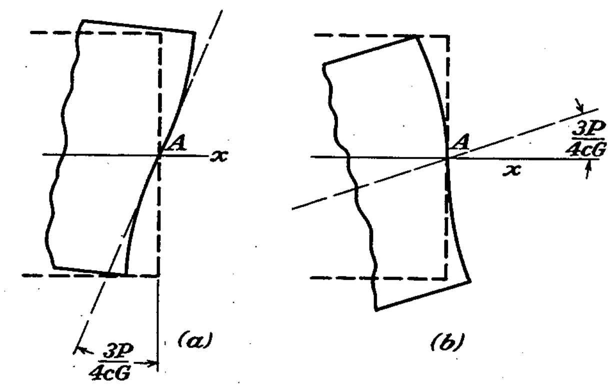


Fig. 27

which gives for the deflection at the loaded end ($x = 0$) the value $Pl^3/3EI$. This coincides with the value usually derived in elementary books on the strength of materials.

To illustrate the distortion of cross sections produced by shearing stresses, let us consider the displacement u at the fixed end ($x = l$). For this end we have from Eqs. (m),

$$\begin{aligned} (u)_{x=l} &= -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} - \frac{Pc^2y}{2IG} \\ \left(\frac{\partial u}{\partial y}\right)_{x=l} &= -\frac{\nu Py^2}{2EI} + \frac{Py^2}{2IG} - \frac{Pc^2}{2IG} \\ \left(\frac{\partial u}{\partial y}\right)_{\substack{x=l\\y=0}} &= -\frac{Pc^2}{2IG} = -\frac{3}{4}\frac{P}{cG} \end{aligned} \tag{o}$$

The shape of the cross section after distortion is as shown in Fig. 27a. Owing to the shearing stress $\tau_{xy} = -3P/4c$ at the point A, an element of the cross section at A rotates in the xy plane about the point A through an angle $3P/4cG$ in the clockwise direction.

If a vertical element of the cross section is fixed at A (Fig. 27b), instead of a horizontal element of the axis, we find from condition (l) and the first of Eqs. (g)

$$e = \frac{Pl^2}{2EI}$$

and from Eq. (e) we find

$$d = -\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}$$

Substituting in the second of Eqs. (g) we find

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} + \frac{Pc^2}{2IG}(l-x) \qquad (r)$$

Comparing this with Eq. (n) it can be concluded that, owing to rotation of the end of the axis at A (Fig. 27b), the deflections of the axis of the cantilever are increased by the quantity

$$\frac{Pc^2}{2IG}(l-x) = \frac{3P}{4cG}(l-x)$$

This is an estimate[1] of the so-called *effect of shearing force* on the deflection of the beam. In practice, at the built-in end we have conditions different from those shown in Fig. 27. The fixed section[2] is usually not free to distort and the distribution of forces at this end is different from that given by Eqs. (b). However, solution (b) is satisfactory for comparatively long cantilevers at considerable distances from the terminals.

22 | Bending of a Beam by Uniform Load

Let a beam of narrow rectangular cross section of unit width, supported at the ends, be bent by a uniformly distributed load of intensity q, as shown in Fig. 28. The conditions at the upper and lower edges of the beam are:

$$(\tau_{xy})_{y=\pm c} = 0 \qquad (\sigma_y)_{y=+c} = 0 \qquad (\sigma_y)_{y=-c} = -q \qquad (a)$$

The conditions at the ends $x = \pm l$ are

$$\int_{-c}^{c} \tau_{xy}\,dy = \mp ql \qquad \int_{-c}^{c} \sigma_x\,dy = 0 \qquad \int_{-c}^{c} \sigma_x y\,dy = 0 \qquad (b)$$

The last two of Eqs. (b) state that there is no longitudinal force and no bending couple applied at the ends of the beam. All the conditions (a)

[1] Others are indicated in Prob. 3, p. 63, and in the text on p. 49.
[2] The effect of elasticity in the support itself is examined experimentally and analytically by W. J. O'Donnell, *J. Appl. Mech.*, vol. 27, pp. 461–464, 1960.

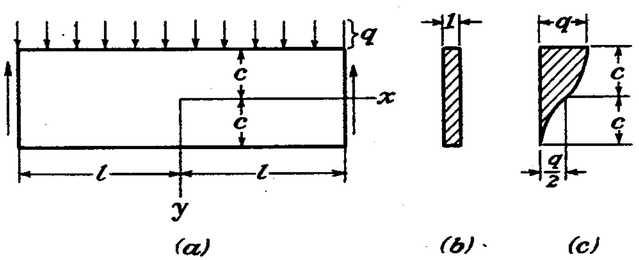


Fig. 28

and (b) can be satisfied by combining certain solutions in the form of polynomials as obtained in Art. 18. We begin with solution (g), illustrated by Fig. 25. To remove the tensile stresses along the side $y = c$ and the shearing stresses along the sides $y = \pm c$, we superpose a simple compression $\sigma_y = a_2$ from solution (b), Art. 18, and the stresses $\sigma_y = b_3 y$ and $\tau_{xy} = -b_3 x$ in Fig. 23. In this manner we find

$$\begin{aligned} \sigma_x &= d_5(x^2y - \tfrac{2}{3}y^3) \\ \sigma_y &= \tfrac{1}{3}d_5y^3 + b_3y + a_2 \\ \tau_{xy} &= -d_5xy^2 - b_3x \end{aligned} \qquad (c)$$

From the conditions (a) we find

$$\begin{aligned} -d_5c^2 - b_3 &= 0 \\ \tfrac{1}{3}d_5c^3 + b_3c + a_2 &= 0 \\ -\tfrac{1}{3}d_5c^3 - b_3c + a_2 &= -q \end{aligned}$$

from which

$$a_2 = -\frac{q}{2} \qquad b_3 = \frac{3}{4}\frac{q}{c} \qquad d_5 = -\frac{3}{4}\frac{q}{c^3}$$

Substituting in Eqs. (c) and noting that $2c^3/3$ is equal to the moment of inertia I of the rectangular cross-sectional area of unit width, we find

$$\begin{aligned} \sigma_x &= -\frac{3}{4}\frac{q}{c^3}\left(x^2y - \frac{2}{3}y^3\right) = -\frac{q}{2I}\left(x^2y - \frac{2}{3}y^3\right) \\ \sigma_y &= -\frac{3q}{4c^3}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right) = -\frac{q}{2I}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right) \\ \tau_{xy} &= -\frac{3q}{4c^3}(c^2 - y^2)x = -\frac{q}{2I}(c^2 - y^2)x \end{aligned} \qquad (d)$$

It can easily be checked that these stress components satisfy not only conditions (a) on the longitudinal sides but also the first two conditions (b) at the ends. To make the couples at the ends of the beam vanish, we superpose on solution (d) a pure bending, $\sigma_x = d_3 y$, $\sigma_y = \tau_{xy} = 0$, shown in Fig. 22, and determine the constant d_3 from the condition at $x = \pm l$

$$\int_{-c}^{c} \sigma_x y\,dy = \int_{-c}^{c}\left[-\frac{3}{4}\frac{q}{c^3}\left(l^2y - \frac{2}{3}y^3\right) + d_3y\right]y\,dy = 0$$

from which

$$d_3 = \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)$$

$$\sigma_x = -\frac{3}{4}\frac{q}{c^3}\left(x^2 y - \frac{2}{3}y^3\right) + \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)y$$

$$= \frac{q}{2I}\left(l^2 - x^2\right)y + \frac{q}{2I}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2 y\right)$$

elementary solution

solution correction since there is a σ in the vertical direction

$$\sigma_x = -\frac{3}{4}\frac{q}{c^3}\left(x^2 y - \frac{2}{3}y^3\right) + \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)y \quad ; \text{ since } I = 2c^3/3$$

$$= \frac{q}{2I}\left(l^2 - x^2\right)y + \frac{q}{2I}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2 y\right)$$

elementary solution (first term)

solution correction (second term) since there is a σ in the vertical direction

correction doesn't depend on x & is small compared to bending stress (max) as long as $l/c \gg 1$

This expression is exact as long as the normal forces at $x = \pm l$ are distributed according to

$$\pm \frac{3}{4}\frac{q}{c^3}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2 y\right)$$

so that resultant force = 0 & resultant moment = 0

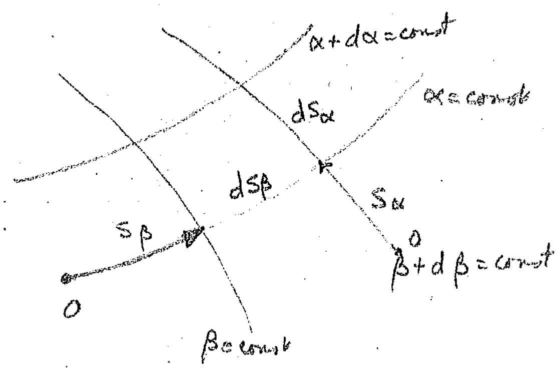
define the displacement vector $\mathbf{u} = u_\gamma \mathbf{e}_\gamma + u_\beta \mathbf{e}_\beta + u_\gamma \mathbf{e}_\gamma$

define a tensor $\mathbb{T} = \sigma_{\alpha\alpha} \mathbf{e}_\alpha \mathbf{e}_\alpha + \sigma_{\alpha\beta} \mathbf{e}_\alpha \mathbf{e}_\beta + \dots = \sigma_{\alpha_i \alpha_j} \mathbf{e}_{\alpha_i} \mathbf{e}_{\alpha_j}$

	x	y	z
α	$\mathbf{e}_\alpha \cdot \mathbf{e}_x$	$\mathbf{e}_\alpha \cdot \mathbf{e}_y$	$\mathbf{e}_\alpha \cdot \mathbf{e}_z$
β	$\mathbf{e}_\beta \cdot \mathbf{e}_x$	$\mathbf{e}_\beta \cdot \mathbf{e}_y$	$\mathbf{e}_\beta \cdot \mathbf{e}_z$
γ	$\mathbf{e}_\gamma \cdot \mathbf{e}_x$	$\mathbf{e}_y \cdot \mathbf{e}_y$	$\mathbf{e}_\gamma \cdot \mathbf{e}_z$

This transforms items in (x,y,z) plane to (α,β,γ) plane

Now look at surface $\gamma = \text{const}$.



stretch function or stretch factor

$$ds_\alpha = h_\alpha(\alpha,\beta,\gamma)\,d\alpha \qquad \frac{d\alpha}{ds_\alpha} = \frac{1}{h_\alpha}$$

$$ds_\beta = h_\beta(\alpha,\beta,\gamma)\,d\beta$$

$$ds_\gamma = h_\gamma(\alpha,\beta,\gamma)\,d\gamma$$

since α, β, or γ may not be a physical length (i.e. θ)

Looking at the position vector in the cartesian plane $\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z$ then

$$d\mathbf{r} = dx\,\mathbf{e}_x + dy\,\mathbf{e}_y + dz\,\mathbf{e}_z = \mathbf{e}_\alpha\,ds_\alpha + \mathbf{e}_\beta\,ds_\beta + \mathbf{e}_\gamma\,ds_\gamma$$

now we note that $\dfrac{\partial \mathbf{r}}{\partial s_\alpha} = \mathbf{e}_\alpha$

We can take

$$\frac{\partial \mathbf{r}}{\partial s_\alpha} = \overset{\mathbf{e}_\alpha\cdot\mathbf{e}_x}{\frac{\partial x}{\partial s_\alpha}}\mathbf{e}_x + \overset{\mathbf{e}_\alpha\cdot\mathbf{e}_y}{\frac{\partial y}{\partial s_\alpha}}\mathbf{e}_y + \overset{\mathbf{e}_z\cdot\mathbf{e}_\alpha}{\frac{\partial z}{\partial s_\alpha}}\mathbf{e}_z = \mathbf{e}_\alpha$$

now take the dot product of

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\alpha = 1 = \left(\frac{\partial x}{\partial s_\alpha}\right)^2 + \left(\frac{\partial y}{\partial s_\alpha}\right)^2 + \left(\frac{\partial z}{\partial s_\alpha}\right)^2$$

$$= \left(\frac{\partial x}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial s_\alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial s_\alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial s_\alpha}\right)^2$$

$$= \frac{1}{h_\alpha^2}\left[\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2\right]$$

$$\therefore \quad h_\alpha = \sqrt{\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2 + \left(\frac{\partial z}{\partial \alpha}\right)^2}$$

$$\therefore \quad \boxed{h_{\alpha_j} = \sqrt{\frac{\partial x_i}{\partial \alpha_j}\frac{\partial x_i}{\partial \alpha_j}} \quad \text{sum over } i, \text{ no sum over } j}$$

$h_r = \sqrt{\cos^2 + \sin^2} = 1$

$h_\theta = \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2} = r$

$h_z = 1$

$ds_\theta = h_\theta \, d\theta = r\,d\theta$
$ds_r = dr$
$ds_z = dz$

for spherical $h_R = 1$ $\therefore$ $ds_R = dr$
$h_\theta = R\sin\varphi$ $ds_\theta = R\sin\varphi \, d\theta$
$h_\varphi = R$ $ds_\varphi = R\,d\varphi$

Direction cosines

$$l_{\alpha y} = e_\alpha \cdot e_y = \frac{\partial y}{\partial s_\alpha} = \frac{1}{h_\alpha} \cdot \frac{\partial y}{\partial \alpha}$$

$$\therefore \quad l_{\alpha_i x_j} = e_{\alpha_i} \cdot e_j = \frac{1}{h_{\alpha_i}} \frac{\partial x_j}{\partial \alpha_i} \qquad i \text{ not summed.}$$

$$e_\alpha = \frac{1}{h_\alpha}\frac{\partial x}{\partial \alpha} e_x + \frac{1}{h_\alpha}\frac{\partial y}{\partial \alpha} e_y + \frac{1}{h_\alpha}\frac{\partial z}{\partial \alpha} e_z$$

for cylindrical coord.

$e_z = e_z$ $\qquad \dfrac{\partial e_r}{\partial \theta} = e_\theta$

$e_r = \cos\theta \, e_x + \sin\theta \, e_y$

$e_\theta = -\sin\theta \, e_x + \cos\theta \, e_y$ $\qquad \dfrac{\partial e_\theta}{\partial \theta} = -e_r$

for spherical coordinates

$e_R = \sin\varphi\cos\theta \, e_x + \sin\varphi\sin\theta \, e_y + \cos\varphi \, e_z$

$e_\theta = -\sin\theta \, e_x + \cos\theta \, e_y$

$e_\varphi = \cos\varphi\cos\theta \, e_x + \cos\varphi\sin\theta \, e_y - \sin\varphi \, e_z$

$$\frac{\partial e_R}{\partial \theta} = \sin\varphi \, e_\theta \, , \quad \frac{\partial e_R}{\partial \varphi} = e_\varphi$$

THUS WE CAN WRITE $\underline{\underline{A}}$ SOLUTION TO $\nabla^4 \phi = 0$ IN POLAR COORDINATES AS: (TIMOSHENKO & GOODIER, p. 133, eq. 80)

(sol for stress distr symm about origin) (Vertical load on straight body) (pure shear)

$$\phi = a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r + d_0 r^2 \theta + a_0' \theta$$

(radial dist along $\theta = 0, \pi$) (Portion of circular ring bent by radial force along $\theta = 0, \pi$)

$$+ a_1 r \theta \sin\theta + \left(b_1 r^3 + a_1' r^{-1} + b_1' r \ln r \right) \cos\theta$$

(entire line – force acting on infinite plate along $\theta = 0, \pi$; for pt force take term 1 and last term) ($\theta = \pm\pi/2$)

(radial dist along $\theta = \pm\frac{\pi}{2}$)

$$+ c_1 r \theta \cos\theta + \left(d_1 r^3 + c_1' r^{-1} + d_1' r \ln r \right) \sin\theta$$

(entire line – force acting on infinite plate along $\theta = \pm\pi/2$; for pt force take term 1 and last term)

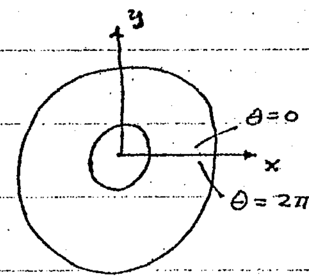
$$+ \sum_{n=2}^{\infty} \left(a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2} \right) \cos n\theta$$

(shearing on ring)

$$+ \sum_{n=2}^{\infty} \left(c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2} \right) \sin n\theta$$

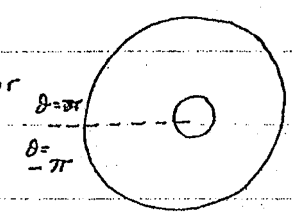
(normal force on ring)

LET US SEE IF WE CAN USE THIS STRESS FUNCTION (OR PIECES OF IT) TO SOLVE PROBLEMS OF LOADING ON A COMPLETE ANGULAR RING.



(1) SOMETIMES WE WILL FIND WE HAVE TO EXAMINE DISPLACEMENTS ALSO TO CHECK ON ALLOWABLE TERMS.

or



(2) SUPPOSE WE ARE INTERESTED IN SOLVING THE TRACTION BOUNDARY VALUE PROBLEM FOR A COMPLETE RING.

SUPPOSE THE BOUNDARY CONDITIONS ARE:

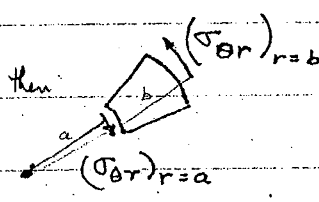
$$(\sigma_{rr})_{r=a} = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta$$

$$(\sigma_{rr})_{r=b} = A_0' + \sum_{n=1}^{\infty} A_n' \cos n\theta + \sum_{n=1}^{\infty} B_n' \sin n\theta$$

$$(\sigma_{r\theta})_{r=a} = C_0 + \sum_{n=1}^{\infty} C_n \cos n\theta + \sum_{n=1}^{\infty} D_n \sin n\theta$$

$$(\sigma_{r\theta})_{r=b} = C_0' + \sum_{n=1}^{\infty} C_n' \cos n\theta + \sum_{n=1}^{\infty} D_n' \sin n\theta$$

I. BOUNDARY LOADS MUST BE SELF-EQUILIBRATING !!



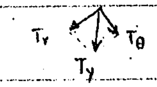
__MOMENT__ __EQUILIBRIUM__ since T_r gives no moments but T_θ does then

about origin $$\int_0^{2\pi} \left\{ b\,\sigma_{\theta r} \right\}_{r=b} b\,d\theta - \int_0^{2\pi} \left\{ a\,\sigma_{\theta r} \right\}_{r=a} a\,d\theta = 0.$$

NOW $$\int_0^{2\pi} \cos n\theta\,d\theta = \int_0^{2\pi} \sin n\theta\,d\theta = 0 \quad \text{FOR } n \geq 1$$

EQUILIBRIUM OF MOMENTS REQUIRES:

$$\boxed{C_0 a^2 = C_0' b^2}$$



__FORCE EQUILIBRIUM__

$$T_x = T_r \cos\theta - T_\theta \sin\theta$$

$$T_y = T_r \sin\theta + T_\theta \cos\theta$$

$$T_r = \sigma_{rr}, \quad T_\theta = \sigma_{r\theta}$$

$$\frac{T_x^{NET}}{\text{unit length}} = 0 = \int_0^{2\pi} \{ \sigma_{rr} \cos\theta - \sigma_{r\theta} \sin\theta \}_{r=b} \, b \, d\theta$$

$$- \int_0^{2\pi} \{ \sigma_{rr} \cos\theta - \sigma_{r\theta} \sin\theta \}_{r=a} \, a \, d\theta$$

Only the $n=1$ terms can contribute: $\int_0^{2\pi} \cos^2\theta \, d\theta = \int_0^{2\pi} \sin^2\theta \, d\theta = \pi$

$\int_0^{2\pi} \sin\theta \cos\theta \, d\theta = 0$

$$\therefore \quad 0 = (A_1' - D_1') b - (A_1 - D_1) a$$

so

$$\boxed{b(A_1' - D_1') = a(A_1 - D_1)}$$

$T_y^{NET} = 0$ REQUIRES

$$\boxed{b(B_1' + C_1') = a(B_1 + C_1)}$$

NOW ASSUME OUR CONSTANTS ARE SUCH THAT THESE RELATIONSHIPS HOLD.

COMPUTE σ_{rr} AND $\sigma_{r\theta}$ FROM $\phi(r,\theta)$.

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

MUST CHOOSE $d_0 = 0$ FOR A COMPLETE RING

$$\sigma_{rr} = \frac{a_0}{r^2} + 2b_0 + c_0(1 + 2\ln r) + 2d_0\theta$$

$$+2\frac{a_1}{r}\cos\theta + \left(2b_1 r - \frac{2a_1'}{r^3} + \frac{b_1'}{r}\right)\cos\theta$$

$$-2\frac{c_1}{r}\sin\theta + \left(2d_1 r - \frac{2c_1'}{r^3} + \frac{d_1'}{r}\right)\sin\theta$$

$$+\sum_{n=2}^{\infty}\left\{a_n n(1-n)r^{n-2} + b_n(2+n-n^2)r^n - a_n' n(1+n)r^{-n-2} + b_n'(2-n-n^2)r^{-n}\right\}\cos n\theta$$

$$+\sum_{n=2}^{\infty}\left\{c_n n(1-n)r^{n-2} + d_n(2+n-n^2)r^n - c_n' n(1+n)r^{-n-2} + d_n'(2-n-n^2)r^{-n}\right\}\sin n\theta$$

BOUNDARY CONDITIONS ON $(\sigma_{rr})_{r=a}$ AND $(\sigma_{rr})_{r=b}$

(1) $\frac{a_0}{a^2} + 2b_0 + c_0(1 + 2\ln a) = A_0$

(2) $\frac{a_0}{b^2} + 2b_0 + c_0(1 + 2\ln b) = A_0'$

} 2 EQNS. — 3 UNKNOWNS

(3) $\frac{2a_1}{a} + 2b_1 a - \frac{2a_1'}{a^3} + \frac{b_1'}{a} = A_1$

(4) $+\frac{2a_1}{b} + 2b_1 b - \frac{2a_1'}{b^3} + \frac{b_1'}{b} = A_1'$

} 2 EQNS. — 4 UNKNOWNS

(5) $-\frac{2c_1}{a} + 2d_1 a - \frac{2c_1'}{a^3} + \frac{d_1'}{a} = B_1$

(6) $-\frac{2c_1}{b} + 2d_1 b - \frac{2c_1'}{b^3} + \frac{d_1'}{b} = B_1'$

} 2 EQNS — 4 UNKNOWNS

$(2-n)(1+n)$

(7) $a_n n(1-n)a^{n-2} + b_n(2+n-n^2)a^n - a_n' n(1+n)a^{-n-2} + b_n'(2-n-n^2)a^{-n} = A_n$

(8) $a_n n(1-n)b^{n-2} + b_n(2+n-n^2)b^n - a_n' n(1+n)b^{-n-2} + b_n'(2-n-n^2)b^{-n} = A_n'$

(9) $c_n n(1-n)a^{n-2} + d_n(2+n-n^2)a^n - c_n' n(1+n)a^{-n-2} + d_n'(2-n-n^2)a^{-n} = B_n$

(10) $c_n n(1-n)b^{n-2} + d_n(2+n-n^2)b^n - c_n' n(1+n)b^{-n-2} + d_n'(2-n-n^2)b^{-n} = B_n'$

(7) – (10): FOR EACH n, 4 EQUATIONS & 8 UNKNOWNS

NOW LOOK AT $\sigma_{r\theta}$ BOUNDARY CONDITIONS: FIRST COMPUTE $\sigma_{r\theta}$ FROM ϕ.

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right\}$$, <u>TAKING $d_0 = 0$.</u>

$$\sigma_{r\theta} = \frac{a_0'}{r^2}$$
$$+ \sin\theta\left\{b_1 \cdot 2r - \frac{2a_1'}{r^3} + \frac{b_1'}{r}\right\}$$

$$- \cos\theta\left\{d_1 \cdot 2r - \frac{2c_1'}{r^3} + \frac{d_1'}{r}\right\}$$

$$+\sum_{n=2}^{\infty} n\sin n\theta\left\{a_n(n-1)r^{n-2} + b_n(n+1)r^n - a_n'(n+1)r^{-n-2} - b_n'(n-1)r^{-n}\right\}$$

$$-\sum_{n=2}^{\infty} n\cos n\theta\left\{c_n(n-1)r^{n-2} + d_n(n+1)r^n - c_n'(n+1)r^{-n-2} - d_n'(n-1)r^{-n}\right\}$$

APPLYING BOUNDARY CONDITIONS ON $\sigma_{r\theta}$ ON $r = a, b$.

$$\frac{a_0'}{a^2} = C_0 \quad ; \quad \frac{a_0'}{b^2} = C_0' \quad \Rightarrow \quad a^2 C_0 = b^2 C_0' \quad \text{WHICH WE FOUND BEFORE}$$

$$\therefore \quad \boxed{a_0' = a^2 C_0 = b^2 C_0'}$$

(11) $$2 b_1 a - \frac{2a_1'}{a^3} + \frac{b_1'}{a} = D_1$$

(12) $$2 b_1 b - \frac{2a_1'}{b^3} + \frac{b_1'}{b} = D_1'$$

(3), (4), (11), (12) GIVES 4 EQNS. FOR FOUR UNKNOWNS b_1, a_1', b_1', a_1

(13) $$2 d_1 a - \frac{2c_1'}{a^3} + \frac{d_1'}{a} = -C_1$$

(14) $$2 d_1 b - \frac{2c_1'}{b^3} + \frac{d_1'}{b} = -C_1'$$

(5), (6), (13), (14) GIVES 4 EQNS FOR FOUR UNKNOWNS d_1, c_1', d_1', c_1

(15) $$n \left\{ a_n (n-1) a^{n-2} + b_n (n+1) a^n - a_n' (n+1) a^{-n-2} - b_n' (n-1) a^{-n} \right\} = D_n$$

(16) $$n \left\{ a_n (n-1) b^{n-2} + b_n (n+1) b^n - a_n' (n+1) b^{-n-2} - b_n' (n-1) b^{-n} \right\} = D_n'$$

(7), (8), (15), (16) GIVES 4 EQNS. FOR a_n, b_n, a_n', b_n'

(17) $$n \left\{ c_n (n-1) a^{n-2} + d_n (n+1) a^n - c_n' (n+1) a^{-n-2} - d_n' (n-1) a^{-n} = -C_n \right.$$

(18) $$n \left\{ c_n (n-1) b^{n-2} + d_n (n+1) b^n - c_n' (n+1) b^{-n-2} - d_n' (n-1) b^{-n} = -C_n' \right.$$

(9), (10), (17), (18) GIVES 4 EQNS. FOR c_n, d_n, c_n', d_n'

TWO PROBLEMS: (1) & (2) STILL YIELD ONLY 2 EQNS. IN 3 UNKNOWNS

ARE WE GUARANTEED THAT:

$$b(A_1' - D_1') = a(A_1 - D_1)$$

$$b(B_1' + C_1') = a(B_1 + C_1)$$

LOOK @ SECOND QUESTION FIRST:

$$\left.\begin{aligned} (3) - (11) &\Rightarrow \frac{2a_1}{a} = A_1 - D_1 \\ (4) - (12) &\Rightarrow \frac{2a_1}{b} = A_1' - D_1' \end{aligned}\right\} \text{OKAY}$$

$$\left.\begin{aligned} (5) - (13) &\Rightarrow -\frac{2c_1}{a} = B_1 + C_1 \\ (6) - (14) &\Rightarrow -\frac{2c_1}{b} = B_1' + C_1' \end{aligned}\right\} \text{OKAY}$$

∴ WE KNOW a_1 AND c_1 FROM LOADING BOUNDARY CONDITIONS AND THIS MEANS THAT (3) & (11) REPRESENT ONLY 1 EQUATION AND (4) & (12) REPRESENT ONLY ONE EQUATION. SIMILARLY FOR (5) & (13) & (6) & (14). THUS

$$\left.\begin{aligned} (3)\ \&\ (11) &\rightarrow 2b_1 a - \frac{2a_1'}{a^3} + \frac{b_1'}{a} = D_1 \\ (4)\ \&\ (12) &\rightarrow 2b_1 b - \frac{2a_1'}{b^3} + \frac{b_1'}{b} = D_1' \end{aligned}\right\} \text{2 EQNS} \leftrightarrow \text{3 UNKNOWNS}$$

$$\left.\begin{aligned} (5)\ \&\ (13) &\rightarrow 2d_1 a - \frac{2c_1'}{a^3} + \frac{d_1'}{a} = -C_1 \\ (6)\ \&\ (14) &\rightarrow 2d_1 b - \frac{2c_1'}{b^3} + \frac{d_1'}{b} = -C_1' \end{aligned}\right\} \text{2 EQNS.} \leftrightarrow \text{3 UNKNOWNS}$$

$$\left.\begin{aligned} (1)\quad & \frac{a_0}{a^2} + 2b_0 + c_0(1 + 2\ln a) = A_0 \\ (2)\quad & \frac{a_0}{b^2} + 2b_0 + c_0(1 + 2\ln b) = A_0' \end{aligned}\right\} \text{2 EQNS.} \leftrightarrow \text{3 UNKNOWNS}$$

TIMOSHENKO AND GOODIER, PAGES 77-78, SHOW THAT $c_0 = 0$ FOR A COMPLETE RING OR ELSE u_θ IS MULTI-VALUED!

DERIVATION OF THE RELATION BETWEEN a_1 & b_1' AND a_1 AND d_1' BY EXAMINING THE MULTI-VALUED NATURE OF THE DISPLACEMENT FIELD (PLANE STRAIN)

LOOK AT THE STRESS FUNCTION

$$\phi = a_1 r\theta \sin\theta + b_1' r \ln r \cos\theta$$

THEN

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = \frac{b_1'}{r}\cos\theta$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \frac{b_1'}{r}\sin\theta$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{2a_1}{r}\cos\theta + \frac{b_1'}{r}\cos\theta = \frac{2a_1+b_1'}{r}\cos\theta$$

$$\sigma_{zz} = \nu(\sigma_{rr}+\sigma_{\theta\theta}) = \nu\frac{2(a_1+b_1')}{r}\cos\theta$$

NOW

$$e_{rr} = \frac{\partial u_r}{\partial r} = \frac{1}{E}\left\{\underbrace{\frac{2a_1+b_1'}{r}\cos\theta}_{\sigma_{rr}} - \nu\left[\underbrace{\frac{b_1'}{r}\cos\theta}_{\sigma_{\theta\theta}} + \nu\underbrace{\frac{2(a_1+b_1')}{r}\cos\theta}_{\sigma_{zz}}\right]\right\}$$

OR

(1) $$\frac{\partial u_r}{\partial r} = \frac{1}{Er}\cos\theta\left\{2a_1 + b_1' - \nu\left(b_1' + 2\nu(a_1+b_1')\right)\right\}$$

THUS

(2) $$\frac{\partial u_r}{\partial r} = \frac{\cos\theta}{Er}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}$$

AND

(3) $$u_r = \frac{\cos\theta}{E}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}\ln r + g(\theta)$$

FURTHERMORE

$$e_{rr}+e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_r}{\partial r} = \frac{1}{E}\left\{\sigma_{rr}+\sigma_{\theta\theta} - \nu\langle\sigma_{rr}+\sigma_{\theta\theta}+2\sigma_{zz}\rangle\right\}$$

$$= \frac{1}{E}\left\{(1-\nu)(\sigma_{rr}+\sigma_{\theta\theta}) - 2\nu^2(\sigma_{rr}+\sigma_{\theta\theta})\right\}$$

$$= \frac{1-\nu-2\nu^2}{E}(\sigma_{rr}+\sigma_{\theta\theta})$$

$$= \frac{1-\nu-2\nu^2}{E}\cdot\frac{2(a_1+b_1')}{r}\cos\theta$$

HENCE

$$\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} = \underbrace{\frac{1-\nu-2\nu^2}{E}\frac{2(a_1+b_1')}{r}\cos\theta}_{e_{rr}+e_{\theta\theta}} - \underbrace{\frac{\cos\theta}{Er}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}}_{\partial u_r/\partial r}$$

$$\underbrace{- \frac{\cos\theta}{Er}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}\ln r - \frac{g(\theta)}{r}}_{u_r/r}$$

OR (3) $$\frac{\partial u_\theta}{\partial \theta} = -g(\theta) + \frac{\cos\theta}{E}\left\{ a_1 \left\langle [-2\nu - 2\nu^2] - 2(1-\nu^2)\ln r \right\rangle + b_1' \langle 1-\nu-2\nu^2\rangle\langle 1-\ln r\rangle \right\}$$

AND

(4) $$u_\theta = -\int_\theta g(t)\,dt + f(r) + \frac{\sin\theta}{E}\left\{ a_1\left[-2\nu-2\nu^2 - 2(1-\nu^2)\ln r\right] + b_1'(1-\nu-2\nu^2)(1-\ln r)\right\}$$

WE HAVE THE ADDITIONAL RELATION $e_{r\theta} = \frac{1}{2\mu}\sigma_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta}$

THIS REQUIRES THAT

(5) $$\frac{1+\nu}{E}\,\frac{b_1'}{r}\sin\theta = \frac{1}{2}\left\{\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right\}$$

HENCE

$$\sin\theta\,\frac{2(1+\nu)}{E}\,\frac{b_1'}{r} = -\frac{\sin\theta}{Er}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}\ln r + \frac{1}{r}\frac{dg(\theta)}{d\theta}$$
$$+ \frac{df(r)}{dr} + \frac{\sin\theta}{E}\left\{a_1\left(\frac{-2(1-\nu^2)}{r}\right) - \frac{b_1'}{r}(1-\nu-2\nu^2)\right\}$$
$$+ \frac{\int_\theta g(t)\,dt}{r} - \frac{f(r)}{r} - \frac{\sin\theta}{Er}\left\{a_1\left[-2\nu-2\nu^2-2(1-\nu^2)\ln r\right] + b_1'(1-\nu-2\nu^2)(1-\ln r)\right.$$

AND

$$\left\{\frac{dg(\theta)}{d\theta} + \int_\theta g(t)\,dt\right\} + \left\{r\frac{df}{dr} - f\right\}$$
$$+ \frac{\sin\theta}{E}\left\{a_1\left[-2(1-\nu^2) + 2\nu + 2\nu^2\right] + b_1'\left[-2(1-\nu-2\nu^2) - 2(1+\nu)\right]\right\} = 0.$$

THIS IS POSSIBLE IF AND ONLY IF ??? (why not = const. yes but only adds rigid body translation)

$$r\frac{df}{dr} - f = 0 \Rightarrow f = \alpha r$$ [THIS IS A RIGID ROTATION TERM]

AND IF

(6) $$g'(\theta) + \int_\theta g(t)\,dt + \frac{\sin\theta}{E}\left\{a_1(-2+2\nu+4\nu^2) + b_1'(-4+4\nu^2)\right\} = 0$$

DIFFERENTIATING (6) WITH RESPECT TO θ YIELDS

$$g''(\theta) + g(\theta) + \frac{\cos\theta}{E}\left\{a_1\left[-2(1-2\nu)(1+\nu)\right] - 4b_1'(1-\nu^2)\right\} = 0$$

OR

$$g''(\theta) + g(\theta) = \frac{2(1+\nu)\cos\theta}{E}\left\{a_1(1-2\nu) + 2b_1'(1-\nu)\right\} = J\cos\theta$$

WHERE

$$J = \frac{2(1+\nu)}{E}\left[a_1(1-2\nu) + 2b_1'(1-\nu)\right]$$

THE SOLUTION TO THIS DIFFERENTIAL EQUATION IS

$$g(\theta) = \alpha_0\cos\theta + \beta_0\sin\theta + \frac{J}{2}\theta\sin\theta$$

SO $g(\theta)$ AND u_r AND u_θ WILL BE MULTI-VALUED FOR A COMPLETE RING UNLESS $J \equiv 0$.

THUS

$$\boxed{b_1' = -\frac{a_1(1-2\nu)}{2(1-\nu)} \quad \text{IN PLANE STRAIN}}$$

SIMILARLY

$$\boxed{d_1' = -\frac{c_1(1-2\nu)}{2(1-\nu)} \quad \text{IN PLANE STRAIN}}$$

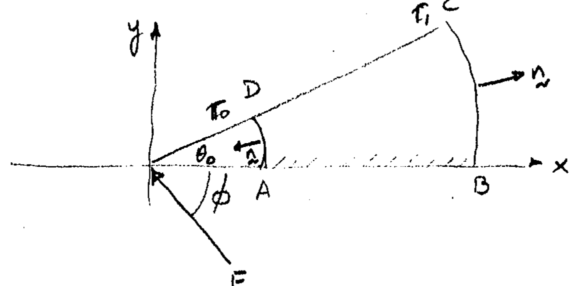
FOR PLANE STRESS REPLACE ν BY $\frac{\nu}{1+\nu}$ SO THAT

$$\frac{1-2\nu}{1-\nu} \rightarrow \frac{1-\frac{2\nu}{1+\nu}}{1-\frac{\nu}{1+\nu}} \rightarrow 1-\nu.$$

SINCE MY a_1 AND c_1 ARE HALF OF TIMOSHENKO AND GOODIER'S a_1 AND c_1, THE PLANE STRESS RELATIONS CHECK WITH THOSE OF T & G ON PAGE 135.

HENCE $a_1, b_1, b_1', c_1, d_1, d_1'$ CAN BE UNIQUELY DETERMINED from last 3 eq

Now look at a sector : We know that Σ Forces $= 0$ for body to be in equilibrium



AB: $T_i = 0$ since only force is F @ origin

BC: $T_{i_{net}} = \int_{\theta=0}^{\theta_0} \sigma_{ij} n_j ds = \int_{\theta=0}^{\theta_0} \frac{F}{r_1} f_{ij}(\phi,\theta)\, n_j(\theta)\, r_1 d\theta = F \int_0^{\theta_0} f_{ij}(\phi,\theta)\, n_j\, d\theta$ fn only of θ

AD: $n_j' = -n_{j\,m_{bc}}$ $\therefore$ $T_{i_{net\,AD}} = -T_{i_{net\,BC}}$ $\quad T_{i_{net\,AD}} = \int \frac{F}{r_0} f_{ij}(\phi,\theta)[-n_j(\theta)] r_0\, d\theta$

DC: $T_{i_{net}} = \int_{r_0}^{r_1} \sigma_{ij} n_j ds = \int_{r_0}^{r_1} \frac{F}{r} f_{ij}(\phi,\theta_0)\, n_j(\theta_0)\, dr = F f_{ij}(\phi,\theta_0)\, n_j(\theta_0) \int_{r_0}^{r_1} \frac{dr}{r}$

$= F \ln(r_1/r_0)\, f_{ij}(\phi,\theta_0)\, n_j(\theta_0) \equiv 0$ since the $\Sigma F_{\text{on AB, BC, AD}} = 0$

$\Rightarrow f_{ij}(\theta_0,\phi)\, n_j(\theta_0) = 0 \quad \forall\, \theta_0$ (since θ_0) is arbitrary

Therefore since ds was arbitrary $\Rightarrow$ T_i on radial lines are zero.

$\sigma_{ij} n_j = 0$ $\forall$ radial planes coming out from the origin and $T_i = 0$ $\forall$ points on all radial plane

in r, θ, z coordinates : on radial plane, $n_j(\theta) \Rightarrow n_\theta = 1,\ n_r = n_z = 0 \Rightarrow \sigma_{i\theta} = 0$ on all radial planes. $\Rightarrow \sigma_{r\theta}, \sigma_{\theta\theta}, \sigma_{z\theta} = 0$ on all radial planes

thus in r,θ,z problems only $\sigma_{rr}, \sigma_{zz} = \nu\sigma_{rr}$ (in plane strain) exist

σ_{rr} exists (in plane stress)

1/24/79

380F : Prof. Barnett → Applied Math Seminar today 4:15 PM

Timoshenko 2-D problems in polar coordinates pg 132-135

① $\sigma_{xx} + \sigma_{yy} + \sigma_{zz} = \sigma_{\theta\theta} + \sigma_{rr} + \sigma_{zz}$ (1st invariant) true no matter what coord.

② Plane strain solution $\nabla^2(\sigma_{xx} + \sigma_{yy}) = 0$ but $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) = \nu(\sigma_{rr} + \sigma_{\theta\theta})$

$\Rightarrow \nabla^2(\sigma_{rr} + \sigma_{\theta\theta}) = 0$ Look at Pg 65-68

Thus we can define a Stress fn. $\phi(r,\theta)$ $\ni$.

$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} \qquad \sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} \qquad \sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)$

thus $\nabla^2(\nabla^2\phi)=0$ where $\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

we will Study: $\phi(r,\theta)$ if $\nabla^2\phi=0 \Rightarrow \nabla^4\phi=0$

(1) if f is harmonic the xf, yf are biharmonic

$$\nabla^2(xf) = f(\nabla^2 x) + x(\nabla^2 f) + 2(\nabla f \cdot \nabla x)$$
$$= 0 + 0 + 2(\nabla f \cdot \underset{\sim}{i}) = 2\frac{\partial f}{\partial x}$$

$$\nabla^4(xf) = 2\nabla^2\left(\frac{\partial f}{\partial x}\right) = 2\frac{\partial}{\partial x}(\nabla^2 f) = 0$$

(2) if f is harmonic $r^2 f = (x^2+y^2)f$ is biharmonic

$$\nabla^2(x^2 f) = f\nabla^2(x^2) + x^2(\nabla^2 f) + 2\nabla f \cdot \nabla(x^2)$$
$$f\cdot 2 \quad + \quad 0 \quad 2(\nabla f \cdot 2x\underset{\sim}{i})$$
$$= 2f + 4x\frac{\partial f}{\partial x}$$

$$\nabla^2(\nabla^2 x^2 f) = 2\nabla^2 f + 4\nabla^2\left(x\frac{\partial f}{\partial x}\right)$$
$$= 0 \quad 4\frac{\partial f}{\partial x}(\nabla^2 x) + 4x\left(\nabla^2\frac{\partial f}{\partial x}\right) + 8\nabla\left(\frac{\partial f}{\partial x}\right)\cdot\nabla x$$
$$\qquad = 0 \qquad\qquad = 0$$
$$8\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 8\frac{\partial^2 f}{\partial x^2}$$

$\therefore \nabla^4(x^2 f) = 8\frac{\partial^2 f}{\partial x^2}$ similarly $\nabla^4(y^2 f) = 8\frac{\partial^2 f}{\partial y^2}$

$$\nabla^4(r^2 f) = 8\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right] = \nabla^4[(x^2+y^2)f] = 8\nabla^2 f = 0$$

Let us look at $r^n\cos n\theta$

$$\nabla^2(r^n\cos n\theta) = \left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right](r^n\cos n\theta)$$
$$= \left[n(n-1)r^{n-2} + nr^{n-2} - n^2 r^{n-2}\right]\cos n\theta = \left[n^2-n+n-n^2\right]r^{n-2}\cos n\theta$$
$$= 0$$

thus $\nabla^2(r^n\cos n\theta)=0$ similarly $\nabla^2(r^n\sin n\theta)=0$
also $\nabla^2(r^{-n}\cos n\theta)=0$ & $\nabla^2(r^{-n}\sin n\theta)=0$ } since $\nabla^2\phi_1=0$ then $\nabla^2(\nabla^2\phi_1)=\nabla^2(0)=0$

thus if $\phi_i = r^{n+2}\cos n\theta,\ r^{n+2}\sin n\theta,\ r^{-n+2}\cos n\theta,\ r^{-n+2}\sin n\theta$
then $\nabla^4\phi=0$ assume $n\geq 2$

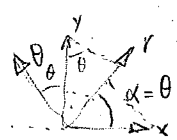
$\uparrow \epsilon_{zr}, \epsilon_{z\theta} = 0$

Plane strain $\epsilon_{zz} = 0$ $\epsilon_{zx} = 0$ or $\epsilon_{zy} = 0$

$\mathbb{t}_n = \mathbb{T} \cdot \underline{n} = \mathbb{T} \underline{e}_r = \sigma_{rr} \underline{e}_r + \sigma_{\theta r} \underline{e}_\theta + \sigma_{zr} \underline{k}$ $\quad \epsilon_{zr} = 0 \Rightarrow \sigma_{zr} = 0$

$\mathbb{t}_r = \mathbb{t}_x \underbrace{\underline{e}_x \cdot \underline{e}_r}_{\cos\alpha} + \mathbb{t}_y \underbrace{\underline{e}_y \cdot \underline{e}_r}_{\cos\beta} + \mathbb{t}_z \underbrace{\underline{e}_z \cdot \underline{e}_r}_{\cos\gamma} = \mathbb{t}_x \cos\theta + \mathbb{t}_y \sin\theta$

here $\gamma = 90°$ since plane problem



$\cos\beta = \sin\theta$

$\mathbb{t}_\theta = \mathbb{t}_x \underbrace{\underline{e}_x \cdot \underline{e}_\theta}_{\substack{\cos\alpha \\ -\sin\theta}} + \mathbb{t}_y \underbrace{\underline{e}_y \cdot e_\theta}_{\substack{\cos\beta \\ \cos\theta}} + \mathbb{t}_z \underbrace{\underline{e}_z \cdot e_\theta}_{\cos\gamma}$

$\gamma = 90°$

$\cos(\theta + 90°) = \cos\alpha = -\sin\theta$ $\quad \cos\beta = \cos\theta$

$\mathbb{t}_\theta = \mathbb{t}_x(-\sin\theta) + \mathbb{t}_y \cos\theta$

$\therefore \quad \mathbb{t}_x = \mathbb{t}_r \cos\theta - \mathbb{t}_\theta \sin\theta$

$\mathbb{t}_y = +\mathbb{t}_r \sin\theta + \mathbb{t}_\theta \cos\theta$

$$\phi^{(1)} = \sum (A_n r^n + B_n r^{-n} + C_n r^{n+2} + D_n r^{-n+2})\cos n\theta + (E_n r^n + F_n r^{-n} + G_n r^{n+2} + H_n r^{-n+2})\sin n\theta$$

θ is harmonic $\therefore \Rightarrow x\theta, y\theta, r^2\theta$ are biharmonic

$$\phi^{(2)} = a_0'\theta + a_1' r\theta\cos\theta + a_2' r\theta\sin\theta + a_3' r^2\theta$$

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look for $\phi = \phi(r)$ only in axi-symmetric case. $\exists \psi \ni \nabla^4\psi = 0$. for use to get ϕ look for

Harmonic fn $\nabla^2\phi = 0$: in axi'sym case $\nabla^2\phi = \phi_{,rr} + \frac{1}{r}\phi_{,r} \Rightarrow (r\phi')' = c$

$\therefore$ $r\phi' = k$ $\quad \phi = k\ln r + k_0$ $\quad$ but $r^2\phi$ is biharmonic

$r^2\phi = \phi^{(2)}$ $\quad \phi = \phi^{(1)}$

$\therefore$ $\psi = k_3 r^2\ln r + k_2 r^2 + k_1 \ln r + k_0 = \phi^{(3)}$ $\quad \nabla^4\psi = \nabla^4(r^2\phi^{(1)}) + \nabla^2(\nabla^2\phi^{(1)}) = 0$

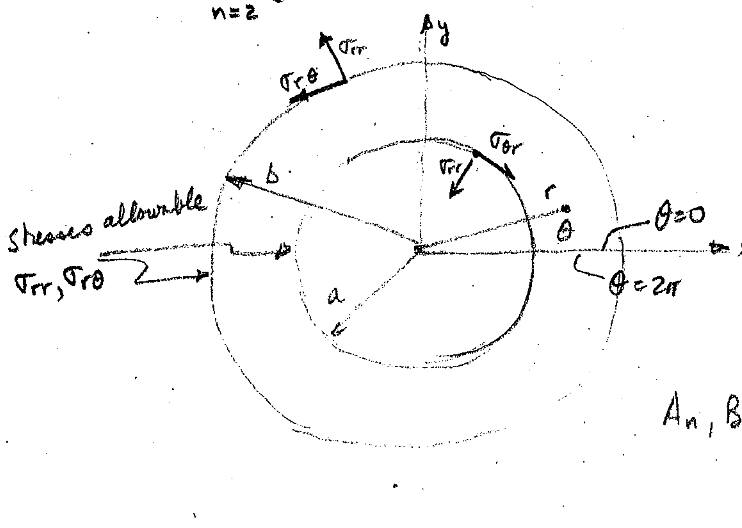
(← k_0: we can drop this, since when we take derivs this makes no difference)

also $x\phi, y\phi$ are biharmonic $\therefore$ $\phi^{(4)} = \alpha_0 r\ln r\cos\theta + \beta_0 r\ln r\sin\theta + k_0 x + ky$

(← $k_0x + ky$: we can drop this also as before)

Our total stress fn is therefore: T&G p 133 (eqn 80)

$$\phi(r,\theta) = a_0\ln r + b_0 r^2 + c_0 r^2\ln r + d_0 r^2\theta + a_0'\theta + a_1 r\theta\sin\theta + (b_1 r^3 + a_1' r^{-1} + b_1' r\ln r)\cos\theta + c_1 r\theta\cos\theta + (d_1 r^3 + c_1' r^{-1} + d_1'\ln r)\sin\theta + \sum_{n=2}^{\infty}(a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2})\cos n\theta + \sum_{n=2}^{\infty}(c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2})\sin n\theta$$



For a complete ring θ is multivalued $\therefore$ take $d_0 = 0$

B.C. $(\sigma_{rr})_{r=a} = A_0 + \sum_{n=1}^{\infty}(A_n\cos n\theta + B_n\sin n\theta)$

$(\sigma_{rr})_{r=b} = A_0' + \sum(A_n'\cos n\theta + B_n'\sin n\theta)$
$\forall\ 0 \le \theta \le 2\pi$

A_n, B_n, A_n', B_n' are known

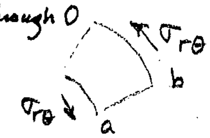
$$(\sigma_{r\theta})_{r=a} = C_0 + \sum_{n=1}^{\infty} (C_n \cos n\theta + D_n \sin n\theta)$$

$$(\sigma_{r\theta})_{r=b} = C_0' + \sum_{n=1}^{\infty} (C_n' \cos n\theta + D_n' \sin n\theta)$$

C_n, D_n, C_n', D_n' are known

To check the well posedness of the problem: now look at moment & force equilib

Moment equil about origin σ_{rr} produces no contrib since σ_{rr} line of action is through O



$\therefore$

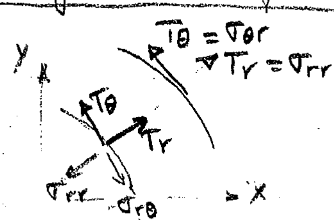
$$\underset{\text{about origin}}{\Sigma M = 0} \Rightarrow \int_0^{2\pi} \{ b\,\sigma_{\theta r} \}_{r=b} \cdot b\,d\theta - \int_0^{2\pi} \{ a\,\sigma_{\theta r} \}_{r=a} \cdot a\,d\theta = 0 \qquad (1)$$

but $\int_0^{2\pi} \cos n\theta \, d\theta$, $\int_0^{2\pi} \sin n\theta \, d\theta = 0 \quad n > 0$

$\therefore$ the eq (1) reduces to $\int (b^2 C_0' - a^2 C_0)\,d\theta = 0 \quad \therefore \quad 2\pi (b^2 C_0' - a^2 C_0) = 0$

$\boxed{b^2 C_0' \equiv a^2 C_0}$ for moment equil

For a plane strain problem: tractions on the boundary



$$T_x = T_r \cos\theta - T_\theta \sin\theta$$

$$T_y = T_r \sin\theta + T_\theta \cos\theta$$

$$F_x^{net} = \int_0^{2\pi} \overset{T_x|_{r=b}}{[\sigma_{rr}\cos\theta - \sigma_{\theta r}\sin\theta]}_{r=b} \, b\,d\theta - \int_0^{2\pi} \overset{T_x|_{r=a}}{[\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta]}_{r=a} \, a\,d\theta$$

for equilib $F_x^{net} \equiv 0$

$T_r = \sigma_{rj} n_j \therefore T_r = \sigma_{rr} \quad T_\theta = -\sigma_{\theta r}$

$T_r = \sigma_{rr} n_r = -\sigma_{rr}$

$T_\theta = \sigma_{\theta j} n_j = \sigma_{\theta r} n_r = -\sigma_{\theta r}$

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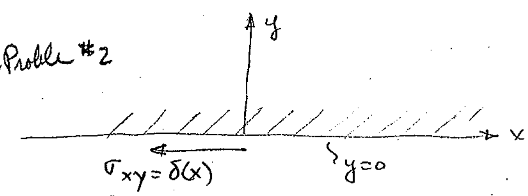
Homework Problem #2



$$\phi(x,y) = \int_{-\infty}^{\infty} e^{-i\lambda x} \{ A(\lambda) e^{-|\lambda| y} + B(\lambda)\, y\, e^{-|\lambda| y} \} d\lambda$$

$$\sigma_{yy} = \left.\frac{\partial^2 \phi}{\partial x^2}\right|_{y=0} = \int_{-\infty}^{\infty} -\lambda^2 e^{-i\lambda x} A(\lambda)\, d\lambda = 0 \Rightarrow A(\lambda) = 0$$

$$\phi(x,y) = y \int_{-\infty}^{\infty} e^{-i\lambda x} B e^{-|\lambda| y} d\lambda$$

$$\sigma_{xy} = -\phi_{,xy}\Big|_{y=0} = \int_{-\infty}^{\infty} i\lambda e^{-i\lambda x} B(\lambda) [e^{-|\lambda| y} - |\lambda| y e^{-|\lambda| y}] d\lambda = \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} d\lambda$$

define $\phi^{(1)} = \int_0^\infty e^{-\lambda y} \frac{\cos\lambda x}{\lambda} d\lambda = -\int_0^\infty \left(\frac{1-\cos\lambda x}{\lambda}\right) e^{-\lambda y} + \int_0^\infty d\lambda e^{-\lambda y} \frac{1}{\lambda}$

let $\xi = \lambda y$

$\int_0^\infty e^{-\xi} \xi^{-1} d\xi \to \infty$

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1. Midterm on the 12/14th will be take home.

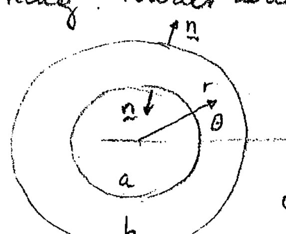
$$Q = \int_0^\infty \frac{1-\cos\lambda x}{\lambda} e^{-\lambda y} d\lambda \qquad \frac{\partial Q}{\partial x} = \int_0^\infty \sin\lambda x \, e^{-\lambda y} d\lambda = \frac{x}{x^2+y^2}$$

$$\therefore \quad Q = \frac{1}{2}\ln(x^2+y^2) + g(y) \qquad \text{as } y\to\infty \quad Q \to 0 \text{ since } e^{-\lambda y} \to 0$$

$$Q\Big|_{\substack{y\to\infty \\ x \text{ fixed}}} \to \frac{1}{2}\ln(y^2) + g(y) = 0 \quad \therefore \quad g(y) = -\frac{1}{2}\ln y^2$$

$$\therefore \quad Q = \frac{1}{2}\ln\left(\frac{x^2+y^2}{y^2}\right)$$

Annular Ring - Fourier Loading



we had found the stress fn for the annular ring to be

$$\phi = a_0\ln r + b_0 r^2 + c_0 r^2\ln r + d_0 r^2\theta + a_0'\theta + a_1 r\theta\sin\theta + (b_1 r^3 + a_1' r^{-1} + b_1' r\ln r)\cos\theta + c_1 r\theta\cos\theta + (d_1 r^3 + c_1' r^{-1} + d_1' r\ln r)\sin\theta + \sum_{n=2}^\infty (a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2})\cos n\theta + \sum_{n=2}^\infty (c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2})\sin n\theta$$

$(\sigma_{rr})_{r=a} = A_0 + \sum_{n=1}^\infty (A_n\cos n\theta + B_n\sin n\theta)$ $\quad (\sigma_{rr})_{r=b}$ same with primes

$(\sigma_{r\theta})_{r=a} = C_0 + \sum_{n=1}^\infty (C_n\cos n\theta + D_n\sin n\theta)$ $\quad (\sigma_{r\theta})_{r=a}$ " " "

From moment equil we found $C_0 a^2 = C_0' b^2$

For force equil we found $T_x = T_r\cos\theta - T_\theta\sin\theta$

$T_y = T_r\sin\theta + T_\theta\cos\theta$

on $r=b$ $T_r = \sigma_{rr}$, $T_\theta = \sigma_{r\theta}$ $\qquad$ $r=a$: $T_r = -\sigma_{rr}$, $T_\theta = -\sigma_{r\theta}$

$$\frac{F_x^{net}}{\text{unit length}} = \int_0^{2\pi} (\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)\Big|_{r=b} b\,d\theta - \int_0^{2\pi} (\sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta)\Big|_{r=a} a\,d\theta = 0$$

only contrib for σ_{rr} are $\cos\theta$ terms ; for $\sigma_{r\theta}$ are $\sin\theta$ terms

$$\Rightarrow \quad \underline{b(A_1' - D_1') = a(A_1 - D_1)}$$

for F_y^{net} a similar relation only terms involving $\sin\theta$ for σ_{rr} and $\cos\theta$ for $\sigma_{r\theta}$ give contribution $\Rightarrow$ $\underline{b(B_1' + C_1') = a(B_1 + C_1)}$

computing σ_{rr} from ϕ ie $\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$

(for complete ring)

$$\sigma_{rr} = \frac{a_0}{r^2} + 2b_0 + c_0(1+2\ln r) + \cancel{2d_0\theta} + \frac{2a_1}{r}\cos\theta + \left(2b_1 r - \frac{2a_1'}{r^3} + \frac{b_1'}{r}\right)\cos\theta - \frac{2c_1}{r}\sin\theta$$
$$+ \left(2d_1 r - \frac{2c_1'}{r^3} + \frac{d_1'}{r}\right)\sin\theta + \sum_{n=2}^{\infty}\Big[a_n n(1-n) r^{n-2} + b_n(2+n-n^2)r^n + a_n'(n^2+n)r^{-n-2}$$
$$+ b_n'(2-n-n^2)r^{-n}\Big]\cos n\theta + \sum_{n=2}^{\infty}\Big[c_n n(1-n)r^{n-2} + d_n(2+n-n^2)r^n - c_n'(n^2+n)r^{-n-2}$$
$$+ d_n'(2-n-n^2)r^{-n}\Big]\sin\theta$$

Since $\sigma_{rr}\Big|_{r=a} = A_0 + \Sigma(A_n\cos n\theta + B_n\sin n\theta)$ then equate term by term.

independent of θ terms

$$\therefore\ A_0 = \frac{a_0}{a^2} + 2b_0 + c_0(1+2\ln a) \ \ \& \ \ A_0' = \frac{a_0}{b^2} + 2b_0 + c_0(1+2\ln b) \qquad \text{2 eqn, 3 unkn.}$$

Look at coeff of $\cos\theta$, $\sin\theta$

$$A_1 = \frac{2a_1}{a} + 2b_1 a - \frac{2a_1'}{a^3} + \frac{b_1'}{a}\ ; \quad A_1' = \frac{2a_1}{b} + 2b_1 b - \frac{2a_1'}{b^3} + \frac{b_1'}{b} \qquad \text{2 eqs, 4 unks}$$

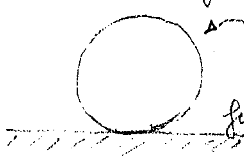
$$B_1 = -\frac{2c_1}{a} + 2d_1 a - \frac{2c_1'}{a^3} + \frac{d_1'}{a} \quad B_1' = -\frac{2c_1}{b} + 2d_1 b - \frac{2c_1'}{b^3} + \frac{d_1'}{b} \qquad \text{2 eqs, 4 unk.}$$

now doing $\sigma_{r\theta}$ we get relations for C_1, C_1' and D_1, D_1'

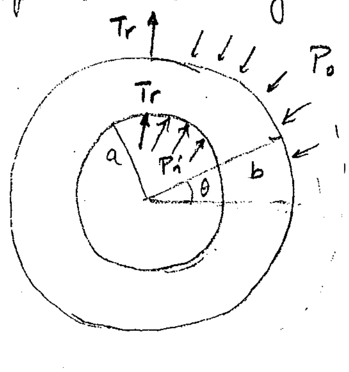
Using these we will find that there is a redundancy and $F_x^{net} = F_y^{net} = 0$ is identically satisfied; so no new info is obtained we must look at the compatabilit

2/2/79

One problem of the midterm



Example: continuation of the derivation is in the handout for rings — example might be pressurized airplane cabin



Axisymmetric deformation :

P_o, P_i goes all way around — by symmetry no θ dependence

@ $r=b$ $\sigma_{rr} = -P_o$, $\sigma_{r\theta}=0$ — $T_r = \sigma_{rj} n_j$ since $n_j = n_r$ & $T_r = -P_o$

@ $r=a$ $\sigma_{rr} = -P_i$ $\sigma_{r\theta}=0$ since $n_j = -n_r$ & $T_r = p_i$

Fourier loading $A_o = -P_i$, $A_o' = -P_o$ all other fourier coeffs are $=0$

$$\frac{a_o}{a^2} + 2b_o + c_o(1+2\ln a) = A_o$$

since $a_o' = a^2 c_o$ & a_o' is coeff of θ term

Timo & Goodier gives that $c_o \equiv 0$ for single valued displ (Pg 77-78)

$$\frac{a_o}{b^2} + 2b_o + C_o(1+2\ln b) = A_o'$$

$\phi = a_o \ln r + b_o r^2$ (only part of ϕ not a fn of θ) and give singlevalued displ & stresses

$$a_o = \frac{(P_o - P_i)a^2b^2}{b^2-a^2}$$

$$b_o = \frac{1}{2}\,\frac{P_i a^2 - P_o b^2}{b^2-a^2}$$

→ really only had to worry about σ_{rr} since $\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial\theta}\right)$ but $\frac{\partial}{\partial\theta}=0$

$\therefore$ $\sigma_{r\theta}=0$ on $r=a,b$ is identically satisfied

$\therefore$ LAMÉS Solution

$$\sigma_{r\theta} = 0;\quad \sigma_{rr} = \frac{(P_o-P_i)a^2b^2}{b^2-a^2}\frac{1}{r^2} + \frac{P_i a^2 - P_o b^2}{b^2-a^2}$$

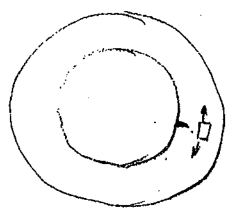
$$\sigma_{\theta\theta} = \frac{-(P_o-P_i)a^2b^2}{b^2-a^2}\frac{1}{r^2} + \frac{P_i a^2 - P_o b^2}{b^2-a^2}$$

$$\sigma_{rr} + \sigma_{\theta\theta} = 2\,\frac{P_i a^2 - P_o b^2}{b^2 - a^2} = \frac{1}{\nu}\sigma_{zz} \quad \text{for plane strain } (= \text{const})$$

in the homework 3rd problem I ask what if I wanted plane stress soln? How could I superpose on this soln another solution ∋ $\sigma_{zz} = 0$

Problem #2

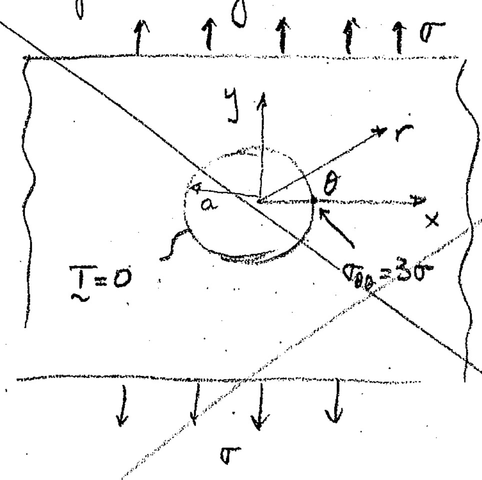
IF $P_o = 0$ then
$$\sigma_{rr} = -\frac{P_i a^2 b^2}{b^2 - a^2}\frac{1}{r^2} + \frac{P_i a^2}{b^2 - a^2} = \frac{P_i a^2}{b^2 - a^2}\left\{1 - \frac{b^2}{r^2}\right\} \le 0 \quad \text{compress.}$$
$$\sigma_{\theta\theta} = +\frac{P_i a^2 b^2}{b^2 - a^2}\frac{1}{r^2} + \frac{P_i a^2}{b^2 - a^2} = \frac{P_i a^2}{b^2 - a^2}\left\{1 + \frac{b^2}{r^2}\right\} > 0 \quad \text{tens}$$



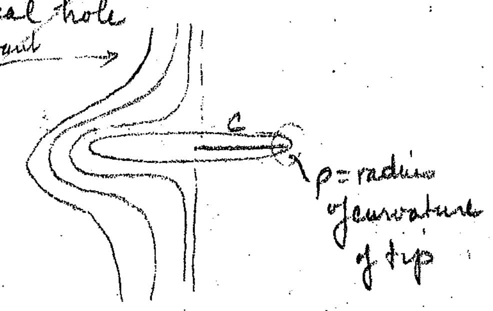
if cracks develop in radial dir } $\sigma_{\theta\theta}$ being tensile cause cracks to propagate

Problem #3

Look at an infinite body with a circular hole in a thick plate

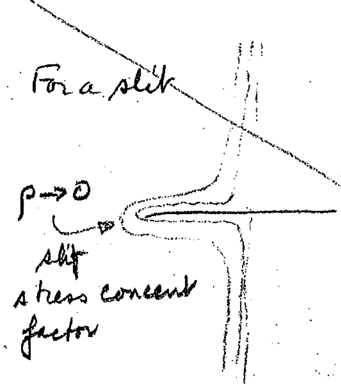


For an elliptical hole



Stress conc: $\sqrt{c/\rho}$

For a slit



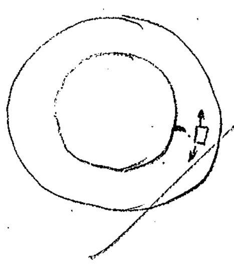
$$\sigma_{rr} + \sigma_{\theta\theta} = 2\,\frac{P_i a^2 - P_o b^2}{b^2 - a^2} = \frac{1}{\nu}\sigma_{zz} \quad \text{for plane strain } (= \text{const})$$

<u>in the homework</u> 3rd problem I ask what if I wanted plane stress soln? How could I superpose on this soln another solution ∋ $\sigma_{zz} = 0$

<u>Problem #2</u>

IF $P_o = 0$ then $\sigma_{rr} = -\frac{P_i a^2 b^2}{b^2 - a^2}\frac{1}{r^2} + \frac{P_i a^2}{b^2 - a^2} = \frac{P_i a^2}{b^2 - a^2}\left\{1 - \frac{b^2}{r^2}\right\} \leq 0$ compress.

$$\sigma_{\theta\theta} = +\frac{P_i a^2 b^2}{b^2 - a^2}\frac{1}{r^2} + \frac{P_i a^2}{b^2 - a^2} = \frac{P_i a^2}{b^2 - a^2}\left\{1 + \frac{b^2}{r^2}\right\} > 0 \quad \text{tens}$$

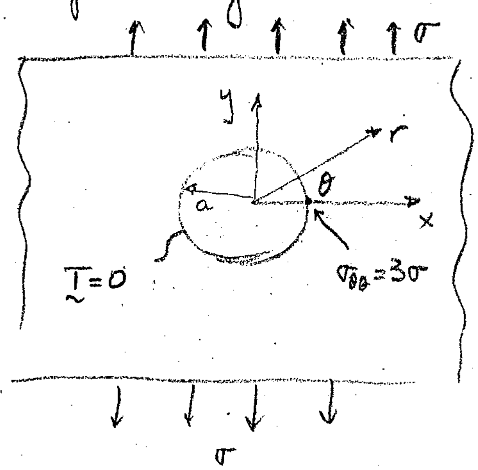


if cracks develop in radial dir } $\sigma_{\theta\theta}$ being tensile cause cracks to propagate

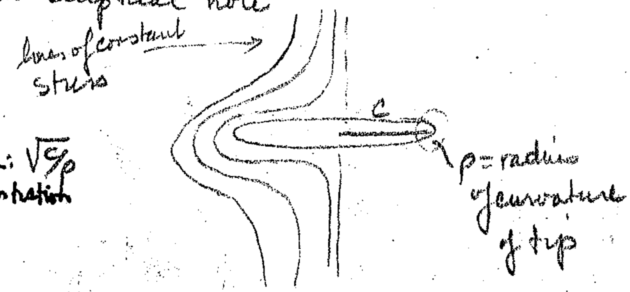
Continue here!

<u>Problem #3</u>

Look at an infinite body with a circular hole in a thick plate

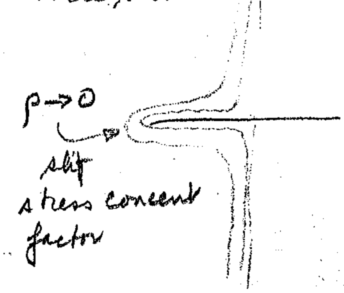


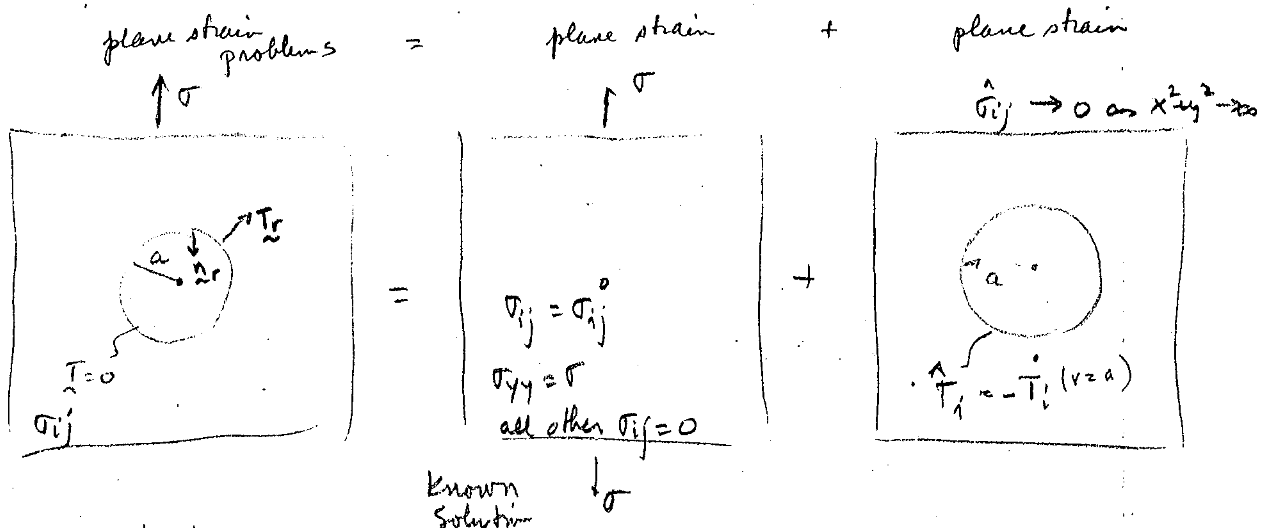
For an elliptical hole



Stress conc: $\sqrt{c/\rho}$ concentration

For a slit





Stress concentration on a Circular Hole.

$$\sigma_{ij} = \sigma^{\circ}_{ij} + \hat\sigma_{ij}$$

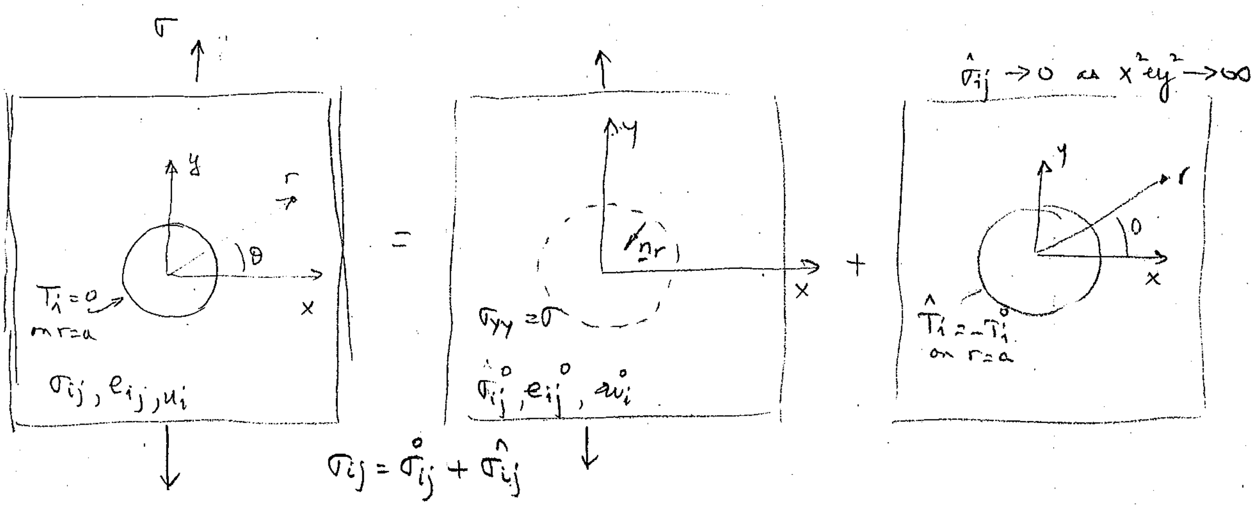
BC: $\hat\sigma_{ij} \to 0$ as $x^2+y^2 \to \infty$

$$T_1 = T_1^{\circ} + \hat T_1 = 0 \implies \hat T_i = -T_1^{\circ} \text{ or } r=a$$

$$\hat T_r = \hat\sigma_{rr} \overset{(-1)}{n_r} = -\hat\sigma_{rr} = -T_r^{\circ} = -\sigma^{\circ}_{rr} n_r = \sigma^{\circ}_{rr} \qquad \therefore \quad \hat\sigma_{rr} = -\sigma^{\circ}_{rr} \text{ on } r=a$$

$$\hat T_\theta : \qquad \hat\sigma_{r\theta} = -\sigma^{\circ}_{r\theta} \text{ on } r=a$$

Stress concent on a Circular Hole 2/5/79



$$\sigma_{ij} = \sigma^{\circ}_{ij} + \hat\sigma_{ij}$$

°FIELDS ⇒ $\sigma^{\circ}_{yy} = \sigma \quad \sigma^{\circ}_{ij} = 0$ all others. from transformation

on $r=a$
$$\begin{cases} T_r^{\circ} = \sigma_{rr} n_r = -\sigma^{\circ}_{rr} = -\sigma^{\circ}_{yy}\sin^2\theta = -\dfrac{\sigma}{2}(1-\cos 2\theta) \\ T_\theta^{\circ} = \sigma_{r\theta} n_r = -\sigma^{\circ}_{\theta r} = -\sigma^{\circ}_{yy}\sin\theta\cos\theta = -\dfrac{\sigma}{2}\sin 2\theta \end{cases}$$

$$\sigma^{\circ}_{\theta\theta} = \frac{\sigma}{2}(1+\cos 2\theta) = \sigma/2 \cos^2\theta$$

$$\therefore \hat T_r = \hat\sigma_{rr} \overset{(-1)}{n_r} = -T_r^{\circ} = \frac{\sigma}{2}(1-\cos 2\theta) \qquad \therefore \quad \hat\sigma_{rr} = -\frac{\sigma}{2}(1-\cos 2\theta)$$

$$\hat T_\theta = \hat\sigma_{r\theta} n_r = -T_\theta^{\circ} = +\frac{\sigma}{2}\sin 2\theta \qquad \therefore \quad \hat\sigma_{r\theta} = -\sigma/2(\sin 2\theta)$$

on $r=a$

$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\cdot 2\sin\theta\cos\theta$

$= \frac{\sigma_x+\sigma_y}{2} + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$

$\sigma_x = 0 \quad \sigma_y = \sigma$

$\sigma_r = \sigma_{y}/_{2}\,(1-\cos 2\theta)$

Now look at the annular ring where at $r=b$ $\sigma_{ij}(r=b)=0$ & let $b\to\infty$

Go back to handout to apply bc.

$\therefore \quad A_0 = -\frac{\sigma}{2} \qquad A_2 = \frac{\sigma}{2} \qquad D_2 = -\frac{\sigma}{2}$ all others are 0.

(from $\sigma_{rr}|_{r=a}$) (from $\sigma_{r\theta}|_{r=a}$)

for the stress fn. when $\sigma_{rr}\Big|_{r=a} = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\Big|_{r=a}$

pg 35 handout

then $\frac{a_0}{a^2} + 2b_0 = A_0 = -\frac{\sigma}{2}$ $\quad$ ~~$+c_0(1+\ln a)$~~ must be dropped for single valued displacement.

$\frac{a_0}{b^2} + 2b_0 = -A_0' = 0$ as $b\to\infty$ $\Rightarrow b_0 = 0 \Rightarrow$ $\boxed{a_0 = -\frac{\sigma}{2}a^2}$

(remember A_0 A_0' are for applied bc at $r=a$, $r=b\to\infty$)

now for A_2, D_2 and A_2' and $D_2' = 0$ since no stresses applied at ∞.

for $n=2$ since $\sin 2\theta$, $\cos 2\theta$ terms in A_2, D_2

from pg 36 handout

$\sigma_{rr}|_{r=a}$ $\quad a_2\cdot 2(1-2)a^{2-2} + b_2(2+2-4)a^2 - a_2'\,2(1+2)a^{-2-2} + b_2'(2-2-4)a^{-2} = A_2 = \frac{\sigma}{2}$

$\sigma_{rr}|_{r=b}$ $\quad a_2\cdot 2(1-2)b^{2-2} + b_2(2+2-4)b^2 - a_2'\,2(1+2)b^{-4} + b_2'(2-2-4)b^{-2} = A_2' = 0$

as $b\to\infty$ b_2' term $\to 0$, a_2' term $\to 0$, b_2 term $\to 0$ since $(2+2-4)=0$

$\Rightarrow \boxed{a_2 = 0} \quad \Rightarrow$ 1st eq must reduce to $\boxed{-a_2'(6a^{-4}) + b_2'(-4)a^{-2} = \frac{\sigma}{2}}$ $*$

pg 37 handout

$\sigma_{r\theta}\Big|_{r=a}$: $2\left\{ \cancel{a_2} \overset{\text{from above}}{\to 0} + b_2(2+1)a^2 - a_2'(2+1)a^{-4} - b_2'(2-1)a^{-2} \right\} = D_2 = -\frac{\sigma}{2}$

$\sigma_{r\theta}\Big|_{r=b}$: $2\left\{ \cancel{a_2} \to 0 + b_2(3)b^2 + a_2'(2+1)b^{-4} - b_2'(2-1)b^{-2} \right\} = D_2' = 0$

$\therefore \boxed{b_2 = 0}$ $\quad \to 0$ as $b\to\infty$ $\quad \to 0$ as $b\to\infty$

$\therefore \quad \boxed{-a_2'(3a^{-4}) - b_2'(1)a^{-2} = -\frac{\sigma}{4}}$ $**$

$\therefore \; * \Rightarrow \; 3\frac{a_2'}{a^4} + 2\frac{b_2'}{a^2} = -\frac{\sigma}{4}$

$** \Rightarrow \; \frac{3a_2'}{a^4} + \frac{b_2'}{a^2} = \frac{\sigma}{4}$

$\Rightarrow \quad \boxed{b_2' = -\frac{\sigma}{2}a^2} \quad \boxed{a_2' = \frac{\sigma a^4}{4}}$

$$\Rightarrow \hat{\sigma}_{rr} = \frac{\sigma}{2}\left[-\frac{a^2}{r^2} + \left\{4\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right\}\cos 2\theta\right]$$

$$\sigma_{rr} = \hat{\sigma}_{rr} + \overset{\circ}{\sigma}_{rr} = \frac{\sigma}{2}(1-\cos 2\theta) + \hat{\sigma}_{rr} = \frac{\sigma}{2}\left[1 - \frac{a^2}{r^2}\right] - \frac{\sigma}{2}\left[1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right]\cos 2\theta$$

T&G pg 91 $\theta_{here} = \frac{\pi}{2} - \theta_{T\&G}$ also look at your books pg 208 there b = to our a

Thus

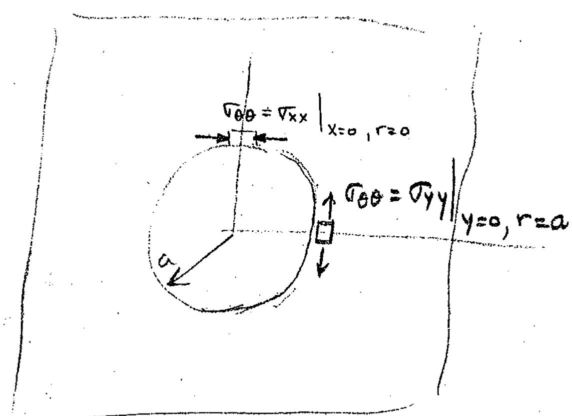
$$\sigma_{\theta\theta} = \overset{\circ}{\sigma}_{\theta\theta} + \hat{\sigma}_{\theta\theta} = \frac{\sigma}{2}\left[1 + \frac{a^2}{r^2}\right] + \frac{\sigma}{2}\left[1 + \frac{3a^4}{r^4}\right]\cos 2\theta$$

$$\sigma_{r\theta} = \frac{\sigma}{2}\left[1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right]\sin 2\theta$$

as $r \to \infty$

$$\left.\begin{aligned} \sigma_{rr} &\to \frac{\sigma}{2}[1-\cos 2\theta] \\ \sigma_{\theta\theta} &\to \frac{\sigma}{2}[1+\cos 2\theta] \\ \sigma_{r\theta} &\to \frac{\sigma}{2}[\sin 2\theta] \end{aligned}\right\} \overset{\circ}{\sigma}_{yy} = \sigma$$

Stress Concentration



$$\sigma_{\theta\theta}\Big|_{\substack{\theta=0\\ r=a}} = \frac{\sigma}{2}(2) + \frac{\sigma}{2}(4) = 3\sigma$$

$$\sigma_{\theta\theta}\Big|_{\substack{\theta=\pi/2\\ r=a}} = \frac{\sigma}{2}(2) - \frac{\sigma}{2}(4) = -\sigma$$

$$\sigma_{rr} + \sigma_{\theta\theta} = 2\,\frac{P_i a^2 - P_o b^2}{b^2 - a^2} = \frac{1}{\nu}\sigma_{zz} \quad \text{for plane strain } (= \text{const})$$

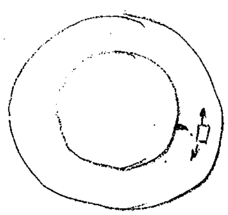
in the homework 3rd problem I ask what if I wanted plane stress soln? How could I superpose on this soln another solution $\ni$ $\sigma_{zz} = 0$

Problem #2

IF $P_o = 0$ then

$$\sigma_{rr} = -\frac{P_i a^2 b^2}{b^2 - a^2}\frac{1}{r^2} + \frac{P_i a^2}{b^2 - a^2} = \frac{P_i a^2}{b^2 - a^2}\left\{1 - \frac{b^2}{r^2}\right\} \leq 0 \quad \text{compressi[ve]}$$

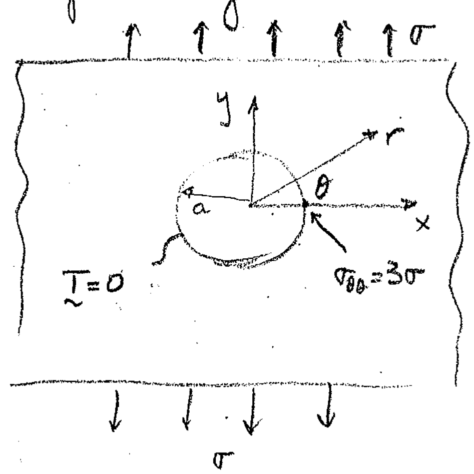
$$\sigma_{\theta\theta} = +\frac{P_i a^2 b^2}{b^2 - a^2}\frac{1}{r^2} + \frac{P_i a^2}{b^2 - a^2} = \frac{P_i a^2}{b^2 - a^2}\left\{1 + \frac{b^2}{r^2}\right\} > 0 \quad \text{tensi[le]}$$



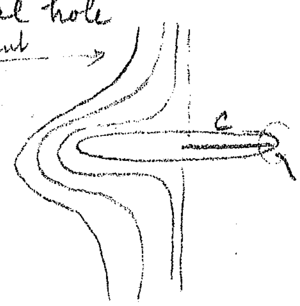
if cracks develop in radial dir } $\sigma_{\theta\theta}$ being tensile cause cracks to propagate

Problem #3

Look at an infinite body with a circular hole in a thick plate.

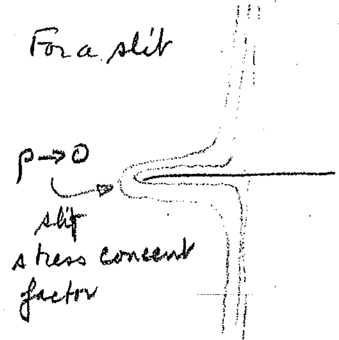


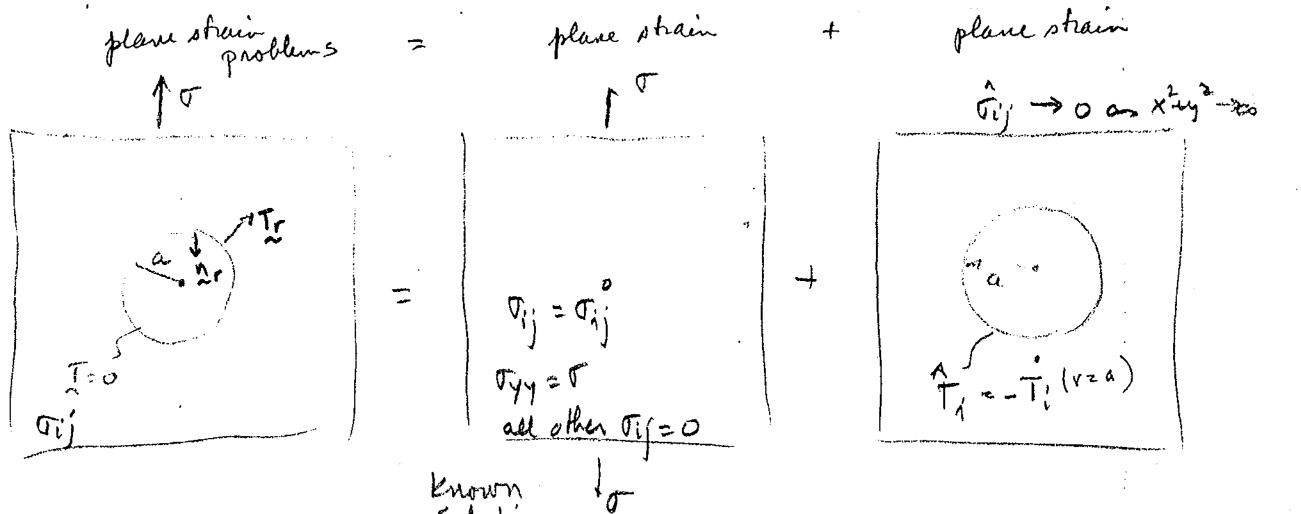
For an elliptical hole



Stress conc: $\sqrt{c/\rho}$

For a slit





Stress Concentration on a Circular Hole.

$$\sigma_{ij} = \sigma_{ij}^o + \hat{\sigma}_{ij}$$

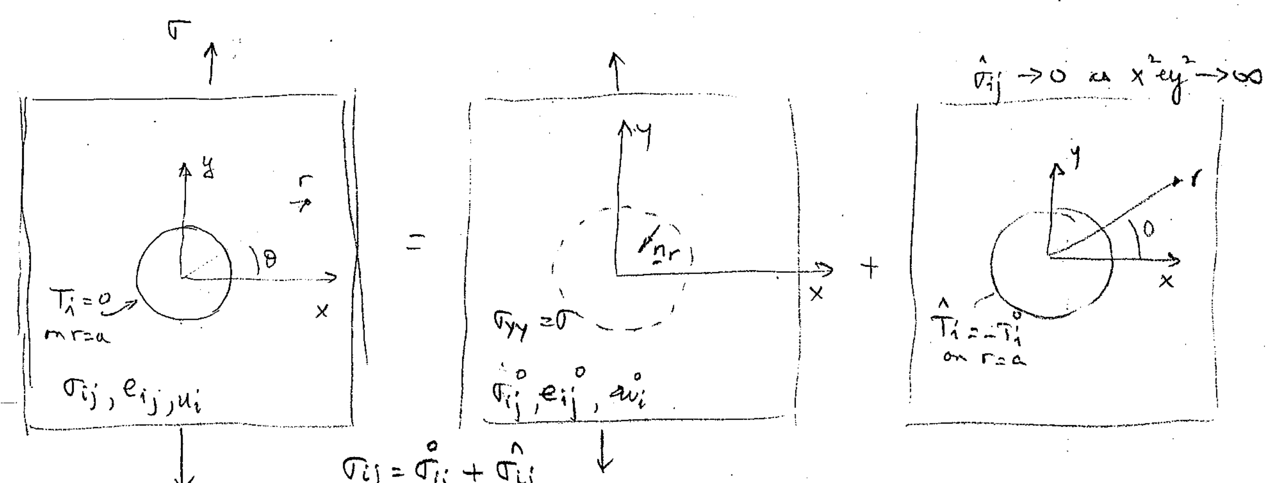
BC: $\hat{\sigma}_{ij} \to 0$ as $x^2+y^2 \to \infty$

$$T_i = T_i^o + \hat{T}_i = 0 \implies \hat{T}_i = -T_i^o \text{ on } r=a$$

$$\hat{T}_r = \hat{\sigma}_{rr} n_r^{(-1)} = -\hat{\sigma}_{rr} = -T_r^o = -\sigma_{rr}^o n_r = \sigma_{rr}^o \qquad \therefore \hat{\sigma}_{rr} = -\sigma_{rr}^o \text{ on } r=a$$

$$\hat{T}_\theta : \qquad \hat{\sigma}_{r\theta} = -\sigma_{r\theta}^o \text{ on } r=a$$

Stress concent on a Circular Hole 2/5/79



$$\sigma_{ij} = \sigma_{ij}^o + \hat{\sigma}_{ij}$$

°FIELDS ⟹ $\sigma_{yy}^o = \sigma \quad \sigma_{ij}^o = 0$ all others. from transformation

on r=a
$$\begin{cases} T_r^o = \sigma_{rr} n_r = -\sigma_{rr}^o = -\sigma_{yy}^o \sin^2\theta = -\frac{\sigma}{2}(1-\cos 2\theta) \\ T_\theta^o = \sigma_{r\theta} n_r = -\sigma_{\theta r}^o = -\sigma_{yy}^o \sin\theta\cos\theta = -\frac{\sigma}{2}\sin 2\theta \end{cases}$$

$\sigma_{\theta\theta}^o = \frac{\sigma}{2}(1+\cos 2\theta) = \sigma/2 \cos^2\theta$

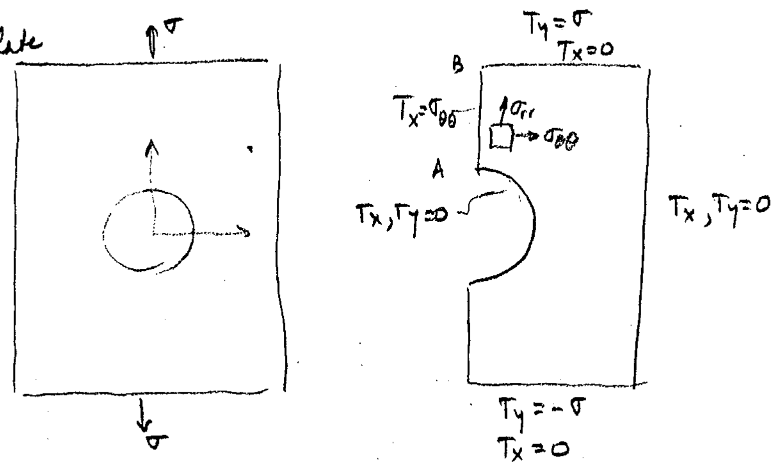
$$\therefore \hat{T}_r = \hat{\sigma}_{rr} n_r^{(-1)} = -T_r^o = \frac{\sigma}{2}(1-\cos 2\theta) \qquad \therefore \hat{\sigma}_{rr} = -\frac{\sigma}{2}(1-\cos 2\theta)$$

$$\hat{T}_\theta = \hat{\sigma}_{r\theta} n_r = -T_\theta^o = +\frac{\sigma}{2}\sin 2\theta \qquad \therefore \hat{\sigma}_{r\theta} = -\sigma/2(\sin 2\theta)$$

on r=a

2/7/79

cut a plate w/ hole

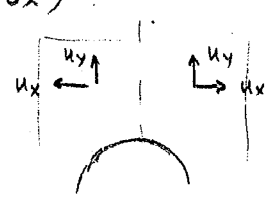


since there are no horiz. stresses on bdy ⇒ no forces in x direction

$\therefore \int_A^B T_x\,ds = 0 \Rightarrow \sigma_{\theta\theta}$ is not of the same sign on $\theta = \pm\frac{\pi}{2}$

to prove $\sigma_{xy} = \mu\left(\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial x}\right)$ is odd

$$\sigma_{xy} = \mu\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right)$$



u_y is even fn in x

u_x is odd fn in x

$\frac{\partial u_y}{\partial x}$ = is odd fn in x

$\frac{\partial u_x}{\partial y}$ is odd fn in x

$\therefore \sigma_{xy}$ is an odd fn in x

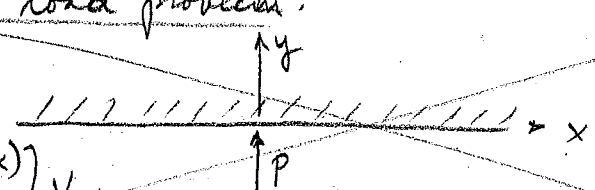
$$\text{now } \sigma_{\theta\theta}\Big|_{\theta=\pi/2} = \frac{\sigma}{2}\left(1 + \frac{a^2}{r^2} - 1 + 3\frac{a^4}{r^4}\right) = \frac{\sigma}{2}\left(\frac{a^2}{r^2}\right)\left(1 - 3\frac{a^2}{r^2}\right)$$

fn depends on sign of this term $\therefore \frac{1}{\sqrt{3}} = \frac{a}{r}$ is pt where sign changes.

pg 83-88 in Timoshenko & Goodier: Bending of Bar w/ curvature

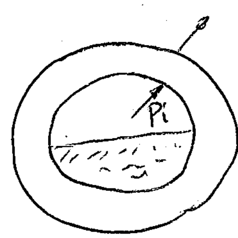
Look at it!

~~Return to~~ point load problem.



y > 0 B.C.

$$\left.\begin{aligned} \sigma_{yy}\big|_{y=0} &= -P\delta(x) \\ \sigma_{xy}\big)_{y=0} &= 0 \end{aligned}\right\} \forall x$$



$$t_n = -p_o e_r = \sigma_r e_r + \sigma_{r\theta} e_\theta$$

$$\sigma_{r\theta} = 0$$

$$\sigma_r = -p_o$$

$$t_n = p_i e_r = -\sigma_r e_r - \sigma_{r\theta} e_\theta$$

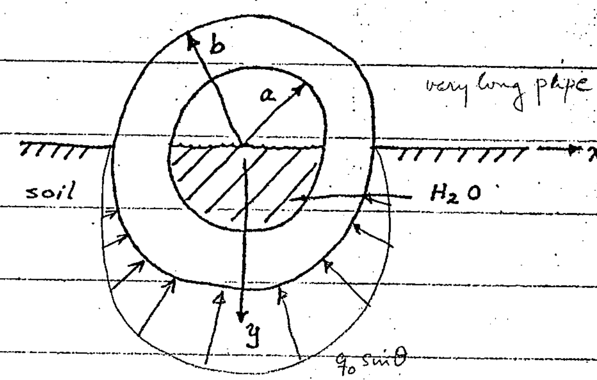
$$\sigma_r = -p_i$$

when $x > c$. Show that when $x = c + \epsilon$ (where ϵ is an arbitrarily small distance)

$$\sigma_{yy}(x, y=0) \sim \frac{K_I}{\sqrt{2\pi\epsilon}}$$

where $K_I = \sigma\sqrt{\pi c}$. K_I is known as the stress intensity factor.

2. A conduit is buried as shown and is half full of water. Ignore the weight of the conduit and assume the soil reaction varies from a maximum at the deepest part of the conduit to zero at ground level; a soil reaction of the form $q_0 \sin\theta$ might be a reasonable assumption to make. Assuming the conduit to be isotropic and linear elastic, solve for the stresses in the conduit.



3. The plane strain solution to Lame's problem of a cylinder under internal and external pressure requires the presence of an axial stress σ_{zz}. By superimposing an appropriate simple solution show that σ_{zz} may be reduced to zero. What does this say about the plane stress Lame' solution?

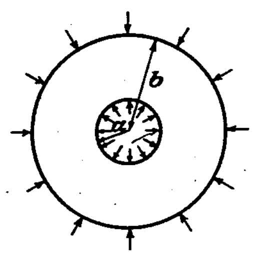


Fig. 41

Substituting in the first of Eqs. (43), we obtain the following equations to determine A and C:

$$\frac{A}{a^2} + 2C = -p_i$$
$$\frac{A}{b^2} + 2C = -p_o$$

from which

$$A = \frac{a^2b^2(p_o - p_i)}{b^2 - a^2}$$
$$2C = \frac{p_ia^2 - p_ob^2}{b^2 - a^2}$$

Substituting these in Eqs. (43) the following expressions for the stress components are obtained:

$$\sigma_r = \frac{a^2b^2(p_o - p_i)}{b^2 - a^2}\frac{1}{r^2} + \frac{p_ia^2 - p_ob^2}{b^2 - a^2}$$
$$\sigma_\theta = -\frac{a^2b^2(p_o - p_i)}{b^2 - a^2}\frac{1}{r^2} + \frac{p_ia^2 - p_ob^2}{b^2 - a^2} \qquad (44)$$

The radial displacement u is easily found since here $\epsilon_\theta = u/r$, and for plane stress

$$E\epsilon_\theta = \sigma_\theta - \nu\sigma_r$$

It is interesting to note that the sum $\sigma_r + \sigma_\theta$ is constant through the thickness of the wall of the cylinder. Hence the stresses σ_r and σ_θ produce a uniform extension or contraction in the direction of the axis of the cylinder, and cross sections perpendicular to this axis remain plane. Hence the deformation produced by the stresses (44) in an element of the cylinder cut out by two adjacent cross sections does not interfere with the deformation of the neighboring elements, and it is justifiable to consider the element in the condition of plane stress as we did in the above discussion.

In the particular case when $p_o = 0$ and the cylinder is submitted to internal pressure only, Eqs. (44) give

$$\sigma_r = \frac{a^2p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right)$$
$$\sigma_\theta = \frac{a^2p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \qquad (45)$$

These equations show that σ_r is always a compressive stress and σ_θ a tensile stress. The latter is greatest at the inner surface of the cylinder, where

$$(\sigma_\theta)_{\max} = \frac{p_i(a^2 + b^2)}{b^2 - a^2} \qquad (46)$$

$(\sigma_\theta)_{\max}$ is always numerically greater than the internal pressure and approaches this quantity as b increases, so that it can never be reduced below p_i, however much material is added on the outside. Various applications of Eqs. (45) and (46) in machine design are usually discussed in elementary books on the strength of materials.[1]

The corresponding problem for a cylinder with an eccentric bore was solved by G. B. Jeffery.[2] If the radius of the bore is a and that of the external surface b, and if the distance between their centers is e, the maximum stress, when the cylinder is under an internal pressure p_i, is the tangential stress at the internal surface at the thinnest part, if $e < \frac{1}{2}a$, and is of the magnitude

$$\sigma = p_i\left[\frac{2b^2(b^2 + a^2 - 2ae - e^2)}{(a^2 + b^2)(b^2 - a^2 - 2ae - e^2)} - 1\right]$$

If $e = 0$, this coincides with Eq. (46).

29 | Pure Bending of Curved Bars

Let us consider a curved bar with a constant narrow rectangular cross section[3] and a circular axis bent in the plane of curvature by couples M applied at the ends (Fig. 42). The bending moment in this case is constant along the length of the bar and it is natural to expect that the stress distribution is the same in all radial cross sections, and that the solution of the problem can therefore be obtained by using expression (41).

[1] See, for instance, S. Timoshenko, "Strength of Materials," 3d ed., vol. 2, chap. 6, D. Van Nostrand Company, Inc., Princeton, N.J., 1956.

[2] *Trans. Roy. Soc. (London)*, ser. A, vol. 221, p. 265, 1921. See also *Brit. Assoc. Advan. Sci. Rept.*, 1921. A complete solution by a different method is given in Art. 66 of the present book.

[3] From the general discussion of the two-dimensional problem, Art. 16, it follows that the solution obtained below for the stress holds also for plane strain.

Investigate $\phi = \frac{3F}{4c}\left(xy - \frac{xy^3}{3c^2}\right) + \frac{Py^2}{2}$

& $\phi = \frac{q}{8c^3}\left[x^2(y^3 - 3c^2 y + 2c^3) - \frac{1}{5}y^3(y^2 - 2c^2)\right]$

$y = \pm c,\ x \geq 0$

&

if ϕ satisfies $\nabla^4 \phi = 0$

what problem it satisfies

$\frac{\partial^2 \phi}{\partial x^2} = \frac{3F}{4c}(0) + 0 = 0 = \sigma_y$

$\frac{\partial^2 \phi}{\partial y^2} = \frac{3F}{4c}\left(-\frac{xy}{2c^2}\right) + \frac{P}{[illegible]} = \sigma_x$

$-\frac{\partial^2 \phi}{\partial x \partial y} = \frac{3F}{4c}\left(1 - \frac{y^2}{c^2}\right) = \tau_{xy}$

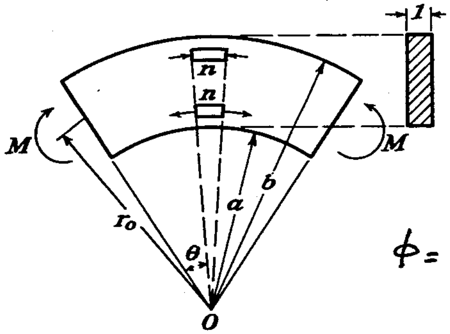


Fig. 42

Denoting by a and b the inner and the outer radii of the boundary and taking the width of the rectangular cross section as unity, the boundary conditions are

$$
\begin{aligned}
&(1) \qquad \sigma_r = 0 \qquad \text{for } r = a \text{ and } r = b \\
&(2) \qquad \int_a^b \sigma_\theta\, dr = 0 \qquad \int_a^b \sigma_\theta r\, dr = -M \qquad (a)\\
&(3) \qquad \tau_{r\theta} = 0 \qquad \text{at the boundary}
\end{aligned}
$$

Condition (1) means that the convex and concave boundaries of the bar are free from normal forces; condition (2) indicates that the normal stresses at the ends give rise to the couple M only, and condition (3) indicates that there are no tangential forces applied at the boundary. Using the first of Eqs. (42) with (1) of the boundary conditions (a) we obtain

$$
\begin{aligned}
\frac{A}{a^2} + B(1 + 2\log a) + 2C = 0 \\
\frac{A}{b^2} + B(1 + 2\log b) + 2C = 0
\end{aligned}
\qquad (b)
$$

Condition (2) in (a) is now necessarily satisfied. The use of a stress function guarantees equilibrium. A nonzero force-resultant on each end would violate equilibrium. To have the bending couple equal to M, the condition

$$\int_a^b \sigma_\theta r\, dr = \int_a^b \frac{\partial^2\phi}{\partial r^2} r\, dr = -M \qquad (d)$$

must be fulfilled. We have

$$\int_a^b \frac{\partial^2\phi}{\partial r^2} r\, dr = \left|\frac{\partial\phi}{\partial r} r\right|_a^b - \int_a^b \frac{\partial\phi}{\partial r} dr = \left|\frac{\partial\phi}{\partial r} r\right|_a^b - |\phi|_a^b$$

and noting that on account of (b),

$$\left|\frac{\partial\phi}{\partial r} r\right|_a^b = 0$$

we find from (d),

$$|\phi|_a^b = M$$

or substituting expression (41) for ϕ,

$$A \log \frac{b}{a} + B(b^2 \log b - a^2 \log a) + C(b^2 - a^2) = M \qquad (e)$$

This equation, together with the two Eqs. (b), completely determines the constants A, B, C, and we find

$$
\begin{aligned}
A &= -\frac{4M}{N} a^2b^2 \log\frac{b}{a} \qquad B = -\frac{2M}{N}(b^2 - a^2) \\
C &= \frac{M}{N}[b^2 - a^2 + 2(b^2 \log b - a^2 \log a)]
\end{aligned}
\qquad (f)
$$

where for simplicity we have put

$$N = (b^2 - a^2)^2 - 4a^2b^2\left(\log\frac{b}{a}\right)^2 \qquad (g)$$

Substituting the values (f) of the constants into the expressions (42) for the stress components, we find

$$
\begin{aligned}
\sigma_r &= -\frac{4M}{N}\left(\frac{a^2b^2}{r^2}\log\frac{b}{a} + b^2\log\frac{r}{b} + a^2\log\frac{a}{r}\right) \\
\sigma_\theta &= -\frac{4M}{N}\left(-\frac{a^2b^2}{r^2}\log\frac{b}{a} + b^2\log\frac{r}{b} + a^2\log\frac{a}{r} + b^2 - a^2\right) \qquad (47)\\
\tau_{r\theta} &= 0
\end{aligned}
$$

This gives the stress distribution satisfying all the boundary conditions[1] (a) for pure bending and represents the exact solution of the problem, provided the distribution of the normal forces at the ends is that given by the second of Eqs. (47). If the forces giving the bending couple M are distributed over the ends of the bar in some other manner, the stress distribution at the ends will be different from that of the solution (47). But, as Saint-Venant's principle suggests, the deviations from solution (47) may be negligible away from the ends, say at distances greater than the depth of the bar. This is illustrated by Fig. 102.

[1] This solution is due to H. Golovin, *Trans. Inst. Tech.*, St. Petersburg, 1881. The paper, published in Russian, remained unknown in other countries, and the same problem was solved later by M. C. Ribière (*Compt. Rend.*, vol. 108, 1889, and vol. 132, 1901) and by L. Prandtl. See A. Föppl, "Vorlesungen uber Technische Mechanik," vol. 5, p. 72, 1907; also A. Timpe, *Z. Math. Physik*, vol. 52, p. 348, 1905.

It is of practical interest to compare solution (47) with the elementary solutions usually given in books on the strength of materials. If the depth of the bar, $b - a$, is small in comparison with the radius of the central axis, $(b + a)/2$, the same stress distribution as that for straight bars is usually assumed. If this depth is not small, it is usual in practice to assume that cross sections of the bar remain plane during the bending, from which it can be shown that the distribution of the normal stresses σ_θ over any cross sections follows a hyperbolic law.[1] In all cases the maximum[2] and minimum values of the stress σ_θ can be presented in the form

$$\sigma_\theta = m \frac{M}{a^2} \tag{h}$$

The following table gives the values of the numerical factor m calculated by the two elementary methods, referred to above, and by the exact formula (47).[3] It can be seen from this table that the elementary solution based on the hypothesis of plane cross sections gives very accurate results.

Coefficient m of Eq. (h)

$\frac{b}{a}$	Linear stress distribution	Hyperbolic stress distribution		Exact solution	
1.3	±66.67	+72.98,	−61.27	+73.05,	−61.35
2	± 6.000	+ 7.725,	− 4.863	+ 7.755,	− 4.917
3	± 1.500	+ 2.285,	− 1.095	+ 2.292,	− 1.130

It will be shown later that, in the case of pure bending, the cross sections actually do remain plane, and the discrepancy between the elementary and the exact solutions comes from the fact that in the elementary solution the stress component σ_r is neglected and it is assumed that longitudinal fibers of the bent bar are in simple tension or compression.

[1] This approximate theory was developed by H. Résal, *Ann. Mines*, p. 617, 1862, and by E. Winkler, *Zivilingenieur*, vol. 4, p. 232, 1858; see also his book "Die Lehre von der Elastizität und Festigkeit," chap. 15, Prag, 1867. Further development of the theory was made by F. Grashof, "Elastizität und Festigkeit," p. 251, 1878, and by K. Pearson, "History of the Theory of Elasticity," vol. 2, pt. 1, p. 422, 1893.

[2] The greatest value of σ_θ in (47) always occurs at the inside ($r = a$). A proof is given by J. E. Brock, *J. Appl. Mech.*, vol. 31, p. 559, 1964.

[3] The results are taken from the doctorate thesis, University of Michigan, 1931, of V. Billevicz.

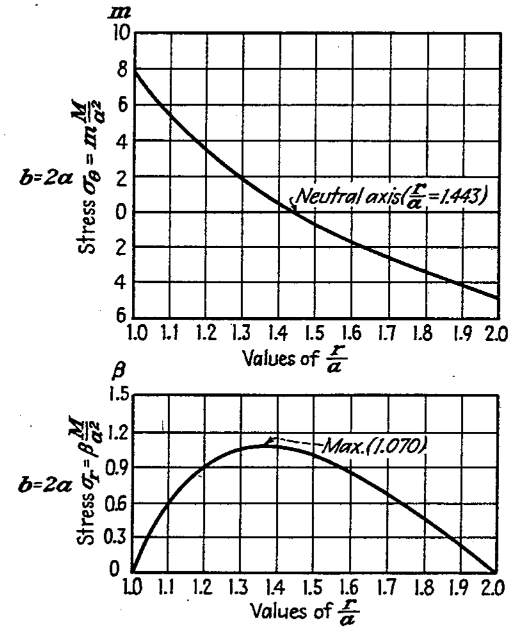
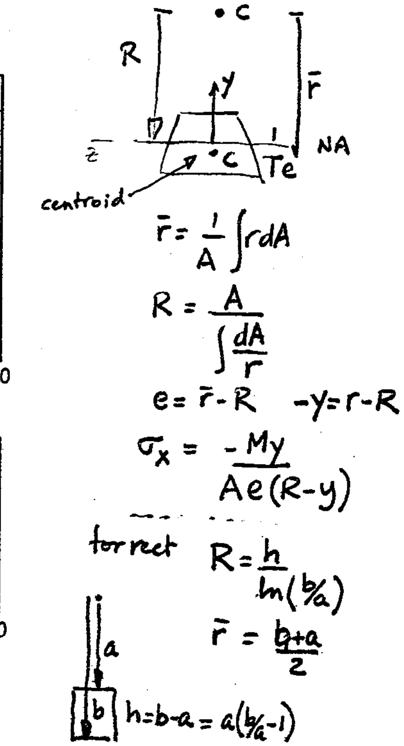


Fig. 43



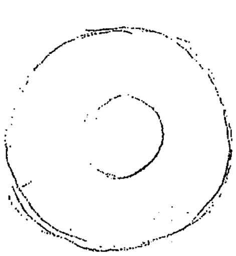
From the first of Eqs. (47) it can be shown that the stress σ_r is always positive for the direction of bending shown in Fig. 42. The same can be concluded at once from the direction of stresses σ_θ acting on the elements $n - n$ in Fig. 42. The corresponding tangential forces give resultants in the radial direction tending to separate longitudinal fibers and producing tensile stress in the radial direction. This stress increases toward the neutral surface and becomes a maximum near this surface. This maximum is always much smaller than $(\sigma_\theta)_{max}$. For instance, for $b/a = 1.3$, $(\sigma_r)_{max} = 0.060(\sigma_\theta)_{max}$; for $b/a = 2$, $(\sigma_r)_{max} = 0.138(\sigma_\theta)_{max}$; for $b/a = 3$, $(\sigma_r)_{max} = 0.193(\sigma_\theta)_{max}$. In Fig. 43 the distribution of σ_θ and σ_r for $b/a = 2$ is given. From this figure we see that the point of maximum stress σ_r is somewhat displaced from the neutral axis in the direction of the center of curvature.

30 | Strain Components in Polar Coordinates

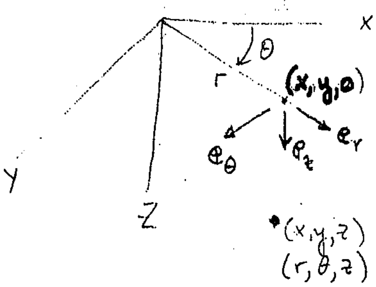
In considering the displacement in polar coordinates let us denote by u and v the components of the displacement in the radial and tangential directions, respectively. If u is the radial displacement of the side ad

Do this first before starting on cylindrical

Orthogonal Curvilinear Coordinates will be discussed in order to do torsional problem of a hollow surface.



Cylindrical Coordinate systems.



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

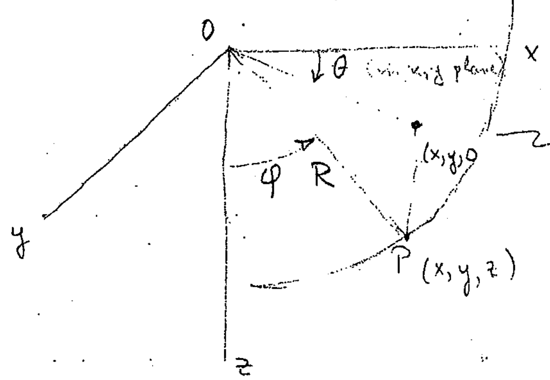
invertible

$r = \sqrt{x^2+y^2}$

$\theta = \arctan(y/x)$

$z = z$

Spherical Coordinate System.



$x = R\sin\varphi\cos\theta$

$y = R\sin\varphi\sin\theta$

$z = R\cos\varphi$

$R = \sqrt{x^2+y^2+z^2}$

$\theta = \tan^{-1}(y/x)$

$\varphi = \arctan\left(\dfrac{\sqrt{x^2+y^2}}{z}\right)$

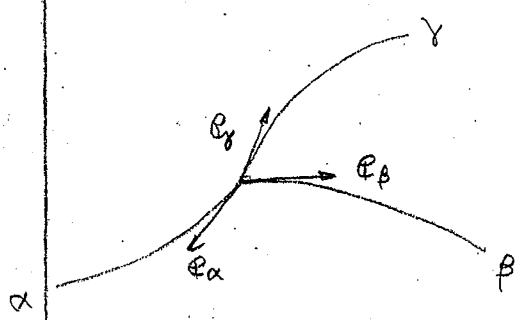
General orthogonal curvilinear coordinates (α, β, γ)

$\alpha = \alpha(x,y,z)$ Look at α = const this defines a surface

$\beta = \beta(x,y,z)$ Look at β = " " " " "

$\gamma = \gamma(x,y,z)$ " " γ = " " " " "

the intersection of these 3 surfaces defines a point p.



$e_{\alpha_i} \cdot e_{\alpha_j} = \delta_{ij}$, $e_{\alpha_i} \times e_{\alpha_j} = e_{\alpha_k} e_{ijk}$

in r, θ, z coord.

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta r} + \frac{\partial}{\partial z}\sigma_{zr} + \frac{(\sigma_{rr} + \sigma_{\theta\theta})}{r} + f_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2}{r}\sigma_{r\theta} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + f_\theta = 0$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = 0$$

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{\theta z} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right)$$

$$\varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)$$

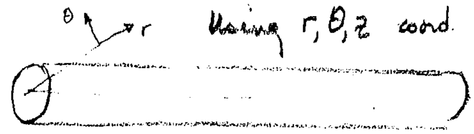
$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$

We next take up torsion of beams.

Coulomb Torsion

Check this again

Using r, θ, z coord.



Coulomb assumed a displacement solution & checked the compat, equil, etc.

He assumed $u_r = 0$, $u_z = 0$, $u_\theta = \alpha r z$ where α = proportionality factor.

$\Rightarrow$ $\varepsilon_{rr} = 0$, $\varepsilon_{\theta\theta} = 0$, $\varepsilon_{zz} = 0$, $\varepsilon_{r\theta} = 0$, $\varepsilon_{zr} = 0$, $\varepsilon_{\theta z} = \frac{1}{2}\alpha r$

let's get the ~~strains~~ stresses

$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu \varepsilon_{ij} \Rightarrow \sigma_{\theta z} = \mu \alpha r$ only non vanishing shear

with this $\sigma_{\theta z}$ we satisfy equil. We also assume torsion is end loaded not surface area loaded.

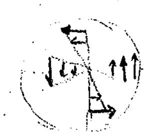
BC Cylindrical surface free of traction ie $e_r \cdot \sigma \overset{?}{=} 0$

but $e_r \cdot \sigma = (\sigma_{r\theta}, \sigma_{rz}, \sigma_{rr}) = 0$ thus we satisfy the BC on surface

on the ends $T = \int_0^a\int_0^{2\pi} \sigma_{\theta z} \cdot r \cdot r\,dr\,d\theta = \int_0^a\int_0^{2\pi} \mu\alpha r^3\,dr\,d\theta = \frac{\pi a^4}{2}\mu\alpha$

define polar moment of inertia $J = \iint r^2\,dA = \frac{\pi a^4}{2}$ $\quad \therefore \quad T = J\mu\alpha$ (torque)

load on the end $e_z \cdot \sigma = (\sigma_{zr}, \sigma_{z\theta}, \sigma_{zz}) = \sigma_{\theta z}$



shear go in opposite direction opposite sides ∴ they cancel out

$\omega_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial(r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right] = \alpha z$ $\quad$ α (is the twist) rotation/unit of length of cylinder

$\alpha = \frac{\partial}{\partial z}(\omega_{r\theta})$

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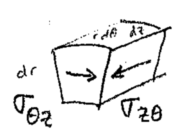
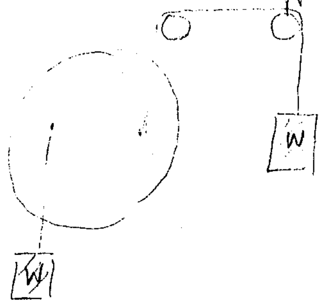
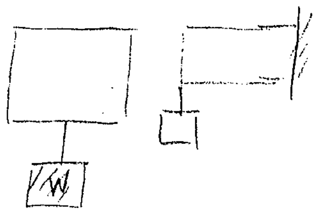
each cross section rotates as a body in Coulomb torsion **No warping**



Suppose we look at different problems



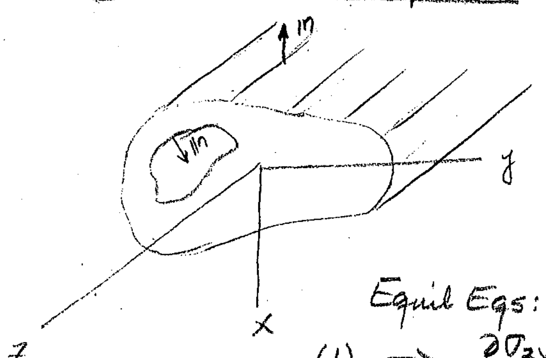
such that this system ∋. T. Is this system the same as the last torsion problem



If $W = P$ is this system the same as a uniformly end loaded beam such that $\int \sigma_{xz} dA = P$.

The answer is given by St. Venant's principle. If the two systems are statically equivalent then the stress distribution will be the same everywhere except near the loading point (points, faces etc) where the stress distribution produces a local effect. (This has not been proven - there are contradictive case.

St. Venant Torsion problem. No tractions on surface of the body except on the ends.



Let $\sigma_{zz}, \sigma_{xx}, \sigma_{yy}, \sigma_{xy} = 0$ and no body forces
Let $\sigma_{zx} \neq 0$; $\sigma_{zy} \neq 0$

Equil Eqs:

(1) $\Longrightarrow \dfrac{\partial \sigma_{zx}}{\partial z} = 0 \Longrightarrow \sigma_{zx} = \sigma_{zx}(x,y)$

(2) $\Rightarrow \dfrac{\partial \sigma_{zy}}{\partial z} = 0 \Longrightarrow \sigma_{zy} = \sigma_{zy}(x,y)$

(3) $\Rightarrow \dfrac{\partial \sigma_{zx}}{\partial x} + \dfrac{\partial \sigma_{zy}}{\partial y} = 0$

Consider $\phi = \phi(x,y)$ such that $\sigma_{xz} = \dfrac{\partial \phi}{\partial y}$ $\sigma_{zy} = -\dfrac{\partial \phi}{\partial x}$

Hence we let $\phi(x,y)$ Stress function for torsion.

~~on cylindrical surface~~

B.C. on free surface, for any surface: $\mathbb{n}\cdot\boldsymbol{\sigma} = 0$

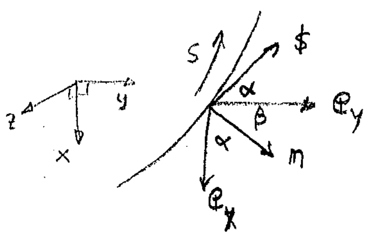
$$\mathbb{n}\cdot\left[\sigma_{zx}\,\mathbb{e}_z\mathbb{e}_x + \sigma_{zy}\,\mathbb{e}_z\mathbb{e}_y + \sigma_{zx}\,\mathbb{e}_x\mathbb{e}_z + \sigma_{zy}\,\mathbb{e}_y\mathbb{e}_z\right] = 0$$

BC for cylindrical surface: for a cylindrical surface $\mathbb{n}\cdot\mathbb{e}_z = 0$ ($\mathbb{n} \perp \mathbb{e}_z$)

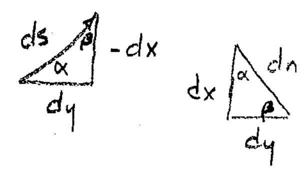
$$\therefore\ \mathbb{n}\cdot\boldsymbol{\sigma} = 0 \implies \left(\mathbb{n}\cdot\mathbb{e}_x\,\sigma_{zx} + \mathbb{n}\cdot\mathbb{e}_y\,\sigma_{zy}\right)\mathbb{e}_z = 0$$

or $\quad \mathbb{n}\cdot\mathbb{e}_x\,\sigma_{zx} + \mathbb{n}\cdot\mathbb{e}_y\,\sigma_{zy} = 0$

Look at the following section of the surface



$$\mathbb{n}\cdot\mathbb{e}_x = \mathbb{s}\cdot\mathbb{e}_y = \frac{dy}{ds} = \frac{dx}{dn} \qquad (=\cos\alpha)$$



$$\mathbb{n}\cdot\mathbb{e}_y = \mathbb{s}\cdot(-\mathbb{e}_x) = -\frac{dx}{ds} = \frac{dy}{dn} \qquad (=\cos\beta)$$

$$\therefore\ \mathbb{n}\cdot\mathbb{e}_x\,\sigma_{xz} + \mathbb{n}\cdot\mathbb{e}_y\,\sigma_{zy} = 0$$

$$\frac{dy}{ds}\cdot\frac{\partial\phi}{\partial y} + -\frac{dx}{ds}\,\frac{-\partial\phi}{\partial x} = \frac{d\phi}{ds} = 0 \implies \phi = \text{constant along the}$$

boundary or along a contour line.

BC. at the end faces $\quad \mathbb{n}\cdot\boldsymbol{\sigma} = \pm\,\mathbb{e}_z\cdot\boldsymbol{\sigma} = \pm\left[\sigma_{zx}\mathbb{e}_x + \sigma_{zy}\mathbb{e}_y\right]$

Now on each end we want the resultant forces — we will show them to be $\equiv 0$

$$\therefore\ \iint_A \sigma_{zx}\,dA = \iint_A \frac{\partial\phi}{\partial y}\,dA = F_x$$

$$\iint_A \sigma_{zy}\,dA = -\iint \frac{\partial\phi}{\partial x}\,dA = F_y$$

Recalling Green's theorem

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \oint_{\text{boundary}} P\,dx + Q\,dy$$

$\implies$ let $Q = 0 \quad -P = \phi \quad$ for F_x

$\phi = Q \quad P = 0 \quad$ for F_y

Evaluation of

$F's \Rightarrow -\oint_{boundary} \phi\, dx = A_1 \cdot \oint dx;\ -\oint_{boundary} \phi\, dy = A_1 \cdot \oint dy$ but $\oint dx = 0$ $\oint dy = 0$ since ϕ = const on surface $\Longrightarrow F_x = F_y = 0$

On the end faces there will be no resultant force

Resultant torque

$$\mathbb{T} = \iint \mathbf{r} \times (\mathbf{n} \cdot \sigma)\, dA \qquad \mathbf{r} \times (\mathbf{n}\cdot\sigma) = \begin{vmatrix} e_x & e_y & e_z \\ x & y & 0 \\ \sigma_{zx} & \sigma_{zy} & 0 \end{vmatrix}$$

$$\mathbf{r} = x e_x + y e_y$$

$$\mathbf{r} \times (\mathbf{n} \cdot \sigma) = \pm \left[x \sigma_{zy} - y \sigma_{zx} \right] e_z$$

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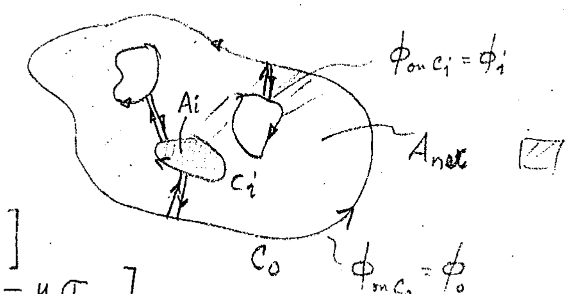
Resultant Torque

$$\mathbb{T} = \iint_A \mathbf{r} \times (\mathbf{n} \cdot \sigma\, dA)$$



$$\mathbf{r} = x e_x + y e_y$$

$$\pm e_z \cdot \sigma = \mathbf{n} \cdot \sigma = \pm \left[e_x \sigma_{zx} + e_y \sigma_{zy} \right]$$

$$\mathbf{r} \times (\mathbf{n} \cdot \sigma) = \pm e_z \left[x \sigma_{zy} - y \sigma_{zx} \right]$$

$$\therefore\ T = \int_A (x \sigma_{zy} - y \sigma_{zx})\, dA = -\iint \left(x \frac{\partial \phi}{\partial x} + y \frac{\partial \phi}{\partial y} \right) dx\, dy$$

Now $\frac{\partial}{\partial x}(\phi x) = 1 \cdot \phi + x \frac{\partial \phi}{\partial x}$

we rewrite

$$T = \iint_A \left[2\phi - \frac{\partial}{\partial x}(\phi x) - \frac{\partial}{\partial y}(\phi y) \right] dx\, dy = \iint_A \left[2\phi + \frac{\partial}{\partial x}(-x\phi) - \frac{\partial}{\partial y}(y\phi) \right] dA$$

now we use green's theorem $\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \oint_C P\, dx + Q\, dy$

$$T = \iint_{A_{net}} 2\phi\, dA - \oint_{C_0} \phi_0\, x\, dy + \oint_{C_0} \phi_0\, y\, dx - \sum_i \phi_i \oint_{C_i} x\, dy + \sum_i \phi_i \oint_{C_i} y\, dx$$

$$- \phi_0 \oint_{C_0} [x\, dy - y\, dx] \qquad - \sum \phi_i \left[\oint_{C_i} (x\, dy - y\, dx) \right]$$

and by using greens theorem in reverse — since this is a clockwise integration of Green's theorem

$$- \phi_0 \iint_{A_{gross}} 2\, dA \qquad - \sum \phi_i \iint_{A_i} -2\, dA$$

$$T = 2\left[\iint_{A_{net}} \phi\, dA - \phi_0 A_{gross} + \sum_i \phi_i A_i \right] \quad \text{where } A_{gross} = A_{net} + \sum A_i$$

Strains and Displacements – we will now get strain and displacement solutions

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = 0 \quad \text{and} \quad \varepsilon_{yz} = \frac{1+\nu}{E}\sigma_{zy}, \quad \varepsilon_{zx} = \frac{1+\nu}{E}\sigma_{zx} = f(x,y) \text{ only}$$

since $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$ & $\sigma_{xy} = 0$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \qquad \varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0 \Rightarrow u_x = u_x(y,z)$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0 \Rightarrow u_y = u_y(x,z)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0 \Rightarrow u_z = u_z(x,y)$$

$$\gamma_{xy} = 2\varepsilon_{xy} = (u_{x,y} + u_{y,x}) = 0 \Rightarrow u_{x,y} = -f(z) \text{ only} \qquad u_{y,x} = +f(z)$$

since $\frac{\partial u_x}{\partial y}$ & $\frac{\partial u_y}{\partial x}$ only has z as common term then let $u_{x,y} =$

integration of $u_{x,y}$ and $u_{y,x}$ gives

$$\therefore \quad u_y(x,z) = x f(z) + g_y(z) \qquad u_x(y,z) = -y f(z) + g_x(z)$$

look at $\frac{\partial u_x}{\partial z} = 2\varepsilon_{zx} - \frac{\partial u_z}{\partial x}$ (note that $\varepsilon_{zx} = \frac{1}{2}(u_{x,z} + u_{z,x})$) Now take $\frac{\partial}{\partial z}$ of this equation

$$\frac{\partial^2 u_x}{\partial z^2} = \frac{\partial}{\partial z}\left(2\varepsilon_{zx} - \frac{\partial u_z}{\partial x}\right)$$

since $u_z = u_z(x,y)$ and $\frac{\partial}{\partial z}u_z = 0$ and since $\sigma_{zx} = \sigma_{zx}(x,y)$ only $\Rightarrow \varepsilon_{zx} = \varepsilon_{zx}(x,y)$ only and $\frac{\partial}{\partial z}\varepsilon_{zx} \equiv 0$.

true for any y ∴

$$\therefore \frac{\partial^2 u_x}{\partial z^2} = 0 = -y f'' + g_x'' = 0 \Rightarrow f'' = 0 \text{ and } g_x'' = 0 \quad \forall y$$

$$\text{hence } f(z) = az + b \qquad g_x(z) = cz + d.$$

We can get an analogous result for ε_{yz}: since $\varepsilon_{zy} = \varepsilon_{zy}(x,y)$ from fact that $\sigma_{zy}(x,y)$ only ∴ $\frac{\partial \varepsilon_{zy}}{\partial z} = 0$ and $\frac{\partial^2 u_z}{\partial y \partial z} = 0 \Rightarrow \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$

$$u_y = axz + bx + ez + f \qquad u_x = -ayz - by + cz + d.$$

B.C. are used to get the unknowns a, b, c, d, e & f

At the origin as a reference point $u_x, u_y = 0 \Rightarrow f, d = 0$ No r.b. transl.

At origin slopes wrt $z = 0$ $\quad u_{x,z} = u_{y,z} = 0 \Rightarrow c, e = 0$ No. r.b. rot

Specify that rotation about the z-axis will be measured from the origin $\omega_{xy}\big|_{z=0} = 0$

$$\omega_{xy}\Big|_{z=0} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \frac{1}{2}\left(az+b+az+b\right) = az+b\Big|_{z=0} = 0 \Rightarrow b=0$$

Let $a = \alpha$ (the twist) hence we go to displ. eqns and finally get.

$$u_y = \alpha x z \qquad u_x = -\alpha y z$$

note that $\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = \overset{\gamma_{yz}}{\overset{\parallel}{2\varepsilon_{yz}}} = -2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial x} = \frac{\sigma_{yz}}{G}$ $\qquad -\frac{\partial\phi}{\partial x} = \tau_{yz} = G\gamma_{yz} = 2G\varepsilon_{yz}$

$$\frac{\partial u_z}{\partial y} = -2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial x} - \alpha x \qquad \left(\text{since } -\frac{\partial u_y}{\partial z} = -\alpha x\right) \text{ from } \gamma_{yz}$$

also $\frac{\partial u_z}{\partial x} = 2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial y} + \alpha y$ $\qquad \left[\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)\right]$ for γ_{zx}

$\frac{\partial u_z}{\partial z} = 0$ since $u_z = u_z(x,y)$ only

we can now get u_z by integrating

Since we have a multiply connected region we must use Cesaro's theorem. ie displ must be singlevalued

$\therefore$ $\qquad u_x = u_x(y,z)$

$$\oint du_1 = 0 \Rightarrow \oint \cancelto{0}{\frac{\partial u_x}{\partial x}} dx + \frac{\partial u_x}{\partial y}dy + \frac{\partial u_x}{\partial z}dz = \oint du_x = 0$$

$$-\alpha z\,dy - \alpha y\,dz = -\alpha\oint z\,dy + y\,dz = -\alpha\cdot 0 = 0$$

$\iint (1-1)\,dx\,dy$ by Green's theorem

hence by Cesaro's theorem the displ are single valued in x direction

we can do same in y direction $\qquad u_y = u_y(x,z)$

$$\oint \underset{\alpha z\,dx + \alpha x\,dz}{\frac{\partial u_y}{\partial x}dx} + \oint \cancel{\frac{\partial u_y}{\partial y}}dy + \oint \frac{\partial u_y}{\partial z}dz = +\alpha\oint(z\,dx + x\,dz) = 0 = du_y = 0$$

11/29/78

we have found so far $u_x = -\alpha y z$ and $u_y = \alpha x z$

now $\frac{\partial u_z}{\partial y} = -2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial x} - \alpha x$

$$\frac{\partial u_z}{\partial x} = 2\frac{(1+\nu)}{E}\frac{\partial \phi}{\partial y} + \alpha y$$

$$\frac{\partial u_z}{\partial z} = 0$$

For single valuedness of u_z

$$u_z = u_z(x,y)$$

$$\oint_c du_z = \oint_c \left(\frac{\partial u_z}{\partial x} dx + \frac{\partial u_z}{\partial y} dy + \frac{\partial u_z}{\partial z} dz \right) = \oint_{C_0} du_z + \sum_i \oint_{C_i} du_z$$

$$= \oint_c \left[\left(\frac{1}{\mu} \frac{\partial \phi}{\partial y} + \alpha y \right) dx - \left(\frac{1}{\mu} \frac{\partial \phi}{\partial x} + \alpha x \right) dy \right]$$

(P and Q marked over the two brackets)

using Green's Thm. the line integral is $\qquad \oint_c Q dy + P dx = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

$$- \iint_{A_{gross}} \left[\left(\frac{1}{\mu} \frac{\partial^2 \phi}{\partial x^2} + \alpha \right) + \left(\frac{1}{\mu} \frac{\partial^2 \phi}{\partial y^2} + \alpha \right) \right] dA - \sum_{i=0} \oint_{c_i} du_z$$

($\partial Q/\partial x$ and $\partial P/\partial y$ marked over the two terms; "cavities" under the sum)

For compatibility we must have that integrand $= 0$

$$\frac{1}{\mu} \nabla_1^2 \phi + 2\alpha = 0 \quad \text{and} \quad \oint_{c_i} du_z = 0$$

or $\boxed{\nabla_1^2 \phi = -2\mu\alpha}$ $\qquad \nabla_1^2 \phi = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

now let us look at $\oint_{C_i} du_z = 0$ $\qquad$ but $\nabla^2\phi = \nabla_1^2 \phi$ since $\phi = \phi(x,y)$ only

$$\oint_{C_i} du_z = \frac{1}{\mu} \oint \left(\frac{\partial \phi}{\partial y} dx - \frac{\partial \phi}{\partial x} dy \right) + \alpha \oint (y dx - x dy) = 0$$

($\frac{\partial u_z}{\partial x} dx$, $\frac{\partial u_z}{\partial y} dy$ marked above)

From our previous lectures we related m, s, e_x, e_y at the boundary so that

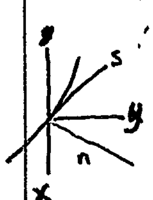
$$\frac{dy}{ds} = \frac{dx}{dn} \;①, \qquad -\frac{dx}{ds} = \frac{dy}{dn} \;②$$

solve for dy & dx in ① & ② respectively

hence the first part of the integral

$$\oint \frac{\partial \phi}{\partial y} dx - \frac{\partial \phi}{\partial x} dy = \oint \left(-\frac{\partial \phi}{\partial y} \frac{dy}{dn} - \frac{\partial \phi}{\partial x} \frac{dx}{dn} \right) ds = \oint -\frac{\partial \phi}{\partial n} ds$$

$$\oint y dx - x dy = -\oint (y dx - x dy) = 2 \oint_{A_i} dA = 2 A_i$$



$$\therefore \oint_{c_i} du_z = -\frac{1}{\mu} \oint \frac{\partial \phi}{\partial n} ds + 2\alpha A_i = 0 \qquad \therefore \boxed{\oint_{C_i} \frac{\partial \phi}{\partial n} ds = 2\alpha A_i \mu}$$

on cavity bdy

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} \frac{dx}{dn} + \frac{\partial \phi}{\partial y} \frac{dy}{dn}$$

$$-\sigma_{zy} \frac{dx}{dn} + \sigma_{zx} \frac{dy}{dn} = -s \cdot t_z$$

shear stress along the contour

$m \cdot e_x \qquad m \cdot e_y$

$s \cdot e_y \qquad -s \cdot e_x$

$$\oint \frac{\partial \phi}{\partial n} ds = \oint \tau ds = 2\alpha A_i \mu$$

$$\tau = \sqrt{\sigma_{zy}^2 + \sigma_{zx}^2}$$

Summary of the St Venant Torsion

<u>Equilib</u> - automatically satisfied by introducing $\phi(x,y)$ ∋. $\sigma_{zx} = \frac{\partial \phi}{\partial y}$, $\sigma_{zy} = -\frac{\partial \phi}{\partial x}$

<u>B.C.</u> we assumed that surface tractions on the surface = 0

$\phi = K_i$ (const) $(i = 0, 1, \ldots, N)$ — K_0 on outer bdy, K_1 on 1st cavity etc.

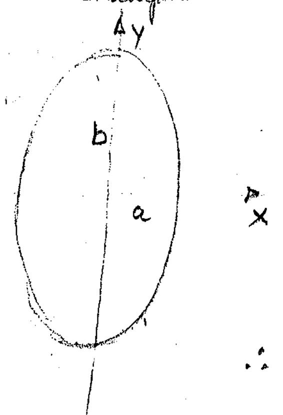
<u>Torque</u> $T = 2\left[\iint_{A_{net}} \phi\, dA - K_0 A_0 + \sum_{i=1}^{n} K_i A_i\right]$ ∋. $A_0 - \sum_i A_i = A_{net}$ (A_0: gross area)

<u>Compatibility</u> $\nabla^2 \phi = -2\mu\alpha$

<u>Additional conditions for shaft w/ cavities</u> $\oint_{C_i} \frac{\partial \phi}{\partial n}\, ds = 2\mu A_i \alpha$

<u>Displacements</u> $u_x = -\alpha y z$ $u_y = \alpha x z$ $u_z = f(x,y)$ this warping fn.

- 1st Example Torsion of a bar of elliptical cross section w/ <u>NO</u> cavities



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Eqn of contour w/ $b > a$

since ϕ = const on the ellipse surface and since $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ on contour let us take $\phi = B\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

∴ Try $\phi = B\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$ const on bdy

∴ since $\nabla^2 \phi = -2\mu\alpha = B\left[\frac{2}{a^2} + \frac{2}{b^2}\right]$ ∴ $\boxed{B = -\frac{\mu\alpha a^2 b^2}{a^2 + b^2}}$

$T = 2\left[\iint \phi\, dA - k_0 A_0\right] = 2\iint \phi\, dA$ since $k_0 = 0$ on bdy

$T = 2\iint \phi\, dA = -B\pi a b = \frac{\pi a^3 b^3 \mu \alpha}{a^2 + b^2} = D\alpha$ — D: torsional stiffness $= \mu\alpha J$

J = polar moment of inertia about z axis

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Final Exam : is due Monday the 11^{th} of Dec will be given out next week

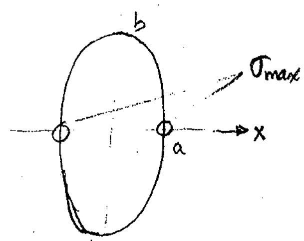
$$\sigma_{zx} = \frac{\partial \phi}{\partial y} = \frac{-2Ty}{\pi a b^3} \qquad \left(-\frac{2y\beta}{b^2} = \frac{2y}{b^2}\cdot\frac{T}{\pi ab}\right)$$

$$\sigma_{zy} = \frac{\partial \phi}{\partial x} = \frac{2Tx}{\pi a^3 b} \qquad \left(-2x\beta/a^2 = \frac{2x}{a^2}\cdot\frac{T}{\pi ab}\right)$$

$$\sigma_{res} = \left(\sigma_{zx}^2 + \sigma_{zy}^2\right)^{1/2} = \frac{2T}{\pi ab}\left(\frac{y^2}{b^4} + \frac{x^2}{a^4}\right)^{1/2} = \frac{2T}{\pi a^3 b^3}\left(b^4x^2 + a^4y^2\right)^{1/2}$$

$$\boxed{\sigma_{max} \ (@\ x = \pm a,\ y = 0) = \frac{2T}{\pi a^2 b}}$$

∴ if yield occurs: at pts on surface where it is closest to origin



Now $u_x = -\alpha y z = -\dfrac{T(a^2+b^2)}{\pi a^3 b^3 \mu}\, yz$

$u_y = \alpha x z = \dfrac{T(a^2+b^2)}{\pi a^3 b^3 \mu}\, xz$

from our summary $\dfrac{\partial u_z}{\partial x} = \dfrac{1}{\mu}\dfrac{\partial \phi}{\partial y} + \alpha y = \dfrac{1}{\mu}\left(\dfrac{-2Ty}{\pi a b^3}\right) + \left(\dfrac{+T(a^2+b^2)}{\pi a^3 b^3 \mu}\right) y$ (the last term $= \alpha$)

$$\boxed{\frac{\partial u_z}{\partial x} = \frac{b^2 - a^2}{b^2 + a^2}\,\alpha y}$$

$$\frac{\partial u_z}{\partial y} = -\left(\frac{1}{\mu}\frac{\partial \phi}{\partial x} + \alpha x\right) = \frac{b^2-a^2}{b^2+a^2}\,\alpha x$$

integrate

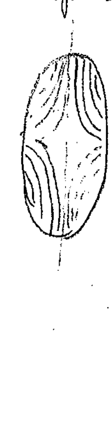
Now $u_z = \dfrac{b^2-a^2}{b^2+a^2}\,\alpha x y + f_1(y)$

$u_z = \dfrac{b^2-a^2}{b^2+a^2}\,\alpha x y + f_2(x)$

∴ $f_1(y) = f_2(x) = $ const. represents a rigid body disp in the z direct

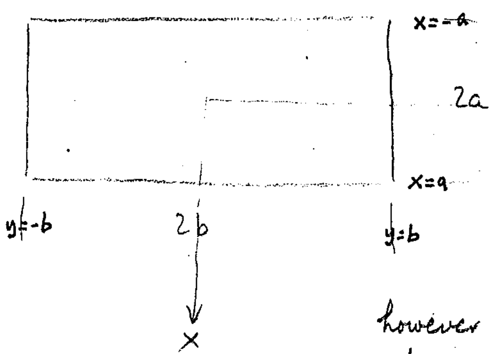
∴ $u_z = \dfrac{b^2-a^2}{b^2+a^2}\,\alpha x y$

warping function



$u_x = u_x(x,y)$
$u_y = u_y(x,y)$ } plane strain problems.
$u_z = 0$

Torsion of a bar of rectangular crossection



can we define a stress function ϕ .э. $\nabla^2\phi = -2\mu\alpha$?
we cannot find a single fn. of ϕ constant on body satisfying eq.

however we can construct solutions $\phi_0 = \mu\alpha(a^2 - x^2)$
.э. $\phi_0 = 0$ on $x = \pm a$
and we can construct a second ϕ_1 $\therefore$ $\nabla^2\phi = \nabla^2\phi_0 + \nabla^2\phi_1 = -2\mu\alpha$
thus $\nabla^2\phi_1 = 0$ since $\nabla^2\phi_0 = -2\mu\alpha$

The B.C. @ $x = \pm a$ $\therefore$ $\phi(\pm a, y) = \phi_1(\pm a, y) = 0$
$y = \pm b$ $\phi(x, \pm b) = \mu\alpha(a^2 - x^2) + \phi_1(x, \pm b) = 0$

since we want ϕ on bdy $= 0$

if we assume $\phi_1(x,y) = F_1(x)\,F_2(y)$ then $\nabla^2\phi_1 = F_1''F_2 + F_1F_2'' = 0$

thus $\dfrac{F_1''}{F_1} = -\dfrac{F_2''}{F_2} = \text{const} = -\beta^2$

$\therefore$ $F_1'' + \beta^2 F_1 = 0 \implies F_1 = A_1\cos\beta x + B_1\sin\beta x$ $\forall y$
$F_2'' - \beta^2 F_2 = 0 \implies F_2 = A_2\cosh\beta y + B_2\sinh\beta y$ $\forall x$

Symmetry w/t $x, y \implies B_1 = 0$ and $B_2 = 0$ $\quad$ $(\sin\beta(-x) = -\sin\beta x)$ $(\sinh\beta(-x) = -\sinh\beta x)$

$\therefore$ $\phi_1 = C\cos\beta x\cosh\beta y$

$$\phi_1(\pm a, y) = C\cos\beta a \cosh\beta y = 0 \qquad \therefore \quad \beta a = \frac{(2k+1)\pi}{2} \qquad (k = 0, 1, 2, \dots)$$

$$\therefore \quad \phi_1(x,y) = \sum_{k=0}^{\infty} C_k \cos\frac{(2k+1)\pi}{2a}x \cosh\frac{(2k+1)\pi}{2a}y$$

other BC. $\phi_1(x, \pm b) = -\mu\alpha(a^2 - x^2)$

$$\therefore \quad -\mu\alpha(a^2 - x^2) = \sum C_k \cos\frac{(2k+1)\pi}{2a}x \cosh\frac{(2k+1)\pi}{2a}b$$

now
$$C_k \cosh\frac{(2k+1)\pi b}{2a} = \frac{2}{a}\int_0^a -\mu\alpha(a^2 - x^2)\cos\frac{(2k+1)\pi}{2a}x\,dx$$

$$= \frac{-32\mu a^2(-1)^k}{(2k+1)^3\pi^3}$$

since only terms which will give results in fourier series is when $m = 2k+1$

$$\therefore \quad C_k = \frac{-32\mu\alpha a^2(-1)^k}{(2k+1)^3\pi^3}$$

$$\therefore \quad \phi = \phi_0 + \phi_1 = -\mu\alpha\left[(x^2 - a^2) + \frac{32a^2}{\pi^3}\sum_{k=0}^{\infty}(-1)^k \frac{\cos\frac{(2k+1)\pi x}{2a}\cosh\frac{(2k+1)\pi y}{2a}}{(2k+1)^3\cosh\frac{(2k+1)\pi b}{2a}}\right]$$

τ_{max} occurs at $x = \pm a$, $y = 0$

$$\tau_{yz} = -\left.\frac{\partial\phi}{\partial y}\right|_{\substack{x=a\\y=0}} = 2\mu\alpha a\zeta \qquad \text{where } \zeta = 1 - \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2\cosh\frac{(2k+1)\pi b}{2a}}$$

b/a	ζ
1.0	.675
1.2	.759
1.5	.848
2.0	.930
2.5	.968
3.0	.985
5.0	.999
∞	1.0

for engineering purposes if $b/a > 2.5$ $\zeta \approx 1$

$$\therefore \quad \tau_{yz} = 2\mu\alpha a$$

$u_x = u_x(x,y)$
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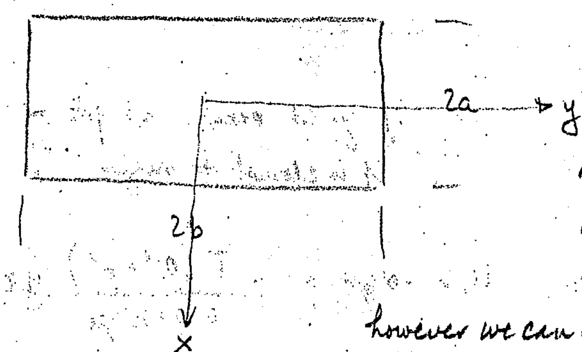
$u_z = 0$

Torsion of a bar of rectangular cross-section



can we define a stress function $\phi \ni \nabla^2\phi = -2\mu\alpha$?

we cannot find a single fn. of ϕ constant on body satisfying eq.

however we can construct solutions $\phi_0 = \mu\alpha(a^2 - x^2)$

$\ni \phi_0 = 0$ on $x = \pm a$

and we can construct a second $\phi_1 \quad \therefore \nabla^2\phi = \nabla^2\phi_0 + \nabla^2\phi_1 = -2\mu\alpha$

thus $\nabla^2\phi_1 = 0$ since $\nabla^2\phi_0 = -2\mu\alpha$

The B.C. @ $x = \pm a \quad \therefore \phi(\pm a, y) = \phi_1(\pm a, y) = 0$

$y = \pm b \quad \phi(x, \pm b) = \mu\alpha(a^2 - x^2) + \phi_1(x, \pm b) = 0$

since we want ϕ on bdy $= 0$

if we assume $\phi_1(x,y) = F_1(x)F_2(y)$ then $\nabla^2\phi_1 = F_1''F_2 + F_1F_2'' = 0$

thus $\dfrac{F_1''}{F_1} = -\dfrac{F_2''}{F_2} = \text{const} = -\beta^2$

$\therefore \quad F_1'' + \beta^2 F_1 = 0 \implies F_1 = A_1 \cos\beta x + B_1 \sin\beta x \quad \forall y$

$F_2'' - \beta^2 F_2 = 0 \implies F_2 = A_2 \cosh\beta y + B_2 \sinh\beta y \quad \forall x$

Symmetry w/t $x, y \implies B_1 = 0$ and $B_2 = 0$ $\quad$ ($\sin\beta(-x) = -\sin\beta x$) ($\sinh\beta(-x) = -\sinh\beta x$)

$\therefore \phi_1 = C\cos\beta x \cosh\beta y$

$F_1(a) = A_1\cos\beta a + B_2\sin\beta a = 0$
$F_1(-a) = A_1\cos\beta a - B_2\sin\beta a = 0$] add $2A_1\cos\beta a = 0$ $A_1 \neq 0 \Rightarrow \beta a = \frac{n\pi}{2}$ n odd
if so $\sin\beta a = \sin\frac{n\pi}{2}$ (n odd) $\neq 0 \Rightarrow B_2 = 0$

$$\phi_1(\pm a, y) = C\cos\beta a\cosh\beta y = 0 \quad \therefore \quad \beta a = (2k+1)\frac{\pi}{2} \quad (k = 0,1,2,\dots)$$

$$\therefore \quad \phi_1(x,y) = \sum_{k=0}^{\infty} C_k \cos\frac{(2k+1)\pi}{2a}x \cosh\frac{(2k+1)\pi}{2a}y$$

other BC. $\phi_1(x, \pm b) = -\mu\alpha(a^2 - x^2)$

$$\therefore \quad -\mu\alpha(a^2 - x^2) = \sum C_k \cos\frac{(2k+1)\pi}{2a}x \cosh\frac{(2k+1)\pi}{2a}b$$

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$$C_k \cosh\frac{(2k+1)\pi b}{2a} = \frac{2}{a}\int_0^a -\mu\alpha(a^2 - x^2)\cos\frac{(2k+1)\pi}{2a}x\,dx$$
$$= \frac{-32\mu a^2(-1)^k}{(2k+1)^3\pi^3},$$

since only terms which will give results in fourier series is when $m = 2k+1$

$$\therefore \quad C_k = \frac{-32\mu\alpha a^2(-1)^k}{(2k+1)^3\pi^3} \cdot \frac{1}{\cosh\frac{(2k+1)\pi b}{2a}}$$

$$\therefore \quad \phi = \phi_0 + \phi_1 = -\mu\alpha\left[(x^2 - a^2) + \frac{32a^2}{\pi^3}\sum_{k=0}^{\infty}(-1)^k \frac{\cos\frac{(2k+1)\pi x}{2a}\cosh\frac{(2k+1)\pi y}{2a}}{(2k+1)^3\cosh\frac{(2k+1)\pi b}{2a}}\right]$$

$$\sigma_{yz} = -\frac{\partial\phi}{\partial x} = \mu\alpha\left[2x - \frac{16a}{\pi^2}\sum_{k=0}^{\infty}(-1)^k \frac{\sin\frac{(2k+1)\pi x}{2a}\cosh\frac{(2k+1)\pi y}{2a}}{(2k+1)^2\cosh\frac{(2k+1)\pi b}{2a}}\right]$$

σ_{max} occurs at $x = \pm a$, $y = 0$ now $(-1)^k\sin\left(\frac{2k+1}{2a}\pi a\right) = (-1)^k(-1)^k = 1$

$$\sigma_{yz} = -\frac{\partial\phi}{\partial x}\bigg|_{\substack{x=a\\y=0}} = 2\mu\alpha a\zeta \quad \text{where } \zeta = 1 - \frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^2\cosh\frac{(2k+1)\pi b}{2a}}$$

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∞	1.0

for engineering purposes if $b/a > 2.5$ $\zeta \approx 1$

$$\therefore \quad \sigma_{yz} = 2\mu\alpha a\zeta \qquad \alpha = \frac{T}{JG}$$

$$= 2\frac{Ta}{J}\zeta = \frac{2Ta\zeta}{\frac{\pi a^3b^3}{a^2+b^2}} = \frac{2T\zeta}{\frac{\pi b^3}{(b/a)^2+1}}$$

For single valuedness of u_z

$$\oint_c du_z = \oint_c \left(\frac{\partial u_z}{\partial x} dx + \frac{\partial u_z}{\partial y} dy + \frac{\partial u_z}{\partial z} dz \right) = \oint_{C_0} du_z + \sum_i \oint_{C_i} du_z \qquad (u_z = u_z(x,y))$$

$$= \oint_c \left[\left(\frac{1}{\mu} \frac{\partial \phi}{\partial y} + \alpha y \right) dx - \left(\frac{1}{\mu} \frac{\partial \phi}{\partial x} + \alpha x \right) dy \right]$$

using Stokes Thm. the line integral is

$$-\iint_{A_{gross}} \left[\left(\frac{1}{\mu} \frac{\partial^2 \phi}{\partial x^2} + \alpha \right) + \left(\frac{1}{\mu} \frac{\partial^2 \phi}{\partial y^2} + \alpha \right) \right] dA - \sum_{i=0} \oint_{C_i} du_z$$

cavities

For compatibility we must have

$$\frac{1}{\mu} \nabla_1^2 \phi + 2\alpha = 0 \quad \text{and} \quad \oint_{C_i} du_z = 0$$

or $\boxed{\nabla_1^2 \phi = -2\mu\alpha}$ now let us look at $\oint_{C_i} du_z = 0$

$$\oint_{C_i} du_z = \frac{1}{\mu} \oint \left(\frac{\partial \phi}{\partial y} dx - \frac{\partial \phi}{\partial x} dy \right) + \alpha \oint (y\,dx - x\,dy)$$

($\frac{\partial u_z}{\partial x} dx$, $\frac{\partial u_z}{\partial y} dy$)

From our previous lectures we related m, s, e_x, e_y at the boundary so that

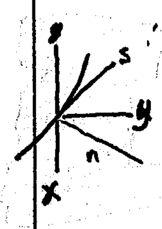
$$\frac{dy}{ds} = \frac{dx}{dn} \;①, \quad -\frac{dx}{ds} = \frac{dy}{dn} \;②$$

solve for dy & dx in ① & ② respectively

hence the first part of the integral

$$\oint \frac{\partial \phi}{\partial y} dx - \frac{\partial \phi}{\partial x} dy = \oint \left(-\frac{\partial \phi}{\partial y} \frac{dy}{dn} - \frac{\partial \phi}{\partial x} \frac{dx}{dn} \right) ds = \oint -\frac{\partial \phi}{\partial n} ds$$

$$\oint y\,dx - x\,dy = -\oint (y\,dx - x\,dy) = 2 \oint_{A_i} dA = 2A_i$$



$$\therefore \oint_{C_i} du_z = -\frac{1}{\mu} \oint \frac{\partial \phi}{\partial n} ds + 2\alpha A_i = 0 \quad \therefore \quad \boxed{\oint_{C_i} \frac{\partial \phi}{\partial n} ds = 2\alpha A_i \mu \text{ on cavity bdy}}$$

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} \frac{dx}{dn} + \frac{\partial \phi}{\partial y} \frac{dy}{dn}$$

$$-\sigma_{zy} \frac{dx}{dn} + \sigma_{zx} \frac{dy}{dn} = -s \cdot t_z$$

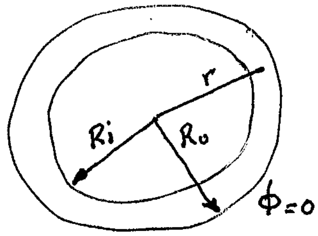
shear stress along the contour

$m \cdot e_x \quad m \cdot e_y$

$s \cdot e_y \quad -s \cdot e_x$

$$\oint \frac{\partial \phi}{\partial n} ds = \oint \tau\, ds = 2\alpha A_i \mu$$

$$\tau = \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2}$$



$$B\left(\frac{x^2+y^2}{R_o^2}-1\right)=\phi$$

$$\nabla^2\phi=\frac{B}{R_o^2}\cdot 4=-2\mu\alpha \quad \therefore \quad B=-\frac{2\mu\alpha R_o^2}{4}$$

$$\phi\big|_{r=R_o}=0=K_o \qquad \phi_{r=R_i}=B\left(\frac{R_i^2}{R_o^2}-1\right)=K_1$$

$$T=2\left[\iint_{A_{net}} B\left(\frac{r^2}{R_o^2}-1\right) r\,dr\,d\theta - \cancel{K_o}\cdot\pi R_o^2+\cancel{K_1}\,\pi R_i^2\right]$$

$$\cancel{K_o}\to 0 \qquad \cancel{K_1}\,\pi R_i^2 \to B\left(\frac{R_i^2}{R_o^2}-1\right)\pi R_i^2$$

$$=2\left[B\left\{\iint \frac{r^3\,dr\,d\theta}{R_o^2}-\iint r\,dr\,d\theta\right\}-0+B\,\frac{\pi R_i^4-\pi R_i^2R_o^2}{R_o^2}\right]$$

$$=2\left[-\frac{2\mu\alpha R_o^2}{4}\,\frac{r^4\cdot 2\pi}{4R_o^2}\Bigg|_{R_i}^{R_o}+\frac{2\mu\alpha R_o^2}{4}\,\frac{r^2}{2}\cdot 2\pi\Bigg|_{R_i}^{R_o}+B\,\frac{\pi R_i^4-\pi R_i^2R_o^2}{R_o^2}\right]$$

$$=2\left[-\frac{\pi\mu\alpha(R_o^4-R_i^4)}{4}+\frac{\pi}{2}\mu\alpha\left(R_o^4-R_o^2R_i^2\right)+\frac{-2\mu\alpha R_o^2}{4\cdot 2}\left[\frac{\pi R_i^4-\pi R_i^2R_o^2}{R_o^2}\right]\right]$$

$$2\left[-\frac{\pi\mu\alpha}{4}\left(R_o^4-R_i^4-2R_o^4+\cancel{2R_o^2R_i^2}+2R_i^4-\cancel{2R_i^2R_o^2}\right)\right]$$

$$R_i^4-R_o^4$$

$$T=\frac{2\pi\mu\alpha}{4}\left(R_o^4-R_i^4\right) \qquad J=\frac{\pi(R_o^4-R_i^4)}{2}$$

$$\alpha=\frac{T}{JG}$$

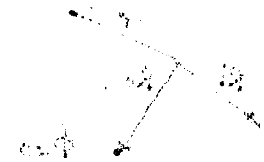
$$\phi=B\left(\frac{r^2}{R_o^2}-1\right) \qquad \frac{\partial\phi}{\partial n}=\frac{\partial\phi}{\partial r}=B\cdot\frac{2r}{R_o^2}$$

$$\oint\frac{\partial\phi}{\partial n}ds=\oint B\left(\frac{2r}{R_o^2}-\cancel{1}\right)r\,d\theta=\oint B\cdot\left(\frac{2R_i}{R_o^2}\right)\cdot R_i\,d\theta$$

$$B\,2\frac{R_i^2}{R_o^2}\cdot 2\pi=-2\mu\cdot\pi R_i^2\alpha=-2GA_i\alpha$$

$$-\frac{2\mu\alpha R_o^2}{4}\cdot\frac{2R_i^2}{R_o^2}\,2\pi=-\mu\alpha R_i^2\cdot 2\pi=-2\mu\pi R_i^2\alpha$$

✓



in which the integration must be extended over the length of the ring. Denoting by A the area bounded by the ring and observing that τ is the slope, so that $\tau\delta$ is the difference in level h of the two adjacent contour lines, we find, from (f),

$$dM_t = 2hA \qquad (g)$$

i.e., the torque corresponding to the elemental ring is given by twice the volume shaded in the figure. The total torque is given by the sum of these volumes, i.e., twice the volume between AB, the membrane AC and DB, and the flat plate CD. The conclusion follows similarly for several holes.

116 | Torsion of Thin Tubes

An approximate solution of the torsional problem for thin tubes can easily be obtained by using the membrane analogy. Let AB and CD (Fig. 172) represent the levels of the outer and the inner boundaries, and AC and DB be the cross section of the membrane stretched between these boundaries. In the case of a thin wall, we can neglect the variation in the slope of the membrane across the thickness and assume that AC and BD are straight lines. This is equivalent to the assumption that the shearing stresses are uniformly distributed over the thickness of the wall. Then denoting by h the difference in level of the two boundaries and by δ the variable thickness of the wall, the stress at any point, given by the slope of the membrane, is

$$\tau = \frac{h}{\delta} \qquad (a)$$

It is inversely proportional to the thickness of the wall and thus greatest where the thickness of the tube is least.

To establish the relation between the stress and the torque M_t, we apply again the membrane analogy and calculate the torque from the volume

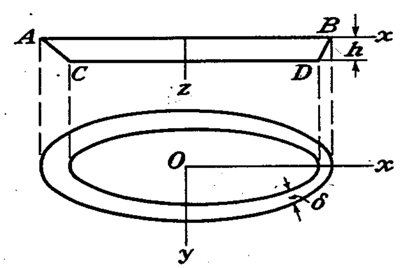


Fig. 172

$ACDB$. Then

$$M_t = 2Ah = 2A\delta\tau \qquad (b)$$

in which A is the mean of the areas enclosed by the outer and the inner boundaries of the cross section of the tube. From (b) we obtain a simple formula for calculating shearing stresses:

$$\tau = \frac{M_t}{2A\delta} \qquad (176)$$

For determining the angle of twist θ, we apply Eq. (160). Then

$$\int \tau\, ds = \frac{M_t}{2A}\int\frac{ds}{\delta} = 2G\theta A \qquad (c)$$

from which[1]

$$\theta = \frac{M_t}{4A^2G}\int\frac{ds}{\delta} \qquad (177)$$

In the case of a tube of uniform thickness, δ is constant and (177) gives

$$\theta = \frac{M_t s}{4A^2G\delta} \qquad (178)$$

in which s is the length of the centerline of the ring section of the tube.

If the tube has reentrant corners, as in the case represented in Fig. 173, a considerable stress concentration may take place at these corners. The maximum stress is larger than the stress given by Eq. (176) and depends on the radius a of the fillet of the reentrant corner (Fig. 173b). In calculating this maximum stress, we shall use the membrane analogy as we

[1] Equations (176) and (177) for thin tubular sections were obtained by R. Bredt, *VDI*, vol. 40, p. 815, 1896.

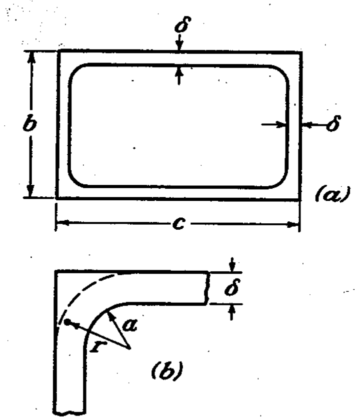


Fig. 173

Substituting $r = a$ we obtain the stress at the reentrant corner. This is plotted in Fig. 174. The other curve[1] (A in Fig. 174) was obtained by the method of finite differences, without the assumption that the membrane at the corner has the form of a surface of revolution. It confirms the accuracy of Eq. (i) for small fillets—say up to $a/\delta = \frac{1}{4}$. For larger fillets the values given by Eq. (i) are too high.

Let us consider now the case when the cross section of a tubular member has more than two boundaries. Taking, for example, the case shown in Fig. 175 and assuming that the thickness of the wall is very small, the shearing stresses in each portion of the wall, from the membrane analogy, are

$$\tau_1 = \frac{h_1}{\delta_1} \qquad \tau_2 = \frac{h_2}{\delta_2} \qquad \tau_3 = \frac{h_1 - h_2}{\delta_3} = \frac{\tau_1\delta_1 - \tau_2\delta_2}{\delta_3} \tag{j}$$

in which h_1 and h_2 are the levels of the inner boundaries CD and EF.[2]

The magnitude of the torque, determined by the volume $ACDEFB$, is

$$M_t = 2(A_1h_1 + A_2h_2) = 2A_1\delta_1\tau_1 + 2A_2\delta_2\tau_2 \tag{k}$$

where A_1 and A_2 are areas indicated in the figure by dotted lines.

Further equations for the solution of the problem are obtained by applying Eq. (160) to the closed curves indicated in the figure by dotted lines. Assuming that the thicknesses δ_1, δ_2, δ_3 are constant and denoting

[1] Huth, *op. cit.*
[2] It is assumed that the plates are guided so as to remain horizontal (see p. 331).

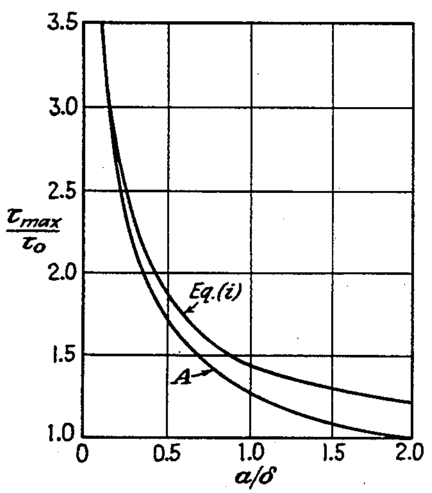


Fig. 174

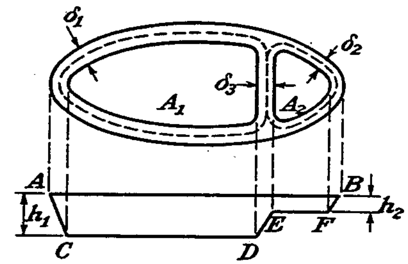


Fig. 175

by s_1, s_2, s_3 the lengths of corresponding dotted curves, we find, from Fig. 175,

$$\begin{aligned} \tau_1 s_1 + \tau_3 s_3 &= 2G\theta A_1 \\ \tau_2 s_2 - \tau_3 s_3 &= 2G\theta A_2 \end{aligned} \tag{l}$$

By using the last of the Eqs. (j) and Eqs. (k) and (l), we find the stresses τ_1, τ_2, τ_3 as functions of the torque:

$$\tau_1 = \frac{M_t[\delta_3 s_2 A_1 + \delta_2 s_3(A_1 + A_2)]}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3(A_1 + A_2)^2]} \tag{m}$$

$$\tau_2 = \frac{M_t[\delta_3 s_1 A_2 + \delta_1 s_3(A_1 + A_2)]}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3(A_1 + A_2)^2]} \tag{n}$$

$$\tau_3 = \frac{M_t(\delta_1 s_2 A_1 - \delta_2 s_1 A_2)}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3(A_1 + A_2)^2]} \tag{o}$$

In the case of a symmetrical cross section, $s_1 = s_2$, $\delta_1 = \delta_2$, $A_1 = A_2$, and $\tau_3 = 0$. In this case the torque is taken by the outer wall of the tube, and the web remains unstressed.[1]

To get the twist for any section like that shown in Fig. 175, one substitutes the values of the stresses in one of the Eqs. (l). Thus θ can be obtained as a function of the torque M_t.

117 | Screw Dislocations

In the two preceding articles, we have observed the requirement that w must be a single-valued function if the solution is to represent correctly a state of torsion. On reexamining Eqs. (149), (150), and (151), and the boundary condition (152), we can quickly see that it is possible to find states of stress corresponding to $\theta = 0$. The stress function ϕ is to satisfy Laplace's equation and to be constant on each boundary curve of the sec-

[1] The small stresses corresponding to the change in slope of the membrane across the thickness of the web are neglected in this derivation.

of material or a crack at such a corner. In the case of a rectangular *keyway*, therefore, a high stress concentration takes place at the reentrant corners at the bottom of the keyway. These high stresses can be reduced by rounding the corners.[1]

115 | Torsion of Hollow Shafts

So far the discussion has been limited to shafts whose cross sections are bounded by single curves. Let us consider now hollow shafts whose cross sections have two or more boundaries. The simplest problem of this kind is a hollow shaft with an inner boundary coinciding with one of the *stress lines* (see page 305) of the solid shaft, having the same boundary as the outer boundary of the hollow shaft.

Take, for instance, an elliptic cross section (Fig. 153). The stress function for the solid shaft is

$$\phi = \frac{a^2b^2F}{2(a^2+b^2)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right) \tag{a}$$

The curve

$$\frac{x^2}{(ak)^2}+\frac{y^2}{(bk)^2}=1 \tag{b}$$

is an ellipse that is geometrically similar to the outer boundary of the cross section. Along this ellipse the stress function (*a*) remains constant, and hence, for k less than unity, *this ellipse* is a stress line for the solid elliptic shaft. The shearing stress at any point of this line is in the direction of the tangent to the line. Imagine now a cylindrical surface generated by this stress line with its axis parallel to the axis of the shaft. Then, from the above conclusion regarding the direction of the shearing stresses, it follows that there will be no stresses acting across this cylindrical surface. We can imagine the material bounded by this surface removed without changing the stress distribution in the outer portion of the shaft. Hence the stress function (*a*) applies to the hollow shaft also.

For a given angle θ of twist the stresses in the hollow shaft are the same as in the corresponding solid shaft. But the torque will be smaller by the amount which in the case of the solid shaft is carried by the portion of the cross section corresponding to the hole. From Eq. (156) we see that the latter portion is in the ratio $k^4:1$ to the total torque. Hence, for the hollow shaft, instead of Eq. (156), we will have

$$\theta = \frac{M_t}{1-k^4}\frac{a^2+b^2}{\pi a^3b^3G}$$

and the stress function (*a*) becomes

$$\phi = -\frac{M_t}{\pi ab(1-k^4)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

The formula for the maximum stress will be

$$\tau_{\max} = \frac{2M_t}{\pi ab^2}\frac{1}{1-k^4}$$

In the membrane analogy the middle portion of the membrane, corresponding to the hole in the shaft (Fig. 171), must be replaced by the horizontal plate CD. We note that the uniform pressure distributed over the portion CFD of the membrane is statically equivalent to the pressure of the same magnitude uniformly distributed over the plate CD and the tensile forces S in the membrane acting along the edge of this plate are in equilibrium with the uniform load on the plate. Hence, in the case under consideration the same experimental soap-film method as before can be employed because the replacement of the portion CFD of the membrane by the plate CD causes no changes in the configuration and equilibrium conditions of the remaining portion of the membrane.

Let us consider now the more general case when the boundaries of the holes are no longer stress lines of the solid shaft. From the general theory of torsion we know (see Art. 104) that the stress function must be constant along each boundary, but these constants cannot be chosen arbitrarily. In discussing multiply-connected boundaries in two-dimensional problems, it was shown that recourse must be had to the expressions for the displacements, and the constants of integration should be found in such

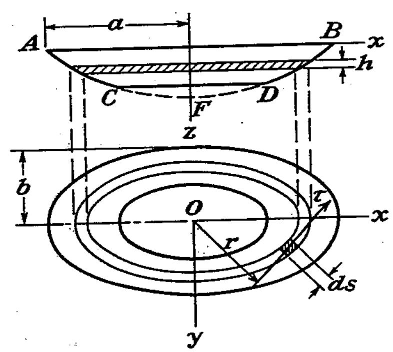


Fig. 171

[1] The stresses at the keyway were investigated by the soap-film method. See the paper by Griffith and Taylor, *loc. cit.*, p. 938.
For design formulas and charts, see R. E. Peterson, "Stress Concentration Design Factors," John Wiley & Sons, Inc., New York, 1953; also M. Nisida and M. Hondo, *Proc. Japan Nat. Congr. Appl. Mech.*, vol. 2, pp. 129–132, 1959.

a manner as to make these expressions single-valued. An analogous procedure is necessary in dealing with the torsion of hollow shafts. The constant values of the stress function along the boundaries should be determined in such a manner as to make the displacements single-valued. A sufficient number of equations for determining these constants will then be obtained.

From Eqs. (*b*) and (*d*) of Art. 104 we have

$$\tau_{xz} = G\left(\frac{\partial w}{\partial x} - \theta y\right) \qquad \tau_{yz} = G\left(\frac{\partial w}{\partial y} + \theta x\right) \tag{c}$$

Let us now calculate the integral

$$\int \tau \, ds \tag{d}$$

along each boundary. Using (*c*) and resolving the total stress into its components we find

$$\int \tau \, ds = \int \left(\tau_{xz}\frac{dx}{ds} + \tau_{yz}\frac{dy}{ds}\right) ds$$
$$= G \int \left(\frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy\right) - \theta G \int (y\,dx - x\,dy) \tag{174}$$

The first integral must vanish from the condition that the integration is taken round a closed curve and that w is a single-valued function. Hence,

$$\int \tau \, ds = \theta G \int (x\,dy - y\,dx)$$

The integral on the right side is equal to double the area enclosed. Then,

$$\int \tau \, ds = 2G\theta A \tag{175}$$

Thus, we must determine the constant values of the stress function along the boundaries of the holes so as to satisfy Eq. (175) for each boundary.

For any closed curve drawn in the cross section (lying wholly in the material) the first and second members of (174) represent the line integral of the tangential component of shear stress τ taken round the curve, and this may be called the shear circulation, by analogy with circulation in fluid dynamics. Then (175) still holds and may be called the shear-circulation theorem.

The significance of (175) for the membrane analogy was discussed on page 306. It indicates that in the membrane the level of each plate, such as the plate CD (Fig. 171), must be taken so that the vertical load on the plate is equal and opposite to the vertical component of the resultant of the tensile forces on the plate produced by the membrane. If the boundaries of the holes coincide with the stress lines of the corresponding solid shaft, the above condition is sufficient to ensure the equilibrium of the plates. In the general case this condition is not sufficient, and to keep the plates in equilibrium in a horizontal position special guiding devices become necessary. This makes the soap-film experiments for hollow shafts more complicated.

To remove this difficulty the following procedure may be adopted.[1] We make a hole in the plate corresponding to the outer boundary of the shaft. The interior boundaries, corresponding to the holes, are mounted each on a vertical sliding column so that their heights can be easily adjusted. Taking these heights arbitrarily and stretching the film over the boundaries, we obtain a surface that satisfies Eq. (150) and boundary conditions (152), but the Eq. (175) above generally will not be satisfied and the film does not represent the stress distribution in the hollow shaft. Repeating such an experiment as many times as the number of boundaries, each time with another adjustment of heights of the interior boundaries and taking measurements on the film each time, we obtain sufficient data for determining the correct values of the heights of the interior boundaries and can finally stretch the soap film in the required manner. This can be proved as follows: If i is the number of boundaries and $\phi_1, \phi_2, \ldots, \phi_i$ are the film surfaces obtained with i different adjustments of the heights of the boundaries, then a function

$$\phi = m_1\phi_1 + m_2\phi_2 + \cdots + m_i\phi_i \tag{e}$$

in which $m_1, m_2, \ldots, m_i$ are numerical factors, is also a solution of Eq. (150), provided that

$$m_1 + m_2 + \cdots + m_i = 1$$

Observing now that the shearing stress is equal to the slope of the membrane, and substituting (*e*) into Eqs. (175) we obtain i equations of the form

$$\int \frac{\partial \phi}{\partial n} ds = 2G\theta A_i$$

from which the i factors $m_1, m_2, \ldots, m_i$ can be obtained as functions of θ. Then the true stress function is obtained from (*e*).[2] This method was applied by Griffith and Taylor in determining stresses in a hollow circular shaft having a keyway in it. It was shown in this manner that the maximum stress can be considerably reduced and the strength of the shaft increased by throwing the bore in the shaft off center.

The torque in the shaft with one or more holes is obtained using twice the volume under the membrane and the flat plates. To see this we calculate the torque produced by the shearing stresses distributed over an elemental ring between two adjacent stress lines, as in Fig. 171 (now taken to represent an arbitrary hollow section). Denoting by δ the variable width of the ring and considering an element such as that shaded in the figure, the shearing force acting on this element is $\tau\delta\, ds$ and its moment with respect to O is $r\tau\delta\, ds$. Then the torque on the elemental ring is

$$dM_t = \int r\tau\delta \, ds \tag{f}$$

[1] Griffith and Taylor, *loc. cit.*, p. 938.

[2] Griffith and Taylor concluded from their experiments that instead of *constant-pressure* films it is more convenient to use *zero-pressure* films (see p. 306) in studying the stress distribution in hollow shafts. A detailed discussion of the calculation of factors $m_1, m_2, \ldots$ is given in their paper.

in which the integration must be extended over the length of the ring. Denoting by A the area bounded by the ring and observing that τ is the slope, so that $\tau\delta$ is the difference in level h of the two adjacent contour lines, we find, from (f),

$$dM_t = 2hA \tag{g}$$

i.e., the torque corresponding to the elemental ring is given by twice the volume shaded in the figure. The total torque is given by the sum of these volumes, i.e., twice the volume between AB, the membrane AC and DB, and the flat plate CD. The conclusion follows similarly for several holes.

116 | Torsion of Thin Tubes

An approximate solution of the torsional problem for thin tubes can easily be obtained by using the membrane analogy. Let AB and CD (Fig. 172) represent the levels of the outer and the inner boundaries, and AC and DB be the cross section of the membrane stretched between these boundaries. In the case of a thin wall, we can neglect the variation in the slope of the membrane across the thickness and assume that AC and BD are straight lines. This is equivalent to the assumption that the shearing stresses are uniformly distributed over the thickness of the wall. Then denoting by h the difference in level of the two boundaries and by δ the variable thickness of the wall, the stress at any point, given by the slope of the membrane, is

$$\tau = \frac{h}{\delta} \tag{a}$$

It is inversely proportional to the thickness of the wall and thus greatest where the thickness of the tube is least.

To establish the relation between the stress and the torque M_t, we apply again the membrane analogy and calculate the torque from the volume

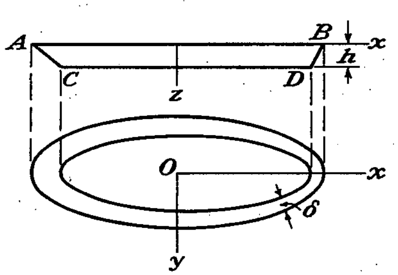


Fig. 172

$\tau = \frac{\partial\phi}{\partial n} = \frac{\phi_t - \phi_o}{n_t - n_o} = \frac{h}{\delta}$ Since δ is ⊥ to contours & $\phi_o = 0$ since $\phi = K\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$ & at x & y $\phi_o = 0$

$ACDB$. Then

$$M_t = 2Ah = 2A\delta\tau \tag{b}$$

in which A is the mean of the areas enclosed by the outer and the inner boundaries of the cross section of the tube. From (b) we obtain a simple formula for calculating shearing stresses:

$$\tau = \frac{M_t}{2A\delta} \tag{176}$$

For determining the angle of twist θ, we apply Eq. (160). Then

$$\int \tau\, ds = \frac{M_t}{2A}\int \frac{ds}{\delta} = 2G\theta A \tag{c}$$

from which[1]

$$\theta = \frac{M_t}{4A^2G}\int \frac{ds}{\delta} \tag{177}$$

In the case of a tube of uniform thickness, δ is constant and (177) gives

$$\theta = \frac{M_t s}{4A^2G\delta} \tag{178}$$

in which s is the length of the centerline of the ring section of the tube.

If the tube has reentrant corners, as in the case represented in Fig. 173, a considerable stress concentration may take place at these corners. The maximum stress is larger than the stress given by Eq. (176) and depends on the radius a of the fillet of the reentrant corner (Fig. 173b). In calculating this maximum stress, we shall use the membrane analogy as we

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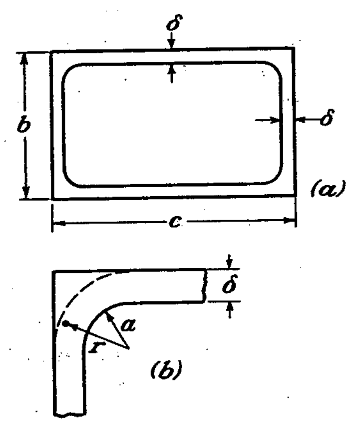

Fig. 173

did for the reentrant corners of rolled sections (Art. 112). The equation of the membrane at the reentrant corner may be taken in the form

$$\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} = -\frac{q}{S}$$

Replacing q/S by $2G\theta$ and noting that $\tau = -dz/dr$ (see Fig. 172), we find

$$\frac{d\tau}{dr} + \frac{1}{r}\tau = 2G\theta \qquad (d)$$

Assuming that we have a tube of constant thickness δ and denoting by τ_0 the stress at a considerable distance from the corner calculated from Eq. (176), we find, from (*c*),

$$2G\theta = \frac{\tau_0 s}{A}$$

Substituting in (*d*),

$$\frac{d\tau}{dr} + \frac{1}{r}\tau = \frac{\tau_0 s}{A} \qquad (e)$$

The general solution of this equation is

$$\tau = \frac{C}{r} + \frac{\tau_0 sr}{2A} \qquad (f)$$

Assuming that the projecting angles of the cross section have fillets with the radius a, as indicated in the figure, the constant of integration C can be determined from the equation

$$\int_a^{a+\delta} \tau\, dr = \tau_0\delta \qquad (g)$$

which follows from the hydrodynamical analogy (Art. 114), viz.: if an ideal fluid circulates in a channel having the shape of the ring cross section of the tubular member, the quantity of fluid passing each cross section of the channel must remain constant. Substituting expression (*f*) for τ into Eq. (*g*), and integrating, we find that

$$C = \tau_0\delta \frac{1 - (s/4A)(2a + \delta)}{\log_e (1 + \delta/a)}$$

and, from Eq. (*f*), that

$$\tau = \frac{\tau_0\delta}{r}\frac{1 - (s/4A)(2a + \delta)}{\log_e (1 + \delta/a)} + \frac{\tau_0 sr}{2A} \qquad (h)$$

For a thin-walled tube the ratios $s(2a + \delta)/A$, sr/A, will be small, and (*h*) reduces to

$$\tau = \frac{\tau_0\delta/r}{\log_e (1 + \delta/a)} \qquad (i)$$

Substituting $r = a$ we obtain the stress at the reentrant corner. This is plotted in Fig. 174. The other curve[1] (A in Fig. 174) was obtained by the method of finite differences, without the assumption that the membrane at the corner has the form of a surface of revolution. It confirms the accuracy of Eq. (*i*) for small fillets—say up to $a/\delta = \frac{1}{4}$. For larger fillets the values given by Eq. (*i*) are too high.

Let us consider now the case when the cross section of a tubular member has more than two boundaries. Taking, for example, the case shown in Fig. 175 and assuming that the thickness of the wall is very small, the shearing stresses in each portion of the wall, from the membrane analogy, are

$$\tau_1 = \frac{h_1}{\delta_1} \qquad \tau_2 = \frac{h_2}{\delta_2} \qquad \tau_3 = \frac{h_1 - h_2}{\delta_3} = \frac{\tau_1\delta_1 - \tau_2\delta_2}{\delta_3} \qquad (j)$$

in which h_1 and h_2 are the levels of the inner boundaries CD and EF.[2]

The magnitude of the torque, determined by the volume $ACDEFB$, is

$$M_t = 2(A_1h_1 + A_2h_2) = 2A_1\delta_1\tau_1 + 2A_2\delta_2\tau_2 \qquad (k)$$

where A_1 and A_2 are areas indicated in the figure by dotted lines.

Further equations for the solution of the problem are obtained by applying Eq. (160) to the closed curves indicated in the figure by dotted lines. Assuming that the thicknesses δ_1, δ_2, δ_3 are constant and denoting

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[2] It is assumed that the plates are guided so as to remain horizontal (see p. 331).

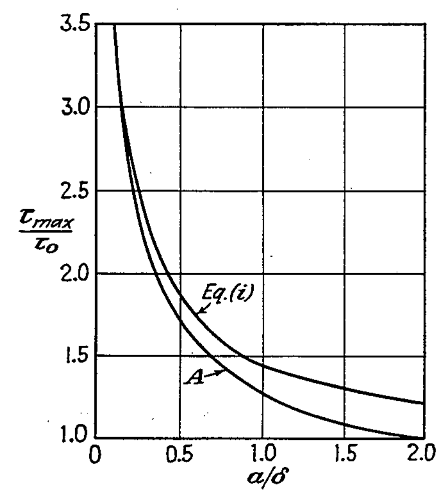

Fig. 174

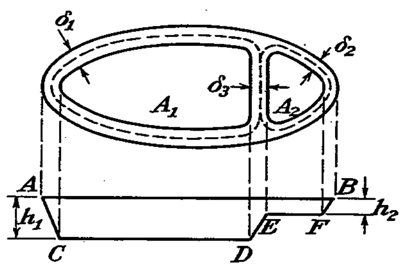

Fig. 175

by s_1, s_2, s_3 the lengths of corresponding dotted curves, we find, from Fig. 175,

$$\begin{aligned} \tau_1 s_1 + \tau_3 s_3 &= 2G\theta A_1 \\ \tau_2 s_2 - \tau_3 s_3 &= 2G\theta A_2 \end{aligned} \qquad (l)$$

By using the last of the Eqs. (j) and Eqs. (k) and (l), we find the stresses τ_1, τ_2, τ_3 as functions of the torque:

$$\tau_1 = \frac{M_t[\delta_3 s_2 A_1 + \delta_2 s_3(A_1 + A_2)]}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3(A_1 + A_2)^2]} \qquad (m)$$

$$\tau_2 = \frac{M_t[\delta_3 s_1 A_2 + \delta_1 s_3(A_1 + A_2)]}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3(A_1 + A_2)^2]} \qquad (n)$$

$$\tau_3 = \frac{M_t(\delta_1 s_2 A_1 - \delta_2 s_1 A_2)}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3(A_1 + A_2)^2]} \qquad (o)$$

In the case of a symmetrical cross section, $s_1 = s_2$, $\delta_1 = \delta_2$, $A_1 = A_2$, and $\tau_3 = 0$. In this case the torque is taken by the outer wall of the tube, and the web remains unstressed.[1]

To get the twist for any section like that shown in Fig. 175, one substitutes the values of the stresses in one of the Eqs. (l). Thus θ can be obtained as a function of the torque M_t.

117 | Screw Dislocations

In the two preceding articles, we have observed the requirement that w must be a single-valued function if the solution is to represent correctly a state of torsion. On reexamining Eqs. (149), (150), and (151), and the boundary condition (152), we can quickly see that it is possible to find states of stress corresponding to $\theta = 0$. The stress function ϕ is to satisfy Laplace's equation and to be constant on each boundary curve of the section.

[1] The small stresses corresponding to the change in slope of the membrane across the thickness of the web are neglected in this derivation.

But we must use w rather than the form $\theta\psi(x,y)$ of Eq. (b) on page 293. Then Eqs. (f) of page 295 are replaced by

$$\frac{\partial \phi}{\partial y} = G\frac{\partial w}{\partial x} \qquad -\frac{\partial \phi}{\partial x} = G\frac{\partial w}{\partial y} \qquad (a)$$

These are Cauchy-Riemann equations (see page 171) for the functions Gw and ϕ. Therefore, $Gw + i\phi$ is an analytic function of $x + iy$. Thus,

$$Gw + i\phi = f(x + iy) \qquad (b)$$

Once the function f is chosen, we have a definite state, in which w will be the only nonzero displacement component.

Let r, ψ now represent polar coordinates in the cross section. The choice

$$f(x + iy) = -iA \log (x + iy) = A\psi - iA \log r \qquad (c)$$

where A is a real constant, is of particular interest in the dislocation theory of plastic deformation (see Art. 34). From (b), we now have

$$Gw = A\psi \qquad \phi = -A \log r \qquad (d)$$

The corresponding shear stress is in the circumferential direction and is given by the polar components

$$\tau_{z\psi} = -\frac{\partial \phi}{\partial r} = \frac{A}{r} \qquad \tau_{zr} = 0 \qquad (e)$$

Any cylindrical boundary surface r = constant is free from loading. But the displacement w is not continuous. We can apply the solution to a hollow circular cylinder $a < r < b$ as in Fig. 176, which has an axial cut. One face is moved axially along the other by the uniform relative displacement

$$w(r,2\pi) - w(r,0) = \frac{2\pi A}{G} \qquad (f)$$

obtained from the first of (d). The stress (e) can be regarded as induced

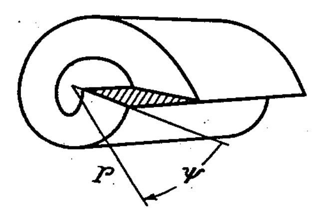

Fig. 176

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The shear stress on the $\frac{1}{2}$-in wall is

$$(\tau)_{1/2\,\text{in}} = \frac{T}{2\bar{A}t} = \frac{50(10^3)}{(2)(43.8)(1/2)} = 1140 \text{ psi}$$

and on the $\frac{1}{4}$-in walls

$$\tau_{1/4\,\text{in}} = \frac{50(10^3)}{(2)(43.8)(1/4)} = 2280 \text{ psi}$$

In evaluating the integral $\int ds/t$ note that t is constant for each wall; thus

$$\int \frac{ds}{t} = \frac{2(8 - \frac{1}{8} - \frac{1}{4})}{\frac{1}{4}} + \frac{6 - \frac{1}{8} - \frac{1}{8}}{\frac{1}{4}} + \frac{6 - \frac{1}{8} - \frac{1}{8}}{\frac{1}{2}} = 95.5$$

5.2.2 MULTIPLE CELL SECTIONS IN TORSION

Figure 5.2-5 shows a closed, thin-walled, multiple cellular cross section transmitting a torque T. Conventional practice is to treat the torsion contribution of each cell separately and combine the results using superposition. This is done subject to the condition that the angle of twist of each cell is the same.

The torsion contribution of the ith cell is given by Eq. (5.2-3) as $T_i = 2q_i\bar{A}_i$. Summing the contribution of the n cells yields

$$T = \sum_{i=1}^{n} T_i = 2\sum_{i=1}^{n} q_i\bar{A}_i \qquad [5.2\text{-}8]$$

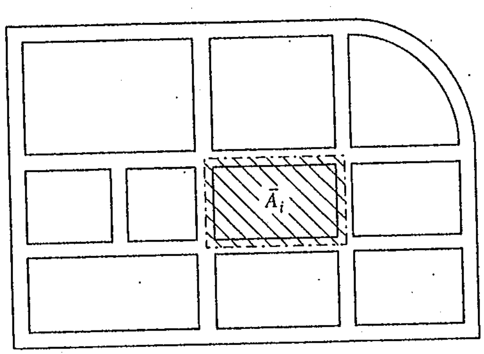

Figure 5.2-5 Multiple cell cross section.

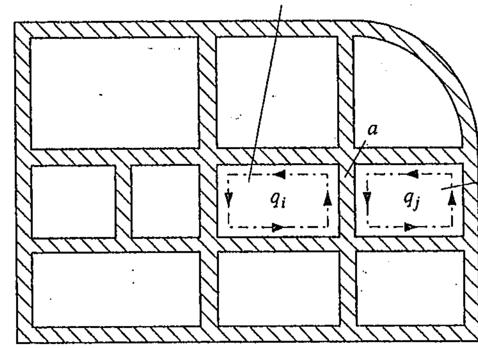

Figure 5.2-6 Positive shear flow in cells i and j.

where we have n unknown values of q_i. The condition that each cell has the same angle of twist θ establishes n equations of twist for each cell. Equation (5.2-5) applies to a single cell where q is the same for all walls of the cell. For a cell within a multiple cell structure, the shear flow q in each wall is the superposition of the shear flow of adjacent cells. Thus in this case, q can vary throughout the walls of a given cell and q must be placed within the integral of Eq. (5.2-5). As an example, the angle of twist of cell i is

$$\theta = \frac{(1+\nu)L}{E\bar{A}_i}\left(\int \frac{q}{t}\,ds\right)_i \qquad \textbf{[5.2-9]}$$

The sign of the shear flow is extremely important. Here we will consider counterclockwise flow within a cell and its walls to be positive. For example, the twist equation of wall a of cell i shown in Fig. 5.2-6 is $(q_a)_i = q_i - q_j$. For the twist equation of wall a of cell j, $(q_a)_j = q_j - q_i$.

Equation (5.2-9) represents n equations, one for each cell. This, together with Eq. (5.2-8), represents $n + 1$ equations in terms of the $n + 1$ unknowns; $\theta, q_1, q_2, \ldots, q_n$. Once the q_i are found the shear flow in each wall can be determined. Dividing the shear flow in a specific wall by the wall thickness yields the shear stress in that wall.

Example 5.2-3 A steel tube 1.5 m long has the cross section shown in Fig. 5.2-7. The tube is transmitting a torque of 200 N·m. Determine the average shear stress in each wall and the angle of twist of the tube. E= 210 GPa, $\nu = 0.29$.

Solution:

From Eq. (5.2-8),

$$T = 2\left[q_1\left(\frac{\pi}{4}a^2\right) + q_2(a)(c-a) + q_3(bc)\right]$$

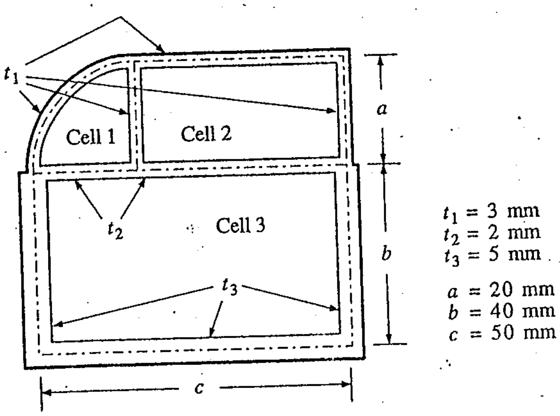

Figure 5.2-7

Substitution of values gives

$$200 = 2\left[q_1\left(\frac{\pi}{4}0.02^2\right) + q_2(0.02)(0.05 - 0.02) + q_3(0.04)(0.05)\right]$$

Simplifying results in

$$3.142q_1 + 6q_2 + 20q_3 = 1(10^6) \qquad \textbf{[a]}$$

For the angle of twist of each cell, consider cell 1 first. Starting at the radial wall and considering each wall of the cell in a clockwise order, Eq. (5.2-9) for cell 1 is

$$\theta = \frac{(1+\nu)L}{E[(\pi/4)a^2]}\left[\frac{q_1}{t_1}\left(\frac{\pi}{2}\right)(a) + \frac{q_1 - q_2}{t_1}(a) + \frac{q_1 - q_3}{t_2}(a)\right]$$

$$= \frac{(1+\nu)L}{E}\frac{4}{\pi(0.02^2)}\left[\frac{q_1}{0.003}\left(\frac{\pi}{2}\right)(0.02) + \frac{q_1 - q_2}{0.003}(0.02) + \frac{q_1 - q_3}{0.002}(0.02)\right]$$

Placing the term involving ν, L, and E on the left side of this equation together with θ and simplifying gives

$$\frac{\theta E}{(1+\nu)L} = (8.638q_1 - 2.122q_2 - 3.183q_3)(10^4)$$

Let

$$C = \frac{\theta E}{(1+\nu)L}(10^{-4}) \qquad \textbf{[b]}$$

Thus Eq. (5.2-9) for cell 1 reduces to

$$8.638q_1 - 2.122q_2 - 3.183q_3 - C = 0 \qquad \textbf{[c]}$$

For cell 2, starting at the top wall and moving clockwise around the cell

$$\frac{\theta E}{(1+\nu)L} = \frac{1}{(a)(c-a)}\left[\frac{q_2}{t_1}(c-a) + \frac{q_2}{t_1}(a) + \frac{q_2 - q_3}{t_2}(c-a) + \frac{q_2 - q_1}{t_1}(a)\right]$$

$$= \frac{1}{(0.02)(0.05-0.02)}\left[\frac{q_2}{0.003}(0.05-0.02) + \frac{q_2}{0.003}(0.02) + \frac{q_2 - q_3}{0.002}(0.05-0.02) + \frac{q_2 - q_1}{0.003}(0.02)\right]$$

Using the term C as before, we obtain

$$-1.111q_1 + 6.389q_2 - 2.5q_3 - C = 0 \qquad \textbf{[d]}$$

For cell 3, combining the right, bottom, and left walls, and then separating the top two sections gives

$$\frac{\theta E}{(1+\nu)L} = \frac{1}{bc}\left[\frac{q_3}{t_3}(2b+c) + \frac{q_3 - q_1}{t_2}(a) + \frac{q_3 - q_2}{t_2}(c-a)\right]$$

$$= \frac{1}{(0.04)(0.05)}\left[\frac{q_3}{0.005}(0.08+0.05) + \frac{q_3 - q_1}{0.002}(0.02) + \frac{q_3 - q_2}{0.002}(0.05-0.02)\right]$$

As with the other cells, simplification yields

$$-0.5q_1 - 0.75q_2 + 1.754q_3 - C = 0 \qquad \textbf{[e]}$$

Solving Eqs. (*a*) and (*c*) to (*e*) simultaneously yields

$$q_1 = 24.62 \text{ kN/m}$$

$$q_2 = 25.11 \text{ kN/m}$$

$$q_3 = 38.60 \text{ kN/m}$$

$$C = 36.56 \text{ kN/m}^3$$

Figure 5.2-8 shows the shear flow and shear stress definitions for this example. The shear stresses are:

$$\tau_1 = \frac{q_1}{t_1} = \frac{24.62}{0.003} = 8.21(10^3)\text{kN/m}^2 = 8.21\text{MPa}$$

$$\tau_2 = \frac{q_2}{t_1} = \frac{25.11}{0.003} = 8.37(10^3)\text{kN/m}^2 = 8.37\text{MPa}$$

$$\tau_3 = \frac{q_3}{t_3} = \frac{38.60}{0.005} = 7.72(10^3)\text{kN/m}^2 = 7.72\text{MPa}$$

$$\tau_4 = \frac{q_3 - q_1}{t_2} = \frac{38.60 - 24.62}{0.002} = 6.99(10^3)\text{kN/m}^2 = 6.99\text{MPa}$$

$$\tau_5 = \frac{q_2 - q_1}{t_1} = \frac{25.11 - 24.62}{0.003} = 0.163(10^3)\text{kN/m}^2 = 0.163\text{MPa}$$

$$\tau_6 = \frac{q_3 - q_2}{t_2} = \frac{38.60 - 25.11}{0.002} = 6.75(10^3)\text{kN/m}^2 = 6.75\text{MPa}$$

The angle of twist is found from Eq. (*b*) as

$$\theta = \frac{(1+\nu)L(10^4)}{E}C$$

$$= \frac{(1+0.29)(1.5)(10^4)}{210(10^9)}36.56(10^3) = 3.37(10^{-3})\text{ rad} = 0.193°$$

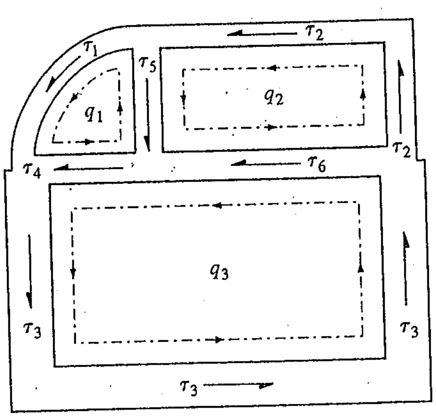

Figure 5.2-8 Shear flow for Example 5.2-3.

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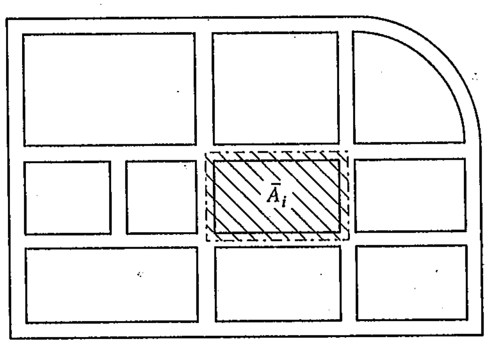

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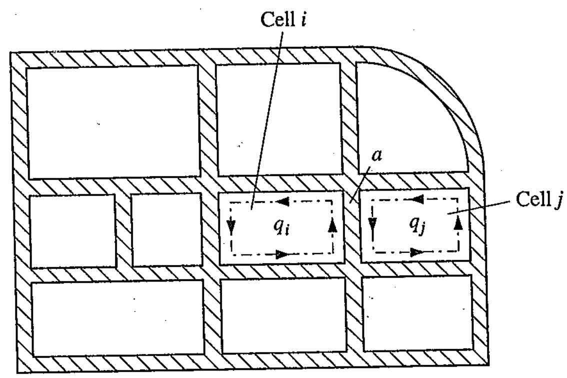

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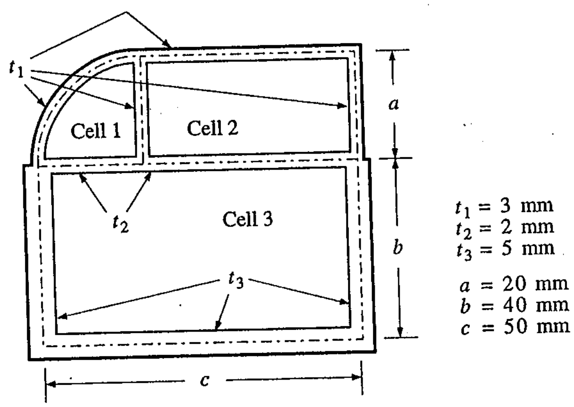

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$$\frac{\theta E}{(1+\nu)L} = \frac{1}{(a)(c-a)}\left[\frac{q_2}{t_1}(c-a) + \frac{q_2}{t_1}(a) + \frac{q_2-q_3}{t_2}(c-a) + \frac{q_2-q_1}{t_1}(a)\right]$$

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$$\left. + \frac{q_2-q_3}{0.002}(0.05-0.02) + \frac{q_2-q_1}{0.003}(0.02)\right]$$

Using the term C as before, we obtain

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$$= \frac{1}{(0.04)(0.05)}\left[\frac{q_3}{0.005}(0.08+0.05) + \frac{q_3-q_1}{0.002}(0.02) + \frac{q_3-q_2}{0.002}(0.05-0.02)\right]$$

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Solving Eqs. (*a*) and (*c*) to (*e*) simultaneously yields

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Figure 5.2-8 shows the shear flow and shear stress definitions for this example. The shear stresses are:

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The angle of twist is found from Eq. (b) as

$$\theta = \frac{(1+\nu)L(10^4)}{E}C$$

$$= \frac{(1+0.29)(1.5)(10^4)}{210(10^9)}36.56(10^3) = 3.37(10^{-3})\text{ rad} = 0.193°$$

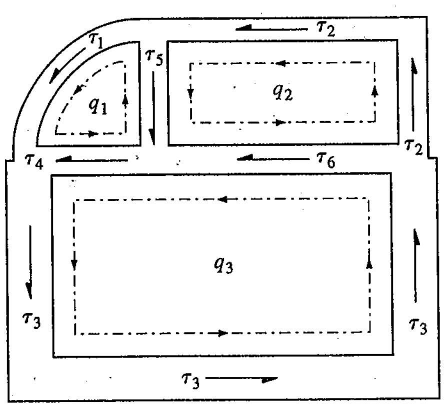

Figure 5.2-8 Shear flow for Example 5.2-3.

of material or a crack at such a corner. In the case of a rectangular *keyway*, therefore, a high stress concentration takes place at the reentrant corners at the bottom of the keyway. These high stresses can be reduced by rounding the corners.[1]

115 | Torsion of Hollow Shafts

So far the discussion has been limited to shafts whose cross sections are bounded by single curves. Let us consider now hollow shafts whose cross sections have two or more boundaries. The simplest problem of this kind is a hollow shaft with an inner boundary coinciding with one of the *stress lines* (see page 305) of the solid shaft, having the same boundary as the outer boundary of the hollow shaft.

Take, for instance, an elliptic cross section (Fig. 153). The stress function for the solid shaft is

$$\phi = \frac{a^2b^2F}{2(a^2+b^2)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right) \tag{a}$$

The curve

$$\frac{x^2}{(ak)^2}+\frac{y^2}{(bk)^2}=1 \tag{b}$$

is an ellipse that is geometrically similar to the outer boundary of the cross section. Along this ellipse the stress function (a) remains constant, and hence, for k less than unity, *this ellipse* is a stress line for the solid elliptic shaft. The shearing stress at any point of this line is in the direction of the tangent to the line. Imagine now a cylindrical surface generated by this stress line with its axis parallel to the axis of the shaft. Then, from the above conclusion regarding the direction of the shearing stresses, it follows that there will be no stresses acting across this cylindrical surface. We can imagine the material bounded by this surface removed without changing the stress distribution in the outer portion of the shaft. Hence the stress function (a) applies to the hollow shaft also.

For a given angle θ of twist the stresses in the hollow shaft are the same as in the corresponding solid shaft. But the torque will be smaller by the amount which in the case of the solid shaft is carried by the portion of the cross section corresponding to the hole. From Eq. (156) we see that the latter portion is in the ratio $k^4:1$ to the total torque. Hence, for the hollow shaft, instead of Eq. (156), we will have

$$\theta = \frac{M_t}{1-k^4}\,\frac{a^2+b^2}{\pi a^3b^3G}$$

and the stress function (a) becomes

$$\phi = -\frac{M_t}{\pi ab(1-k^4)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

The formula for the maximum stress will be

$$\tau_{\max} = \frac{2M_t}{\pi ab^2}\,\frac{1}{1-k^4}$$

In the membrane analogy the middle portion of the membrane, corresponding to the hole in the shaft (Fig. 171), must be replaced by the horizontal plate CD. We note that the uniform pressure distributed over the portion CFD of the membrane is statically equivalent to the pressure of the same magnitude uniformly distributed over the plate CD and the tensile forces S in the membrane acting along the edge of this plate are in equilibrium with the uniform load on the plate. Hence, in the case under consideration the same experimental soap-film method as before can be employed because the replacement of the portion CFD of the membrane by the plate CD causes no changes in the configuration and equilibrium conditions of the remaining portion of the membrane.

Let us consider now the more general case when the boundaries of the holes are no longer stress lines of the solid shaft. From the general theory of torsion we know (see Art. 104) that the stress function must be constant along each boundary, but these constants cannot be chosen arbitrarily. In discussing multiply-connected boundaries in two-dimensional problems, it was shown that recourse must be had to the expressions for the displacements, and the constants of integration should be found in such

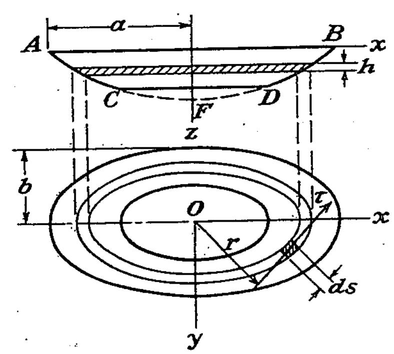

Fig. 171

[1] The stresses at the keyway were investigated by the soap-film method. See the paper by Griffith and Taylor, *loc. cit.*, p. 938.

For design formulas and charts, see R. E. Peterson, "Stress Concentration Design Factors," John Wiley & Sons, Inc., New York, 1953; also M. Nisida and M. Hondo, *Proc. Japan Nat. Congr. Appl. Mech.*, vol. 2, pp. 129–132, 1959.

a manner as to make these expressions single-valued. An analogous procedure is necessary in dealing with the torsion of hollow shafts. The constant values of the stress function along the boundaries should be determined in such a manner as to make the displacements single-valued. A sufficient number of equations for determining these constants will then be obtained.

From Eqs. (b) and (d) of Art. 104 we have

$$\tau_{xz} = G\left(\frac{\partial w}{\partial x} - \theta y\right) \qquad \tau_{yz} = G\left(\frac{\partial w}{\partial y} + \theta x\right) \tag{c}$$

Let us now calculate the integral

$$\int \tau \, ds \tag{d}$$

along each boundary. Using (c) and resolving the total stress into its components we find

$$\int \tau \, ds = \int \left(\tau_{xz}\frac{dx}{ds} + \tau_{yz}\frac{dy}{ds}\right) ds$$
$$= G \int \left(\frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy\right) - \theta G \int (y\,dx - x\,dy) \tag{174}$$

The first integral must vanish from the condition that the integration is taken round a closed curve and that w is a single-valued function. Hence,

$$\int \tau \, ds = \theta G \int (x\,dy - y\,dx)$$

The integral on the right side is equal to double the area enclosed. Then,

$$\int \tau \, ds = 2G\theta A \tag{175}$$

Thus, we must determine the constant values of the stress function along the boundaries of the holes so as to satisfy Eq. (175) for each boundary.

For any closed curve drawn in the cross section (lying wholly in the material) the first and second members of (174) represent the line integral of the tangential component of shear stress τ taken round the curve, and this may be called the shear circulation, by analogy with circulation in fluid dynamics. Then (175) still holds and may be called the shear-circulation theorem.

The significance of (175) for the membrane analogy was discussed on page 306. It indicates that in the membrane the level of each plate, such as the plate CD (Fig. 171), must be taken so that the vertical load on the plate is equal and opposite to the vertical component of the resultant of the tensile forces on the plate produced by the membrane. If the boundaries of the holes coincide with the stress lines of the corresponding solid shaft, the above condition is sufficient to ensure the equilibrium of the plates. In the general case this condition is not sufficient, and to keep the plates in equilibrium in a horizontal position special guiding devices become necessary. This makes the soap-film experiments for hollow shafts more complicated.

To remove this difficulty the following procedure may be adopted.[1] We make a hole in the plate corresponding to the outer boundary of the shaft. The interior boundaries, corresponding to the holes, are mounted each on a vertical sliding column so that their heights can be easily adjusted. Taking these heights arbitrarily and stretching the film over the boundaries, we obtain a surface that satisfies Eq. (150) and boundary conditions (152), but the Eq. (175) above generally will not be satisfied and the film does not represent the stress distribution in the hollow shaft. Repeating such an experiment as many times as the number of boundaries, each time with another adjustment of heights of the interior boundaries and taking measurements on the film each time, we obtain sufficient data for determining the correct values of the heights of the interior boundaries and can finally stretch the soap film in the required manner. This can be proved as follows: If i is the number of boundaries and $\phi_1, \phi_2, \ldots, \phi_i$ are the film surfaces obtained with i different adjustments of the heights of the boundaries, then a function

$$\phi = m_1\phi_1 + m_2\phi_2 + \cdots + m_i\phi_i \tag{e}$$

in which $m_1, m_2, \ldots, m_i$ are numerical factors, is also a solution of Eq. (150), provided that

$$m_1 + m_2 + \cdots + m_i = 1$$

Observing now that the shearing stress is equal to the slope of the membrane, and substituting (e) into Eqs. (175) we obtain i equations of the form

$$\int \frac{\partial \phi}{\partial n} ds = 2G\theta A_i$$

from which the i factors $m_1, m_2, \ldots, m_i$ can be obtained as functions of θ. Then the true stress function is obtained from (e).[2] This method was applied by Griffith and Taylor in determining stresses in a hollow circular shaft having a keyway in it. It was shown in this manner that the maximum stress can be considerably reduced and the strength of the shaft increased by throwing the bore in the shaft off center.

The torque in the shaft with one or more holes is obtained using twice the volume under the membrane and the flat plates. To see this we calculate the torque produced by the shearing stresses distributed over an elemental ring between two adjacent stress lines, as in Fig. 171 (now taken to represent an arbitrary hollow section). Denoting by δ the variable width of the ring and considering an element such as that shaded in the figure, the shearing force acting on this element is $\tau\delta\, ds$ and its moment with respect to O is $r\tau\delta\, ds$. Then the torque on the elemental ring is

$$dM_t = \int r\tau\delta \, ds \tag{f}$$

[1] Griffith and Taylor, *loc. cit.*, p. 938.

[2] Griffith and Taylor concluded from their experiments that instead of *constant-pressure* films it is more convenient to use *zero-pressure* films (see p. 306) in studying the stress distribution in hollow shafts. A detailed discussion of the calculation of factors $m_1, m_2, \ldots$ is given in their paper.

in which the integration must be extended over the length of the ring. Denoting by A the area bounded by the ring and observing that τ is the slope, so that $\tau\delta$ is the difference in level h of the two adjacent contour lines, we find, from (f),

$$dM_t = 2hA \tag{g}$$

i.e., the torque corresponding to the elemental ring is given by twice the volume shaded in the figure. The total torque is given by the sum of these volumes, i.e., twice the volume between AB, the membrane AC and DB, and the flat plate CD. The conclusion follows similarly for several holes.

116 | Torsion of Thin Tubes

An approximate solution of the torsional problem for thin tubes can easily be obtained by using the membrane analogy. Let AB and CD (Fig. 172) represent the levels of the outer and the inner boundaries, and AC and DB be the cross section of the membrane stretched between these boundaries. In the case of a thin wall, we can neglect the variation in the slope of the membrane across the thickness and assume that AC and BD are straight lines. This is equivalent to the assumption that the shearing stresses are uniformly distributed over the thickness of the wall. Then denoting by h the difference in level of the two boundaries and by δ the variable thickness of the wall, the stress at any point, given by the slope of the membrane, is

$$\tau = \frac{h}{\delta} \tag{a}$$

It is inversely proportional to the thickness of the wall and thus greatest where the thickness of the tube is least.

To establish the relation between the stress and the torque M_t, we apply again the membrane analogy and calculate the torque from the volume

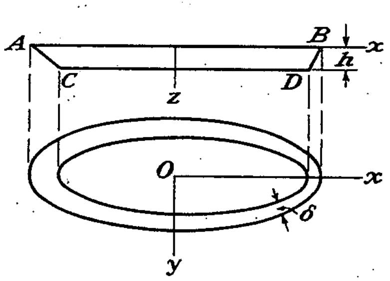

Fig. 172

$ACDB$. Then

$$M_t = 2Ah = 2A\delta\tau \tag{b}$$

in which A is the mean of the areas enclosed by the outer and the inner boundaries of the cross section of the tube. From (b) we obtain a simple formula for calculating shearing stresses:

$$\tau = \frac{M_t}{2A\delta} \tag{176}$$

For determining the angle of twist θ, we apply Eq. (160). Then

$$\tau\, ds = \frac{M_t}{2A}\int\frac{ds}{\delta} = 2G\theta A \tag{c}$$

from which[1]

$$\theta = \frac{M_t}{4A^2G}\int\frac{ds}{\delta} \tag{177}$$

In the case of a tube of uniform thickness, δ is constant and (177) gives

$$\theta = \frac{M_t s}{4A^2G\delta} \tag{178}$$

in which s is the length of the centerline of the ring section of the tube.

If the tube has reentrant corners, as in the case represented in Fig. 173, a considerable stress concentration may take place at these corners. The maximum stress is larger than the stress given by Eq. (176) and depends on the radius a of the fillet of the reentrant corner (Fig. 173b). In calculating this maximum stress, we shall use the membrane analogy as we

[1] Equations (176) and (177) for thin tubular sections were obtained by R. Bredt, *VDI*, vol. 40, p. 815, 1896.

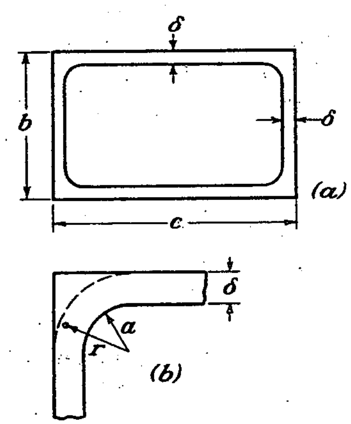

Fig. 173

did for the reentrant corners of rolled sections (Art. 112). The equation of the membrane at the reentrant corner may be taken in the form

$$\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} = -\frac{q}{S}$$

Replacing q/S by $2G\theta$ and noting that $\tau = -dz/dr$ (see Fig. 172), we find

$$\frac{d\tau}{dr} + \frac{1}{r}\tau = 2G\theta \tag{d}$$

Assuming that we have a tube of constant thickness δ and denoting by τ_0 the stress at a considerable distance from the corner calculated from Eq. (176), we find, from (c),

$$2G\theta = \frac{\tau_0 s}{A}$$

Substituting in (d),

$$\frac{d\tau}{dr} + \frac{1}{r}\tau = \frac{\tau_0 s}{A} \tag{e}$$

The general solution of this equation is

$$\tau = \frac{C}{r} + \frac{\tau_0 sr}{2A} \tag{f}$$

Assuming that the projecting angles of the cross section have fillets with the radius a, as indicated in the figure, the constant of integration C can be determined from the equation

$$\int_a^{a+\delta} \tau\, dr = \tau_0\delta \tag{g}$$

which follows from the hydrodynamical analogy (Art. 114), viz.: if an ideal fluid circulates in a channel having the shape of the ring cross section of the tubular member, the quantity of fluid passing each cross section of the channel must remain constant. Substituting expression (f) for τ into Eq. (g), and integrating, we find that

$$C = \tau_0\delta \frac{1 - (s/4A)(2a + \delta)}{\log_e (1 + \delta/a)}$$

and, from Eq. (f), that

$$\tau = \frac{\tau_0\delta}{r}\frac{1 - (s/4A)(2a + \delta)}{\log_e (1 + \delta/a)} + \frac{\tau_0 sr}{2A} \tag{h}$$

For a thin-walled tube the ratios $s(2a + \delta)/A$, sr/A, will be small, and (h) reduces to

$$\tau = \frac{\tau_0\delta/r}{\log_e (1 + \delta/a)} \tag{i}$$

Substituting $r = a$ we obtain the stress at the reentrant corner. This is plotted in Fig. 174. The other curve[1] (A in Fig. 174) was obtained by the method of finite differences, without the assumption that the membrane at the corner has the form of a surface of revolution. It confirms the accuracy of Eq. (i) for small fillets—say up to $a/\delta = \frac{1}{4}$. For larger fillets the values given by Eq. (i) are too high.

Let us consider now the case when the cross section of a tubular member has more than two boundaries. Taking, for example, the case shown in Fig. 175 and assuming that the thickness of the wall is very small, the shearing stresses in each portion of the wall, from the membrane analogy, are

$$\tau_1 = \frac{h_1}{\delta_1} \qquad \tau_2 = \frac{h_2}{\delta_2} \qquad \tau_3 = \frac{h_1 - h_2}{\delta_3} = \frac{\tau_1\delta_1 - \tau_2\delta_2}{\delta_3} \tag{j}$$

in which h_1 and h_2 are the levels of the inner boundaries CD and EF.[2]

The magnitude of the torque, determined by the volume $ACDEFB$, is

$$M_t = 2(A_1h_1 + A_2h_2) = 2A_1\delta_1\tau_1 + 2A_2\delta_2\tau_2 \tag{k}$$

where A_1 and A_2 are areas indicated in the figure by dotted lines.

Further equations for the solution of the problem are obtained by applying Eq. (160) to the closed curves indicated in the figure by dotted lines. Assuming that the thicknesses δ_1, δ_2, δ_3 are constant and denoting

[1] Huth, *op. cit.*

[2] It is assumed that the plates are guided so as to remain horizontal (see p. 331).

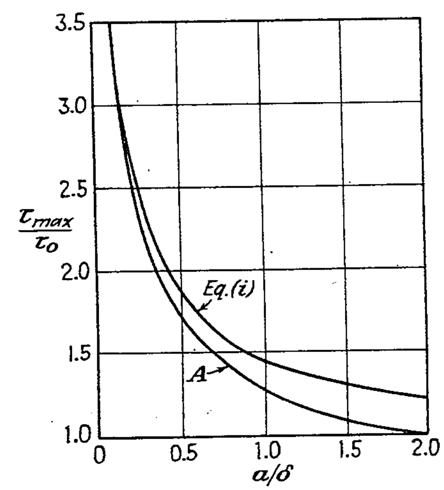

Fig. 174

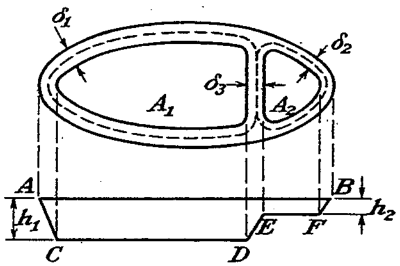

Fig. 175

by s_1, s_2, s_3 the lengths of corresponding dotted curves, we find, from Fig. 175,

$$\begin{aligned} \tau_1 s_1 + \tau_3 s_3 &= 2G\theta A_1 \\ \tau_2 s_2 - \tau_3 s_3 &= 2G\theta A_2 \end{aligned} \qquad (l)$$

By using the last of the Eqs. (*j*) and Eqs. (*k*) and (*l*), we find the stresses τ_1, τ_2, τ_3 as functions of the torque:

$$\tau_1 = \frac{M_t[\delta_3 s_2 A_1 + \delta_2 s_3 (A_1 + A_2)]}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3 (A_1 + A_2)^2]} \qquad (m)$$

$$\tau_2 = \frac{M_t[\delta_3 s_1 A_2 + \delta_1 s_3 (A_1 + A_2)]}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3 (A_1 + A_2)^2]} \qquad (n)$$

$$\tau_3 = \frac{M_t(\delta_1 s_2 A_1 - \delta_2 s_1 A_2)}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3 (A_1 + A_2)^2]} \qquad (o)$$

In the case of a symmetrical cross section, $s_1 = s_2$, $\delta_1 = \delta_2$, $A_1 = A_2$, and $\tau_3 = 0$. In this case the torque is taken by the outer wall of the tube, and the web remains unstressed.[1]

To get the twist for any section like that shown in Fig. 175, one substitutes the values of the stresses in one of the Eqs. (*l*). Thus θ can be obtained as a function of the torque M_t.

117 | Screw Dislocations

In the two preceding articles, we have observed the requirement that w must be a single-valued function if the solution is to represent correctly a state of torsion. On reexamining Eqs. (149), (150), and (151), and the boundary condition (152), we can quickly see that it is possible to find states of stress corresponding to $\theta = 0$. The stress function ϕ is to satisfy Laplace's equation and to be constant on each boundary curve of the sec-

[1] The small stresses corresponding to the change in slope of the membrane across the thickness of the web are neglected in this derivation.

tion. But we must use w rather than the form $\theta\psi(x,y)$ of Eq. (*b*) on page 293. Then Eqs. (*f*) of page 295 are replaced by

$$\frac{\partial\phi}{\partial y} = G\frac{\partial w}{\partial x} \qquad -\frac{\partial\phi}{\partial x} = G\frac{\partial w}{\partial y} \qquad (a)$$

These are Cauchy-Riemann equations (see page 171) for the functions Gw and ϕ. Therefore, $Gw + i\phi$ is an analytic function of $x + iy$. Thus,

$$Gw + i\phi = f(x + iy) \qquad (b)$$

Once the function f is chosen, we have a definite state, in which w will be the only nonzero displacement component.

Let r, ψ now represent polar coordinates in the cross section. The choice

$$f(x + iy) = -iA\log(x + iy) = A\psi - iA\log r \qquad (c)$$

where A is a real constant, is of particular interest in the dislocation theory of plastic deformation (see Art. 34). From (*b*), we now have

$$Gw = A\psi \qquad \phi = -A\log r \qquad (d)$$

The corresponding shear stress is in the circumferential direction and is given by the polar components

$$\tau_{z\psi} = -\frac{\partial\phi}{\partial r} = \frac{A}{r} \qquad \tau_{zr} = 0 \qquad (e)$$

Any cylindrical boundary surface r = constant is free from loading. But the displacement w is not continuous. We can apply the solution to a hollow circular cylinder $a < r < b$ as in Fig. 176, which has an axial cut. One face is moved axially along the other by the uniform relative displacement

$$w(r,2\pi) - w(r,0) = \frac{2\pi A}{G} \qquad (f)$$

obtained from the first of (*d*). The stress (*e*) can be regarded as induced

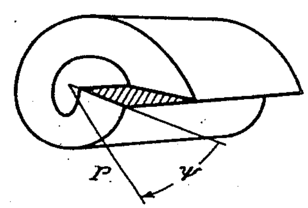

Fig. 176

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The shear stress on the $\frac{1}{2}$-in wall is

$$(\tau)_{1/2\,\text{in}} = \frac{T}{2\bar{A}t} = \frac{50(10^3)}{(2)(43.8)(1/2)} = 1140 \text{ psi}$$

and on the $\frac{1}{4}$-in walls

$$\tau_{1/4\,\text{in}} = \frac{50(10^3)}{(2)(43.8)(1/4)} = 2280 \text{ psi}$$

In evaluating the integral $\int ds/t$ note that t is constant for each wall; thus

$$\int \frac{ds}{t} = \frac{2(8 - \frac{1}{8} - \frac{1}{4})}{\frac{1}{4}} + \frac{6 - \frac{1}{8} - \frac{1}{8}}{\frac{1}{4}} + \frac{6 - \frac{1}{8} - \frac{1}{8}}{\frac{1}{2}} = 95.5$$

5.2.2 MULTIPLE CELL SECTIONS IN TORSION

Figure 5.2-5 shows a closed, thin-walled, multiple cellular cross section transmitting a torque T. Conventional practice is to treat the torsion contribution of each cell separately and combine the results using superposition. This is done subject to the condition that the angle of twist of each cell is the same.

The torsion contribution of the ith cell is given by Eq. (5.2-3) as $T_i = 2q_i\bar{A}_i$. Summing the contribution of the n cells yields

$$T = \sum_{i=1}^{n} T_i = 2\sum_{i=1}^{n} q_i\bar{A}_i \qquad \textbf{[5.2-8]}$$

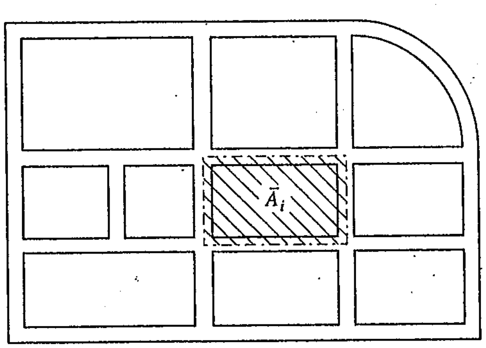


Figure 5.2-5 Multiple cell cross section.

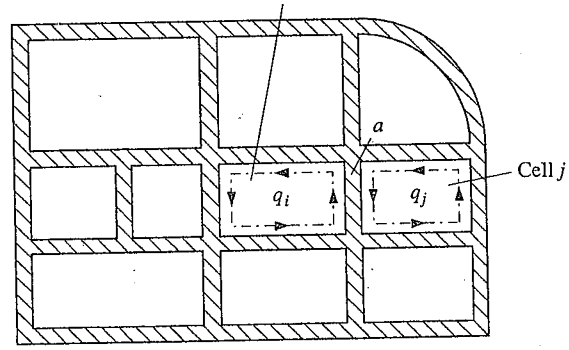


Figure 5.2-6 Positive shear flow in cells i and j.

where we have n unknown values of q_i. The condition that each cell has the same angle of twist θ establishes n equations of twist for each cell. Equation (5.2-5) applies to a single cell where q is the same for all walls of the cell. For a cell within a multiple cell structure, the shear flow q in each wall is the superposition of the shear flow of adjacent cells. Thus in this case, q can vary throughout the walls of a given cell and q must be placed within the integral of Eq. (5.2-5). As an example, the angle of twist of cell i is

$$\theta = \frac{(1+\nu)L}{E\bar{A}_i}\left(\int \frac{q}{t}\,ds\right)_i \qquad \textbf{[5.2-9]}$$

The sign of the shear flow is extremely important. Here we will consider counterclockwise flow within a cell and its walls to be positive. For example, the twist equation of wall a of cell i shown in Fig. 5.2-6 is $(q_a)_i = q_i - q_j$. For the twist equation of wall a of cell j, $(q_a)_j = q_j - q_i$.

Equation (5.2-9) represents n equations, one for each cell. This, together with Eq. (5.2-8), represents $n + 1$ equations in terms of the $n + 1$ unknowns; $\theta, q_1, q_2, \ldots, q_n$. Once the q_i are found the shear flow in each wall can be determined. Dividing the shear flow in a specific wall by the wall thickness yields the shear stress in that wall.

Example 5.2-3 A steel tube 1.5 m long has the cross section shown in Fig. 5.2-7. The tube is transmitting a torque of 200 N·m. Determine the average shear stress in each wall and the angle of twist of the tube. E= 210 GPa, ν = 0.29.

Solution:

From Eq. (5.2-8),

$$T = 2\left[q_1\left(\frac{\pi}{4}a^2\right) + q_2(a)(c-a) + q_3(bc)\right]$$

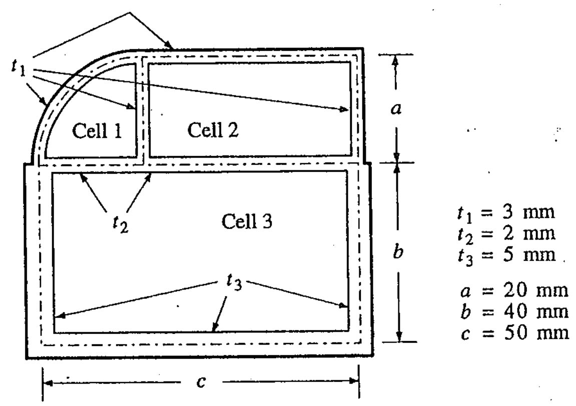


Figure 5.2-7

Substitution of values gives

$$200 = 2\left[q_1\left(\frac{\pi}{4}0.02^2\right) + q_2(0.02)(0.05 - 0.02) + q_3(0.04)(0.05)\right]$$

Simplifying results in

$$3.142q_1 + 6q_2 + 20q_3 = 1(10^6) \qquad \textbf{[a]}$$

For the angle of twist of each cell, consider cell 1 first. Starting at the radial wall and considering each wall of the cell in a clockwise order, Eq. (5.2-9) for cell 1 is

$$\theta = \frac{(1+\nu)L}{E[(\pi/4)a^2]}\left[\frac{q_1}{t_1}\left(\frac{\pi}{2}\right)(a) + \frac{q_1 - q_2}{t_1}(a) + \frac{q_1 - q_3}{t_2}(a)\right]$$

$$= \frac{(1+\nu)L}{E}\frac{4}{\pi(0.02^2)}\left[\frac{q_1}{0.003}\left(\frac{\pi}{2}\right)(0.02) + \frac{q_1 - q_2}{0.003}(0.02) + \frac{q_1 - q_3}{0.002}(0.02)\right]$$

Placing the term involving ν, L, and E on the left side of this equation together with θ and simplifying gives

$$\frac{\theta E}{(1+\nu)L} = (8.638q_1 - 2.122q_2 - 3.183q_3)(10^4)$$

Let

$$C = \frac{\theta E}{(1+\nu)L}(10^{-4}) \qquad \textbf{[b]}$$

Thus Eq. (5.2-9) for cell 1 reduces to

$$8.638q_1 - 2.122q_2 - 3.183q_3 - C = 0 \qquad \textbf{[c]}$$

For cell 2, starting at the top wall and moving clockwise around the cell

$$\frac{\theta E}{(1+\nu)L} = \frac{1}{(a)(c-a)}\left[\frac{q_2}{t_1}(c-a) + \frac{q_2}{t_1}(a) + \frac{q_2-q_3}{t_2}(c-a) + \frac{q_2-q_1}{t_1}(a)\right]$$

$$= \frac{1}{(0.02)(0.05-0.02)}\left[\frac{q_2}{0.003}(0.05-0.02) + \frac{q_2}{0.003}(0.02)\right.$$

$$\left. + \frac{q_2-q_3}{0.002}(0.05-0.02) + \frac{q_2-q_1}{0.003}(0.02)\right]$$

Using the term C as before, we obtain

$$-1.111q_1 + 6.389q_2 - 2.5q_3 - C = 0 \qquad \textbf{[d]}$$

For cell 3, combining the right, bottom, and left walls, and then separating the top two sections gives

$$\frac{\theta E}{(1+\nu)L} = \frac{1}{bc}\left[\frac{q_3}{t_3}(2b+c) + \frac{q_3-q_1}{t_2}(a) + \frac{q_3-q_2}{t_2}(c-a)\right]$$

$$= \frac{1}{(0.04)(0.05)}\left[\frac{q_3}{0.005}(0.08+0.05) + \frac{q_3-q_1}{0.002}(0.02) + \frac{q_3-q_2}{0.002}(0.05-0.02)\right]$$

As with the other cells, simplification yields

$$-0.5q_1 - 0.75q_2 + 1.754q_3 - C = 0 \qquad \textbf{[e]}$$

Solving Eqs. (a) and (c) to (e) simultaneously yields

$$q_1 = 24.62 \text{ kN/m}$$

$$q_2 = 25.11 \text{ kN/m}$$

$$q_3 = 38.60 \text{ kN/m}$$

$$C = 36.56 \text{ kN/m}^3$$

Figure 5.2-8 shows the shear flow and shear stress definitions for this example. The shear stresses are:

$$\tau_1 = \frac{q_1}{t_1} = \frac{24.62}{0.003} = 8.21(10^3)\text{kN/m}^2 = 8.21\text{MPa}$$

$$\tau_2 = \frac{q_2}{t_1} = \frac{25.11}{0.003} = 8.37(10^3)\text{kN/m}^2 = 8.37\text{MPa}$$

$$\tau_3 = \frac{q_3}{t_3} = \frac{38.60}{0.005} = 7.72(10^3)\text{kN/m}^2 = 7.72\text{MPa}$$

$$\tau_4 = \frac{q_3 - q_1}{t_2} = \frac{38.60 - 24.62}{0.002} = 6.99(10^3)\text{kN/m}^2 = 6.99\text{MPa}$$

$$\tau_5 = \frac{q_2 - q_1}{t_1} = \frac{25.11 - 24.62}{0.003} = 0.163(10^3)\text{kN/m}^2 = 0.163\text{MPa}$$

$$\tau_6 = \frac{q_3 - q_2}{t_2} = \frac{38.60 - 25.11}{0.002} = 6.75(10^3)\text{kN/m}^2 = 6.75\text{MPa}$$

The angle of twist is found from Eq. (b) as

$$\theta = \frac{(1+\nu)L(10^4)}{E} C$$

$$= \frac{(1+0.29)(1.5)(10^4)}{210(10^9)} 36.56(10^3) = 3.37(10^{-3}) \text{ rad} = 0.193°$$

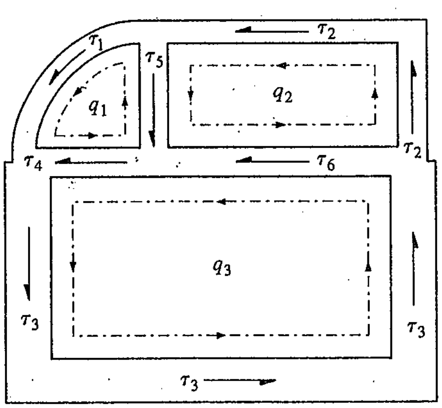


Figure 5.2-8 Shear flow for Example 5.2-3.

Florida International University
Department of Mechanical and Materials Engineering

EGM 5615 FINAL EXAMINATION December 4, 2019

This examination will be a takehome exam. This exam allows you to use your book and notes only as well as one book on fluid mechanics. This exam is due December 10, 2019 at 2 pm in my office EC3442.

Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of this may result in failure of the exam.

PRINT NAME SIGN NAME

This examination consists of ~~four~~ five **problems with several parts to one of the problems. Do all problems.** Read each question carefully. Show all work!!!!!

Problem 1 (30 points).
Consider a bar of thin-walled closed section having the geometry illustrated in the accompanying figure. While the thicknesses t vary from wall to wall as indicated, they are assumed to remain constant along each wall.
If the bar is subjected to a torque T= 9 x 10^5 in-lbs and length a = 9 inches, determine the shear stress distribution in the walls and the angle of twist per unit length of the section. Take G = 4 x 10^6 psi.

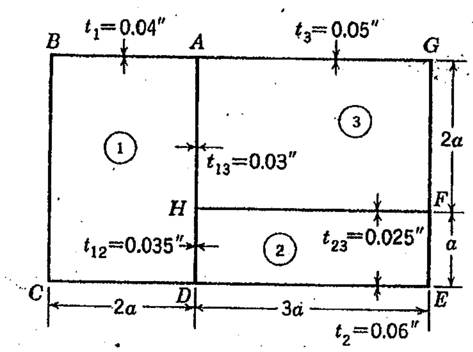
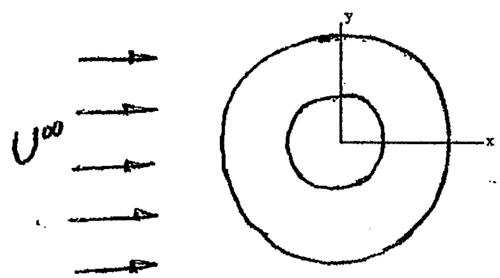


Figure for Problem 1

Problem 2 (15 points).
Do problem 7.12.4 on page 230 AND find expressions for the stress components $\sigma_{\theta\theta}$, $\sigma_{r\theta}$, σ_{rr} and the displacements u_r, and u_θ . Assume this problem is a plane strain problem



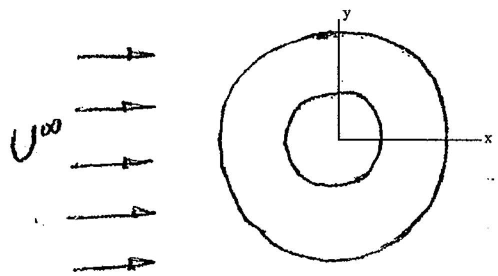
Problem 3 (25 points).
A hollow cylinder of very long length, inner radius Ri and outer radius Ro, is submerged in an infinite, incompressible, *inviscid* fluid flowing with a uniform velocity U^∞ as shown to the left.

Determine the stress state in the cylinder at steady state flow. Clearly state all assumptions you are making in formulating the associated boundary value problem. Indicate which book on Fluid Mechanics you are using.

Hint: Look at potential flow theory. Assume no pressure on the inner surface and find an expression for the pressure on the outer surface as a function of U^∞ and the angle θ. The angle θ is measured positive counterclockwise from the x axis. Show that the pressure is in the form $A+B\bullet f(\theta)$. Write the boundary conditions on σ_{rr} and $\sigma_{r\theta}$ from this information and solve.

Problem 4 (15 points).
Do problem 4.5.2 part (c) only, found on page 121 of Cook and Young.

Problem 5 (15 points).
Do problem 8.2.3 on page 253 of Cook and Young.



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Do problem 4.5.2 part (c) only, found on page 121 of Cook and Young.

Problem 5 (15 points).
Do problem 8.2.3 on page 253 of Cook and Young.

Florida International University
Department of Mechanical and Materials Engineering

EGM 5615 FINAL EXAMINATION 20 July 2006

This examination will be a takehome exam. This exam allows you to use your book and notes only as well as one book on fluid mechanics. This exam is due 25 July at 5 pm in my office EAS3462.

Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of this may result in failure of the exam.

____________________ ____________________

PRINT NAME SIGN NAME

This examination consists of **four problems with several parts to one of the problems. Do all problems.** Read each question carefully. Show all work!!!!!

Problem 1 (30 points).
Consider a bar of thin-walled closed section having the geometry illustrated in the accompanying figure. While the thicknesses t vary from wall to wall as indicated, they are assumed to remain constant along each wall.
If the bar is subjected to a torque T= 9 x 10^5 in-lbs and length a = 9 inches, determine the shear stress distribution in the walls and the angle of twist per unit length of the section. Take G = 4 x 10^6 psi.

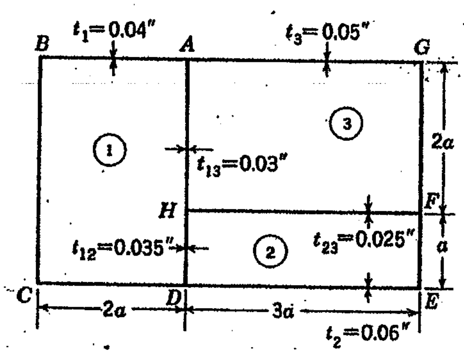
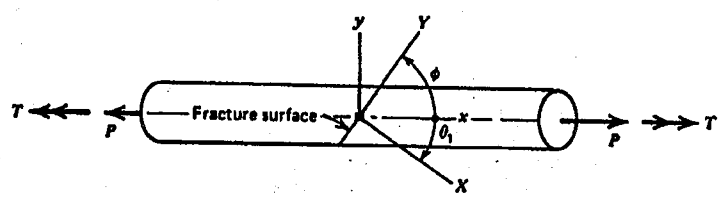


Figure for Problem 1

Problem 2 (15 points).
Do problem 7.12.4 on page 230 AND find expressions for the stress components $\sigma_{\theta\theta}$, $\sigma_{r\theta}$, σ_{rr} and the displacements u_r, and u_θ . Assume this problem is a plane strain problem

ME 440 Spring 2002
Midterm Exam

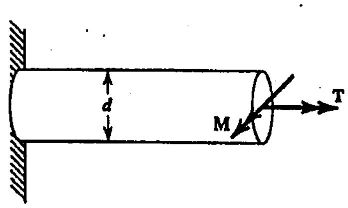
1. A piece of chalk is subjected to combined loading consisting of a tensile load P and torque T. The chalk has an ultimate strength σ_u. The load P remains constant at such a value that it produces a tensile stress 0.51 σ_u on any cross section. The torque T is increased gradually until fracture occurs on some inclined surface. Assuming that the fracture takes place when the maximum principal stress σ_1 reaches the ultimate strength, determine the magnitude of torsional shear stress produced by torque at fracture and determine the orientation of the fracture surface.



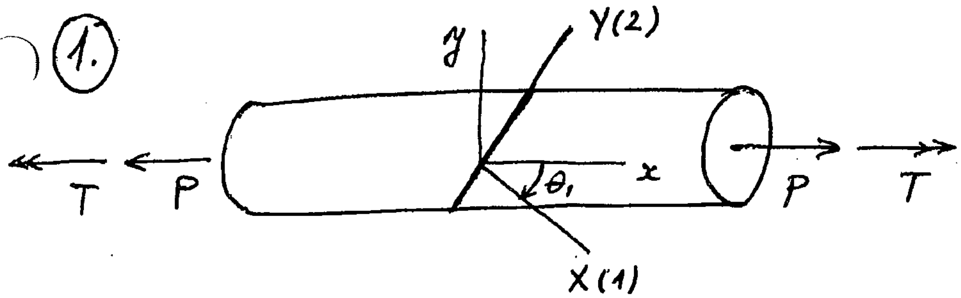
2. For the given stress state in the body, derive expressions for the displacement components u(x,y). v(x,y), where c is a constant.

$$\begin{bmatrix} cy^2 & 0 \\ 0 & -cx^2 \end{bmatrix}$$

3. A circular cylindrical shaft made of steel with (σ_y = 700MPa) is subjected to bending moment M = 13 kN·m and torsional moment T = 30 kN·m. Employing factor of safety SF=2.6, determine minimum required safe diameter for the shaft.



(1.)



$\sigma_x = 0.51\,\sigma_u$

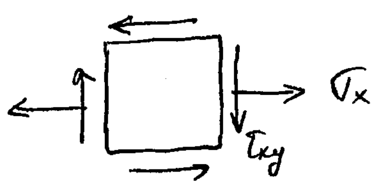
$\sigma_y = 0$

$\tau_{xy} = ?$

Max. principal stress $\sigma_1 = \dfrac{\sigma_x}{2} + \sqrt{\left(\dfrac{\sigma_x}{2}\right)^2 + \tau_{xy}^2}$

$\sigma_1 = \sigma_u$

$\sigma_u = 0.255\,\sigma_u + \sqrt{(0.255\,\sigma_u)^2 + \tau_{xy}^2} \rightarrow |\tau_{xy}| = 0.7\,\sigma_u$



Positive torque (on right hand side) produces negative τ_{xy}, therefore $\boxed{\tau_{xy} = -\,0.7\,\sigma_u}$

Orientation of fracture surface

$$\tan\theta_1 = \frac{2\,\tau_{xy}}{\sigma_x} = \frac{-\,2\cdot 0.7\cdot\sigma_u}{0.51\cdot\sigma_u} = -\,2.7452$$

$$\boxed{\theta_1 = -\,0.6107 \text{ rad}}$$

(2.)

$$[\sigma] = \begin{bmatrix} cy^2 & 0 \\ 0 & -cx^2 \end{bmatrix} \qquad u(x,y) = ? \quad v(x,y) = ?$$

$\sigma_x = cy^2$
$\sigma_y = -cx^2$
$\tau_{xy} = 0$

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] = \frac{c}{E}[y^2 + \nu x^2]$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] = -\frac{c}{E}[x^2 + \nu y^2]$$

$$\varepsilon_x = \frac{\partial u}{\partial x} \rightarrow u = \int \varepsilon_x dx = \frac{c}{E}\left\{y^2 x + \nu \frac{x^3}{3}\right\} + g(y)$$
$$= \frac{c}{3E}\left\{3y^2 x + \nu x^3\right\} + g(y)$$

$$\varepsilon_y = \frac{\partial v}{\partial y} \rightarrow v = \int \varepsilon_y dx = -\frac{c}{3E}\left\{3x^2 y + \nu y^3\right\} + f(x)$$

$\tau_{xy} = 0 \longrightarrow \gamma_{xy} = 0$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

$g' + \frac{c}{E}(2yx + 0) - \frac{c}{3E}(6x \cdot y) + f'$
$g'(y) + f'(x) = 0$
$g'(y) = -f'(x) = K$
$g(y) = Ky + B$
$f(x) = -Kx + C$

This is possible only if:

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = a$$

not true

$\frac{\partial u}{\partial y} = a \qquad u = ay + f(x)$

$$\frac{\partial u}{\partial y} = \frac{2c}{E}xy + \frac{dg}{dy} = a \qquad (1)$$

$$\frac{\partial v}{\partial x} = -\frac{2c}{E}xy + \frac{df}{dx} = -a \qquad (2)$$

From (1) : $$\frac{dg}{dy} = -\frac{2c}{E}xy + a$$

$$g(y) = \int \left\{ -\frac{2c}{E}xy + a \right\} dy$$

$$\underline{g(y) = -\frac{c}{E}xy^2 + ay + b_1}$$

×

From (2) : $$\frac{df}{dx} = \frac{2c}{E}xy - a$$

$$f(x) = \int \left\{ \frac{2c}{E}xy - a \right\} dx$$

not f(x) only

$$\underline{f(x) = \frac{c}{E}yx^2 - ax + b_2}$$

×

Displacement field:

$$u(x,y) = \frac{c}{E}\left\{ y^2x + \frac{\nu}{3}x^3 \right\} + a\cdot y + b_1$$

$$\frac{\partial u}{\partial y} = \frac{c}{E}\left\{ 2yx - 2xy \right\} + a$$

$$v(x,y) = \frac{c}{E}\left\{ -x^2y - \frac{\nu}{3}y^3 \right\} - ax + b_2$$

$$\frac{\partial v}{\partial x} = \frac{c}{E}\left\{ -2xy + 2xy \right\} - a \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

Constants $b_1, b_2 \rightarrow$ from B.C.s

(3.)

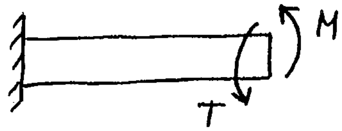
$\sigma_y = 700 MPa$

$M = 13 \ kN \cdot m$

$T = 30 \ kN \cdot m$

$SF = 2.6$

$d_{min} = ?$



Octahedral shear-stress criterion

$$\tau_{oct} = \frac{\sqrt{2}}{3}\sigma_y = \frac{1}{3}\sqrt{2\sigma^2 + 6\tau^2}$$

$$\sigma_y = \sqrt{\sigma^2 + 3\tau^2}$$

$$\sigma = SF \cdot \frac{M \cdot c}{I} = \frac{32 \cdot SF \cdot M}{\pi \cdot d^3}$$

$$\tau = SF \cdot \frac{T \cdot c}{J} = \frac{16 \cdot SF \cdot T}{\pi \cdot d^3}$$

$$\sigma_y = \frac{16 \cdot SF}{\pi d^3}\sqrt{4M^2 + 3T^2}$$

$$d_{min} = \left(\frac{16 \cdot SF}{\pi \cdot \sigma_y}\sqrt{4M^2 + 3T^2}\right)^{\frac{1}{3}}$$

$$\boxed{d_{min} = 103 mm}$$

Maximum shear stress criterion.

$$\tau_{max} = \frac{\sigma_y}{2} = \frac{1}{2}\sqrt{\sigma^2 + 4\tau^2}$$

$$d_{min} = \left(\frac{32 \cdot SF}{\pi \cdot \sigma_y}\sqrt{M^2 + T^2}\right)^{\frac{1}{3}}$$

$$\boxed{d_{min} = 107 mm}$$ ←