EGM 5615 Synthesis of Engineering Mechanics (3)

Professor:

Cesar Levy
Office EAS 3443
Office hours: TBD
Tel: (305) 348-3643 (voice mail)
e-mail: levyez@fiu.edu

Textbook:

Cook and Young, Advanced Mechanics of Materials, 2nd Edition, Macmillan Publishing Co., 1999.

References:

There are plenty of books at the FIU Library on the subject. Here is a list of a few books on the subject available at the library.

Engineering mechanics of solids. Egor P. Popov. Englewood Cliffs, N.J. : Prentice Hall, c1990.

Mechanics of materials. James M. Gere, Stephen P. Timoshenko. 3rd ed. Boston : PWS-KENT Pub. Co., c1990.

Engineering considerations of stress, strain, and strength, Robert C. Juvinall, New York, McGraw-Hill [1967].

Theory of Elasticity Timoshenko and Goodier. 3rd Ed., McGraw-Hill

Course Objectives

1. obtain viable approximations to the solutions of design problems, without "full-field" (e.g., FEA) modeling and computer solution.
2. make the student understand the nature of the approximations of such "strength of materials" solutions and their likely impact on the reliability and robustness of the resulting design.
3. understand common mechanisms of failures, to analyze potential designs for criticality, and produce designs that can be expected to be reasonably safe from such failures.

Topics

1. Stress-strain-temperature relations, stress at a point (2 classes)
   Review of elementary mechanics of materials, Introduction to theory of elasticity
   Principal stresses Octahedral and max shear stress; temperature relations
2. Theories of failure and fracture (2 classes)
   Failure criteria; Fracture mechanics; Fatigue
3. Plane elasticity (2 classes)
   Stress field solution; polynomial solution; Plane stress problems; circular hole;
4. Unsymmetrical bending of straight beams (2 classes)
   Beam deflections in unsymmetric bending; Transverse shear
5. Shear center for thin-walled beam cross-sections (2 classes)
6. Bending of curved beams and rings (2 classes)
   circumferential stress; Radial stress and shear stress; thin flanges; Thin walled section; deflections of sharply curved beams
7. Energy methods for deflections and static indeterminacy (3 classes)
   Strain energy density; Reciprocal theorems; strain energy; Unit load method; Statically indeterminate problems

8. Beams on elastic foundations (3 classes)
9. Thick-walled cylinders (3 classes)
   Pressurized cylinders; Shrink Fits

10. Torsion of non-circular cross-sections and Torsion with restraint of warping (2 classes)
    Torsion of non-circular cross sections; Warping function; Prandtl stress function; Membrane analogy; Thin walled open Sections; Pure twist of single celled hollow cross sections
11. Stress concentration and contact stress (1.5 classes)
    Stress concentration; contact stress;
12. Review and problem sessions (1.5 classes)
13. Tests (2 classes)

Homeworks are due a week after being assigned. These problems should be neatly worked out, preferably on engineering paper. Use the "Given, Required, Solution" format and completely draw appropriate diagrams and coordinate systems. All numerical answers should have the appropriate units. Note in each exam there will be one problem quite similar to the assigned problems. Problems submitted after the class hour the due day or the next day will be penalized with 15% of the total grade for that assignment. <u>No homework will be accepted after two days without a medical or any other documented excuse. You must keep up with the homework in order to do well in class</u>

Grading

Homework	15%
Exams (2 25% each)	50%
Final Exam	35%
Total	100%

Grades will be assigned based on your performance on the activities above. Final letter grades will be assigned as follows:

100 – 95	A	77-79.99	B-	60-64.99	D
90-94.99	A-	73-76.99	C+	Below 60	F
85-89.99	B+	70-72.99	C		
80-84.99	B	65-69.99	C-		

We meet on Wednesdays 10-1050am, and F 10-1150am in room EC3327.
FINAL EXAM (Cumulative): To be announced.

NOTE: ***This is a preliminary syllabus and it might be changed during the semester. Any change will be announced in class.***

EMA 506 Advanced Mechanics of Materials

Fall Semester, 2001

Modified 11/21/01

Date	#	Study Assignment Cook & Young, 2nd ed. *[Gere & Timoshenko, 4th ed.]*	Topic	HW Problems
~~Wed 9/5~~ R 8/29	1		Review of elementary mechanics of materials	
Fri 9/7	2	1.1,1.2,1.3 (skim 1.4-1.10)	Review of elementary mechanics of materials	1.4-3, 1.5-2
Mon 9/10	3	7.1,7.2,7.3	Introduction to theory of elasticity	1.7-3
Wed 9/12	4	2.1,2.2,2.3	Principal stresses	2.3-2(c,g)
Fri 9/14	5	2.4,2.5	Oct and max shear stress; temperature relations	2.5-3
Mon 9/17	6	2.6 *[G&T: 2.7]*	Strain energy density	2.6-2 (for 2.3-2c and g only) HW#1 Due for lectures 1-5
Wed 9/19	7	2.7,2.8 *[G&T: 2.10]*	Stress concentration; contact stress	2.7-7
Fri 9/21	8	3.1,3.2,3.3,3.4	Failure criteria	3.2-5,3.3-3
Mon 9/24	9	3.5	Fracture mechanics	3.5-6
Wed 9/26	10	3.6 *[G&T: 2.9]*	Fracture mechanics	3.5-7
Fri 9/28	11	3.6	Fatigue	3.6-3, 3.6-6
Mon 10/1	12	7.4,7.5	Stress field solution; polynomial solution	7.3-2(a), 7.4-2, 7.5-5 HW #2 Due for lectures 6-11

Wed 10/3	13	7.7,7.8	Plane stress problems; circular hole	7.7-5, 7.7-7, 7.8-5
Fri 10/5	14	8.1 *[G&T: 8.3]*	Pressurized cylinders	
Mon 10/8	15	8.2	Pressurized cylinders	8.2-7
Wed 10/10	16	8.3	Shrink Fits	8.3-2, 8.3-10
Fri 10/12	17	4.1,4.2,4.3 *[G&T: 9.8,9.9]*	Reciprocal theorems; strain energy	4.1-6, 4.2-1, 4.3-2
Mon 10/15	18	4.4,4.5,4.6	Unit load method	4.4-1, 4.5-4, 4.6-5 HW #3 Due for lectures 12-17
Wed 10/17	19	4.7	Statically indeterminate problems	4.7-3
Fri 10/19	20	4.7	Statically indeterminate problems	4.7-4
Mon 10/22	21	In Class Review EVENING EXAM #1	Covers lectures 1-17	
Wed 10/24	22	7.11,9.1 *[G&T: 3.1-3.4]*	Torsion of non-circular cross sections; Warping function	7.11-1(a)
Fri 10/26	23	7.12	Prandtl stress function	7.12-4
Mon 10/29	24	9.2,9.3,9.4 *[G&T: 3.10]*	Membrane analogy; Thin walled open sections	9.2-1, 9.3-3, 9.3-4, 9.4-3 HW #4 Due for lectures 18-23
Wed 10/31	25	9.5 (skim 9.6,9.7)	Pure twist of single celled hollow cross sections	9.5-7
Fri 11/2	26	10.1 *[G&T: 6.5]*	Unsymmetric bending	10.1-1

Mon 11/5	27	Evening Guest Lecture	Aircraft Design Considerations **Jeremy Monnett, General Manager, Sonex, Ltd. (time and location to be announced)**	
Wed 11/7	28	10.2	Unsymmetric bending examples	10.2-5
Fri 11/9	29	10.3	Beam deflections in unsymmetric bending	10.3-4

Mon 11/12	30	10.4	Transverse shear	10.4-3 HW #5 Due for lectures 24-29
Wed 11/14	31		Transverse Shear Examples	
Fri 11/16	32	10.5, 10.6 *[G&T: 6.6-6.8]*	Shear center	10.5-1(a), 10.6-1(b,e,k)
Mon 11/19	33	In Class Review EVENING EXAM #2	Covers lecture 18-29	
Wed 11/21	34	6.1,6.2	Curved beams; circumferential stress	6.1-1, 6.2-6
Fri 11/23		Thanksgiving Holiday	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	
Mon 11/26	35	6.3	Curved beams examples	6.3-5
Wed 11/28	36	6.4,6.5	Radial stress and shear stress; thin flanges	6.4-10, 6.5-4 HW #6 Due for lectures 30-35
Fri 11/30	37	6.6, 6.7	Thin walled section; deflections of sharply curved beams	6.6-1, 6.7-4
Mon 12/3	38		Design Meetings	
Wed 12/5	39		Design Meetings	
Fri 12/7	40		Design Meetings	
Mon 12/10	41		Case Study	
Wed 12/12	42		Case Study	Written Project Reports Due
Fri 12/14	43	In Class Review		HW #7 Due for lectures 36-37
Mon 12/17 at 7:45am		Final Exam		

43 classes $\times \frac{5}{6}$ = 36 hrs

EGM 5615 Synthesis of Engineering Mechanics

Review of elementary formulae

Here we critically re-examine some basic formulae in connection with underlying assumptions.

1. Simple tension of a bar of cross section area A, length L under load P. E is Young's modulus.

Stress, $\sigma = \frac{P}{A}$. Deflection, $\Delta = \frac{PL}{AE}$

2. Twisting of a bar by torque T. r = radial coordinate, J = polar moment of inertia, L = length, G = shear modulus.

Stress $\tau = \frac{Tr}{J}$, twist angle, $\theta = \frac{TL}{JG}$.

3. Beam bending. Moment M is applied to beam of area moment I. y is distance, perpendicular to the long axis of the beam, from neutral axis. x is coordinate along beam. v = displacement.

Stress $\sigma = \frac{\pm My}{I}$, curvature, $\frac{d^2v}{dx^2} = \frac{M}{EI}$.

Geometrical assumptions

Axial load in tension must be centered, otherwise there is superposed bending.

In torsion, cross section must be circular. There can be a hole, but it must be on center. Plane sections perpendicular to the rod axis were assumed to rotate but remain plane. For non-circular sections, there is warp of cross sections.

In bending, moment vector must be along a principal axis of inertia. For the deflection equation to be valid, deflections must be small enough that the second derivative is a good approximation to the curvature.

In bending, depth of cross section should exceed width. Otherwise the beam tends to act as a plate, and structural stiffness is perhaps 10% greater for many structural materials. Why? The Poisson effect is restrained in cylindrical plate bending, while it is free to occur in beam bending.

Material assumptions

Demonstration: stretch or bend viscoelastic putty. Observe time dependent behavior.

We have assumed *elastic* material behavior, specifically linearly elastic.

Demonstration: bend copper wire. It stays bent. Deflection depends not only on applied moment, but also exhibits a threshold effect and hysteresis.

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Demonstration: Bend off-axis honeycomb, observe twist.

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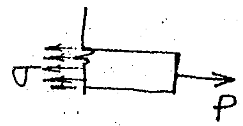
Demonstration: Bend stack of paper, observe slip between sheets. Stack is much easier to bend than to stretch in comparison with a block of wood of similar thickness.

We have assumed *homogeneity*.

All materials are in fact heterogeneous, if only due to their atomic structure. Real materials such as steel or aluminum, have larger scale heterogeneities such as dislocations, grain boundaries, and inclusions. Often, we can get away with an assumption of homogeneity if the heterogeneities are much smaller than any size scale of interest in the deformation field.

1.1 (a)

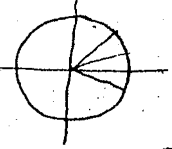
1. Plane Section remain Plain and translate with respect to one another in the load direction
2. Axial strain ϵ constant over cross section and lengthwise.
3. Uniaxial stress $\sigma = E\epsilon$ - constitutive relation



4. $P - \sigma A = 0$; $\sigma = \frac{P}{A}$

$$\frac{P}{A} = E\,\frac{\Delta}{L} \; ; \quad \Delta = \frac{PL}{AE}$$

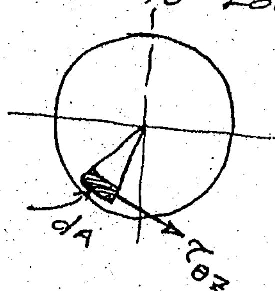
1.1 (b)



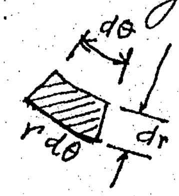
1. Establish geometry of deformation
   - Plane section remain Plane and radial straight lines remain straight.
2. Strain distribution.
   - Shear strain proportional to radial coordinate

$$\gamma_{\theta z} = K r \quad (K = \text{constant})$$

3. Constitutive relation.
   - $\tau_{\theta z} = G\gamma_{\theta z} = GKr$
4. Stress to Load relation and Free Body Diagram.



$$dA = (r\,d\theta)\,dr$$



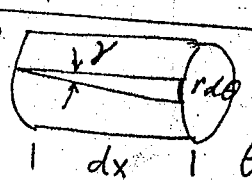
$$T = \int r\,(\tau_{\theta z}\,dA)$$

$$= GK\int_0^{2\pi}\int_0^{R} r^3\,dr\,d\theta = \frac{GK\pi}{2}R^4 = GKJ$$

So $GK = \frac{T}{J}$; applying to step 3

$$\tau_{\theta z} = \frac{Tr}{J} \qquad \gamma_{\theta z} = \frac{\tau_{\theta z}}{G} = \frac{Tr}{JG} = Kr \quad K = \frac{T}{JG}$$

1 (c)

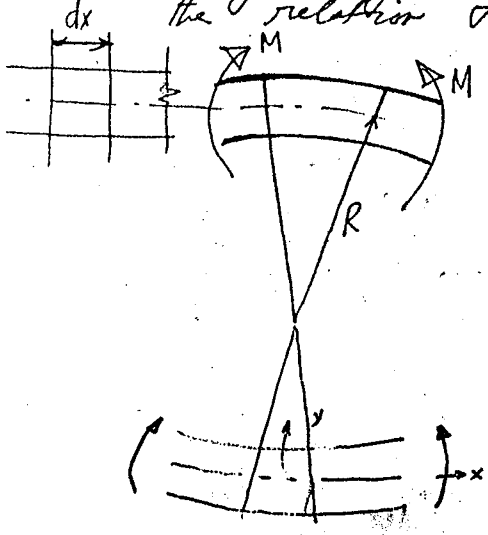


$$\frac{r\,d\theta}{dx} = \gamma = \frac{\tau}{G}$$

$$\theta = \int_0^L d\theta = \int_0^L \frac{\gamma\,dx}{r} = \int_0^L \frac{\tau}{rG}\,dx = \int_0^L \frac{Tr}{rGJ}\,dx$$

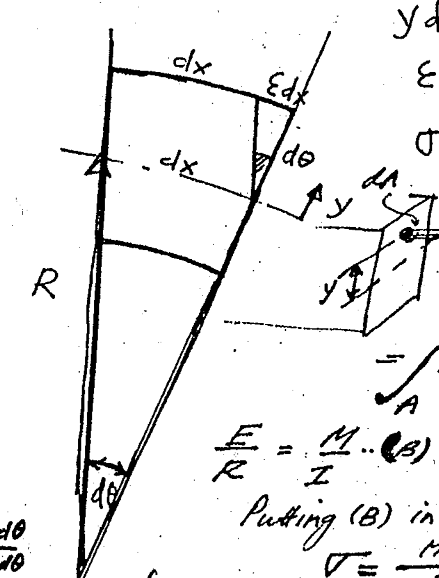
for Constant values $\theta = \frac{TL}{GJ}$

1) Bar must be straight and of constant cross section and have a plane of symmetry that contains the axis. M is the moment vector perpendicular to the plane of symmetry. The material is homogeneous and displays the relation $\sigma = E\varepsilon$



$$\varepsilon = \frac{l_{top}-l}{l} = \frac{(l+y\,d\theta)-l}{l} = \frac{-y\,d\theta}{R\,d\theta}$$

$$\sigma = E\varepsilon = -Ey/R$$



$$-M = \int \sigma dA \cdot y = 0$$

$$y\,d\theta = \varepsilon\,dx = \frac{dx}{R}$$

$$\varepsilon = \frac{y}{R}$$

$$\sigma = E\varepsilon = \frac{Ey}{R} \quad \cdots\cdots (A)$$

$$M = \int_A (\sigma dA)\,y$$

$$= \int_A \frac{E}{R} y^2 dA = \frac{E}{R}\int_A y^2 dA = \frac{EI}{R}$$

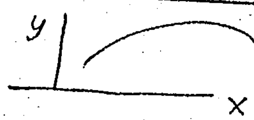
$$\frac{E}{R} = \frac{M}{I} \;..(B)$$

$$M = +\frac{E}{R}I$$

Putting (B) in (A)

$$\sigma = \frac{MY}{I}$$

$$-\frac{1}{R} = \frac{\left(\frac{d^2y}{dx^2}\right)}{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}$$



for any curve $y = f(x)$ from calculus

Small deflection (slope $= \frac{dy}{dx}$) assume $\left|\frac{dy}{dx}\right| \ll 1$

$$\frac{1}{R} = \frac{d^2y}{dx^2} \quad \cdots (c)$$

between (B) in 1.1(d) and (c)

$$\frac{E}{R} = \frac{M}{I} \Rightarrow \frac{1}{R} = \frac{M}{EI}$$

$$\frac{d^2y}{dx^2} = \frac{M}{EI}$$

$$T = \int (\tau dA)\,r = \int_0^{2\pi}\int_0^a \tau\,(r\,dr\,d\theta)\,r$$

$$= 2\pi \int_0^a \tau r^2 dr = 2\pi c \int_0^a \gamma^{1/2} r^2 dr$$

$$= 2\pi c K \int_0^a r^{1/2} r^2 dr \quad \text{(assuming } \gamma = K r^{1/2}\text{)}$$

$$T = 2\pi c K \frac{2}{7} a^{7/2}$$

But $\tau = c\gamma^{1/2} = cKr^{1/2}$

$$\therefore cK = \frac{\tau}{r^{1/2}}$$

$$\tau = \frac{7\,T\,r^{1/2}}{4\pi\,a^{7/2}}$$

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Stress Strain Relations and Elastic Symmetry

References: Sokolnikoff, *Mathematical Theory of Elasticity* pp. 56-71
Malvern, *Introduction To The Mechanics of A Deformable Medium*, pp 273-294

Triclinic Crystal (Most General Anisotropic material)

21 constants.

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix}}_{C \text{ (Stiffness) Matrix}} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{bmatrix} \rightarrow$$

Monoclinic Crystal

One plane of elastic symmetry. e.g. yz plane. Reflection of x axis leaves constants unchanged

Direction cosines of Transformation:

$$\begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x' \\ y' \\ z' \end{array} & \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

"C" Matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix}$$

Orthorhombic (Orthotropic) Material

3 mutually orthogonal planes of symmetry. Reflection of x, y, and z leaves constants unchanged

Direction cosines (reflect y axis)

$$\begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x' \\ y' \\ z' \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

"C" Matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66} \end{bmatrix} \rightarrow$$

Cubic Material

Interchange of 2 axes (i.e. rotate 90° then reflect) leaves constants unchanged.

Direction cosines (interchange Y & Z)

$$\begin{array}{c|c} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} x' \\ y' \\ z' \end{array} & \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

"C" matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & C_{44} \end{bmatrix}$$

Isotropic Material: Any coordinate transformation leaves Elastic Constants unchanged

"C" matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{44}^{*} & & \\ & & & & C_{44}^{*} & \\ & & & & & C_{44}^{*} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \mu \end{bmatrix}$$

$$^{*}C_{44} = \frac{C_{11} - C_{12}}{2}$$

λ, μ, are the Lamé constants

$\underline{\sigma} = C\,\underline{e}$ may be written $\quad \sigma_{ij} = \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}$

where ε_{kk} is the dilatation

$$\tau_{oct} = (1/3)\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \frac{1}{3}\left[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)\right]^{1/2}$$

Strain energy

Consider a spring. F = kx. The work done in compressing the spring is

$W = \int_0^x F\,dx = \frac{1}{2}kx^2$. Since the spring is elastic, this is the strain energy U.

Now consider the strain energy in compressing a cubic block of side L and Young's modulus E.

$F = kx$

$\varnothing\ \sigma A = \sigma L^2 = kx = kxL/L = k\,\varepsilon\,L$, so

$\sigma = \frac{k\varepsilon}{L}$. So $E = k/L$. So $U = \frac{1}{2}kx^2 = \frac{1}{2}(EL)(\varepsilon L)^2 = \frac{\mathbf{1}}{\mathbf{2}}\mathbf{E\varepsilon^2 L^3} = \frac{1}{2E}\sigma_x^2 L^3$.

So $\frac{\mathbf{1}}{\mathbf{2}}\mathbf{E\varepsilon^2}$ represents strain energy per unit **volume**.

Three-dimensional forms for the strain energy density are obtained by superposition. Poisson related strains enter in the energy expression because stresses in an orthogonal direction do work as the material deforms due to Poisson effects.

$$U_0 = \frac{1}{2E}\{\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_x\sigma_z)\} + \frac{1}{2G}\{\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\}.$$

Strain energy of distortion

In the study of **yield** criteria it is expedient to separate the effects of normal and shear stress.

Mean normal stress: $\sigma_a = \frac{1}{3}[\sigma_x + \sigma_y + \sigma_z] = \frac{1}{3}[\sigma_1 + \sigma_2 + \sigma_3]$, since the trace is invariant.

This is also the normal stress on an octahedral plane.

Deviatoric stress: The trace of this is zero by construction. This produces change of shape only, no change of volume. By contrast, a hydrostatic stress produces change of volume only, no change of shape *in an isotropic material*. A general state of stress can be written as a sum of hydrostatic and deviatoric stresses.

$$\mathbf{s} = \begin{pmatrix} \sigma_x - \sigma_a & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y - \sigma_a & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z - \sigma_a \end{pmatrix}.$$

Distortional energy, $U_{0d} = \frac{3}{4G}\tau_{oct}^2 = \frac{1}{6G}\sigma_{eff}^2$.

$$\sigma_{eff} = \frac{1}{\sqrt{2}}\{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\{\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\}\}^{1/2}.$$

Maximum shear stress

Suppose the principal stresses are $\sigma_1 > \sigma_2 > \sigma_3$.

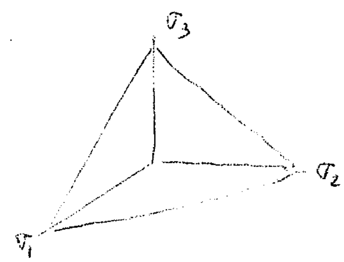
Then the maximum shear stress is

$$\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$$

and it acts on planes 45 degrees from the principal stresses. Proof: Sokolnikoff, Mathematical Theory of Elasticity, page 50-53.

Octahedral shear stress

An octahedral plane makes equal angles with the principal stress directions. It is given that name because there are eight such planes forming an octahedron about the origin. Octahedral shear stress is of interest in the context of failure criteria.



Consider *force* $d\mathbf{R}$ on an arbitrary plane of area dA,

$$d\mathbf{R} = \sigma_1 l \; dA\, \underset{\sim}{i} + \sigma_2 m \, dA\, \underset{\sim}{j} + \sigma_3 n \, dA\, \underset{\sim}{k}\,.$$

The *traction vector* T has dimensions of stress:

$$T = \frac{dR}{dA}$$

The stress normal to this plane is

$$\sigma_n = d\mathbf{R}\cdot \mathbf{n}/dA. \;=\; \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2$$

The shear stress is, by the Pythagorean theorem,

$$\tau_s = d\mathbf{R}_s/dA = (1/dA)\sqrt{dR^2 - dR_n^{\,2}}\;,$$

but $\mathbf{n} = l\,\underset{\sim}{\mathbf{i}} + m\,\underset{\sim}{j} + n\,\underset{\sim}{\mathbf{k}}$, with l, m, n as direction cosines.

For the octahedral plane, $l = m = n = 1/\sqrt{3}$, since it makes equal angles with the axes and since the sum of the squares of the direction cosines equals 1.

$$d\mathbf{R}^2/dA^2 = (d\mathbf{R}\cdot d\mathbf{R})/dA^2 = \sigma_1^2 l^2 + \sigma_2^2 m^2 + \sigma_3^2 n^2 \;=\; \frac{1}{3}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

The octahedral normal stress is, by σ_n, $= \sigma_1 l^2 + \sigma_2 m^2 + \sigma_3 n^2$

$$\sigma_{oct,\,n} = (1/3)(\sigma_1 + \sigma_2 + \sigma_3)$$

The octahedral shear stress is, by τ_s, dR, and $\mathbf{n}$,

$$\tau_{oct} = \{\sigma_n^{\,2} - \sigma_{oct,\,n}^{\,2}\}^{1/2}\,.$$

$\tau_{oct} = (1/3)\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \frac{1}{3}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)\right]^{1/2}$

Strain energy

Consider a spring. F = kx. The work done in compressing the spring is

$W = \int_0^x F\,dx = \frac{1}{2}kx^2$. Since the spring is elastic, this is the strain energy U.

Now consider the strain energy in compressing a cubic block of side L and Young's modulus E.

F = kx

$\varnothing$ $\sigma A = \sigma L^2 = kx = kxL/L = k\,\varepsilon\,L$, so

$\sigma = \frac{k\varepsilon}{L}$. So E = k/L. So $U = \frac{1}{2}kx^2 = \frac{1}{2}(EL)(\varepsilon L)^2 = \frac{1}{2}E\varepsilon^2 L^3 = \frac{1}{2E}\sigma_x^2 L^3$.

So $\frac{1}{2}E\varepsilon^2$ represents strain energy per unit **volume**.

Three-dimensional forms for the strain energy density are obtained by superposition. Poisson related strains enter in the energy expression because stresses in an orthogonal direction do work as the material deforms due to Poisson effects.

$$U_0 = \frac{1}{2E}\{\sigma_x^2+\sigma_y^2+\sigma_z^2-2\nu(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_x\sigma_z)\} + \frac{1}{2G}\{\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\}.$$

Strain energy of distortion

In the study of **yield** criteria it is expedient to separate the effects of normal and shear stress.

Mean normal stress: $\sigma_a = \frac{1}{3}[\sigma_x+\sigma_y+\sigma_z] = \frac{1}{3}[\sigma_1+\sigma_2+\sigma_3]$, since the trace is invariant.

This is also the normal stress on an octahedral plane.

Deviatoric stress: The trace of this is zero by construction. This produces change of shape only, no change of volume. By contrast, a hydrostatic stress produces change of volume only, no change of shape *in an isotropic material.* A general state of stress can be written as a sum of hydrostatic and deviatoric stresses.

$$\mathbf{s} = \begin{pmatrix} \sigma_x-\sigma_a & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y-\sigma_a & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z-\sigma_a \end{pmatrix}.$$

Distortional energy, $U_{0d} = \frac{3}{4G}\tau_{oct}^2 = \frac{1}{6G}\sigma_{eff}^2$.

$$\sigma_{eff} = \frac{1}{\sqrt{2}}\{(\sigma_x-\sigma_y)^2 + (\sigma_y-\sigma_z)^2 + (\sigma_z-\sigma_x)^2 + 6\{\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\}\}^{1/2}.$$

Failure criteria

Failure criteria deal with failure of a material, in contrast with failure of a structure.

Brittle materials

Maximum normal stress criterion

Failure if $\sigma_1 \geq \sigma_t^{ult}$, that is , if the largest tensile principal stress exceeds the ultimate strength. The other principal stresses are ignored. Similarly in compression, the applied stress is compared with the ultimate strength in compression. This criterion is simplistic since if all the principal stresses are compressive, most materials are much stronger than would be expected based on uniaxial tests.

Mohr criterion

This takes into account ultimate tensile, compressive and shear stresses. Represent each state by a Mohr circle. Draw an 'envelope' tangent to the circles. Mohr suggested that, provided an arbitrary state of stress was represented by Mohr circle within that envelope, failure would not occur. Sometimes a simplified form is taken in which the shear test is ignored. Then the envelope is a straight line and failure is predicted if $\sigma_1/\sigma_t^{ult} - \sigma_3/\sigma_c^{ult} \geq 1$. This also is not ery realistic for hydrostatic compression.

Ductile materials: yield criteria

Maximum shear stress criterion (Tresca criterion)

Yield when $\tau_{max} \geq \tau_Y$.

In principal stress space this looks like a hexagon.

Tension test to yield.

Recall that the maximum shear stress is $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$, but the minimum principal stress is zero so for tension, $\tau_Y = 0.5\,\sigma_Y$.

Von Mises

Yield when $\sigma_{eff} \geq \tau_Y$.

Recall $\sigma_{eff} = \frac{1}{\sqrt{2}}\{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\{\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\}\}^{1/2}$.

Distortional energy, $U_{0d} = \frac{3}{4G}\ \tau_{oct}^2 = \frac{1}{6G}\ \sigma_{eff}^2$.

In principal stress space, the Von Mises criterion looks like an ellipse.

Fatigue

A material loaded through multiple cycles will break at a stress considerably less than the ultimate strength for a single application of load. Fatigue is quantified by the S-N curve, in which the number N of cycles is plotted logarithmically.

The effect of cyclic stresses is to initiate microcracks at centers of stress concentration within the material or on the surface resulting in the growth and propagation of cracks leading to failure.

As for fatigue testing, the rate of crack growth can be plotted in a log-log scale versus time. Testing the fatigue properties to generate an S-N curve entails monitoring the number of cycles to failure at various stress levels. This test requires a large number of specimens compared with the crack propagation test.

The *endurance limit* is the stress below which the material will not fail in fatigue no matter how many cycles are applied. Not all materials exhibit an endurance limit. (a practical limit is often chosen as 10^7 cycles).

The presence of a saline environment exacerbates fatigue.
Surface roughness exacerbates fatigue. A polished surface is better.
Rubbing or 'fretting' exacerbates fatigue. Re-design the part or use lubricants.
Heat treatment to introduce residual surface compression can be helpful.

EGM 5615

11/8/2010

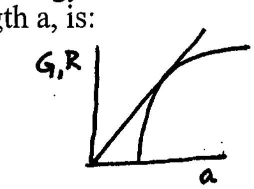
Fracture mechanics

As elliptic hole becomes progressively narrower, the ellipse approaches a crack shape and SCF = K $\rightarrow\infty$. Actual observed stress concentration factors for cracks are not infinite.

Therefore a material with one perfectly sharp crack will have **zero** strength, since the stress concentration factor becomes infinite. Experimentally, even for brittle materials, strength is reduced by cracks but not infinitely.

A criterion based on energy balance rather than on stress has therefore been adopted. The approach is due to Griffith. The energy relations are visualized as follows. The energy required to extend a crack is linear in crack length because the energy is expended in creating new free surfaces. By contrast the strain energy available comes from a roughly semicircular region around the crack and is therefore quadratic in crack length. As the crack propagates, the material in that region is unloaded and its energy is made available to drive the crack. When the crack is long enough that an increment in crack length gives rise to an energy release equal to energy expended, catastrophic crack growth impends.

Griffith proposed an **energy** approach to fracture. The elastic energy stored in a test specimen of unit thickness, in a circular region around a crack of length a, is:



$$2\pi a^2 \frac{1}{2E}\sigma^2 = 2 \cdot \text{strain energy density} \cdot \pi a^2 \cdot 1 \qquad \text{(F1)}$$

Recall that $\frac{1}{2}E\varepsilon^2 = \frac{1}{2E}\sigma^2$ represents a strain energy per unit volume.

The elastic energy for a brittle material is twice the area under the stress strain curve. The elastic energy is used to create two new surfaces as the crack propagates. The surface energy, 4 γa (γ is the surface energy; it is an energy per unit area.) should be smaller than the elastic energy for the crack to grow. Thus, the incremental changes of both energies for the crack to grow can be written,

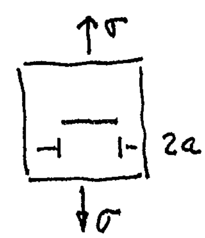
| 2a | 2 surfaces ∴ energy = 2γ·2a

surface ener

$$\frac{d}{da}\left(\frac{\pi(a\sigma)^2}{E}\right) = \frac{d}{da}(4\gamma a) \qquad \text{(F2)}$$

Hence,

$$\frac{2\pi a\sigma^2}{E} = 4\gamma$$



$$\sigma = \sigma_f = \sqrt{\frac{2\gamma E}{\pi a}} \qquad \text{(F3)}$$

Since for a given material E and γ are constants,

$$\sigma_f = \frac{K}{\sqrt{\pi a}} \qquad \text{(F4)}$$

In this case K has the units of psi $\sqrt{\text{in}}$ or MPa $\sqrt{\text{m}}$ and is proportional to the energy required for fracture.

K is a measure of *fracture toughness,* called the stress intensity factor. Cracks and stress concentrations also occur in ductile materials, but their effect is usually not as serious as in brittle ones since local yielding which occurs in the region of peak stress will effectively blunt the crack and alleviate the stress concentration.

Example (adapted from example on page 62-63)

Consider a 1 cm crack in a large plate of steel. Assume the plate is large enough that $f(a/c) \cong 1$.

What stress gives rise to fracture for a weaker or 'mild' steel (σ_y = 500 MPa, K_{1c} = 175 MPa$\sqrt{m}$) and a high strength steel (σ_y = 1410 MPa, K_{1c} = 50 MPa$\sqrt{m}$).

Solution: Use $K_1 = \sigma f(a/c) \sqrt{\pi a}$, so with $f(a/c) \cong 1$, $\sigma = K_{1c}/ \sqrt{\pi a}$.

Weaker steel A, σ = 987 MPa, which exceeds the yield strength, so it undergoes yield at 500 MPa rather than fracture from the crack.

Stronger steel, σ = 282 MPa, which is less than the yield strength 1410 MPa, so it fractures catastrophically.

Consequently, in the presence of 1 cm cracks, the stronger steel actually is weaker than the 'mild' weaker steel. Moreover, the consequence of an overload is more severe for the stronger steel since it catastrophically fractures rather than yields in bulk.

Example (adapted from Gordon, *Structures*)

Suppose we have a large structure such as a ship or a bridge and wish to tolerate a 1 meter long crack without catastrophic failure. Consider 'mild' steel (σ_y = 500 MPa, K_{1c} = 175 MPa$\sqrt{m}$).
Solution-

With $f(a/c) \cong 1$, $\sigma = K_{1c}/ \sqrt{\pi a}$ = 90 MPa or 14,000 psi.

In **foam**, Gibson and Ashby [*Cellular solids*] predict toughness K_{1c} proportional to [√(cell size)](density)$^{3/2}$.

Stress concentrations: appendix

Experimental stress concentrations in composite materials are consistently less than the theoretical ones. The non-classical fracture behavior has been dealt with using point stress and average stress criteria, however that approach cannot account for non-classical strain distributions in objects under small load. Such differences may be accounted for via a generalized continuum approach. Reduced stress concentration factors for small holes are known experimentally in fibrous composite materials. The fracture strength of graphite epoxy plates with holes depends on the size of the hole [1]. Moreover the strain around small holes and notches in fibrous composites well below the yield point is smaller than expected classically [2,3], while for large holes, the strain field follows classical predictions [4].

1. R.F. Karlak, "Hole effects in a related series of symmetrical laminates", in *Proceedings of failure modes in composites, IV*, The metallurgical society of AIME, Chicago, 106-117, (1977)
2. J.M. Whitney, and R.J. Nuismer, "Stress fracture criteria for laminated composites containing stress concentrations", *J. Composite Materials*, **8**, (1974) 253-275.
3. M. Daniel, "Strain and failure analysis of graphite-epoxy plates with cracks", *Experimental Mechanics,* **18** (1978) 246-252, .
4. R. E. Rowlands, I. M. Daniel, and J. B. Whiteside "Stress and failure analysis of a glass-epoxy plate with a circular hole", *Experimental Mechanics*, **13**, (1973) 31-37

nuch lower stresses. Thickness-direction tension at the iist ahead of the crack tip is therefore in a state of *triax-* e more than plastic flow.

ate at great speed after reaching a critical length? The considerations, which are summarized as follows. netry of Fig. 3.5-1a, which is shown again in Fig. 3.5-2a. ack an amount da is independent of crack length a, so ce the crack varies linearly with a (Fig. 3.5-2b). As the ed in material alongside the crack, and stored strain y released varies approximately quadratically with a. e that a crack of length a nullifies the uniaxial state of radius a, shown dashed in Fig. 3.5-2a. Strain energy ional to the volume of this disc, $V = \pi a^2 t/2$.) When a ent of strain energy released to equal the increment e crack, sudden fracture impends. Thus $dU_e = dU_s$ in crack length. Energy needed to drive the crack is sup- material. There is no need (and insufficient time) for k of external forces acting through a distance. The fail- ıld be if the crack were absent because the crack pro- onds to be broken sequentially rather than all at once. as shown in Fig. 3.5-1. In practice, Mode I is most com- For any mode, one can calculate a *stress intensity fac-* e it with an allowable value to determine whether sity factor is *not* a stress concentration factor! Indeed, it is not necessary to use stress concentration data: not be calculated.

er only isotropic materials, and only Mode I cracks ss intensity factor for a Mode I crack is denoted by K_I

$$K_I = \beta\sigma\sqrt{\pi a} \tag{3.5-1}$$

would exist if the crack were absent. Thus, stress σ is ıltiplier β is dimensionless and depends on geometry .5-1). Dimension a is defined as either the full crack geometry. Units of K_I are MPa$\sqrt{\text{m}}$. Fracture impends known as *fracture toughness*. K_{Ic} can be considered a of specimen thickness t, if the specimen is sufficiently sion at the crack tip to develop fully. Also, the crack inimum. Recommended minimum dimensions are

$$\ldots\Big)^2 \quad \text{and} \quad a \geq 2.5\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 \tag{3.5-2}$$

determined by a tension test of the material, and t is thickness is less than the value described by Eq. 3.5-2,

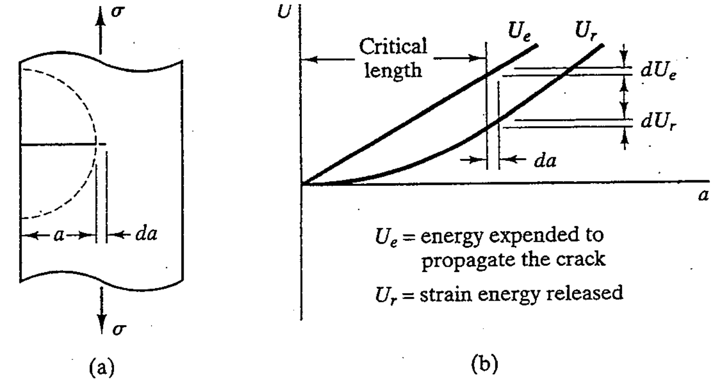
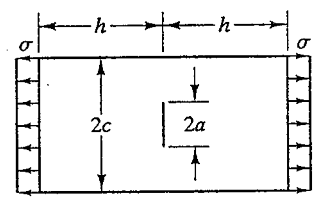


FIGURE 3.5-2 (a) Plate with an edge crack of length a. (b) Energy relations for crack extension.

TABLE 3.5-1 Stress intensity data for flat plates, of isotropic material and uniform thickness, with in-plane loading [3.6].

Tension, central crack of length $2a$

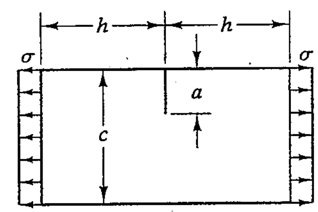


$K_I = \beta\sigma\sqrt{\pi a}$

$$\beta = \frac{1 - 0.5(a/c) + 0.326(a/c)^2}{\sqrt{1-(a/c)}}$$

Accurate to within 1% for all a/c, provided h/c is "large"

Tension, edge crack of length a

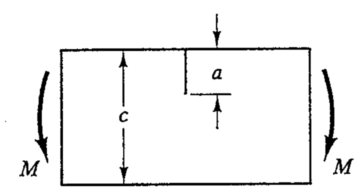


$K_I = \beta\sigma\sqrt{\pi a}$

$\beta = [1.12 - 0.23\,(a/c) + 10.6\,(a/c)^2 - 21.7\,(a/c)^3 + 30.4\,(a/c)^4]$

Accurate to within 1% for $a/c \leqslant 0.6$, provided $h/c > 1$ and sides are free to rotate

Pure bending, edge crack of length a

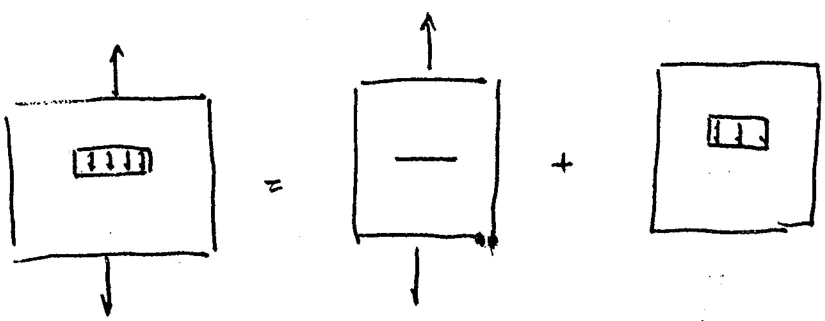


$K_I = \beta\sigma\sqrt{\pi a}$

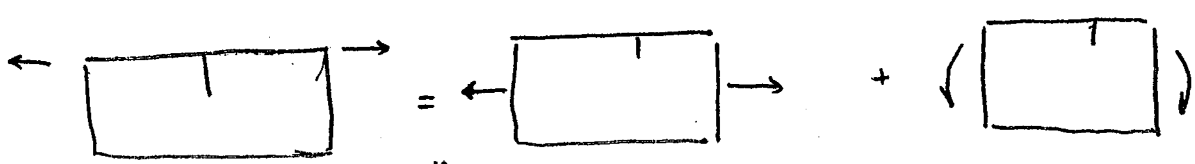
$$\sigma = \frac{M(c/2)}{I} = \frac{6M}{tc^2}$$

$\beta = [1.12 - 1.39\,(a/c) + 7.32\,(a/c)^2 - 13.1\,(a/c)^3 + 14.0\,(a/c)^4]$

Accurate to within 1% for $a/c \leqslant 0.6$

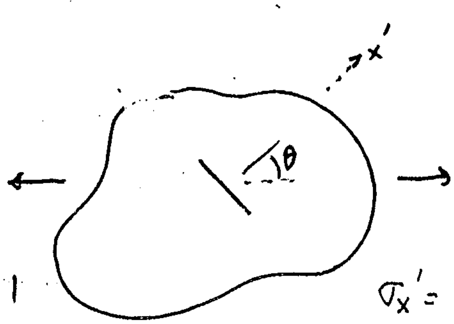


$$K_I = K_{I_{(1)}} + K_{I_{(2)}}$$



$$K_{I_{TOT}} = \sum K_I^{(i)}$$

$$K_{II_{TOT}} = \sum K_{II}^{(i)}$$



find σ_x' & $\tau_{x'y'}$

$K_I = \sigma_x' \sqrt{\pi a}$

σ_x $\quad K_{II} = \tau_{x'y'} \sqrt{\pi a}$

$$\sigma_x' = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = \tau_{xy} \cos 2\theta - \left(\frac{\sigma_x - \sigma_y}{2}\right) \sin 2\theta \; ?$$

$$\left(\frac{K_I}{K_{I_c}}\right)^2 + \left(\frac{K_{II}}{K_{II_c}}\right)^2 \geq 1$$

$$\left(\frac{\sigma_x}{\sigma_{yp}}\right)^2 + \left(\frac{\tau_{xy}}{\tau_{max}}\right)^2 \geq 1$$

$K_I = \sigma_x' \sqrt{\pi a}$

$K_{II} = \tau_{x'y'} \sqrt{\pi a}$

$K_{I_c} = \sigma_y \sqrt{\pi a}$

$K_{II} = \tau_{max} \sqrt{\pi a}$

Fatigue

A material loaded through multiple cycles will break at a stress considerably less than the ultimate strength for a single application of load. Fatigue is quantified by the S-N curve, in which the number N of cycles is plotted logarithmically.

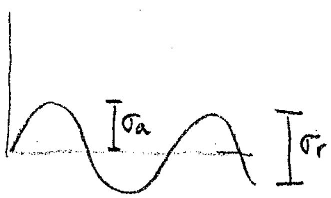
The effect of cyclic stresses is to initiate microcracks at centers of stress concentration within the material or on the surface resulting in the growth and propagation of cracks leading to failure.

As for fatigue testing, the rate of crack growth can be plotted in a log-log scale versus time. Testing the fatigue properties to generate an S-N curve entails monitoring the number of cycles to failure at various stress levels. This test requires a large number of specimens compared with the crack propagation test.

The *endurance limit* is the stress below which the material will not fail in fatigue no matter how many cycles are applied. Not all materials exhibit an endurance limit. (a practical limit is often chosen as 10^7 cycles).

The presence of a saline environment exacerbates fatigue.
Surface roughness exacerbates fatigue. A polished surface is better.
Rubbing or 'fretting' exacerbates fatigue. Re-design the part or use lubricants.
Heat treatment to introduce residual surface compression can be helpful.

FATIGUE DEPENDS ON $R = \frac{\sigma_{min}}{\sigma_{max}}$



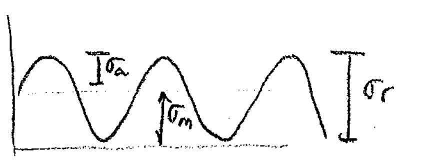
σ_a = stress amplitude $\frac{\sigma_{max} - \sigma_{min}}{2}$

σ_r = stress range $= 2\sigma_a$

$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}$

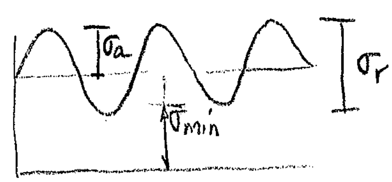
$\sigma_m = 0$

$R = -1$ fully reversed



$\sigma_{min} = 0$

$R = 0$



$\sigma_{min} \neq 0$

$R > 0$

NOTCH GEOMETRY EFFECTS as $\rho\uparrow$ (FOR SAME $\Delta\sigma$) NO OF CYCLES $\uparrow$ UNTIL CRACK STARTS

as $\rho = \text{const}$ if $\Delta\sigma\uparrow$ NO OF CYCLES $\downarrow$

ALSO $\frac{da}{dN}$ has been found to vary as $\frac{(\Delta K)^n}{(1-R)K_c - \Delta K}$

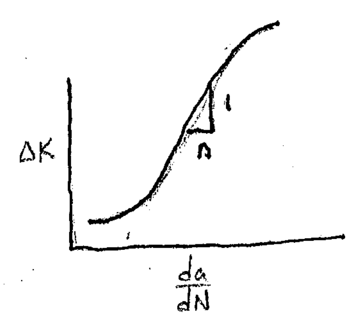
$\Delta K = K_{max} - K_{min}$

and $K_c = \max K$ when $\frac{da}{dN} \to \infty$

$K_{max} = \sigma_{max}\sqrt{\pi a}$

$K_{min} = \sigma_{min}\sqrt{\pi a}$

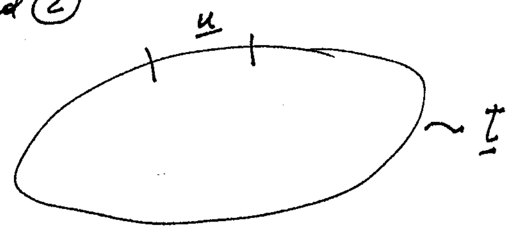
$$\int_{a_0}^{a} da = \int_0^N \frac{(\Delta K)^n}{(1-R)K_c - \Delta K}\, dN$$



INTRO TO ELASTICITY

UNIQUENESS OF SOLUTIONS

1) IF WE SPECIFY A TRACTION VECTOR $\sigma \cdot \underline{n} = \underline{t}_n = \sigma_{in}$ at all points on the boundary then the displacement vector $\underline{u}$ (u, v, w) is unique up to a rigid body motion (Neumann-type problem)
2) IF $\underline{u}$ IS defined everywhere on the boundary, then this gives a unique solution to the problem (Dirichlet-type problem)
3) Mix 1 and 2



$\underline{t}_x, \underline{t}_y, u_z$

u_x, u_y, t_z

Cannot do $\underline{\underline{u_x, t_x}}, u_y$

Failure criteria

Failure criteria deal with failure of a material, in contrast with failure of a structure.

Brittle materials

Maximum normal stress criterion

Failure if $\sigma_1 \geq \sigma_t^{ult}$, that is , if the largest tensile principal stress exceeds the ultimate strength. The other principal stresses are ignored. Similarly in compression, the applied stress is compared with the ultimate strength in compression. This criterion is simplistic since if all the principal stresses are compressive, most materials are much stronger than would be expected based on uniaxial tests.

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This takes into account ultimate tensile, compressive and shear stresses. Represent each state by a Mohr circle. Draw an 'envelope' tangent to the circles. Mohr suggested that, provided an arbitrary state of stress was represented by Mohr circle within that envelope, failure would not occur. Sometimes a simplified form is taken in which the shear test is ignored. Then the envelope is a straight line and failure is predicted if $\sigma_1/\sigma_t^{ult} - \sigma_3/\sigma_c^{ult} \geq 1$. This also is not ery realistic for hydrostatic compression.

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Maximum shear stress criterion (Tresca criterion)

Yield when $\tau_{max} \geq \tau_Y$.

In principal stress space this looks like a hexagon.

Tension test to yield.

Recall that the maximum shear stress is $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$, but the minimum principal stress is zero so for tension, $\tau_Y = 0.5\,\sigma_Y$.

Von Mises

Yield when $\sigma_{eff} \geq \tau_Y$.

Recall $\sigma_{eff} = \frac{1}{\sqrt{2}}\{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\{\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2\}\}^{1/2}$.

Distortional energy, $U_{0d} = \frac{3}{4G}\ \tau_{oct}^2 = \frac{1}{6G}\ \sigma_{eff}^2$.

In principal stress space, the Von Mises criterion looks like an ellipse.

Fatigue

A material loaded through multiple cycles will break at a stress considerably less than the ultimate strength for a single application of load. Fatigue is quantified by the S-N curve, in which the number N of cycles is plotted logarithmically.

The effect of cyclic stresses is to initiate microcracks at centers of stress concentration within the material or on the surface resulting in the growth and propagation of cracks leading to failure.

As for fatigue testing, the rate of crack growth can be plotted in a log-log scale versus time. Testing the fatigue properties to generate an S-N curve entails monitoring the number of cycles to failure at various stress levels. This test requires a large number of specimens compared with the crack propagation test.

The *endurance limit* is the stress below which the material will not fail in fatigue no matter how many cycles are applied. Not all materials exhibit an endurance limit. (a practical limit is often chosen as 10^7 cycles).

The presence of a saline environment exacerbates fatigue.
Surface roughness exacerbates fatigue. A polished surface is better.
Rubbing or 'fretting' exacerbates fatigue. Re-design the part or use lubricants.
Heat treatment to introduce residual surface compression can be helpful.

Stress concentration

Ratio of local maximum stress to applied stress in the absence of the heterogeneity is called the stress concentration factor or SCF.

Stress concentration factors are determined from

- Elasticity theory
- Experiment
- Finite elements.

Stress concentration factors arise from

- Holes
- Notches
- Grooves
- Heterogeneities in loading
- Heterogeneities in material

Some particular values for holes and inclusions

Circular hole in plane uniaxial tension, SCF = K = 3.0

Elliptic hole, with a as major axis, b as minor axis, ρ as radius of curvature

$$\mathrm{SCF} = K = (1 + 2\frac{a}{b}) = 1 + 2\sqrt{\frac{a}{\rho}}$$

Example, consider glass with theoretical strength of $\sigma^{ult} = 14$ GPa, with cracks 2 μm long with radius of curvature 1 A (0.1 nm). Then the strength of glass with these cracks is $\sigma = 14$ GPa/[SCF] = 70 MPa. This is about the strength of common glass.

Spherical cavity in uniaxial tension

$$\mathrm{SCF}|_{polar} = -\frac{3+15\nu}{14-10\nu} \quad \mathrm{SCF}|_{eq\theta\theta} = \frac{27-15\nu}{14-10\nu} \quad \mathrm{SCF}|_{eq\psi\psi} = -\frac{3-15\nu}{14-10\nu}$$

Spherical cavity in biaxial tension

Spherical cavity in pure shear

$$\mathrm{SCF} = \frac{15(1-\nu)}{7-5\nu}$$

Rigid cylindrical inclusion in uniaxial tension

$$\mathrm{SCF}|_{polar} = \frac{1}{2}\left(3-2\nu+\frac{1}{3-4\nu}\right) \qquad \mathrm{SCF}|_{eq} = \frac{1}{2}\left(1+2\nu-\frac{3}{3-4\nu}\right)$$

Rigid spherical inclusion in uniaxial tension

$$\mathrm{SCF}|_{polar} = \frac{2}{1+\nu} + \frac{1}{4-5\nu} \qquad \mathrm{SCF}|_{eq} = \frac{\nu}{1+\nu} - \frac{5\nu}{8-10\nu}$$

Rigid spherical inclusion in hydrostatic tension

$$SCF|_{radial} = 3\,\frac{1-\nu}{1+\nu}$$

Reference: Goodier, J. N., "Concentration of stress around spherical and cylindrical inclusions and flaws", *Trans. ASME* Vol. 55, 1933, 39-44. (later called *J. Applied Mech.*, Vol. 1)

Observe that for the three dimensional cases, the stress concentration factor depends on the Poisson's ratio of the material in question.

A heterogeneous load distribution

Consider a **rigid circular cylindrical indenter** of radius R pressed with load F on a semi infinite solid substrate. This could represent a building erected upon compliant earth, or an industrial press operation. A solution for an elastic solid of Young's modulus E and Poisson's ratio ν is available. The indenter displacement is (Timoshenko, S. P. and Goodier, J. N., *Theory of Elasticity*, McGraw Hill, 1982.)

$$w = \frac{F(1 - \nu^2)}{2RE}\,.$$

The pressure distribution q(r) as a function of radial coordinate r is

$$q(r) = \frac{F}{2\pi R\sqrt{R^2 - r^2}}\,.$$

Observe that the pressure becomes **singular** at the edge. The indenter is idealized as perfectly rigid (much stiffer than the elastic substrate), and with a perfectly sharp edge.

Uses of concept of stress concentration.

- Ø Find stress distribution (nominal) in the absence of holes.
- Ø Multiply nominal stresses by the appropriate stress concentration factors. Many of these may be obtained from a handbook.
- Ø The largest stress will cause failure. It is not necessarily the largest nominal stress.
- Ø In the design process it is sensible to ameliorate stress concentrations by avoiding sharp re-entrant corners, and rounding them off when they are an unavoidable part of a structure.

Demonstrations, by photoelasticity. Circular hole at center of a compressed bar. Circular hole in bar subjected to pure bending. Circular notches in bar subjected to pure bending.

Contact stress

From the theory of elasticity, we have several interesting solutions for spheres and cylinders in contact.

For spheres of radius R of Young's modulus E, Poisson's ratio ν, under force F, contact region radius: $a = 0.880\ (FR/E)^{1/3}$. It increases slowly with force.

peak compressive stress: $p_0 = 0.616\ (FE^2/R^2)^{1/3}$. It increases slowly with force.

Stress vs radial position in region, $\sigma_z = -\, p_0 \dfrac{\sqrt{a^2 - r^2}}{a}$, a parabolic distribution.

Overall 3-D pattern of stress is complex and multiaxial. Cracks may develop below the surface in ball bearings.

Fracture mechanics

As elliptic hole becomes progressively narrower, the ellipse approaches a crack shape and SCF = K $\to\infty$. Actual observed stress concentration factors for cracks are not infinite.

Therefore a material with one perfectly sharp crack will have **zero** strength, since the stress concentration factor becomes infinite. Experimentally, even for brittle materials, strength is reduced by cracks but not infinitely.

A criterion based on energy balance rather than on stress has therefore been adopted. The approach is due to Griffith. The energy relations are visualized as follows. The energy required to extend a crack is linear in crack length because the energy is expended in creating new free surfaces. By contrast the strain energy available comes from a roughly semicircular region around the crack and is therefore quadratic in crack length. As the crack propagates, the material in that region is unloaded and its energy is made available to drive the crack. When the crack is long enough that an increment in crack length gives rise to an energy release equal to energy expended, catastrophic crack growth impends.

Griffith proposed an **energy** approach to fracture. The elastic energy stored in a test specimen of unit thickness, in a circular region around a crack of length a, is:

$$2\pi a^2 \frac{1}{2E}\sigma^2 \qquad \text{(F1)}$$

Recall that $\frac{1}{2}E\varepsilon^2 = \frac{1}{2E}\sigma^2$ represents a strain energy per unit volume.

The elastic energy for a brittle material is twice the area under the stress strain curve. The elastic energy is used to create two new surfaces as the crack propagates. The surface energy, $4\gamma a$ (γ is the surface energy; it is an energy per unit area.) should be smaller than the elastic energy for the crack to grow. Thus, the incremental changes of both energies for the crack to grow can be written,

$$\frac{d}{da}\left(\frac{\pi(a\sigma)^2}{E}\right) = \frac{d}{da}(4\gamma a) \qquad \text{(F2)}$$

Hence,

$$\sigma = \sigma_f = \sqrt{\frac{2\gamma E}{\pi a}} \qquad \text{(F3)}$$

Since for a given material E and γ are constants,

$$\sigma_f = \frac{K}{\sqrt{\pi a}} \qquad \text{(F4)}$$

In this case K has the units of psi $\sqrt{in}$ or MPa $\sqrt{m}$ and is proportional to the energy required for fracture.
K is a measure of *fracture toughness,* called the stress intensity factor. Cracks and stress concentrations also occur in ductile materials, but their effect is usually not as serious as in brittle ones since local yielding which occurs in the region of peak stress will effectively blunt the crack and alleviate the stress concentration.

The **stress intensity factor** K is the criterion for fracture in cracked objects. For a small Mode I crack of length a,

$$K_1 = \sigma \sqrt{\pi a}\ f(a/c).$$

Here f(a/c) is a dimensionless function of loading geometry; it expresses the effect of crack length in relation to block size. σ is the stress required for fracture in the absence of a crack. The units for K are MPa$\sqrt{m}$, in contrast to the stress concentration factor which is dimensionless. Observed that there is no characteristic length scale in the classical theory of elasticity. The length scale must come from other considerations.

Fracture occurs when K_1 exceeds a critical value, K_{1c} determined from experiment. This is the fracture toughness based on a static test. The fracture toughness for a dynamic situation is NOT the same as for a static situation

Formulas for K are valid over a range of geometrical parameters, specifically, thickness $t \geq 2.5\ (K_{1c}/\sigma_y)^2$, and crack length $a \geq 2.5\ (K_{1c}/\sigma_y)^2$.

In a thick block, the stress field around the tip of the crack is triaxial, since the Poisson contraction in the highly stressed region near the crack is restrained by the surrounding material, which is not so highly stressed. This triaxial stress causes brittle behavior in seemingly ductile materials, since shear deformation is suppressed.

If the block is thinner than the above limit, toughness depends on thickness. If the crack length is less than the above limit, then the material may undergo yield before any fracture occurs from the crack.

Be aware that K_{1c} depends on temperature, and often drops precipitously at low temperature.

Example

Estimate the size of the surface flaw in a glass whose modulus of elasticity and surface energy are 70 GPa and 800 erg/cm^2 respectively. Assume that the glass breaks at a tensile stress of 100 MPa.

Answer
From equation (F4), and keeping in mind the transformation from cgs to SI units,

$$a = \frac{2\gamma E}{\pi \sigma_f^2}$$

$$= \frac{2 \times 800 \text{ dyne/cm} \times 70 \text{ GPa}}{\pi (100 \text{ MPa})^2}$$

$$= \underline{3.565 \ \mu m}$$

To two significant figures, $a = 3.6$ µm.
[Note that if the crack is on the surface its length is a, if it is inside the specimen it is 2a.
Remember 1 erg = 1 dyne cm]

Example (adapted from example on page 62-63)

Consider a 1 cm crack in a large plate of steel. Assume the plate is large enough that $f(a/c) \cong 1$.

What stress gives rise to fracture for a weaker or 'mild' steel ($\sigma_y = 500$ MPa, $K_{1c} = 175$ MPa$\sqrt{m}$) and a high strength steel ($\sigma_y = 1410$ MPa, $K_{1c} = 50$ MPa$\sqrt{m}$).

Solution: Use $K_1 = \sigma f(a/c) \sqrt{\pi a}$, so with $f(a/c) \cong 1$, $\sigma = K_{1c}/\sqrt{\pi a}$.

Weaker steel A, $\sigma = 987$ MPa, which exceeds the yield strength, so it undergoes yield at 500 MPa rather than fracture from the crack.

Stronger steel, $\sigma = 282$ MPa, which is less than the yield strength 1410 MPa, so it fractures catastrophically.

Consequently, in the presence of 1 cm cracks, the stronger steel actually is weaker than the 'mild' weaker steel. Moreover, the consequence of an overload is more severe for the stronger steel since it catastrophically fractures rather than yields in bulk.

Example (adapted from Gordon, *Structures*)

Suppose we have a large structure such as a ship or a bridge and wish to tolerate a 1 meter long crack without catastrophic failure. Consider 'mild' steel ($\sigma_y = 500$ MPa, $K_{1c} = 175$ MPa$\sqrt{m}$).

Solution-
With $f(a/c) \cong 1$, $\sigma = K_{1c}/\sqrt{\pi a} = 90$ MPa or 14,000 psi.

In **foam,** Gibson and Ashby [*Cellular solids*] predict toughness K_{1c} proportional to [√(cell size)](density)$^{3/2}$.

Stress concentrations: appendix

Experimental stress concentrations in composite materials are consistently less than the theoretical ones. The non-classical fracture behavior has been dealt with using point stress and average stress criteria, however that approach cannot account for non-classical strain distributions in objects under small load. Such differences may be accounted for via a generalized continuum approach. Reduced stress concentration factors for small holes are known experimentally in fibrous composite materials. The fracture strength of graphite epoxy plates with holes depends on the size of the hole [1]. Moreover the strain around small holes and notches in fibrous composites well below the yield point is smaller than expected classically [2,3], while for large holes, the strain field follows classical predictions [4].

1. R.F. Karlak, "Hole effects in a related series of symmetrical laminates", in *Proceedings of failure modes in composites, IV*, The metallurgical society of AIME, Chicago, 106-117, (1977)
2. J.M. Whitney, and R.J. Nuismer, "Stress fracture criteria for laminated composites containing stress concentrations", *J. Composite Materials*, **8**, (1974) 253-275.
3. M. Daniel, "Strain and failure analysis of graphite-epoxy plates with cracks", *Experimental Mechanics*, **18** (1978) 246-252, .
4. R. E. Rowlands, I. M. Daniel, and J. B. Whiteside "Stress and failure analysis of a glass-epoxy plate with a circular hole", *Experimental Mechanics*, **13**, (1973) 31-37

The surface given by Eq. (119) being drawn with the point O (Fig. 128) as center, we may identify δx, δy, δz in Eqs. (b) with x, y, z in Eqs. (e).

We consider now the special case when ω_x, ω_y, ω_z are zero. Then the right-hand sides of Eqs. (e) are the same as the right-hand sides of Eqs. (b) but for a factor 2. Consequently the displacement given by Eqs. (b) is normal to the surface given by Eq. (119). Considering the point O_1 (Fig. 128) as a point on the surface, this means that the displacement of O_1 is normal to the surface at O_1. Hence if OO_1 is one of the principal axes of strain, that is, one of the principal axes of the surface, the displacement of O_1 is in the direction of OO_1, and therefore OO_1 does not rotate. The displacement in question will correspond to step 1.

In order to complete the displacement, we must restore to Eqs. (b) the terms in ω_x, ω_y, ω_z. But these terms correspond to a small rigid-body rotation having components ω_x, ω_y, ω_z about the x, y, z axes. Consequently these quantities, given by (122), express the rotation of step 3—that is, the rotation of the principal axes of strain at the point O. They are called simply the *components of rotation.*

PROBLEMS

1. What is the equation, of the type $f(x,y,z) = 0$, of the surface with center at O which becomes a sphere $x'^2 + y'^2 + z'^2 = r^2$ *after* the homogeneous deformation of Art. 80? What kind of surface is it?
2. Show that if the rotation is zero throughout the body (irrotational deformation), the displacement vector is the gradient of a scalar potential function.

   Indicate one or more examples of such irrotational deformation from the problems treated in the text.

chapter | 8

Timoshenko & Goodier
Theory of Elasticity

General Theorems

stresses & strains in 3-D

84 | Differential Equations of Equilibrium

In the discussion of Art. 74 we considered the stress at a point of an elastic body. Let us consider now the variation of the stress as we change the position of the point. For this purpose the conditions of equilibrium of a small rectangular parallelepiped with the sides δx, δy, δz (Fig. 129) must be studied. The components of stresses acting on the sides of this small element and their positive directions are indicated in the figure. Here we take into account the small changes of the components of stress due to the small increases δx, δy, δz of the coordinates. Thus designating the midpoints of the sides of the element by 1, 2, 3, 4, 5, 6 as in Fig. 129, we distinguish between the value of σ_x at point 1, and its value at point 2, writing these $(\sigma_x)_1$ and $(\sigma_x)_2$, respectively. The symbol σ_x itself denotes, of course, the value of this stress component at the point x, y, z. In calculating the *forces* acting on the element we consider the sides as very small, and the force is obtained by multiplying the stress at the centroid of a side by the area of this side.

It should be noted that the body force acting on the element, which was neglected as a small quantity of higher order in discussing the equilibrium of a tetrahedron (Fig. 126), must now be taken into account, because it is of the same order of magnitude as the terms due to variations of the stress components, which we are now considering. If we let X, Y, Z denote the components of this force per unit volume of the element, then the equation of equilibrium obtained by summing all the forces acting on the element in the x direction is

$$[(\sigma_x)_1 - (\sigma_x)_2]\,\delta y\,\delta z + [(\tau_{xy})_3 - (\tau_{xy})_4]\,\delta x\,\delta z\,[(\tau_{xz})_5 - (\tau_{xz})_6]\,\delta x\,\delta y + X\,\delta x\,\delta y\,\delta z = 0$$

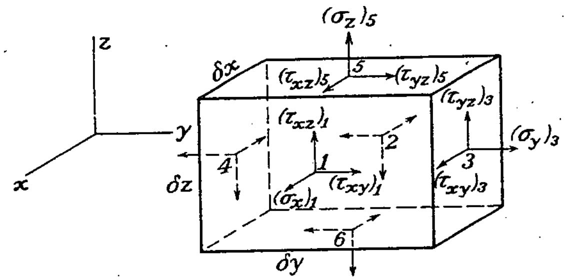

Fig. 129

The two other equations of equilibrium are obtained in the same manner. After dividing by $\delta x\ \delta y\ \delta z$ and proceeding to the limit by shrinking the element down to the point x, y, z, we find

$$\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + X &= 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yz}}{\partial z} + Y &= 0 \\ \frac{\partial \sigma_z}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + Z &= 0 \end{aligned} \tag{123}$$

Equations (123) must be satisfied at all points throughout the volume of the body. The stresses vary over the volume of the body, and when we arrive at the surface they must be such as to be in equilibrium with the external forces on the surface of the body. These conditions of equilibrium at the surface can be obtained from Eqs. (108). Taking a tetrahedron $OBCD$ (Fig. 126) so that the side BCD coincides with the surface of the body, and denoting by $\bar{X}, \bar{Y}, \bar{Z}$ the components of the surface forces per unit area at this point, Eqs. (108) become

$$\begin{aligned} \bar{X} &= \sigma_x l + \tau_{xy} m + \tau_{xz} n \\ \bar{Y} &= \sigma_y m + \tau_{yz} n + \tau_{xy} l \\ \bar{Z} &= \sigma_z n + \tau_{xz} l + \tau_{yz} m \end{aligned} \tag{124}$$

in which l, m, n are the direction cosines of the external normal to the surface of the body at the point under consideration.

If the problem is to determine the state of stress in a body submitted to the action of given forces, it is necessary to solve Eqs. (123), and the solution must be such as to satisfy the boundary conditions (124). These equations, containing six components of stress, $\sigma_x, \ldots, \tau_{yz}$, are not sufficient for the determination of these components. The problem is a statically indeterminate one, and in order to obtain the solution we must

proceed as in the case of two-dimensional problems, i.e., the elastic deformations of the body must also be considered.

85 | Conditions of Compatibility

It should be noted that the six components of strain at each point are completely determined by the three functions u, v, w, representing the components of displacement. Hence, the components of strain cannot be taken arbitrarily as functions of x, y, z but are subject to relations that follow from Eqs. (2).

Thus, from Eqs. (2),

$$\frac{\partial^2 \epsilon_x}{\partial y^2} = \frac{\partial^3 u}{\partial x\,\partial y^2} \qquad \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^3 v}{\partial x^2\,\partial y} \qquad \frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} = \frac{\partial^3 u}{\partial x\,\partial y^2} + \frac{\partial^3 v}{\partial x^2\,\partial y}$$

from which

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} \tag{a}$$

Two more relations of the same kind can be obtained by cyclical interchange of the letters x, y, z.

From the derivatives

$$\frac{\partial^2 \epsilon_x}{\partial y\,\partial z} = \frac{\partial^3 u}{\partial x\,\partial y\,\partial z} \qquad \frac{\partial \gamma_{yz}}{\partial x} = \frac{\partial^2 v}{\partial x\,\partial z} + \frac{\partial^2 w}{\partial x\,\partial y}$$

$$\frac{\partial \gamma_{xz}}{\partial y} = \frac{\partial^2 u}{\partial y\,\partial z} + \frac{\partial^2 w}{\partial x\,\partial y} \qquad \frac{\partial \gamma_{xy}}{\partial z} = \frac{\partial^2 u}{\partial y\,\partial z} + \frac{\partial^2 v}{\partial x\,\partial z}$$

we find that

$$2\frac{\partial^2 \epsilon_x}{\partial y\,\partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \tag{b}$$

Two more relations of the kind (b) can be obtained by interchange of the letters x, y, z. We thus arrive at the following six differential relations between the components of strain, which must be satisfied by virtue of Eqs. (2):

$$\begin{aligned} \frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x\,\partial y} & 2\frac{\partial^2 \epsilon_x}{\partial y\,\partial z} &= \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\ \frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y\,\partial z} & 2\frac{\partial^2 \epsilon_y}{\partial x\,\partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial \gamma_{yz}}{\partial x} - \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right) \\ \frac{\partial^2 \epsilon_z}{\partial x^2} + \frac{\partial^2 \epsilon_x}{\partial z^2} &= \frac{\partial^2 \gamma_{xz}}{\partial x\,\partial z} & 2\frac{\partial^2 \epsilon_z}{\partial x\,\partial y} &= \frac{\partial}{\partial z}\left(\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} - \frac{\partial \gamma_{xy}}{\partial z}\right) \end{aligned} \tag{125}$$

These differential relations[1] are called the *conditions of compatibility*.

[1] Proofs that these six equations are sufficient to ensure the existence of a displacement corresponding to a given set of functions $\epsilon_x, \ldots, \gamma_{xy} \ldots$, may be found in

By using Hooke's law [Eqs. (3)] conditions (125) can be transformed into relations between the components of stress. Take, for instance, the condition

$$\frac{\partial^2 \epsilon_y}{\partial z^2} + \frac{\partial^2 \epsilon_z}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y\, \partial z} \tag{c}$$

From Eqs. (3) and (4), using the notation (7), we find

$$\epsilon_y = \frac{1}{E}[(1+\nu)\sigma_y - \nu\Theta]$$

$\Theta = I_{1\sigma} = \sigma_x + \sigma_y + \sigma_z$

$$\epsilon_z = \frac{1}{E}[(1+\nu)\sigma_z - \nu\Theta]$$

$$\gamma_{yz} = \frac{2(1+\nu)\tau_{yz}}{E}$$

Substituting these expressions in (c), we obtain

$$(1+\nu)\left(\frac{\partial^2 \sigma_y}{\partial z^2} + \frac{\partial^2 \sigma_z}{\partial y^2}\right) - \nu\left(\frac{\partial^2 \Theta}{\partial z^2} + \frac{\partial^2 \Theta}{\partial y^2}\right) = 2(1+\nu)\frac{\partial^2 \tau_{yz}}{\partial y\, \partial z} \tag{d}$$

The right side of this equation can be transformed by using the equations of equilibrium (123). From these equations we find

$$\frac{\partial \tau_{yz}}{\partial y} = -\frac{\partial \sigma_z}{\partial z} - \frac{\partial \tau_{xz}}{\partial x} - Z$$

$$\frac{\partial \tau_{yz}}{\partial z} = -\frac{\partial \sigma_y}{\partial y} - \frac{\partial \tau_{xy}}{\partial x} - Y$$

Differentiating the first of these equations with respect to z and the second with respect to y, and adding them together, we find

$$2\frac{\partial^2 \tau_{yz}}{\partial y\, \partial z} = -\frac{\partial^2 \sigma_z}{\partial z^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial}{\partial x}\left(\frac{\partial \tau_{xz}}{\partial z} + \frac{\partial \tau_{xy}}{\partial y}\right) - \frac{\partial Z}{\partial z} - \frac{\partial Y}{\partial y}$$

or, by using the first of Eqs. (123),

$$2\frac{\partial^2 \tau_{yz}}{\partial y\, \partial z} = \frac{\partial^2 \sigma_x}{\partial x^2} - \frac{\partial^2 \sigma_y}{\partial y^2} - \frac{\partial^2 \sigma_z}{\partial z^2} + \frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}$$

Substituting this in Eq. (d) and using, to simplify the writing, the symbol

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

A. E. H. Love, "Mathematical Theory of Elasticity," 4th ed., p. 49, Cambridge University Press, New York, 1927, and I. S. Sokolnikoff, "Mathematical Theory of Elasticity," p. 25, 1956. The equations themselves were given by B. de Saint-Venant in his edition of the book by C. L. M. H. Navier, "Résumé des Leçons sur l'Application de la Mécanique," app. 3, Carilian-Goeury, Paris, 1864.

we find

$$(1+\nu)\left(\nabla^2\Theta - \nabla^2\sigma_x - \frac{\partial^2 \Theta}{\partial x^2}\right) - \nu\left(\nabla^2\Theta - \frac{\partial^2 \Theta}{\partial x^2}\right) = (1+\nu)\left(\frac{\partial X}{\partial x} - \frac{\partial Y}{\partial y} - \frac{\partial Z}{\partial z}\right) \tag{e}$$

Two analogous equations can be obtained from the two other conditions of compatibility of the type (c).

Adding together all three equations of the type (e) we find

$$(1-\nu)\nabla^2\Theta = -(1+\nu)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) \tag{f}$$

Substituting this expression for $\nabla^2\Theta$ in Eq. (e),

$$\nabla^2\sigma_x + \frac{1}{1+\nu}\frac{\partial^2 \Theta}{\partial x^2} = -\frac{\nu}{1-\nu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z}\right) - 2\frac{\partial X}{\partial x} \tag{g}$$

We can obtain three equations of this kind, corresponding to the first three of Eqs. (125). In the same manner the remaining three conditions (125) can be transformed into equations of the following kind:

$$\nabla^2\tau_{yz} + \frac{1}{1+\nu}\frac{\partial^2 \Theta}{\partial y\, \partial z} = -\left(\frac{\partial Z}{\partial y} + \frac{\partial Y}{\partial z}\right) \tag{h}$$

If there are no body forces or if the body forces are constant, Eqs. (g) and (h) become

$$\begin{aligned}
(1+\nu)\nabla^2\sigma_x + \frac{\partial^2 \Theta}{\partial x^2} &= 0 \qquad & (1+\nu)\nabla^2\tau_{yz} + \frac{\partial^2 \Theta}{\partial y\, \partial z} &= 0 \\
(1+\nu)\nabla^2\sigma_y + \frac{\partial^2 \Theta}{\partial y^2} &= 0 \qquad & (1+\nu)\nabla^2\tau_{xz} + \frac{\partial^2 \Theta}{\partial x\, \partial z} &= 0 \\
(1+\nu)\nabla^2\sigma_z + \frac{\partial^2 \Theta}{\partial z^2} &= 0 \qquad & (1+\nu)\nabla^2\tau_{xy} + \frac{\partial^2 \Theta}{\partial x\, \partial y} &= 0
\end{aligned} \tag{126}$$

We see that in addition to the equations of equilibrium (123) and the boundary conditions (124) the stress components in an isotropic body must satisfy the six conditions of compatibility (g) and (h) or the six conditions (126). This system of equations is generally sufficient for determining the stress components without ambiguity (see Art. 96).

The conditions of compatibility contain only second derivatives of the stress components. Hence, if the external forces are such that the equations of equilibrium (123) together with the boundary conditions (124) can be satisfied by taking the stress components either as constants or as linear functions of the coordinates, the equations of compatibility are satisfied identically and this stress system is the correct solution of the problem. Several examples of such problems will be considered in Chap. 9.

86 | Determination of Displacements

When the components of stress are found from the previous equations, the components of strain can be calculated by using Hooke's law [Eqs. (3) and (6)]. Then Eqs. (2) are used for the determination of the displacements u, v, w. Differentiating Eqs. (2) with respect to x, y, z we can obtain 18 equations containing 18 second derivatives of u, v, w, from which all these derivatives can be determined. For u, for instance, we obtain

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial \epsilon_x}{\partial x} \qquad \frac{\partial^2 u}{\partial y^2} = \frac{\partial \gamma_{xy}}{\partial y} - \frac{\partial \epsilon_y}{\partial x} \qquad \frac{\partial^2 u}{\partial z^2} = \frac{\partial \gamma_{xz}}{\partial z} - \frac{\partial \epsilon_z}{\partial x}$$
$$\frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial \epsilon_x}{\partial y} \qquad \frac{\partial^2 u}{\partial x\, \partial z} = \frac{\partial \epsilon_x}{\partial z} \qquad \frac{\partial^2 u}{\partial y\, \partial z} = \frac{1}{2}\left(\frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z} - \frac{\partial \gamma_{yz}}{\partial x}\right) \tag{a}$$

The second derivatives for the two other components of displacement v and w can be obtained by cyclical interchange in Eqs. (a) of the letters x, y, z.

Now u, v, w can be obtained by double integration of these second derivatives. The introduction of arbitrary constants of integration will result in adding to the values of u, v, w linear functions in x, y, z, as it is evident that such functions can be added to u, v, w without affecting such equations as (a). To have the strain components (2) unchanged by such an addition, the additional linear functions must have the form

$$\begin{aligned} u' &= a + by - cz \\ v' &= d - bx + ez \\ w' &= f + cx - ey \end{aligned} \tag{b}$$

This means that the displacements are not entirely determined by the stresses and strains. On the displacements found from the differential Eqs. (123), (124), and (126) a displacement like that of a rigid body can be superposed. The constants a, d, f in Eqs. (b) represent a translatory motion of the body, and the constants b, c, e are the three rotations of the rigid body around the coordinate axes. When there are sufficient constraints to prevent motion as a rigid body, the six constants in Eqs. (b) can easily be calculated so as to satisfy the conditions of constraint. Several examples of such calculations will be shown later.

87 | Equations of Equilibrium in Terms of Displacements

One method of solution of the problems of elasticity is to eliminate the stress components from Eqs. (123) and (124) by using Hooke's law and to

express the strain components in terms of displacements by using Eqs. (2). In this manner we arrive at three equations of equilibrium containing only the three unknown functions u, v, w. Substituting in the first of Eqs. (123) from (11),

$$\sigma_x = \lambda e + 2G\frac{\partial u}{\partial x} \qquad e = \epsilon_x + \epsilon_y + \epsilon_z \tag{a}$$

and from (6),

$$\begin{aligned} \tau_{xy} &= G\gamma_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\ \tau_{xz} &= G\gamma_{xz} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \end{aligned} \tag{b}$$

we find

$$(\lambda + G)\frac{\partial e}{\partial x} + G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + X = 0$$

The two other equations can be transformed in the same manner. Then, using the symbol ∇^2 (see page 238), the equations of equilibrium (123) become

$$\begin{aligned} (\lambda + G)\frac{\partial e}{\partial x} + G\,\nabla^2 u + X &= 0 \\ (\lambda + G)\frac{\partial e}{\partial y} + G\,\nabla^2 v + Y &= 0 \\ (\lambda + G)\frac{\partial e}{\partial z} + G\,\nabla^2 w + Z &= 0 \end{aligned} \tag{127}$$

and, when there are no body forces,

$$\begin{aligned} (\lambda + G)\frac{\partial e}{\partial x} + G\,\nabla^2 u &= 0 \\ (\lambda + G)\frac{\partial e}{\partial y} + G\,\nabla^2 v &= 0 \\ (\lambda + G)\frac{\partial e}{\partial z} + G\,\nabla^2 w &= 0 \end{aligned} \tag{128}$$

Differentiating these equations, the first with respect to x, the second with respect to y, and the third with respect to z, and adding them together, we find

$$(\lambda + 2G)\,\nabla^2 e = 0$$

i.e., the volume expansion e satisfies the differential equation

$$\frac{\partial^2 e}{\partial x^2} + \frac{\partial^2 e}{\partial y^2} + \frac{\partial^2 e}{\partial z^2} = 0 \tag{129}$$

The same conclusion holds also when body forces are constant throughout the volume of the body.

Substituting from such equations as (a) and (b) into the boundary conditions (124) we find

$$\bar{X} = \lambda e l + G\left(\frac{\partial u}{\partial x} l + \frac{\partial u}{\partial y} m + \frac{\partial u}{\partial z} n\right) + G\left(\frac{\partial u}{\partial x} l + \frac{\partial v}{\partial x} m + \frac{\partial w}{\partial x} n\right) \quad (130)$$

. . . .

Equations (127) together with the boundary conditions (130) define completely the three functions u, v, w. From these the components of strain are obtained from Eqs. (2) and the components of stress from Eqs. (9) and (6). Applications of these equations will be shown in Chap. 14.

88 | General Solution for the Displacements

It is easily verified by substitution that the differential equations (128) of equilibrium in terms of displacement are satisfied by[1]

$$u = \phi_1 - \alpha \frac{\partial}{\partial x}(\phi_0 + x\phi_1 + y\phi_2 + z\phi_3)$$

$$v = \phi_2 - \alpha \frac{\partial}{\partial y}(\phi_0 + x\phi_1 + y\phi_2 + z\phi_3)$$

$$w = \phi_3 - \alpha \frac{\partial}{\partial z}(\phi_0 + x\phi_1 + y\phi_2 + z\phi_3)$$

where $4\alpha = 1/(1 - \nu)$ and the four functions $\phi_0, \phi_1, \phi_2, \phi_3$ are harmonic, i.e.,

$$\nabla^2\phi_0 = 0 \qquad \nabla^2\phi_1 = 0 \qquad \nabla^2\phi_2 = 0 \qquad \nabla^2\phi_3 = 0$$

It can be shown that this solution is general, even when ϕ_0 is omitted.[2]

This form of solution has been adapted to curvilinear coordinates by Neuber and applied by him in the solution of problems of solids of revolution[3] generated by hyperbolas (the hyperbolic groove on a cylinder) and ellipses (cavity in the form of an ellipsoid of revolution) transmitting tension, bending, torsion, or shear force transverse to the axis with accompanying bending.

[1] This solution was given independently by P. F. Papkovitch, *Compt. Rend.*, vol. 195, pp. 513 and 754, 1932, and by H. Neuber, *Z. Angew. Math. Mech.*, vol. 14, p. 203, 1934. Other general solutions were given by B. Galerkin, *Compt. Rend.*, vol. 190, p. 1047, 1930, and by Boussinesq and Kelvin—see Todhunter and Pearson, "History of Elasticity," vol. 2, pt. 2, p. 268. See also R. D. Mindlin, *Bull. Am. Math. Soc.*, 1936, p. 373.

[2] For discussion of the number of functions needed for completeness, see P. M. Naghdi and C. S. Hsu, *J. Math. Mech.*, vol. 10, pp. 233–246, 1961, and references given there.

[3] H. Neuber, "Kerbspannungslehre," 2d ed., Springer-Verlag OHG, Berlin, 1958. This book also contains solutions of two-dimensional problems. See Chap. 6 above.

89 | The Principle of Superposition

The solution of a problem of a given elastic solid with given surface and body forces requires us to determine stress components, or displacements, that satisfy the differential equations and the boundary conditions. If we choose to work with stress components, we have to satisfy: (1) the equations of equilibrium (123); (2) the compatibility conditions (125); (3) the boundary conditions (124). Let σ_x . . . , τ_{xy} . . . , be the stress components so determined, and due to surface forces $\bar{X}$, $\bar{Y}$, $\bar{Z}$ and body forces X, Y, Z.

Let σ_x', . . . , τ_{xy}' . . . be the stress components in the same elastic solid due to surface forces $\bar{X}'$, $\bar{Y}'$, $\bar{Z}'$ and body forces X', Y', Z'. Then the stress components $\sigma_x + \sigma_x'$, . . . , $\tau_{xy} + \tau_{xy}'$, . . . will represent the stress due to the surface forces $\bar{X} + \bar{X}'$, . . . and the body forces $X + X'$, ... This holds because all the differential equations and boundary conditions are *linear*. Thus, adding the first of Eqs. (123) to the corresponding equation

$$\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}'}{\partial y} + \frac{\partial \tau_{xz}'}{\partial z} + X' = 0$$

we find

$$\frac{\partial}{\partial x}(\sigma_x + \sigma_x') + \frac{\partial}{\partial y}(\tau_{xy} + \tau_{xy}') + \frac{\partial}{\partial z}(\tau_{xz} + \tau_{xz}') + X + X' = 0$$

and similarly from the first of (124) and its counterpart we have by addition

$$\bar{X} + \bar{X}' = (\sigma_x + \sigma_x')l + (\tau_{xy} + \tau_{xy}')m + (\tau_{xz} + \tau_{xz}')n$$

The compatibility conditions can be combined in the same manner. The complete set of equations shows that $\sigma_x + \sigma_x'$, . . . , $\tau_{xy} + \tau_{xy}'$ . . . satisfy all the equations and conditions determining the stress due to forces $\bar{X} + \bar{X}'$, . . . , $X + X'$, ... This is an instance of the principle of superposition. It is readily extended to other types of boundary conditions such as given displacements.

In deriving our equations of equilibrium (123) and boundary conditions (124), we made no distinction between the position and form of the element before loading, and its position and form after loading. As a consequence, our equations and the conclusions drawn from them are valid only so long as the small displacements in the deformation do not affect substantially the action of the external forces. There are cases, however, in which the deformation must be taken into account. Then the justification of the principle of superposition given above fails. The beam under simultaneous thrust and lateral load affords an example of this kind,

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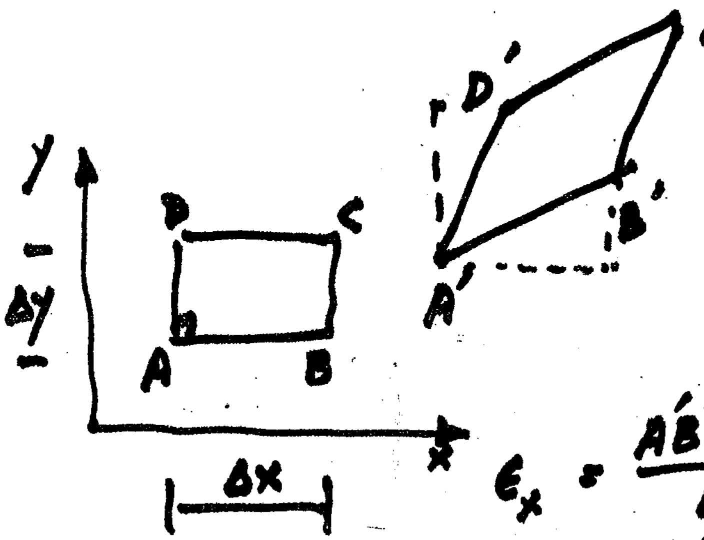


$$\epsilon_x = \frac{A'B' - AB}{AB} = \frac{\frac{\partial u}{\partial x}\Delta x}{\Delta x} \approx \frac{\partial u}{\partial x}$$

$$\epsilon_y = \frac{A'D' - AD}{AD} = \frac{\partial v/\partial y \, \Delta y}{\Delta y} \approx \frac{\partial v}{\partial y}$$

$$\gamma_{yx} = \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

$u(x,y,z)$ - disp in x dir
$v(x,y,z)$ - " " y dir

$$\epsilon_z = \frac{\partial w}{\partial z}$$

w - disp in z dir $\quad w(x,y,z)$

$$\gamma_{xz} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \quad \gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$$

$$\gamma_{xz} = \gamma_{zx} \qquad \gamma_{yz} = \gamma_{zy}$$

STRAIN TENSOR

$$\epsilon = \begin{pmatrix} \epsilon_x & \frac{\gamma_{xy}}{2} & \frac{\gamma_{xz}}{2} \\ \frac{\gamma_{yx}}{2} & \epsilon_y & \gamma_{yz}/2 \\ \frac{\gamma_{zx}}{2} & \frac{\gamma_{zy}}{2} & \epsilon_z \end{pmatrix} \qquad \sigma = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}$$

$\longleftarrow \quad \longrightarrow$
σ, ϵ

$$\sigma = E\epsilon \iff \sigma = C\epsilon$$

$$\tau_{xy} = \tau_{yx} \qquad \gamma_{xy} = \gamma_{yx}$$

Stress Strain Relations and Elastic Symmetry

References: Sokolnikoff, Mathematical Theory of Elasticity pp. 56-71
Malvern, Introduction to the Mechanics of a Deformable Medium, pp 273-294

Triclinic Crystal (Most General Anisotropic material) constants.

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \underbrace{\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46}\\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56}\\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66}\end{bmatrix}}_{C\text{ (Stiffness) Matrix}} \begin{bmatrix}\epsilon_1\\ \epsilon_2\\ \epsilon_3\\ \epsilon_4\\ \epsilon_5\\ \epsilon_6\end{bmatrix}$$

Monoclinic Crystal one plane of elastic symmetry. e.g. yz plane. Reflection of x axis leaves constants unchanged

Direction cosines of Transformation:

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array}\\ \begin{array}{c} x' \\ y' \\ z' \end{array} & \begin{bmatrix} -1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\end{array}$$

"C" matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0\\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0\\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0\\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & C_{55} & C_{56}\\ 0 & 0 & 0 & 0 & C_{56} & C_{66}\end{bmatrix}$$

Orthorhombic (orthotropic) Material 3 mutually orthogonal planes of symmetry. Reflection of x, y, and z leaves constants unchanged

Direction cosines (reflect y axis)

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array}\\ \begin{array}{c} x' \\ y' \\ z' \end{array} & \begin{bmatrix} 1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\end{array}$$

"C" Matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66}\end{bmatrix}$$

Cubic Material Interchange of 2 axes (i.e. rotate 90° then reflect) leaves constants unchanged.

Direction cosines (interchange Y & Z)

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array}\\ \begin{array}{c} x' \\ y' \\ z' \end{array} & \begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}\end{array}$$

"C" matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & C_{44}\end{bmatrix}$$

Isotropic Material: Any coordinate transformation leaves Elastic Constants unchanged

"C" matrix:

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{44}^* & & \\ & & & & C_{44}^* & \\ & & & & & C_{44}^*\end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & 2\mu & & \\ & & & & 2\mu & \\ & & & & & 2\mu\end{bmatrix}$$

$$^*C_{44} = \frac{C_{11}-C_{12}}{2}$$

λ, μ, are the Lamé constants $\qquad \mu = G \qquad \lambda = \dfrac{E\nu}{(1+\nu)(1-2\nu)}$

$\sigma = C\,\epsilon$ may be written $\sigma_{ij} = \lambda\,\epsilon_{kk}\,\delta_{ij} + 2\mu\,\epsilon_{ij}$ $\qquad G = \dfrac{E}{2(1+\nu)}$

where ϵ_{kk} is the dilatation

$\sigma_1 = \sigma_x \quad \sigma_2 = \sigma_y \quad \sigma_3 = \sigma_z \quad \tau_{xy} = \sigma_4 \quad \tau_{yz} = \sigma_5 \quad \tau_{zx} = \sigma_6$

$\epsilon_1 = \epsilon_x \quad \epsilon_2 = \epsilon_y \quad \epsilon_3 = \epsilon_z \quad \epsilon_4 = \gamma_{xy}/2 \quad \dfrac{\gamma_{yz}}{2} = \epsilon_5 \quad \dfrac{\gamma_{zx}}{2} = \epsilon_6$

$\epsilon_4 = 2\epsilon_{xy}$

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1} \qquad \epsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{\gamma_{12}}{2}$$

if σ_x is gradually applied, work $= \frac{1}{2}\sigma_x \epsilon_x = \frac{1}{2} C_{11}\sigma_x^2$

Now apply σ_y while σ_x is present. Causes work $\frac{1}{2}\sigma_y \epsilon_y + \sigma_x \epsilon_x' = \frac{1}{2} C_{22}\sigma_y^2 + C_{12}\sigma_x\sigma_y$ since $\epsilon_x' = C_{12}\sigma_y$

total work $= \frac{1}{2} C_{11}\sigma_x^2 + \frac{1}{2} C_{22}\sigma_y^2 + C_{12}\sigma_x\sigma_y$

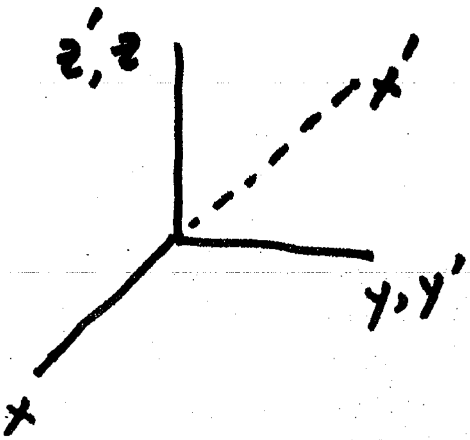
now change order & apply σ_y first. work $= \frac{1}{2}\sigma_y \epsilon_y = \frac{1}{2} C_{22}\sigma_y^2$

Now apply σ_x while σ_y is present. work $= \frac{1}{2}\sigma_x \epsilon_x + \sigma_y \epsilon_y' = \frac{1}{2} C_{11}\sigma_x^2 + C_{21}\sigma_y\sigma_x$ since $\epsilon_y' = C_{21}\sigma_x$

total work $= \frac{1}{2} C_{22}\sigma_y^2 + \frac{1}{2}\sigma_x^2 C_{11} + C_{21}\sigma_x\sigma_y$

since work must be equal $\Rightarrow C_{12} = C_{21}$

Do same for other directions



$$\sigma'_{mn} = \sigma_{ij}\, l_{mi}\, l_{nj} \qquad l_{\alpha\beta} = \cos(x_\alpha, x_\beta)$$

	x	y	z
x'	-1	0	0
y'	0	1	0
z'	0	0	1

$j, i = 1 \Rightarrow x$
$\quad = 2 \quad y$
$\quad\; 3 \quad z$

$$\sigma'_1 = \sigma_1 \qquad \sigma'_4 = \sigma_4$$
$$\sigma'_2 = \sigma_2 \qquad \sigma'_5 = -\sigma_5 \qquad \sigma'_5 = \tau_{y'z'} \quad \sigma_5 = \tau_{yz}$$
$$\sigma'_3 = \sigma_3 \qquad \sigma'_6 = -\sigma_6$$

$$\varepsilon'_1 = \varepsilon_1 \qquad \varepsilon'_4 = \varepsilon_4$$
$$\varepsilon'_2 = \varepsilon_2 \qquad \varepsilon'_5 = -\varepsilon_5$$
$$\varepsilon'_3 = \varepsilon_3 \qquad \varepsilon'_6 = -\varepsilon_6$$

$$\sigma'_1 = C_{11}\varepsilon'_1 + C_{12}\varepsilon'_2 + \dots + C_{15}\varepsilon'_5 + C_{16}\varepsilon'_6$$
$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + \dots + C_{15}\varepsilon_5 + C_{16}\varepsilon_6 \quad \Big] \Rightarrow C_{16}, C_{15} = 0$$
$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + \dots - C_{15}\varepsilon_5 - C_{16}\varepsilon_6$$

for $\sigma_2, \sigma'_2 \Rightarrow C_{25} = C_{26} = 0$

$\sigma_3, \sigma'_3 \Rightarrow C_{35} = C_{36} = 0$

$\sigma_4, \sigma'_4 \Rightarrow C_{45} = C_{46} = 0$

$\sigma_5, \sigma'_5 \Rightarrow C_{55}, C_{56} \neq 0$

$\sigma_6, \sigma'_6 \Rightarrow C_{56}, C_{66} \neq 0$

$\sigma_{12} = (C_{12} + C_{21})\,\varepsilon_1 \varepsilon_2$

$$U = \tfrac{1}{2} C_{11}\varepsilon_1^2 + C_{12}\varepsilon_1\varepsilon_2 + C_{13}\varepsilon_1\varepsilon_3 + \dots + C_{16}\varepsilon_1\varepsilon_6$$
$$+ \tfrac{1}{2} C_{22}\varepsilon_2^2 + C_{23}\varepsilon_2\varepsilon_3 + \dots + C_{26}\varepsilon_2\varepsilon_6$$
$$\vdots$$
$$+ \tfrac{1}{2} C_{66}\varepsilon_6^2$$

Now we note that

$\sigma_1 = \dfrac{\partial U}{\partial \varepsilon_1} = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + \dots C_{16}\varepsilon_6$ and this is true for any σ_r $(r = 1, \dots, 6)$

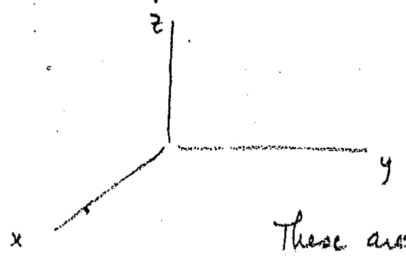
Now $\sigma_r = C_{rp}\,\varepsilon_p \quad (r, p = 1 \cdots 6) = \dfrac{\partial U}{\partial \varepsilon_p}$

Most general stress-strain relation of a linearly elastic solid involves 21 independent constants C_{pr}

Material scientists & physicists provided theory of crystals in order to determine (

Crystal theory	Triclinic	involves 21 constants	
	Monoclinic	13	1 symmetry
	Orthotropic / Orthorhombic	9	2 symmetries
	Tetragonal / Trigonal	7, 6	
	Hexagonal	5	
	Cubic	3	
	Isotropic	2	material properties in all directions are exactly the same.

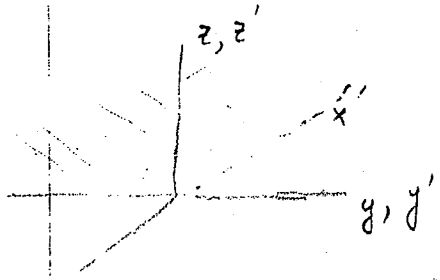
To reduce from <u>21 to 2</u> we introduce cartesian plane & use symmetries of the crystal



These are original relations involving σ_i, c_{ij} & ε_j

$$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 + \dots + C_{16}\varepsilon_6$$
$$\sigma_2 = C_{12}\varepsilon_1 + \dots \qquad + C_{26}\varepsilon_6$$
$$\vdots$$
$$\sigma_6 = C_{16}\varepsilon_1 + \dots \qquad + C_{66}\varepsilon_6$$

Now let the yz plane be the plane of symmetry ie (x, x')



$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & -1 & 0 & 0 \\ y' & 0 & 1 & 0 \\ z' & 0 & 0 & 1 \end{array}$$

$l_{\alpha\beta} = \cos(x_\alpha, x_\beta)$

remember $\sigma'_{mn} = \sigma_{ij}\, l_{mi}\, l_{nj}$ $\qquad \varepsilon'_{mn} = \varepsilon_{ij}\, l_{mi}\, l_{nj}$

$\therefore \quad \sigma_1' = \sigma_1 \quad \sigma_4' = \sigma_4 \qquad \varepsilon_1' = \varepsilon_1 \quad \varepsilon_4' = \varepsilon_4$

$\sigma_2' = \sigma_2 \quad \sigma_5' = -\sigma_5 \qquad \varepsilon_2' = \varepsilon_2 \quad \varepsilon_5' = -\varepsilon_5$

$\sigma_3' = \sigma_3 \quad \sigma_6' = -\sigma_6 \qquad \varepsilon_3' = \varepsilon_3 \quad \varepsilon_6' = -\varepsilon_6$

now $\sigma_1' = C_{11}\varepsilon_1' + \dots + C_{16}\varepsilon_6'$ $\quad$ or C_{ij} is invariant under transformation.

$\sigma_1 = C_{11}\varepsilon_1 + \dots - C_{15}\varepsilon_5 - C_{16}\varepsilon_6$ $\quad$ Substituting for $\sigma_1', \varepsilon_1', \dots, \varepsilon_6'$

but originally $\sigma_1 = C_{11}\varepsilon_1 + \dots + C_{15}\varepsilon_5 + C_{16}\varepsilon_6$ $\quad$ must also be true $\therefore C_{15} = C_{16} = 0$

$\sigma_2 = \sigma_2'$ from 2^{nd} relation $\Rightarrow C_{25} = C_{26} = 0$

$\sigma_3 = \sigma_3'$ from 3^{rd} relation $\Rightarrow C_{35} = C_{36} = 0$

$\sigma_4 = \sigma_4'$ from 4^{th} relation $\Rightarrow C_{45} = C_{46} = 0$

$\sigma_5' = -\sigma_5$ $\quad$ no change $\Rightarrow -\sigma_5 = -C_{55}\varepsilon_5 - C_{56}\varepsilon_6$

$\sigma_6' = -\sigma_6$ $\quad$ no change $\quad -\sigma_6 = -C_{65}\varepsilon_5 - C_{66}\varepsilon_6$

reduced const C_{ij} by 8

These are the coeff ($\neq 0$) for <u>monoclinic</u> $C_{56}, C_{66}, C_{11}, C_{12}, C_{13}, C_{14}, C_{55}$
$C_{22}, C_{23}, C_{24}, C_{33}, C_{34}, C_{44}$

Now we look at symmetry in the (y, y')
then for orthorhombic $C_{14} = C_{24} = C_{34} = C_{56} = 0$ $\quad$ Non zero coeff $C_{11}, C_{12}, C_{13}, C_{22}, C_{23}, C_{33}$
C_{66}, C_{55}, C_{44}

$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & 1 & 0 & 0 \\ y' & 0 & -1 & 0 \\ z' & 0 & 0 & 1 \end{array}$$

$$\begin{pmatrix} C_{11} & C_{12} & C_{13} & C_{14} & & \\ C_{12} & C_{22} & C_{23} & C_{24} & & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & & \\ C_{14} & C_{24} & C_{34} & C_{44} & & \\ & 0 & & & C_{55} & C_{56} \\ & & & & C_{56} & C_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \sigma_6 \end{pmatrix}$$

monoclinic

$\sigma_1 = C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3$

$\sigma_2 = C_{12}\varepsilon_2 + C_{22}\varepsilon_2 + C_{23}\varepsilon_3 \qquad 0$

$\sigma_3 = C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{33}\varepsilon_3$

$\sigma_4 = \qquad C_{44}\varepsilon_4 \quad 0 \quad 0$

$\sigma_5 = \quad 0 \quad 0 \quad C_{55}\varepsilon_5 \quad 0$

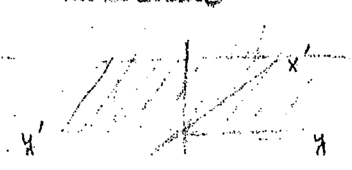
$\sigma_6 = \qquad 0 \quad 0 \quad C_{66}\varepsilon_6$

orthorhombic

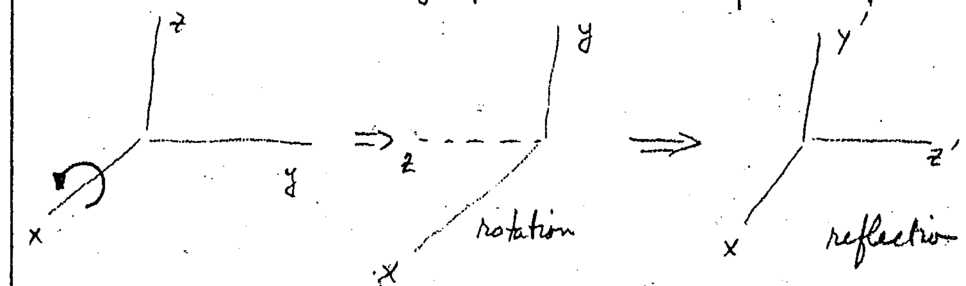
$\sigma_1' = \sigma_1 \quad \sigma_4' = -\sigma_4 \qquad \varepsilon_1' = \varepsilon_1 \quad \varepsilon_4' = -\varepsilon_4$

$\sigma_2' = \sigma_2 \quad \sigma_5' = \sigma_5 \qquad \varepsilon_2' = \varepsilon_2 \quad \varepsilon_5' = \varepsilon_5$

$\sigma_3' = \sigma_3 \quad \sigma_6' = -\sigma_6 \qquad \varepsilon_3' = \varepsilon_3 \quad \varepsilon_6' = -\varepsilon_6$



cubic material - interchange of 2 axes   rotate by 90° & reflect.   11/9/78



$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & 1 & 0 & 0 \\ y' & 0 & 0 & 1 \\ z' & 0 & 1 & 0 \end{array}$$

$\sigma_1' = \sigma_1$ $\sigma_4' = \sigma_4$ $\epsilon_1' = \epsilon_1$ $\epsilon_4' = \epsilon_4$

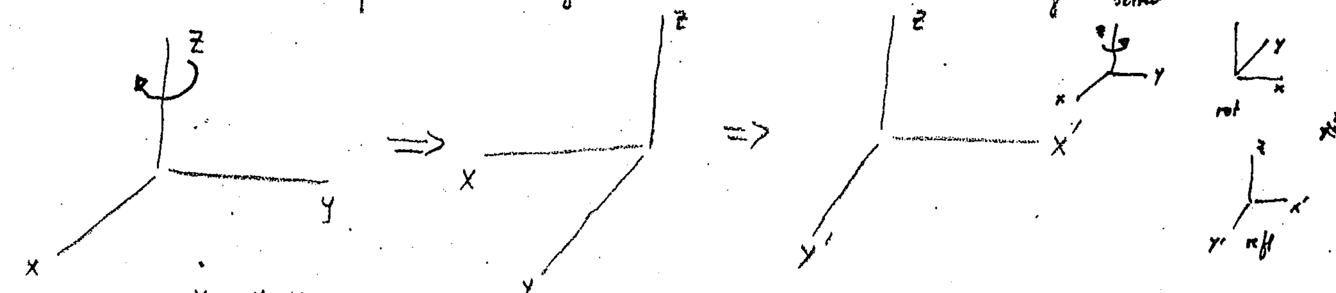
$\sigma_2' = \sigma_3$ $\sigma_5' = \sigma_6$ $\epsilon_2' = \epsilon_3$ $\epsilon_5' = \epsilon_6$

$\sigma_3' = \sigma_2$ $\sigma_6' = \sigma_6$ $\epsilon_3' = \epsilon_2$ $\epsilon_6' = \epsilon_6$

$$\begin{pmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_3' \\ \sigma_4' \\ \sigma_5' \\ \sigma_6' \end{pmatrix} = \begin{pmatrix} \sigma_1 \\ \sigma_3 \\ \sigma_2 \\ \sigma_4 \\ \sigma_6 \\ \sigma_5 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{13} & C_{12} & & & \\ C_{12} & C_{23} & C_{22} & & & \\ C_{13} & C_{33} & C_{23} & & & \\ & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66} \end{pmatrix} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \\ \epsilon_5 \\ \epsilon_6 \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & & & \\ C_{12} & C_{22} & C_{23} & & & \\ C_{13} & C_{23} & C_{33} & & & \\ & & & C_{44} & & \\ & & & & C_{55} & \\ & & & & & C_{66} \end{pmatrix}$$

$\Rightarrow \quad C_{13} = C_{12}, \quad C_{22} = C_{33}, \quad C_{55} = C_{66}$

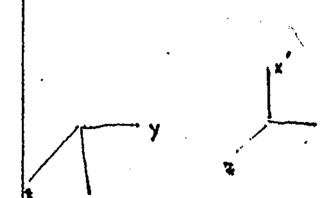
now since this is only 1 interchange we can do another interchange   better



$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & 0 & 1 & 0 \\ y' & 1 & 0 & 0 \\ z' & 0 & 0 & 1 \end{array}$$

using this we will get $C_{11} = C_{22}$, $C_{12} = C_{23}$, $C_{44} = C_{55}$ such that

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & & \\ C_{12} & C_{12} & C_{11} & & & \\ & & & C_{44} & & \\ & & & & C_{44} & \\ & & & & & C_{44} \end{pmatrix}$$



to go to isotropic just rotate about the x axis

$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & 1 & 0 & 0 \\ y' & 0 & \cos\theta & \sin\theta \\ z' & 0 & -\sin\theta & \cos\theta \end{array}$$

$\sigma_4' = \frac{(\sigma_3 - \sigma_2)}{2}\sin 2\theta + \sigma_4 (\cos 2\theta)$ but $\sigma_4' = C_{44}\varepsilon_4'$
and $\varepsilon_4' = (\varepsilon_3 - \varepsilon_2)\sin 2\theta + \varepsilon_4 \cos 2\theta$

from the cubic $\sigma_3 - \sigma_2 = (C_{11} - C_{12})(\varepsilon_3 - \varepsilon_2)$ and $\sigma_4 = C_{44}\varepsilon_4$ then we obtain $\sigma_4' = \frac{C_{11} - C_{12}}{2}(\varepsilon_3 - \varepsilon_2)\sin 2\theta + C_{44}\varepsilon_4 \cos 2\theta = C_{44}(\varepsilon_3 - \varepsilon_2)\sin 2\theta + C_{44}\varepsilon_4 \cos 2\theta = C_{44}\varepsilon_4'$

since θ is arbitrary $\Rightarrow$ $C_{44} = \frac{C_{11} - C_{12}}{2}$

if we define $C_{44} = \mu$, $C_{12} = \lambda \Rightarrow C_{11} = \lambda + 2\mu$ λ, μ are Lamé constants.

and $\boxed{\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}}$ Generalized Hooke's Law

ε_{kk} is the dilatation change in volume.

We can go directly to isotropic from the general strain energy density
U being a scalar it is invariant under any transformation for an isotropic material

Normally $U = f(\varepsilon_{ij}') = \frac{1}{2} c_{ijkl}\varepsilon_{ij}'\varepsilon_{kl}'$ for anisotropic
but $U = f(\varepsilon_{ij})$ for isotropic and since it is invariant under any transformation we can pick U as a fn of the strain invariants or

$U = f(I, II, III)$; since III involves cubic terms of ε_{ij} we can drop it to the first approx $\therefore$

$U = f(I, II) \cong \bar{A} I^2 + B\, II$

where $I = \varepsilon_{qq}$

$II = \varepsilon_{rs}\varepsilon_{rs} - \varepsilon_{qq}\varepsilon_{rr}$

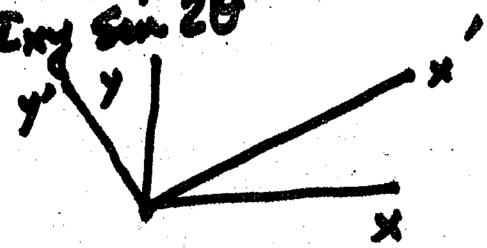
$U \cong \bar{A}\varepsilon_{qq}\varepsilon_{rr} + B\varepsilon_{rs}\varepsilon_{rs} - B\varepsilon_{qq}\varepsilon_{rr}$

$= A\varepsilon_{qq}\varepsilon_{rr} + B\varepsilon_{rs}\varepsilon_{rs}$

$$\sigma_x' = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_y' = (\quad) - (\quad)\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\sigma_x' = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\sin 2\theta$$



$$\begin{array}{c|ccc} & x & y & z \\ \hline x' & \cos\theta & \sin\theta & 0 \\ y' & -\sin\theta & \cos\theta & 0 \\ z' & 0 & 0 & 1 \end{array}$$

$l_{\alpha\beta}$ $\quad$ $\sigma_x', \sigma_y', \tau_{xy}'$

$$\sigma_{mn}' = \sigma_{ij}\, l_{mi}\, l_{nj} \qquad \begin{array}{l} j\ 1,2,3 \\ i\ 1,2,3 \end{array}$$

λ, μ LAME CONST.

$$\sigma_{ij} = \lambda \varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}$$

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$\varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

a manner as to make these expressions single-valued. An analogous procedure is necessary in dealing with the torsion of hollow shafts. The constant values of the stress function along the boundaries should be determined in such a manner as to make the displacements single-valued. A sufficient number of equations for determining these constants will then be obtained.

From Eqs. (*b*) and (*d*) of Art. 104 we have

$$\tau_{xz} = G\left(\frac{\partial w}{\partial x} - \theta y\right) \qquad \tau_{yz} = G\left(\frac{\partial w}{\partial y} + \theta x\right) \tag{c}$$

Let us now calculate the integral

$$\int \tau \, ds \tag{d}$$

along each boundary. Using (*c*) and resolving the total stress into its components we find

$$\begin{aligned}\int \tau \, ds &= \int \left(\tau_{xz}\frac{dx}{ds} + \tau_{yz}\frac{dy}{ds}\right) ds \\ &= G\int\left(\frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy\right) - \theta G \int (y\,dx - x\,dy)\end{aligned} \tag{174}$$

The first integral must vanish from the condition that the integration is taken round a closed curve and that w is a single-valued function. Hence,

$$\int \tau \, ds = \theta G \int (x\,dy - y\,dx)$$

The integral on the right side is equal to double the area enclosed. Then,

$$\int \tau \, ds = 2G\theta A \tag{175}$$

Thus, we must determine the constant values of the stress function along the boundaries of the holes so as to satisfy Eq. (175) for each boundary.

For any closed curve drawn in the cross section (lying wholly in the material) the first and second members of (174) represent the line integral of the tangential component of shear stress τ taken round the curve, and this may be called the shear circulation, by analogy with circulation in fluid dynamics. Then (175) still holds and may be called the shear-circulation theorem.

The significance of (175) for the membrane analogy was discussed on page 306. It indicates that in the membrane the level of each plate, such as the plate CD (Fig. 171), must be taken so that the vertical load on the plate is equal and opposite to the vertical component of the resultant of the tensile forces on the plate produced by the membrane. If the boundaries of the holes coincide with the stress lines of the corresponding solid shaft, the above condition is sufficient to ensure the equilibrium of the plates. In the general case this condition is not sufficient, and to keep the plates in equilibrium in a horizontal position special guiding devices become necessary. This makes the soap-film experiments for hollow shafts more complicated.

To remove this difficulty the following procedure may be adopted.[1] We make a hole in the plate corresponding to the outer boundary of the shaft. The interior boundaries, corresponding to the holes, are mounted each on a vertical sliding column so that their heights can be easily adjusted. Taking these heights arbitrarily and stretching the film over the boundaries, we obtain a surface that satisfies Eq. (150) and boundary conditions (152), but the Eq. (175) above generally will not be satisfied and the film does not represent the stress distribution in the hollow shaft. Repeating such an experiment as many times as the number of boundaries, each time with another adjustment of heights of the interior boundaries and taking measurements on the film each time, we obtain sufficient data for determining the correct values of the heights of the interior boundaries and can finally stretch the soap film in the required manner. This can be proved as follows: If i is the number of boundaries and $\phi_1, \phi_2, \ldots, \phi_i$ are the film surfaces obtained with i different adjustments of the heights of the boundaries, then a function

$$\phi = m_1\phi_1 + m_2\phi_2 + \cdots + m_i\phi_i \tag{e}$$

in which $m_1, m_2, \ldots, m_i$ are numerical factors, is also a solution of Eq. (150), provided that

$$m_1 + m_2 + \cdots + m_i = 1$$

Observing now that the shearing stress is equal to the slope of the membrane, and substituting (*e*) into Eqs. (175) we obtain i equations of the form

$$\int \frac{\partial \phi}{\partial n} ds = 2G\theta A_i$$

from which the i factors $m_1, m_2, \ldots, m_i$ can be obtained as functions of θ. Then the true stress function is obtained from (*e*).[2] This method was applied by Griffith and Taylor in determining stresses in a hollow circular shaft having a keyway in it. It was shown in this manner that the maximum stress can be considerably reduced and the strength of the shaft increased by throwing the bore in the shaft off center.

The torque in the shaft with one or more holes is obtained using twice the volume under the membrane and the flat plates. To see this we calculate the torque produced by the shearing stresses distributed over an elemental ring between two adjacent stress lines, as in Fig. 171 (now taken to represent an arbitrary hollow section). Denoting by δ the variable width of the ring and considering an element such as that shaded in the figure, the shearing force acting on this element is $\tau\delta\, ds$ and its moment with respect to O is $r\tau\delta\, ds$. Then the torque on the elemental ring is

$$dM_t = \int r\tau\delta \, ds \tag{f}$$

[1] Griffith and Taylor, *loc. cit.*, p. 938.

[2] Griffith and Taylor concluded from their experiments that instead of *constant-pressure* films it is more convenient to use *zero-pressure* films (see p. 306) in studying the stress distribution in hollow shafts. A detailed discussion of the calculation of factors $m_1, m_2, \ldots$ is given in their paper.

of material or a crack at such a corner. In the case of a rectangular *keyway*, therefore, a high stress concentration takes place at the reentrant corners at the bottom of the keyway. These high stresses can be reduced by rounding the corners.[1]

115 | Torsion of Hollow Shafts

So far the discussion has been limited to shafts whose cross sections are bounded by single curves. Let us consider now hollow shafts whose cross sections have two or more boundaries. The simplest problem of this kind is a hollow shaft with an inner boundary coinciding with one of the *stress lines* (see page 305) of the solid shaft, having the same boundary as the outer boundary of the hollow shaft.

Take, for instance, an elliptic cross section (Fig. 153). The stress function for the solid shaft is

$$\phi = \frac{a^2b^2F}{2(a^2+b^2)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right) \tag{a}$$

The curve

$$\frac{x^2}{(ak)^2}+\frac{y^2}{(bk)^2}=1 \tag{b}$$

is an ellipse that is geometrically similar to the outer boundary of the cross section. Along this ellipse the stress function (a) remains constant, and hence, for k less than unity, *this ellipse* is a stress line for the solid elliptic shaft. The shearing stress at any point of this line is in the direction of the tangent to the line. Imagine now a cylindrical surface generated by this stress line with its axis parallel to the axis of the shaft. Then, from the above conclusion regarding the direction of the shearing stresses, it follows that there will be no stresses acting across this cylindrical surface. We can imagine the material bounded by this surface removed without changing the stress distribution in the outer portion of the shaft. Hence the stress function (a) applies to the hollow shaft also.

For a given angle θ of twist the stresses in the hollow shaft are the same as in the corresponding solid shaft. But the torque will be smaller by the amount which in the case of the solid shaft is carried by the portion of the cross section corresponding to the hole. From Eq. (156) we see that the latter portion is in the ratio k^4:1 to the total torque. Hence, for the hollow shaft, instead of Eq. (156), we will have

$$\theta = \frac{M_t}{1-k^4}\frac{a^2+b^2}{\pi a^3b^3G}$$

and the stress function (a) becomes

$$\phi = -\frac{M_t}{\pi ab(1-k^4)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

The formula for the maximum stress will be

$$\tau_{\max} = \frac{2M_t}{\pi ab^2}\frac{1}{1-k^4}$$

In the membrane analogy the middle portion of the membrane, corresponding to the hole in the shaft (Fig. 171), must be replaced by the horizontal plate CD. We note that the uniform pressure distributed over the portion CFD of the membrane is statically equivalent to the pressure of the same magnitude uniformly distributed over the plate CD and the tensile forces S in the membrane acting along the edge of this plate are in equilibrium with the uniform load on the plate. Hence, in the case under consideration the same experimental soap-film method as before can be employed because the replacement of the portion CFD of the membrane by the plate CD causes no changes in the configuration and equilibrium conditions of the remaining portion of the membrane.

Let us consider now the more general case when the boundaries of the holes are no longer stress lines of the solid shaft. From the general theory of torsion we know (see Art. 104) that the stress function must be constant along each boundary, but these constants cannot be chosen arbitrarily. In discussing multiply-connected boundaries in two-dimensional problems, it was shown that recourse must be had to the expressions for the displacements, and the constants of integration should be found in such

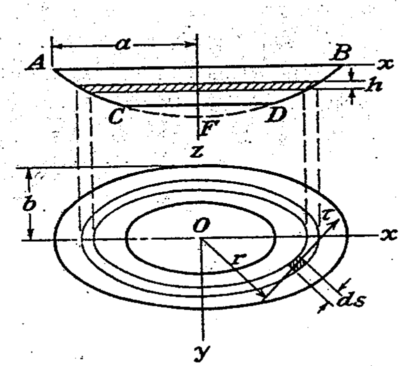


Fig. 171

[1] The stresses at the keyway were investigated by the soap-film method. See the paper by Griffith and Taylor, *loc. cit.*, p. 938.

For design formulas and charts, see R. E. Peterson, "Stress Concentration Design Factors," John Wiley & Sons, Inc., New York, 1953; also M. Nisida and M. Hondo, *Proc. Japan Nat. Congr. Appl. Mech.*, vol. 2, pp. 129–132, 1959.

in which the integration must be extended over the length of the ring. Denoting by A the area bounded by the ring and observing that τ is the slope, so that $\tau\delta$ is the difference in level h of the two adjacent contour lines, we find, from (f),

$$dM_t = 2hA \tag{g}$$

i.e., the torque corresponding to the elemental ring is given by twice the volume shaded in the figure. The total torque is given by the sum of these volumes, i.e., twice the volume between AB, the membrane AC and DB, and the flat plate CD. The conclusion follows similarly for several holes.

116 | Torsion of Thin Tubes

An approximate solution of the torsional problem for thin tubes can easily be obtained by using the membrane analogy. Let AB and CD (Fig. 172) represent the levels of the outer and the inner boundaries, and AC and DB be the cross section of the membrane stretched between these boundaries. In the case of a thin wall, we can neglect the variation in the slope of the membrane across the thickness and assume that AC and BD are straight lines. This is equivalent to the assumption that the shearing stresses are uniformly distributed over the thickness of the wall. Then denoting by h the difference in level of the two boundaries and by δ the variable thickness of the wall, the stress at any point, given by the slope of the membrane, is

$$\tau = \frac{h}{\delta} \tag{a}$$

It is inversely proportional to the thickness of the wall and thus greatest where the thickness of the tube is least.

To establish the relation between the stress and the torque M_t, we apply again the membrane analogy and calculate the torque from the volume

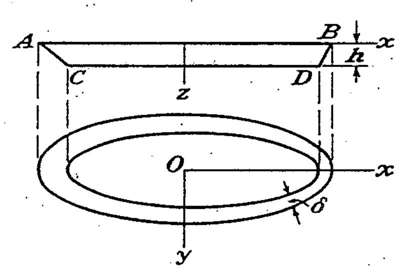


Fig. 172

$\tau = \frac{\partial\phi}{\partial n} = \frac{\phi_1 - \phi_0}{n_1 - n_0} = \frac{h}{\delta}$

Since δ is $\perp$ to contours

& $\phi_0 = 0$ since

$\phi = K\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

& at x & y $\phi_0 = 0$

$ACDB$. Then

$$M_t = 2Ah = 2A\delta\tau \tag{b}$$

in which A is the mean of the areas enclosed by the outer and the inner boundaries of the cross section of the tube. From (b) we obtain a simple formula for calculating shearing stresses:

$$\tau = \frac{M_t}{2A\delta} \tag{176}$$

For determining the angle of twist θ, we apply Eq. (160). Then

$$\tau\, ds = \frac{M_t}{2A}\int \frac{ds}{\delta} = 2G\theta A \tag{c}$$

from which[1]

$$\theta = \frac{M_t}{4A^2G}\int \frac{ds}{\delta} \tag{177}$$

In the case of a tube of uniform thickness, δ is constant and (177) gives

$$\theta = \frac{M_t s}{4A^2G\delta} \tag{178}$$

in which s is the length of the centerline of the ring section of the tube.

If the tube has reentrant corners, as in the case represented in Fig. 173, a considerable stress concentration may take place at these corners. The maximum stress is larger than the stress given by Eq. (176) and depends on the radius a of the fillet of the reentrant corner (Fig. 173b). In calculating this maximum stress, we shall use the membrane analogy as we

[1] Equations (176) and (177) for thin tubular sections were obtained by R. Bredt, *VDI*, vol. 40, p. 815, 1896.

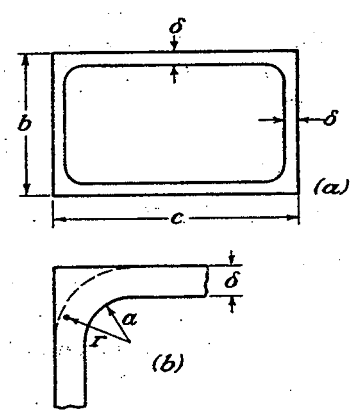


Fig. 173

did for the reentrant corners of rolled sections (Art. 112). The equation of the membrane at the reentrant corner may be taken in the form

$$\frac{d^2z}{dr^2} + \frac{1}{r}\frac{dz}{dr} = -\frac{q}{S}$$

Replacing q/S by $2G\theta$ and noting that $\tau = -dz/dr$ (see Fig. 172), we find

$$\frac{d\tau}{dr} + \frac{1}{r}\tau = 2G\theta \qquad (d)$$

Assuming that we have a tube of constant thickness δ and denoting by τ_0 the stress at a considerable distance from the corner calculated from Eq. (176), we find, from (c),

$$2G\theta = \frac{\tau_0 s}{A}$$

Substituting in (d),

$$\frac{d\tau}{dr} + \frac{1}{r}\tau = \frac{\tau_0 s}{A} \qquad (e)$$

The general solution of this equation is

$$\tau = \frac{C}{r} + \frac{\tau_0 sr}{2A} \qquad (f)$$

Assuming that the projecting angles of the cross section have fillets with the radius a, as indicated in the figure, the constant of integration C can be determined from the equation

$$\int_a^{a+\delta} \tau\, dr = \tau_0\delta \qquad (g)$$

which follows from the hydrodynamical analogy (Art. 114), viz.: if an ideal fluid circulates in a channel having the shape of the ring cross section of the tubular member, the quantity of fluid passing each cross section of the channel must remain constant. Substituting expression (f) for τ into Eq. (g), and integrating, we find that

$$C = \tau_0\delta \frac{1 - (s/4A)(2a + \delta)}{\log_e (1 + \delta/a)}$$

and, from Eq. (f), that

$$\tau = \frac{\tau_0\delta}{r}\frac{1 - (s/4A)(2a + \delta)}{\log_e (1 + \delta/a)} + \frac{\tau_0 sr}{2A} \qquad (h)$$

For a thin-walled tube the ratios $s(2a + \delta)/A$, sr/A, will be small, and (h) reduces to

$$\tau = \frac{\tau_0\delta/r}{\log_e (1 + \delta/a)} \qquad (i)$$

Substituting $r = a$ we obtain the stress at the reentrant corner. This is plotted in Fig. 174. The other curve[1] (A in Fig. 174) was obtained by the method of finite differences, without the assumption that the membrane at the corner has the form of a surface of revolution. It confirms the accuracy of Eq. (i) for small fillets—say up to $a/\delta = 1/4$. For larger fillets the values given by Eq. (i) are too high.

Let us consider now the case when the cross section of a tubular member has more than two boundaries. Taking, for example, the case shown in Fig. 175 and assuming that the thickness of the wall is very small, the shearing stresses in each portion of the wall, from the membrane analogy, are

$$\tau_1 = \frac{h_1}{\delta_1} \qquad \tau_2 = \frac{h_2}{\delta_2} \qquad \tau_3 = \frac{h_1 - h_2}{\delta_3} = \frac{\tau_1\delta_1 - \tau_2\delta_2}{\delta_3} \qquad (j)$$

in which h_1 and h_2 are the levels of the inner boundaries CD and EF.[2]

The magnitude of the torque, determined by the volume $ACDEFB$, is

$$M_t = 2(A_1h_1 + A_2h_2) = 2A_1\delta_1\tau_1 + 2A_2\delta_2\tau_2 \qquad (k)$$

where A_1 and A_2 are areas indicated in the figure by dotted lines.

Further equations for the solution of the problem are obtained by applying Eq. (160) to the closed curves indicated in the figure by dotted lines. Assuming that the thicknesses δ_1, δ_2, δ_3 are constant and denoting

[1] Huth, *op. cit.*

[2] It is assumed that the plates are guided so as to remain horizontal (see p. 331).

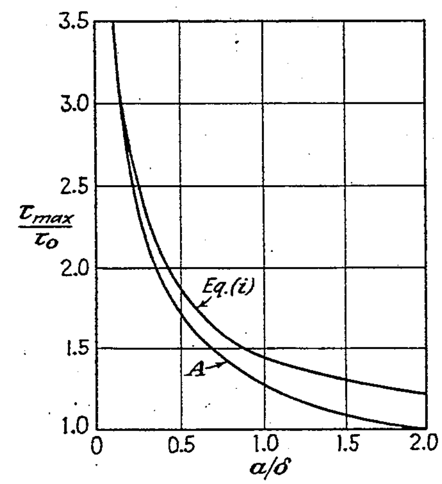


Fig. 174

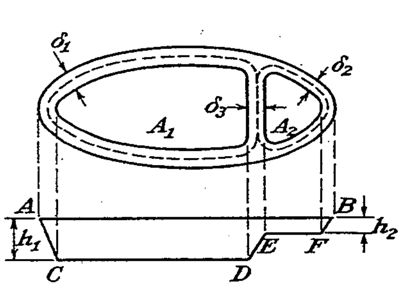


Fig. 175

by s_1, s_2, s_3 the lengths of corresponding dotted curves, we find, from Fig. 175,

$$\begin{aligned} \tau_1 s_1 + \tau_3 s_3 &= 2G\theta A_1 \\ \tau_2 s_2 - \tau_3 s_3 &= 2G\theta A_2 \end{aligned} \tag{l}$$

By using the last of the Eqs. (j) and Eqs. (k) and (l), we find the stresses τ_1, τ_2, τ_3 as functions of the torque:

$$\tau_1 = \frac{M_t[\delta_3 s_2 A_1 + \delta_2 s_3 (A_1 + A_2)]}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3 (A_1 + A_2)^2]} \tag{m}$$

$$\tau_2 = \frac{M_t[\delta_3 s_1 A_2 + \delta_1 s_3 (A_1 + A_2)]}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3 (A_1 + A_2)^2]} \tag{n}$$

$$\tau_3 = \frac{M_t(\delta_1 s_2 A_1 - \delta_2 s_1 A_2)}{2[\delta_1\delta_3 s_2 A_1^2 + \delta_2\delta_3 s_1 A_2^2 + \delta_1\delta_2 s_3 (A_1 + A_2)^2]} \tag{o}$$

In the case of a symmetrical cross section, $s_1 = s_2$, $\delta_1 = \delta_2$, $A_1 = A_2$, and $\tau_3 = 0$. In this case the torque is taken by the outer wall of the tube, and the web remains unstressed.[1]

To get the twist for any section like that shown in Fig. 175, one substitutes the values of the stresses in one of the Eqs. (l). Thus θ can be obtained as a function of the torque M_t.

117 | Screw Dislocations

In the two preceding articles, we have observed the requirement that w must be a single-valued function if the solution is to represent correctly a state of torsion. On reexamining Eqs. (149), (150), and (151), and the boundary condition (152), we can quickly see that it is possible to find states of stress corresponding to $\theta = 0$. The stress function ϕ is to satisfy Laplace's equation and to be constant on each boundary curve of the section. But we must use w rather than the form $\theta\psi(x,y)$ of Eq. (b) on page 293. Then Eqs. (f) of page 295 are replaced by

$$\frac{\partial \phi}{\partial y} = G\frac{\partial w}{\partial x} \qquad -\frac{\partial \phi}{\partial x} = G\frac{\partial w}{\partial y} \tag{a}$$

These are Cauchy-Riemann equations (see page 171) for the functions Gw and ϕ. Therefore, $Gw + i\phi$ is an analytic function of $x + iy$. Thus,

$$Gw + i\phi = f(x + iy) \tag{b}$$

Once the function f is chosen, we have a definite state, in which w will be the only nonzero displacement component.

Let r, ψ now represent polar coordinates in the cross section. The choice

$$f(x + iy) = -iA \log (x + iy) = A\psi - iA \log r \tag{c}$$

where A is a real constant, is of particular interest in the dislocation theory of plastic deformation (see Art. 34). From (b), we now have

$$Gw = A\psi \qquad \phi = -A \log r \tag{d}$$

The corresponding shear stress is in the circumferential direction and is given by the polar components

$$\tau_{z\psi} = -\frac{\partial \phi}{\partial r} = \frac{A}{r} \qquad \tau_{zr} = 0 \tag{e}$$

Any cylindrical boundary surface r = constant is free from loading. But the displacement w is not continuous. We can apply the solution to a hollow circular cylinder $a < r < b$ as in Fig. 176, which has an axial cut. One face is moved axially along the other by the uniform relative displacement

$$w(r,2\pi) - w(r,0) = \frac{2\pi A}{G} \tag{f}$$

obtained from the first of (d). The stress (e) can be regarded as induced

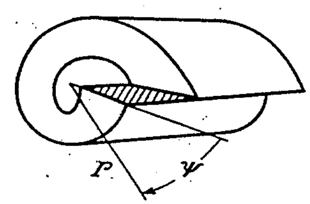


Fig. 176

[1] The small stresses corresponding to the change in slope of the membrane across the thickness of the web are neglected in this derivation.

Concept of superposition

The effect of a compound cause, say a loading configuration, is the sum of the effects of the individual causes.

Motivation: solve problems involving complex load configurations based on simpler solutions.

Caveat: Superposition depends on linearity, both material and geometrical. Geometrical nonlinear situations include large deformation of a bar in bending, and contact between two spheres.

Safety factor

Safety factor $SF = \frac{\text{failure load}}{\text{working load}}$.

Failure does not necessarily mean fracture. It may mean excessive deformation, damage, or any effect which causes the structure or structural element to no longer function as intended.

Example: What if airplane wings could be made of an infinitely strong but not infinitely stiff material?

§2.1 Plane elasticity

Method of solution.

Recall in the mechanics of materials method, we began with an assumption about the deformation field. We only checked the stresses to make sure they agreed with the applied loads in terms of *resultants* .

In the elasticity method, one must simultaneously satisfy:
1. equilibrium conditions in a continuum sense at each point,
2. continuity of the displacement field,
3. boundary conditions at the surface.

If they are all satisfied exactly, we have an exact solution.

The theory of elasticity permits one to deal with problems which are not necessarily geometrically simple.

Few new elasticity solutions are now being discovered. Even so, study of elasticity aids in the development of physical insight.

Stress-strain relations

Elementary form of **Hooke's law** for a linear, isotropic, elastic solid:

$$\varepsilon_{xx} = \frac{1}{E}\ \{\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\}$$

$$\varepsilon_{yy} = \frac{1}{E}\ \{\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}\},$$

$$\varepsilon_{zz} = \frac{1}{E}\ \{\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}\}.$$

These three are complete, but sometimes a shear relation is also presented,

$$\gamma_{xy} = \frac{1}{G}\ \tau_{xy}\ .$$

Plane stress in xy plane means σ_{zz}, σ_{yz}, and σ_{zx} are zero. Then,

$$\varepsilon_{xx} = \frac{1}{E}\ \{\sigma_{xx} - \nu\sigma_{yy}\}$$

$$\varepsilon_{yy} = \frac{1}{E}\ \{\sigma_{yy} - \nu\sigma_{xx}\}$$

$$\gamma_{xy} = \frac{1}{G}\ \tau_{xy}\ .$$

Although they are simpler in the compliance formulation, one can solve for stress and present them in the modulus formulation, for plane stress.

$$\sigma_{xx} = \frac{E}{1 - \nu^2}\ \{\varepsilon_{xx} + \nu\overset{\varepsilon}{\cancel{\sigma}}_{yy}\}$$

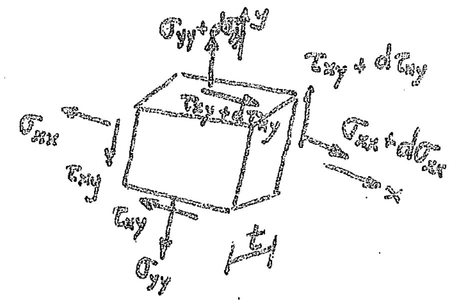
$$\sigma_{yy} = \frac{E}{1 - \nu^2}\ \{\varepsilon_{yy} + \nu\overset{\varepsilon}{\cancel{\sigma}}_{xx}\}$$

$$\tau_{xy} = G\ \gamma_{xy}\ .$$

§2.2 Equilibrium equations, boundary conditions, Saint Venant's principle

We are familiar with the application of Newton's first law of equilibrium to macroscopic objects. In solving problems on a continuum scale, a differential form of the equations of **equilibrium** is needed.

Consider a free-body diagram of a differential element of thickness t of material.



From sum of forces in the x direction,

$$-t\,\sigma_{xx}\,dy - \tau_{xy}\,t\,dx + \{\sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}dx\}t\,dy + \{\tau_{xy} + \frac{\partial\tau_{xy}}{\partial y}dy\}\ t\,dx = 0$$

From sum of forces in the y direction,

$$-t\,\tau_{xy}\,dy - \sigma_{yy}\,t\,dx + \{\tau_{xy} + \frac{\partial\tau_{xy}}{\partial x}dx\}t\,dy + \{\sigma_{yy} + \frac{\partial\sigma_{yy}}{\partial y}dy\}\ t\,dx = 0$$

Simplifying, the equilibrium equations are:

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0 \text{ in x direction}$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0 \text{ in y direction.}$$

The equilibrium relations in the index notation are as follows for force and moment respectively: The Einstein summation convention assumed in which repeated indices are summed over. The comma represents differentiation with respect to the spatial coordinate corresponding to the index after the comma.

σ_{ji} is stress
G_i is a body force, or force per unit volume.
e_{ijk} is the permutation symbol
m_{ji} is a moment per unit area or couple stress. It is neglected in classical elasticity.
C_i is a body moment, or couple per unit volume.

$$\sigma_{ji,j} + G_i = 0 \qquad (1)$$

$$e_{ijk}\,\sigma_{jk} + m_{ji,j} + C_i = 0 \qquad (2)$$

Body forces arise due to gravitation.
Body moments arise due to electromagnetic interactions in magnetic materials.

Couple stresses represent a distributed average of moments upon fibers, ribs, layers, or other structural elements in composite materials.

In classical elasticity, in the absence of body couples or surface couples, Eq. 2 reduces to
$\sigma_{jk} = \sigma_{kj}$,

that is, the stress is symmetric.
If body couples or surface couples are permitted, the stress can become asymmetric.

Boundary conditions.

Boundary conditions entail prescription of stress or displacement upon the surface of the object in question. In many problems, the surface tractions (stresses at the surface) are zero over much of the surface.

Saint Venant's principle

Saint-Venant's principle is important in the application of elasticity solutions in many practical situations in which boundary conditions are satisfied in the sense of **resultants** rather than pointwise. For example, a bending moment may be applied to a beam via a complex array of bolted joints, which generate a locally complex stress pattern. In view of Saint-Venant's principle, one expects to observe bending type stresses far from the ends.

Saint-Venant's principle states that a localized **self-equilibrated** load system produces stresses which decay with distance more rapidly than stresses due to forces and moments. It is applicable in many situations of interest in engineering.

Demonstration: stress fields for concentrated loads which give rise to compression or bending, as seen with a photoelastic demonstrator.

There are some counter-examples. Consider a sandwich panel with rigid face sheets and an elastic material of Poisson's ratio ν sandwiched between them. For Poisson's ratios in the vicinity of 0.5, stresses applied to the end will decay with distance z as $\sigma(z) \propto e^{-\gamma z}$. The decay rate is

$$\gamma \propto \sqrt{\frac{3(1-2\nu)}{3-4\nu}} \; .$$

The distance $1/\gamma$, over which there is significant stress, diverges as Poisson's ratio approaches $\frac{1}{2}$.

In some thin-walled structures, localized self-equilibrated loads may propagate a significant distance. Saint-Venant's principle is inapplicable for such structures.

Constitutive relations

We mostly deal with linear isotropic elastic materials in this class. Many other possibilities exist.

Anisotropic: Dependent upon direction, referring to the material properties of composites, aggregates, single crystals, and oriented polycrystalline materials.

Creep: Time dependent strain in response to step stress; a manifestation of viscoelastic behavior.

Cubic: A type of anisotropic symmetry in which the unit cells are cube shaped. There are three independent elastic constants. Material is invariant to 90 degree rotations.

Elastic: Stress-strain path for loading is identical to the path for unloading, with immediate recovery to zero upon unloading.

Elastic-perfectly plastic: Elastic up to yield point after which strain increases with no increase in stress.

Elastic-plastic with work hardening: Beyond yield point, stress increases with strain.

Hexagonal: A type of anisotropic symmetry in which the unit cells are hexagonally shaped. Material is invariant to 60 degree rotations about an axis. There are five independent elastic constants. **Transverse isotropy** is mechanically equivalent to hexagonal although the structure may be random in the transverse direction.

Homogeneous: Material properties are identical at every point in the body. Concept of symmetry is expressed here as translational symmetry: material is invariant to translations. Homogeneous materials may be isotropic or anisotropic. At the atomic scale all materials are heterogeneous, but for many engineering applications we may view them as continuous media.

Isotropic: Independent of direction, referring to material properties. There are two independent elastic constants for a linearly elastic material. Engineering constants are E, G, B, ν, but they are interrelated.

Linear: Stress is proportional to strain, assuming all other variables upon which stress or strain might depend are held constant.

Orthotropic: A type of anisotropic symmetry in which the unit cells are shaped like rectangular parallelepipeds. In crystallography, this is called orthorhombic. There are nine independent elastic constants. Principal directions are mutually orthogonal. Material is invariant to reflections in two or three orthogonal planes.

Piezoelectric: In some crystalline or polycrystalline materials which lack a center of symmetry, there is coupling in which both stress and electric field contribute to the strain.

Thermoelastic: In all materials with a nonzero coefficient of thermal expansion, there is thermoelastic coupling in which both stress and temperature changes contribute to the strain.

Triclinic: A type of anisotropic symmetry in which the unit cells are oblique parallelepipeds with unequal sides and angles. There are 21 independent elastic constants.

Viscoelastic: Relation between stress and strain depends upon time or upon frequency.

Principal stress

Principal stresses are normal stresses which act on mutually perpendicular planes. They include the absolute largest and smallest normal stresses at a given point.

Recall Mohr's circle is a tool for 2-D transformations of stress and special 3-D transformations. It can be used to determine principal stresses under those special circumstances. It is not applicable to general 3-D transformations.

Consider a free body diagram of a cut corner of a unit cube. This is called a Cauchy tetrahedron.

Fatigue

A material loaded through multiple cycles will break at a stress considerably less than the ultimate strength for a single application of load. Fatigue is quantified by the S-N curve, in which the number N of cycles is plotted logarithmically.

The effect of cyclic stresses is to initiate microcracks at centers of stress concentration within the material or on the surface resulting in the growth and propagation of cracks leading to failure.

As for fatigue testing, the rate of crack growth can be plotted in a log-log scale versus time. Testing the fatigue properties to generate an S-N curve entails monitoring the number of cycles to failure at various stress levels. This test requires a large number of specimens compared with the crack propagation test.

The *endurance limit* is the stress below which the material will not fail in fatigue no matter how many cycles are applied. Not all materials exhibit an endurance limit. (a practical limit is often chosen as 10^7 cycles).

The presence of a saline environment exacerbates fatigue.
Surface roughness exacerbates fatigue. A polished surface is better.
Rubbing or 'fretting' exacerbates fatigue. Re-design the part or use lubricants.
Heat treatment to introduce residual surface compression can be helpful.

Stress concentration

Ratio of local maximum stress to applied stress in the absence of the heterogeneity is called the stress concentration factor or SCF.

Stress concentration factors are determined from

- Elasticity theory
- Experiment
- Finite elements.

Stress concentration factors arise from

- Holes
- Notches
- Grooves
- Heterogeneities in loading
- Heterogeneities in material

Some particular values for holes and inclusions

Circular hole in plane uniaxial tension, SCF = K = 3.0

Elliptic hole, with a as major axis, b as minor axis, ρ as radius of curvature

$$SCF = K = (1 + 2\frac{a}{b}) = 1 + 2\sqrt{\frac{a}{\rho}}$$

Example, consider glass with theoretical strength of σ^{ult} = 14 GPa, with cracks 2 μm long with radius of curvature 1 A (0.1 nm). Then the strength of glass with these cracks is σ = 14 GPa/[SCF] = 70 MPa. This is about the strength of common glass.

Spherical cavity in uniaxial tension

$$SCF|_{polar} = -\frac{3+15\nu}{14-10\nu} \quad SCF|_{eq\theta\theta} = \frac{27-15\nu}{14-10\nu} \quad SCF|_{eq\psi\psi} = -\frac{3-15\nu}{14-10\nu}$$

Spherical cavity in biaxial tension

Spherical cavity in pure shear

$$SCF = \frac{15(1-\nu)}{7-5\nu}$$

Rigid cylindrical inclusion in uniaxial tension

$$SCF|_{polar} = \frac{1}{2}\left(3-2\nu+\frac{1}{3-4\nu}\right) \qquad SCF|_{eq} = \frac{1}{2}\left(1+2\nu-\frac{3}{3-4\nu}\right)$$

Rigid spherical inclusion in uniaxial tension

$$SCF|_{polar} = \frac{2}{1+\nu} + \frac{1}{4-5\nu} \qquad SCF|_{eq} = \frac{\nu}{1+\nu} - \frac{5\nu}{8-10\nu}$$

Rigid spherical inclusion in hydrostatic tension contact region radius: $a = 0.880\,(FR/E)^{1/3}$. It increases slowly with force.

peak compressive stress: $p_0 = 0.616\,(FE^2/R^2)^{1/3}$. It increases slowly with force.

Stress vs radial position in region, $\sigma_z = -\,p_0 \dfrac{\sqrt{a^2 - r^2}}{a}$, a parabolic distribution.

Overall 3-D pattern of stress is complex and multiaxial. Cracks may develop below the surface in ball bearings.

$$SCF|_{radial} = 3\,\frac{1-\nu}{1+\nu}$$

Reference: Goodier, J. N., "Concentration of stress around spherical and cylindrical inclusions and flaws", *Trans. ASME* Vol. 55, 1933, 39-44. (later called *J. Applied Mech.*, Vol. 1)

Observe that for the three dimensional cases, the stress concentration factor depends on the Poisson's ratio of the material in question.

A heterogeneous load distribution

Consider a **rigid circular cylindrical indenter** of radius R pressed with load F on a semi infinite solid substrate. This could represent a building erected upon compliant earth, or an industrial press operation. A solution for an elastic solid of Young's modulus E and Poisson's ratio ν is available. The indenter displacement is (Timoshenko, S. P. and Goodier, J. N., *Theory of Elasticity*, McGraw Hill, 1982.)

$$w = \frac{F(1 - \nu^2)}{2RE}\ .$$

The pressure distribution q(r) as a function of radial coordinate r is

$$q(r) = \frac{F}{2\pi R\sqrt{R^2 - r^2}}\ .$$

Observe that the pressure becomes **singular** at the edge. The indenter is idealized as perfectly rigid (much stiffer than the elastic substrate), and with a perfectly sharp edge.

Uses of concept of stress concentration.

- Find stress distribution (nominal) in the absence of holes.
- Multiply nominal stresses by the appropriate stress concentration factors. Many of these may be obtained from a handbook.
- The largest stress will cause failure. It is not necessarily the largest nominal stress.
- In the design process it is sensible to ameliorate stress concentrations by avoiding sharp re-entrant corners, and rounding them off when they are an unavoidable part of a structure.

Demonstrations, by photoelasticity. Circular hole at center of a compressed bar. Circular hole in bar subjected to pure bending. Circular notches in bar subjected to pure bending.

Contact stress

From the theory of elasticity, we have several interesting solutions for spheres and cylinders in contact.

For spheres of radius R of Young's modulus E, Poisson's ratio ν, under force F,

I. ASTM model for K_{Ic} testing

A. designed to produce valid K_{Ic} results - How?

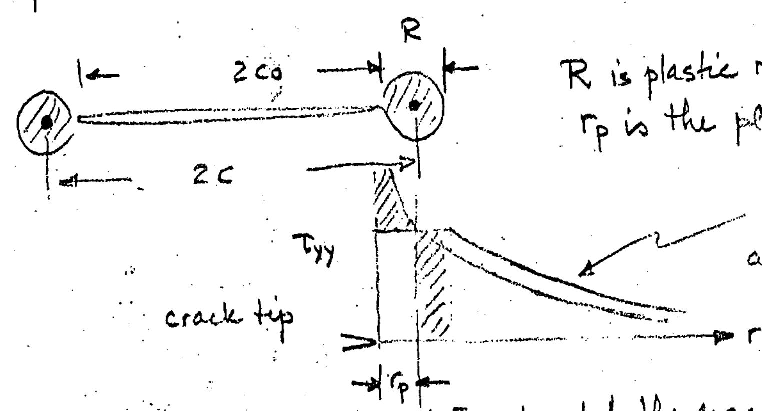
1. must meet $c_0 \geq 2.5\left(K_{Ic}/\sigma_y\right)^2$
2. " " $B \geq 2.5\left(K_{Ic}/\sigma_y\right)^2$; $W/B \cong 2$
3. starting crack length must be 0.45 - 0.55 W (width of specimen)
4. crack must be sharp and must be introduced via a fatigue crack starting from a V-notch
5. The fatigue crack must be introduced by <u>low</u> type cycling
6. A displacement gage will be used to accurately measure the <u>relative</u> displacement of two precisely located gages positions
7. Temperature and load rate requirements

B. Why these requirements -

1. $c_0 \geq 2.5\left(K_{Ic}/\sigma_y\right)^2$. this is a requirement that is necessary and sufficient in order for LEFM to hold

<u>Proof</u>:

Consider a plate loaded in tension



R is plastic region
r_p is the plastic zone

- We assume that the stresses are redistributed ahead of the crack so that the load bearing capacity in front of the crack is unchanged when yielding occurs. We assume that the shaded areas under the graph are the same.
- Thus $2c = 2c_0 + 2r_p = 2c_0 + R$ is the <u>effective</u> length of the crack
- In plane strain mode I $R = \frac{1}{6\pi}\left(K_I/\sigma_y\right)^2$

if the stress $\sigma \uparrow$ $K_I \uparrow$ also $K_I \uparrow$ due to the plastic zone correction.

Thus $$K_I = \sigma\sqrt{\pi c_0}\left\{1 - \frac{1}{12}\left(\sigma/\sigma_y\right)^2\right\}^{-1/2} \qquad (1)$$

- In a test as $\sigma \to \sigma_y$, $K_I \to K_{Ic}$

- If σ reaches σ_y before $K_I = K_{Ic}$ we get yielding and by our elastic-plastic model r_p (and R) $\to \infty$. Hence we violate the LEFM assumption of small scale yielding

- We want $K_I = K_{Ic}$ before $\sigma = \sigma_y$. Thus let $K_I = K_{Ic}$ in (1) and solve for the crack length $2c_0$

$$2c_0 = \frac{2}{\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2\left\{\left(\sigma_y/\sigma\right)^2 - \frac{1}{12}\right\} \qquad K_I = K_{Ic}$$

This will cause unstable crack growth

- The crack length that produces yielding is when $\sigma_y = \sigma$

or $$2c_0 = \frac{11}{12}\cdot\frac{2}{\pi}\left(K_{Ic}/\sigma_y\right)^2 \sim \frac{1}{2}\left(\frac{K_{Ic}}{\sigma_y}\right)^2$$

if $\sigma > \sigma_y$ then $2c_0 < \frac{1}{2}\left(K_{Ic}/\sigma_y\right)^2$ unacceptable

$\sigma < \sigma_y$ then $2c_0 > \frac{1}{2}\left(K_{Ic}/\sigma_y\right)^2$ or $c_0 > \frac{1}{4}\left(K_{Ic}/\sigma_y\right)^2$

- Because we want to make adequate measurements of K_{Ic}

we want $c_0 \gg \frac{1}{4}\left(K_{Ic}/\sigma_y\right)^2$

Srawley and Brown suggested that $c_0 \geq 2.5\left(K_{Ic}/\sigma_y\right)^2$ and this is accepted as the standard.

2. $B \geq 2.5 \left(K_{Ic}/\sigma_y \right)^2$: This requirement arises from the consideration that we want only MODE I type fracture

Proof:

- As has been shown to you in class in order for cracks to propagate for perfectly brittle materials, the crack extension force $G_I = 2\gamma_s$, where γ_s is the surface energy. However for materials that deform plastically, then crack extension will only occur when $G_I = 2\gamma_s + p$ where p is the plastic work of crack extension. p is not a consta and depends on the size of the plastic zone, σ_y, the work hardening rate, etc AND they all in turn depend on the crack length.

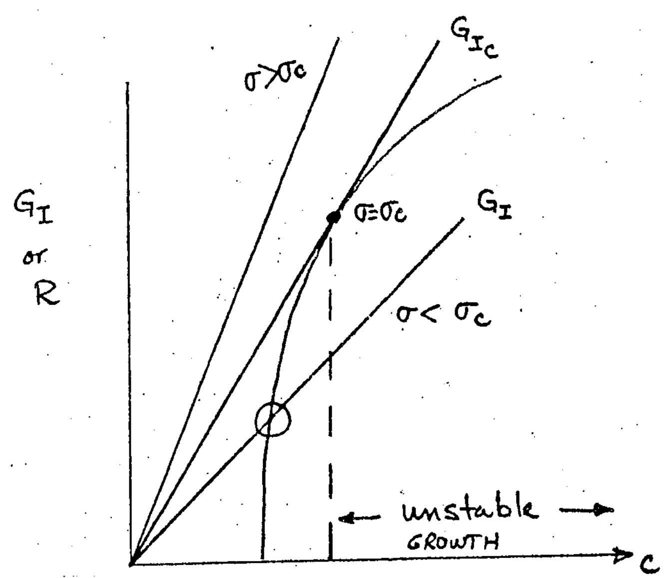
- If we define $R \equiv$ crack extension resistance $= 2\gamma_s + p$, then for unstable growth we must have that

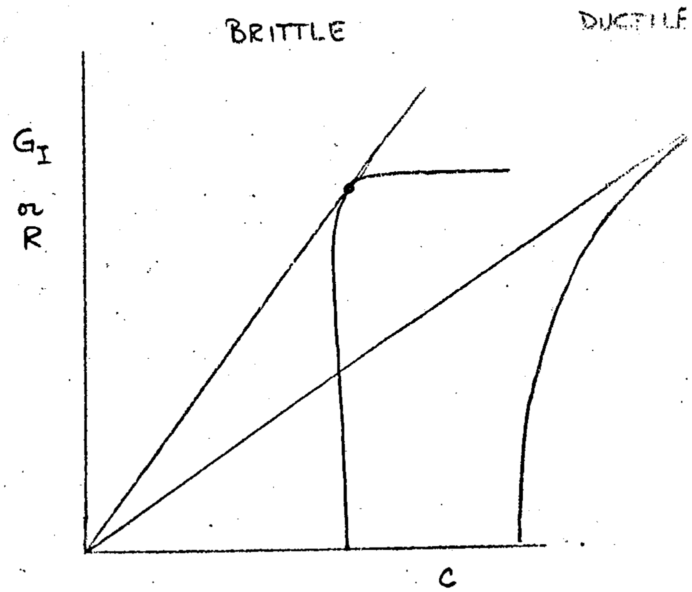
$$G_I \geq R$$

and also

$$\partial G_I / \partial c \geq \frac{\partial R}{\partial c}$$

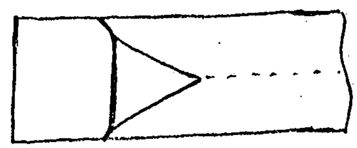
Thus if we remember that $G_I = \frac{\sigma^2 \pi c}{2\mu}(1-\nu)$ and look at a typical G_I versus c curve,



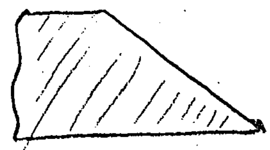


Note that the brittle material shows little plastic deformation and

What has been found is that as the plate is made thinner the R curve will vary and will no longer have a distinct intersection point. The reason for this is the growth of "shear lips" from the free surface and the thinness of the plate (plane stress conditions).



DIR. OF crack propagation →

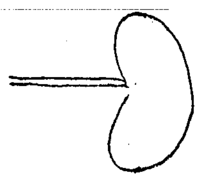


SLANT

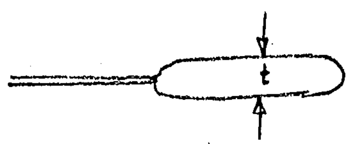
or

V SLANT

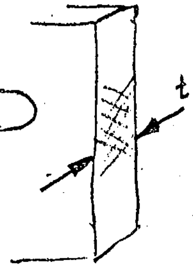
The growth of the shear lips is due to the plastic zone being constrained in the thickness direction. So it will spread in front of the crack tip. The mechanism that will cause crack extension will be due to failure in shear (mode III); hence we see the slant formation.



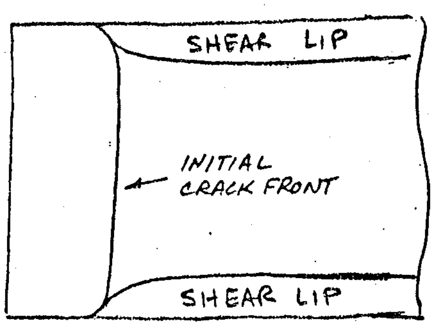
unrealistic character of plastic zone.

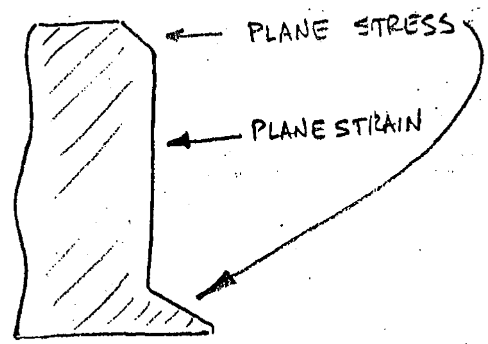


realistic

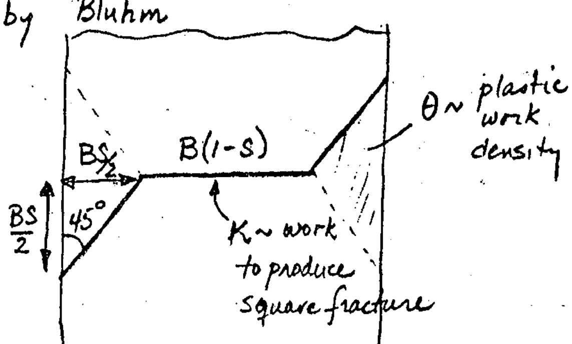


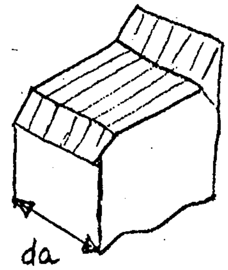
As the plate width is increased, the formation of the shear lips is reduced due to the plane strain effect and the cross-section will look like this





Many have proposed models to describe what occurs here. One such model is that of Krafft, Sullivan and Boyle (1961) modified by Bluhm



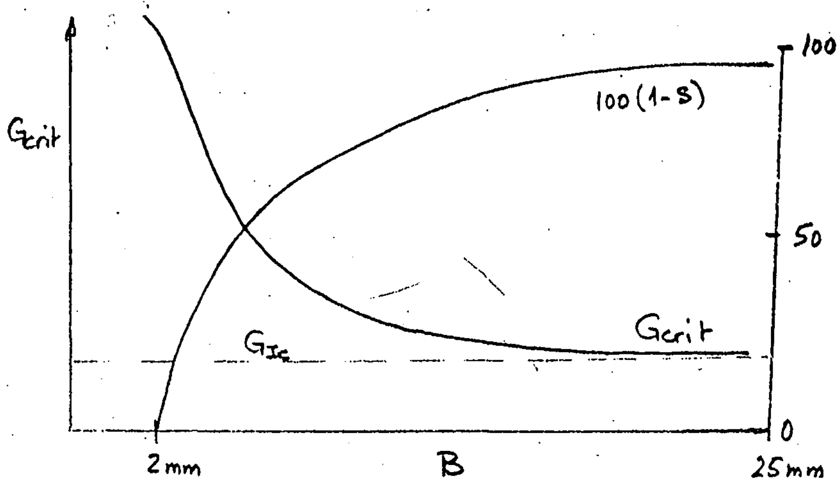


1. square fracture $\neq f(c_0)$
2. shear lips are assumed to occur at 45°
3. flat fracture is a surface phenomenon
4. Shear lip is volumetric

The work done to create the crack surface da is:

Now $G_I = \frac{1}{B}\frac{dW}{da} = K(1-S) + \frac{Bs^2\theta}{2}$. Note that S is picked so that BS = constant as crack length ↑.

Thus as $B \to \infty$ $S \to 0$ and $G_I \to K$.

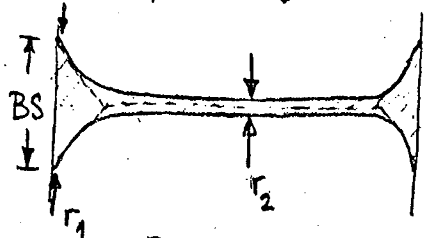


From data for Aluminum 7075-T6

$K \sim 200$ kJ/m²

$\theta \sim 20$ kJ/m²

Look at the plastic zone and superpose the model of Krafft:



$BS \gg r_{\text{plane stress}} \sim r_p = \frac{1}{2\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2$ (*)

since $r_1 > r_2$ (plane strain). If (*) is true then plain strain conditions will extend over most of the cross-section and we will have essentially mode I fracture.

$r_p/B \ll 1$ or $\frac{r_p}{B} < S \ll 1$

hence $B \gtrsim \left(\frac{K_{Ic}}{\sigma_y}\right)^2$. To determine the real equation, tests were done on many types of metals and here are some of the results.

Example: Maraging Steel $\sigma_y = 259$ KSi

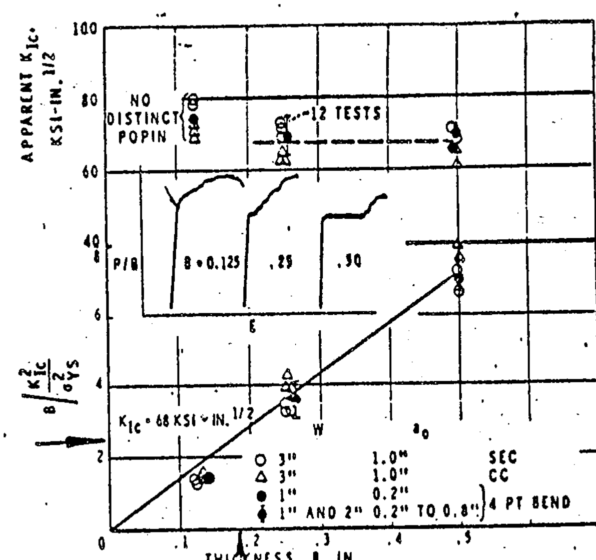


FIG. 14—Effect of thickness on popin behavior and apparent K_{Ic} for 259 ksi

conclusion:

$B \geq 2.5\left(\frac{K_{Ic}}{\sigma_Y}\right)^2$

Specimen Size Requirements

We have argued that to limit yielding we must make large samples with long cracks. Thus $K_I \to K_{I_c}$ before $\sigma \to \sigma_Y$. From our analysis we expect

$$C_o \geq 2.5 \left(\frac{K_{I_c}}{\sigma_Y} \right)^2$$

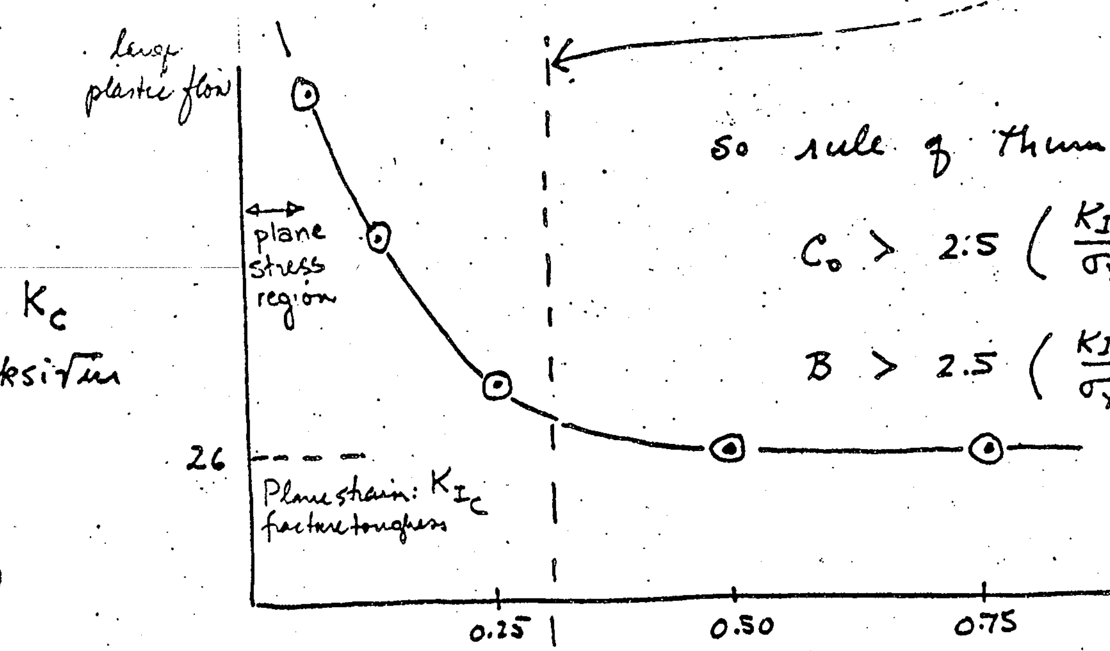
this needs to be checked. Also, How thick must sample be for plane strain conditions?

Consider 7075-T6 (MS&E 202C experiment).

$\sigma_y = 75$ ksi

$K_{I_c} = 26$ ksi $\sqrt{in}$

$$2.5 \left(\frac{K_{I_c}}{\sigma_Y} \right)^2 = 0.3 \text{ in}$$



so rule of thumb:

$$C_o > 2.5 \left(\frac{K_{I_c}}{\sigma_Y} \right)^2$$

$$B > 2.5 \left(\frac{K_{I_c}}{\sigma_Y} \right)^2$$

usually more difficult to achieve due to how the specimen is machi

Fracture mechanics

As elliptic hole becomes progressively narrower, the ellipse approaches a crack shape and SCF = K $\rightarrow\infty$. Actual observed stress concentration factors for cracks are not infinite.

Therefore a material with one perfectly sharp crack will have **zero** strength, since the stress concentration factor becomes infinite. Experimentally, even for brittle materials, strength is reduced by cracks but not infinitely.
A criterion based on energy balance rather than on stress has therefore been adopted. The approach is due to Griffith. The energy relations are visualized as follows. The energy required to extend a crack is linear in crack length because the energy is expended in creating new free surfaces. By contrast the strain energy available comes from a roughly semicircular region around the crack and is therefore quadratic in crack length. As the crack propagates, the material in that region is unloaded and its energy is made available to drive the crack. When the crack is long enough that an increment in crack length gives rise to an energy release equal to energy expended, catastrophic crack growth impends.

Griffith proposed an **energy** approach to fracture. The elastic energy stored in a test specimen of unit thickness, in a circular region around a crack of length a, is:

$$2\pi a^2 \frac{1}{2E}\,\sigma^2 \qquad \text{(F1)}$$

Recall that $\frac{1}{2}\,E\varepsilon^2 = \frac{1}{2E}\,\sigma^2$ represents a strain energy per unit volume.
The elastic energy for a brittle material is twice the area under the stress strain curve. The elastic energy is used to create two new surfaces as the crack propagates. The surface energy, 4 γa (γ is the surface energy; it is an energy per unit area.) should be smaller than the elastic energy for the crack to grow. Thus, the incremental changes of both energies for the crack to grow can be written,

$$\frac{d}{da}\left(\frac{\pi(a\sigma)^2}{E}\right) = \frac{d}{da}(4\gamma a) \qquad \text{(F2)}$$

Hence,

$$\sigma = \sigma_f = \sqrt{\frac{2\gamma E}{\pi a}} \qquad \text{(F3)}$$

Since for a given material E and γ are constants,

$$\sigma_f = \frac{K}{\sqrt{\pi a}} \qquad \text{(F4)}$$

In this case K has the units of psi $\sqrt{in}$ or MPa $\sqrt{m}$ and is proportional to the energy required for fracture.
K is a measure of *fracture toughness,* called the stress intensity factor. Cracks and stress concentrations also occur in ductile materials, but their effect is usually not as serious as in brittle ones since local yielding which occurs in the region of peak stress will effectively blunt the crack and alleviate the stress concentration.

The **stress intensity factor** K is the criterion for fracture in cracked objects. For a small Mode I crack of length a,

$K_1 = \sigma \sqrt{\pi a}$ f(a/c).

Here f(a/c) is a dimensionless function of loading geometry; it expresses the effect of crack length in relation to block size. σ is the stress required for fracture in the absence of a crack. The units for K are MPa$\sqrt{}$m, in contrast to the stress concentration factor which is dimensionless. Observed that there is no characteristic length scale in the classical theory of elasticity. The length scale must come from other considerations.

Fracture occurs when K_1 exceeds a critical value, K_{1c} determined from experiment. This is the fracture toughness based on a static test. The fracture toughness for a dynamic situation is NOT the same as for a static situation

Formulas for K are valid over a range of geometrical parameters, specifically,
thickness $t \geq 2.5\ (K_{1c}/\sigma_y)^2$, and crack length $a \geq 2.5\ (K_{1c}/\sigma_y)^2$.

In a thick block, the stress field around the tip of the crack is triaxial, since the Poisson contraction in the highly stressed region near the crack is restrained by the surrounding material, which is not so highly stressed. This triaxial stress causes brittle behavior in seemingly ductile materials, since shear deformation is suppressed.

If the block is thinner than the above limit, toughness depends on thickness. If the crack length is less than the above limit, then the material may undergo yield before any fracture occurs from the crack.

Be aware that K_{1c} depends on temperature, and often drops precipitously at low temperature.

Example

Estimate the size of the surface flaw in a glass whose modulus of elasticity and surface energy are 70 GPa and 800 erg/cm^2 respectively. Assume that the glass breaks at a tensile stress of 100 MPa.

Answer
From equation (F4), and keeping in mind the transformation from cgs to SI units,

$$a = \frac{2\gamma E}{\pi\sigma_f^2}$$

$$= \frac{2 \times 800 \text{ dyne/ cm} \times 70 \text{ GPa}}{\pi\,(100 \text{ MPa})^2}$$

$$= \underline{3.565\ \mu m}$$

To two significant figures, a = 3.6 μm.
[Note that if the crack is on the surface its length is a, if it is inside the specimen it is 2a.
Remember 1 erg = 1 dyne cm]

Example (adapted from example on page 62-63)

Consider a 1 cm crack in a large plate of steel. Assume the plate is large enough that $f(a/c) \cong 1$.

What stress gives rise to fracture for a weaker or 'mild' steel (σ_y = 500 MPa, K_{1c} = 175 MPa√m) and a high strength steel (σ_y = 1410 MPa, K_{1c} = 50 MPa√m).

Solution: Use $K_1 = \sigma\, f(a/c)\, \sqrt{\pi a}$, so with $f(a/c) \cong 1$, $\sigma = K_{1c}/\sqrt{\pi a}$.

Weaker steel A, σ = 987 MPa, which exceeds the yield strength, so it undergoes yield at 500 MPa rather than fracture from the crack.

Stronger steel, σ = 282 MPa, which is less than the yield strength 1410 MPa, so it fractures catastrophically.

Consequently, in the presence of 1 cm cracks, the stronger steel actually is weaker than the 'mild' weaker steel. Moreover, the consequence of an overload is more severe for the stronger steel since it catastrophically fractures rather than yields in bulk.

Example (adapted from Gordon, *Structures*)

Suppose we have a large structure such as a ship or a bridge and wish to tolerate a 1 meter long crack without catastrophic failure. Consider 'mild' steel (σ_y = 500 MPa, K_{1c} = 175 MPa√m) .
Solution-
With $f(a/c) \cong 1$, $\sigma = K_{1c}/\sqrt{\pi a}$ = 90 MPa or 14,000 psi.

In **foam**, Gibson and Ashby [*Cellular solids*] predict toughness K_{1c} proportional to [√(cell size)](density)$^{3/2}$.

Stress concentrations: appendix

Experimental stress concentrations in composite materials are consistently less than the theoretical ones. The non-classical fracture behavior has been dealt with using point stress and average stress criteria, however that approach cannot account for non-classical strain distributions in objects under small load. Such differences may be accounted for via a generalized continuum approach. Reduced stress concentration factors for small holes are known experimentally in fibrous composite materials. The fracture strength of graphite epoxy plates with holes depends on the size of the hole [1]. Moreover the strain around small holes and notches in fibrous composites well below the yield point is smaller than expected classically [2,3], while for large holes, the strain field follows classical predictions [4].

1. R.F. Karlak, "Hole effects in a related series of symmetrical laminates", in *Proceedings of failure modes in composites, IV*, The metallurgical society of AIME, Chicago, 106-117, (1977)
2. J.M. Whitney, and R.J. Nuismer, "Stress fracture criteria for laminated composites containing stress concentrations", *J. Composite Materials*, **8**, (1974) 253-275.
3. M. Daniel, "Strain and failure analysis of graphite-epoxy plates with cracks", *Experimental Mechanics,* **18** (1978) 246-252, .
4. R. E. Rowlands, I. M. Daniel, and J. B. Whiteside "Stress and failure analysis of a glass-epoxy plate with a circular hole", *Experimental Mechanics*, **13**, (1973) 31-37

CONCEPT OF ELASTIC STRAIN ENERGY

- DEFORMATION OF BODY UNDER EXTERNAL LOADS - LOADS DO WORK
- ASSUME - NO KINETIC OR HEAT EXCHANGE
  - GRADUAL INCREASE IN LOAD FROM INITIAL TO FINAL STATE
  - CONSERVATION OF ENERGY $\Rightarrow$ STRAIN ENERGY IS POTENTIAL ENERGY
- FROM STATICS REMEMBER IF $\vec{F}$ CONSTANT OR NOT

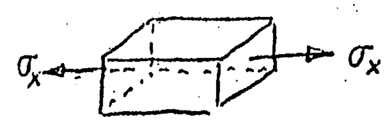
  WORK DONE ON UNIT VOLUME IS $\vec{F} \cdot d\vec{x} = \bar{\sigma} A \cdot d\bar{\epsilon}\, l = \bar{\sigma} \cdot d\bar{\epsilon}\, (Al)$
- TOTAL WORK DONE BY FORCE IS STORED AS STRAIN ENERGY GIVEN BY

$$\int_V \left[\int \bar{\sigma} \cdot d\bar{\epsilon}\right] dV = \int_V \left[\frac{\sigma^2}{2E}\right] dV = \int_V \left[\frac{E}{2}\epsilon^2\right] dV = \int \frac{\bar{\sigma}\cdot\bar{\epsilon}}{2}\, dV$$

- REMEMBER WORK IS ADDITIVE AND DEPENDS ON FINAL & INITIAL STATES AND NOT ON PATH BETWEEN STATES
- IF WE HAVE A BODY THAT OBEYS HOOKE'S LAW AND WE APPLY A FORCE IN THE X - DIRECTION ONLY

$$\sigma_x = E\epsilon_{x_1}, \quad \epsilon_{y_1} = -\nu\epsilon_{x_1}, \quad \epsilon_{z_1} = -\nu\epsilon_{x_1} \qquad \nu - \text{POISSON RATIO}$$

- SINCE σ_x, ϵ_{x_1} ARE PARALLEL TO EACH OTHER, WORK IS DONE
- SINCE σ_x, ϵ_{y_1} OR ϵ_{z_1} ARE $\perp$ TO EACH OTHER, NO WORK DONE
- WORK DUE TO σ_x IS $\int \frac{\sigma_x \epsilon_{x_1}}{2}\, dV$



- IF WE NOW APPLY TO THIS STATE AN ADDITIONAL FORCE IN THE Y-DIRECTION KEEPING σ_x FIXED

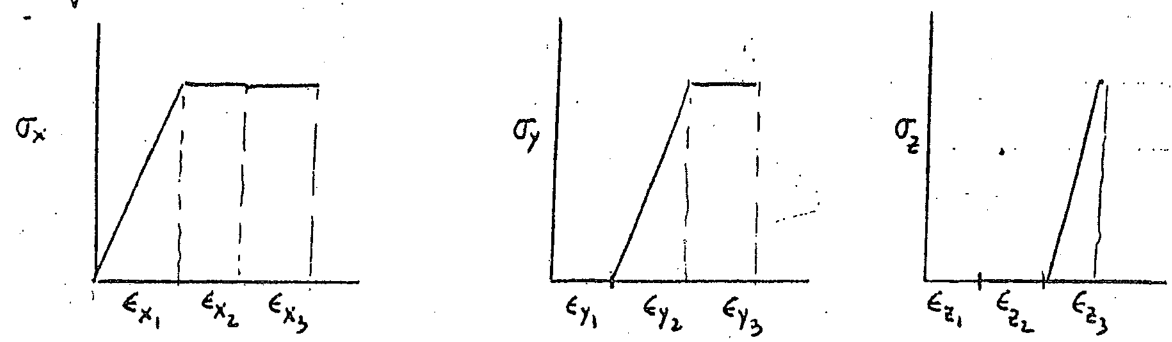
$$\sigma_y = E\epsilon_{y_2} \qquad \epsilon_{x_2} = -\nu\epsilon_{y_2} \qquad \epsilon_{z_2} = -\nu\epsilon_{y_2}$$

- SINCE σ_y, ϵ_{y_2} ARE PARALLEL, WORK IS DONE
- SINCE σ_y, ϵ_{x_2} OR ϵ_{z_2} ARE $\perp$ TO EACH OTHER, NO WORK DONE
- BUT

  σ_x FROM BEFORE DOES DO WORK DUE TO ϵ_{x_2}

- THUS ADDITIVE WORK DUE TO σ_y IS

$$\int_V \left[\frac{\sigma_y \epsilon_{y_2}}{2} + \sigma_x \epsilon_{x_2}\right] dV$$



- SIMILARLY IF WE APPLY A FORCE IN THE Z DIRECTION ONLY KEEPING σ_x, σ_y FIXED : $\sigma_z = E\epsilon_{z_3}$, $\epsilon_{x_3} = -\nu\epsilon_{z_3}$, $\epsilon_{y_3} = -\nu\epsilon_{z_3}$
- A SIMILAR ARGUMENT YIELDS THE ADDITIVE WORK DONE

$$\int_V \left[\frac{\sigma_z \epsilon_{z_3}}{2} + \sigma_x \epsilon_{x_3} + \sigma_y \epsilon_{y_3}\right] dV$$

- BY ADDING THE THREE TERMS WE GET THE TOTAL WORK DONE...

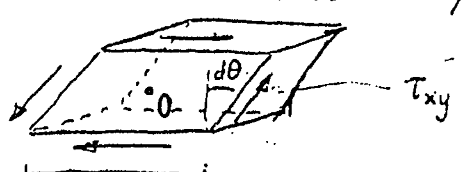
$$\int \left[\frac{\sigma_x \epsilon_{x_1}}{2} + \frac{\sigma_y \epsilon_{y_2}}{2} + \sigma_x \epsilon_{x_2} + \frac{\sigma_z \epsilon_{z_3}}{2} + \sigma_x \epsilon_{x_3} + \sigma_y \epsilon_{y_3}\right] dV$$

- NOTE THAT $\frac{1}{2}\sigma_x \epsilon_{x_2} = \frac{1}{2}E\epsilon_{x_1} \cdot (-\nu \epsilon_{y_2}) = \frac{1}{2}E\epsilon_{y_2} \cdot (-\nu\epsilon_{x_1}) = \sigma_y \epsilon_{y_1}/2$
- SIMILARLY $\frac{1}{2}\sigma_x \epsilon_{x_3} = \frac{1}{2}\sigma_z \epsilon_{z_1}$ & $\frac{1}{2}\sigma_y \epsilon_{y_3} = \frac{1}{2}\sigma_z \epsilon_{z_2}$

- THUS TOTAL WORK DONE IS $\frac{1}{2}\int \left[\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z\right] dV$

  WHERE $\epsilon_x = \epsilon_{x_1} + \epsilon_{x_2} + \epsilon_{x_3}$, $\epsilon_y = \epsilon_{y_1} + \epsilon_{y_2} + \epsilon_{y_3}$ ...

- LOOK AT SHEAR FORCES THEY DO WORK THRU SHEAR STRAINS $d\gamma$



- CAN DO WORK BY FORCE COUPLE CAUSING BODY TO ROTATE ABOUT AN AXIS THROUGH 0
- FROM STATICS WORK DONE BY MOMENT IS $\vec{M}\cdot d\bar{\theta} = \bar{T}l \cdot d\bar{\theta} = \bar{\tau}Al \cdot d\bar{\theta}$

$$= \bar{\tau} \cdot d\bar{\gamma}\ Al$$

- WORK DONE $= \int_V \left[\int \bar{\tau} \cdot d\bar{\gamma}\right] dV$
- FOR A BODY OBEYING HOOKE'S LAW $\int \bar{\tau} \cdot d\bar{\gamma} = \frac{\tau^2}{2G} = \frac{G\gamma^2}{2} = \frac{\tau \cdot \gamma}{2}$

- BY SIMILAR MANNER WORK DONE BY SHEAR FORCES ARE

$$\int_V \left[\frac{\tau_{xy}\gamma_{xy}}{2} + \frac{\tau_{xz}\gamma_{xz}}{2} + \frac{\tau_{yz}\gamma_{yz}}{2} \right] dV$$

- TOTAL WORK DONE DUE TO ALL STRESSES IS SUM OF THE TWO

$$\int_V \frac{1}{2}\left[\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{xz}\gamma_{xz}\right] dV = \int_V U_{s_0}\, dV = U_s$$

- U_{s_0} IS THE STRAIN ENERGY DENSITY ; ALWAYS ≥ 0
- U_{s_0} IS ZERO ONLY WHEN ALL σ's ; ϵ's , τ's AND γ's $= 0$

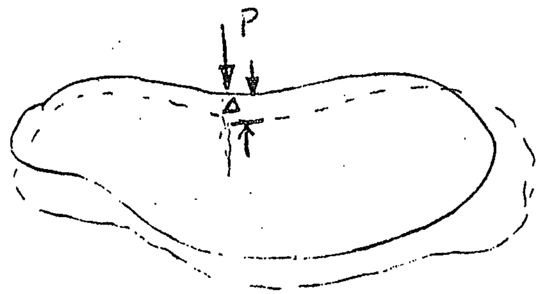
- NOTE THAT SINCE $dU_{s_0} = \left(\sigma_x\, d\epsilon_x + \sigma_y\, d\epsilon_y + \sigma_z\, d\epsilon_z + \tau_{xy}\, d\gamma_{xy} + \tau_{xz}\, d\gamma_{xz} + \tau_{yz}\, d\gamma_{yz}\right)$

AND dU_{s_0} is perfect DIFFERENTIAL , MEANING THAT

$$dU_{s_0} = \frac{\partial U_{s_0}}{\partial \epsilon_x}\, d\epsilon_x + \frac{\partial U_{s_0}}{\partial \epsilon_y}\, d\epsilon_y + \frac{\partial U_{s_0}}{\partial \epsilon_z}\, d\epsilon_z + \frac{\partial U_{s_0}}{\partial \gamma_{xy}}\, d\gamma_{xy} + \frac{\partial U_{s_0}}{\partial \gamma_{xz}}\, d\gamma_{xz} + \frac{\partial U_{s_0}}{\partial \gamma_{yz}}$$

$\Rightarrow \frac{\partial U_{s_0}}{\partial \epsilon_x} = \sigma_x$ AND $\frac{\partial U_{s_0}}{\partial \epsilon_y} = \sigma_y$ ETC.

- SINCE σ IS RELATED TO A LOAD AND ϵ IS RELATED TO A DISPLACEMENT IN THE DIRECTION OF THAT LOAD $\Rightarrow$ WE CAN DETERMINE THE LOAD IF WE KNOW HOW THE STRAIN ENERGY VARIES WITH THE DISPLACEMENT IN THE DIRECTION OF THAT LOAD
- REMEMBER $U = U_{s_0} \cdot V_0 l$



$$U_s = U_s(\Delta) \Rightarrow \frac{\partial U_s}{\partial \Delta} = P$$

THIS FACT WILL BE USED LATER

- WE HAVE CONSIDERED WORK DONE $= \vec{F} \cdot d\vec{x}$ ...... MOST GENERAL EXPRESSION FOR WORK

  IF $\vec{F}$ is CONSTANT $\vec{F} \cdot d\vec{x} = d(\vec{F} \cdot \vec{x})$

- WHAT IF $\underset{\sim}{x}$ IS NOW CONSTANT AND $\underset{\sim}{F}$ CHANGES ie $d(\vec{F} \cdot \vec{x}) = \vec{x} \cdot d\vec{F}$
  - THIS ALSO CAUSE WORK TO BE DONE

$$d\vec{F} = d\vec{\sigma} \cdot A \qquad \vec{x} = l\vec{\epsilon} \qquad \Rightarrow \quad \vec{x} \cdot d\vec{F} = \vec{\epsilon} \cdot d\vec{\sigma} \quad (lA)$$

- JUST AS BEFORE WE CAN GO THROUGH THE PROCESS AND SHOW THAT

$$\text{TOTAL WORK DONE} = \int_V \int (\bar{\epsilon}_x \cdot d\bar{\sigma}_x + \bar{\epsilon}_y \cdot d\bar{\sigma}_y + \bar{\epsilon}_z \cdot d\bar{\sigma}_z + \bar{\gamma}_{xy} \cdot d\bar{\tau}_{xy} + \bar{\gamma}_{xz} \cdot d\bar{\tau}_{xz} + \bar{\gamma}_{yz} \cdot d\bar{\tau}_{yz})\, dV$$

- THE INNER INTEGRAL IS THE COMPLEMENTARY ENERGY OF THE BODY : U_{co}
  - $dU_{co} = \left(\bar{\epsilon}_x \cdot d\bar{\sigma}_x + \bar{\epsilon}_y \cdot d\bar{\sigma}_y + \bar{\epsilon}_z \cdot d\bar{\sigma}_z + \bar{\gamma}_{xy} \cdot d\bar{\tau}_{xy} + \bar{\gamma}_{xz} \cdot d\bar{\tau}_{xz} + \bar{\gamma}_{yz} \cdot d\bar{\tau}_{yz}\right)$
  - IT IS ALSO A PERFECT DIFFERENTIAL SO THAT

$$dU_{co} = \left(\frac{\partial U_{co}}{\partial \sigma_x} d\sigma_x + \frac{\partial U_{co}}{\partial \sigma_y} d\sigma_y + \frac{\partial U_{co}}{\partial \sigma_z} d\sigma_z + \frac{\partial U_{co}}{\partial \tau_{xy}} d\tau_{xy} + \frac{\partial U_{so}}{\partial \tau_{xz}} d\tau_{xz} + \frac{\partial U_{so}}{\partial \tau_{yz}} d\tau_{yz}\right)$$

AND $\dfrac{\partial U_{co}}{\partial \sigma_x} = \epsilon_x \quad \dfrac{\partial U_{co}}{\partial \sigma_y} = \epsilon_y \quad$ etc. $\quad \dfrac{\partial U_{co}}{\partial \tau_{xy}} = \gamma_{xy}$

- HERE IF WE KNOW HOW THE COMPLEMENTARY ENERGY CHANGES AS LOAD CHANGES THE WE CAN FIND THE DISPLACEMENT, DUE TO THAT LOAD, IN DIRECTION OF LOAD

- REMEMBER σ_x CAN BE RELATED TO LOAD IN X-DIRECTION

  ϵ_x CAN BE RELATED TO DISPLACEMENT IN DIRECTION OF LOAD

- WHAT WE SAID ABOUT U_{S_0} & U_{C_0} IS TRUE EVEN IF BODY DOESN'T OBEY HOOKE'S LAW.
- IF BODY IS LINEALY ELASTIC : $U_{C_0} = U_{S_0}$

- NOTE : ALWAYS WRITE U_S IN TERMS OF STRAINS / DISPLACEMENTS
  U_C IN TERMS OF STRESSES / LOADS

- CLARIFY SOME PTS
  - FROM STATICS: BODY WAS RIGID - NO WORK DONE DUE TO INTERNAL LOADS
  - FOR A BODY THAT DEFORMS WORK IS DONE BY INTERNAL FORCES (U_C , U_S)
  - WHAT WE'VE JUST DISCUSSED IS WORK DONE BY INTERNAL FORCES
  - BOTH FOR NON DEFORMABLE & DEFORMABLE BODIES, THE EXTERNAL FORCES ALSO DO WORK
  - THE EXPRESSIONS FOR U_C, U_{C_0} , U_S, U_{S_0} HOLD FOR NON-LINEARLY ELASTIC BODIES AS WELL
  - TOTAL WORK DONE BY BODY THAT HAS EXTERNAL LOADS APPLIED AND UNDERGOES DEFORMATION IS $W_i + W_e = \Pi$

- $$W_e = \int_S (\bar{X} u + \bar{Y} v + \bar{Z} w)\, dS$$

  S - surface of body
  u, v, w - displacements undergone by forces $\bar{X}, \bar{Y}, \bar{Z}$ applied to surface

- ASSUMPTION - BODY FORCES (LIKE WEIGHT) CAN BE ACCOUNTED FOR THROUGH W_e TERM.
- WHEN AN ELASTIC BODY IS AT REST, THE EXTERNAL FORCES + BODY FORCES + INTERNAL FORCES ARE IN A STATE OF EQUILIBRIUM

- WORK DONE BY THESE THREE SET OF FORCES IS AT A MINIMUM
- THUS $\delta\pi = \delta W_i + \delta W_e = 0$
- ALSO $\delta W_i = -\delta U_s$ REMEMBER FROM STATICS POTENTIAL ENERGY IS NEGATIVE OF WORK DONE.

- THUS FOR ANY CHANGE IN DISPLACEMENTS OF THE BODY THAT KEEPS IT IN EQUILIBRIUM

$$\delta\pi / \delta \text{ displacements} = 0 = -\frac{\delta U_s}{\delta \text{displ.}} + \frac{\delta W_e}{\delta \text{displ.}}$$

- BUT $\dfrac{\delta U_s}{\delta \text{displ}} = \dfrac{\delta W_e}{\delta \text{displ}}$ = load, due to that displ, IN DIRECTION OF DISPLACEMENTS.

- THIS IS CASTIGLIANO'S THEOREM (FIRST)
- SIMILARLY SINCE $\delta W_i = -\delta U_s = -\delta U_c$

FOR ANY CHANGE IN LOADS OF THE BODY THAT KEEPS IT IN EQUILIBRIUM

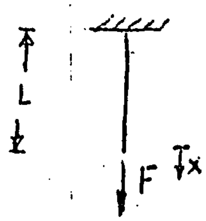
$$\delta\pi / \delta_{\text{LOADS}} = 0 = -\frac{\delta U_c}{\delta \text{LOAD}} + \frac{\delta W_e}{\delta \text{LOAD}}$$

- BUT $\dfrac{\delta U_c}{\delta \text{LOAD}} = \dfrac{\delta W_e}{\delta \text{LOAD}}$ = DISPL, DUE TO THAT LOAD, IN DIR. OF LOAD

- THIS IS CASTIGLIANO'S THEOREM (SECOND)

→ USING U_c TO DETERMINE DISPLACEMENTS ←

EXAMPLE #1 EXTENSIBLE ROD



$\sigma = F/A \qquad \epsilon = \dfrac{\sigma}{E} = \dfrac{F}{AE} = \dfrac{x}{L}$

$$\underbrace{U_c}_{U_{c_0}\cdot \text{Vol}} = \frac{\sigma^2}{2E}\cdot AL = \frac{F^2L}{2EA} \qquad \underbrace{U_s}_{U_{s_0}\cdot\text{Vol}} = \frac{E\epsilon^2}{2}\cdot AL = \frac{Ex^2}{2L}A$$

THUS $U_c = \dfrac{F^2L}{2EA} \qquad U_s = \dfrac{Ex^2}{2L}A$

$W_e = Fx$

TO FIND F, ASSUMING x IS KNOWN, DEFINE Π IN TERMS OF U_s & W_e

$$W_i + W_e = \Pi = -\frac{Ex^2}{2L}A + Fx = -U_s + W_e$$

$$\frac{\partial \Pi}{\partial x} = -\frac{2ExA}{2L} + F = 0 \qquad F = x\frac{EA}{L} \qquad \left(\frac{\partial U_s}{\partial x} = F\right)$$

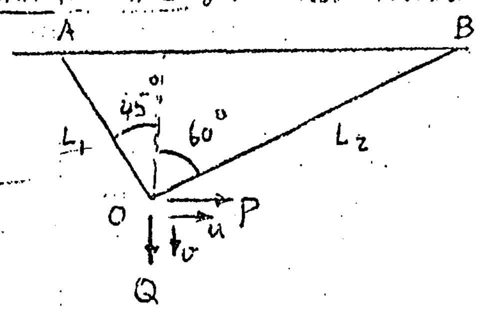
TO FIND x, ASSUMING F IS KNOWN, DEFINE Π IN TERMS OF U_c & W_e

$$W_i + W_e = -U_c + W_e = \Pi = -\frac{F^2 L}{2EA} + Fx$$

$$\frac{\partial \Pi}{\partial F} = -\frac{2FL}{2EA} + x = 0 \qquad x = \frac{FL}{AE} \qquad \left(\frac{\partial U_c}{\partial F} = x\right)$$

- WE WILL LOOK AT TRUSSES WHERE EXTENSION/COMPRESSION IS PRIMARY LOADING

EXAMPLE #2 : TWO-ROD TRUSS

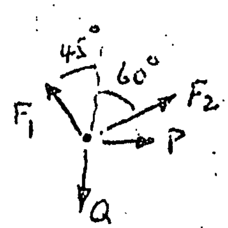


STATICALLY DETERMINATE SYSTEM

LOOK AT TWO BARS CONNECTED AT O, BARS ARE EXTENSIBLE — HAVE THE SAME CROSS SECTION, A, AND YOUNG'S MODULUS, E.

→ WANT TO FIND DISPLACEMENTS u, v GIVEN P AND Q

1) USE STATICS TO FIND FORCES IN OB & OA
2) DETERMINE U_c
3) $\dfrac{\partial U_c}{\partial P} = u \qquad \dfrac{\partial U_c}{\partial Q} = v$



$$\left.\begin{aligned} Q &= F_2\cos 60^\circ + F_1\cos 45^\circ \\ P &= F_1 \sin 45^\circ - F_2 \sin 60^\circ \end{aligned}\right\} \quad \begin{aligned} F_1 &= \frac{\sqrt{3}-1}{\sqrt{2}}(P + \sqrt{3}Q) \\ F_2 &= (\sqrt{3}-1)(Q-P) \end{aligned}$$

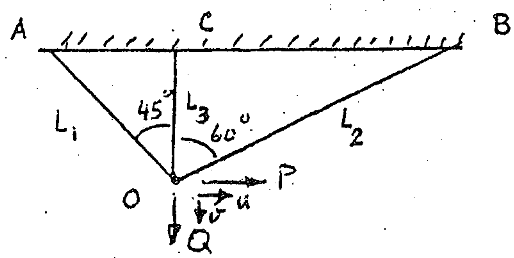
NOTE F_1 & F_2 ARE FNS OF P & Q

Now $$U_c = \sum \frac{F^2 L}{2AE} = \frac{F_1^2 L_1}{2AE} + \frac{F_2^2 L_2}{2AE} = \frac{1}{2AE}\left\{\left[\frac{\sqrt{3}-1}{\sqrt{2}}(P+\sqrt{3}Q)\right]^2 L_1 + \left[(\sqrt{3}-1)(Q-P)\right]^2 L_2\right.$$

$$u = \frac{\partial U_c}{\partial P} = \frac{2F_1 L_1}{2AE}\frac{\partial F_1}{\partial P} + \frac{2F_2 L_2}{2AE}\frac{\partial F_2}{\partial P} = \frac{\sqrt{3}-1}{\sqrt{2}}(P+\sqrt{3}Q)\frac{L_1}{AE}\cdot\frac{\sqrt{3}-1}{\sqrt{2}} + (\sqrt{3}-1)(Q-P)\frac{L_2}{AE}\left\{-(\sqrt{3}-1)\right\}$$

$$v = \frac{\partial U_c}{\partial Q} = \frac{2F_1 L_1}{2AE}\frac{\partial F_1}{\partial Q} + \frac{2F_2 L_2}{2AE}\frac{\partial F_2}{\partial Q} = \frac{L_1}{AE}\left\{\frac{\sqrt{3}-1}{\sqrt{2}}(P+\sqrt{3}Q)\cdot\frac{\sqrt{3}}{\sqrt{2}}(\sqrt{3}-1)\right\} + \frac{L_2}{AE}\left\{(\sqrt{3}-1)^2(Q-P)\right\}$$

EXAMPLE #3 - INDETERMINATE (STATICALLY) TRUSS



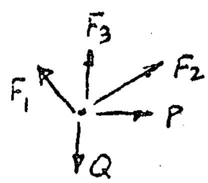
$L_3 = L \qquad L_1 = \sqrt{2}\,L \qquad L_2 = 2L$

GIVEN: A, E SAME FOR ALL THREE

WANT TO FIND DISPLACEMENTS u, v

GIVEN P & Q

STATICALLY INDETERMINATE: 3 FORCES, 2 EQS.



$$Q = F_3 + F_1\cos 45^\circ + F_2 \cos 60^\circ$$
$$P = F_1 \sin 45^\circ - F_2 \sin 60^\circ$$

NOTE P & Q ARE FNS OF F_1, F_2, F_3

- NOTE THIS WILL GIVE SAME SOLUTION FOR F_1 & F_2 IF $Q - F_3$ REPLACES Q

TO FIND F_3 : 1) FIND U_c FIRST $\quad U_c = \sum \frac{F^2 L}{2AE} = \frac{F_1^2 L_1}{2AE} + \frac{F_2^2 L_2}{2AE} + \frac{F_3^2 L_3}{2AE}$

2) TAKE $\frac{\partial U_c}{\partial F_3} = 0$ THIS GIVES 3rd EQ. NEEDED

$$U_c = \frac{1}{2AE}\left\{\left[\frac{\sqrt{3}-1}{\sqrt{2}}\left(P+\sqrt{3}(Q-F_3)\right)\right]^2 L_1^2 + \left[(\sqrt{3}-1)(Q-F_3-P)\right]^2 L_2^2 + F_3^2 L_3^2\right\}$$

$$\frac{\partial U_c}{\partial F_3} = \frac{F_1 L_1}{AE}\frac{\partial F_1}{\partial F_3} + \frac{F_2 L_2}{AE}\frac{\partial F_2}{\partial F_3} + \frac{F_3 L_3}{AE} = \frac{\sqrt{3}-1}{\sqrt{2}}\left[P+\sqrt{3}(Q-F_3)\right]\frac{L_1}{AE}\left[-\sqrt{3}\frac{(\sqrt{3}-1)}{\sqrt{2}}\right] -$$

$$+ (\sqrt{3}-1)(Q-F_3-P)\frac{L_2}{AE}\left[-(\sqrt{3}-1)\right] + \frac{F_3 L_3}{AE} = 0$$

3) SOLVE FOR F_3 IN TERMS OF KNOWN FORCES P & Q

$$F_3 = -0.01295\,P + 0.6883\,Q$$

4) PUT THIS INTO $F_1 = \frac{\sqrt{3}-1}{\sqrt{2}}\left(P+\sqrt{3}(Q-F_3)\right)$; $F_2 = (\sqrt{3}-1)(Q-F_3-P)$

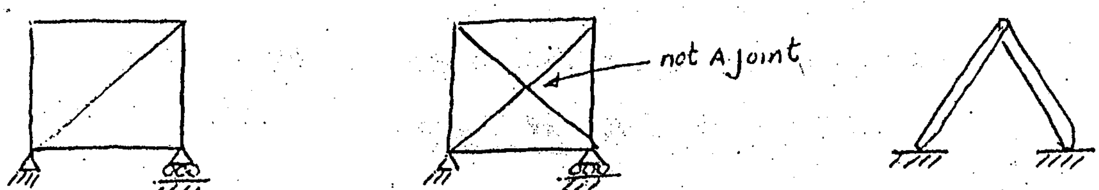
TO FIND F_1 & F_2 IN TERMS OF P & Q

5) TAKE $\frac{\partial U_c}{\partial P}$ TO GET u & $\frac{\partial U_c}{\partial Q}$ TO GET v

ASIDE

- WHEN IS TRUSS STATICALLY DETERMINATE
  - MUST CHECK BOTH EXTERNAL & INTERNAL CONDITIONS
  - FOR A TRUSS HAVING j JOINTS & n BARS

EQNS OF EQUILIB.
- IF $2j-3=n$ STATICALLY DETERMINATE INTERNALLY
- IF $<n$ STATICALLY INDETERMINATE INTERNALLY
- IF $>n$ IT IS A MECHANISM AND IT IS STATICALLY DETERMINATE



DETERMINATE INDETERMINATE MECHANISM

- FOR A <u>SPACE TRUSS</u> $3j-6=n$ (3-D TRUSS)

- <u>EXTERNAL</u> EQNS OF EQUILIB. $\Sigma F_x=0 \quad \Sigma F_y=0 \quad \Sigma M=0$

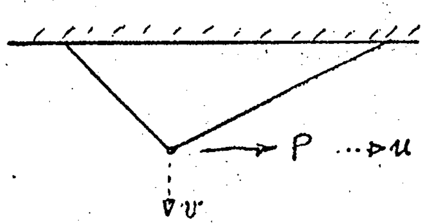
  IF NO. OF UNKNOWNS > NO. OF EQNS STATICALLY INDETERMINATE

  IF " " " $\leq$ " " " DETERMINATE

- FOR A <u>SPACE TRUSS</u> EQS OF EQUILIB. $\Sigma F_x, \Sigma F_y, \Sigma F_z, \Sigma M_x, \Sigma M_y, \Sigma M_z = 0$

- WE SEE WE CAN FIND DISPLACEMENT IN DIRECTION OF FORCE USING CASTIGLIANO'S THEOREM $\left(\frac{\partial U_c}{\partial P} = u\right)$

- WHAT IF WE WANT DISPLACEMENT OF A POINT WHERE THERE IS <u>NO</u> FORCE APPLIED?

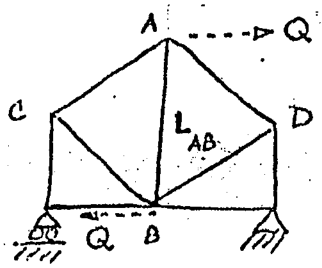


$$\frac{\partial U_c}{\partial P} = u$$

HOW DO WE GET v?

- PUT A FICTITIOUS FORCE Q IN DIRECTION OF v; FIND U_c; TAKE $\lim_{Q\to 0} \frac{\partial U_c}{\partial Q} = v$

- CAN WE USE CASTIGLIANO'S THEOREM TO DETERMINE ROTATIONS OF A BAR IN A TRUSS? YES
- HOW? 1) APPLY A COUPLE WHOSE FORCES ARE $\perp$ TO BAR
  2) FIND U_c DUE TO THAT COUPLE, AFTER FINDING FORCES IN BARS OF TRUSS USING EQUILIB
  3) TAKE $\dfrac{\partial U_c}{\partial(\text{FORCE OF COUPLE})}$; THEN TAKE limit AS THAT FORCE GOES TO ZERO
  4) TAKE THE RESULT AND DIVIDE BY LENGTH OF BAR. THIS GIVES ROTATION OF BAR IN RADIANS



FIND $U_c = \sum \dfrac{F_i^2 L_i}{2A_i E_i}$

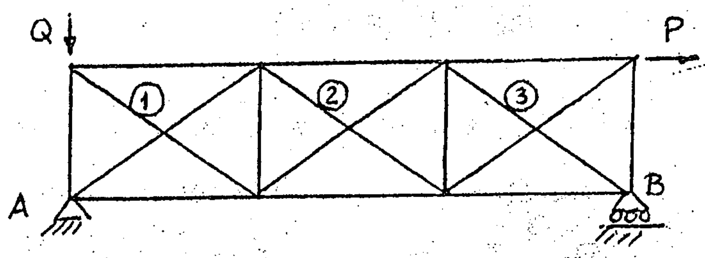
- HERE THE F_i's WOULD BE FUNCTIONS OF Q
- NOW TAKE $\dfrac{\partial U_c}{\partial Q}$; TAKE $\lim\limits_{Q\to 0} \dfrac{\partial U_c}{\partial Q} = L_{AB}\,\theta_{AB}$
- NOW TAKE $\dfrac{1}{L_{AB}} \cdot (L_{AB}\,\theta_{AB}) = \theta_{AB}$ IN RADIANS

- ALSO NOTE THAT $L_{AB}Q$ REPRESENTS THE MOMENT OF THE COUPLE
- WHAT IF AT A A FORCE P EXISTED ALREADY AND YOU WANTED THE ROTATION OF BAR AB? MUST ADD THE COUPLE IN ADDITION TO THE EXISTING FORCE SYSTEM.

- THE ABOVE SYSTEM IS DETERMINATE. WHAT IF A BAR WERE PLACED ACROSS CD MAKING THE SYSTEM INDETERMINATE. HOW WOULD YOU PROCEED? 1) WRITE EQUILIBRIUM EQS AND FIND FORCES IN TERMS OF THE INDETERMINATE FORCE F_{CD} (AS WE DID BEFORE)
  2) FIND U_c ; TAKE $\dfrac{\partial U_c}{\partial F_{CD}} = 0$ TO GET F_{CD}
  3) SUBSTITUTE TH RESULT INTO THE FORCES FOUND FROM EQUILIBRIUM; FORM U_c ; TAKE $\dfrac{\partial U_c}{\partial Q}\Big|_{Q\to 0} = L_{AB}\,\theta_{AB}$, ETC.

- THE DEGREE OF INDETERMINACY IS THE DIFFERENCE BETWEEN n AND $2j-3$ FOR A PLANE TRUSS OR n AND $3j-6$ FOR A SPACE TRUSS. THIS IS THE NUMBER OF EQUATIONS NEEDED VIA CASTIGLIANO'S THEOREM

EXAMPLE



8 JOINTS

$2j-3=13$

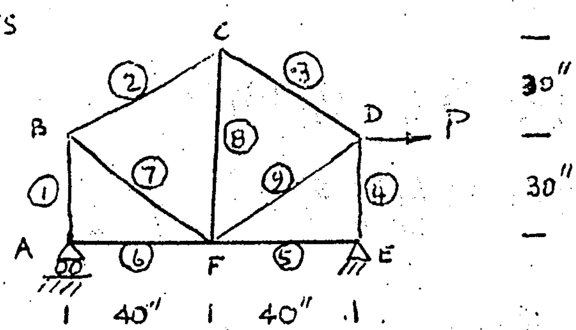
$n=16$

$n-(2j-3)=3$

- THIS IS DEGREE OF INDETERMINACY 3
- THUS AFTER YOU FIND U_c THEN $\frac{\partial U_c}{\partial F_1}=0$; $\frac{\partial U_c}{\partial F_2}=0$; $\frac{\partial U_c}{\partial F_3}=0$
- THESE GIVE THE REQUIRED EQ'S TO SOLVE FOR F_1, F_2, F_3

- TO FIND THE REACTION FORCES AT A & B : USE TRUSS METHODS LEARNED IN STATICS OR USE CASTIGLIANO'S THEOREM WITH $u_A = v_A = 0$ AND $v_B = 0$

EXAMPLES



$P = 4000$ lb

$A = .1\ in^2$

$E = 30\times10^6$ psi

$CF = 60''$ $BC, BF, FD, CD = 50''$

$BA = DE = 30'' = L$

THIS IS STATICALLY DETERMINATE : EXTERNAL/INTERNAL

USE EXTERNAL EQUILIBRIUM TO FIND AT A $V_A = \frac{3P}{8} \downarrow$

E $H_E = P \leftarrow$ $V_E = \frac{3P}{8} \uparrow$

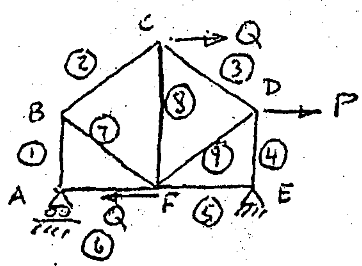
USE JOINT METHOD OF STATICS TO FIND

$F_1 = -F_4 = \frac{3P}{8} = -F_8$; $F_2 = F_3 = -F_7 = \frac{5P}{16}$; $F_6 = 0$; $F_5 = -P$; $F_9 = \frac{15P}{16}$

$$U_c = \sum \frac{F_i^2 L_i}{2A_iE_i} = \frac{L}{2AE}\left\{\left(\frac{3P}{8}\right)^2\cdot 1 + \left(\frac{5P}{16}\right)^2\cdot \frac{5}{3} + \left(\frac{5P}{16}\right)^2\cdot\frac{5}{3} + \left(-\frac{3P}{8}\right)^2\cdot 1 + (-P)^2\cdot\frac{4}{3}\right.$$
$$\left. + (0)^2\cdot\frac{4}{3} + \left(-\frac{5P}{16}\right)^2\cdot\frac{5}{3} + \left(-\frac{3P}{8}\right)^2\cdot 2 + \left(\frac{15P}{16}\right)^2\cdot\frac{5}{3}\right\} = \frac{739}{384}\frac{P^2L}{AE}$$

$$u_d = \frac{\partial U_c}{\partial P} = \frac{739}{192}\frac{PL}{AE} = \frac{739\,(4000)(30'')}{192\ \ .1\ \ (30\times10^6)} = .154\,in \qquad \text{DISPL OF D DUE TO P}$$

TO FIND ROTATION OF CF, ASSUME LOADS Q AT C & F



USE EQUILIB. EQNS TO FIND

$$V_E = \frac{3P}{8} + \frac{3Q}{4}\uparrow \qquad V_A = \frac{3P}{8} + \frac{3Q}{4}\downarrow$$
$$H_E = P \leftarrow$$

USE JOINT EQUIL. TO FIND: $F_1 = -F_4 = \frac{3P}{8} + \frac{3Q}{4}$; $F_2 = -F_7 = \frac{5P}{16} + \frac{5Q}{8}$; $F_3 = \frac{5P}{16} - \frac{5Q}{8}$

$F_5 = -P + 0$; $F_6 = 0 + 0$; $F_8 = -\frac{3P}{8} + 0$; $F_9 = \frac{15P}{16} + \frac{5Q}{8}$

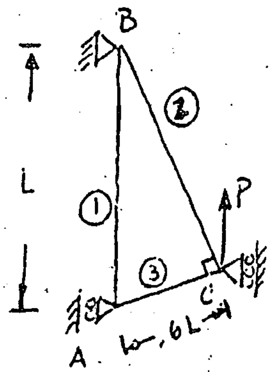
$$U_c = \sum \frac{F_i^2 L_i}{2A_iE_i} = \frac{L}{2AE}\left\{\left(\frac{3P}{8} + \frac{3Q}{4}\right)^2\cdot 1 + \left(\frac{5P}{16} + \frac{5Q}{8}\right)^2\cdot\frac{5}{3} + \left(\frac{5P}{16} - \frac{5Q}{8}\right)^2\cdot\frac{5}{3} + \left(-\frac{3P}{8} - \frac{3Q}{4}\right)^2\cdot 1\right.$$
$$+ (-P+0)^2\cdot\frac{4}{3} + (0+0)^2\cdot\frac{4}{3} + \left(-\frac{5P}{16} - \frac{5Q}{8}\right)^2\cdot\frac{5}{3} + \left(-\frac{3P}{8} + 0\right)^2\cdot 2$$
$$\left. + \left(\frac{15P}{16} + \frac{5Q}{8}\right)^2\cdot\frac{5}{3}\right\}$$

$$\left.\frac{\partial U_c}{\partial Q}\right|_{Q=0} = \frac{L}{AE}\left\{\left(\frac{3P}{8}\right)\cdot 1\cdot\frac{3}{4} + \left(\frac{5P}{16}\right)\cdot\frac{5}{3}\cdot\frac{5}{8} + \left(\frac{5P}{16}\right)\cdot\frac{5}{3}\cdot -\frac{5}{8} + \left(-\frac{3P}{8}\right)\cdot 1\cdot -\frac{3}{4} + 0 + 0 +\right.$$
$$\left.\left(-\frac{5P}{16}\right)\cdot\frac{5}{3}\cdot -\frac{5}{8} + 0 + \left(\frac{15P}{16}\right)\cdot\frac{5}{3}\cdot\frac{5}{8}\right\} = \frac{179}{96}\frac{PL}{AE}$$

Now

$$\theta_{CF} = \frac{1}{L_{CF}}\left.\frac{\partial U_c}{\partial Q}\right|_{Q=0} = 0.00124 \text{ radians} \quad \text{or } 0.0712^\circ$$

- REMEMBER: TRUSSES ASSUME LOAD AT JOINTS, WEIGHTLESS AND ONLY EXTEND BUT **DO** NOT BEND
- WHAT ABOUT IF WE WANT TO FIND LOADS GIVEN THE DISPLACEMENTS



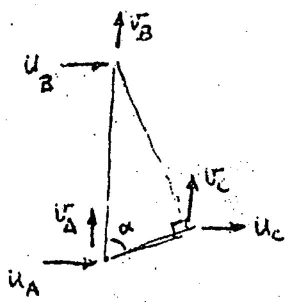
$P = 2500$ LB $\quad A = .1\ in^2 \quad E = 30 \times 10^6$ psi

$L_1 = L, \quad L_2 = .8L, \quad L_3 = .6L$

DFN: ELONGATION - CHANGE IN LENGTH OF BAR ALONG ITS LINE OF ACTION DUE TO LOAD IN BAR

EXAMPLE: $e_1 = \epsilon_1 L_1 = \dfrac{\sigma_1}{E_1} L_1 = \dfrac{F_1 L_1}{A_1 E_1}$

- CONSIDER ELONGATION POSITIVE IN DIRECTION OF POSITIVE (TENSILE FORCE)



$u_B, v_B;\ u_C, u_A = 0$ BY BOUNDARY CONDITIONS

$$\left.\begin{aligned} e_1 &= -v_A \\ e_2 &= -v_C \sin\alpha = -.8v_C \\ e_3 &= v_C \cos\alpha - v_A \cos\alpha = (v_C - v_A).6 \end{aligned}\right\}$$

MUST WRITE ELONGATIONS IN TERMS OF DISPLACEMENTS

- HERE WE ASSUME POSITIVE DISPLACEMENT IN POSITIVE X & Y DIRECTION

$$U_s = \sum \frac{A_i E_i}{2} \frac{e_i^2}{L_i} = \sum \frac{AE}{2} \left\{ \frac{(-v_A)^2}{L} + \frac{(-.8v_C)^2}{.8L} + \frac{(.6[v_C - v_A])^2}{.6L} \right\}$$

- BUT AT A THERE IS NO VERTICAL LOAD ∴ $\dfrac{\partial U_s}{\partial v_A} = 0 = \dfrac{AE}{L}\left[(-v_A)(-1) + .6[v_C - v_A](-1)\right]$

$$\boxed{0 = \frac{AE}{L}\left(1.6v_A - .6v_C\right)}$$

BUT AT C $\dfrac{\partial U_s}{\partial v_C} = P$

$$P = AE\left[\left(\frac{-.8v_C}{.8L}\right)(-.8) + \frac{.6(v_C - v_A)}{.6L}(.6)\right]$$

$$\boxed{P = \frac{AE}{L}\left[1.4v_C - .6v_A\right]}$$

SOLUTION OF THESE GIVE

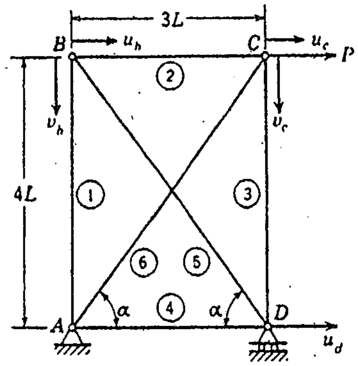
$$v_A = \frac{15PL}{47AE} \qquad v_C = \frac{40}{47}\frac{PL}{AE}$$

Now: $e_1 = -v_A = -\dfrac{15PL}{47AE} \qquad e_2 = -.8v_C = \dfrac{-32PL}{47AE} \qquad e_3 = .6(v_C - v_A) = \dfrac{15PL}{47AE}$

THUS $F_1 = \dfrac{AEe_1}{L_1} = -\dfrac{15P}{47}; \quad F_2 = \dfrac{AEe_2}{L_2} = \dfrac{-32PL}{47(.8L)} = -\dfrac{40P}{47}; \quad F_3 = \dfrac{AEe_3}{L_3} = \dfrac{15PL}{47(.6L)} = \dfrac{25P}{47}$

Example 15.4 For the six-bar truss supported and loaded in its own plane as shown in Fig. 15.6*a*, determine the forces in the bars and the displacement components of the joints. All the bars of the truss have the same cross-sectional area A and the same elastic modulus E.

STATICALLY INDETERMINATE TRUSS. $2j-3<n$



(a)

$$\frac{4LF_1}{AE} = e_1 = -v_b \qquad \frac{3LF_4}{AE} = e_4 = u_d$$

$$\frac{3LF_2}{AE} = e_2 = u_c - u_b \qquad \frac{5LF_5}{AE} = e_5 = (u_d - u_b)\cos\alpha - v_b \sin\alpha$$

$$\frac{4LF_3}{AE} = e_3 = -v_c \qquad \frac{5LF_6}{AE} = e_6 = u_c \cos\alpha - v_c \sin\alpha$$

NOTE: $u_A, v_A, v_d = 0$ BOUNDARY CONDITIONS

$$U_s = \sum \frac{A_i E_i e_i^2}{2L_i} \; ; \quad \frac{\partial U_s}{\partial u_c} = P \; ; \quad \frac{\partial U_s}{\partial u_b} = \frac{\partial U_s}{\partial v_b} = \frac{\partial U_s}{\partial u_d} = 0 = \frac{\partial U_s}{\partial v_c}$$

$$U_s = \frac{AE}{2}\left\{ \frac{v_b^2}{4L} + \frac{(u_c-u_b)^2}{3L} + \frac{v_c^2}{4L} + \frac{(u_d)^2}{3L} + \frac{[(u_d-u_b)\cos\alpha - v_b \sin\alpha]^2}{5L} + \frac{[u_c\cos\alpha - v_c\sin\alpha]^2}{5L} \right\}$$

$$\frac{\partial U_s}{\partial u_c} = P = \frac{AE}{L}\left\{ \frac{(u_c-u_b)\cdot 1}{3} + \frac{(u_c\cos\alpha - v_c\sin\alpha)\cos\alpha}{5} \right\} = \frac{-AE}{375L}(125u_b - 152u_c + v_c\cdot 36)$$

$$\frac{\partial U_s}{\partial u_d} = 0 = \frac{AE}{L}\left\{ \frac{u_d}{3}\cdot 1 + \frac{[((u_d-u_b)\cos\alpha - v_b\sin\alpha]\cos\alpha}{5} \right\} = \frac{-AE}{375L}(27u_b + 36v_b - 152u_d)$$

$$\frac{\partial U_s}{\partial u_b} = 0 = \frac{AE}{L}\left\{ \frac{(u_c-u_b)(-1)}{3} + \frac{[(u_d-u_b)\cos\alpha - v_b\sin\alpha](-\cos\alpha)}{5} \right\} = \frac{-AE}{375L}(27u_d + 125u_c - 36v_b - 152u_b)$$

$$\frac{\partial U_s}{\partial v_b} = 0 = \frac{AE}{L}\left\{ \frac{v_b}{4}\cdot 1 + \frac{[(u_d-u_b)\cos\alpha - v_b\sin\alpha](-\sin\alpha)}{5} \right\} = \frac{-AE}{375L}\left(-36u_b + \frac{567}{4}v_b + 36u_d\right)$$

$$\frac{\partial U_s}{\partial v_c} = 0 = \frac{AE}{L}\left\{ \frac{v_c}{4}\cdot 1 + \frac{[u_c\cos\alpha - v_c\sin\alpha](-\sin\alpha)}{5} \right\} = \frac{-AE}{375L}\left(\frac{567}{4}v_c - 36u_c\right)$$

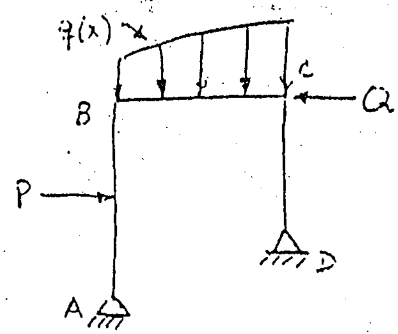
FROM THESE WE GET $u_b = \frac{21PL}{2AE}$ $\quad u_c = \frac{189}{16}\frac{PL}{AE}$ $\quad u_d = \frac{21PL}{16AE}$ $\quad v_b = -\frac{7PL}{3AE}$

$v_c = +\frac{3PL}{AE}$;

AND $F_1 = \frac{7P}{12}$, $F_2 = \frac{7P}{16}$, $F_3 = -3P$, $F_4 = \frac{7P}{16}$, $F_5 = \frac{15P}{16}$; $F_6 = -\frac{35P}{48}$

GO TO NEXT PAGE

- WHAT IF STRUCTURE BENDS ? BEAM UNDER END LOADING
  FRAME UNDER TRANSVERSE LOADING
- HERE BENDING IS PRIMARY MODE OF LOADING .



ACTUALLY TOTAL SOLUTION WILL INVOLVE EXTENSIONAL , BENDING AND SHEARING EFFECTS

REMEMBER $U_{S_e} = \int_V \frac{E\epsilon^2}{2} dV \qquad U_{C_e} = \int \frac{\sigma^2}{2E} dV$

DUE TO TENSION

7.3 PRINCIPLE OF STATIONARY POTENTIAL ENERGY

TABLE 7.2.1 Factors k_y and k_z for use in Eq. 7.2.3, where y and z are centroidal principal axes of the cross section.

Cross-Sectional Type	k_y	k_z
Rectangle	1.20	1.20
Solid circle	≈1.11	≈1.11
Thin-walled cylinder	2.00	2.00
I section, web parallel to z axis	≈1.20[a]	≈1.00[b]
Closed thin-walled box section	≈1.00[b]	≈1.00[b]

[a]For area A in this calculation use the combined cross-sectional areas of the flanges.
[b]For area A in this calculation use the cross-sectional area of the web (or webs in the box section).

- SIMILARLY $U_{s_s} = \int_V \frac{G\gamma^2}{2} dV$ and $U_{c_s} = \int_V \frac{\tau^2}{2G} dV$ DUE TO SHEAR

- FOR BENDING ENERGY $\sigma = \frac{-My}{I}$ AND $\epsilon = -Ky = +\frac{d^2v}{dx^2} y$ K is CURVATURE

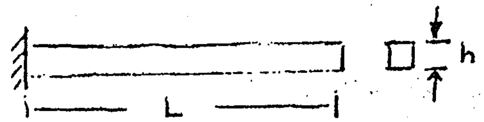
FOR A BEAM

$$U_{s_b} = \int_V \frac{E}{2}\epsilon^2 dV = \int_V \frac{E}{2}\left(+\frac{d^2v}{dx^2} y\right)^2 dV = \int_0^L \frac{E}{2}(v'')^2\left(\int_A y^2 dA\right)dx$$

$$= \int_0^L \frac{EI}{2}(v'')^2 dx$$

$$U_{c_b} = \int_V \frac{\sigma^2}{2E} dV = \int_V \frac{M^2 y^2}{2EI^2} dV = \int_0^L \frac{M^2}{2EI^2}\left(\int_A y^2 dA\right)dx = \int_0^L \frac{M^2}{2EI} dx$$

- LOOK AT SHEAR EFFECTS – IF BEAM IS <u>NOT LONG</u> (SHORT BEAM), MUST ACCOUNT FOR THEM



IF $h/L \sim 1$ MUST ACCOUNT FOR SHEAR
OTHER WISE – NO.

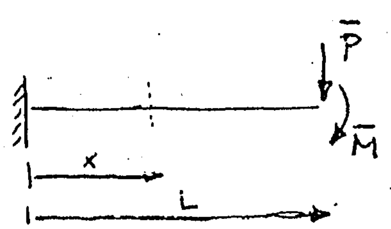
- IF NECESSARY $\int_V \frac{\tau^2}{2G} dV = \int_0^L \int_A \frac{\tau^2 dA}{2G} dx = \int_0^L k\frac{V^2}{2GA} dx$

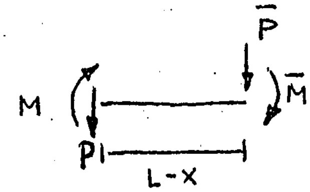
k IS REPORTED IN TABLE ~~7.2.1 PG 231~~ 4.4 pg 87 2^nd ed.; V IS SHEAR FORCE FOUND FROM

$$-\frac{dM}{dx} = V$$

A IS CROSS-SECTIONAL AREA

- LOOK AT SEVERAL CASES





$M + \bar{P}(L-x) + \bar{M} = 0$

$M = -\left[\bar{M} + \bar{P}(L-x)\right]$

AT POSITION X

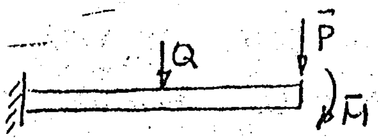
BY $U_{c_b} = \int_0^L \frac{M^2}{2EI}\,dx = \int_0^L \frac{\left(-[\bar{M} + \bar{P}(L-x)]\right)^2}{2EI}\,dx$

$$\frac{\partial U_{c_b}}{\partial \bar{P}} = \text{DISPL (VERTICAL) WHERE } \bar{P} \text{ IS APPLIED} = \int_0^L \frac{1}{EI}\left(-[\bar{M}+\bar{P}(L-x)]\right)\underbrace{\left(-(L-x)\right)}_{\partial M/\partial \bar{P}}dx$$

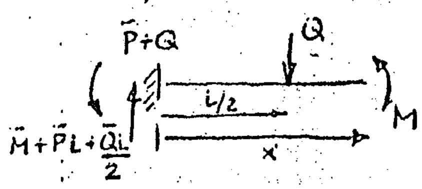
$$\frac{\partial U_{c_b}}{\partial \bar{M}} = \text{ROTATION WHERE } \bar{M} \text{ IS APPLIED} = \int_0^L \frac{1}{EI}\left(-[\bar{M}+\bar{P}(L-x)]\right)\underbrace{(-1)}_{\partial M/\partial \bar{M}}dx$$

- THIS IS A STATICALLY DETERMINATE PROBLEM. NOTE THAT LOAD & MOMENT GIVE THE TWO EQUATIONS OF EQUILIBRIUM

- WHAT IF WE WANTED TO FIND DISPLACEMENT AT $x = L/2$?
- PUT LOAD Q THERE, FIND U_{c_b} ; THEN TAKE $\left.\frac{\partial U_{c_b}}{\partial Q}\right|_{Q=0}$ ....

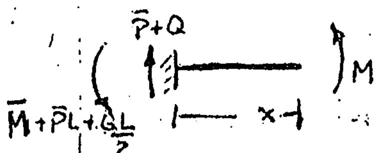


DEFINE M $\quad 0 \le x \le L/2 \quad \& \quad L/2 \le x \le L$



$$+M + (Q)(x-L/2) - (\bar{P}+Q)x + (\bar{M}+\bar{P}L+\tfrac{QL}{2}) = M - \bar{P}(x-L) + \bar{M} = 0$$

$$M = +\bar{P}(x-L) - \bar{M} \qquad \text{FOR} \quad L/2 \le x \le L$$



$$+M - (\bar{P}+Q)x + (\bar{M}+\bar{P}L+\tfrac{QL}{2}) = M + \bar{P}(L-x) + Q(L/2-x) + \bar{M} = 0$$

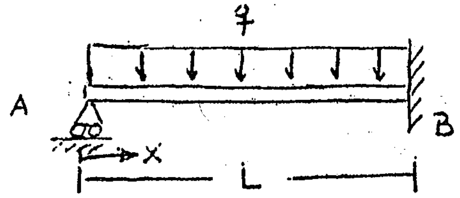
$$M = -[\bar{M} + \bar{P}(L-x) + Q(L/2-x)] \qquad \text{FOR} \quad 0 \le x \le L/2$$

$$U_{c_b} = \int_0^L \frac{M^2}{2EI}\,dx = \int_0^{L/2} \frac{\left(-[\bar{M}+\bar{P}(L-x)+Q(L/2-x)]\right)^2}{2EI}dx + \int_{L/2}^L \frac{\left(-[\bar{M}+\bar{P}(L-x)]\right)^2}{2EI}dx$$

$$\frac{\partial U_{c_b}}{\partial Q} = \int_0^{L/2} \frac{\cancel{2}\left(-[\bar{M}+\bar{P}(L-x)+Q(L/2-x)]\right)}{\cancel{2}EI}\underbrace{\left(-[L/2-x]\right)}_{\partial M/\partial Q}dx$$

$$v\Big|_{x=L/2} = \left.\frac{\partial U_{c_b}}{\partial Q}\right|_{Q=0} = \int_0^{L/2} \frac{\cancel{2}\left(-[\bar{M}+\bar{P}(L-x)]\right)}{\cancel{2}EI}\left(-[L/2-x]\right)dx$$

STATICALLY INDETERMINATE BEAMS

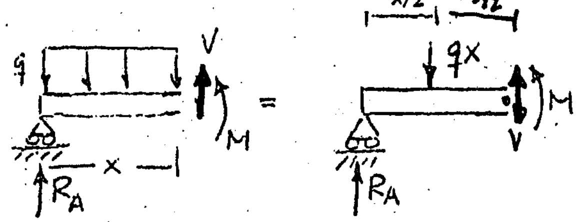


AT A REACTION UPWARD

B MOMENT + REACTION

} 4 UNKNOWNS, 3 EQNS

LOOK AT BEAM AT DISTANCE x FROM LEFT END



$$V = +qx + R_A$$
$$M + \frac{qx^2}{2} - R_A x = 0$$
$$M = R_A x - \frac{qx^2}{2} \qquad 0 \le x \le L$$

$$U_{c_b} = \int_0^L \frac{M^2}{2EI}\,dx = \int_0^L \frac{\left(R_A x - \frac{qx^2}{2}\right)^2}{2EI}\,dx$$

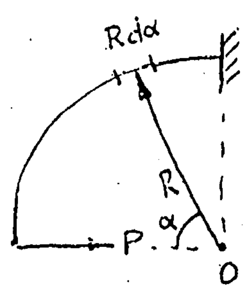
- WHAT IS $\dfrac{\partial U_{c_b}}{\partial R_A} = U_A = 0 = \displaystyle\int_0^L \frac{\left(R_A x - \frac{qx^2}{2}\right)}{EI}\,\underbrace{x}_{\frac{\partial M}{\partial R_A}}\,dx \implies R_A = \frac{3qL}{8}$

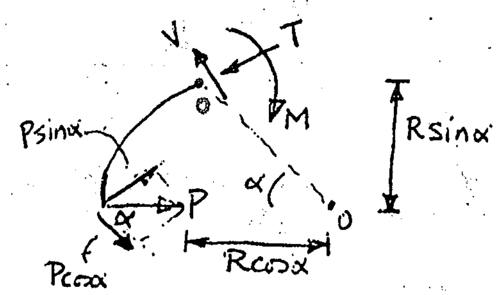
- WHAT IF WE WANTED TO FIND THE ROTATION OF THE BEAM AT $x = L/4$?

  SUPERPOSE A MOMENT M_1 AT $x = L/4$, FIND U_{c_b}, TAKE $\left.\dfrac{\partial U_{c_b}}{\partial M_1}\right|_{M_1=0} = \theta\Big|_{L/4}$

- WHAT IF THERE ARE TWO LOADS CALLED P? WHAT DOES $\dfrac{\partial U_{c_b}}{\partial P}$ MEAN?
- HOW WOULD I FIND THE DISPLACEMENT UNDER ONE OF THESE LOADS?

LET'S LOOK AT A CIRCULAR CANTILEVERED BEAM UNDER HORIZONTAL LOAD

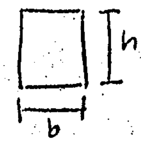




$T = P\sin\alpha$
$V = P\cos\alpha$
$M = PR\sin\alpha$

FROM EQUILIBRIUM -

IF THE CROSS-SECTION IS RECTANGULAR



$A = bh \quad I = \frac{bh^3}{12}$

$$\sigma = T/A = \frac{P\sin\alpha}{A} \qquad \tau = \frac{3V(h^2-4y^2)}{2bh^3} = \frac{3P\cos\alpha\,(h^2-4y^2)}{2bh^3}$$

$$U_{c_t} = U_{c_e} + U_{c_s} + U_{c_b} \qquad U_{c_e} = \int_V \frac{\sigma^2}{2E}dV = \int_0^{\pi/2} \frac{P^2\sin^2\alpha}{2A^2E}A\cdot R\,d\alpha$$

$$= \int_0^{\pi/2} \frac{P^2R}{2AE}\sin^2\alpha\,d\alpha = \frac{\pi P^2R}{8AE}$$

$$U_{c_s} = \int_V \frac{\tau^2}{2G}dV = \int_0^{\pi/2}\frac{R\,d\alpha}{2G}\int_{-h/2}^{h/2}\frac{9V^2}{(2bh^3)^2}b(h^2-4y^2)^2dy$$

$dV = dx\,dy\,dz = (R\,d\alpha)(dy)b$

$$= \frac{3\pi P^2R}{20GA}$$ and multiply by $k = 1.2$

$$U_{c_b} = \int_V \frac{\sigma^2}{2E}dV = \int_0^{\pi/2}\frac{M^2}{2EI}R\,d\alpha = \int_0^{\pi/2}\frac{P^2R^3\sin^2\alpha\,d\alpha}{2EI} = \frac{\pi P^2R^3}{8EI}$$

$$U_{c_t} = \frac{\pi P^2R^3}{8EI}\left[1 + \frac{1}{12}\left(\frac{h}{R}\right)^2 + \frac{E}{10G}\left(\frac{h}{R}\right)^2\right]$$

BENDING EXTENSION SHEAR

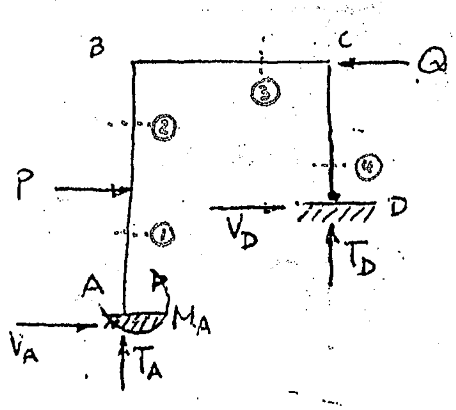
AND $\frac{\partial U_{c_t}}{\partial P}$ = DISPL IN DIRECTION OF $P = \frac{\pi PR^3}{4EI}\left[1+\frac{1}{12}\left(\frac{h}{R}\right)^2+\frac{E}{10G}\left(\frac{h}{R}\right)^2\right]$

IF $\frac{E}{G} = 2(1+\nu)$ AND $\nu = .3$ $\quad \frac{E}{10G} = .26$

IF $h/R = \frac{1}{10}$ $\quad \frac{1}{12}\left(\frac{h}{R}\right)^2 = .00083$ $\quad \frac{E}{10G}\left(\frac{h}{R}\right)^2 = .0026$

NOTE EXTENSION & SHEAR CONTRIB. MUCH SMALLER

WHAT ABOUT A FRAME?



1) CONSIDER FREE AS FREE BODY
2) THE REACTION COMPONENTS ARE V_A, T_A, M_A AT A AND V_D, T_D (PINNED END) AT D
3) SINCE 3 EQNS OF EQUILIB ONLY
$\Rightarrow$ 2 OF THESE UNKNOWNS CAN BE WRITTEN IN TERMS OF OTHER 3

e.g. $V_A = V_A(V_D, T_A, T_D)$ $\quad M_A = M_A(V_D, T_A, T_D)$

4) NOW FIND M IN TERMS OF THESE QUANTITIES OVER EACH RANGE

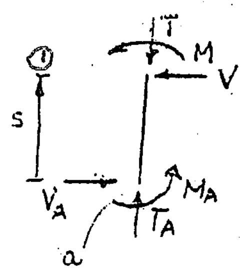
FROM A TO P P TO B B TO C C TO D

5) USE $$U_{c_b} = \Sigma \int \frac{M^2}{2EI}\,ds = \int_A^P \frac{M^2}{2EI}ds + \int_P^B \frac{M^2}{2EI}ds + \int_B^C \frac{M^2 ds}{2EI} + \int_C^D \frac{M^2 ds}{2EI}$$

6) THEN TAKE $\frac{\partial U_{c_b}}{\partial M_A} = 0$ $\frac{\partial U_{c_b}}{\partial V_A} = 0$ SINCE THESE ARE THE INDETERMINATE QUANTITIES

7) THESE TWO EQUATIONS AND THE EQUILIBRIUM EQUATIONS GIVE 5 EQNS & 5 UNKN

EXAMPLE : BETWEEN A - P



$\Sigma F_H = 0 \Rightarrow V = V_A$

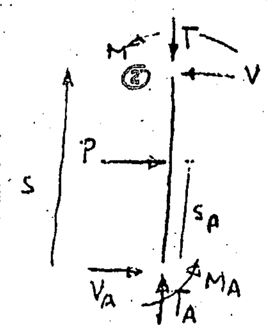
$\Sigma F_V = 0 \Rightarrow T = T_A$

$\Sigma M_a = 0 = M + Vs + M_A = 0$

$M = -(V_A s + M_A)$

$$\therefore \int_A^P \frac{M^2 ds}{2EI} = \int_0^{S_P} \frac{[-(V_A s + M_A)]^2 ds}{2EI}$$

BETWEEN P - B



$V = P + V_A$

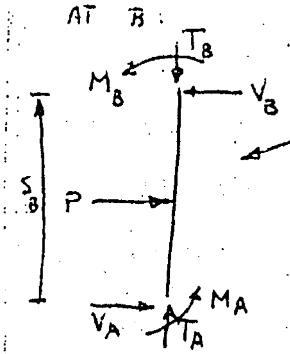
$T = T_A$

$M = -P(s - s_P) - V_A s - M_A$

$$\int_P^B \frac{M^2 ds}{2EI} = \int_{S_P}^{S} \frac{[-(P\{s-s_P\} + V_A s + M_A)]^2 ds}{2EI}$$

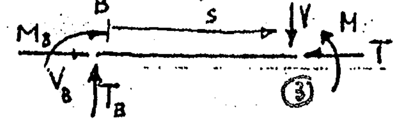
BETWEEN B - C

AT B :



$$\begin{cases} V_B = P + V_A \\ T_B = T_A \\ M_B = -P(s_B - s_P) - V_A s_B - M_A \end{cases}$$

SECTION B - C



$T = V_B = P + V_A$

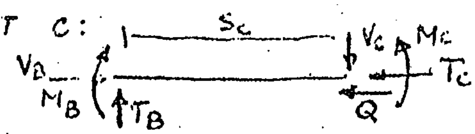
$V = T_B = T_A$

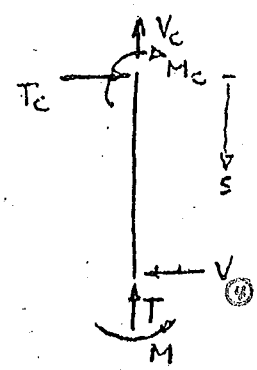
$M = T_B s + M_B = T_A s - [P(s_B - s_P) + V_A s_B + M_A]$

$$\int_B^C \frac{M^2 ds}{2EI} = \int_0^S \frac{(T_A s + M_B)^2 ds}{2EI}$$

BETWEEN C – D

AT C:





$$T_c = V_B - Q = P + V_A - Q$$

$$V_c = T_B = T_A$$

$$M_c = T_B s_c + M_B = \underbrace{T_A s_c - \left[P(s_B - s_P) + V_A s_B + M_A\right]}$$

$$T = -V_c = -T_A$$

$$V = T_c = P + V_A - Q$$

$$M = M_c + T_c s = M_c \cancel{+ V_A s} + (P + V_A - Q)s$$

$$\int_c^D \frac{M^2 ds}{2EI} = \int_0^{s_D} \frac{(M_c + T_c s)^2}{2EI} ds$$

NOW $\quad U_{c_b} = \sum \int \frac{M^2}{2EI} ds \quad$ AND TAKE $\quad \frac{\partial U_{c_b}}{\partial M_A} = 0 \quad \& \quad \frac{\partial U_{c_b}}{\partial V_A} = 0$

FROM EQUILIB.

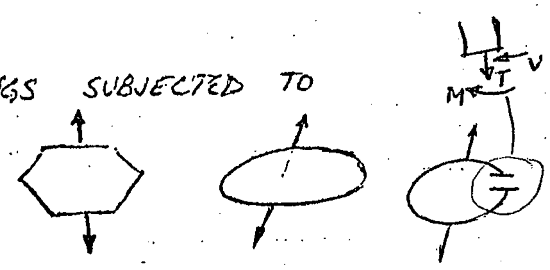
$\sum F_H \qquad V_A + P + Q + V_D = 0$

$\sum F_V \qquad T_A + T_D = 0$

$\sum M \qquad M_A - P s_p + Q s_B - V_D (s_B - s_D) + T_D S_c = 0$

GIVE 5 EQNS
5 UNKNOWNS V_A, V_D, T_A, M_A

- WHAT ABOUT CLOSED PLANE FRAMES & RINGS SUBJECTED TO LOADS THAT ARE SELF EQUILIBRATING?



- YOU DO SAME THING AS WITH A FRAME. THERE WILL BE 3 STATICALLY INDETERMINATE REACTIONS THEY ARE INTERNAL REACTIONS M, T, V. THESE REACTIONS MUST MINIMIZE $\Pi = W_i = -U_{c_b} = -U_{c_s}$
- FORM $U_{c_b} = \sum \int \frac{M^2}{2EI} ds$ AROUND THE CLOSED PERIMETER
- THEN TAKE $\frac{\partial U_{c_b}}{\partial M} = 0 \quad \frac{\partial U_{c_b}}{\partial V} = 0 \quad \frac{\partial U_{c_b}}{\partial T} = 0$

Structural Mechanics w/ Intro to Elasticity & Plasticity by Patel & Venkatraman

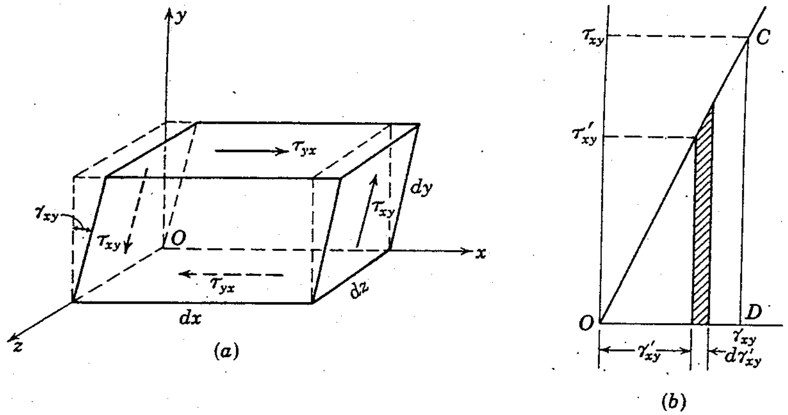


Fig. 4.2

Substitution for τ'_{xy} on the right side of this equation from Eq. (3.8) and integration yield the relations

$$dU_{1\tau} = \frac{G\gamma_{xy}^2}{2}\,dV = \frac{\tau_{xy}^2}{2G}\,dV = \frac{\tau_{xy}\gamma_{xy}}{2}\,dV \tag{4.11}$$

Here, the terms $G\gamma_{xy}^2/2 = \tau_{xy}^2/2G = \tau_{xy}\gamma_{xy}/2$ represent the area of the triangle OCD in Fig. 4.2*b*.

By considering the gradual application of the shear stresses τ_{yz} and τ_{zx} on the appropriate faces of the parallelepiped in a manner analogous to that of τ_{xy}, the corresponding shear-strain energies stored in its volume are easily seen to be $dU_{2\tau} = (\tau_{yz}\gamma_{yz}/2)\,dx\,dy\,dz$ and $dU_{3\tau} = (\tau_{zx}\gamma_{zx}/2)\,dx\,dy\,dz$. Therefore, the total shear-strain energy dU_τ accumulated in the parallelepiped in its deformed shape because of the action τ_{xy}, τ_{yz}, and τ_{zx} is

$$dU_\tau = dU_{1\tau} + dU_{2\tau} + dU_{3\tau} = \tfrac{1}{2}(\tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})\,dV \tag{4.12}$$

In the general three-dimensional state of stress, the total strain energy stored in an elementary volume dV is the sum

$$\begin{aligned} dU_s &= dU_\sigma + dU_\tau = U_{s0}\,dV \\ &= \tfrac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y + \sigma_z\epsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})\,dV \end{aligned} \tag{4.13}$$

Here, by implicit definition, U_{s0} is the total strain energy stored in a unit volume of the material. It is known as the *strain-energy density*. Integration of this density over the volume V yields the total strain energy stored in the body as

$$U_s = \int_V U_{s0}\,dV \tag{4.14}$$

Now, with the stress-strain relations in Eqs. (3.5*a*) and (3.8) and the strain and stress invariants defined by Eqs. (1.38), (2.6), and (2.7), the strain-energy density U_{s0} can be expressed, after some algebraic manipulations, in the following alternate forms:

$$\begin{aligned} U_{s0} &= \frac{E}{2(1+\nu)(1-2\nu)}\{(1-\nu)(\epsilon_x+\epsilon_y+\epsilon_z)^2 - 2(1-2\nu) \\ &\qquad \times [\epsilon_x\epsilon_y + \epsilon_y\epsilon_z + \epsilon_z\epsilon_x - \tfrac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2)]\} \\ &= \frac{1}{2}\left[\frac{E(1-\nu)}{(1+\nu)(1-2\nu)} I_{1\epsilon}^2 - 4GI_{2\epsilon}\right] \end{aligned} \tag{4.15a}$$

$$\begin{aligned} U_{s0} &= \frac{1}{2E}[(\sigma_x+\sigma_y+\sigma_z)^2 \\ &\qquad - 2(1+\nu)(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2)] \\ &= \frac{1}{2}\left(\frac{I_{1\sigma}^2}{E} - \frac{I_{2\sigma}}{G}\right) \end{aligned} \tag{4.15b}$$

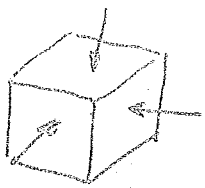
mention this !!

These expressions show that the strain-energy density is invariant under orthogonal rotations of coordinates. Therefore, the function U_{s0} can be expressed solely in terms of the principal strains, or the principal stresses. Thus, by denoting the principal directions by 1, 2, and 3, Eq. (4.15) reduces to

$$\begin{aligned} U_{s0} &= \frac{E}{2(1+\nu)(1-2\nu)}[(1-\nu)(\epsilon_1+\epsilon_2+\epsilon_3)^2 \\ &\qquad - 2(1-2\nu)(\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_3\epsilon_1)] \\ &= \frac{E}{2(1+\nu)(1-2\nu)}[(1-2\nu)(\epsilon_1^2+\epsilon_2^2+\epsilon_3^2) + \nu(\epsilon_1+\epsilon_2+\epsilon_3)^2] \end{aligned} \tag{4.16a}$$

$$\begin{aligned} U_{s0} &= \frac{1}{2E}[(\sigma_1+\sigma_2+\sigma_3)^2 - 2(1+\nu)(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] \\ &= \frac{1}{2E}\{(1-2\nu)(\sigma_1^2+\sigma_2^2+\sigma_3^2) \\ &\qquad + \nu[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2]\} \end{aligned} \tag{4.16b}$$

For values of Poisson's ratio in the range of $-1 \le \nu \le 0.5$, these equations show that the strain-energy density U_{s0} is always positive except in the single instance when all stresses and strains vanish simultaneously, in which case its value is zero. While additional properties of the density function U_{s0} could be discussed here on the basis of Eqs. (4.15) and (4.16), it is more instructive to deal with these in the context of the less restrictive discussion of strain energy and its relation to the generalized Hooke's law [Eq. (3.14)] presented in the next section.



$P = \sigma_x + \sigma_y + \sigma_z$

$\sigma_x = \sigma_y = \sigma_z = P/3$

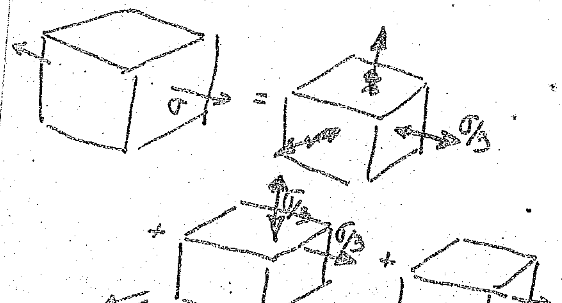
$\epsilon_x = \frac{1-2\nu}{E}\frac{P}{3} = \epsilon_y = \epsilon_z$

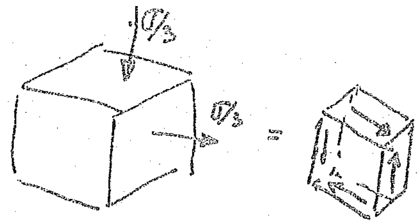
$\therefore\ e = \frac{\Delta V}{V} = 3\left(\frac{1-2\nu}{E}\right)\frac{P}{3}$

So hydrostatic produces volume changes
or dilatational change

$\epsilon_x \frac{\sigma_x}{2} = \frac{1-2\nu}{2E}\frac{P^2}{9}$ or $\frac{1-2\nu}{2E}\left(\frac{P}{3}\right)^2 = U_{0\,dilat}$

$$U_{S_0} = \frac{1}{2E}(\sigma_x^2+\sigma_y^2+\sigma_z^2) - \frac{\nu}{E}(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x) + \frac{1}{2G}(\tau_{xy}^2+\dots)$$





$\tau = \frac{\sigma_1 - \sigma_3}{2} = \sigma/3$

Here $U = U_o$ because the structure has unit volume and is uniformly stressed. Similarly, for a state of pure shear, Fig. 2.6-1b,

$$U_o = \frac{G\gamma_{xy}^2}{2} \quad \text{or} \quad U_o = \frac{\tau_{xy}^2}{2G} \tag{2.6-3}$$

Multiaxial States of Stress. Consider first an isotropic and linearly elastic body in a state of plane stress. Nonzero stresses are σ_x, σ_y, and τ_{xy}. Let these stresses be applied one after another. If σ_x is applied first, with σ_y and τ_{xy} both zero, U_o is given by Eq. 2.6-2. If σ_y is now added, it produces the strains

$$\epsilon_y = \frac{\sigma_y}{E} \quad \text{and} \quad \epsilon_x = -\nu\epsilon_y = -\nu\frac{\sigma_y}{E} \tag{2.6-4}$$

and the following contribution to U_o.

$$\int_0^{\epsilon_y} \sigma_y \, d\epsilon_y + \sigma_x \epsilon_x = \frac{\sigma_y^2}{2E} - \nu\frac{\sigma_x \sigma_y}{E} \tag{2.6-5}$$

No integration is needed to obtain $\sigma_x \epsilon_x$ in Eq. 2.6-5 because σ_x remains constant as x-direction strain is produced by σ_y. Another contribution to U_o comes from Eq. 2.6-3. The final result for a state of plane stress, from Eqs. 2.6-2, 2.6-3, and 2.6-5, is

$$U_o = \frac{1}{2E}[\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y] + \frac{\tau_{xy}^2}{2G} \tag{2.6-6}$$

The foregoing argument can be extended to the fully three-dimensional case by adding the stresses σ_z, τ_{yz}, and τ_{zx}. The result is

$$U_o = \frac{1}{2E}[\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)] + \frac{1}{2G}[\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2] \tag{2.6-7}$$

Strain Energy of Distortion. In an arbitrary state of stress, the average normal stress is

$$\sigma_a = \frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \tag{2.6-8}$$

which, incidentally, is the normal stress on an octahedral plane. *Deviatoric stresses* are given the symbol s and are defined as follows.

$$\begin{aligned} s_x &= \sigma_x - \sigma_a & s_y &= \sigma_y - \sigma_a & s_z &= \sigma_z - \sigma_a \\ s_{xy} &= \tau_{xy} & s_{yz} &= \tau_{yz} & s_{zx} &= \tau_{zx} \end{aligned} \tag{2.6-9}$$

An arbitrary state of stress can be represented as the sum of two states: (1) a hydrostatic state in which principal stresses are $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_a$, and (2) a state in which all ... hydrostatic state. No change in volume due

Strain energy of distortion, associate... predict the onset of yielding. An expressio... volume, U_{od}, is obtained from Eq. 2.6-7 by ... toric stresses. Also, we eliminate E by using

$$U_{od} = \frac{1}{12G}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\ldots$$

Alternative expressions for U_{od} are

$$U_{od} = \frac{3}{4G}\tau_{oct}^2 \quad \text{a...}$$

where τ_{oct} is given by Eq. 2.4-3, 2.4-4, or 2.4...

$$\sigma_e = \frac{1}{\sqrt{2}}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma\ldots$$

From Eqs. 2.4-5 and 2.6-12, $\tau_{oct} = \sqrt{2}\,\sigma_e/$... is a uniaxial state of stress. If the state of ... sible that $\sigma_e > \sigma_1$: for example, if princi... $\sigma_e = \sqrt{3}\sigma_1$. The von Mises failure criterion...

2.7 STRESS CONCENTRATION

Stress in a solid is rarely uniform. It rises ... geneity or abrupt changes in geometry. ... boundaries in metal, small inclusions of fo... stituents of concrete, and the cell struc... changes in geometry include tool marks ... geometry are common, such as threads on ... way in a shaft. Peak stress associated with ... the geometry is accurately known and th... been tabulated.

Stress Concentration Factors. Conside... sion, Fig. 2.7-1a. It is obvious that stress m... section containing the hole, because there ... a reduced area. However, the mechanics ... mula for σ_{max} because we have no reliable ... tion. The problem can be solved (with d... Most stress concentration problems are to ... been solved experimentally. Results have ... elastic material, in the form of *stress conc*... the problem in Fig. 2.7-1a,

$\sigma \ldots = K \ldots \sigma$ where

ıe structure has unit volume and is uniformly stressed. re shear, Fig. 2.6-1b,

$$U_o = \frac{G\gamma_{xy}^2}{2} \quad \text{or} \quad U_o = \frac{\tau_{xy}^2}{2G} \tag{2.6-3}$$

ss. Consider first an isotropic and linearly elastic body in a zero stresses are σ_x, σ_y, and τ_{xy}. Let these stresses be applied applied first, with σ_y and τ_{xy} both zero, U_o is given by Eq. 2.6- oduces the strains

$$= \frac{\sigma_y}{E} \quad \text{and} \quad \epsilon_x = -\nu\epsilon_y = -\nu\frac{\sigma_y}{E} \tag{2.6-4}$$

ıtion to U_o.

$$\int_0^{\epsilon_y} \sigma_y\, d\epsilon_y + \sigma_x\epsilon_x = \frac{\sigma_y^2}{2E} - \nu\frac{\sigma_x\sigma_y}{E} \tag{2.6-5}$$

o obtain $\sigma_x\epsilon_x$ in Eq. 2.6-5 because σ_x remains constant as x- d by σ_y. Another contribution to U_o comes from Eq. 2.6-3. of plane stress, from Eqs. 2.6-2, 2.6-3, and 2.6-5, is

$$= \frac{1}{2E}[\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y] + \frac{\tau_{xy}^2}{2G} \tag{2.6-6}$$

ın be extended to the fully three-dimensional case by adding The result is

$$\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - 2\nu(\sigma_x\sigma_y + \sigma_y\sigma_z + \sigma_z\sigma_x)]$$
$$[\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2] \tag{2.6-7}$$

tion. In an arbitrary state of stress, the average normal

$$\frac{1}{3}(\sigma_x + \sigma_y + \sigma_z) = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \tag{2.6-8}$$

ıormal stress on an octahedral plane. *Deviatoric stresses* are defined as follows.

$$-\sigma_a \qquad s_y = \sigma_y - \sigma_a \qquad s_z = \sigma_z - \sigma_a$$
$$s_{yz} = \tau_{yz} \qquad s_{zx} = \tau_{zx} \tag{2.6-9}$$

can be represented as the sum of two states: (1) a hydrosta- stresses are $\sigma_1 = \sigma_2 = \sigma_3 = \sigma_a$, and (2) a state in which all change of shape is produced by the hydrostatic state. No

Strain energy of distortion, associated with the deviatoric state, can be used to predict the onset of yielding. An expression for the strain energy of distortion per unit volume, U_{od}, is obtained from Eq. 2.6-7 by replacing the written stresses by the deviatoric stresses. Also, we eliminate E by using Eq. 2.5-4. Thus

$$U_{od} = \frac{1}{12G}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)] \tag{2.6-10}$$

Alternative expressions for U_{od} are

$$U_{od} = \frac{3}{4G}\tau_{\text{oct}}^2 \quad \text{and} \quad U_{od} = \frac{1}{6G}\sigma_e^2 \tag{2.6-11}$$

where τ_{oct} is given by Eq. 2.4-3, 2.4-4, or 2.4-5, and σ_e is an "effective" stress defined as

$$\sigma_e = \frac{1}{\sqrt{2}}[(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2)]^{1/2} \tag{2.6-12}$$

From Eqs. 2.4-5 and 2.6-12, $\tau_{\text{oct}} = \sqrt{2}\,\sigma_e/3$. Conveniently, σ_e reduces to $\sigma_e = \sigma_x$ if σ_x is a uniaxial state of stress. If the state of stress is hydrostatic, then $\sigma_e = 0$. It is possible that $\sigma_e > \sigma_1$: for example, if principal stresses are $\sigma_1 = -\sigma_3$ and $\sigma_2 = 0$, then $\sigma_e = \sqrt{3}\sigma_1$. The von Mises failure criterion is often stated in terms of σ_e.

2.7 STRESS CONCENTRATION

Stress in a solid is rarely uniform. It rises to local peaks because of material inhomogeneity or abrupt changes in geometry. Material inhomogeneities include crystal boundaries in metal, small inclusions of foreign material, small voids, the various constituents of concrete, and the cell structure of wood. Unintentional and random changes in geometry include tool marks and surface scratches. *Intentional* changes in geometry are common, such as threads on a bolt, teeth on a gear, an oil hole, and a keyway in a shaft. Peak stress associated with a change in geometry is easy to calculate if the geometry is accurately known and the associated stress concentration factor has been tabulated.

Stress Concentration Factors. Consider a central circular hole in a plate under tension, Fig. 2.7-1a. It is obvious that stress must be greater than σ_o somewhere on a cross section containing the hole, because there the axial force $P = \sigma_o Dt$ must be carried by a reduced area. However, the mechanics of materials method cannot provide a formula for $\sigma_{\max}$ because we have no reliable way of predicting the geometry of deformation. The problem can be solved (with difficulty) by the theory of elasticity method. Most stress concentration problems are too complicated for either method. Many have been solved experimentally. Results have been tabulated for an isotropic and linearly elastic material, in the form of *stress concentration factors* K_t. Using them is simple. For the problem in Fig. 2.7-1a,

Failure criteria

Failure criteria deal with failure of a material, in contrast with failure of a structure.

Brittle materials

Maximum normal stress criterion

Failure if $\sigma_1 \geq \sigma_t^{ult}$, that is , if the largest tensile principal stress exceeds the ultimate strength. The other principal stresses are ignored. Similarly in compression, the applied stress is compared with the ultimate strength in compression. This criterion is simplistic since if all the principal stresses are compressive, most materials are much stronger than would be expected based on uniaxial tests.

Mohr criterion

This takes into account ultimate tensile, compressive and shear stresses. Represent each state by a Mohr circle. Draw an 'envelope' tangent to the circles. Mohr suggested that, provided an arbitrary state of stress was represented by Mohr circle within that envelope, failure would not occur. Sometimes a simplified form is taken in which the shear test is ignored. Then the envelope is a straight line and failure is predicted if $\sigma_1/\sigma_t^{ult} - \sigma_3/\sigma_c^{ult} \geq 1$. This also is not ery realistic for hydrostatic compression.

Ductile materials: yield criteria

Maximum shear stress criterion (Tresca criterion)

Yield when $\tau_{max} \geq \tau_Y$.

In principal stress space this looks like a hexagon.

Tension test to yield.

Recall that the maximum shear stress is $\tau_{max} = \frac{1}{2}(\sigma_1 - \sigma_3)$, but the minimum principal stress is zero so for tension, $\tau_Y = 0.5\ \sigma_Y$.

Von Mises

Yield when $\sigma_{eff} \geq \tau_Y$.

Recall $\sigma_{eff} = \frac{1}{\sqrt{2}}\{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\{\tau_{xy}{}^2 + \tau_{yz}{}^2 + \tau_{zx}{}^2\}\}^{1/2}$.

Distortional energy, $U_{0d} = \frac{3}{4G}\ \tau_{oct}{}^2 = \frac{1}{6G}\ \sigma_{eff}{}^2$.

In principal stress space, the Von Mises criterion looks like an ellipse.

Most failure criteria for composite materials address failure of an individual ply. Failure of the laminated structure is more complicated because of unresolved questions about interlaminar stresses, how degraded plies unload, and how failure of some plies influences the remaining intact plies. At present, no failure criterion intended for a laminated structure is reliable enough to be used without experimental confirmation.

3.5 INTRODUCTION TO FRACTURE MECHANICS

Cracks and Brittle Fracture. One expects that materials such as glass and rock will fail in a brittle manner. A normally ductile material such as structural steel may also fail in a brittle manner if it contains a crack in a region of tensile stress. Typically a crack begins at a stress raiser and grows gradually, due to cyclic loading or due to corrosion under steady loading. When a crack reaches a "critical length" it suddenly propagates as a brittle fracture, and the part or structure breaks, perhaps completely in two. Complete separation may be prevented by progagation of the crack into a "crack arrester" such as an existing hole, or by deformations that happen to relieve the mechanism that causes the crack to propagate. Crack propagation speeds may exceed 1000 m/s.

The Liberty cargo ships of World War II are classic examples of this kind of failure. Of some 2700 built, more than 100 broke in two. Part of the trouble was welded construction, in which edges of adjacent plates were welded together. (Previously, ships were constructed of overlapping plates connected by rivets, thus incorporating "crack arresters" in the structure.) Also, the material itself was made more susceptible to brittle fracture by heat of the welding process and by cold conditions in which these ships often operated.

The state of stress at a crack tip causes material there to lose ductility. Consider, for example, a flat plate with a crack oriented perpendicular to the direction of load (Fig. 3.5-1a). Near the crack tip, normal stresses in the plane of the plate are tensile and very large. Consequently, due to the Poisson effect, material around the crack tip tries to contract in the thickness direction (normal to the plate surface). However, the

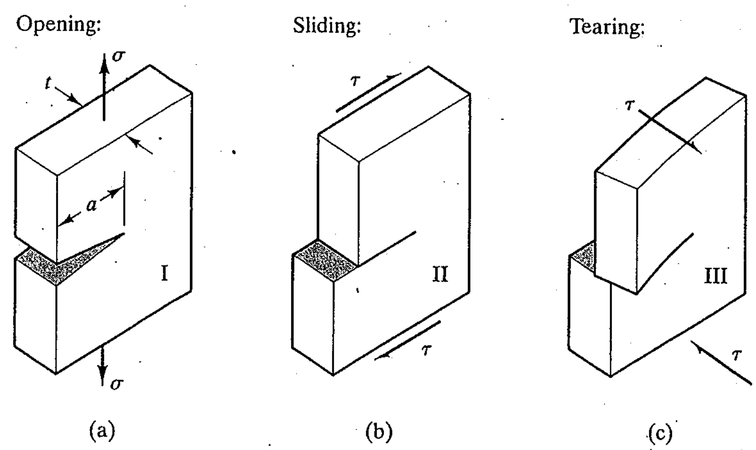


FIGURE 3.5-1 The three crack modes, commonly named I, II, and III.

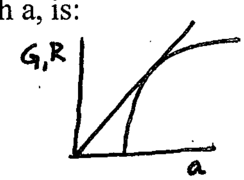
Fracture mechanics

As elliptic hole becomes progressively narrower, the ellipse approaches a crack shape and SCF = K $\to\infty$. Actual observed stress concentration factors for cracks are not infinite.

Therefore a material with one perfectly sharp crack will have **zero** strength, since the stress concentration factor becomes infinite. Experimentally, even for brittle materials, strength is reduced by cracks but not infinitely.
A criterion based on energy balance rather than on stress has therefore been adopted. The approach is due to Griffith. The energy relations are visualized as follows. The energy required to extend a crack is linear in crack length because the energy is expended in creating new free surfaces. By contrast the strain energy available comes from a roughly semicircular region around the crack and is therefore quadratic in crack length. As the crack propagates, the material in that region is unloaded and its energy is made available to drive the crack. When the crack is long enough that an increment in crack length gives rise to an energy release equal to energy expended, catastrophic crack growth impends.

Griffith proposed an **energy** approach to fracture. The elastic energy stored in a test specimen of unit thickness, in a circular region around a crack of length a, is:

$$2\pi a^2 \frac{1}{2E}\sigma^2 \approx 2\cdot \text{strain energy density} \cdot \pi a^2 \cdot 1 \cdot \qquad \text{(F1)}$$



Recall that $\frac{1}{2}\,E\varepsilon^2 = \frac{1}{2E}\,\sigma^2$ represents a strain energy per unit volume.
The elastic energy for a brittle material is twice the area under the stress strain curve. The elastic energy is used to create two new surfaces as the crack propagates. The surface energy, 4 γa (γ is the surface energy; it is an energy per unit area.) should be smaller than the elastic energy for the crack to grow. Thus, the incremental changes of both energies for the crack to grow can be written,

2a

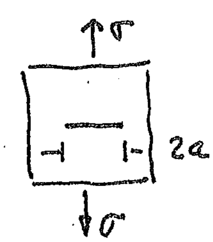
2 surfaces ∴ energy = $2\gamma\cdot 2a$
surface energy

$$\frac{d}{da}\left(\frac{\pi(a\sigma)^2}{E}\right) = \frac{d}{da}(4\gamma a) \qquad \text{(F2)}$$

$$\frac{2\pi a\sigma^2}{E} = 4\gamma$$

Hence,

$$\sigma = \sigma_f = \sqrt{\frac{2\gamma E}{\pi a}} \qquad \text{(F3)}$$



Since for a given material E and γ are constants,

$$\sigma_f = \frac{K}{\sqrt{\pi a}} \qquad \text{(F4)}$$

In this case K has the units of psi $\sqrt{\text{in}}$ or MPa $\sqrt{\text{m}}$ and is proportional to the energy required for fracture.
K is a measure of *fracture toughness,* called the stress intensity factor. Cracks and stress concentrations also occur in ductile materials, but their effect is usually not as serious as in brittle ones since local yielding which occurs in the region of peak stress will effectively blunt the crack and alleviate the stress concentration.

The **stress intensity factor** K is the criterion for fracture in cracked objects. For a small Mode I crack of length a,

$K_1 = \sigma \sqrt{\pi a}\ f(a/c).$

Here f(a/c) is a dimensionless function of loading geometry; it expresses the effect of crack length in relation to block size. σ is the stress required for fracture in the absence of a crack. The units for K are MPa√m, in contrast to the stress concentration factor which is dimensionless. Observed that there is no characteristic length scale in the classical theory of elasticity. The length scale must come from other considerations.

Fracture occurs when K_1 exceeds a critical value, K_{1c} determined from experiment. This is the fracture toughness based on a static test. The fracture toughness for a dynamic situation is NOT the same as for a static situation

Formulas for K are valid over a range of geometrical parameters, specifically, thickness $t \geq 2.5\ (K_{1c}/\sigma_y)^2$, and crack length $a \geq 2.5\ (K_{1c}/\sigma_y)^2$.

In a thick block, the stress field around the tip of the crack is triaxial, since the Poisson contraction in the highly stressed region near the crack is restrained by the surrounding material, which is not so highly stressed. This triaxial stress causes brittle behavior in seemingly ductile materials, since shear deformation is suppressed.

If the block is thinner than the above limit, toughness depends on thickness. If the crack length is less than the above limit, then the material may undergo yield before any fracture occurs from the crack.

Be aware that K_{1c} depends on temperature, and often drops precipitously at low temperature.

Example

Estimate the size of the surface flaw in a glass whose modulus of elasticity and surface energy are 70 GPa and 800 erg/cm^2 respectively. Assume that the glass breaks at a tensile stress of 100 MPa.

Answer

From equation (F4), and keeping in mind the transformation from cgs to SI units,

$$a = \frac{2\gamma E}{\pi \sigma_f^2}$$

$$= \frac{2 \times 800 \text{ dyne/cm} \times 70 \text{ GPa}}{\pi\ (100 \text{ MPa})^2}$$

$$= \underline{3.565\ \mu\text{m}}$$

To two significant figures, a = 3.6 μm.

[Note that if the crack is on the surface its length is a, if it is inside the specimen it is 2a. Remember 1 erg = 1 dyne cm]

I. ASTM model for K_{Ic} testing

A. designed to produce valid K_{Ic} results - How?

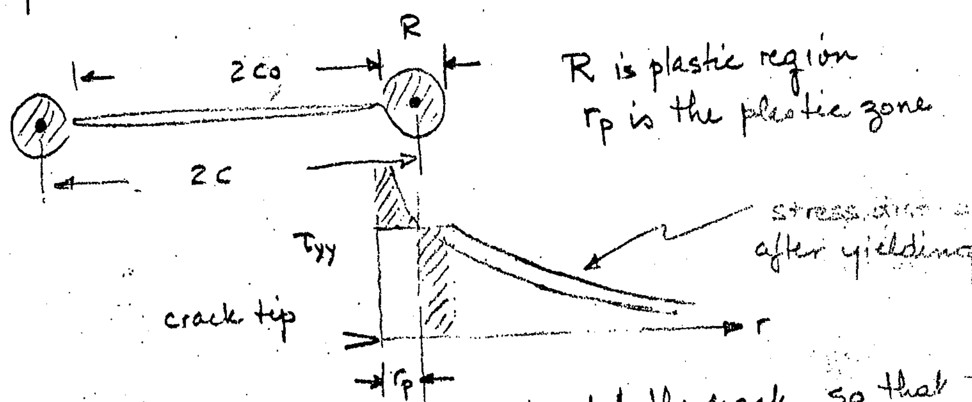
1. must meet $c_0 \geq 2.5 \left(K_{Ic}/\sigma_y \right)^2$
2. " " $B \geq 2.5 \left(K_{Ic}/\sigma_y \right)^2$; $W/B \cong 2$
3. starting crack length must be 0.45 - 0.55 W (width of specimen)
4. crack must be sharp and must be introduced via a fatigue crack starting from a V-notch
5. The fatigue crack must be introduced by low type cycling
6. A displacement gage will be used to accurately measure the relative displacement of two precisely located gages positions
7. Temperature and load rate requirements

B. Why these requirements -

1. $c_0 \geq 2.5 \left(K_{Ic}/\sigma_y \right)^2$. this is a requirement that is necessary and sufficient in order for LEFM to hold

Proof:

Consider a plate loaded in tension



- We assume that the stresses are redistributed ahead of the crack so that the load bearing capacity in front of the crack is unchanged when yielding occurs. We assume that the shaded areas under the graph are the same.
- Thus $2c = 2c_0 + 2r_p = 2c_0 + R$ is the effective length of the crack
- In plane strain mode I $R = \frac{1}{6\pi} \left(K_I/\sigma_y \right)^2$

And $c = c_0 + \frac{1}{6\pi} \left(K_I/\sigma_y \right)^2$

if the stress $\sigma \uparrow$ $K_I \uparrow$ also $K_I \uparrow$ due to the plastic zone correction.

Thus $K_I = \sigma\sqrt{\pi c_0}\left\{1 - \frac{1}{12}\left(\sigma/\sigma_y\right)^2\right\}^{-1/2}$ (1)

- In a test as $\sigma \to \sigma_y$, $K_I \to K_{Ic}$

- If σ reaches σ_y before $K_I = K_{Ic}$ we get yielding and by our elastic-plastic model r_p (and R) $\to \infty$. Hence we violate the LEFM assumption of <u>small scale yielding</u>

- We want $K_I = K_{Ic}$ before $\sigma = \sigma_y$. Thus let $K_I = K_{Ic}$ in (1) and solve for the crack length $2c_0$

$$2c_0 = \frac{2}{\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2\left\{\left(\sigma_y/\sigma\right)^2 - \frac{1}{12}\right\} \qquad K_I = K_{Ic}$$

This will cause <u>unstable</u> crack growth

- The crack length that produces yielding is when $\sigma_y = \sigma$

or $2c_0 = \frac{11}{12}\cdot\frac{2}{\pi}\left(K_{Ic}/\sigma_y\right)^2 \sim \frac{1}{2}\left(\frac{K_{Ic}}{\sigma_y}\right)^2$

if $\sigma > \sigma_y$ then $2c_0 < \frac{1}{2}\left(K_{Ic}/\sigma_y\right)^2$ unacceptable

$\sigma < \sigma_y$ then $2c_0 > \frac{1}{2}\left(K_{Ic}/\sigma_y\right)^2$ or $c_0 > \frac{1}{4}\left(K_{Ic}/\sigma_y\right)^2$

- Because we want to make adequate measurements of K_{Ic} we want $c_0 >> \frac{1}{4}\left(K_{Ic}/\sigma_y\right)^2$

Srawley and Brown suggested that $c_0 \geqslant 2.5\left(K_{Ic}/\sigma_y\right)^2$ and this is accepted as the <u>standard</u>.

2. $B \geq 2.5 \left(K_{Ic}/\sigma_y \right)^2$: This requirement arises from the consideration that we want only MODE I type fracture

<u>Proof:</u>

- As has been shown to you in class in order for cracks to propagate for <u>perfectly</u> brittle materials, the crack extension force $G_I = 2\gamma_s$, where γ_s is the surface energy. However for materials that deform plastically, then crack extension will only occur when $G_I = 2\gamma_s + p$ where p is the plastic work of crack extension. p is <u>not</u> a constant and depends on the size of the plastic zone, σ_y, the work hardening rate, etc <u>AND</u> they all in turn depend on the crack length.

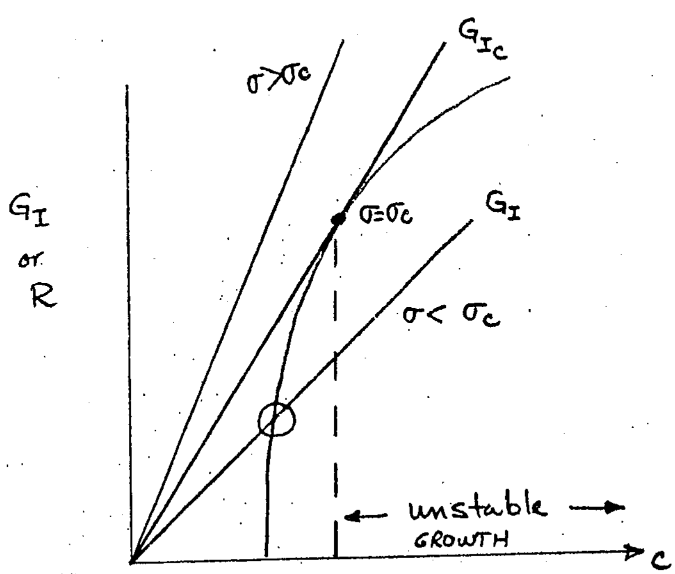
- If we define $R \equiv$ crack extension resistance $= 2\gamma_s + p$, then for unstable growth we must have that

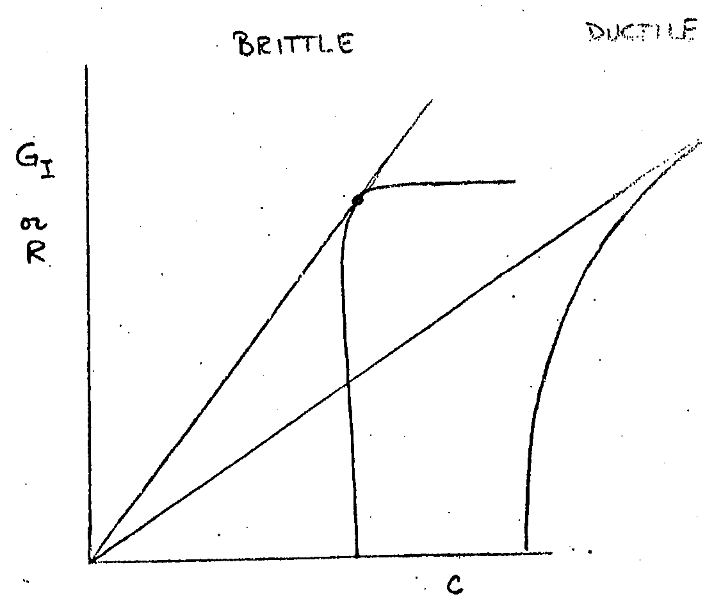
$$G_I \geq R$$

and also

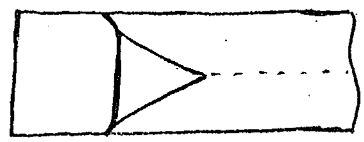
$$\partial G_I/\partial c \geq \frac{\partial R}{\partial c}$$

Thus if we remember that $G_I = \dfrac{\sigma^2 \pi c}{2\mu}(1-\nu)$ and look at a typical G_I versus c curve,

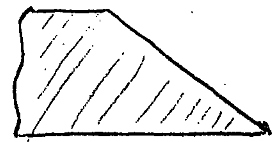




Note that the brittle material shows little plastic deformation and ... $G = R$ point of intersection and occurs below the what has been found is that as the plate is made thinner the R curve will vary and will no longer have a distinct intersection point. The reason for this is the growth of "shear lips" from the free surface and the thinness of the plate (plane stress conditions).



DIR. OF crack propagation →

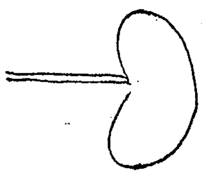


SLANT

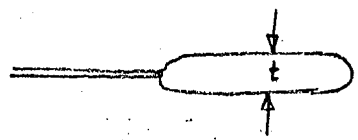
or

V SLANT

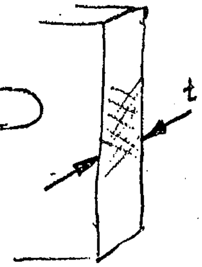
The growth of the shear lips is due to the plastic zone being constrained in the thickness direction. So it will spread in front of the crack tip. The mechanism that will cause crack extension will be due to failure in <u>shear</u> (mode III); hence we see the slant formation.



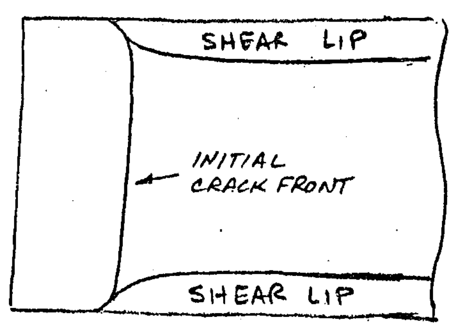
unrealistic character of plastic zone.

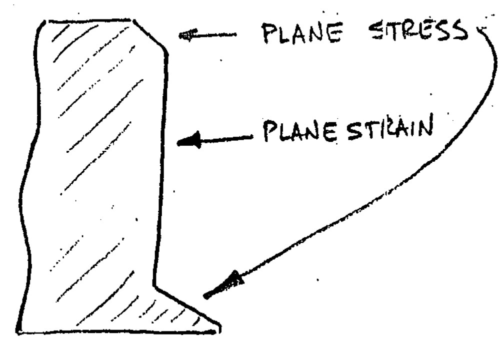


realistic

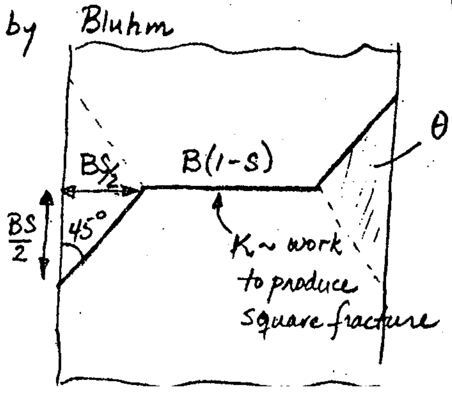


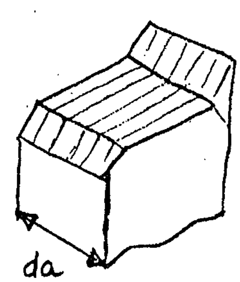
As the plate width is increased, the formation of the shear lips is reduced due to the plane strain effect and the cross-section will look like this





Many have proposed models to describe what occurs here. One such model is that of Krafft, Sullivan and Boyle (1961) modified by Bluhm



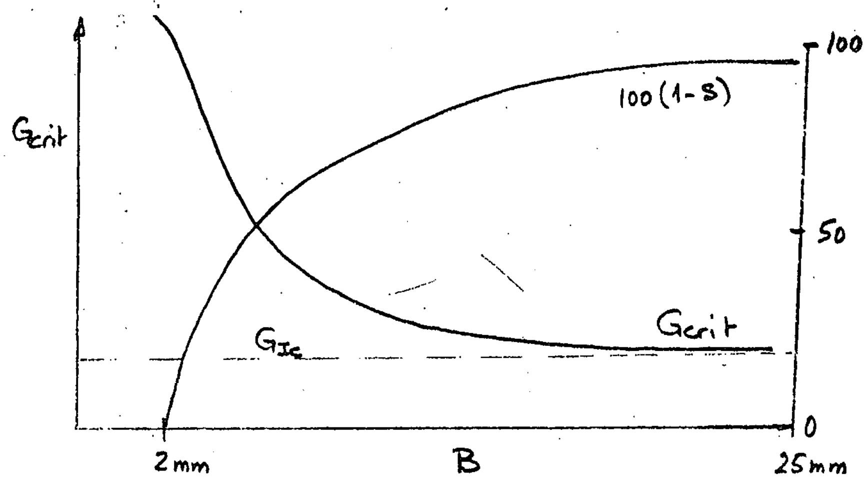


1. square fracture $\neq f(c_0)$
2. shear lips are assumed to occur at $45°$
3. flat fracture is a surface phenomenon
4. Shear lip is volumetric

The work done to create the crack surface da is:

Now $G_I = \frac{1}{B}\frac{dW}{da} = \kappa(1-S) + \frac{Bs^2\theta}{2}$. Note that S is picked so that $Bs = $ constant as crack length ↑

Thus as $B \to \infty$ $S \to 0$ and $G_I \to \kappa$.

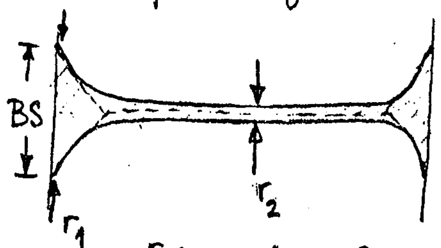


From data for Aluminum 7075-T6

$K \sim 200$ kJ/m²

$\theta \sim 20$ kJ/m²

Look at the plastic zone and superpose the model of Krafft:



$$BS \gg r_{\text{plane stress}} \sim r_p = \frac{1}{2\pi}\left(\frac{K_{Ic}}{\sigma_y}\right)^2 \qquad (*)$$

since $r_1 > r_2$ (plane strain). If $(*)$ is true then plain strain conditions will extend over most of the cross-section and we will have essentially mode I fracture.

$r_p/B \ll 1$ or $\frac{r_p}{B} < S \ll 1$

hence $B \gtrsim \left(\frac{K_{Ic}}{\sigma_y}\right)^2$. To determine the real equation, tests were done on many types of metals and here are some of the results.

Example: Maraging Steel $\sigma_y = 259$ KSI

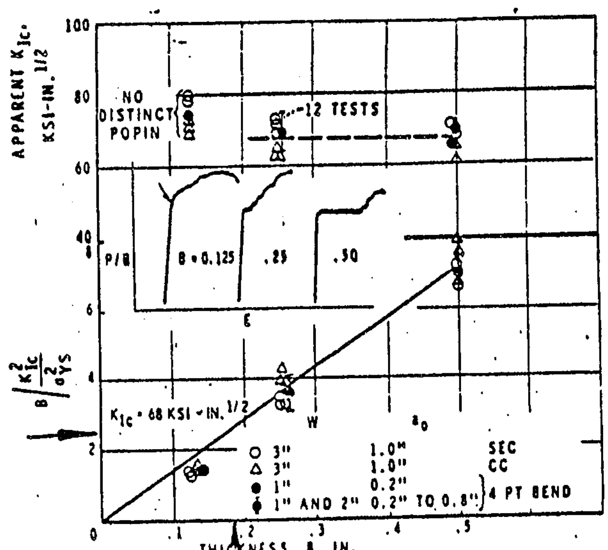


FIG. 14—Effect of thickness on popin behavior and apparent K_{Ic} for 259 ksi

conclusion:

$$B \geq 2.5\left(\frac{K_{Ic}}{\sigma_y}\right)^2$$

Specimen Size Requirements

We have argued that to limit yielding we must make large samples with long cracks. Thus $K_I \to K_{I_c}$ before $\sigma \to \sigma_y$. From our analysis we expect

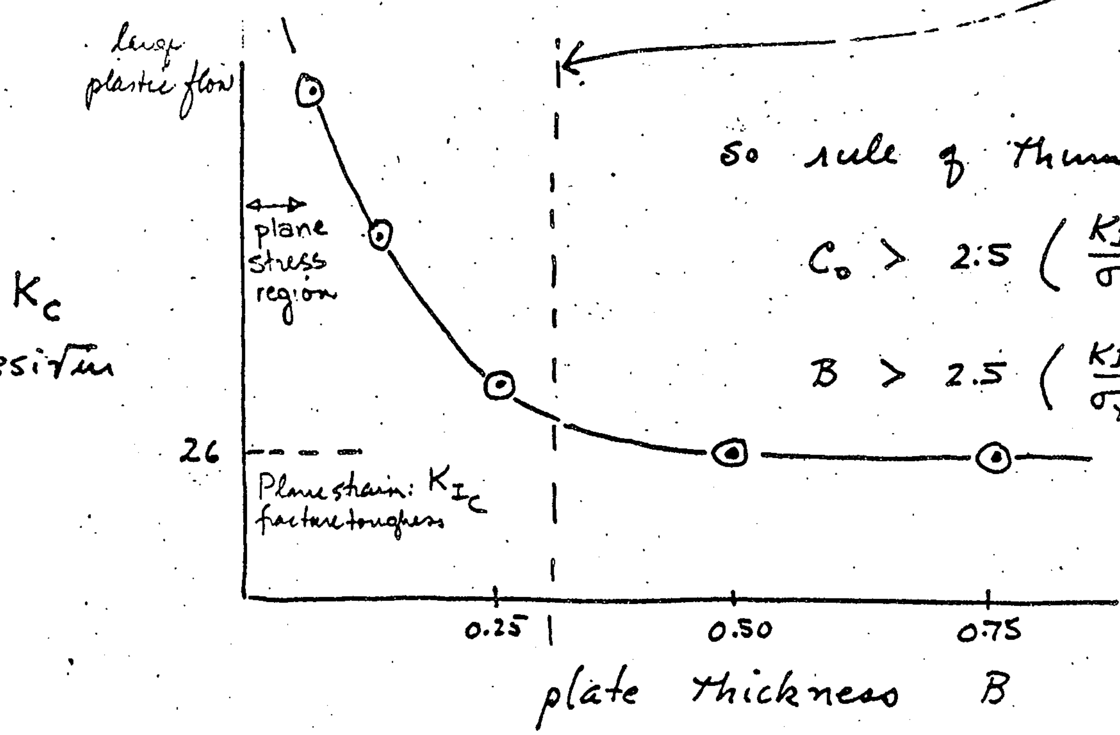
$$C_0 \geq 2.5 \left(\frac{K_{I_c}}{\sigma_y} \right)^2$$

this needs to be checked. Also, How thick must sample be for plane strain conditions?

<u>Consider 7075-T6</u> (MS+E 202C experiment).

$\sigma_y = 75$ ksi

$K_{I_c} = 26$ ksi $\sqrt{in}$ $\qquad 2.5 \left(\frac{K_{I_c}}{\sigma_y} \right)^2 = \qquad 0.3$ in



so rule of thumb:

$$C_0 > 2.5 \left(\frac{K_{I_c}}{\sigma_y} \right)^2$$

$$B > 2.5 \left(\frac{K_{I_c}}{\sigma_y} \right)^2$$

usually more difficult to achieve due to how the specimen is machi

Example (adapted from example on page 62-63)

Consider a 1 cm crack in a large plate of steel. Assume the plate is large enough that $f(a/c) \cong 1$.

What stress gives rise to fracture for a weaker or 'mild' steel ($\sigma_y = 500$ MPa, $K_{1c} = 175$ MPa$\sqrt{m}$) and a high strength steel ($\sigma_y = 1410$ MPa, $K_{1c} = 50$ MPa$\sqrt{m}$).

Solution: Use $K_1 = \sigma f(a/c) \sqrt{\pi a}$, so with $f(a/c) \cong 1$, $\sigma = K_{1c} / \sqrt{\pi a}$.

Weaker steel A, $\sigma = 987$ MPa, which exceeds the yield strength, so it undergoes yield at 500 MPa rather than fracture from the crack.

Stronger steel, $\sigma = 282$ MPa, which is less than the yield strength 1410 MPa, so it fractures catastrophically.

Consequently, in the presence of 1 cm cracks, the stronger steel actually is weaker than the 'mild' weaker steel. Moreover, the consequence of an overload is more severe for the stronger steel since it catastrophically fractures rather than yields in bulk.

Example (adapted from Gordon, *Structures*)

Suppose we have a large structure such as a ship or a bridge and wish to tolerate a 1 meter long crack without catastrophic failure. Consider 'mild' steel ($\sigma_y = 500$ MPa, $K_{1c} = 175$ MPa$\sqrt{m}$) .

Solution-

With $f(a/c) \cong 1$, $\sigma = K_{1c} / \sqrt{\pi a} = 90$ MPa or 14,000 psi.

In **foam**, Gibson and Ashby [*Cellular solids*] predict toughness K_{1c} proportional to [$\sqrt{}$(cell size)](density)$^{3/2}$.

Stress concentrations: appendix

Experimental stress concentrations in composite materials are consistently less than the theoretical ones. The non-classical fracture behavior has been dealt with using point stress and average stress criteria, however that approach cannot account for non-classical strain distributions in objects under small load. Such differences may be accounted for via a generalized continuum approach. Reduced stress concentration factors for small holes are known experimentally in fibrous composite materials. The fracture strength of graphite epoxy plates with holes depends on the size of the hole [1]. Moreover the strain around small holes and notches in fibrous composites well below the yield point is smaller than expected classically [2,3], while for large holes, the strain field follows classical predictions [4].

1. R.F. Karlak, "Hole effects in a related series of symmetrical laminates", in *Proceedings of failure modes in composites, IV*, The metallurgical society of AIME, Chicago, 106-117, (1977)
2. J.M. Whitney, and R.J. Nuismer, "Stress fracture criteria for laminated composites containing stress concentrations", *J. Composite Materials*, **8**, (1974) 253-275.
3. M. Daniel, "Strain and failure analysis of graphite-epoxy plates with cracks", *Experimental Mechanics,* **18** (1978) 246-252, .
4. R. E. Rowlands, I. M. Daniel, and J. B. Whiteside "Stress and failure analysis of a glass-epoxy plate with a circular hole", *Experimental Mechanics*, **13**, (1973) 31-37

much lower stresses. Thickness-direction tension at the just ahead of the crack tip is therefore in a state of *triax-* re more than plastic flow.

gate at great speed after reaching a critical length? The gy considerations, which are summarized as follows. metry of Fig. 3.5-1a, which is shown again in Fig. 3.5-2a. rack an amount da is independent of crack length a, so ice the crack varies linearly with a (Fig. 3.5-2b). As the ced in material alongside the crack, and stored strain gy released varies approximately quadratically with a. ne that a crack of length a nullifies the uniaxial state of f radius a, shown dashed in Fig. 3.5-2a. Strain energy tional to the volume of this disc, $V = \pi a^2 t/2$.) When a ient of strain energy released to equal the increment ie crack, sudden fracture impends. Thus $dU_e = dU_s$ in crack length. Energy needed to drive the crack is sup- material. There is no need (and insufficient time) for rk of external forces acting through a distance. The fail- uld be if the crack were absent because the crack pro- onds to be broken sequentially rather than all at once. as shown in Fig. 3.5-1. In practice, Mode I is most com- . For any mode, one can calculate a *stress intensity fac-* e it with an allowable value to determine whether sity factor is *not* a stress concentration factor! Indeed, it is not necessary to use stress concentration data: not be calculated.

ler only isotropic materials, and only Mode I cracks ss intensity factor for a Mode I crack is denoted by K_I

$$K_I = \beta\sigma\sqrt{\pi a} \tag{3.5-1}$$

t would exist if the crack were absent. Thus, stress σ is ltiplier β is dimensionless and depends on geometry 3.5-1). Dimension a is defined as either the full crack geometry. Units of K_I are MPa$\sqrt{\text{m}}$. Fracture impends known as *fracture toughness*. K_{Ic} can be considered a of specimen thickness t, if the specimen is sufficiently sion at the crack tip to develop fully. Also, the crack inimum. Recommended minimum dimensions are

$$\Big)^2 \quad \text{and} \quad a \geq 2.5\left(\frac{K_{Ic}}{\sigma_Y}\right)^2 \tag{3.5-2}$$

determined by a tension test of the material, and t is thickness is less than the value described by Eq. 3.5-2,

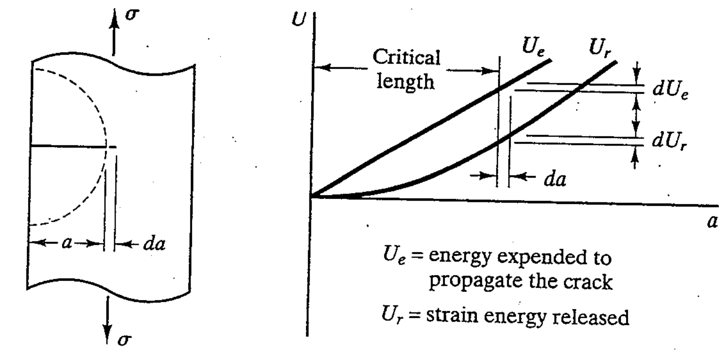
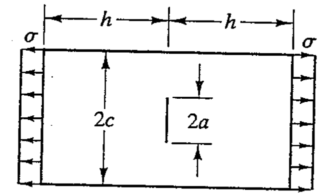


FIGURE 3.5-2 (a) Plate with an edge crack of length a. (b) Energy relations for crack extension.

TABLE 3.5-1 Stress intensity data for flat plates, of isotropic material and uniform thickness, with in-plane loading [3.6].

Tension, central crack of length $2a$

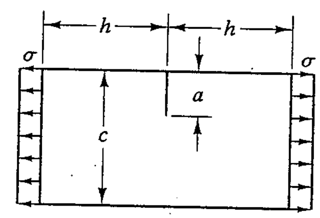


$K_I = \beta\sigma\sqrt{\pi a}$

$\beta = \dfrac{1 - 0.5\,(a/c) + 0.326\,(a/c)^2}{\sqrt{1-(a/c)}}$

Accurate to within 1% for all a/c, provided h/c is "large"

Tension, edge crack of length a

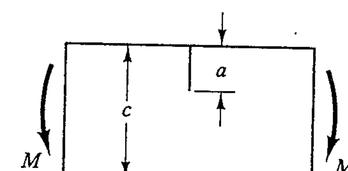


$K_I = \beta\sigma\sqrt{\pi a}$

$\beta = [1.12 - 0.23\,(a/c) + 10.6\,(a/c)^2 - 21.7\,(a/c)^3 + 30.4\,(a/c)^4]$

Accurate to within 1% for $a/c \leqslant 0.6$, provided $h/c > 1$ and sides are free to rotate

Pure bending, edge crack of length a

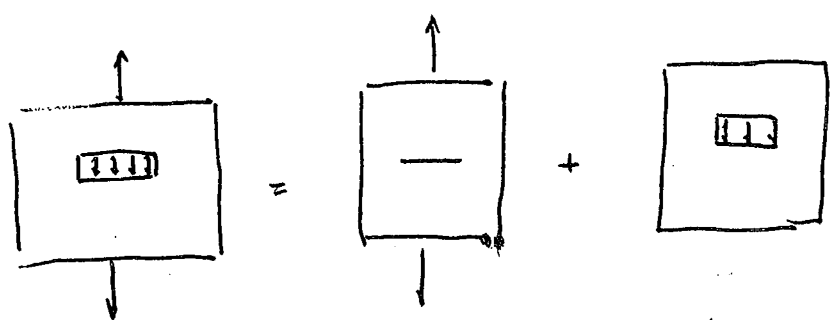


$K_I = \beta\sigma\sqrt{\pi a}$

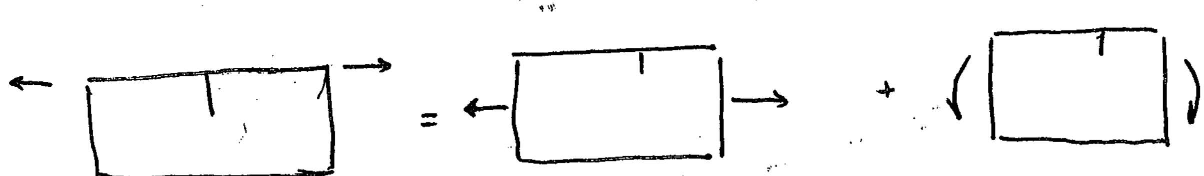
$\sigma = \dfrac{M(c/2)}{I} = \dfrac{6M}{tc^2}$

$\beta = [1.12 - 1.39\,(a/c) + 7.32\,(a/c)^2 - 13.1\,(a/c)^3 + 14.0\,(a/c)^4]$

Accurate to within 1% for $a/c \leqslant 0.6$

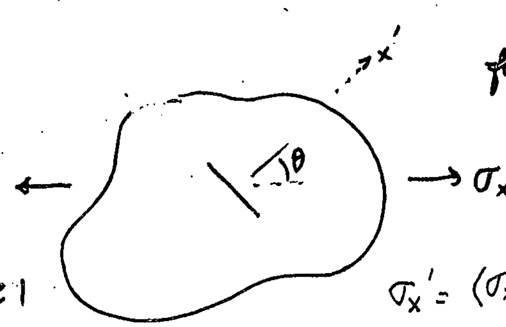


$K_I \quad = \quad K_{I_{(1)}} \quad + \quad K_{I_{(2)}}$



$K_{I_{TOT}} = \sum K_I^{(i)}$

$K_{II_{TOT}} = \sum K_{II}^{(i)}$



find $\sigma_{x'}$ & $\tau_{x'y'}$

$K_I = \sigma_{x'}\sqrt{\pi a}$

$K_{II} = \tau_{x'y'}\sqrt{\pi a}$

$$\left(\frac{K_I}{K_{I_c}}\right)^2 + \left(\frac{K_{II}}{K_{II_c}}\right)^2 \geq 1$$

$$\sigma_{x'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = \tau_{xy}\cos 2\theta - \left(\frac{\sigma_x - \sigma_y}{2}\right)^{?}\sin 2\theta$$

$K_I = \sigma_{x'}\sqrt{\pi a}$

$K_{II} = \tau_{x'y'}\sqrt{\pi a}$

$K_{I_c} = \sigma_y\sqrt{\pi a}$

$$\left(\frac{\sigma_x}{\sigma}\right)^2 + \left(\frac{\tau_{xy}}{\tau_{max}}\right)^2 \geq 1$$

EGM 5615 Synthesis of Engineering Mechanics

Fatigue

A material loaded through multiple cycles will break at a stress considerably less than the ultimate strength for a single application of load. Fatigue is quantified by the S-N curve, in which the number N of cycles is plotted logarithmically.

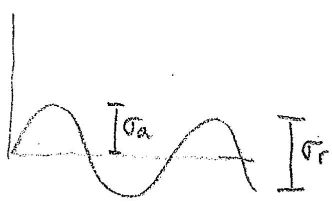
The effect of cyclic stresses is to initiate microcracks at centers of stress concentration within the material or on the surface resulting in the growth and propagation of cracks leading to failure.

As for fatigue testing, the rate of crack growth can be plotted in a log-log scale versus time. Testing the fatigue properties to generate an S-N curve entails monitoring the number of cycles to failure at various stress levels. This test requires a large number of specimens compared with the crack propagation test.

The *endurance limit* is the stress below which the material will not fail in fatigue no matter how many cycles are applied. Not all materials exhibit an endurance limit. (a practical limit is often chosen as 10^7 cycles).

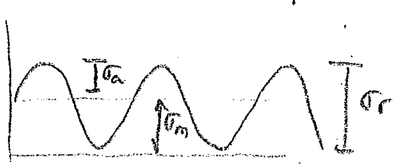
The presence of a saline environment exacerbates fatigue.
Surface roughness exacerbates fatigue. A polished surface is better.
Rubbing or 'fretting' exacerbates fatigue. Re-design the part or use lubricants.
Heat treatment to introduce residual surface compression can be helpful.

FATIGUE DEPENDS ON $R = \frac{\sigma_{min}}{\sigma_{max}}$

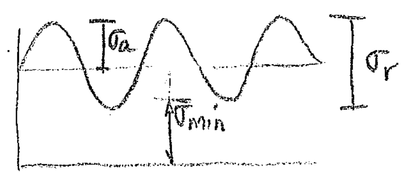


$\sigma_m = 0$
$R = -1$ fully reversed

σ_a = stress amplitude $\frac{\sigma_{max} - \sigma_{min}}{2}$
σ_r = stress range $= 2\sigma_a$
$\sigma_m = \frac{\sigma_{max} + \sigma_{min}}{2}$



$\sigma_{min} = 0$
$R = 0$



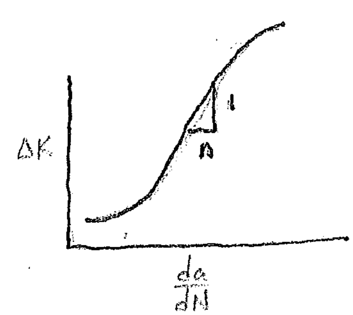
$\sigma_{min} \neq 0$
$R > 0$

<u>NOTCH GEOMETRY EFFECTS</u> as $\rho \uparrow$ (FOR SAME $\Delta\sigma$) NO OF CYCLES $\uparrow$ UNTIL CRACK STARTS
as $\rho = $ const if $\Delta\sigma \uparrow$ NO OF CYCLES $\downarrow$

ALSO $\frac{da}{dN}$ has been found to vary as $\frac{(\Delta K)^n}{(1-R)K_c - \Delta K}$

$\Delta K = K_{max} - K_{min}$

and $K_c = \max K$ when $\frac{da}{dN} \rightarrow \infty$

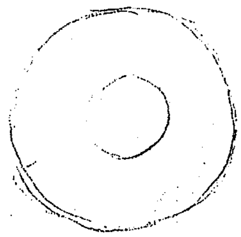


$K_{max} = \sigma_{max}\sqrt{\pi a}$
$K_{min} = \sigma_{min}\sqrt{\pi a}$

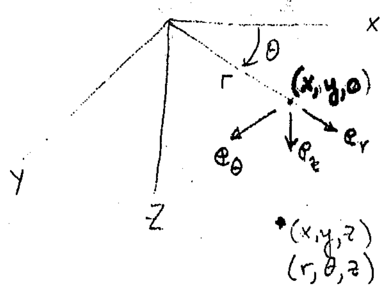
$$\int^a da = \int^N \frac{(\Delta K)^n}{(1-R)K_c - \Delta K}\, dN$$

Do this first before starting on cylindrical

Orthogonal Curvilinear Coordinates will be discussed in order to do torsional problem of a hollow surface.



Cylindrical Coordinate systems.

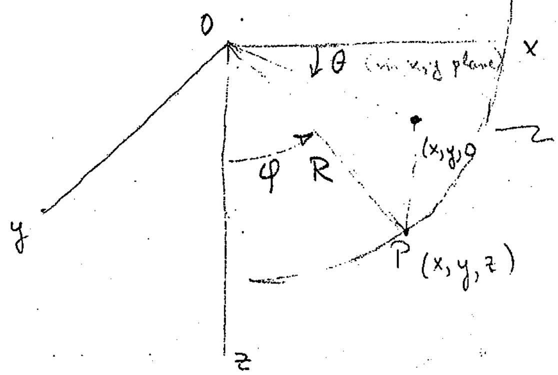


$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

invertible

$r = \sqrt{x^2+y^2}$
$\theta = \arctan(y/x)$
$z = z$

Spherical Coordinate System.

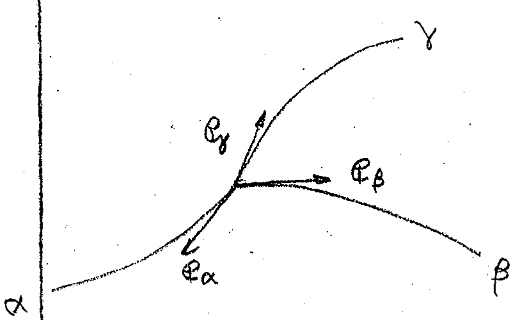


$x = R\sin\varphi\cos\theta$
$y = R\sin\varphi\sin\theta$
$z = R\cos\varphi$
$R = \sqrt{x^2+y^2+z^2}$
$\theta = \tan^{-1}(y/x)$
$\varphi = \arctan\left(\dfrac{\sqrt{x^2+y^2}}{z}\right)$

General orthogonal curvilinear coordinates (α, β, γ)

$\alpha = \alpha(x, y, z)$ Look at $\alpha = \text{const}$ this defines a surface
$\beta = \beta(x, y, z)$ Look at $\beta =$ " " " " "
$\gamma = \gamma(x, y, z)$ " " $\gamma =$ " " " " "

the intersection of these 3 surfaces defines a point p.



$e_{\alpha_i} \cdot e_{\alpha_j} = \delta_{ij}$, $e_{\alpha_i} \times e_{\alpha_j} = e_{\alpha_k}\, e_{ijk}$

in r, θ, z coord.

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \theta}\sigma_{\theta r} + \frac{\partial}{\partial z}\sigma_{zr} + \left(\frac{\sigma_{rr} + \sigma_{\theta\theta}}{r}\right) + f_r = 0$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{2}{r}\sigma_{r\theta} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + f_\theta = 0$$

$$\frac{\partial \sigma_{rz}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{rz}}{r} + f_z = 0$$

$$\varepsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \varepsilon_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} \qquad \varepsilon_{zz} = \frac{\partial u_z}{\partial z}$$

$$\varepsilon_{\theta z} = \frac{1}{z}\left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} + \frac{\partial u_\theta}{\partial z}\right)$$

$$\varepsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right)$$

$$\varepsilon_{r\theta} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right)$$

Rough Outline

a. 2-D Problems

Fourier Series, Integral transforms

Complex variable methods

Cracks, Inclusions

b. 3-D Problems

Green's functions methods

Boundary Integral Eqn. Method

Will try to take a physical notion and derive p.d.e. for it.

a. $C_{ijkl} \dfrac{\partial^2 u_k}{\partial x_i \partial x_l} = 0$ (static, no. boundary forces)

or $\dfrac{\partial \sigma_{ij}}{\partial x_i} = 0 \Rightarrow \sigma_{ij} = C_{ijkl}\, e_{kl}$; $e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$

uses the stress function approach; requires use of compat eqn

In 2-D we will have 2 P.D.E. and 2 unknowns $\rightarrow$ 2 B.C.

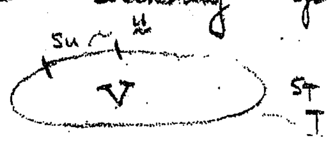
3-D " " " 3 " " 3 " $\rightarrow$ 3 B.C.

Uniqueness Thm (a review)

(1) We specify traction vector $\underset{\sim}{T}$ at all points on the boundary (T_x, T_y, T_z). This is a "dead loading" problem. $\underset{\sim}{u}$ is unique only to a rigid body motion.

(2) Specify $\underset{\sim}{u}$ everywhere on boundary - gives a unique solution

(3) Mix (1) and (2)



a. T_x, u_y, u_z

b. T_x, T_y, u_z

cannot specify T_x, u_y, u_x (cannot specify u_x & T_x)

c. for elastic foundation $T_x + k u_x = 0$

The uniqueness proof assumes that the volume is finite and "No elastic singularities".

i.e. (1) $\underset{\sim}{u}$ discontinuous (shrink fit, dislocations, cracks)

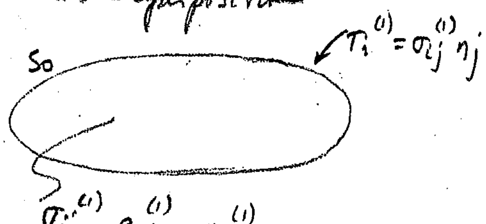
(2) $|\sigma_{ij}| \to \infty$

Semi Inverse Method

Obtain by any fashion a solution to the governing PDE (equil eqns) w/out regard for Boundary conditions. Then try to figure out what problem you have solved.

If solution is not what you need, generate another. Then linearly superpose these problems to get different solutions.

Linear Superposition



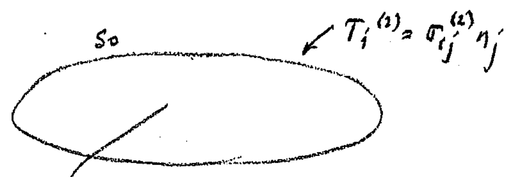
$\sigma_{ij}^{(1)}, e_{ij}^{(1)}, u_i^{(1)}$

Equil. $\sigma_{ij,i}^{(1)} = 0$

$e_{ij}^{(1)} = \frac{1}{2}(u_{i,j}^{(1)} + u_{j,i}^{(1)})$

$\sigma_{ij}^{(1)} = C_{ijkl}\, e_{kl}^{(1)}$

$T_i^{(1)} = \sigma_{ij}^{(1)} n_j$ on S_o



$\sigma_{ij}^{(2)}, e_{ij}^{(2)}, u_i^{(1)}$

$\sigma_{ij,i}^{(2)} = 0$

$e_{ij}^{(2)} = \frac{1}{2}(u_{i,j}^{(2)} + u_{j,i}^{(2)})$

$\sigma_{ij}^{(2)} = C_{ijkl}\, e_{kl}^{(2)}$

$T_i^{(2)} = \sigma_{ij}^{(2)} n_j$ on S_o

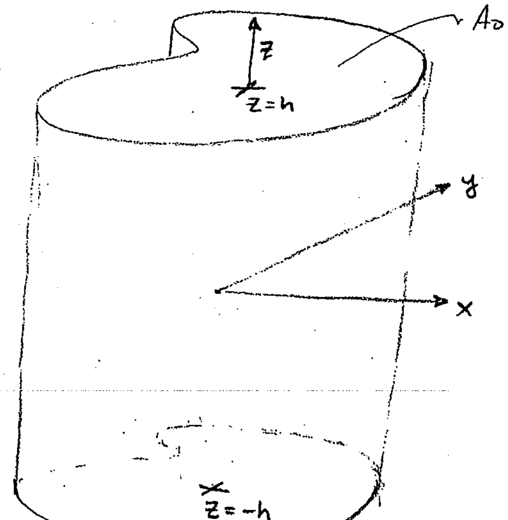
if $\Sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \Rightarrow \Sigma_{ij,i} = 0$ ⇐ this works no matter what

if $E_{ij} = e_{ij}^{(1)} + e_{ij}^{(2)}$ and $U_i = u_i^{(1)} + u_i^{(2)} \Rightarrow E_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$
this works only for infinitesimal strains

$$\Downarrow$$

$\Sigma_{ij} = C_{ijkl} E_{ij}$ and $T_i = T_i^{(1)} + T_i^{(2)} = \Sigma_{ij} n_j$ on S_0

Simple extension problem



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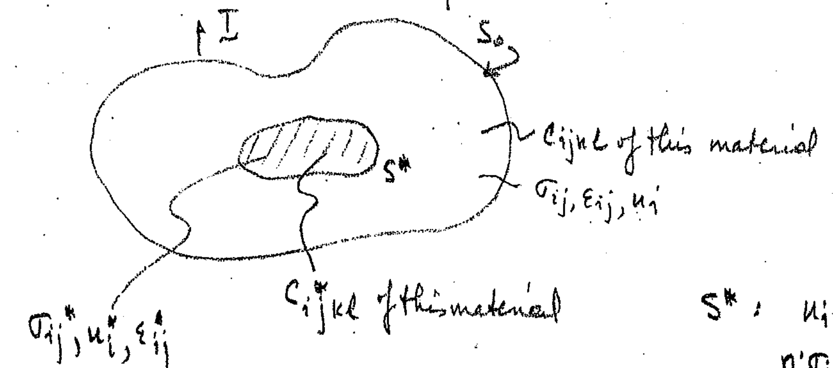
One problem not mentioned in class last time
Inclusion Boundary Conditions



S_0 : $\sigma_{ij} n_j = T_i$

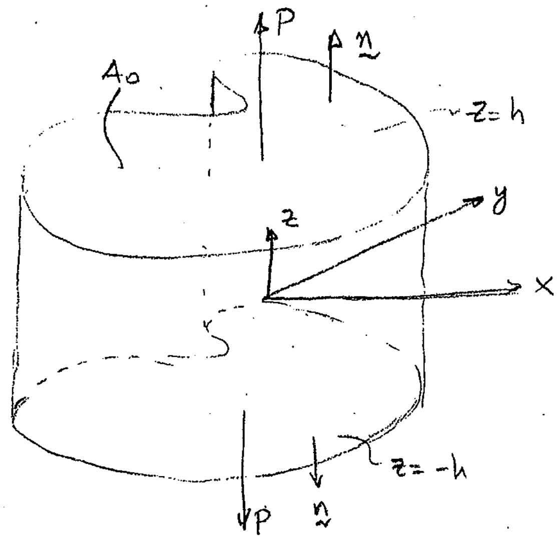
S^* : $u_i^* = u_i$ perfectly bonded
$n_j \sigma_{ij}^* = \sigma_{ij} n_j$

Back to the problem we started last time

Simple Extension

(Any constant state of stress satisfies equil eqns)

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0$$



Traction BC.

@ $z=h$ $\quad T_z = \frac{P}{A_0} = \sigma_{zj} n_j \quad n_x = n_y = 0;\ n_z = +1$

$\boxed{T_z = \frac{P}{A_0} = \sigma_{zz} \text{ @ } z=h}$ $\quad \boxed{T_x = \sigma_{xj} n_j \Rightarrow \sigma_{xz} = 0 \text{ @ } z=h}$

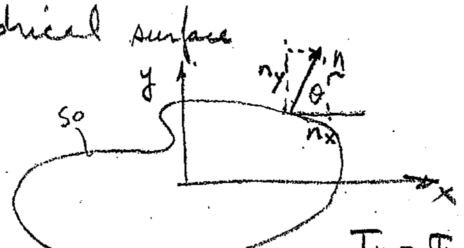
$T_y = 0 \Rightarrow \boxed{\sigma_{yz} = 0 \text{ @ } z=h}$

@ $z=-h$ $\quad n_x = n_y = 0;\ n_z = -1$

$T_z = \frac{-P}{A_0} = \sigma_{zj} n_j = \sigma_{zz}(-1) \Rightarrow \boxed{\sigma_{zz} = \frac{P}{A_0} \text{ on } z=-h}$

$T_x, T_y = 0 \quad \boxed{\sigma_{xz} = \sigma_{yz} = 0 \text{ on } z=-h}$

on the cylindrical surface



$T_x = T_y = T_z = 0$

$n_x = \cos\theta \quad n_y = \sin\theta \quad n_z = 0$

$T_x = \sigma_{xx}\cos\theta + \sigma_{xy}\sin\theta = 0$

$T_y = \sigma_{yx}\cos\theta + \sigma_{yy}\sin\theta = 0$

$$T_z = \sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta = 0$$

if $T_x, T_y, T_z = 0$ for any $\theta \Rightarrow$

pick $\sigma_{xx} = \sigma_{xy} = \sigma_{yy} = \sigma_{zx} = \sigma_{zy} = 0$ on S_0

Hence if we pick $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{zx} = \sigma_{zy} = 0$ everywhere and $\sigma_{zz} = \frac{P}{A}$ on $z = th$

Assume for this problem $\sigma_{zz} = \frac{P}{A_0}$; all others $\sigma_{ij} = 0$

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right) = -\frac{\nu}{E}\frac{P}{A_0} = \frac{\partial u_x}{\partial x}$$

$$e_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right) = -\frac{\nu P}{E A_0} = \frac{\partial u_y}{\partial y}$$

$$e_{zz} = \frac{\sigma_{zz}}{E} = \frac{P}{E A_0} = \frac{\partial u_z}{\partial z}$$

$$\sigma_{xz} = \sigma_{yz} = \sigma_{xy} = 0 \Rightarrow e_{xz} = e_{yz} = e_{xy} = 0$$

$$u_x = -\frac{\nu P}{A_0 E} x + f(y,z)$$

$$u_y = -\frac{\nu P}{E A_0} y + g(x,z)$$

$$u_z = \frac{P}{E A_0} z + h(y,x)$$

$$e_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \Rightarrow \frac{\partial f}{\partial z} + \frac{\partial h}{\partial x} = 0$$

must use compatibility to show $\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 g}{\partial x \partial z} = \frac{\partial^2 h}{\partial y \partial x} = 0$

$\cancel{\Rightarrow \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x} = k(y)}$

$\cancel{\therefore f = k_1(y) z + k_2(y)}$

$\cancel{h = k_1(y) x + k_3(y)}$

→ HW #1 complete and solve showing solution is/includes a rigid body rotation/translat

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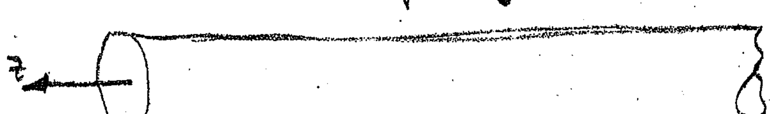
2-D elastostatic problems (isotropic materials)

Plain strain

Elastic solid very long in 1 direction



$$e_{zz} = 0$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2\mu}\sigma_{yz} = 0$$

$e_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{2\mu}\sigma_{xz} = 0$

$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq 0$ and also not fns of z

$\therefore \quad u_x = u_x(x,y) \; ; \; u_y = u_y(x,y) \; ; \; u_z = 0$
$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq 0 \; ; \; \neq$ fns of z

Since $e_{zz} = 0 = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx}+\sigma_{yy})\right) \Rightarrow \sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy}) \neq$ fn of z

each cross section has same thing happening as any other cross section. Plane strain normally simulates the effects at center of a very thick plate

The Equil Eqns reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \qquad \text{since } \sigma_{zx} = 0 \qquad (1)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \qquad \text{since } \sigma_{zy} = 0 \qquad (2)$$

since $\sigma_{zz} \neq$ fn of $z \Longrightarrow \frac{\partial \sigma_{zz}}{\partial z} \equiv 0$ in third eq

<u>Solution by Airy Stress fn</u>

Define a fn ϕ $\ni$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \; ; \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \; ; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

put into equil (1) $\therefore$ $\frac{\partial^3 \phi}{\partial x \partial y^2} - \frac{\partial^3 \phi}{\partial x \partial y^2} = 0$

also same for (2) $\frac{\partial^3 \phi}{\partial x^2 \partial y} - \frac{\partial^3 \phi}{\partial x^2 \partial y} = 0$

But what does ϕ satisfy? Look at hooke's law

$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$ assume a displ field exists $\ni e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

now $\sigma_{ij,i} = \lambda \delta_{ij} e_{kk,i} + 2\mu e_{ij,i} = \lambda e_{kk,j} + 2\mu e_{ij,i}$

subst. the displ. gradient relationship $= \lambda (u_{k,kj}) + \mu (u_{j,ii} + u_{i,ji})$

$$= (\lambda+\mu) u_{k,kj} + \mu u_{j,ii}$$

now differentiate once

$$(\lambda+\mu)(u_{k,kjj}) + \mu (u_{j,iij}) = 0$$

$$\text{or} \quad (\lambda+2\mu)(u_{k,kjj}) = 0$$

$$\text{or} \quad (\lambda+2\mu) \nabla^2 e_{kk} = 0 \implies \nabla^2 \sigma_{kk} = 0 \implies \nabla^4 \phi = 0$$

now $\sigma_{ii} = \overset{3}{\delta_{ii}} \lambda e_{kk} + 2\mu e_{ii} \Rightarrow \sigma_{kk} = (3\lambda+2\mu) e_{kk}$ or $\nabla^2 \sigma_{kk} = (3\lambda+2\mu)\nabla^2 e_{kk} = 0$. Next time will prove
no summation

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Plain Strain From last time

$$\left.\begin{aligned} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0 \end{aligned}\right\} \quad \begin{aligned} \sigma_{xx} &= \phi_{,yy} \\ \sigma_{xy} &= -\phi_{,xy} \\ \sigma_{yy} &= \phi_{,xx} \end{aligned}$$ where ϕ is the airy stress fn

using $\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$ material relation w/

Equil $\sigma_{ij,i} = 0$ and $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$(\lambda+\mu) u_{i,ij} + \mu u_{j,ii} = 0$$

now take $\frac{\partial}{\partial x_j}$ $\quad (\lambda+\mu) u_{i,ijj} + \mu u_{j,iij} = 0$ since dummy indices swap $i \to j$, $j \to i$ in 2nd relation but $i,jji = i,ijj$

hence $(\lambda+2\mu)(u_{i,i})_{,jj} = (\lambda+2\mu)(e_{ii})_{,jj} = 0$

$$\therefore \nabla^2 e_{ii} = 0 \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i} = (\ \ )_{,ii}$$

now $\sigma_{ii} = \lambda \overset{3}{\delta_{ii}} e_{kk} + 2\mu e_{ii} = (3\lambda+2\mu) e_{ii}$

$$\therefore \nabla^2 e_{ii} \Rightarrow \nabla^2 \sigma_{ii} = 0$$

Now in plane strain $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ from $\epsilon_{zz} = 0$

$$\nabla^2 \sigma_{ii} = \nabla^2 (1+\nu)(\sigma_{xx} + \sigma_{yy}) = (1+\nu) \nabla^2 (\sigma_{xx} + \sigma_{yy}) = 0$$

using Airy stress fns

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = \nabla^2(\phi_{,yy} + \phi_{,xx}) = \nabla^2(\nabla^2 \phi) = \nabla^4 \phi = 0$$

for these two to be same then the two conditions needed are the $\overset{\text{stress}}{\epsilon_{xx} = A\sigma_{xx} + B\sigma_{yy}} = \overset{\text{strain}}{\epsilon_{xx} = C\sigma_{xx} + D\sigma_{yy}}$

$$\frac{1}{1+\nu_\sigma} = 1 - \nu_\varepsilon \quad \text{and} \quad \nu_\varepsilon = \frac{\nu_\sigma}{1+\nu_\sigma} \quad \Rightarrow \quad \nu_\sigma = \frac{\nu_\varepsilon}{1-\nu_\varepsilon}$$

(i) Given a complete plain strain soln, get the plane stress solution by leaving μ fix and replace ν_ε by $\frac{\nu}{1+\nu}$

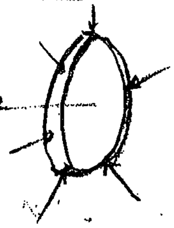
in Plane stress $\sigma_{zz}=0 \Rightarrow \epsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx}+\sigma_{yy})$

$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy})$ $\quad \epsilon_{kk} = (\sigma_{xx}+\sigma_{yy})\frac{1-2\nu}{E}$

$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx})$

<u>Plane Stress</u> Plane $\Rightarrow \sigma_{zx} = \sigma_{zy} = 0$ Plane Stress also $\Rightarrow \sigma_{zz} = 0$

If it is small thickness & hence must vary from 0 to 0 over a small thickness: assume 0 everywhere



$0 = \begin{pmatrix}\sigma_{zz}\\ \sigma_{zx}\\ \sigma_{zy}\end{pmatrix}$ on flat faces

$\sigma_{zz}=0 \Rightarrow e_{zz} = -\frac{\nu}{E}(\sigma_{xx}+\sigma_{yy})$

$\sigma_{zx}=0 \Rightarrow e_{zx}=0$

$\sigma_{zy}=0 \Rightarrow e_{zy}=0$

Tentatively Assume $\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq f(z)$ or $u_x = u_x(x,y)$

$u_y = u_y(x,y)$

Timoshenko & Goodier
Rd 274-277 $\boxed{\text{HW \#1b prove that this assumption is inconsistent}}$

however we can see that u_x, u_y have z^2 component & that for $z \ll 1$ then we can assume the above w/o loss in accuracy

hence, we can define generalized displ for disc w/ thickness h

$$\bar{U}_x(x,y) = \frac{1}{h}\int_0^h u_x(x,y,z)\,dz$$

We will now prove that $\nabla^4\phi = 0$ is DE for plane strain & plane stress for certain conditions

<u>Plane strain</u>

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \underbrace{\sigma_{zz}}_{\nu(\sigma_{xx}+\sigma_{yy})})\right)$$

$$= \frac{1}{E}\left\{\sigma_{xx}(1-\nu^2) - \sigma_{yy}\,\nu(1+\nu)\right\}$$

$$= \frac{1+\nu}{E}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$$

now $\mu = \frac{E}{2(1+\nu)}$ $\quad \therefore e_{xx} = \frac{1}{2\mu}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$

$$e_{xy} = \frac{\sigma_{xy}}{2\mu}$$

<u>Plane stress</u>

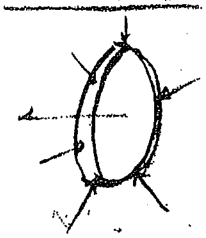
$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \cancel{\sigma_{zz}}^{\,0})\right) = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) = \frac{1}{2\mu}\left\{\frac{\sigma_{xx}}{1+\nu} - \frac{\nu}{1+\nu}\sigma_{yy}\right\}$$

$$e_{xy} = \frac{1}{2\mu}\sigma_{xy} \qquad e_{ii} = \frac{1}{2\mu}\frac{(1-\nu)}{1+\nu}(\sigma_{xx}+\sigma_{yy})$$

in Plane stress $\sigma_{zz}=0 \Rightarrow \epsilon_{zz} = \frac{-\nu}{E}(\sigma_{xx}+\sigma_{yy})$
$\epsilon_{xx} = \frac{1}{E}(\sigma_{xx} - \nu\sigma_{yy})$ $\epsilon_{kk} = (\sigma_{xx}+\sigma_{yy})\frac{1-\nu}{E}$
$\epsilon_{yy} = \frac{1}{E}(\sigma_{yy} - \nu\sigma_{xx})$

Plane Stress   Plane $\Rightarrow \sigma_{zx} = \sigma_{zy} = 0$ Plane Stress also $\Rightarrow \sigma_{zz} = 0$

If it is small thickness & hence must vary from 0 to 0 over a small thickness: assume 0 everywhere



$0 = \begin{pmatrix} \sigma_{zz} \\ \sigma_{zx} \\ \sigma_{zy} \end{pmatrix}$ on flat faces

$\sigma_{zz} = 0 \Rightarrow e_{zz} = -\frac{\nu}{E}(\sigma_{xx}+\sigma_{yy})$
$\sigma_{zx} = 0 \Rightarrow e_{zx} = 0$
$\sigma_{zy} = 0 \Rightarrow e_{zy} = 0$

Tentatively Assume $\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq f(z)$ or $u_x = u_x(x,y)$
$u_y = u_y(x,y)$

Timoshenko & Goodier Rd 274-277 $\boxed{\text{HW \#1b prove that this assumption is inconsistent}}$

however we can see that u_x, u_y have z^2 component & that for $z \ll 1$ then we can assume the above w/o loss in accuracy
hence, we can define generalized displ for disc w/ thickness h

$$U_x(x,y) = \frac{1}{h}\int_0^h u_x(x,y,z)\,dz$$

We will now prove that $\nabla^4\phi = 0$ is DE for plane strain & plane stress for certain conditions

Plane strain

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right) \qquad \sigma_{zz} = \nu(\sigma_{xx}+\sigma_{yy})$$

$$= \frac{1}{E}\left\{\sigma_{xx}(1-\nu^2) - \sigma_{yy}\,\nu(1+\nu)\right\}$$

$$= \frac{1+\nu}{E}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$$

now $\mu = \frac{E}{2(1+\nu)}$ $\therefore\ e_{xx} = \frac{1}{2\mu}\left\{(1-\nu)\sigma_{xx} - \nu\sigma_{yy}\right\}$

$$e_{xy} = \frac{\sigma_{xy}}{2\mu}$$

Plane stress

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \cancel{\sigma_{zz}}^{\,0})\right) = \frac{1}{E}\left(\sigma_{xx} - \nu\sigma_{yy}\right) = \frac{1}{2\mu}\left\{\frac{\sigma_{xx}}{1+\nu} - \frac{\nu}{1+\nu}\sigma_{yy}\right\}$$

$e_{xy} = \frac{1}{2\mu}\sigma_{xy}$ $e_{ii} = \frac{1}{2\mu}\frac{(1-\nu)}{1+\nu}(\sigma_{xx}+\sigma_{yy})$

now $\sigma_{ij,i} = \lambda\delta_{ij} e_{kk,i} + 2\mu e_{ij,i} = \lambda e_{kk,j} + 2\mu e_{ij,i}$

subst. the displ gradient relationships $= \lambda(u_{k,kj}) + \mu(u_{j,ii} + u_{i,ji})$

$$= (\lambda+\mu) u_{k,kj} + \mu u_{j,ii}$$

now differentiate once

$$(\lambda+\mu)(u_{k,kjj}) + \mu(u_{j,iij}) = 0$$

$$\text{or} \quad (\lambda+2\mu)(u_{k,kjj}) = 0$$

$$\text{or} \quad (\lambda+2\mu)\nabla^2 e_{kk} = 0 \Rightarrow \nabla^2\sigma_{kk} = 0 \Rightarrow \nabla^4\phi = 0$$

now $\sigma_{ii} = \overset{3}{\delta_{ii}}\lambda e_{kk} + 2\mu e_{ii}$ (no summation) $\Rightarrow \sigma_{kk} = (3\lambda+2\mu)e_{kk}$ or $\nabla^2\sigma_{kk} = (3\lambda+2\mu)\nabla^2 e_{kk} = 0$. Next time will prove

Plain Strain — From last time

$$\left.\begin{aligned} \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\sigma_{xy}}{\partial y} &= 0 \\ \frac{\partial\sigma_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} &= 0 \end{aligned}\right\} \quad \begin{aligned} \sigma_{xx} &= \phi_{,yy} \\ \sigma_{xy} &= -\phi_{,xy} \\ \sigma_{yy} &= \phi_{,xx} \end{aligned} \quad \text{where } \phi \text{ is the airy stress fn}$$

using $\sigma_{ij} = \lambda\delta_{ij} e_{kk} + 2\mu e_{ij}$ material relation w/

Equil $\sigma_{ij,i} = 0$ and $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$(\lambda+\mu) u_{i,ij} + \mu u_{j,ii} = 0$$

now take $\frac{\partial}{\partial x_j}$ $\quad (\lambda+\mu) u_{i,ijj} + \mu u_{j,iij} = 0 \quad$ since dummy indices rep $i \to j$, $j \to i$ in 2nd relation but $i,jji = i,ijj$

hence $(\lambda+2\mu)(u_{i,i})_{,jj} = (\lambda+2\mu)(e_{ii})_{,jj} = 0$

$$\therefore \quad \nabla^2 e_{ii} = 0 \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i} = (\quad)_{,ii}$$

$$\text{now} \quad \sigma_{ii} = \lambda\overset{3}{\delta_{ii}} e_{kk} + 2\mu e_{ii} = (3\lambda+2\mu) e_{ii}$$

$$\therefore \quad \nabla^2 e_{ii} \Rightarrow \nabla^2\sigma_{ii} = 0$$

Now in plane strain $\sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy})$ from $\epsilon_{zz} = 0$

$$\nabla^2\sigma_{ii} = \nabla^2(1+\nu)(\sigma_{xx} + \sigma_{yy}) = (1+\nu)\nabla^2(\sigma_{xx} + \sigma_{yy}) = 0$$

using Airy stress fns

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = \nabla^2(\phi_{,yy} + \phi_{,xx}) = \nabla^2(\nabla^2\phi) = \nabla^4\phi = 0$$

$e_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = \frac{1}{2\mu}\sigma_{xz} = 0$

$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq 0$ and also not fns of z

$\therefore$ $u_x = u_x(x,y)$; $u_y = u_y(x,y)$; $u_z = 0$
$\sigma_{xx}, \sigma_{xy}, \sigma_{yy} \neq 0$; $\neq$ fns of z

Since $e_{zz} = 0 = \frac{1}{E}\left(\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})\right) \Rightarrow \sigma_{zz} = \nu(\sigma_{xx} + \sigma_{yy}) \neq$ fn of z

each cross section has same thing happening as any other cross section. Plane strain normally simulates the effects at center of a very thick plate

The Equil Eqns reduce to

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0 \qquad \text{since } \sigma_{zx} = 0 \qquad (1)$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \qquad \text{since } \sigma_{zy} = 0 \qquad (2)$$

since $\sigma_{zz} \neq$ fn of z $\Longrightarrow$ $\frac{\partial \sigma_{zz}}{\partial z} \equiv 0$ in third eq

Solution by Airy Stress fn.

Define a fn ϕ $\ni$

$$\sigma_{xx} = \frac{\partial^2 \phi}{\partial y^2} \; ; \quad \sigma_{yy} = \frac{\partial^2 \phi}{\partial x^2} \; ; \quad \sigma_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y}$$

put into equil (1) $\therefore$ $\frac{\partial^3 \phi}{\partial x \partial y^2} - \frac{\partial^3 \phi}{\partial x \partial y^2} = 0$

also same for (2) $\frac{\partial^3 \phi}{\partial x^2 \partial y} - \frac{\partial^3 \phi}{\partial x^2 \partial y} = 0$

But what does ϕ satisfy? Look at hooke's law

$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij}$ assume a displ field exists $\Rightarrow$ $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$$T_z = \sigma_{zx}\cos\theta + \sigma_{zy}\sin\theta = 0$$

if $T_x, T_y, T_z = 0$ for any $\theta \Rightarrow$

pick $\sigma_{xx} = \sigma_{xy} = \sigma_{yy} = \sigma_{zx} = \sigma_{zy} = 0$ on S_0

Hence if we pick $\sigma_{xx} = \sigma_{yy} = \sigma_{xy} = \sigma_{zx} = \sigma_{zy} = 0$ everywhere and $\sigma_{zz} = \frac{P}{A}$ on $z = $ th

Assume for this problem $\sigma_{zz} = \frac{P}{A_0}$; all others $\sigma_{ij} = 0$

$$e_{xx} = \frac{1}{E}\left(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})\right) = -\frac{\nu}{E}\frac{P}{A_0} = \frac{\partial u_x}{\partial x}$$

$$e_{yy} = \frac{1}{E}\left(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})\right) = -\frac{\nu P}{E A_0} = \frac{\partial u_y}{\partial y}$$

$$e_{zz} = \frac{\sigma_{zz}}{E} = \frac{P}{E A_0} = \frac{\partial u_z}{\partial z}$$

$$\sigma_{xz} = \sigma_{yz} = \sigma_{xy} = 0 \Rightarrow e_{xz} = e_{yz} = e_{xy} = 0$$

$$u_x = -\frac{\nu P}{A_0 E}x + f(y,z)$$

$$u_y = -\frac{\nu P}{E A_0}y + g(x,z)$$

$$u_z = \frac{P}{E A_0}z + h(y,x)$$

must use compatibility to show $\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 g}{\partial x \partial z} = \frac{\partial^2 h}{\partial y \partial x} = 0$

$\cancel{\Rightarrow \frac{\partial f}{\partial z} = \frac{\partial h}{\partial x} = k(y)}$

$\cancel{\therefore f = k_1(y) z + k_2(y)}$

$\cancel{h = k_1(y) x + k_3(y)}$

$$e_{xz} = 0 = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right) = 0 \Rightarrow \frac{\partial f}{\partial z} + \frac{\partial h}{\partial x} = 0$$

HW #1 complete and solve showing solution is/includes a rigid body rotation/translation

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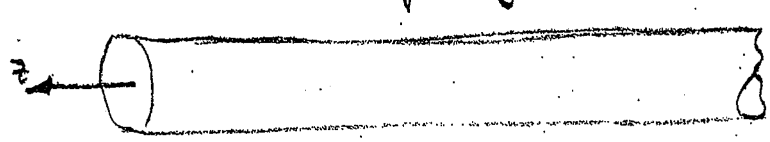
2-D elastostatic problems (isotropic materials)

Plain strain

Elastic solid very long in 1 direction



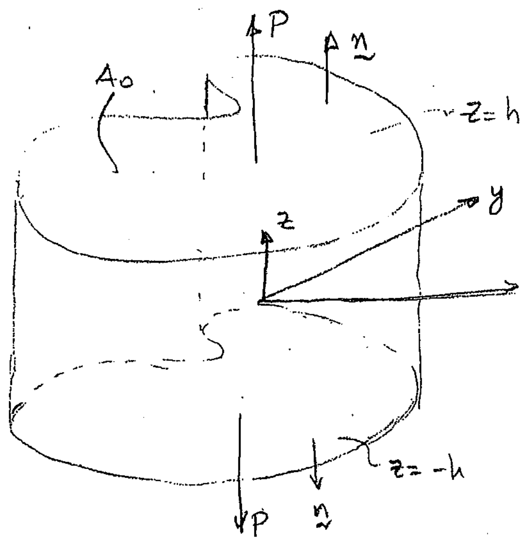
$$e_{zz} = 0$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right) = \frac{1}{2\mu}\sigma_{yz} = 0$$

Simple Extension

(Any constant state of stress satisfies equil eqns)

$$\frac{\partial \sigma_{ij}}{\partial x_i} = 0$$



Traction BC.

@ $z = h$ $\quad T_z = \frac{P}{A_0} = \sigma_{zj} n_j \qquad n_x = n_y = 0;\; n_z = +1$

$$\boxed{T_z = \frac{P}{A_0} = \sigma_{zz} \;@\; z = h} \qquad \boxed{T_x = \sigma_{xj} n_j \Rightarrow \sigma_{xz} = 0 \;@\; z = h}$$

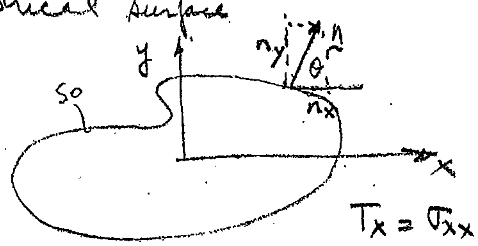
$$T_y = 0 \Rightarrow \boxed{\sigma_{yz} = 0 \;@\; z = h}$$

@ $z = -h$ $\qquad n_x = n_y = 0 \;;\; n_z = -1$

$$T_z = \frac{-P}{A_0} = \sigma_{zj} n_j = \sigma_{zz}(-1) \Rightarrow \boxed{\sigma_{zz} = \frac{P}{A_0} \text{ on } z = -h}$$

$$T_x, T_y = 0 \qquad \boxed{\sigma_{xz} = \sigma_{yz} = 0 \text{ on } z = -h}$$

on the cylindrical surface



$$T_x = T_y = T_z = 0$$

$$n_x = \cos\theta \quad n_y = \sin\theta \quad n_z = 0$$

$$T_x = \sigma_{xx}\cos\theta + \sigma_{xy}\sin\theta = 0$$

$$T_y = \sigma_{yx}\cos\theta + \sigma_{yy}\sin\theta = 0$$

if $\Sigma_{ij} = \sigma_{ij}^{(1)} + \sigma_{ij}^{(2)} \Rightarrow \Sigma_{ij,i} = 0$ ⇐ this works no matter what

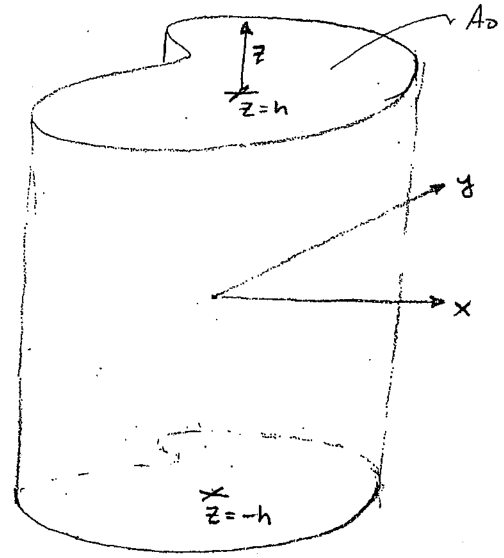
if $E_{ij} = e_{ij}^{(1)} + e_{ij}^{(2)}$ and $U_i = u_i^{(1)} + u_i^{(2)} \Rightarrow E_{ij} = \frac{1}{2}(U_{i,j} + U_{j,i})$

this works <u>only</u> for infinitesimal strains

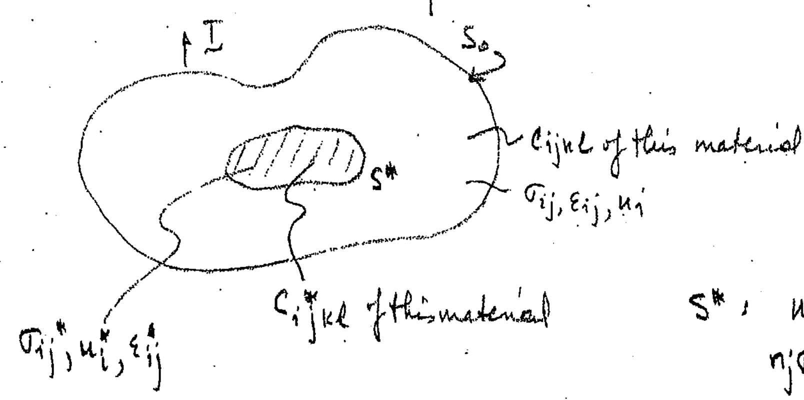
$$\Downarrow$$

$\Sigma_{ij} = C_{ijkl} E_{ij}$ and $T_i = T_i^{(1)} + T_i^{(2)} = \Sigma_{ij} n_j$ on S_0

<u>Simple extension problem</u>



One problem not mentioned in class last time
Inclusion Boundary Conditions



S_0 : $\sigma_{ij} n_j = T_i$

S^* : $u_i^* = u_i$ perfectly bonded
$n_j \sigma_{ij}^* = \sigma_{ij} n_j$

<u>Back to the problem we started last time</u>

a. T_x, u_y, u_z

b. T_x, T_y, u_z

cannot specify T_x, u_y, u_x (cannot specify u_x & T_x)

c. for elastic foundation $T_x + k u_x = 0$

The uniqueness proof assumes that the volume is finite and "No elastic singularities".

i.e. (1) $\underset{\sim}{u}$ discontinuous (shrink fit, dislocations, cracks)

(2) $|\sigma_{ij}| \to \infty$

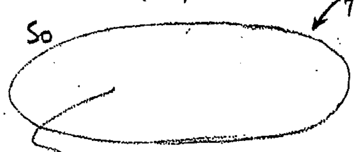
<u>Semi Inverse Method</u>

Obtain by any fashion a solution to the governing PDE (equil eqns) w/out regard for Boundary conditions. Then try to figure out what problem you have solved.

If solution is not what you need, generate another. Then linearly superpose these problems to get different solutions.

Linear Superposition

S_o $\quad T_i^{(1)} = \sigma_{ij}^{(1)} n_j$



$\sigma_{ij}^{(1)}, e_{ij}^{(1)}, u_i^{(1)}$

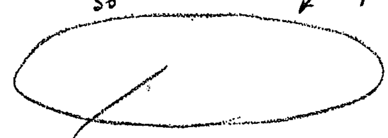
Equil. $\sigma_{ij,i}^{(1)} = 0$

$e_{ij}^{(1)} = \frac{1}{2}\left(u_{i,j}^{(1)} + u_{j,i}^{(1)}\right)$

$\sigma_{ij}^{(1)} = C_{ijkl}\, e_{kl}^{(1)}$

$T_i^{(1)} = \sigma_{ij}^{(1)} n_j$ on S_o

S_o $\quad T_i^{(2)} = \sigma_{ij}^{(2)} n_j$



$\sigma_{ij}^{(2)}, e_{ij}^{(2)}, u_i^{(1)}$

$\sigma_{ij,i}^{(2)} = 0$

$e_{ij}^{(2)} = \frac{1}{2}\left(u_{i,j}^{(2)} + u_{j,i}^{(2)}\right)$

$\sigma_{ij}^{(2)} = C_{ijkl}\, e_{kl}^{(2)}$

$T_i^{(2)} = \sigma_{ij}^{(2)} n_j$ on S_o

Rough Outline

a. 2-D Problems

Fourier Series, Integral transforms

Complex variable methods

Cracks, Inclusions

b. 3-D Problems

Green's functions methods

Boundary Integral Eqn. Method

Will try to take a physical notion and derive p.d.e. for it.

a. $C_{ijkl} \dfrac{\partial^2 u_k}{\partial x_i \partial x_l} = 0$ (static, no boundary forces)

or $\dfrac{\partial \sigma_{ij}}{\partial x_i} = 0 \Rightarrow \sigma_{ij} = C_{ijkl}\, e_{kl}$; $e_{kl} = \frac{1}{2}(u_{k,l} + u_{l,k})$

uses the stress function approach; requires use of compat eqn

In 2-D we will have 2 P.D.E. and 2 unknowns → 2 B.C.

3-D " " " 3 " " 3 " → 3 B.C.

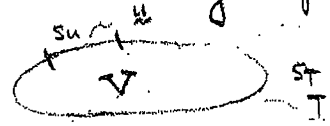
Uniqueness Thm (a review)

(1) We specify traction vector $\underset{\sim}{T}$ at all points on the boundary (T_x, T_y, T_z)

This is a "dead loading" problem. $\underset{\sim}{u}$ is unique only to a rigid body motion.

(2) Specify $\underset{\sim}{u}$ everywhere on boundary - gives a unique solution

(3) Mix (1) and (2)



$$\sigma_{rr} + \sigma_{\theta\theta} = 2\,\frac{P_i a^2 - P_o b^2}{b^2 - a^2} = \frac{1}{\nu}\,\sigma_{zz} \quad \text{for plane strain } (= \text{const})$$

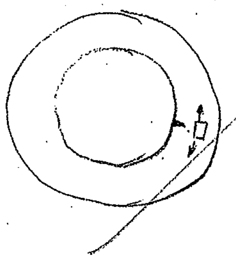
in the homework 3rd problem I ask what if I wanted plane stress soln? How could I superpose on this soln another solution $\ni$ $\sigma_{zz} = 0$

Problem #2

IF $P_o = 0$ then

$$\sigma_{rr} = -\frac{P_i a^2 b^2}{b^2 - a^2}\,\frac{1}{r^2} + \frac{P_i a^2}{b^2 - a^2} = \frac{P_i a^2}{b^2 - a^2}\left\{1 - \frac{b^2}{r^2}\right\} \le 0 \quad \text{compress.}$$

$$\sigma_{\theta\theta} = +\frac{P_i a^2 b^2}{b^2 - a^2}\,\frac{1}{r^2} + \frac{P_i a^2}{b^2 - a^2} = \frac{P_i a^2}{b^2 - a^2}\left\{1 + \frac{b^2}{r^2}\right\} > 0 \quad \text{tens.}$$

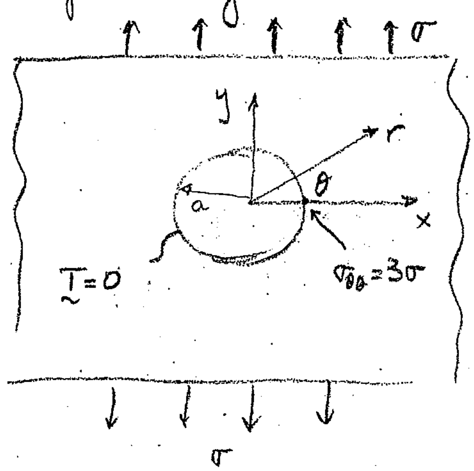


if cracks develop in radial dir } $\sigma_{\theta\theta}$ being tensile cause cracks to propagate

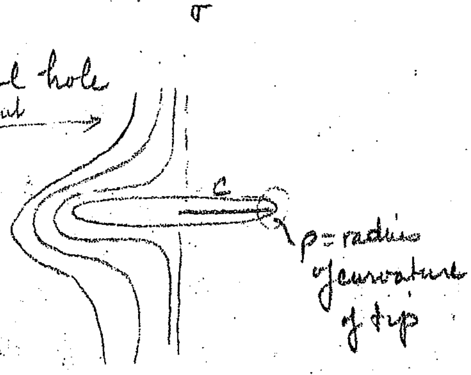
Continue here!

Problem #3

Look at an infinite body with a circular hole in a thick plate

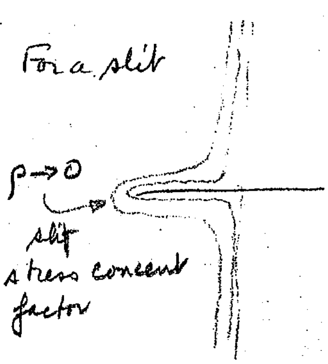


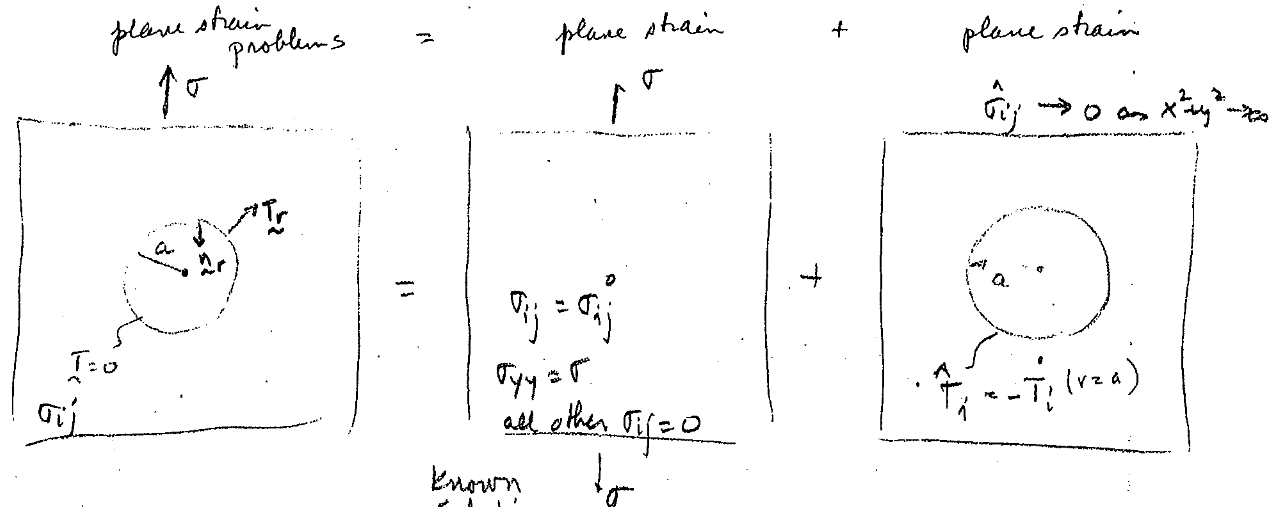
For an elliptical hole



Stress conc: $\sqrt{c/\rho}$ concentration

For a slit





Stress concentration on a Circular Hole.

$$\sigma_{ij} = \overset{\circ}{\sigma}_{ij} + \hat{\sigma}_{ij}$$

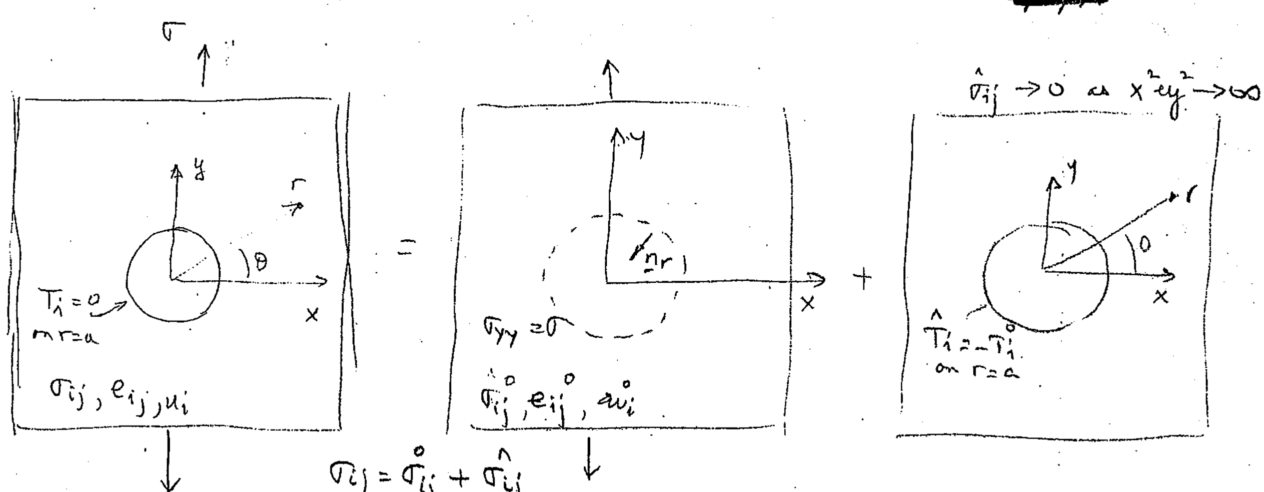
BC: $\hat{\sigma}_{ij} \to 0$ as $x^2+y^2 \to \infty$

$$T_i = T_i^{\circ} + \hat{T}_i = 0 \implies \hat{T}_i = -T_i^{\circ} \text{ or } r=a$$

$$\hat{T}_r = \hat{\sigma}_{rr} \overset{(-1)}{n_r} = -\hat{\sigma}_{rr} = -T_r^{\circ} = -\sigma_{rr}^{\circ} n_r = \sigma_{rr}^{\circ} \qquad \therefore \; \hat{\sigma}_{rr} = -\sigma_{rr}^{\circ} \text{ on } r=a$$

$$\hat{T}_\theta : \qquad \hat{\sigma}_{r\theta} = -\overset{\circ}{\sigma}_{r\theta} \text{ on } r=a$$

Stress concent on a Circular Hole [redacted]



$$\sigma_{ij} = \overset{\circ}{\sigma}_{ij} + \hat{\sigma}_{ij}$$

$^{\circ}$FIELDS $\Rightarrow$ $\sigma_{yy}^{\circ} = \sigma \qquad \sigma_{ij}^{\circ} = 0$ all others. from transformation

$$\text{or } r=a \left\{ \begin{array}{l} T_r^{\circ} = \sigma_{rr} n_r = -\sigma_{rr}^{\circ} = -\sigma_{yy}^{\circ} \sin^2\theta = -\frac{\sigma}{2}(1-\cos 2\theta) \\ T_\theta^{\circ} = \sigma_{r\theta} n_r = -\sigma_{\theta r}^{\circ} = -\sigma_{yy}^{\circ} \sin\theta\cos\theta = -\frac{\sigma}{2}\sin 2\theta \end{array} \right.$$

$$\sigma_{\theta\theta}^{\circ} = \tfrac{\sigma}{2}(1+\cos 2\theta) = \sigma_{/2}\cos^2\theta$$

$$\therefore \hat{T}_r = \hat{\sigma}_{rr} \overset{(-1)}{n_r} = -T_r^{\circ} = \frac{\sigma}{2}(1-\cos 2\theta) \qquad \therefore \; \hat{\sigma}_{rr} = -\frac{\sigma}{2}(1-\cos 2\theta)$$

$$\hat{T}_\theta = \hat{\sigma}_{r\theta} n_r = -T_\theta^{\circ} = +\frac{\sigma}{2}\sin 2\theta \qquad \therefore \; \hat{\sigma}_{r\theta} = -\sigma_{/2}(\sin 2\theta) \quad \Bigg\| \text{ on } r=a$$

$$\sigma_{rr} + \sigma_{\theta\theta} = 2\,\frac{P_i a^2 - P_o b^2}{b^2 - a^2} = \frac{1}{\nu}\sigma_{zz} \quad \text{for plane strain } (= \text{const})$$

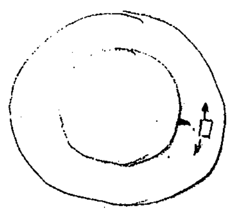
<u>in the homework</u> 3rd problem I ask what if I wanted plane stress soln? How could I superpose on this soln another solution ∋. $\sigma_{zz} = 0$

<u>Problem #2</u>

IF $P_o = 0$ then

$$\sigma_{rr} = -\frac{P_i a^2 b^2}{b^2 - a^2}\frac{1}{r^2} + \frac{P_i a^2}{b^2 - a^2} = \frac{P_i a^2}{b^2 - a^2}\left\{1 - \frac{b^2}{r^2}\right\} \le 0 \quad \text{compressi}$$

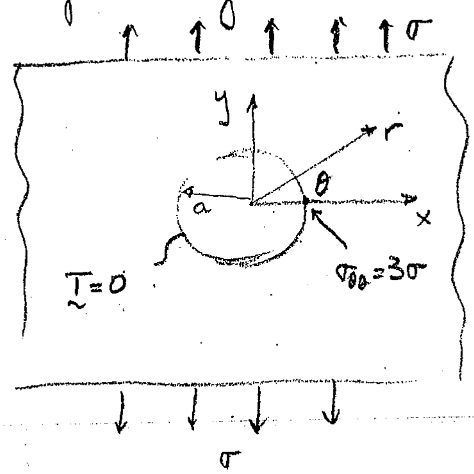
$$\sigma_{\theta\theta} = +\frac{P_i a^2 b^2}{b^2 - a^2}\frac{1}{r^2} + \frac{P_i a^2}{b^2 - a^2} = \frac{P_i a^2}{b^2 - a^2}\left\{1 + \frac{b^2}{r^2}\right\} > 0 \quad \text{tensi}$$



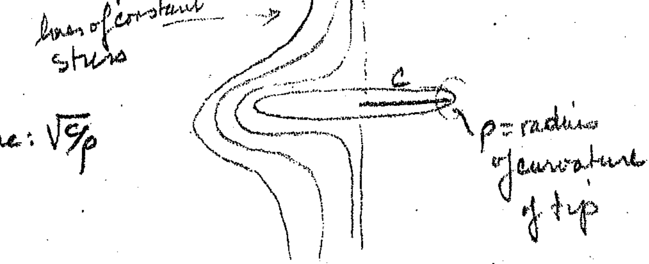
if cracks develop in radial dir } $\sigma_{\theta\theta}$ being tensile cause cracks to propagate

<u>Problem #3</u>

Look at an infinite body with a circular hole in a thick plate

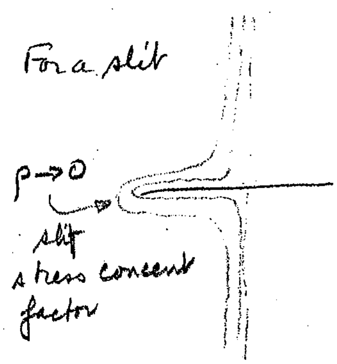


For an elliptical hole

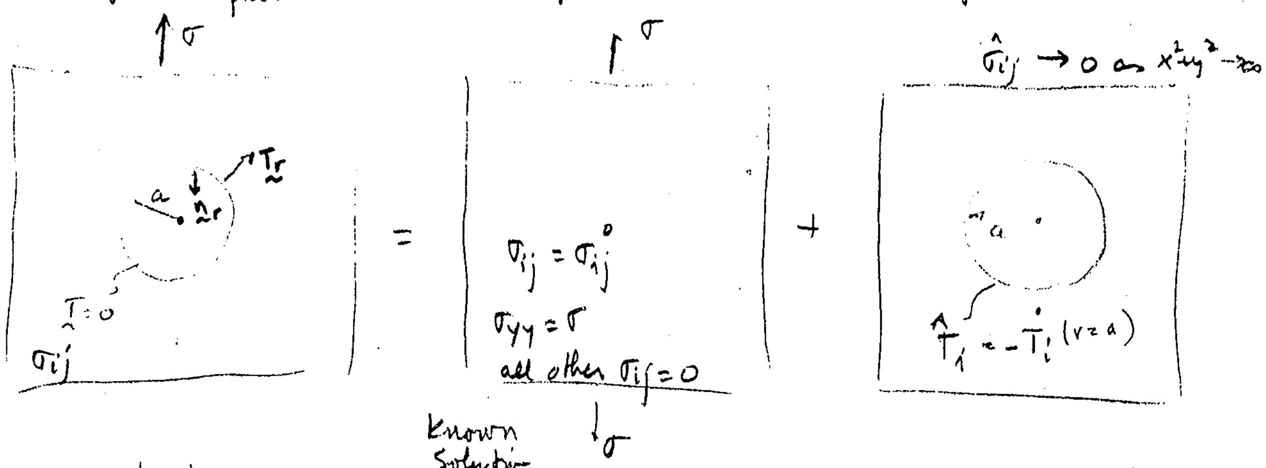


stress conc: $\sqrt{c/\rho}$

For a slit



plane strain problems = plane strain + plane strain



Stress concentration on a Circular Hole.

$$\sigma_{ij} = \sigma_{ij}^{0} + \hat{\sigma}_{ij}$$

BC: $\hat{\sigma}_{ij} \to 0$ as $x^2+y^2 \to \infty$

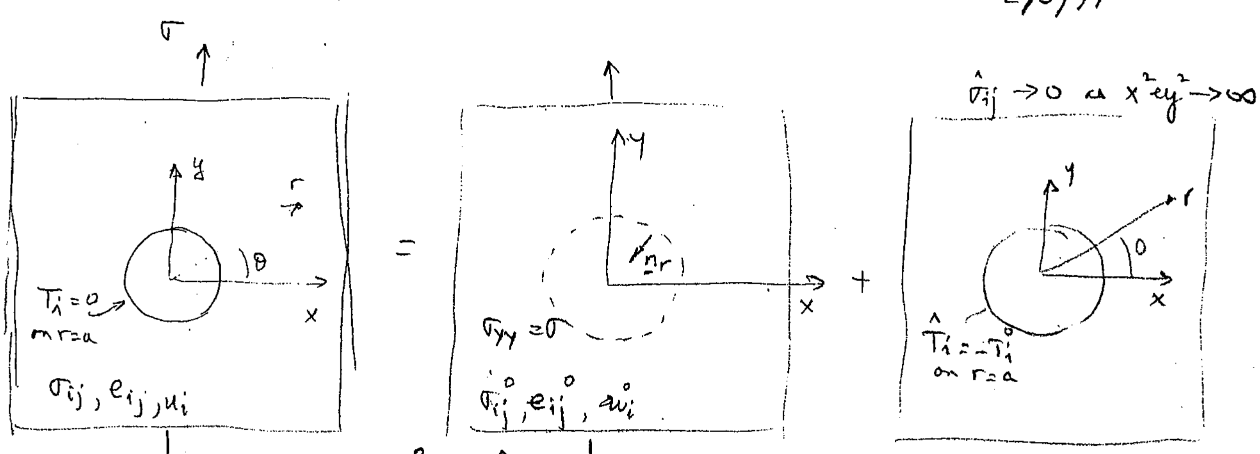
$$T_i = T_i^{0} + \hat{T}_i = 0 \implies \hat{T}_i = -T_i^{0} \text{ on } r=a$$

$\hat{T}_r = \hat{\sigma}_{rr} \overset{(-1)}{n_r} = -\hat{\sigma}_{rr} = -T_r^{0} = -\sigma_{rr}^{0} n_r = \sigma_{rr}^{0}$ $\quad\therefore\quad \hat{\sigma}_{rr} = -\sigma_{rr}^{0}$ on $r=a$

$\hat{T}_\theta$: $\quad\quad \hat{\sigma}_{r\theta} = -\sigma_{r\theta}^{0}$ on $r=a$

Stress concent on a Circular Hole

2/5/79



$$\sigma_{ij} = \sigma_{ij}^{0} + \hat{\sigma}_{ij}$$

^{0}FIELDS $\Rightarrow$ $\sigma_{yy}^{0} = \sigma \quad \sigma_{ij}^{0} = 0$ all others. $\quad$ from transformation

on $r=a$ $\left\{ \begin{aligned} T_r^{0} &= \sigma_{rr} n_r = -\sigma_{rr}^{0} = -\sigma_{yy}^{0} \sin^2\theta = -\frac{\sigma}{2}(1-\cos 2\theta) \\ T_\theta^{0} &= \sigma_{r\theta} n_r = -\sigma_{\theta r}^{0} = -\sigma_{yy}^{0} \sin\theta\cos\theta = -\frac{\sigma}{2}\sin 2\theta \end{aligned} \right.$

$\sigma_{\theta\theta}^{0} = \frac{\sigma}{2}(1+\cos 2\theta) = \sigma/2 \cos^2\theta$

$\therefore \hat{T}_r = \hat{\sigma}_{rr} \overset{(-1)}{n_r} = -T_r^{0} = \frac{\sigma}{2}(1-\cos 2\theta) \quad\quad \therefore\ \hat{\sigma}_{rr} = -\frac{\sigma}{2}(1-\cos 2\theta)$

$\hat{T}_\theta = \hat{\sigma}_{r\theta} n_r = -T_\theta^{0} = +\frac{\sigma}{2}\sin 2\theta \quad\quad \therefore\ \hat{\sigma}_{r\theta} = -\sigma/2(\sin 2\theta)$ $\Big\|$ on $r=a$

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\cdot 2\sin\theta\cos\theta$$
$$= \frac{\sigma_x+\sigma_y}{2} + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$
$$\sigma_x = 0 \quad \sigma_y = \sigma$$

Now look at the annular ring where at $r=b$ $\sigma_{ij}(r=b)=0$ & let $b\to\infty$

Go back to handout to apply bc.

$\therefore \quad \underbrace{A_0 = -\frac{\sigma}{2} \qquad A_2 = \frac{\sigma}{2}}_{\text{from } \sigma_{rr}|_{r=a}} \qquad \underbrace{D_2 = -\frac{\sigma}{2}}_{\text{from } \sigma_{r\theta}|_{r=a}}$ all others are 0.

for the stress fn. when $\sigma_{rr}\Big|_{r=a} = \dfrac{\partial^2\phi}{\partial r^2} + \dfrac{1}{r^2}\dfrac{\partial^2\phi}{\partial\theta^2}\Big|_{r=a}$

then $\quad \dfrac{a_0}{a^2} + 2b_0 = A_0 = -\dfrac{\sigma}{2}$ $\qquad \cancel{+C_0(1+\ln a)}$ must be dropped for single valued displacement.

pg 35 handout

$\dfrac{a_0}{b^2} + 2b_0 = -A_0' = 0$ $\quad$ as $b\to\infty$ $\Rightarrow b_0 = 0 \Rightarrow \boxed{a_0 = -\frac{\sigma}{2}a^2}$

(remember A_0 A_0' are for applied bc at $r=a$, $r=b\to\infty$)

now for A_2, D_2 and A_2' and $D_2' = 0$ since no stresses applied at ∞.

~~for~~

$n=2$ since $\sin 2\theta$, $\cos 2\theta$ terms in A_2, D_2

from pg 36 handout

$\sigma_{rr}|_{r=a} \quad a_2\cdot 2(1-2)a^{2-2} + b_2(2+2-4)a^2 - a_2'\,2(1+2)a^{-2-2} + b_2'(2-2-4)a^{-2} = A_2 = \frac{\sigma}{2}$

$\sigma_{rr}|_{r=b} \quad a_2\cdot 2(1-2)b^{2-2} + b_2(2+2-4)b^2 - a_2'\,2(1+2)b^{-4} + b_2'(2-2-4)b^{-2} = A_2' = 0$

as $b\to\infty$ $\quad b_2'$ term $\to 0$, a_2' term $\to 0$, b_2 term $\to 0$ since $(2+2-4)=0$

$\Rightarrow \boxed{a_2 = 0} \quad \Rightarrow$ 1st eq must reduce to $\boxed{-a_2'(6a^{-4}) + b_2'(-4)a^{-2} = \frac{\sigma}{2}}$ *

pg 37 handout

$\sigma_{r\theta}\Big|_{r=a}: \quad 2\left\{\overset{\text{from above}}{\cancel{a_2}}{}^{0} + b_2(2+1)a^2 - a_2'(2+1)a^{-4} - b_2'(2-1)a^{-2}\right\} = D_2 = -\frac{\sigma}{2}$

$\sigma_{r\theta}\Big|_{r=b}: \quad 2\left\{\cancel{a_2}{}^{0} + b_2(3)b^2 + a_2'(2+1)b^{-4} - b_2'(2-1)b^{-2}\right\} = D_2' = 0$

$\therefore \boxed{b_2 = 0}$ $\quad \to 0$ as $b\to\infty$ $\quad \to 0$ as $b\to\infty$

$\therefore \quad \boxed{-a_2'(3a^{-4}) - b_2'(1)a^{-2} = -\frac{\sigma}{4}}$ **

$$\left.\begin{aligned} \therefore\ * \Rightarrow \quad & 3\frac{a_2'}{a^4} + 2\frac{b_2'}{a^2} = -\frac{\sigma}{4} \\ ** \Rightarrow \quad & \frac{3a_2'}{a^4} + \frac{b_2'}{a^2} = \frac{\sigma}{4} \end{aligned}\right\} \Longrightarrow \boxed{b_2' = -\frac{\sigma}{2}a^2} \quad \boxed{a_2' = \frac{\sigma a^4}{4}}$$

$$\Rightarrow \hat{\sigma}_{rr} = \frac{\sigma}{2}\left[-\frac{a^2}{r^2} + \left\{4\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right\}\cos 2\theta\right]$$

$$\sigma_{rr} = \hat{\sigma}_{rr} + \overset{o}{\sigma}_{rr} = \frac{\sigma}{2}(1-\cos 2\theta) + \hat{\sigma}_{rr} = \frac{\sigma}{2}\left[1-\frac{a^2}{r^2}\right] - \frac{\sigma}{2}\left[1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right]\cos 2\theta$$

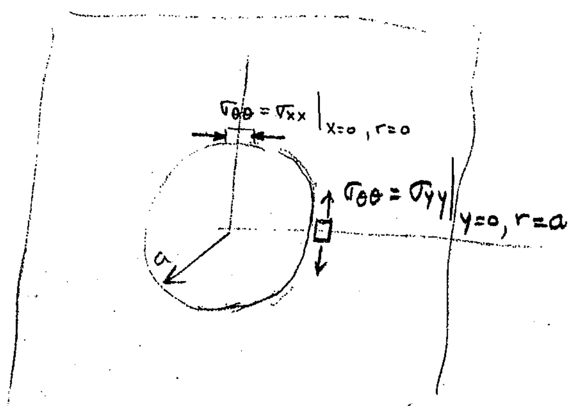
T&G pg 91 $\quad \theta_{\text{here}} = \frac{\pi}{2} - \theta_{T\&G}$

Thus $\quad \sigma_{\theta\theta} = \overset{o}{\sigma}_{\theta\theta} + \hat{\sigma}_{\theta\theta} = \frac{\sigma}{2}\left[1+\frac{a^2}{r^2}\right] + \frac{\sigma}{2}\left[1+\frac{3a^4}{r^4}\right]\cos 2\theta$

$$\sigma_{r\theta} = \frac{\sigma}{2}\left[1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right]\sin 2\theta$$

as $r \to \infty$
$$\left.\begin{aligned} \sigma_{rr} &\to \frac{\sigma}{2}[1-\cos 2\theta] \\ \sigma_{\theta\theta} &\to \frac{\sigma}{2}[1+\cos 2\theta] \\ \sigma_{r\theta} &\to \frac{\sigma}{2}[\sin 2\theta] \end{aligned}\right\} \overset{o}{\sigma}_{yy} = \sigma$$

Stress Concentration



$$\sigma_{\theta\theta}\Big|_{\substack{\theta=0\\ r=a}} = \frac{\sigma}{2}(2) + \frac{\sigma}{2}(4) = 3\sigma$$

$$\sigma_{\theta\theta}\Big|_{\substack{\theta=\pi/2\\ r=a}} = \frac{\sigma}{2}(2) - \frac{\sigma}{2}(4) = -\sigma$$

$$\sigma_r = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + \tau_{xy}\cdot 2\sin\theta\cos\theta$$
$$= \frac{\sigma_x+\sigma_y}{2} + \left(\frac{\sigma_x-\sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$
$\sigma_x = 0 \quad \sigma_y = \sigma$

$\sigma_r = \sigma_{y/2}(1-\cos 2\theta)$

Now look at the annular ring where at $r=b$ $\sigma_{ij}(r=b)=0$ & let $b\to\infty$

Go back to handout to apply bc.

$\therefore \quad \underbrace{A_0 = -\frac{\sigma}{2} \qquad A_2 = \frac{\sigma}{2}}_{\text{from } \sigma_{rr}|_{r=a}} \qquad \underbrace{D_2 = -\frac{\sigma}{2}}_{\text{from } \sigma_{r\theta}|_{r=a}}$ all others are 0.

for the stress fn. when $\sigma_{rr}\Big|_{r=a} = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\Big|_{r=a}$

pg 35 handout then $\frac{a_0}{a^2} + 2b_0 = A_0 = -\frac{\sigma}{2}$ $\cancel{+c_0(1+\ln a)}$ must be dropped for single valued displacement.

$\frac{a_0}{b^2} + 2b_0 = -A_0' = 0$ as $b\to\infty \Rightarrow b_0 = 0 \Rightarrow \boxed{a_0 = -\frac{\sigma}{2}a^2}$

(remember A_0 A_0' are from applied bc at $r=a$, $r=b\to\infty$)

now for A_2, D_2 and A_2' and $D_2' = 0$ since no stresses applied at ∞.

for $n=2$ since $\sin 2\theta$, $\cos 2\theta$ terms in A_2, D_2

from pg 36 handout

$\sigma_{rr}|_{r=a} \quad a_2\cdot 2(1-2)a^{2-2} + b_2(2+2-4)a^2 - a_2'\,2(1+2)a^{-2-2} + b_2'(2-2-4)a^{-2} = A_2 = \frac{\sigma}{2}$

$\sigma_{rr}|_{r=b} \quad a_2\cdot 2(1-2)b^{2-2} + b_2(2+2-4)b^2 - a_2'\,2(1+2)b^{-4} + b_2'(2-2-4)b^{-2} = A_2' = 0$

as $b\to\infty$ b_2' term $\to 0$, a_2' term $\to 0$, b_2 term $\to 0$ since $(2+2-4)=0$

$\Rightarrow \boxed{a_2 = 0} \Rightarrow 1^{st}$ eq must reduce to $\boxed{-a_2'(6a^{-4}) + b_2'(-4)a^{-2} = \frac{\sigma}{2}}$ *

pg 37 handout

$\sigma_{r\theta}\Big|_{r=a}: \quad 2\left\{\overset{\text{from above}}{\cancel{a_2}}{}^{0} + b_2(2+1)a^2 - a_2'(2+1)a^{-4} - b_2'(2-1)a^{-2}\right\} = D_2 = -\frac{\sigma}{2}$

$\sigma_{r\theta}\Big|_{r=b}: \quad 2\left\{\cancel{a_2}{}^{0} + b_2(3)b^2 + a_2'(2+1)b^{-4} - b_2'(2-1)b^{-2}\right\} = D_2' = 0$

$\therefore \boxed{b_2 = 0}$ $\to 0$ as $b\to\infty$ $\to 0$ as $b\to\infty$

$\therefore \quad \boxed{-a_2'(3a^{-4}) - b_2'(1)a^{-2} = -\frac{\sigma}{4}}$ **

$$\left.\begin{aligned} \therefore * &\Rightarrow 3\frac{a_2'}{a^4} + 2\frac{b_2'}{a^2} = -\frac{\sigma}{4} \\ ** &\Rightarrow \frac{3a_2'}{a^4} + \frac{b_2'}{a^2} = \frac{\sigma}{4} \end{aligned}\right\} \Rightarrow \boxed{b_2' = -\frac{\sigma}{2}a^2} \quad \boxed{a_2' = \frac{\sigma a^4}{4}}$$

$$\Rightarrow \hat{\sigma}_{rr} = \frac{\sigma}{2}\left[-\frac{a^2}{r^2} + \left\{4\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right\}\cos 2\theta\right]$$

$$\sigma_{rr} = \hat{\sigma}_{rr} + \overset{\circ}{\sigma}_{rr} = \frac{\sigma}{2}(1-\cos 2\theta) + \hat{\sigma}_{rr} = \frac{\sigma}{2}\left[1 - \frac{a^2}{r^2}\right] - \frac{\sigma}{2}\left[1 - 4\frac{a^2}{r^2} + 3\frac{a^4}{r^4}\right]\cos 2\theta$$

T&G pg 91 ... $\theta_{here} = \frac{\pi}{2} - \theta_{T\&G}$ also look at your books pg 208

there b = to our a

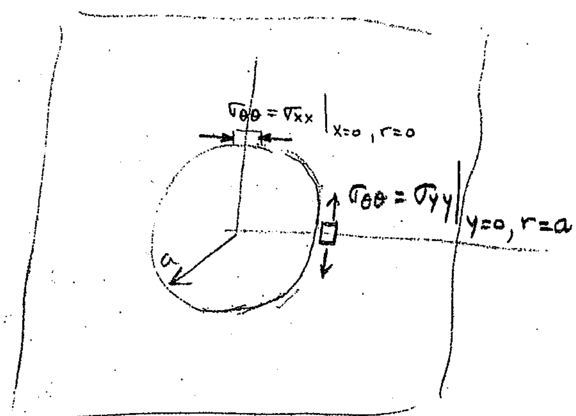
~~thete~~ θ = our $\theta - \pi/2$ there

Thus $\sigma_{\theta\theta} = \overset{\circ}{\sigma}_{\theta\theta} + \hat{\sigma}_{\theta\theta} = \frac{\sigma}{2}\left[1 + \frac{a^2}{r^2}\right] + \frac{\sigma}{2}\left[1 + \frac{3a^4}{r^4}\right]\cos 2\theta$

$$\sigma_{r\theta} = \frac{\sigma}{2}\left[1 + 2\frac{a^2}{r^2} - 3\frac{a^4}{r^4}\right]\sin 2\theta$$

as $r \to \infty$
$$\left.\begin{array}{l} \sigma_{rr} \to \frac{\sigma}{2}\left[1 - \cos 2\theta\right] \\ \sigma_{\theta\theta} \to \frac{\sigma}{2}\left[1 + \cos 2\theta\right] \\ \sigma_{r\theta} \to \frac{\sigma}{2}\left[\sin 2\theta\right] \end{array}\right\} \overset{\circ}{\sigma}_{yy} = \sigma$$

Stress Concentration



$$\sigma_{\theta\theta}\Big|_{\substack{\theta=0 \\ r=a}} = \frac{\sigma}{2}(2) + \frac{\sigma}{2}(4) = 3\sigma$$

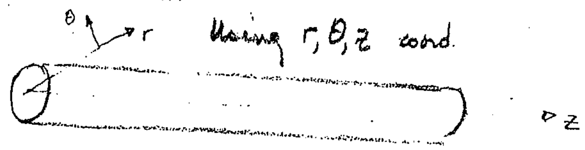
$$\sigma_{\theta\theta}\Big|_{\substack{\theta=\pi/2 \\ r=a}} = \frac{\sigma}{2}(2) - \frac{\sigma}{2}(4) = -\sigma$$

We next take up torsion of beams.

<u>Coulomb Torsion</u>

Check this again

Using r, θ, z coord.



Coulomb assumed a displacement solution & checked the compat, equil, etc.

He assumed $u_r = 0$, $u_z = 0$, $u_\theta = \alpha r z$ where α = proportionality factors.

$\Rightarrow$ $\varepsilon_{rr} = 0$, $\varepsilon_{\theta\theta} = 0$, $\varepsilon_{zz} = 0$, $\varepsilon_{r\theta} = 0$, $\varepsilon_{zr} = 0$, $\varepsilon_{\theta z} = \frac{1}{2}\alpha r$

let's get the ~~strains~~ stresses

$\sigma_{ij} = \lambda \varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij} \Rightarrow \sigma_{\theta z} = \mu\alpha r$ only non vanishing shear

with this $\sigma_{\theta z}$ we satisfy equil. We also assume torsion is <u>end</u> loaded <u>not</u> surface area loaded.

BC Cylindrical surface free of traction ie $e_r \cdot \sigma \overset{?}{=} 0$

but $e_r \cdot \sigma = (\sigma_{r\theta}, \sigma_{rz}, \sigma_{rr}) = 0$ thus we satisfy the BC on surface

on the ends $T = \int_0^a\int_0^{2\pi} \sigma_{\theta z} \cdot r \cdot r\,dr\,d\theta = \int_0^a\int_0^{2\pi} \mu\alpha r^3\,dr\,d\theta = \frac{\pi a^4}{2}\mu\alpha$

define polar moment of inertia $J = \iint r^2\,dA = \frac{\pi a^4}{2}$ $\therefore$ $T = J\mu\alpha$ (torque)

load on the end $e_z \cdot \sigma = (\sigma_{zr}, \sigma_{z\theta}, \sigma_{zz}) = \sigma_{\theta z}$

shear go in opposite direction opposite sides $\therefore$ they cancel out

$\omega_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial(r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] = \alpha z$ α (is the twist) rotation / unit of length of cylinder

$\alpha = \frac{\partial}{\partial z}(\omega_{r\theta})$

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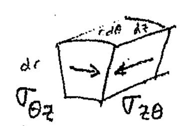
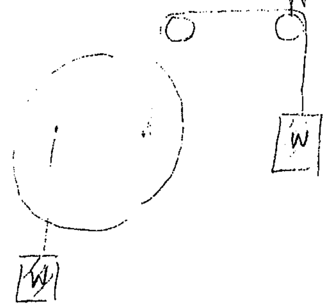
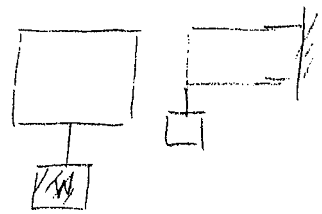
each cross section rotates as a body in Coulomb torsion **No warping**



Suppose we look at different problems



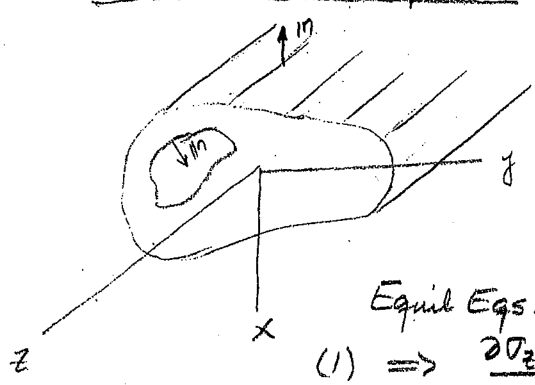
such that this system $\ni$. T. Is this system the same as the last torsion problem



If $W = P$ is this system the same as a uniformly end loaded beam such that $\int \sigma_{xz} dA = P$.

The answer is given by St. Venant's principle. If the two systems are statically equivalent then the stress distribution will be the same everywhere *except* near the loading point (points, faces etc) where the stress distribution produces a local effect. (This has not been proven – there are contradictive case.

St. Venant Torsion problem. No tractions on surface of the body except on the ends.



Let $\sigma_{zz}, \sigma_{xx}, \sigma_{yy}, \sigma_{xy} = 0$ and no body forces

Let $\sigma_{zx} \neq 0$; $\sigma_{zy} \neq 0$

Equil Eqs:

$$(1) \Rightarrow \frac{\partial \sigma_{zx}}{\partial z} = 0 \Rightarrow \sigma_{zx} = \sigma_{zx}(x,y)$$

$$(2) \Rightarrow \frac{\partial \sigma_{zy}}{\partial z} = 0 \Rightarrow \sigma_{zy} = \sigma_{zy}(x,y)$$

$$(3) \Rightarrow \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} = 0$$

Consider $\phi = \phi(x,y)$ such that $\sigma_{xz} = \dfrac{\partial \phi}{\partial y}$ $\sigma_{zy} = -\dfrac{\partial \phi}{\partial x}$

Hence we let $\phi(x,y)$ Stress function for torsion.

~~on cylindrical surface~~

B.C. on free surface
for any surface $\mathbb{n}\cdot\sigma = 0$

$$\mathbb{n}\cdot\left[\sigma_{zx}\,\mathbb{e}_z\mathbb{e}_x + \sigma_{zy}\,\mathbb{e}_z\mathbb{e}_y + \sigma_{zx}\,\mathbb{e}_x\mathbb{e}_z + \sigma_{zy}\,\mathbb{e}_y\mathbb{e}_z\right] = 0$$

BC for cylindrical surface

for a cylindrical surface $\mathbb{n}\cdot\mathbb{e}_z = 0$

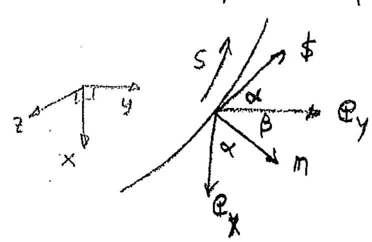
$\uparrow \mathbb{n}$

$\mathbb{e}_z$

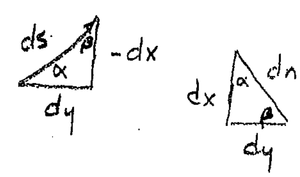
$$\therefore\ \mathbb{n}\cdot\sigma = 0 \Rightarrow \left(\mathbb{n}\cdot\mathbb{e}_x\,\sigma_{zx} + \mathbb{n}\cdot\mathbb{e}_y\,\sigma_{zy}\right)\mathbb{e}_z = 0$$

or $\quad \mathbb{n}\cdot\mathbb{e}_x\,\sigma_{zx} + \mathbb{n}\cdot\mathbb{e}_y\,\sigma_{zy} = 0$

Look at the following section of the surface



$$\mathbb{n}\cdot\mathbb{e}_x = \$\cdot\mathbb{e}_y = \frac{dy}{ds} = \frac{dx}{dn}$$
$= \cos\alpha$



$$\mathbb{n}\cdot\mathbb{e}_y = \$\cdot(-\mathbb{e}_x) = -\frac{dx}{ds} = \frac{dy}{dn}$$
$= \cos\beta$

$$\therefore\ \mathbb{n}\cdot\mathbb{e}_x\,\sigma_{xz} + \mathbb{n}\cdot\mathbb{e}_y\,\sigma_{zy} = 0$$

$$\frac{dy}{ds}\cdot\frac{\partial\phi}{\partial y} + -\frac{dx}{ds}\,-\frac{\partial\phi}{\partial x} = \frac{d\phi}{ds} = 0 \Rightarrow \phi = \text{constant along the}$$

boundary or along a contour line.

BC. at the end faces $\quad \mathbb{n}\cdot\sigma = \pm\,\mathbb{e}_z\cdot\sigma = \pm\left[\sigma_{zx}\mathbb{e}_x + \sigma_{zy}\mathbb{e}_y\right]$

Now on each end we want the resultant forces. we will show them to be $\equiv 0$

$$\therefore\ \iint_A \sigma_{zx}\,dA = \iint_A \frac{\partial\phi}{\partial y}\,dA = F_x$$

$$\iint_A \sigma_{zy}\,dA = -\iint \frac{\partial\phi}{\partial x}\,dA = F_y$$

Recalling green's theorem

$$\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \oint_{\text{boundary}} P\,dx + Q\,dy$$

$\Rightarrow$ let $Q = 0 \quad -P = \phi \quad$ for F_x

$\phi = Q \quad P = 0 \quad$ for F_y

Evaluation of

$F's \Rightarrow -\oint_{\text{boundary}} \phi\, dx = A_1 \cdot \oint dx;\ -\oint_{\text{boundary}} \phi\, dy = A_1 \cdot \oint dy$ but $\oint dx = 0$ $\oint dy = 0$ since ϕ = const on surface $\Longrightarrow F_x = F_y = 0$

On the end faces there will be no resultant force

Resultant torque

$$\mathbb{T} = \iint \mathbb{r} \times (\mathbb{n}\cdot\sigma)\, dA \qquad \mathbb{r}\times(\mathbb{n}\cdot\sigma) = \begin{vmatrix} e_x & e_y & e_z \\ x & y & 0 \\ \sigma_{zx} & \sigma_{zy} & 0 \end{vmatrix}$$

$$\mathbb{r} = x e_x + y e_y$$

$$\mathbb{r}\times(\mathbb{n}\cdot\sigma) = \pm\left[x\sigma_{zy} - y\sigma_{zx}\right] e_z$$

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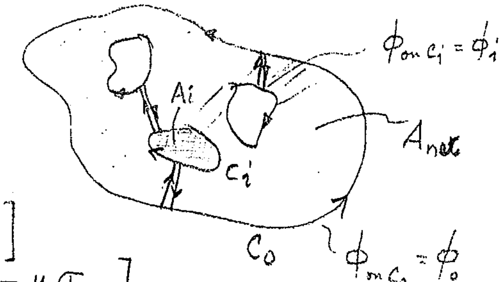
Resultant Torque

$$\mathbb{T} = \iint_A \mathbb{r}\times(\mathbb{n}\cdot\sigma\, dA)$$



$$\mathbb{r} = x e_x + y e_y$$

$$\pm e_z \cdot \mathbb{T} = \quad \mathbb{n}\cdot\sigma = \pm\left[e_x \sigma_{zx} + e_y \sigma_{zy}\right]$$

$$\mathbb{r}\times(\mathbb{n}\cdot\sigma) = \pm e_z\left[x\sigma_{zy} - y\sigma_{zx}\right]$$

$$\therefore\ T = \int_A (x\sigma_{zy} - y\sigma_{zx})\, dA = -\iint\left(x\frac{\partial\phi}{\partial x} + y\frac{\partial\phi}{\partial y}\right)dx\,dy$$

Now $\frac{\partial}{\partial x}(\phi x) = 1\cdot\phi + x\frac{\partial\phi}{\partial x}$

we rewrite

$$T = \iint_A \left[2\phi - \frac{\partial}{\partial x}(\phi x) - \frac{\partial}{\partial y}(\phi y)\right]dx\,dy = \iint_A \left[2\phi + \frac{\partial}{\partial x}(-x\phi) - \frac{\partial}{\partial y}(y\phi)\right]dA$$

now we use green's theorem $\iint_A\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \oint_C P\,dx + Q\,dy$

$$T = \iint_{A_{net}} 2\phi\, dA - \oint_{C_0}\phi_0\, x\, dy + \oint_{C_0}\phi_0\, y\, dx - \sum_i \phi_i \oint_{C_i} x\, dy + \sum_i \phi_i \oint_{C_i} y\, dx$$

$$-\phi_0\oint_{C_0}\left[x\,dy - y\,dx\right] \qquad -\sum\phi_i\left[\oint_{C_i}(x\,dy - y\,dx)\right]$$

and by using greens theorem in reverse

$$-\phi_0\iint_{A_{gross}} 2\, dA \qquad -\sum\phi_i\iint_{A_i} -2\, dA$$

since this is a clockwise integration of Green's theorem

$$T = 2\left[\iint_{A_{net}}\phi\, dA - \phi_0 A_{gross} + \sum_i \phi_i A_i\right]$$ where $A_{gross} = A_{net} + \sum A_i$

Strains and Displacements – we will now get strain and displacement solutions

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij}$$

$$\varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = \varepsilon_{xy} = 0 \quad \text{and} \quad \varepsilon_{yz} = \frac{1+\nu}{E}\sigma_{zy}, \quad \varepsilon_{zx} = \frac{1+\nu}{E}\sigma_{zx} = f(x,y) \text{ only}$$

since $\sigma_{xx} = \sigma_{yy} = \sigma_{zz} = 0$ & $\sigma_{xy} = 0$

$$\varepsilon_{ij} = \tfrac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad \varepsilon_{xx} = \frac{\partial u_x}{\partial x} = 0 \Rightarrow u_x = u_x(y,z)$$

$$\varepsilon_{yy} = \frac{\partial u_y}{\partial y} = 0 \Rightarrow u_y = u_y(x,z)$$

$$\varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0 \Rightarrow u_z = u_z(x,y)$$

$$\gamma_{xy} = 2\varepsilon_{xy} = \left(u_{x,y} + u_{y,x}\right) = 0 \Rightarrow u_{x,y} = -f(z) \text{ only} \qquad u_{y,x} = +f(z)$$

since $\frac{\partial u_x}{\partial y}$ & $\frac{\partial u_y}{\partial x}$ only has z as common term then let $u_{xy} =$

integration of $u_{x,y}$ and $u_{y,x}$ gives

$$\therefore \quad u_y(x,z) = x f(z) + g_y(z) \qquad u_x(y,z) = -y f(z) + g_x(z)$$

look at $\frac{\partial u_x}{\partial z} = 2\varepsilon_{zx} - \frac{\partial u_z}{\partial x}$ (note that $\varepsilon_{zx} = \frac{1}{2}\left(u_{x,z} + u_{z,x}\right)$) Now take $\frac{\partial}{\partial z}$ of this equation

$$\frac{\partial^2 u_x}{\partial z^2} = \frac{\partial}{\partial z}\left(2\varepsilon_{zx} - \frac{\partial u_z}{\partial x}\right)$$

since $u_z = u_z(x,y)$ and $\frac{\partial}{\partial z}u_z = 0$ and since $\sigma_{zx} = \sigma_{zx}(x,y)$ only $\Rightarrow \varepsilon_{zx} = \varepsilon_{zx}(x,y)$ only and $\frac{\partial}{\partial z}\varepsilon_{zx} \equiv 0$.

$$\therefore \frac{\partial^2 u_x}{\partial z^2} = 0 = -y f'' + g_x'' = 0 \xRightarrow{\text{true for any } y\ \therefore} f'' = 0 \text{ and } g_x'' = 0 \quad \forall y$$

hence $f(z) = az + b \qquad g_x(z) = cz + d$.

We can get an analogous result for ε_{yz}: since $\varepsilon_{zy} = \varepsilon_{zy}(x,y)$ from fact that $\sigma_{zy}(x,y)$ only $\therefore \frac{\partial \varepsilon_{zy}}{\partial z} = 0$ and $\frac{\partial^2 u_z}{\partial y \partial z} = 0 \Rightarrow \qquad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)$

$$u_y = axz + bx + ez + f \qquad u_x = -ayz - by + cz + d.$$

true no matter what X-section looks like

B.C. are used to get the unknowns a, b, c, d, e & f

At the origin as a reference point $u_x, u_y = 0 \Rightarrow f, d = 0$ No r.b. transl.

At origin slopes wrt $z = 0$ $\quad u_{x,z} = u_{y,z} = 0 \Rightarrow c, e = 0$ No. r.b. rot

Specify that rotation about the z-axis will be measured from the origin $\omega_{xy}\big|_{z=0} = 0$

$$\omega_{xy}\Big|_{z=0} = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) = \frac{1}{2}\left(az+b+az+b\right) = az+b\Big|_{z=0} = 0 \Rightarrow b=0$$

Let $a = \alpha$ (the twist) hence we go to displ. eqns and finally get.

$$u_y = \alpha x z \qquad u_x = -\alpha y z$$

$$-\frac{\partial \phi}{\partial x} = \tau_{yz} = G\gamma_{yz} = 2G\varepsilon_{yz}$$

note that $\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} = \overset{\gamma_{yz}}{\overset{\parallel}{2\varepsilon_{yz}}} = -2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial x} = \frac{\sigma_{yz}}{G}$

$$\frac{\partial u_z}{\partial y} = -2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial x} - \alpha x \qquad \left(\text{since } -\frac{\partial u_y}{\partial z} = -\alpha x\right) \quad \text{from } \gamma_{yz}$$

also $\frac{\partial u_z}{\partial x} = 2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial y} + \alpha y \qquad \left[\epsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)\right]$ for γ_{zx}

$\frac{\partial u_z}{\partial z} = 0$ since $u_z = u_z(x,y)$ only

we can now get u_z by integrating

Since we have a multiply connected region we must use Cesaro's theorem. ie displ must be singlevalued!

$\therefore$ $u_x = u_x(y,z)$

$$\oint du = 0 \Rightarrow \oint \underbrace{\frac{\partial u_x}{\partial x}}_{0} dx + \frac{\partial u_x}{\partial y} dy + \frac{\partial u_x}{\partial z} dz = \oint du_x = 0$$

$$-\alpha z\, dy - \alpha y\, dz = -\alpha \oint z\,dy + y\,dz = -\alpha \cdot 0 = 0$$

$\iint (1-1)\,dx\,dy$ by Green's theorem

hence by Cesaro's theorem the displ are single valued in x direction

we can do same in y direction $\quad u_y = u_y(x,z)$

$$\oint \frac{\partial u_y}{\partial x} dx + \oint \cancel{\frac{\partial u_y}{\partial y}} dy + \oint \frac{\partial u_y}{\partial z} dz = +\alpha \oint (z\,dx + x\,dz) = 0 = du_y = 0$$

$\alpha z\,dx + \alpha x\,dz$

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We have found so far $u_x = -\alpha y z$ and $u_y = \alpha x z$

now
$$\frac{\partial u_z}{\partial y} = -2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial x} - \alpha x$$

$$\frac{\partial u_z}{\partial x} = 2\left(\frac{1+\nu}{E}\right)\frac{\partial \phi}{\partial y} + \alpha y$$

$$\frac{\partial u_z}{\partial z} = 0$$

For single valuedness of u_z

$$\oint_c du_z = \oint_c \left(\frac{\partial u_z}{\partial x} dx + \frac{\partial u_z}{\partial y} dy + \frac{\partial u_z}{\partial z} dz \right) = \oint_{C_0} du_z + \sum_i \oint_{C_i} du_z$$

$u_z = u_z(x,y)$

$$= \oint_c \left[\left(\frac{1}{\mu} \frac{\partial \phi}{\partial y} + \alpha y \right) dx - \left(\frac{1}{\mu} \frac{\partial \phi}{\partial x} + \alpha x \right) dy \right]$$

(P, Q)

using ~~[illegible]~~ Green's Thm the line integral is $\oint_c Q dy + P dx = \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$

$$-\iint_{A_{gross}} \left[\left(\frac{1}{\mu} \frac{\partial^2 \phi}{\partial x^2} + \alpha \right) + \left(\frac{1}{\mu} \frac{\partial^2 \phi}{\partial y^2} + \alpha \right) \right] dA - \underbrace{\sum_{i=0} \oint_{C_i} du_z}_{\text{cavities}}$$

($\partial Q/\partial x$, $\partial P/\partial y$)

For compatibility we must have that integrand $= 0$

$$\frac{1}{\mu} \nabla_1^2 \phi + 2\alpha = 0 \quad \text{and} \quad \oint_{C_i} du_z = 0$$

or $\boxed{\nabla_1^2 \phi = -2\mu\alpha}$

$\nabla_1^2 \phi = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$
but $\nabla^2 \phi = \nabla_1^2 \phi$ since $\phi = \phi(x,y)$ only

Now let us look at $\oint_{C_i} du_z = 0$

$$\oint_{C_i} du_z = \frac{1}{\mu} \oint \left(\frac{\partial \phi}{\partial y} dx - \frac{\partial \phi}{\partial x} dy \right) + \alpha \oint (y dx - x dy) = 0$$

($\frac{\partial u_z}{\partial x} dx$, $\frac{\partial u_z}{\partial y} dy$)

From our previous lectures we related m, s, e_x, e_y at the boundary so that

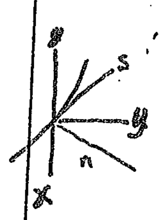
$$\frac{dy}{ds} = \frac{dx}{dn} \ ①, \quad -\frac{dx}{ds} = \frac{dy}{dn} \ ②$$

solve for dy & dx in ① & ② respectively

hence the first part of the integral

$$\oint \frac{\partial \phi}{\partial y} dx - \frac{\partial \phi}{\partial x} dy = \oint \left(-\frac{\partial \phi}{\partial y} \frac{dy}{dn} - \frac{\partial \phi}{\partial x} \frac{dx}{dn} \right) ds = \oint -\frac{\partial \phi}{\partial n} ds$$

$$\oint y dx - x dy = -\oint (y dx - x dy) = 2 \oint_{A_i} dA = 2A_i$$



$$\therefore \oint_{C_i} du_z = -\frac{1}{\mu} \oint \frac{\partial \phi}{\partial n} ds + 2\alpha A_i = 0 \quad \therefore \quad \boxed{\oint_{C_i} \frac{\partial \phi}{\partial n} ds = 2\alpha A_i \mu}$$

on cavity bdy

$$\frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial x} \frac{dx}{dn} + \frac{\partial \phi}{\partial y} \frac{dy}{dn}$$

$$-\sigma_{zy} \frac{dx}{dn} + \sigma_{zx} \frac{dy}{dn} = -s \cdot u_z$$

shear stress along the contour

$m \cdot e_x \quad m \cdot e_y$

$s \cdot e_y \quad -s \cdot e_x$

$$\oint \frac{\partial \phi}{\partial n} ds \quad \oint \tau ds = 2\alpha A_i \mu$$

$\tau = \sqrt{\sigma_{zx}^2 + \sigma_{zy}^2}$

Summary of the St Venant Torsion

<u>Equilib</u> - automatically satisfied by introducing $\phi(x,y) \ni \sigma_{zx} = \frac{\partial \phi}{\partial y}$, $\sigma_{zy} = -\frac{\partial \phi}{\partial x}$

<u>B.C.</u> we assumed that surface tractions on the surface = 0

$\phi = K_i$ (const) $(i = 0, 1, \ldots, N)$ — K_0 on outer bdy, K_1 on 1st cavity etc.

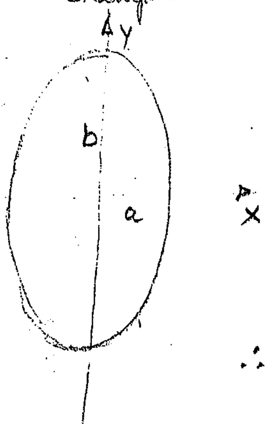
<u>Torque</u> $T = 2\left[\iint_{A_{net}} \phi \, dA - \underbrace{K_0 A_0}_{\text{gross area}} + \sum_{i=1}^{N} K_i A_i\right] \ni A_0 - \sum_i A_i = A_{net}$

<u>Compatibility</u> $\nabla^2 \phi = -2\mu\alpha$

<u>Additional conditions for shaft w/ cavities</u> $\oint_{C_i} \frac{\partial \phi}{\partial n} ds = 2\mu A_i \alpha$

<u>Displacements</u> $u_x = -\alpha y z \quad u_y = \alpha x z \quad u_z = f(x,y)$ this warping fn.

1st Example Torsion of a bar of elliptical cross section w/ <u>NO</u> cavities



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ Eqn of contour w/ $b > a$

since ϕ = const on the ellipse surface and since $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ on contour let us take $\phi = B\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

$\therefore$ Try $\phi = B\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$ const on bdy

$\therefore$ since $\nabla^2 \phi = -2\mu\alpha = B\left[\frac{2}{a^2} + \frac{2}{b^2}\right] \quad \therefore \quad \boxed{B = -\frac{\mu\alpha a^2 b^2}{a^2 + b^2}}$

$T = 2\left[\iint \phi \, dA - k_0 A_0\right] = 2\iint \phi \, dA$ since $k_0 = 0$ on bdy

$$T \overset{(*2)}{=} 2\iint \phi \, dA = -B\pi ab = \overset{(*1)}{\frac{\pi a^3 b^3 \mu\alpha}{a^2 + b^2}} = D\alpha$$

D ~ torsional stiffness $= \mu\alpha J$

J = polar moment of inertia about z axis

12/1/78

Final Exam : is due Monday the 11^{th} of Dec — will be given out next week

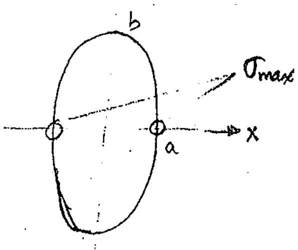
$-\frac{2y\beta}{b^2} = \frac{2y}{b^2}\cdot\frac{T}{\pi ab}$ $\qquad$ $-2x\beta/a^2 = \frac{2x}{a^2}\cdot\frac{T}{\pi ab}$

$$\sigma_{zx} = \frac{\partial\phi}{\partial y} = \frac{-2Ty}{\pi a b^3} \qquad \sigma_{zy} = \frac{\partial\phi}{\partial x} = \frac{2Tx}{\pi a^3 b}$$

$$\sigma_{res} = \left(\sigma_{zx}^2 + \sigma_{zy}^2\right)^{1/2} = \frac{2T}{\pi ab}\left(\frac{y^2}{b^4} + \frac{x^2}{a^4}\right)^{1/2} = \frac{2T}{\pi a^3 b^3}\left(b^4x^2 + a^4y^2\right)^{1/2}$$

$$\boxed{\sigma_{max}\ \left(@\ \begin{matrix} x = \pm a \\ y = 0\end{matrix}\right) = \frac{2T}{\pi a^2 b}}$$

$\therefore$ if yield occurs: at pts on surface where it is closest to origin



Now $\quad u_x = -\alpha yz = -\dfrac{T(a^2+b^2)}{\pi a^3 b^3 \mu}\, yz$ $\qquad$ α obtained from (#1), (#

$\quad u_y = \alpha xz = \dfrac{T(a^2+b^2)}{\pi a^3 b^3 \mu}\, xz$

$\gamma_{zx} = \frac{\tau_{zx}}{G} = \frac{1}{G}\frac{\partial\phi}{\partial y}$

$\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}$

$+ (-\alpha y)$

solve for $\frac{\partial u_z}{\partial x} = \frac{1}{G}\frac{\partial\phi}{\partial y} + \alpha y$

from our summary $\quad \dfrac{\partial u_z}{\partial x} = \dfrac{1}{\mu}\dfrac{\partial\phi}{\partial y} + \alpha y = \dfrac{1}{\mu}\left(\dfrac{-2Ty}{\pi a b^3}\right) + \left(+\underbrace{\dfrac{T(a^2+b^2)}{\pi a^3 b^3 \mu}}_{=\alpha}\right) y$

$$\boxed{\frac{\partial u_z}{\partial x} = \frac{b^2-a^2}{b^2+a^2}\,\alpha y}$$

$$\frac{\partial u_z}{\partial y} = -\left(\frac{1}{\mu}\frac{\partial\phi}{\partial x} + \alpha x\right) = \frac{b^2-a^2}{b^2+a^2}\,\alpha x$$

integrate

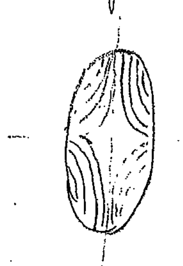
Now $\quad u_z = \dfrac{b^2-a^2}{b^2+a^2}\,\alpha xy + f_1(y)$ $\qquad$ since u_z is $f(x,y)$ integrate

$\quad u_z = \dfrac{b^2-a^2}{b^2+a^2}\,\alpha xy + f_2(x)$

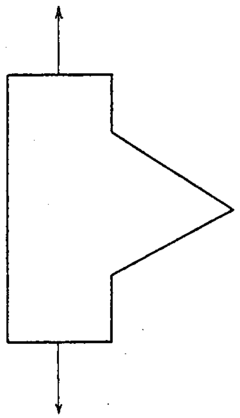
$\therefore \quad f_1(y) = f_2(x) = \text{const.}$ $\quad$ represents a rigid body disp in the z direct

$\therefore \quad u_z = \dfrac{b^2-a^2}{b^2+a^2}\,\alpha xy$

warping function



8. The figure represents a "tooth" on a plate in a state of plane stress in the plane of the paper. The faces of the tooth (the two straight lines) are free from force. On the supposition that the stress components are all finite and continuous throughout the region, prove that there is no stress at all at the apex of the tooth.



chapter 3

Two-dimensional Problems in Rectangular Coordinates

18 | Solution by Polynomials

It has been shown that the solution of two-dimensional problems, when body forces are absent or are constant, is reduced to the integration of the differential equation

$$\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0 \tag{a}$$

having regard to boundary conditions (20). In the case of long rectangular strips, solutions of Eq. (a) in the form of polynomials are of interest. By taking polynomials of various degrees, and suitably adjusting their coefficients, a number of practically important problems can be solved.[1]

Beginning with a polynomial of the second degree

$$\phi_2 = \frac{a_2}{2}x^2 + b_2xy + \frac{c_2}{2}y^2 \tag{b}$$

which evidently satisfies Eq. (a), we find from Eqs. (29), putting $\rho g = 0$,

$$\sigma_x = \frac{\partial^2 \phi_2}{\partial y^2} = c_2 \qquad \sigma_y = \frac{\partial^2 \phi_2}{\partial x^2} = a_2 \qquad \tau_{xy} = -\frac{\partial^2 \phi_2}{\partial x \, \partial y} = -b_2$$

All three stress components are constant throughout the body, i.e., the stress function (b) represents a combination of uniform tensions or compressions[2] in two perpendicular directions and a uniform shear. The

[1] A. Mesnager, *Compt. Rend.*, vol. 132, p. 1475, 1901. See also A. Timpe, *Z. Math. Physik*, vol. 52, p. 348, 1905.

[2] The arrows in Fig. 21 are all drawn in the standard sense, as defined in Art. 3. The numbers a_2, $-b_2$, c_2 attached to them may be positive or negative. Thus all possibilities can be covered without changing the directions of the arrows. In Fig. 22, however, the arrows show directly the intended directions of the applied forces.

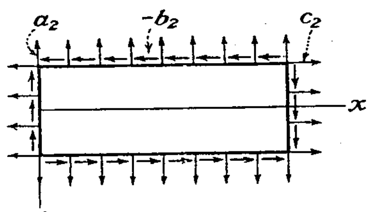


Fig. 21

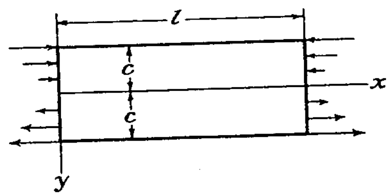


Fig. 22

forces on the boundaries must equal the stresses at these points as discussed on page 28; in the case of a rectangular plate with sides parallel to the coordinate axes, these forces are shown in Fig. 21.

Let us consider now a stress function in the form of a polynomial of the third degree:

$$\phi_3 = \frac{a_3}{3(2)} x^3 + \frac{b_3}{2} x^2 y + \frac{c_3}{2} x y^2 + \frac{d_3}{3(2)} y^3 \qquad (c)$$

This also satisfies Eq. (*a*). Using Eqs. (29) and putting $\rho g = 0$, we find

$$\sigma_x = \frac{\partial^2 \phi_3}{\partial y^2} = c_3 x + d_3 y$$

$$\sigma_y = \frac{\partial^2 \phi_3}{\partial x^2} = a_3 x + b_3 y$$

$$\tau_{xy} = -\frac{\partial^2 \phi_3}{\partial x\, \partial y} = -b_3 x - c_3 y$$

For a rectangular plate, taken as in Fig. 22, assuming all coefficients except d_3 equal to zero, we obtain pure bending. If only coefficient a_3 is different from zero, we obtain pure bending by normal stresses applied to the sides $y = \pm c$ of the plate. If coefficient b_3 or c_3 is taken different from zero, we obtain not only normal but also shearing stresses acting on the sides of the plate. Figure 23 represents, for instance, the case in

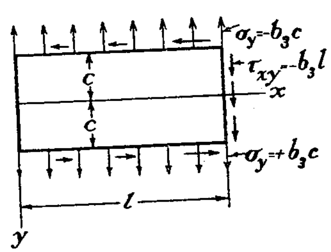


Fig. 23

which all coefficients except b_3 in function (*c*) are equal to zero. Along t sides $y = \pm c$ we have uniformly distributed tensile and compressi stresses, respectively, and shearing stresses proportional to x. On tl side $x = l$ we have only the constant shearing stress $-b_3 l$, and there are ı stresses acting on the side $x = 0$. An analogous stress distribution obtained if coefficient c_3 is taken different from zero.

In taking the stress function in the form of polynomials of the secoı and third degrees we are completely free in choosing the magnitudes the coefficients, since Eq. (*a*) is satisfied whatever values they may hav In the case of polynomials of higher degrees Eq. (*a*) is satisfied only if ce tain relations between the coefficients are satisfied. Taking, for instanc the stress function in the form of a polynomial of the fourth degree,

$$\phi_4 = \frac{a_4}{4(3)} x^4 + \frac{b_4}{3(2)} x^3 y + \frac{c_4}{2} x^2 y^2 + \frac{d_4}{3(2)} x y^3 + \frac{e_4}{4(3)} y^4 \qquad (\text{(}$$

and substituting it into Eq. (*a*), we find that the equation is satisfie only if

$$e_4 = -(2c_4 + a_4)$$

The stress components in this case are

$$\sigma_x = \frac{\partial^2 \phi_4}{\partial y^2} = c_4 x^2 + d_4 xy - (2c_4 + a_4) y^2$$

$$\sigma_y = \frac{\partial^2 \phi_4}{\partial x^2} = a_4 x^2 + b_4 xy + c_4 y^2$$

$$\tau_{xy} = -\frac{\partial^2 \phi_4}{\partial x\, \partial y} = -\frac{b_4}{2} x^2 - 2c_4 xy - \frac{d_4}{2} y^2$$

Coefficients $a_4, \ldots, d_4$ in these expressions are arbitrary, and by suitа bly adjusting them we obtain various conditions of loading of a rectangı lar plate. For instance, taking all coefficients except d_4 equal to zer we find

$$\sigma_x = d_4 xy \qquad \sigma_y = 0 \qquad \tau_{xy} = -\frac{d_4}{2} y^2 \qquad (\epsilon$$

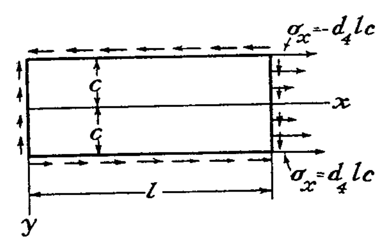


Fig. 24

Assuming d_4 positive, the forces acting on the rectangular plate shown in Fig. 24 and producing the stresses (e) are as given. On the longitudinal sides, $y = \pm c$ are uniformly distributed shearing forces; on the ends, shearing forces are distributed according to a parabolic law. The shearing forces acting on the boundary of the plate reduce to the couple[1]

$$M = \frac{d_4c^2l}{2}2c - \frac{1}{3}\frac{d_4c^2}{2}2cl = \frac{2}{3}d_4c^3l$$

This couple balances the couple produced by the normal forces along the side $x = l$ of the plate.

Let us consider a stress function in the form of a polynomial of the fifth degree.

$$\phi_5 = \frac{a_5}{5(4)}x^5 + \frac{b_5}{4(3)}x^4y + \frac{c_5}{3(2)}x^3y^2 + \frac{d_5}{3(2)}x^2y^3 + \frac{e_5}{4(3)}xy^4 + \frac{f_5}{5(4)}y^5 \qquad (f)$$

Substituting in Eq. (a) we find that this equation is satisfied if

$$e_5 = -(2c_5 + 3a_5)$$
$$f_5 = -\tfrac{1}{3}(b_5 + 2d_5)$$

The corresponding stress components are:

$$\sigma_x = \frac{\partial^2\phi_5}{\partial y^2} = \frac{c_5}{3}x^3 + d_5x^2y - (2c_5 + 3a_5)xy^2 - \frac{1}{3}(b_5 + 2d_5)y^3$$
$$\sigma_y = \frac{\partial^2\phi_5}{\partial x^2} = a_5x^3 + b_5x^2y + c_5xy^2 + \frac{d_5}{3}y^3$$
$$\tau_{xy} = -\frac{\partial^2\phi_5}{\partial x\,\partial y} = -\frac{1}{3}b_5x^3 - c_5x^2y - d_5xy^2 + \frac{1}{3}(2c_5 + 3a_5)y^3$$

Again coefficients $a_5, \ldots, d_5$ are arbitrary, and in adjusting them we obtain solutions for various loading conditions of a plate. Taking, for instance, all coefficients, except d_5, equal to zero we find

$$\sigma_x = d_5(x^2y - \tfrac{2}{3}y^3)$$
$$\sigma_y = \tfrac{1}{3}d_5y^3 \qquad (g)$$
$$\tau_{xy} = -d_5xy^2$$

The normal forces are uniformly distributed along the longitudinal sides of the plate (Fig. 25a). Along the side $x = l$, the normal forces consist of two parts, one following a linear law and the other following the law of a

[1] The thickness of the plate is taken equal to unity.

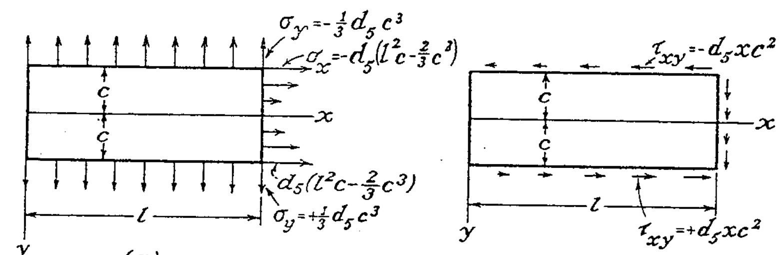


Fig. 25

cubic parabola. The shearing forces are proportional to x on the longitudinal sides of the plate and follow a parabolic law along the side $x = l$. The distribution of these stresses is shown in Fig. 25b.

Since Eq. (a) is a linear differential equation, a sum of several solutions of this equation is also a solution. We can superpose the elementary solutions considered in this article to arrive at new solutions of practical interest. Several examples of the application of this method of superposition will be considered.

19 | End Effects. Saint-Venant's Principle

In the previous article several solutions for rectangular plates were obtained from very simple forms of the stress function ϕ. In each case the boundary forces must be distributed exactly as the solution itself requires. In the case of pure bending, for instance (Fig. 22), the loading on the ends must consist of normal traction (σ_x, at $x = 0$ or $x = l$) proportional to y. If the couples on the ends are applied in any other manner, the solution given in Art. 18 is no longer correct. Another solution must be found if the changed boundary conditions on the ends are to be exactly satisfied. Many such solutions have been obtained (some are referred to later) not only for rectangular regions but for prismatic, cylindrical, and tapered shapes. These show that a change in the distribution of the load on an end, without change of the resultant, alters the stress significantly only near the end. In such cases then, simple solutions such as those of the present chapter can give sufficiently accurate results except near the ends.

The change of distribution of the load is equivalent to the superposition of a system of forces statically equivalent to zero force and zero couple. The expectation that such a system, applied to a small part of the surface of the body, would give rise to localized stress and strain only, was enunciated by Saint-Venant[1] in 1855 and came to be

[1] B. de Saint-Venant, "Mémoires des Savants Etrangers," vol. 14, 1855.

known as *Saint-Venant's principle*. It accords with common experience in a variety of circumstances not confined to small strains in elastic materials obeying Hooke's law—for instance, the application of a small clamp to a length of thick rubber tube causes appreciable strain only in the immediate neighborhood of the clamp.

For bodies extended in two or three dimensions, such as disks, spheres, or the semi-infinite solid, the stress or strain due to loading on a small part of the body may be expected to diminish with distance on account of "geometrical divergence," whether or not the resultant is zero. It has been shown[1] that vanishing of the resultant is not an adequate criterion for the degree of localization.

20 | Determination of Displacements

When the components of stress are found from the previous equations, the components of strain can be obtained by using Hooke's law, Eqs. (3) and (6). Then the displacements u and v can be obtained from the equations

$$\frac{\partial u}{\partial x} = \epsilon_x \qquad \frac{\partial v}{\partial y} = \epsilon_y \qquad \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy} \tag{a}$$

The integration of these equations in each particular case does not present any difficulty, and we shall have several examples of their application. It may be seen at once that the strain components (*a*) remain unchanged if we add to u and v the linear functions

$$u_1 = a + by \qquad v_1 = c - bx \tag{b}$$

in which a, b, and c are constants. This means that the displacements are not entirely determined by the stresses and strains. A displacement like that of a rigid body can be superposed on the displacements due to the internal strains. The constants a and c in Eqs. (*b*) represent a translatory motion of the body and the constant b is a small angle of rotation of the rigid body about the z axis.

It has been shown (see page 31) that in the case of constant body forces the stress distribution is the same for plane stress distribution or plane strain. The displacements are different for these two problems, however, since in the case of plane stress distribution the components of strain, entering into Eqs. (*a*), are given by equations

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \qquad \gamma_{xy} = \frac{1}{G}\tau_{xy}$$

[1] R. von Mises, *Bull. Am. Math. Soc.*, vol. 51, p. 555, 1945; E. Sternberg, *Quart. Appl. Math.*, vol. 11, p. 393, 1954; E. Sternberg and W. T. Koiter, *J. Appl. Mech.*, vol. 25, pp. 575–581, 1958.

and in the case of plane strain the strain components are:

$$\epsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] = \frac{1}{E}[(1 - \nu^2)\sigma_x - \nu(1 + \nu)\sigma_y]$$

$$\epsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_x + \sigma_z)] = \frac{1}{E}[(1 - \nu^2)\sigma_y - \nu(1 + \nu)\sigma_x]$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

It is easily verified that these equations can be obtained from the preceding set for plane stress by replacing F in the latter by $E/(1 - \nu^2)$, and ν by $\nu/(1 - \nu)$. These substitutions leave G, which is $E/2(1 + \nu)$, unchanged. The integration of Eqs. (*a*) will be shown later in discussing particular problems.

21 | Bending of a Cantilever Loaded at the End

Consider a cantilever having a narrow rectangular cross section of unit width bent by a force P applied at the end (Fig. 26). The upper and lower edges are free from load, and shearing forces, having a resultant P, are distributed along the end $x = 0$. These conditions can be satisfied by a proper combination of pure shear with the stresses (*e*) of Art. 18 represented in Fig. 24. Superposing the pure shear $\tau_{xy} = -b_2$ on the stresses (*e*), we find

$$\sigma_x = d_4 xy \qquad \sigma_y = 0$$
$$\tau_{xy} = -b_2 - \frac{d_4}{2}y^2 \tag{a}$$

To have the longitudinal sides $y = \pm c$ free from forces we must have

$$(\tau_{xy})_{y=\pm c} = -b_2 - \frac{d_4}{2}c^2 = 0$$

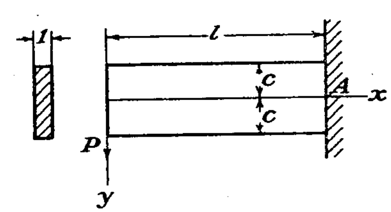


Fig. 26

from which

$$d_4 = -\frac{2b_2}{c^2}$$

To satisfy the condition on the loaded end the sum of the shearing forces distributed over this end must be equal to P. Hence[1]

$$-\int_{-c}^{c} \tau_{xy}\, dy = \int_{-c}^{c} \left(b_2 - \frac{b_2}{c^2} y^2\right) dy = P$$

from which

$$b_2 = \frac{3}{4}\frac{P}{c}$$

Substituting these values of d_4 and b_2 in Eqs. (a) we find

$$\sigma_x = -\frac{3}{2}\frac{P}{c^3} xy \qquad \sigma_y = 0$$

$$\tau_{xy} = -\frac{3P}{4c}\left(1 - \frac{y^2}{c^2}\right)$$

Noting that $\frac{2}{3}c^3$ is the moment of inertia I of the cross section of the cantilever, we have

$$\sigma_x = -\frac{Pxy}{I} \qquad \sigma_y = 0$$

$$\tau_{xy} = -\frac{P}{I}\frac{1}{2}(c^2 - y^2) \tag{b}$$

This coincides completely with the elementary solution as given in books on the strength of materials. It should be noted that this solution represents an exact solution only if the shearing forces on the ends are distributed according to the same parabolic law as the shearing stress τ_{xy} and the intensity of the normal forces at the built-in end is proportional to y. If the forces at the ends are distributed in any other manner, the stress distribution (b) is not a correct solution for the ends of the cantilever, but, by virtue of Saint-Venant's principle, it can be considered satisfactory for cross sections at a considerable distance from the ends.

Let us consider now the displacement corresponding to the stresses (b). Applying Hooke's law we find

$$\epsilon_x = \frac{\partial u}{\partial x} = \frac{\sigma_x}{E} = -\frac{Pxy}{EI} \qquad \epsilon_y = \frac{\partial v}{\partial y} = -\frac{\nu\sigma_x}{E} = \frac{\nu Pxy}{EI} \tag{c}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\tau_{xy}}{G} = -\frac{P}{2IG}(c^2 - y^2) \tag{d}$$

[1] The minus sign before the integral follows from the rule for the sign of shearing stresses. Stress τ_{xy} on the end $x = 0$ is positive if it is upward (see p. 4).

The procedure for obtaining the components u and v of the displacement consists in integrating Eqs. (c) and (d). By integration of Eqs. (c) we find

$$u = -\frac{Px^2y}{2EI} + f(y) \qquad v = \frac{\nu Pxy^2}{2EI} + f_1(x)$$

in which $f(y)$ and $f_1(x)$ are as yet unknown functions of y only and x only. Substituting these values of u and v in Eq. (d) we find

$$-\frac{Px^2}{2EI} + \frac{df(y)}{dy} + \frac{\nu Py^2}{2EI} + \frac{df_1(x)}{dx} = -\frac{P}{2IG}(c^2 - y^2)$$

In this equation some terms are functions of x only, some are functions of y only, and one is independent of both x and y. Denoting these groups by $F(x)$, $G(y)$, K, we have

$$F(x) = -\frac{Px^2}{2EI} + \frac{df_1(x)}{dx} \qquad G(y) = \frac{df(y)}{dy} + \frac{\nu Py^2}{2EI} - \frac{Py^2}{2IG}$$

$$K = -\frac{Pc^2}{2IG}$$

and the equation may be written

$$F(x) + G(y) = K$$

Such an equation means that $F(x)$ must be some constant d and $G(y)$ some constant e. Otherwise $F(x)$ and $G(y)$ would vary with x and y, respectively, and by varying x alone, or y alone, he equality would be violated. Thus

$$e + d = -\frac{Pc^2}{2IG} \tag{e}$$

and

$$\frac{df_1(x)}{dx} = \frac{Px^2}{2EI} + d \qquad \frac{df(y)}{dy} = -\frac{\nu Py^2}{2EI} + \frac{Py^2}{2IG} + e$$

Functions $f(y)$ and $f_1(x)$ are then

$$f(y) = -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$

$$f_1(x) = \frac{Px^3}{6EI} + dx + h$$

Substituting in the expressions for u and v we find

$$u = -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + ey + g$$

$$v = \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} + dx + h \tag{g}$$

The constants d, e, g, h may now be determined from Eq. (e) and from the three conditions of constraint that are necessary to prevent the beam from moving as a rigid body in the xy plane. Assume that the point A, the centroid of the end cross section, is fixed. Then u and v are zero for $x = l$, $y = 0$, and we find from Eqs. (g)

$$g = 0 \qquad h = -\frac{Pl^3}{6EI} - dl$$

The deflection curve is obtained by substituting $y = 0$ into the second of Eqs. (g). Then

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^3}{6EI} - d(l - x) \tag{h}$$

For determining the constant d in this equation, we must use the third condition of constraint, eliminating the possibility of rotation of the beam in the xy plane about the fixed point A. This constraint can be realized in various ways. Let us consider two cases: (1) When an element of the axis of the beam is fixed at the end A. Then the condition of constraint is

$$\left(\frac{\partial v}{\partial x}\right)_{\substack{x=l\\y=0}} = 0 \tag{k}$$

(2) When a vertical element of the cross section at the point A is fixed. Then the condition of constraint is

$$\left(\frac{\partial u}{\partial y}\right)_{\substack{x=l\\y=0}} = 0 \tag{l}$$

In the first case we obtain from Eq. (h)

$$d = -\frac{Pl^2}{2EI}$$

and from Eq. (e) we find

$$e = \frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}$$

Substituting all the constants in Eqs. (g), we find

$$\begin{aligned} u &= -\frac{Px^2y}{2EI} - \frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} + \left(\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}\right)y \\ v &= \frac{\nu Pxy^2}{2EI} + \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} \end{aligned} \tag{m}$$

The equation of the deflection curve is

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} \tag{n}$$

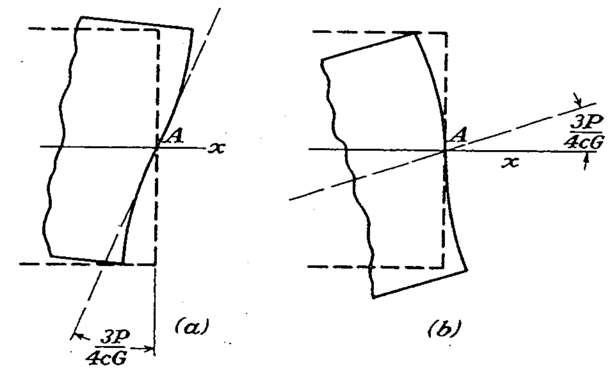


Fig. 27

which gives for the deflection at the loaded end ($x = 0$) the value $Pl^3/3EI$. This coincides with the value usually derived in elementary books on the strength of materials.

To illustrate the distortion of cross sections produced by shearing stresses, let us consider the displacement u at the fixed end ($x = l$). For this end we have from Eqs. (m),

$$\begin{aligned} (u)_{x=l} &= -\frac{\nu Py^3}{6EI} + \frac{Py^3}{6IG} - \frac{Pc^2y}{2IG} \\ \left(\frac{\partial u}{\partial y}\right)_{x=l} &= -\frac{\nu Py^2}{2EI} + \frac{Py^2}{2IG} - \frac{Pc^2}{2IG} \\ \left(\frac{\partial u}{\partial y}\right)_{\substack{x=l\\y=0}} &= -\frac{Pc^2}{2IG} = -\frac{3}{4}\frac{P}{cG} \end{aligned} \tag{o}$$

The shape of the cross section after distortion is as shown in Fig. 27a. Owing to the shearing stress $\tau_{xy} = -3P/4c$ at the point A, an element of the cross section at A rotates in the xy plane about the point A through an angle $3P/4cG$ in the clockwise direction.

If a vertical element of the cross section is fixed at A (Fig. 27b), instead of a horizontal element of the axis, we find from condition (l) and the first of Eqs. (g)

$$e = \frac{Pl^2}{2EI}$$

and from Eq. (e) we find

$$d = -\frac{Pl^2}{2EI} - \frac{Pc^2}{2IG}$$

Substituting in the second of Eqs. (g) we find

$$(v)_{y=0} = \frac{Px^3}{6EI} - \frac{Pl^2x}{2EI} + \frac{Pl^3}{3EI} + \frac{Pc^2}{2IG}(l - x) \qquad (r)$$

Comparing this with Eq. (n) it can be concluded that, owing to rotation of the end of the axis at A (Fig. 27b), the deflections of the axis of the cantilever are increased by the quantity

$$\frac{Pc^2}{2IG}(l - x) = \frac{3P}{4cG}(l - x)$$

This is an estimate[1] of the so-called *effect of shearing force* on the deflection of the beam. In practice, at the built-in end we have conditions different from those shown in Fig. 27. The fixed section[2] is usually not free to distort and the distribution of forces at this end is different from that given by Eqs. (b). However, solution (b) is satisfactory for comparatively long cantilevers at considerable distances from the terminals.

22 | Bending of a Beam by Uniform Load

Let a beam of narrow rectangular cross section of unit width, supported at the ends, be bent by a uniformly distributed load of intensity q, as shown in Fig. 28. The conditions at the upper and lower edges of the beam are:

$$(\tau_{xy})_{y=\pm c} = 0 \qquad (\sigma_y)_{y=+c} = 0 \qquad (\sigma_y)_{y=-c} = -q \qquad (a)$$

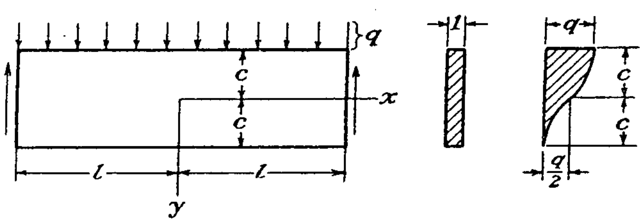
The conditions at the ends $x = \pm l$ are

$$\int_{-c}^{c} \tau_{xy}\,dy = \mp ql \qquad \int_{-c}^{c} \sigma_x\,dy = 0 \qquad \int_{-c}^{c} \sigma_x y\,dy = 0 \qquad (b)$$

The last two of Eqs. (b) state that there is no longitudinal force and no bending couple applied at the ends of the beam. All the conditions (a)

[1] Others are indicated in Prob. 3, p. 63, and in the text on p. 49.

[2] The effect of elasticity in the support itself is examined experimentally and analytically by W. J. O'Donnell, *J. Appl. Mech.*, vol. 27, pp. 461–464, 1960.



(a) (b) (c)

Fig. 28

and (b) can be satisfied by combining certain solutions in the form of polynomials as obtained in Art. 18. We begin with solution (g), illustrated by Fig. 25. To remove the tensile stresses along the side $y = c$ and the shearing stresses along the sides $y = \pm c$, we superpose a simple compression $\sigma_y = a_2$ from solution (b), Art. 18, and the stresses $\sigma_y = b_3 y$ and $\tau_{xy} = -b_3 x$ in Fig. 23. In this manner we find

$$\begin{aligned} \sigma_x &= d_5(x^2 y - \tfrac{2}{3}y^3) \\ \sigma_y &= \tfrac{1}{3}d_5 y^3 + b_3 y + a_2 \\ \tau_{xy} &= -d_5 x y^2 - b_3 x \end{aligned} \qquad (c)$$

From the conditions (a) we find

$$\begin{aligned} -d_5c^2 - b_3 &= 0 \\ \tfrac{1}{3}d_5c^3 + b_3c + a_2 &= 0 \\ -\tfrac{1}{3}d_5c^3 - b_3c + a_2 &= -q \end{aligned}$$

from which

$$a_2 = -\frac{q}{2} \qquad b_3 = \frac{3}{4}\frac{q}{c} \qquad d_5 = -\frac{3}{4}\frac{q}{c^3}$$

Substituting in Eqs. (c) and noting that $2c^3/3$ is equal to the moment of inertia I of the rectangular cross-sectional area of unit width, we find

$$\begin{aligned} \sigma_x &= -\frac{3}{4}\frac{q}{c^3}\left(x^2y - \frac{2}{3}y^3\right) = -\frac{q}{2I}\left(x^2y - \frac{2}{3}y^3\right) \\ \sigma_y &= -\frac{3q}{4c^3}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right) = -\frac{q}{2I}\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right) \\ \tau_{xy} &= -\frac{3q}{4c^3}(c^2 - y^2)x = -\frac{q}{2I}(c^2 - y^2)x \end{aligned} \qquad (d)$$

It can easily be checked that these stress components satisfy not only conditions (a) on the longitudinal sides but also the first two conditions (b) at the ends. To make the couples at the ends of the beam vanish, we superpose on solution (d) a pure bending, $\sigma_x = d_3 y$, $\sigma_y = \tau_{xy} = 0$, shown in Fig. 22, and determine the constant d_3 from the condition at $x = \pm l$

$$\int_{-c}^{c} \sigma_x y\,dy = \int_{-c}^{c}\left[-\frac{3}{4}\frac{q}{c^3}\left(l^2y - \frac{2}{3}y^3\right) + d_3 y\right] y\,dy = 0$$

from which

$$d_3 = \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)$$

Hence, finally,

$$\sigma_x = -\frac{3}{4}\frac{q}{c^3}\left(x^2y - \frac{2}{3}y^3\right) + \frac{3}{4}\frac{q}{c}\left(\frac{l^2}{c^2} - \frac{2}{5}\right)y$$
$$= \frac{q}{2I}(l^2 - x^2)y + \frac{q}{2I}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2y\right) \tag{33}$$

The first term in this expression represents the stresses given by the usual elementary theory of bending, and the second term gives the necessary correction. This correction does not depend on x and is small in comparison with the maximum bending stress, provided the span of the beam is large in comparison with its depth. For such beams the elementary theory of bending gives a sufficiently accurate value for the stresses σ_x. It should be noted that expression (33) is an exact solution only if at the ends $x = \pm l$ the normal forces are distributed according to the law

$$\bar{X} = \pm\frac{3}{4}\frac{q}{c^3}\left(\frac{2}{3}y^3 - \frac{2}{5}c^2y\right)$$

i.e., if the normal forces at the ends are the same as σ_x for $x = \pm l$ from Eq. (33). These forces have zero resultant force and zero resultant couple. Hence, from Saint-Venant's principle we can conclude that their effects on the stresses at considerable distances from the ends, say at distances larger than the depth of the beam, can be neglected. Solution (33) at such points is therefore accurate enough when no such forces $\bar{X}$ are applied.

The discrepancy between the exact solution (33) and the approximate solution, given by the first term of (33), appears because in deriving the approximate solution it is assumed that the longitudinal fibers of the beam are in a condition of simple tension. From solution (d) it can be seen that there are compressive stresses σ_y between the fibers. These stresses are responsible for the correction represented by the second term of solution (33). The distribution of the compressive stresses σ_y over the depth of the beam is shown in Fig. 28c. The distribution of shearing stress τ_{xy}, given by the third of Eqs. (d), over a cross section of the beam coincides with that given by the usual elementary theory.

When the beam is loaded by its own weight instead of the distributed load q, the solution must be modified by putting $q = 2\rho gc$ in (33) and the last two of Eqs. (d) and adding the stresses

$$\sigma_x = 0 \qquad \sigma_y = \rho g(c - y) \qquad \tau_{xy} = 0 \tag{e}$$

For the stress distribution, (e) can be obtained from Eqs. (29) by taking

$$\phi = \tfrac{1}{6}\rho g(y^3 + 3cx^2)$$

and therefore represents a possible state of stress due to weight and boundary forces. On the upper edge $y = -c$ we have $\sigma_y = 2\rho gc$, and on the lower edge $y = c$, $\sigma_y = 0$. Thus, when the stresses (e) are added to the previous solution, with $q = 2\rho gc$, the stress on both horizontal edges is zero, and the load on the beam consists only of its own weight.

The displacements u and v can be calculated by the method indicated in the previous article. Assuming that at the centroid of the middle cross section ($x = 0$, $y = 0$) the horizontal displacement is zero and the vertical displacement is equal to the deflection δ, we find, using solutions (d) and (33),

$$u = \frac{q}{2EI}\left[\left(l^2x - \frac{x^3}{3}\right)y + x\left(\frac{2}{3}y^3 - \frac{2}{5}c^2y\right) + \nu x\left(\frac{1}{3}y^3 - c^2y + \frac{2}{3}c^3\right)\right]$$
$$v = -\frac{q}{2EI}\left\{\frac{y^4}{12} - \frac{c^2y^2}{2} + \frac{2}{3}c^3y + \nu\left[(l^2 - x^2)\frac{y^2}{2} + \frac{y^4}{6} - \frac{1}{5}c^2y^2\right]\right\}$$
$$- \frac{q}{2EI}\left[\frac{l^2x^2}{2} - \frac{x^4}{12} - \frac{1}{5}c^2x^2 + \left(1 + \frac{1}{2}\nu\right)c^2x^2\right] + \delta$$

It can be seen from the expression for u that the neutral surface of the beam is not at the centerline. Owing to the compressive stress

$$(\sigma_y)_{y=0} = -\frac{q}{2}$$

the centerline has a tensile strain $\nu q/2E$, and we find

$$(u)_{y=0} = \frac{\nu qx}{2E}$$

From the expression for v we find the equation of the deflection curve,

$$(v)_{y=0} = \delta - \frac{q}{2EI}\left[\frac{l^2x^2}{2} - \frac{x^4}{12} - \frac{1}{5}c^2x^2 + \left(1 + \frac{1}{2}\nu\right)c^2x^2\right] \tag{f}$$

Assuming that the deflection is zero at the ends ($x = \pm l$) of the centerline, we find

$$\delta = \frac{5}{24}\frac{ql^4}{EI}\left[1 + \frac{12}{5}\frac{c^2}{l^2}\left(\frac{4}{5} + \frac{\nu}{2}\right)\right] \tag{34}$$

The factor before the brackets is the deflection that is derived by the elementary analysis, assuming that cross sections of the beam remain plane during bending. The second term in the brackets represents the correction usually called the *effect of shearing force*.

By differentiating Eq. (f) for the deflection curve twice with respect to x, we find the following expression for the curvature:

$$\left(\frac{d^2v}{dx^2}\right)_{y=0} = \frac{q}{EI}\left[\frac{l^2 - x^2}{2} + c^2\left(\frac{4}{5} + \frac{\nu}{2}\right)\right] \tag{35}$$

It will be seen that the curvature is not exactly proportional to the bending moment[1] $q(l^2 - x^2)/2$. The additional term in the brackets represents the necessary correction to the usual elementary formula. A more general investigation of the curvature of beams shows[2] that the correction term given in expression (35) can also be used for any case of continuously varying intensity of load. The effect of shearing force on the deflection in the case of a concentrated load will be discussed later (page 122).

An elementary derivation of the effect of the shearing force on the curvature of the deflection curve of beams was given by Rankine[3] in England and by Grashof[4] in Germany. Taking the maximum shearing strain at the neutral axis of a rectangular beam of unit width as $\frac{3}{2}(Q/2cG)$, where Q is the shearing force, the corresponding increase in curvature is given by the derivative of the above shearing strain with respect to x, which gives $\frac{3}{2}(q/2cG)$. The corrected expression for the curvature by elementary analysis then becomes

$$\frac{q}{EI}\frac{l^2 - x^2}{2} + \frac{3}{2}\frac{q}{2cG} = \frac{q}{EI}\left[\frac{l^2 - x^2}{2} + c^2(1 + \nu)\right]$$

Comparing this with expression (35), it is seen that the elementary solution gives an exaggerated value[5] for the correction.

The correction term in expression (35) for the curvature cannot be attributed to the shearing force alone. It is produced partially by the compressive stresses σ_y. These stresses are not uniformly distributed over the depth of the beam. The lateral expansion in the x direction produced by these stresses diminishes from the top to the bottom of the beam, and in this way a reversed curvature (convex upwards) is produced. This curvature together with the effect of shearing force accounts for the correction term in Eq. (35).

23 | Other Cases of Continuously Loaded Beams

By increasing the degree of polynomials representing solutions of the two-dimensional problem (Art. 18), we may obtain solutions of bending problems with various types of continuously varying load.[6] By taking, for instance, a solution in the form of a polynomial of the sixth degree and combining it with the previous solutions of Art. 18, we may obtain the stresses in a vertical cantilever loaded by hydrostatic pressure, as shown in Fig. 29. In this manner it can be shown that all conditions on the longitudinal sides of the cantilever are satisfied by the following

[1] This was pointed out first by K. Pearson, *Quart. J. Math.*, vol. 24, p. 63, 1889.

[2] See paper by T. v. Kármán, *Abhandl. Aerodynam. Inst., Tech. Hochschule, Aachen*, vol. 7, p. 3, 1927.

[3] Rankine, "Applied Mechanics," 14th ed., p. 344, 1895.

[4] Grashof, "Elastizität und Festigkeit," 2d ed., 1878.

[5] A better approximation is given by elementary strain-energy considerations. See S. Timoshenko, "Strength of Materials," 3d ed., vol. 1, p. 318.

[6] See paper by Timpe, *loc. cit.*; W. R. Osgood, *J. Res. Nat. Bur. Std.*, ser. B, vol. 28, p. 159, 1942.

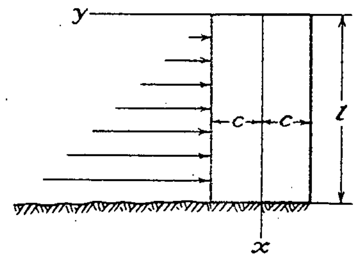


Fig. 29

system of stresses:

$$\sigma_x = \frac{qx^3y}{4c^3} + \frac{q}{4c^3}\left(-2xy^3 + \frac{6}{5}c^2xy\right)$$

$$\sigma_y = -\frac{qx}{2} + qx\left(\frac{y^3}{4c^3} - \frac{3y}{4c}\right) \tag{a}$$

$$\tau_{xy} = \frac{3qx^2}{8c^3}(c^2 - y^2) - \frac{q}{8c^3}(c^4 - y^4) + \frac{q}{4c^3}\frac{3}{5}c^2(c^2 - y^2)$$

Here q is the weight of unit volume of the fluid, so that the intensity of the load at a depth x is qx. The shearing force and the bending moment at the same depth are $qx^2/2$ and $qx^3/6$, respectively. It is evident that the first terms in the expressions for σ_x and τ_{xy} are the values of the stresses calculated by the usual elementary formulas.

On the top end of the beam ($x = 0$) the normal stress is zero. The shearing stress is

$$\tau_{xy} = -\frac{q}{8c^3}(c^4 - y^4) + \frac{q}{4c^3}\frac{3}{5}c^2(c^2 - y^2)$$

Although this is not zero, it is small all over the cross section and the resultant is zero, so that the condition approaches that of an end free from external forces.

By adding to σ_x in Eqs. (a) the term $-q_1x$, in which q_1 is the weight of unit volume of the material of the cantilever, the effect of the weight of the beam on the stress distribution is taken into account. It has been proposed[1] to use the solution obtained in this way for calculating the stresses in masonry dams of rectangular cross section. It should be noted that this solution does not satisfy the conditions at the bottom of the dam. Solution (a) is exact if, at the bottom, forces are acting which are

[1] M. Levy, *Compt. Rend.*, vol. 126, p. 1235, 1898.

distributed in the same manner as σ_x and τ_{xy} in solution (*a*). In an actual case the bottom of the dam is connected with the foundation, and the conditions are different from those represented by this solution. From Saint-Venant's principle it can be stated that the effect of the constraint at the bottom is negligible at large distances from the bottom, but in the case of a masonry dam the cross-sectional dimension $2c$ is usually not small in comparison with the height l and this effect cannot be neglected.[1]

By taking for the stress function a polynomial of the seventh degree, the stresses in a beam loaded by a parabolically distributed load may be obtained. In Chap. 6 (page 178) it is shown how, by use of the complex variable, the polynomial stress function of any degree may be written down at once.

In the general case of a continuous distribution of load q (Fig. 30) the stresses at any cross section at a considerable distance from the ends, say at a distance larger than the depth of the beam, can be approximately calculated from the following equations:[2]

$$\begin{aligned}
\sigma_x &= \frac{My}{I} + q\left(\frac{y^3}{2c^3} - \frac{3}{10}\frac{y}{c}\right) \\
\sigma_y &= -\frac{q}{2} + q\left(\frac{3y}{4c} - \frac{y^3}{4c^3}\right) \\
\tau_{xy} &= \frac{Q}{2I}(c^2 - y^2)
\end{aligned} \tag{36}$$

in which M and Q are the bending moment and shearing forces calculated in the usual

[1] The problem of stresses in masonry dams is of great practical interest and has been discussed by various authors. See K. Pearson, On Some Disregarded Points in the Stability of Masonry Dams, *Drapers' Co. Research Mem.*, 1904; K. Pearson and C. Pollard, An Experimental Study of the Stresses in Masonry Dams, *Drapers' Co. Research Mem.*, 1907. See also papers by L. F. Richardson, *Trans. Roy. Soc. (London)*, ser. A, vol. 210, p. 307, 1910; and S. D. Carothers, *Proc. Roy. Soc. Edinburgh*, vol. 33, p. 292, 1913. I. Muller, *Publ. Lab. Photoélasticité*, Zürich, 1930. Fillunger, *Oesterr. Wochschr. Öffentl. Baudienst*, 1913, No. 35. K. Wolf, *Sitzber. Akad. Wiss. Wien*, vol. 123, 1914.

[2] F. Seewald, *Abhandl. Aerodynam. Inst., Tech. Hochschule, Aachen*, vol. 7, p. 11, 1927. Concerning further development of such approximations see B. E. Gatewood and R. Dale, *J. Appl. Mech.*, vol. 29, 1962, pp. 747–749.

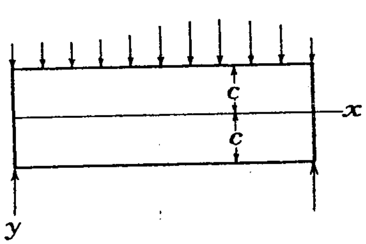


Fig. 30

way and q is the intensity of load at the cross section under consideration. These equations agree with those previously obtained for a uniformly loaded beam (see Art. 22).

If the load of intensity q, in the downward direction, is distributed along the lower edge ($y = +c$) of the beam, the expressions for the stresses are obtained from Eqs. (36) by superposing a uniform tensile stress, $\sigma_y = q$, and

$$\begin{aligned}
\sigma_x &= \frac{My}{I} + q\left(\frac{y^3}{2c^3} - \frac{3}{10}\frac{y}{c}\right) \\
\sigma_y &= \frac{q}{2} + q\left(\frac{3y}{4c} - \frac{y^3}{4c^3}\right) \\
\tau_{xy} &= \frac{Q}{2I}(c^2 - y^2)
\end{aligned} \tag{36'}$$

24 | Solution of the Two-dimensional Problem in the Form of a Fourier Series[1]

It has been shown that if the load is continuously distributed along the length of a rectangular beam of narrow cross section, a stress function in the form of a polynomial may be used in certain simple cases. A much greater degree of generality is attained by taking the function as a Fourier series (in x). Each component of load on the upper and lower edges can then have the generality possible in such series. For instance, it may have discontinuities.

The equation for the stress function,

$$\frac{\partial^4\phi}{\partial x^4} + 2\frac{\partial^4\phi}{\partial x^2\,\partial y^2} + \frac{\partial^4\phi}{\partial y^4} = 0 \tag{a}$$

may be satisfied by taking the function ϕ in the form

$$\phi = \sin\frac{m\pi x}{l} f(y) \tag{b}$$

in which m is an integer and $f(y)$ a function of y only. Substituting (*b*) into Eq. (*a*) and using the notation $m\pi/l = \alpha$, we find the following equation for determining $f(y)$:

$$\alpha^4 f(y) - 2\alpha^2 f''(y) + f^{IV}(y) = 0 \tag{c}$$

The general integral of this linear differential equation with constant coefficients is

$$f(y) = C_1\cosh\alpha y + C_2\sinh\alpha y + C_3 y\cosh\alpha y + C_4 y\sinh\alpha y$$

The stress function then is

$$\phi = \sin\alpha x(C_1\cosh\alpha y + C_2\sinh\alpha y + C_3 y\cosh\alpha y + C_4 y\sinh\alpha y) \tag{d}$$

[1] Perhaps the earliest investigation of Fourier solutions, and still one of the most thorough, is given by E. Mathieu, "Théorie de l'Elasticité des Corps Solides," seconde partie, chap. 10, pp. 140–178, Gauthier-Villars, Paris, 1890. Single Fourier series in x and y are superposed to solve problems of finite rectangles. Convergence in the determination of the Fourier coefficients from an infinite set of simultaneous algebraic equations is examined.

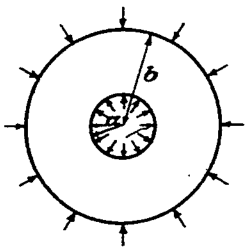


Fig. 41

Substituting in the first of Eqs. (43), we obtain the following equations to determine A and C:

$$\frac{A}{a^2} + 2C = -p_i$$

$$\frac{A}{b^2} + 2C = -p_o$$

from which

$$A = \frac{a^2b^2(p_o - p_i)}{b^2 - a^2}$$

$$2C = \frac{p_ia^2 - p_ob^2}{b^2 - a^2}$$

Substituting these in Eqs. (43) the following expressions for the stress components are obtained:

$$\begin{aligned} \sigma_r &= \frac{a^2b^2(p_o - p_i)}{b^2 - a^2}\frac{1}{r^2} + \frac{p_ia^2 - p_ob^2}{b^2 - a^2} \\ \sigma_\theta &= -\frac{a^2b^2(p_o - p_i)}{b^2 - a^2}\frac{1}{r^2} + \frac{p_ia^2 - p_ob^2}{b^2 - a^2} \end{aligned} \tag{44}$$

The radial displacement u is easily found since here $\epsilon_\theta = u/r$, and for plane stress

$$E\epsilon_\theta = \sigma_\theta - \nu\sigma_r$$

It is interesting to note that the sum $\sigma_r + \sigma_\theta$ is constant through the thickness of the wall of the cylinder. Hence the stresses σ_r and σ_θ produce a uniform extension or contraction in the direction of the axis of the cylinder, and cross sections perpendicular to this axis remain plane. Hence the deformation produced by the stresses (44) in an element of the cylinder cut out by two adjacent cross sections does not interfere with the deformation of the neighboring elements, and it is justifiable to consider the element in the condition of plane stress as we did in the above discussion.

In the particular case when $p_o = 0$ and the cylinder is submitted to internal pressure only, Eqs. (44) give

$$\begin{aligned} \sigma_r &= \frac{a^2p_i}{b^2 - a^2}\left(1 - \frac{b^2}{r^2}\right) \\ \sigma_\theta &= \frac{a^2p_i}{b^2 - a^2}\left(1 + \frac{b^2}{r^2}\right) \end{aligned} \tag{45}$$

These equations show that σ_r is always a compressive stress and σ_θ a tensile stress. The latter is greatest at the inner surface of the cylinder, where

$$(\sigma_\theta)_{\max} = \frac{p_i(a^2 + b^2)}{b^2 - a^2} \tag{46}$$

$(\sigma_\theta)_{\max}$ is always numerically greater than the internal pressure and approaches this quantity as b increases, so that it can never be reduced below p_i, however much material is added on the outside. Various applications of Eqs. (45) and (46) in machine design are usually discussed in elementary books on the strength of materials.[1]

The corresponding problem for a cylinder with an eccentric bore was solved by G. B. Jeffery.[2] If the radius of the bore is a and that of the external surface b, and if the distance between their centers is e, the maximum stress, when the cylinder is under an internal pressure p_i, is the tangential stress at the internal surface at the thinnest part, if $e < \frac{1}{2}a$, and is of the magnitude

$$\sigma = p_i\left[\frac{2b^2(b^2 + a^2 - 2ae - e^2)}{(a^2 + b^2)(b^2 - a^2 - 2ae - e^2)} - 1\right]$$

If $e = 0$, this coincides with Eq. (46).

29 | Pure Bending of Curved Bars

Let us consider a curved bar with a constant narrow rectangular cross section[3] and a circular axis bent in the plane of curvature by couples M applied at the ends (Fig. 42). The bending moment in this case is constant along the length of the bar and it is natural to expect that the stress distribution is the same in all radial cross sections, and that the solution of the problem can therefore be obtained by using expression (41).

$\phi = a_0 \ln r + c_0 r^2 \ln r + b_0 r^2 + D$

[1] See, for instance, S. Timoshenko, "Strength of Materials," 3d ed., vol. 2, chap. 6, D. Van Nostrand Company, Inc., Princeton, N.J., 1956.

[2] *Trans. Roy. Soc. (London)*, ser. A, vol. 221, p. 265, 1921. See also *Brit. Assoc. Advan. Sci. Rept.*, 1921. A complete solution by a different method is given in Art. 66 of the present book.

[3] From the general discussion of the two-dimensional problem, Art. 16, it follows that the solution obtained below for the stress holds also for plane strain.

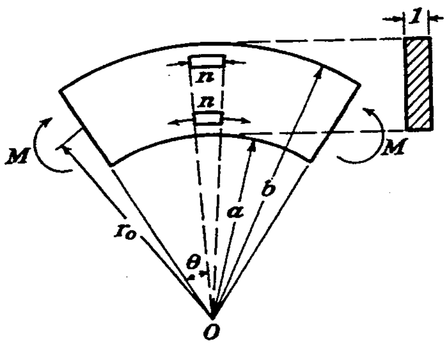


Fig. 42

Denoting by a and b the inner and the outer radii of the boundary and taking the width of the rectangular cross section as unity, the boundary conditions are

$$
\begin{array}{llll}
(1) & \sigma_r = 0 & \text{for } r = a \text{ and } r = b & \\
(2) & \displaystyle\int_a^b \sigma_\theta\, dr = 0 & \displaystyle\int_a^b \sigma_\theta r\, dr = -M & (a) \\
(3) & \tau_{r\theta} = 0 & \text{at the boundary} &
\end{array}
$$

Condition (1) means that the convex and concave boundaries of the bar are free from normal forces; condition (2) indicates that the normal stresses at the ends give rise to the couple M only, and condition (3) indicates that there are no tangential forces applied at the boundary. Using the first of Eqs. (42) with (1) of the boundary conditions (a) we obtain

$$
\begin{aligned}
\frac{A}{a^2} + B(1 + 2\log a) + 2C &= 0 \\
\frac{A}{b^2} + B(1 + 2\log b) + 2C &= 0
\end{aligned} \qquad (b)
$$

Condition (2) in (a) is now necessarily satisfied. The use of a stress function guarantees equilibrium. A nonzero force-resultant on each end would violate equilibrium. To have the bending couple equal to M, the condition

$$\int_a^b \sigma_\theta r\, dr = \int_a^b \frac{\partial^2\phi}{\partial r^2} r\, dr = -M \qquad (d)$$

must be fulfilled. We have

$$\int_a^b \frac{\partial^2\phi}{\partial r^2} r\, dr = \left|\frac{\partial\phi}{\partial r} r\right|_a^b - \int_a^b \frac{\partial\phi}{\partial r}\, dr = \left|\frac{\partial\phi}{\partial r} r\right|_a^b - |\phi|_a^b = -M$$

and noting that on account of (b),

$$\left|\frac{\partial\phi}{\partial r} r\right|_a^b = 0$$

we find from (d),

$$|\phi|_a^b = M$$

or substituting expression (41) for ϕ,

$$A \log\frac{b}{a} + B(b^2\log b - a^2\log a) + C(b^2 - a^2) = M \qquad (e)$$

This equation, together with the two Eqs. (b), completely determines the constants A, B, C, and we find

$$
\begin{aligned}
A &= -\frac{4M}{N} a^2b^2 \log\frac{b}{a} \qquad B = -\frac{2M}{N}(b^2 - a^2) \\
C &= \frac{M}{N}[b^2 - a^2 + 2(b^2\log b - a^2\log a)]
\end{aligned} \qquad (f)
$$

where for simplicity we have put

$$N = (b^2 - a^2)^2 - 4a^2b^2\left(\log\frac{b}{a}\right)^2 \qquad (g)$$

Substituting the values (f) of the constants into the expressions (42) for the stress components, we find

$$
\begin{aligned}
\sigma_r &= -\frac{4M}{N}\left(\frac{a^2b^2}{r^2}\log\frac{b}{a} + b^2\log\frac{r}{b} + a^2\log\frac{a}{r}\right) \\
\sigma_\theta &= -\frac{4M}{N}\left(-\frac{a^2b^2}{r^2}\log\frac{b}{a} + b^2\log\frac{r}{b} + a^2\log\frac{a}{r} + b^2 - a^2\right) \\
\tau_{r\theta} &= 0
\end{aligned} \qquad (47)
$$

This gives the stress distribution satisfying all the boundary conditions[1] (a) for pure bending and represents the exact solution of the problem, provided the distribution of the normal forces at the ends is that given by the second of Eqs. (47). If the forces giving the bending couple M are distributed over the ends of the bar in some other manner, the stress distribution at the ends will be different from that of the solution (47). But, as Saint-Venant's principle suggests, the deviations from solution (47) may be negligible away from the ends, say at distances greater than the depth of the bar. This is illustrated by Fig. 102.

[1] This solution is due to H. Golovin, *Trans. Inst. Tech.*, St. Petersburg, 1881. The paper, published in Russian, remained unknown in other countries, and the same problem was solved later by M. C. Ribière (*Compt. Rend.*, vol. 108, 1889, and vol. 132, 1901) and by L. Prandtl. See A. Föppl, "Vorlesungen uber Technische Mechanik," vol. 5, p. 72, 1907; also A. Timpe, *Z. Math. Physik*, vol. 52, p. 348, 1905.

It is of practical interest to compare solution (47) with the elementary solutions usually given in books on the strength of materials. If the depth of the bar, $b - a$, is small in comparison with the radius of the central axis, $(b + a)/2$, the same stress distribution as that for straight bars is usually assumed. If this depth is not small, it is usual in practice to assume that cross sections of the bar remain plane during the bending, from which it can be shown that the distribution of the normal stresses σ_θ over any cross sections follows a hyperbolic law.[1] In all cases the maximum[2] and minimum values of the stress σ_θ can be presented in the form

$$\sigma_\theta = m\frac{M}{a^2} \tag{h}$$

The following table gives the values of the numerical factor m calculated by the two elementary methods, referred to above, and by the exact formula (47).[3] It can be seen from this table that the elementary solution based on the hypothesis of plane cross sections gives very accurate results.

Coefficient m of Eq. (h)

$\frac{b}{a}$	Linear stress distribution	Hyperbolic stress distribution	Exact solution
1.3	±66.67	+72.98, −61.27	+73.05, −61.35
2	± 6.000	+ 7.725, − 4.863	+ 7.755, − 4.917
3	± 1.500	+ 2.285, − 1.095	+ 2.292, − 1.130

It will be shown later that, in the case of pure bending, the cross sections actually do remain plane, and the discrepancy between the elementary and the exact solutions comes from the fact that in the elementary solution the stress component σ_r is neglected and it is assumed that longitudinal fibers of the bent bar are in simple tension or compression.

[1] This approximate theory was developed by H. Résal, *Ann. Mines*, p. 617, 1862, and by E. Winkler, *Zivilingenieur*, vol. 4, p. 232, 1858; see also his book "Die Lehre von der Elastizität und Festigkeit," chap. 15, Prag, 1867. Further development of the theory was made by F. Grashof, "Elastizität und Festigkeit," p. 251, 1878, and by K. Pearson, "History of the Theory of Elasticity," vol. 2, pt. 1, p. 422, 1893.

[2] The greatest value of σ_θ in (47) always occurs at the inside ($r = a$). A proof is given by J. E. Brock, *J. Appl. Mech.*, vol. 31, p. 559, 1964.

[3] The results are taken from the doctorate thesis, University of Michigan, 1931, of V. Billevicz.

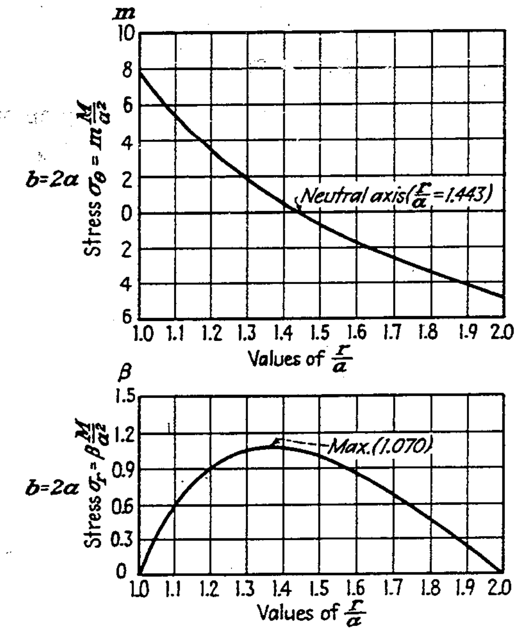
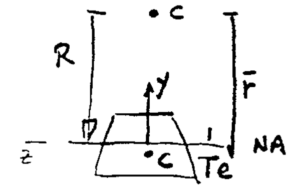


Fig. 43



$\bar{r} = \frac{1}{A}\int r\,dA$

$R = \frac{A}{\int \frac{dA}{r}}$

$e = \bar{r} - R \quad -y = r - R$

$\sigma_x = \frac{-My}{Ae(R-y)}$

for rect $R = \frac{h}{\ln(b/a)}$

$\bar{r} = \frac{b+a}{2}$

From the first of Eqs. (47) it can be shown that the stress σ_r is always positive for the direction of bending shown in Fig. 42. The same can be concluded at once from the direction of stresses σ_θ acting on the elements $n - n$ in Fig. 42. The corresponding tangential forces give resultants in the radial direction tending to separate longitudinal fibers and producing tensile stress in the radial direction. This stress increases toward the neutral surface and becomes a maximum near this surface. This maximum is always much smaller than $(\sigma_\theta)_{max}$. For instance, for $b/a = 1.3$, $(\sigma_r)_{max} = 0.060(\sigma_\theta)_{max}$; for $b/a = 2$, $(\sigma_r)_{max} = 0.138(\sigma_\theta)_{max}$; for $b/a = 3$, $(\sigma_r)_{max} = 0.193(\sigma_\theta)_{max}$. In Fig. 43 the distribution of σ_θ and σ_r for $b/a = 2$ is given. From this figure we see that the point of maximum stress σ_r is somewhat displaced from the neutral axis in the direction of the center of curvature.

30 | Strain Components in Polar Coordinates

In considering the displacement in polar coordinates let us denote by u and v the components of the displacement in the radial and tangential directions, respectively. If u is the radial displacement of the side ad

THUS WE CAN WRITE $\underline{\underline{A}}$ SOLUTION TO $\nabla^4\phi = 0$ IN POLAR COORDINATES AS: (TIMOSHENKO & GOODIER, p. 133, eq. 80)

$$\phi = \overbrace{a_0 \ln r + b_0 r^2 + c_0 r^2 \ln r}^{\text{sol for stress distr symm about origin}} + \overbrace{d_0 r^2\theta}^{\text{Vertical load on straight body}} + \overbrace{a_0'\theta}^{\text{pure shear}}$$

$$+ \overbrace{a_1 r\theta\sin\theta}^{\text{radial dist along } \theta = 0,\pi} + \overbrace{(b_1 r^3 + a_1' r^{-1} + b_1' r\ln r)\cos\theta}^{\text{Portion of circular ring bent by radial force along } \theta = 0,\pi}$$

entire line = force acting on infinite plate along $\theta = 0,\pi$
for pt force take term 1 and last term

$\theta = \pm\pi/2$

$$+ \overbrace{c_1 r\theta\cos\theta}^{\text{radial dist along } \theta = \pm\frac{\pi}{2}} + (d_1 r^3 + c_1' r^{-1} + d_1' r\ln r)\sin\theta$$

entire line = force acting on infinite plate along $\theta = \pm\pi/2$
for pt force take term 1 and last term

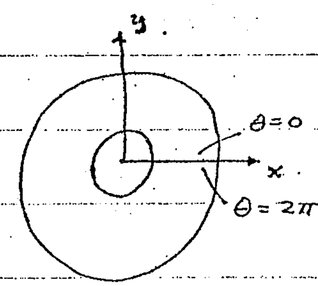
$$+ \sum_{n=2}^{\infty} (a_n r^n + b_n r^{n+2} + a_n' r^{-n} + b_n' r^{-n+2})\cos n\theta$$

shearing on ring

$$+ \sum_{n=2}^{\infty} (c_n r^n + d_n r^{n+2} + c_n' r^{-n} + d_n' r^{-n+2})\sin n\theta$$

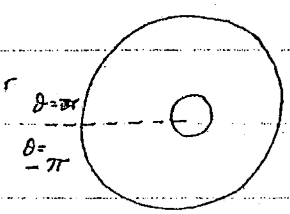
normal force on ring

LET US SEE IF WE CAN USE THIS STRESS FUNCTION (OR PIECES OF IT) TO SOLVE PROBLEMS OF LOADING ON A COMPLETE ANGULAR RING.



(1) SOMETIMES WE WILL FIND WE HAVE TO EXAMINE DISPLACEMENTS ALSO TO CHECK ON ALLOWABLE TERMS.

or



(2) SUPPOSE WE ARE INTERESTED IN SOLVING THE TRACTION BOUNDARY VALUE PROBLEM FOR A COMPLETE RING.

SUPPOSE THE BOUNDARY CONDITIONS ARE:

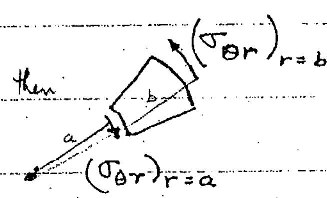
$$(\sigma_{rr})_{r=a} = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta$$

$$(\sigma_{rr})_{r=b} = A_0' + \sum_{n=1}^{\infty} A_n' \cos n\theta + \sum_{n=1}^{\infty} B_n' \sin n\theta$$

$$(\sigma_{r\theta})_{r=a} = C_0 + \sum_{n=1}^{\infty} C_n \cos n\theta + \sum_{n=1}^{\infty} D_n \sin n\theta$$

$$(\sigma_{r\theta})_{r=b} = C_0' + \sum_{n=1}^{\infty} C_n' \cos n\theta + \sum_{n=1}^{\infty} D_n' \sin n\theta$$

I. BOUNDARY LOADS MUST BE SELF-EQUILIBRATING !!



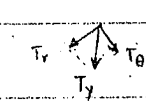
MOMENT EQUILIBRIUM since T_r gives no moments but T_θ does then

about origin $$\int_0^{2\pi} \{ b\,\sigma_{\theta r} \}_{r=b}\, b\, d\theta - \int_0^{2\pi} \{ a\,\sigma_{\theta r} \}_{r=a}\, a\, d\theta = 0$$

NOW $$\int_0^{2\pi} \cos n\theta\, d\theta = \int_0^{2\pi} \sin n\theta\, d\theta = 0 \qquad \text{FOR } n \geq 1$$

EQUILIBRIUM OF MOMENTS REQUIRES:

$$\boxed{C_0 a^2 = C_0' b^2}$$



FORCE EQUILIBRIUM

$$T_x = T_r \cos\theta - T_\theta \sin\theta$$

$$T_y = T_r \sin\theta + T_\theta \cos\theta$$

$$T_r = \sigma_{rr}, \quad T_\theta = \sigma_{r\theta}$$

$$\frac{T_x^{NET}}{\text{unit length}} = 0 = \int_0^{2\pi} \left\{ \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta \right\}_{r=b} b\,d\theta$$

$$- \int_0^{2\pi} \left\{ \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta \right\}_{r=a} a\,d\theta$$

Only the $n=1$ terms can contribute: $\int_0^{2\pi} \cos^2\theta\,d\theta = \int_0^{2\pi} \sin^2\theta\,d\theta = \pi$

$\int_0^{2\pi} \sin\theta\cos\theta\,d\theta = 0$

$$\therefore \quad 0 = (A_1' - D_1')\,b - (A_1 - D_1)\,a$$

so

$$\boxed{b\,(A_1' - D_1') = a\,(A_1 - D_1)}$$

$T_y^{NET} = 0$ REQUIRES

$$\boxed{b\,(B_1' + C_1') = a\,(B_1 + C_1)}$$

NOW ASSUME OUR CONSTANTS ARE SUCH THAT THESE RELATIONSHIPS HOLD.

COMPUTE σ_{rr} AND $\sigma_{r\theta}$ FROM $\phi(r,\theta)$.

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

MUST CHOOSE $d_0 = 0$ FOR A COMPLETE RING

$$\sigma_{rr} = \frac{a_0}{r^2} + 2b_0 + c_0(1+2\ln r) + 2d_0\theta$$

$$+2\frac{a_1}{r}\cos\theta + \left(2b_1 r - \frac{2a_1'}{r^3} + \frac{b_1'}{r}\right)\cos\theta$$

$$-2\frac{c_1}{r}\sin\theta + \left(2d_1 r - \frac{2c_1'}{r^3} + \frac{d_1'}{r}\right)\sin\theta$$

$$+\sum_{n=2}^{\infty}\left\{a_n n(1-n)r^{n-2} + b_n(2+n-n^2)r^n - a_n' n(1+n)r^{-n-2} + b_n'(2-n-n^2)r^{-n}\right\}\cos n\theta$$

$$+\sum_{n=2}^{\infty}\left\{c_n n(1-n)r^{n-2} + d_n(2+n-n^2)r^n - c_n' n(1+n)r^{-n-2} + d_n'(2-n-n^2)r^{-n}\right\}\sin n\theta$$

BOUNDARY CONDITIONS ON $(\sigma_{rr})_{r=a}$ AND $(\sigma_{rr})_{r=b}$

$$\left.\begin{aligned} &(1)\quad \frac{a_0}{a^2} + 2b_0 + c_0(1+2\ln a) = A_0 \\ &(2)\quad \frac{a_0}{b^2} + 2b_0 + c_0(1+2\ln b) = A_0' \end{aligned}\right\} \quad \text{2 EQNS. — 3 UNKNOWNS}$$

$$\left.\begin{aligned} &(3)\quad \frac{2a_1}{a} + 2b_1 a - \frac{2a_1'}{a^3} + \frac{b_1'}{a} = A_1 \\ &(4)\quad +\frac{2a_1}{b} + 2b_1 b - \frac{2a_1'}{b^3} + \frac{b_1'}{b} = A_1' \end{aligned}\right\} \quad \text{2 EQNS. — 4 UNKNOWNS}$$

$$\left.\begin{aligned} &(5)\quad -\frac{2c_1}{a} + 2d_1 a - \frac{2c_1'}{a^3} + \frac{d_1'}{a} = B_1 \\ &(6)\quad -\frac{2c_1}{b} + 2d_1 b - \frac{2c_1'}{b^3} + \frac{d_1'}{b} = B_1' \end{aligned}\right\} \quad \text{2 EQNS — 4 UNKNOWNS}$$

$(2-n)(1+n)$

(7) $a_n n(1-n) a^{n-2} + b_n (2+n-n^2) a^n - a_n' n(1+n) a^{-n-2} + b_n' (2-n-n^2) a^{-n} = A_n$

(8) $a_n n(1-n) b^{n-2} + b_n (2+n-n^2) b^n - a_n' n(1+n) b^{-n-2} + b_n' (2-n-n^2) b^{-n} = A_n'$

(9) $c_n n(1-n) a^{n-2} + d_n (2+n-n^2) a^n - c_n' n(1+n) a^{-n-2} + d_n' (2-n-n^2) a^{-n} = B_n$

(10) $c_n n(1-n) b^{n-2} + d_n (2+n-n^2) b^n - c_n' n(1+n) b^{-n-2} + d_n' (2-n-n^2) b^{-n} = B_n'$

(7) – (10): FOR EACH n, 4 EQUATIONS & 8 UNKNOWNS

NOW LOOK AT $\sigma_{r\theta}$ BOUNDARY CONDITIONS: FIRST COMPUTE $\sigma_{r\theta}$ FROM ϕ.

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left\{ \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right\}$$, TAKING $d_0 = 0$.

$$\begin{aligned} \sigma_{r\theta} = {} & \frac{a_0'}{r^2} \\ & + \sin\theta \left\{ b_1 \cdot 2r - \frac{2a_1'}{r^3} + \frac{b_1'}{r} \right\} \\ & - \cos\theta \left\{ d_1 \cdot 2r - \frac{2c_1'}{r^3} + \frac{d_1'}{r} \right\} \\ & + \sum_{n=2}^{\infty} n \sin n\theta \left\{ a_n (n-1) r^{n-2} + b_n (n+1) r^n - a_n' (n+1) r^{-n-2} - b_n' (n-1) r^{-n} \right\} \\ & - \sum_{n=2}^{\infty} n \cos n\theta \left\{ c_n (n-1) r^{n-2} + d_n (n+1) r^n - c_n' (n+1) r^{-n-2} - d_n' (n-1) r^{-n} \right\} \end{aligned}$$

APPLYING BOUNDARY CONDITIONS ON $\sigma_{r\theta}$ ON $r=a, b$.

$$\frac{a_0'}{a^2} = C_0 \quad ; \quad \frac{a_0'}{b^2} = C_0' \quad \Rightarrow \quad a^2 C_0 = b^2 C_0'$$ WHICH WE FOUND BEFOR

$$\therefore \boxed{a_0' = a^2 C_0 = b^2 C_0'}$$

$$(11) \quad 2 b_1 a - \frac{2a_1'}{a^3} + \frac{b_1'}{a} = D_1$$

$$(12) \quad 2 b_1 b - \frac{2a_1'}{b^3} + \frac{b_1'}{b} = D_1'$$

(3), (4), (11), (12) GIVES 4 EQNS. FOR FOUR UNKNOWNS b_1, a_1', b_1', a_1

$$(13) \quad 2 d_1 a - \frac{2c_1'}{a^3} + \frac{d_1'}{a} = -C_1$$

$$(14) \quad 2 d_1 b - \frac{2c_1'}{b^3} + \frac{d_1'}{b} = -C_1'$$

(5), (6), (13), (14) GIVES 4 EQNS FOR FOUR UNKNOWNS d_1, c_1', d_1', c_1

$$(15) \quad n \left\{ a_n (n-1) a^{n-2} + b_n (n+1) a^n - a_n' (n+1) a^{-n-2} - b_n' (n-1) a^{-n} \right\} = D_n$$

$$(16) \quad n \left\{ a_n (n-1) b^{n-2} + b_n (n+1) b^n - a_n' (n+1) b^{-n-2} - b_n' (n-1) b^{-n} \right\} = D_n'$$

(7), (8), (15), (16) GIVES 4 EQNS. FOR a_n, b_n, a_n', b_n'

$$(17) \quad n \left\{ c_n (n-1) a^{n-2} + d_n (n+1) a^n - c_n' (n+1) a^{-n-2} - d_n' (n-1) a^{-n} = -C_n \right.$$

$$(18) \quad n \left\{ c_n (n-1) b^{n-2} + d_n (n+1) b^n - c_n' (n+1) b^{-n-2} - d_n' (n-1) b^{-n} = -C_n' \right.$$

(9), (10), (17), (18) GIVES 4 EQNS. FOR c_n, d_n, c_n', d_n'

TWO PROBLEMS: (1) & (2) STILL YIELD ONLY 2 EQNS. IN 3 UNKNOWNS

ARE WE GUARANTEED THAT:

$$b(A_1'-D_1') = a(A_1-D_1)$$

$$b(B_1'+C_1') = a(B_1+C_1)$$

LOOK @ SECOND QUESTION FIRST:

$$\left.\begin{array}{ll} (3)-(11) \Rightarrow \dfrac{2a_1}{a} = A_1-D_1 \\ (4)-(12) \Rightarrow \dfrac{2a_1}{b} = A_1'-D_1' \end{array}\right\} \text{OKAY}$$

$$\left.\begin{array}{ll} (5)-(13) \Rightarrow -\dfrac{2c_1}{a} = B_1+C_1 \\ (6)-(14) \Rightarrow -\dfrac{2c_1}{b} = B_1'+C_1' \end{array}\right\} \text{OKAY}$$

$\therefore$ WE KNOW a_1 AND c_1 FROM LOADING BOUNDARY CONDITIONS AND THIS MEANS THAT (3) & (11) REPRESENT ONLY $\underline{\underline{1}}$ EQUATION AND (4) & (12) REPRESENT ONLY $\underline{\underline{\text{ONE}}}$ EQUATION. SIMILARLY FOR (5) & (13) & (6) & (14). THUS

$$\left.\begin{array}{ll} (3)\ \&\ (11) \rightarrow 2b_1a - \dfrac{2a_1'}{a^3} + \dfrac{b_1'}{a} = D_1 \\ (4)\ \&\ (12) \rightarrow 2b_1b - \dfrac{2a_1'}{b^3} + \dfrac{b_1'}{b} = D_1' \end{array}\right\} \text{2 EQNS} \leftrightarrow \text{3 UNKNOWNS}$$

$$\left.\begin{array}{ll} (5)\ \&\ (13) \rightarrow 2d_1a - \dfrac{2c_1'}{a^3} + \dfrac{d_1'}{a} = -C_1 \\ (6)\ \&\ (14) \rightarrow 2d_1b - \dfrac{2c_1'}{b^3} + \dfrac{d_1'}{b} = -C_1' \end{array}\right\} \text{2 EQNS.} \leftrightarrow \text{3 UNKNOWNS}$$

$$\left.\begin{array}{ll} (1) \quad \dfrac{a_0}{a^2} + 2b_0 + c_0(1+2\ln a) = A_0 \\ (2) \quad \dfrac{a_0}{b^2} + 2b_0 + c_0(1+2\ln b) = A_0' \end{array}\right\} \text{2 EQNS.} \leftrightarrow \text{3 UNKNOWN.}$$

TIMOSHENKO AND GOODIER, PAGES 77-78, SHOW THAT $c_0=0$ FOR A COMPLETE RING OR ELSE u_θ IS MULTI-VALUED!

DERIVATION OF THE RELATION BETWEEN a_1 & b_1' AND c_1 AND d_1' BY EXAMINING THE MULTI-VALUED NATURE OF THE DISPLACEMENT FIELD (PLANE STRA

LOOK AT THE STRESS FUNCTION

$$\phi = a_1 r\theta\sin\theta + b_1' r\ln r\cos\theta$$

THEN

$$\sigma_{\theta\theta} = \frac{\partial^2\phi}{\partial r^2} = \frac{b_1'}{r}\cos\theta$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) = \frac{b_1'}{r}\sin\theta$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = \frac{2a_1}{r}\cos\theta + \frac{b_1'}{r}\cos\theta = \frac{2a_1+b_1'}{r}\cos\theta$$

$$\sigma_{zz} = \nu(\sigma_{rr}+\sigma_{\theta\theta}) = \nu\,\frac{2(a_1+b_1')}{r}\cos\theta$$

NOW

$$e_{rr} = \frac{\partial u_r}{\partial r} = \frac{1}{E}\left\{\overset{\sigma_{rr}}{\frac{2a_1+b_1'}{r}\cos\theta} - \nu\left[\overset{\sigma_{\theta\theta}}{\frac{b_1'}{r}\cos\theta} + \nu\,\overset{\sigma_{zz}}{\frac{2(a_1+b_1')}{r}\cos\theta}\right]\right\}$$

OR

$$(1)\quad \frac{\partial u_r}{\partial r} = \frac{1}{Er}\cos\theta\left\{2a_1 + b_1' - \nu\left(b_1' + 2\nu(a_1+b_1')\right)\right\}$$

THUS

$$(2)\quad \frac{\partial u_r}{\partial r} = \frac{\cos\theta}{Er}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}$$

AND

$$(3)\quad u_r = \frac{\cos\theta}{E}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}\ln r + g(\theta)$$

FURTHERMORE

$$e_{rr}+e_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{\partial u_r}{\partial r} = \frac{1}{E}\left\{\sigma_{rr}+\sigma_{\theta\theta} - \nu\langle\sigma_{rr}+\sigma_{\theta\theta}+2\sigma_{zz}\rangle\right\}$$

$$= \frac{1}{E}\left\{(1-\nu)(\sigma_{rr}+\sigma_{\theta\theta}) - 2\nu^2(\sigma_{rr}+\sigma_{\theta\theta})\right\}$$

$$= \frac{1-\nu-2\nu^2}{E}(\sigma_{rr}+\sigma_{\theta\theta})$$

$$= \frac{1-\nu-2\nu^2}{E}\cdot\frac{2(a_1+b_1')}{r}\cos\theta$$

HENCE

$$\frac{1}{r}\frac{\partial u_\theta}{\partial\theta} = \overset{e_{rr}+e_{\theta\theta}}{\frac{1-\nu-2\nu^2}{E}\,\frac{2(a_1+b_1')}{r}\cos\theta} - \overset{\frac{\partial u_r}{\partial r}}{\frac{\cos\theta}{Er}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}}$$

$$\overset{\longleftarrow\ \frac{u_r}{r}\ \longrightarrow}{-\frac{\cos\theta}{Er}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}\ln r - \frac{g(\theta)}{r}}$$

OR (3) $$\frac{\partial u_\theta}{\partial \theta} = -g(\theta) + \frac{\cos\theta}{E}\left\{ a_1 \left\langle [-2\nu - 2\nu^2] - 2(1-\nu^2)\ln r \right\rangle + b_1' \langle 1-\nu-2\nu^2\rangle \langle 1-\ln r\rangle \right\}$$

AND

(4) $$u_\theta = -\int_\theta g(t)\,dt + f(r) + \frac{\sin\theta}{E}\left\{ a_1 \left[-2\nu - 2\nu^2 - 2(1-\nu^2)\ln r\right] + b_1' (1-\nu-2\nu^2)(1-\ln r)\right\}$$

WE HAVE THE ADDITIONAL RELATION $e_{r\theta} = \frac{1}{2\mu}\sigma_{r\theta} = \frac{1+\nu}{E}\sigma_{r\theta}$

THIS REQUIRES THAT

(5) $$\frac{1+\nu}{E}\frac{b_1'}{r}\sin\theta = \frac{1}{2}\left\{\frac{1}{r}\frac{\partial u_r}{\partial\theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right\}$$

HENCE

$$\sin\theta\,\frac{2(1+\nu)}{E}\frac{b_1'}{r} = -\frac{\sin\theta}{Er}\left\{2(1-\nu^2)a_1 + b_1'(1-\nu-2\nu^2)\right\}\ln r + \frac{1}{r}\frac{dg(\theta)}{d\theta}$$
$$+ \frac{df(r)}{dr} + \frac{\sin\theta}{E}\left\{a_1\left(\frac{-2(1-\nu^2)}{r}\right) - \frac{b_1'}{r}(1-\nu-2\nu^2)\right\}$$
$$+ \frac{\int_\theta g(t)\,dt}{r} - \frac{f(r)}{r} - \frac{\sin\theta}{Er}\left\{a_1\left[-2\nu-2\nu^2-2(1-\nu^2)\ln r\right] + b_1'(1-\nu-2\nu^2)(1-\ln r)\right\}$$

AND

$$\left\{\frac{dg(\theta)}{d\theta} + \int_\theta g(t)\,dt\right\} + \left\{r\frac{df}{dr} - f\right\}$$
$$+ \frac{\sin\theta}{E}\left\{a_1\left[-2(1-\nu^2) + 2\nu + 2\nu^2\right] + b_1'\left[-2(1-\nu-2\nu^2) - 2(1+\nu)\right]\right\} = 0.$$

THIS IS POSSIBLE IF AND ONLY IF ??? (why not = const. yes but only adds rigid body translation

$$r\frac{df}{dr} - f = 0 \Rightarrow f = \alpha r$$ [THIS IS A RIGID ROTATION TERM]

AND IF

(6) $$g'(\theta) + \int_\theta g(t)\,dt + \frac{\sin\theta}{E}\left\{a_1(-2+2\nu+4\nu^2) + b_1'(-4+4\nu^2)\right\} = 0$$

DIFFERENTIATING (6) WITH RESPECT TO θ YIELDS

$$g''(\theta) + g(\theta) + \frac{\cos\theta}{E}\left\{ a_1\left[-2(1-2\nu)(1+\nu)\right] - 4b_1'(1-\nu^2)\right\} = 0$$

OR

$$g''(\theta) + g(\theta) = \frac{2(1+\nu)\cos\theta}{E}\left\{ a_1(1-2\nu) + 2b_1'(1-\nu)\right\} = J\cos\theta$$

WHERE

$$J = \frac{2(1+\nu)}{E}\left[a_1(1-2\nu) + 2b_1'(1-\nu)\right]$$

THE SOLUTION TO THIS DIFFERENTIAL EQUATION IS

$$g(\theta) = \alpha_0\cos\theta + \beta_0\sin\theta + \frac{J}{2}\theta\sin\theta$$

SO $g(\theta)$ AND u_r AND u_θ WILL BE MULTI-VALUED FOR A COMPLETE RING UNLESS $J \equiv 0$.

THUS

$$\boxed{b_1' = -\frac{a_1(1-2\nu)}{2(1-\nu)} \quad \text{IN PLANE STRAIN}}$$

SIMILARLY

$$\boxed{d_1' = -\frac{c_1(1-2\nu)}{2(1-\nu)} \quad \text{IN PLANE STRAIN}}$$

FOR PLANE STRESS REPLACE ν BY $\frac{\nu}{1+\nu}$ SO THAT

$$\frac{1-2\nu}{1-\nu} \rightarrow \frac{1-\frac{2\nu}{1+\nu}}{1-\frac{\nu}{1+\nu}} \rightarrow 1-\nu.$$

SINCE MY a_1 AND c_1 ARE HALF OF TIMOSHENKO AND GOODIER'S a_1 AND c_1, THE PLANE STRESS RELATIONS CHECK WITH THOSE OF T & G ON PAGE 135.

HENCE $a_1, b_1, b_1', c_1, d_1, d_1'$ CAN BE UNIQUELY DETERMINED from last 3 eq

pg 38

Florida International University
Department of Mechanical and Materials Engineering

EGM 5615 EXAMINATION 2A 22 November 2019

This examination is a take-home examination and is due by 4pm on 25 November 2019. This exam allows you to use your notes and book only.

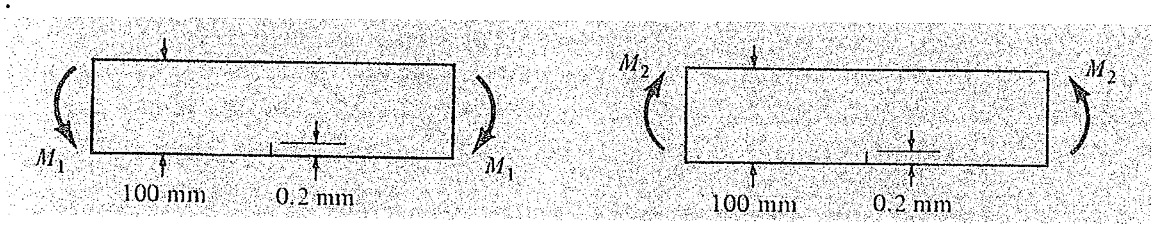
Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of this may result in failure of the exam.

____________________ ____________________
PRINT NAME SIGN NAME

This examination consists of **three problems with several parts to two of the problems. Do all problems.** Read each question carefully. Show all work!!!!!

Problem 1 (30 points). The beam shown is 30 mm thick and is made of glass. It contains a crack 0.2 mm emanating from one edge. What bending moment will produce fracture in each case? Compare the ratio of M_1 to M_2. Use K_{IC} =0.25 MPa$\sqrt{m}$ and σ_y = 100 MPa for glass.
Also, check to see if this is a valid test specimen.



Problem 2 (40 points). Show that $\Phi = a_1(x^4 - 3x^2y^2)$ where a_1 is a constant, satisfies $\nabla^4\Phi=0$. Sketch the stresses that act on the boundaries of the region $0<x<1$ and $0<y<1$. Show that moments about the point x=y=0 sum to zero. Also find u_x, u_y from knowledge of stresses.

Problem 3 (30 points). Given the following stress field

$$\sigma_x = -a_1x^2y, \quad \sigma_y = -a_1y^3/3, \quad \tau_{xy} = a_1xy^2$$

with a_1 being a constant and the stresses do not depend on z, i.e., a plane strain problem,

DETERMINE if this is a valid solution of an elasticity problem. If these stresses are not valid, give the reason why not. Lastly, if this is a valid elasticity solution, FIND the strains and displacements.

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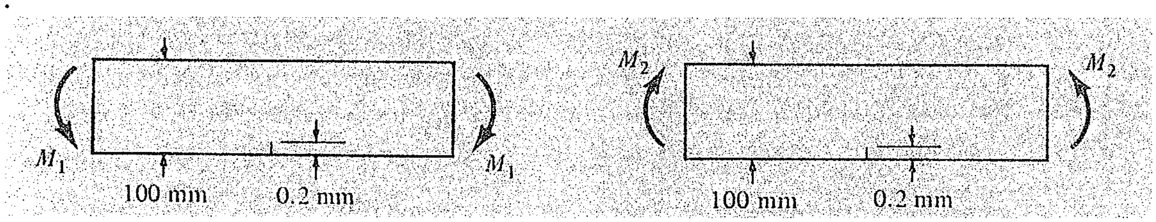

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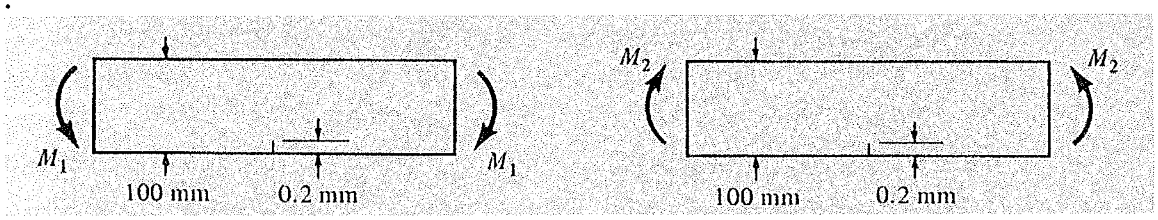

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II For an infinite incompressible inviscid flow I assumed: steady state, 2D, irrotational flow, ρ = constant, solid walls are streamlines, we neglect the no slip condition (ie $\sigma_{r\theta} = 0$ at wall). We thus obtain the pressure on the cylinder by using Bernoulli's equation. We define a stream function ψ ∋. $u = \frac{\partial \psi}{\partial y}$ $\quad v = -\frac{\partial \psi}{\partial x}$ $\quad$ or $\quad v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$, $v_\theta = -\frac{\partial \psi}{\partial r}$

The steady state Bernoulli equation is $p/\rho - U + \frac{1}{2}(v_r^2 + v_\theta^2) = \text{const}$ where $\nabla U = f_b$ (the body force which we assume is conservative).

The stream function $\psi = V_\infty r \sin\theta \left[1 - b^2/r^2\right]$. We note that the $\left\{ \begin{matrix}\text{inner}\\ \text{outer}\end{matrix} \right\}$ surfaces can be considered as streamlines and no fluid thus passes through the cylinder; thus

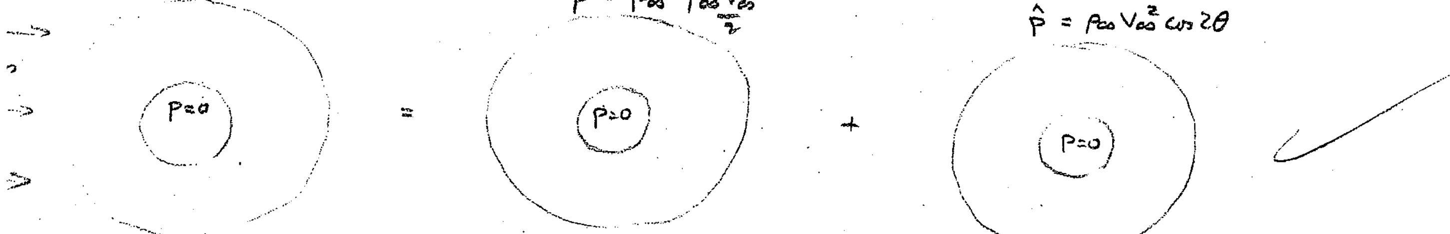
$$v_r\Big|_{r=b} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = V_\infty \cos\theta\left(1 - \frac{b^2}{r^2}\right)\Bigg|_{r=b} = 0 \qquad v_\theta\Big|_{r=b} = -\frac{\partial \psi}{\partial r} = -V_\infty \sin\theta\left(1 + \frac{b^2}{r^2}\right)\Bigg|_{r=b} = -2V_\infty \sin\theta$$

We will neglect the body forces and evaluate the Bernoulli constant at ∞ so that

$$p = p_\infty + \frac{\rho_\infty V_\infty^2}{2} - \frac{\rho_\infty}{2} 4 V_\infty^2 \sin^2\theta = p_\infty + \frac{\rho_\infty V_\infty^2}{2}\left[1 - 4\sin^2\theta\right] = p_\infty - \frac{\rho_\infty V_\infty^2}{2} + \rho_\infty V_\infty^2 \cos 2\theta$$

We can solve this by superposition of solutions w/ $p_{inner} = p\big|_{r=a} = 0$

$p = p^\circ + \hat{p}$ $\qquad p^\circ = p_\infty - \frac{\rho_\infty V_\infty^2}{2}$ $\qquad \hat{p} = \rho_\infty V_\infty^2 \cos 2\theta$



p° can be obtained from the plane strain solution w/ $p_{inner} = 0$

$$T_r^\circ = -p^\circ = \sigma_{rr_o} \qquad \therefore\ \sigma_{rr_o} = -p^\circ,\ \sigma_{r\theta_o} = 0 \qquad @\ r = b$$
$$T_r^i = T_\theta^i = 0 \qquad \therefore\ \sigma_{rr_i} = 0 \qquad \sigma_{r\theta_i} = 0 \qquad @\ r = a$$

$$\sigma_{rr}^\circ = \frac{\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right)a^2 b^2}{b^2 - a^2}\cdot\frac{1}{r^2} - \frac{\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right)b^2}{b^2 - a^2}$$

$$\sigma_{r\theta}^\circ = 0$$

$$\sigma_{\theta\theta}^\circ = -\frac{\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right)a^2 b^2}{b^2 - a^2}\cdot\frac{1}{r^2} - \frac{\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right)b^2}{b^2 - a^2}$$

$$\sigma_{zz}^\circ = \nu(\sigma_{rr}^\circ + \sigma_{\theta\theta}^\circ) = -2\nu\left(p_\infty - \frac{\rho_\infty V_\infty^2}{2}\right)\frac{b^2}{}$$

The $\hat{p}$ solution can be obtained as in class $\hat{\sigma}_{r\theta}\big|_{r=a} = \hat{\sigma}_{rr}\big|_{r=a} = \hat{\sigma}_{r\theta}\big|_{r=b} = 0$ $\hat{\sigma}_{rr}\big|_{r=b} = \hat{T}_r = -\hat{p}$

by use of the stress fn and the stresses at the boundary as found in the handout

$A_0' = 0$, $A_1' = 0$, $A_2' = -\rho_\infty V_\infty^2$, $A_3' = A_4' = \ldots = A_\infty' = 0$, $B_n' = 0$

$A_0 = A_n = B_n = 0$, $C_0 = C_n = D_n = C_0' = C_n' = D_n' = 0$

We note the due to the above coefficients the momentum equilib eqn. is satisfied identically

As in the notes for single valued displacements take $d_0 = c_0 = 0$. From the above coeffs we also get $a_0' = a_1 = c_1 = b_1' = d_1' = b_1 = a_1' = c_1' = d_1 = a_0 = b_0 = 0$

Since the only non zero term occurs for a_2, b_2, a_2', b_2' then $\forall n \geq 3$ $a_n = b_n = a_n' = b_n' = 0$

Since the traction on the inner boundary $= 0 \Rightarrow c_n = d_n = c_n' = d_n' = 0$ $\forall n \geq 2$.

Thus the non zero terms give rise to the following equations

$$\begin{bmatrix} -2 & 0 & -6a^{-4} & -4a^{-2} \\ -2 & 0 & -6b^{-4} & -4b^{-2} \\ 1 & 3a^2 & -3a^{-4} & -a^{-2} \\ 1 & 3b^2 & -3b^{-4} & -b^{-2} \end{bmatrix} \begin{pmatrix} a_2 \\ b_2 \\ a_2' \\ b_2' \end{pmatrix} = \begin{pmatrix} 0 \\ -\rho_\infty V_\infty^2 \\ 0 \\ 0 \end{pmatrix}$$

hence $\hat{\phi} = (a_2 r^2 + b_2 r^4 + a_2' r^{-2} + b_2')\cos 2\theta$

Now I don't know whether Prof. Barnett said not to solve this or not but here is what I found after reducing the matrix by dividing lines 1, 2 by -2.

D = denominator in the cramer rule $= (b^{-2} - a^{-2})(3a^{-4}b^2 - 9a^{-2} - 3a^2b^{-4} + 9b^{-2})$

N_1 = numerator for $a_2 = -\rho_\infty V_\infty^2 \cdot \frac{3b^{-2}}{2}(a^{-6}b^4 - 2b^{-2} + a^{-2})$

N_2 = " " $b_2 = \rho_\infty V_\infty^2 \cdot \frac{a^{-2}}{2}(a^{-2} - b^{-2})(a^{-2} + 3b^{-2})$

N_3 = " " $a_2' = \frac{\rho_\infty V_\infty^2}{2} \cdot (-3a^{-2}b^2 + a^2b^{-2} + 2)$

N_4 = " " $b_2' = \frac{\rho_\infty V_\infty^2}{2} \cdot 3(a^2b^{-4} - a^{-2} + 2a^{-4}b^2)$

$\therefore$ $a_2 = N_1/D$ $b_2 = N_2/D$ $a_2' = N_3/D$ $b_2' = N_4/D$

$\therefore$ $\hat{\sigma}_{rr} = (-2a_2 - 6a_2' r^{-4} - 4b_2' r^{-2})\cos 2\theta$

$\hat{\sigma}_{r\theta} = 2\sin 2\theta\,(a_2 + 3b_2 r^2 - 3a_2' r^{-4} - b_2' r^{-2})$

$\hat{\sigma}_{\theta\theta} = (2a_2 + 12b_2 r^2 + 6a_2' r^{-4})\cos 2\theta$

20/20

thus $\sigma_{rr} = \sigma_{rr}^0 + \hat{\sigma}_{rr}$ $\sigma_{r\theta} = \hat{\sigma}_{r\theta}$ and $\sigma_{\theta\theta} = \sigma_{\theta\theta}^0 + \hat{\sigma}_{\theta\theta}$

Stress Strain Relations and Elastic Symmetry

References: Sokolnikoff, *Mathematical Theory of Elasticity* pp. 56-91
Malvern, *Introduction To The Mechanics of A Deformable Medium*, pp 273-294

Triclinic Crystal (Most General Anisotropic material)

21 constants.

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \underbrace{\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16}\\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26}\\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36}\\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46}\\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56}\\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66}\end{bmatrix}}_{C \text{ (Stiffness) Matrix}} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix}$$

Monoclinic Crystal

One plane of elastic symmetry. e.g. ~~yz plane~~. Reflection of x axis leaves constants unchanged

Direction cosines of Transformation:

$$\begin{array}{c|ccc} & x & y & z\\ \hline x' & -1 & 0 & 0\\ y' & 0 & 1 & 0\\ z' & 0 & 0 & 1\end{array}$$

"C" Matrix: 13 unknowns

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0\\ c_{12} & c_{22} & c_{23} & c_{24} & 0 & 0\\ c_{13} & c_{23} & c_{33} & c_{34} & 0 & 0\\ c_{14} & c_{24} & c_{34} & c_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & c_{55} & c_{56}\\ 0 & 0 & 0 & 0 & c_{56} & c_{66}\end{bmatrix}$$

Orthorhombic (Orthotropic) Material

3 mutually orthogonal planes of symmetry. Reflection of x, y, and z leaves constants unchanged

Direction cosines (reflect y axis)

$$\begin{array}{c|ccc} & x & y & z\\ \hline x' & 1 & 0 & 0\\ y' & 0 & -1 & 0\\ z' & 0 & 0 & 1\end{array}$$

"C" Matrix: 9

$$\begin{bmatrix} c_{11} & c_{12} & c_{13} & & & \\ c_{12} & c_{22} & c_{23} & & & \\ c_{13} & c_{23} & c_{33} & & & \\ & & & c_{44} & & \\ & & & & c_{55} & \\ & & & & & c_{66}\end{bmatrix}$$

Cubic Material

Interchange of 2 axes (i.e. rotate 90° then reflect) leaves constants unchanged.

Direction cosines (interchange Y & Z)

$$\begin{array}{c|ccc} & x & y & z\\ \hline x' & 1 & 0 & 0\\ y' & 0 & 0 & 1\\ z' & 0 & 1 & 0\end{array}$$

"C" matrix: 3

$$\begin{bmatrix} c_{11} & c_{12} & c_{12} & & & \\ c_{12} & c_{11} & c_{12} & & & \\ c_{12} & c_{12} & c_{11} & & & \\ & & & c_{44} & & \\ & & & & c_{44} & \\ & & & & & c_{44}\end{bmatrix}$$

Isotropic Material: Any coordinate transformation leaves Elastic Constants unchanged

"C" matrix:

$$\begin{bmatrix} c_{11} & c_{12} & c_{12} & & & \\ c_{12} & c_{11} & c_{12} & & & \\ c_{12} & c_{12} & c_{11} & & & \\ & & & c_{44}^* & & \\ & & & & c_{44}^* & \\ & & & & & c_{44}^*\end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & \mu & & \\ & & & & \mu & \\ & & & & & \mu\end{bmatrix}$$

$$^*c_{44} = \frac{c_{11}-c_{12}}{2}$$

λ, μ, are the Lamé constants

$\underline{\sigma} = C\underline{\varepsilon}$ may be written $\sigma_{ij} = \lambda\,\varepsilon_{kk}\,\delta_{ij} + 2\mu\,\varepsilon_{ij}$

where ε_{kk} is the dilatation

Let's apply couple C and load F simultaneously:

$M = \frac{qx^2}{2} + C$ for $0 < x < \frac{L}{2}$

$M = \frac{qx^2}{2} + C + F\left(x - \frac{L}{2}\right)$ for $\frac{L}{2} < x < L$

End rotation due to q only:

$$\theta = \frac{\partial U^*}{\partial C} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial C}\,dx \quad \text{with } C=0,\ F=0$$

$$\frac{\partial M}{\partial C} = 1, \text{ so } \theta = \int_0^L \frac{M\,dx}{EI} = \int_0^L \frac{qx^2}{2EI}\,dx = \frac{qL^3}{6EI}$$

Midpoint lateral displacement due to q only:

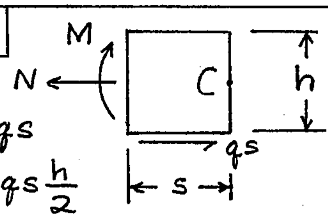
$$v = \frac{\partial U^*}{\partial F} = \int_0^L \frac{M}{EI}\frac{\partial M}{\partial F}\,dx \quad \text{with } C=0,\ F=0$$

$\frac{\partial M}{\partial F} = 0$ for $0 < x < \frac{L}{2}$

$\frac{\partial M}{\partial F} = x - \frac{L}{2}$ for $\frac{L}{2} < x < L$

$$v = \int_{L/2}^{L} \frac{qx^2/2}{EI}\left(x - \frac{L}{2}\right)dx = \frac{17qL^4}{384EI}$$

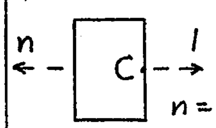
4.5-1



$N = qs$

$M = qs\frac{h}{2}$

Horizontal displacement u_c at C:

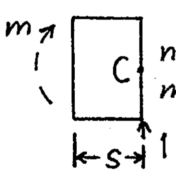


$n = 1$
$m = 0$

$$u_c = \int_0^L \frac{Nn}{EA}\,ds = \int_0^L \frac{qs\,(1)}{EA}\,ds$$

$$u_c = \frac{qL^2}{2EA} = \frac{qL^2}{2Ebh}$$

Vertical displacement v_c at C:



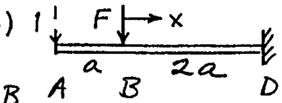
$n = 0$
$m = s$

$$v_c = \int_0^L \frac{Mm}{EI}\,ds$$

$$v_c = \int_0^L \frac{qsh/2}{EI}\,s\,ds$$

$$v_c = \frac{qh}{2EI}\frac{L^3}{3} = \frac{qhL^3}{6E}\frac{12}{bh^3} = \frac{2qL^3}{Ebh^2}$$

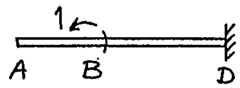
4.5-2 (a)



$M = 0$ from A to B.

For $0 < x < 2a$:
$M = -Fx$
$m = -(a+x)$

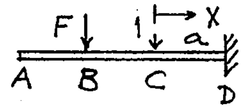
$$v_A = \int \frac{Mm}{EI}\,dx = \int_0^{2a} \frac{-Fx}{EI}\left[-(a+x)\right]dx = \frac{14Fa^3}{3EI}$$
(down)



$m = -1$

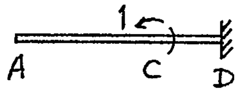
$$\theta_A = \int \frac{Mm}{EI}\,dx = \int_0^{2a} \frac{-Fx}{EI}(-1)\,dx = \frac{2Fa^2}{EI} \quad \text{(CCW)}$$

(b) $m = 0$ from A to C



For $0 < x < a$:
$M = -F(a+x)$
$m = -x$

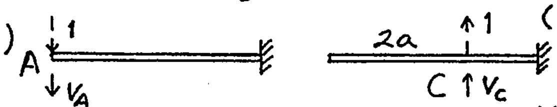
$$v_C = \int \frac{Mm}{EI}\,dx = \int_0^a \frac{-F(a+x)(-x)}{EI}\,dx = \frac{5Fa^3}{6EI}$$
(down)



$m = -1$

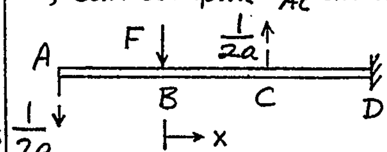
$$\theta_C = \int \frac{Mm}{EI}\,dx = \int_0^a \frac{-F(a+x)}{EI}(-1)\,dx = \frac{3Fa^2}{2EI}$$
(CCW)

(c)



Can compute v_A and v_c, then $\theta_{Ac} = \frac{v_c - v_A}{2a}$

Or, can compute θ_{Ac} all at once, as follows.



$$\theta_{Ac} = \int \frac{Mm}{EI}\,dx$$

From A to B: $M = 0$

From B to C: $M = -Fx$, $m = -\frac{a+x}{2a}$

From C to D: $M = -Fx$, $m = -1$

$$\theta_{Ac} = \int_0^a \frac{-Fx}{EI}\left(-\frac{a+x}{2a}\right)dx + \int_a^{2a} \frac{-Fx}{EI}(-1)\,dx$$

$$\theta_{Ac} = \frac{23Fa^2}{12EI} \quad \text{(CCW)}$$

(b) $M_y = 0$, $M_z = -M$. Eq. 10.1-6 gives

$$\lambda = \arctan \frac{I_y}{I_{yz}} = 38.1°$$

Max. $|\sigma_x|$ appears at points B.

Eq. 10.1-5: $y = 65$ mm and $z = -5$ mm, so

$$|\sigma_{xB}| = \frac{-I_y y + I_{yz} z}{I_y I_z - I_{yz}^2}(-M) = 102 \text{ MPa}$$

(c) $M_y = 0$, $M_z = -2500(60) = -0.15(10^6)$ N·mm

Scale answer of part (b) & add direct σ_x:

$$|\sigma_{xB}| = 102\frac{0.15}{4.00} + \frac{2500}{A}, \text{ where}$$

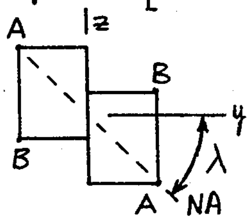
$$A = 10(130) + 2(65)10 = 2600 \text{ mm}^2$$

Hence $|\sigma_{xB}| = 3.83 + 0.96 = 4.79$ MPa

10.2-7 $I_y = 2\left[\frac{(30)40^3}{12} + 30(40)10^2\right] = 560{,}000$ mm^4

$$I_z = 2\left[\frac{(40)30^3}{3}\right] = 720{,}000 \text{ mm}^4$$

$$I_{yz} = -2\left[30(40)10(15)\right] = -360{,}000 \text{ mm}^4$$



With points A stress-free, $\tan\lambda = -1$. Now apply Eq. 10.1-7 to point B in first quadrant:

$$200 = \frac{[10 - 30(-1)]M_y}{560{,}000 - (-360{,}000)(-1)}$$

from which $M_y = 1.00(10^6)$ N·mm

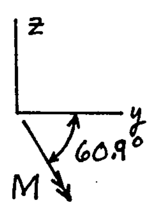
From Eq. 10.1-6 with $\tan\lambda = -1$,

$$M_z = -\frac{I_{yz} + I_z}{I_{yz} + I_y} M_y = -\frac{-36+72}{-36+56}10^6 = -1.80(10^6) \text{ N·mm}$$

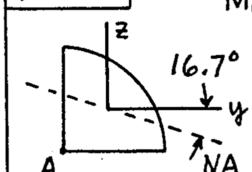
Resultant: $M = \sqrt{M_y^2 + M_z^2} = 2.06(10^6)$ N·mm

$$\beta = \arctan\frac{M_z}{M_y} = -60.9°$$

M acts in the direction shown if $\sigma_x > 0$ at B in the first quadrant.



10.2-8 $M_y = -\frac{qL^2}{8} = -\frac{6.5(2000)^2}{8}$



$M_y = -3.25(10^6)$ N·mm

$M_z = 0$ Eq. 10.1-6:

$$\lambda = \arctan\frac{I_{yz}}{I_z}$$

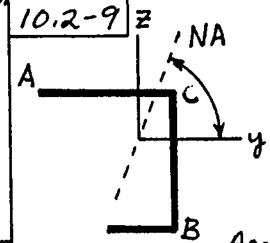
$$\lambda = \arctan\frac{-1647}{5488} = -16.7°$$

Max. tensile stress is at A, where $y = z = -4R/3\pi$. Thus, and with $\tan\lambda = -0.300$, Eq. 10.1-7 becomes

$$24 = \frac{(-4R/3\pi) - (-4R/3\pi)(-0.300)}{[0.05488 - (-0.01647)(-0.3)]R^4}(-3.25)10^6$$

from which $R^3 = 1.496(10^6)$, $R = 114.4$ mm

10.2-9



At the fixed end,

$M_y = -5000(1600) = -8.00(10^6)$ N·mm

$M_z = -6000(1200) = -7.20(10^6)$ N·mm

Apply Eq. 10.1-6:

$$\tan\lambda = \frac{-8(-4.8) + (-7.2)(16.6)}{-8(8.4) + (-7.2)(-4.8)} = 2.485$$

$\lambda = 68.1°$ So extreme σ_x values are at A, B.

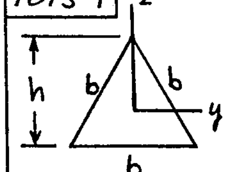
Can use Eq. 10.1-7.

$$I_y - I_{yz}\tan\lambda = [16.6 - (-4.8)2.485]10^6 = 28.53(10^6) \text{ mm}^4$$

$$\sigma_{xA} = \frac{80 - (-150)2.485}{28.53(10^6)}(-8.00)10^6 = -127 \text{ MPa}$$

$$\sigma_{xB} = \frac{-120 - (50)2.485}{28.53(10^6)}(-8.00)10^6 = 68.5 \text{ MPa}$$

10.3-1



About any centroidal axis of x-sec., moment of inertia I has the same value.

$$I = I_y = \frac{bh^3}{36} = \frac{b}{36}\left(\frac{\sqrt{3}}{2}b\right)^3$$

(a) Apply P in negative z direction: on top,

$$\sigma_x = \frac{PL(2h/3)}{I} = \frac{24PL}{bh^2} = \frac{24PL}{3b^3/4} = \frac{32PL}{b^3}$$

Apply P in negative y direction: on rt. edge,

$$\sigma_x = \frac{PL(b/2)}{I} = \frac{18PL}{h^3} = \frac{48PL}{\sqrt{3}b^3} = 27.7\frac{PL}{b^3}$$

(b) For any orientation of P in the tip cross section, tip deflection has magnitude

$$\Delta = \frac{PL^3}{3EI} = \frac{12PL^3}{Ebh^3} = \frac{32PL^3}{\sqrt{3}Eb^4} = 18.48\frac{PL^3}{Eb^4}$$

10.2.9. $q = \dfrac{(-V_z I_{yz} + V_y I_y)\bar{y} + (V_z I_z - V_y I_{yz})\bar{z}}{I_y I_z - I_{yz}^2} A_s$

$V_y = -6$ kN $A_s = 200t$
$V_z = 5$ kN
@ C $\bar{z} = 80$ mm $\bar{y} = -50$

$\therefore q = 8.31t$ $\tau_c = 8.31 \times 10^6$ Pa

With the use of Eqs. (14.21) and (14.22), the four independent shear flows can be evaluated explicitly and the unit angle of twist α is given by the expression on any one of the sides of the equality signs in Eq. (14.22). The following numerical example will illustrate the procedure.

Example 14.2 Consider a bar of thin-walled closed section having the geometry illustrated in Fig. 14.7*a*. While the thicknesses t vary from wall to wall as indicated, they are assumed to remain constant along each wall. If the bar is subjected to a torque $T = 900{,}000$ in.-lb and length $a = 9$ in., determine the shear-stress distribution in the walls and the unit angle of twist of the section ($G = 4 \times 10^6$ psi).

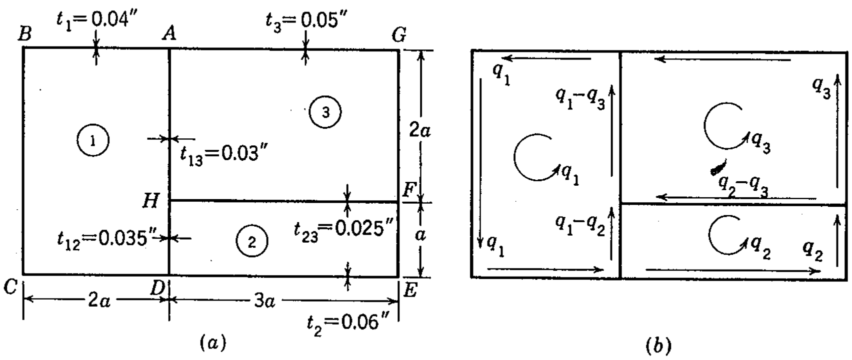


Fig. 14.7

Solution Subject to the condition that the total shear flow away from a junction of the section be equal to the total shear flow toward the junction, the shear-flow distribution in the walls is as indicated in Fig. 14.7*b*. With the use of the given dimensions, the areas enclosed by the three cells are found to be

$$A_1 = A_3 = 6a^2 \qquad A_2 = 3a^2 \tag{a}$$

Further, since t is assumed to remain constant along each wall, it can be shifted out of the integral signs from equations analogous to those in Eq. (14.22) and the resulting line integrals $\int dS$ can be readily evaluated. Thus, for Fig. 14.7*a*

$$\begin{aligned} S_1 &= ABCD = 7a & S_2 &= DEF = 4a & S_3 &= FGA = 5a \\ S_{12} &= DH = a & S_{23} &= HF = 3a & S_{31} &= HA = 2a \end{aligned} \tag{b}$$

Next, with the use of the areas and the lengths of the wall segments in Eqs. (*a*) and (*b*), respectively, the three equations for the determination of the independent shear flows q_1, q_2, and q_3 are found by analogy with Eqs. (14.21) and (14.22) to be

$$6a^2q_1 + 3a^2q_2 + 6a^2q_3 = \frac{T}{2}$$

$$\begin{aligned} \alpha_1 &= \frac{1}{12Ga^2}\left[\frac{7aq_1}{t_1} + \frac{a(q_1-q_2)}{t_{12}} + \frac{2a(q_1-q_3)}{t_{31}}\right] \\ &= \alpha_2 = \frac{1}{6Ga^2}\left[\frac{4aq_2}{t_2} + \frac{3a(q_2-q_3)}{t_{23}} - \frac{a(q_1-q_2)}{t_{12}}\right] \\ &= \alpha_3 = \frac{1}{12Ga^2}\left[\frac{5aq_3}{t_3} - \frac{3a(q_2-q_3)}{t_{23}} - \frac{2a(q_1-q_3)}{t_{31}}\right] \end{aligned} \tag{c}$$

For the given values of the thicknesses (Fig. 14.7*a*), the solution to these three independent equations is

$$q_1 = 0.0292\frac{T}{a^2} \qquad q_2 = 0.0347\frac{T}{a^2} \qquad q_3 = 0.0368\frac{T}{a^2} \tag{d}$$

= 324.44   385.56   408.89

Hence, the shear flows in the remaining walls of the section are (Fig. 14.7*b*)

$$q_{12} = q_1 - q_2 = -0.0055\frac{T}{a^2} \qquad q_{23} = q_2 - q_3 = -0.0021\frac{T}{a^2}$$

-61.12   -23.33

$$q_{31} = q_1 - q_3 = -0.0076\frac{T}{a^2} \tag{e}$$

-84.45

The negative signs indicate that the directions are opposite to those assumed in Fig. 14.7*b*. For the given values of T and a, use of the relation $\tau = q/t$ in Fig. 14.7*b* yields the shear-stress distribution

$$\begin{aligned} &\tau_1 = 8110 \text{ psi} \qquad &\tau_2 = 6420 \text{ psi} \qquad &\tau_3 = 8180 \text{ psi} \\ &\tau_{12} = -1750 \text{ psi} \qquad &\tau_{23} = -930 \text{ psi} \qquad &\tau_{31} = -2810 \text{ psi} \end{aligned} \tag{f}$$

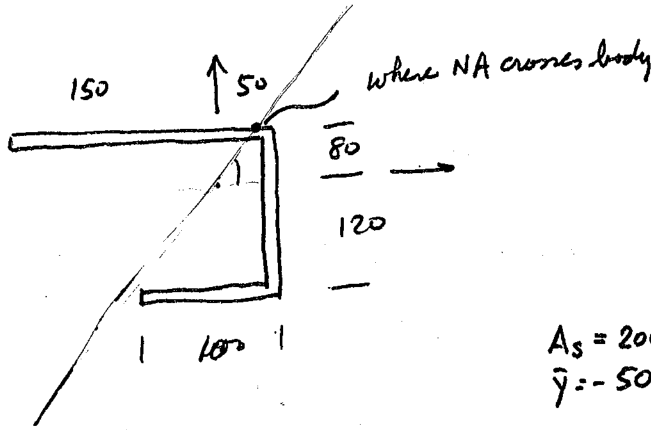
Finally, the unit angle of twist α of the entire section must be equal to the unit angles of twist of each cell individually. Hence, substitution of Eq. (*d*) in any one of the expressions for the angles in Eq. (*c*) and use of the given values of T, a, and G yield

$$\alpha = 0.3700\frac{T}{Ga^3} = \frac{0.3700 \times 900{,}000}{4 \times 10^6 \times 9^3} = 0.1142 \times 10^{-3} \text{ rad/in.} \tag{g}$$

14.4 THE EFFECT OF RESTRAINED WARPING

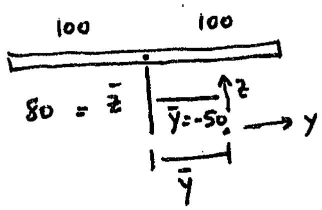
The discussion in the present chapter has thus far been restricted to the pure twisting of bars whose cross sections warp freely. In many practical cases, however, external constraints cause some sections of a bar to remain plane. Thus, when the bar is subjected to pure torsion, the prevention of out-of-plane displacements at the constrained sections curtails the free warping of the other sections. Such a deformation pattern of zero warping of the constrained planes and restrained warping of the others must give rise to normal stresses in the axial direction which produce a bending of the bar. Since these normal stresses at a section depend upon the amount of restraint to warping at the section, they generally vary along the axis of the bar. Therefore, they must be accompanied by transverse shear stresses caused by bending and these, too, generally vary from section to section along the axis. Since the transverse shear stresses and the shear stresses due to twisting at a section must jointly give rise to the torque at the section, it is evident that the unit angle of twist no longer remains constant in the axial direction. The type of mechanical behavior discussed thus far is also exhibited by a bar without cross-sectional constraints when it is subjected to an axially varying torque which produces a differential warping of the cross sections.

In a bar of solid cross section subjected to pure torsion, the effect of constraining a plane is usually restricted to the vicinity of the plane. Therefore, by St. Venant's principle, a greater portion of the bar a few cross-sectional dimensions away from the plane can be assumed to warp freely.



$A_s = 200t$

$\bar{y} = -50\,\text{mm} \quad \bar{z} = 80\,\text{mm}$



$$q = \frac{(-V_z I_{yz} + V_y I_y)\bar{y} + (V_z I_z - V_y I_{yz})\bar{z}}{I_y I_z - I_{yz}^2} A_s$$

$V_y = -6000\,N \qquad I_y = 16.6 \times 10^{-6}\,m^4$

$V_z = 5000\,N \qquad I_z = 8.4 \times 10^{-6}\,m^4$

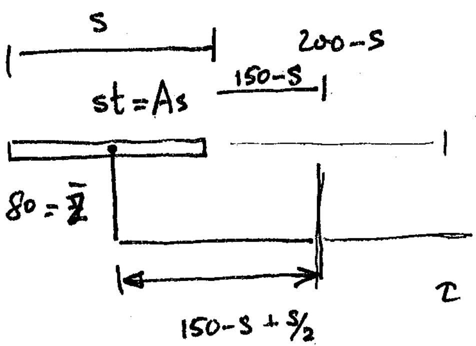
$I_{yz} = -4.8 \times 10^{-6}\,m^4$

$$q = \frac{-.0756\,\bar{y} + (.0132)\bar{z}}{116.4 \times 10^{-12}}\, .2t$$

$$\tau = \frac{-.0756(-.05) + .0132(.08)}{116.4 \times 10^{-12}}\, .2$$

$8.309 \times 10^6 \frac{N}{m^2}$

$+8.31\,MPa$ or $+8.31 \frac{N}{mm^2}$



$\bar{y} = -(150 - s/2)$

$$\tau = \frac{\left[-(-.0756)(.15 - \frac{s}{2}) + .0132(+.08)\right]s}{\text{denom}}$$

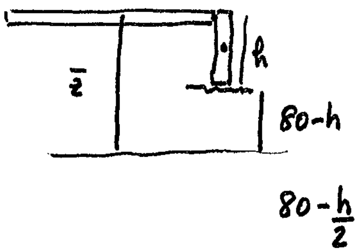
$.0124s \; - .0378s^2$

$$\tau = \cancel{\frac{-.0102428 + .0271s^2}{\text{denom}}}$$

$$\frac{d\tau}{ds} = 0 = \cancel{\frac{-.010242 + .0542s}{\text{denom}}} \quad +0.0124 - .0756s \quad \checkmark$$

$\Rightarrow s = \cancel{.189} \qquad s = .164$

$\tau = +8.734\,MPa$



$$\bar{z}A_s = \bar{z}_1 A_1 + \bar{z}_2 A_2$$
$$= .080(.2t) + (.08-\tfrac{h}{2})ht$$

$$\bar{y}A_s = \bar{y}_1 A_1 + \bar{y}_2 A_2$$
$$= (-.05).2t + .05(ht)$$

$$\tau = \frac{(-.0576)\left[-.05(.2)+.05\cdot h\right] + .0132\left[.08(.2)+(.08-\tfrac{h}{2})h\right]}{\text{denom}}$$

$$\tau = \frac{.0004872 - .001824h - .0066h^2}{\text{den}} \quad \cancel{-.0009649 + .000598h + .0132h}$$

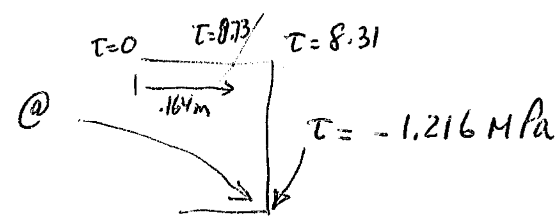
$$\frac{d\tau}{dh} = 0 = -.001824 - .0132h \quad \cancel{+.000898 - .0264h}$$

$h = -.138$

$h = -\cancel{.02268}$ impossible since $h>0$

$h<0$ is off body

implies $\tau = +8.31$ MPa at top.



Florida International University
Department of Mechanical Engineering

EGM 5615 FINAL EXAMINATION 6 December 2002

This examination will be a takehome exam. This exam allows you to use your book and notes only as well as one book on fluid mechanics. This exam is due 9 December at 5 pm in my office EAS3462

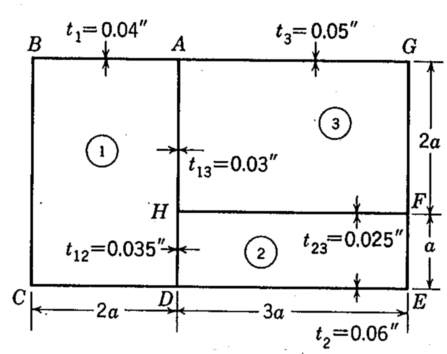
Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of this may result in failure of the exam.

PRINT NAME SIGN NAME

This examination consists of **four problems with several parts to one of the problems. Do all problems.** Read each question carefully. Show all work!!!!!

Problem 1 (30 points). Consider a bar of thin-walled closed section having the geometry illustrated in the accompanying figure. While the thicknesses t vary from wall to wall as indicated, they are assumed to remain constant along each wall.
If the bar is subjected to a torque T= 9 x 10^5 in-lbs and length a = 9 inches, determine the shear stress distribution in the walls and the angle of twist per unit length of the section. Take G = 4 x 10^6 psi.

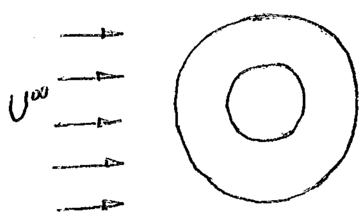


Problem 2 (20 points). Please do problem 10.2.9 on page 339 of Cook and Young.

(b) Also determine the shear stress at a point a little to the left of the upper right hand corner of the figure.
(c) Where will the maximum shear stress occur? What will its value be? Take t= 4.8 mm

Problem 3 (30 points). A hollow cylinder of very long length, inner radius Ri and outer radius Ro, is submerged in an infinite, incompressible, *inviscid* fluid flowing with a uniform velocity U^{∞} as shown.
Determine the stress state in the cylinder at steady state flow. Clearly state all assumptions you are making in formulating the associated boundary value problem. Indicate which book on Fluid Mechanics you are using.

Hint: Look at potential flow theory. Assume no pressure on the inner surface and find an expression for the pressure on the outer surface as a function of U^{∞} and the angle θ. The angle θ is measured positive counterclockwise from the x axis. Show that the pressure is in the form $A+B \bullet f(\theta)$. Write the boundary conditions on σ_{rr} and $\sigma_{r\theta}$ from this information and solve.



Problem 4 (20 points). Do problem 4.5.2 part (c) only, found on page 121 of Cook and Young.

2.32c,g ∮2.34, 1.4.2, 1.

CHAPTER 1

Orientation. Review of Elementary Mechanics of Materials

This chapter includes a selective and *brief* review of important assumptions, procedures, and results from a first course in mechanics of materials. Some items of importance are incorporated in subsequent chapters rather than appearing here. The reader is encouraged to consult a textbook of elementary mechanics of materials for detailed treatment of material reviewed in this chapter.

1.1 METHODS OF STRESS ANALYSIS

Typical questions posed in stress analysis are: Given the geometry of a body or structure, as well as its material properties, support conditions, and time-independent loads applied to it, what are the stresses and what are the displacements? A solution may be obtained by analytical, numerical, or experimental methods. Analytical methods include *mechanics of materials* and *theory of elasticity*. This book considers both, and places emphasis on the first.

Mechanics of materials is the engineer's way of doing stress analysis. The method involves the following steps.

1. Consider deformations produced by load, and establish (or approximate) how they are distributed over the body. This may be done by experiment, intuition, symmetry arguments, and/or prior knowledge of similar situations.
2. Analyze the geometry of deformation to determine how strains are distributed over a cross section.
3. Determine how stresses are distributed over a cross section by applying the stress-strain relation of the material to the strain distribution.
4. Relate stress to load. This step involves drawing a free-body diagram and writing equations of static equilibrium. The result is a formula for stress, typically in terms of applied loading and geometric parameters of the body.
5. Similarly, relate load to displacement, either by integration of the strain distribution determined in step 2 or by using energy arguments that relate work done by applied loads to elastic strain energy stored.

Results of a mechanics of materials analysis may be exact, or good approximations, or rough estimates, depending mainly on the accuracy of assumptions made in the first step. Examples of the foregoing analysis are reviewed in subsequent sections, which point out that a substantial list of restrictions is needed if the resulting formulas are to be valid.

Theory of elasticity is the mathematician's way of doing stress analysis. In this method, one seeks stresses and displacements that simultaneously satisfy the requirements of equilibrium at every point, compatibility of all displacements, and boundary conditions on stress and displacement. In contrast to the mechanics of materials method, this method does not operate under any initial assumption or approximation about the geometry of deformation. Therefore theory of elasticity can solve a problem for which deformations cannot be reliably anticipated, such as the problem of determining stresses around a hole in a plate. However, the technique is more difficult than the mechanics of materials method and cannot be successfully applied to as great a variety of practical problems. Often, a practical problem is treated by a mixture of elasticity and mechanics of materials techniques.

Many problems of stress analysis are best solved numerically, on computers that range from PCs to supercomputers. Numerical analysis software is powerful and versatile; it has become comparatively easy to use and presents results graphically with great polish. None of this analytical power assures that results are even approximately correct. An analyst might easily blunder in deciding what simplifications are appropriate, in choosing the specific computational procedures to use, or in preparing input data. Computed results may contain large errors and, in any case, must be checked against results obtained in some other way. Mechanics of materials analysis serves well for checking, even in cases where it provides only a rough approximation. Regardless of the analysis method, success in solving a problem depends mainly upon the analyst's having clear insight into the phenomenon under study.

An analysis, by any method other than experiment, is applied to a model of reality rather than to reality itself. One cannot possibly take full account of the numerous details of the actual problem. Accordingly, the model is an idealization, in which geometry, loads, and/or support conditions are simplified, based on the analyst's understanding of which aspects of the actual problem are unimportant for the purpose at hand. Thus, a stress raiser may be temporarily neglected, weight of the body may be ignored, or a distributed load may be regarded as acting at a point. (As a practical matter, even the magnitude of loading is not usually known with much precision.) After devising a model, one must do all appropriate analyses. For example, one must not stop with stresses if buckling is also a possible mode of failure. Accordingly, a goal of studying stress analysis is to learn what idealizations and analysis goals are appropriate, which implies that one must learn how bodies of various shapes and support conditions respond to various loads.

Finally, some words about derivations. Why study the derivation of a formula? First, it makes the formula plausible. A more important reason is that a derivation makes clear the assumptions and restrictions needed in order to obtain the formula. Thus, by knowing the derivation, one can recognize situations in which a formula should *not* be applied.

1.2 TERMINOLOGY

The following list is far from exhaustive. Terms listed are used throughout this book.

Beam: An elongated member, usually slender, intended to resist lateral loads by bending.

Body force: A loading that acts throughout a body rather than only on its surface. Self-weight and the inertia force of spinning about an axis are instances of body force.

Boundary conditions: Prescribed displacements at certain locations; for example, the stipulation that the supported end of a cantilever beam neither translates nor rotates. These boundary conditions may also be called *support conditions.* The term "boundary conditions" may also indicate prescribed stresses, forces, or moments. For example, at the unsupported end of a cantilever beam loaded only by its own weight, transverse shear force and bending moment must both vanish.

Brittle behavior: A material failure in which fracture surfaces show little or no evidence that failure has produced permanent deformation.

Cold working: Deformation that results in residual stresses. (In contrast, *hot working* is deformation at high enough temperature that stresses quickly dissipate by annealing.) Cold working by *shot peening* is the bombarding of an object by metal shot (roughly 0.2 mm to 4 mm in diameter) thrown at substantial velocity (roughly 70 m/s), the purpose being to produce residual compressive stresses in the surface layer.

Curvature: The reciprocal of the radius of curvature ρ, that is, $\kappa = 1/\rho$; used in beam theory.

Ductile behavior: Material behavior in which appreciable permanent deformation is possible without fracture.

Elastic: Material behavior in which deformations produced by load disappear when load is removed.

Elastic limit: The largest uniaxial normal stress for which material behavior is elastic. (Compare *yield strength.*)

Elastic modulus: The ratio of axial stress σ_a to axial strain ϵ_a in uniaxial loading; $E = \sigma_a/\epsilon_a$. Restricted to a linear relation between σ_a and ϵ_a. Also called *modulus of elasticity* or *Young's modulus.*

Fixed: A boundary condition in which all motion is prevented. Also called *built-in, clamped,* or *encastre.*

Flexure: Bending.

Frame: A structure built of bars, in which relative rotation between bars is prevented at joints, as by welding bars together where they meet. Bending of the bars is usually important in the calculation of stresses. (Compare *truss.*)

Homogeneous: Having the same material properties at all locations.

Isotropic: Having the same properties (stiffness, strength, conductivity, etc.) in every direction. As examples, glass is isotropic, wood is not isotropic. (Compare *orthotropic.*)

Lateral: Directed to the side; thus, directed normal to the axis of a beam or normal to the surface of a plate or a shell.

Nonlinear problem: A problem in which deflections or stresses are not directly proportional to the load that produces them. An example is the contact stress where a train wheel meets the rail. The area of contact grows as load increases. Another example is an initially flat membrane, like a trampoline. Lateral load is resisted by forces in the membrane that are functions of both the amount of deflection and the deflected shape.

Orthotropic: Having different stiffness (or other properties) in different directions, with the directions of maximum and minimum stiffness being mutually perpendicular. (Compare *isotropic.*)

Permanent set: Deformation that remains after removal of the load that produced it.

Plastic: A state of stress or deformation that results in permanent set if the load is removed.

Poisson's ratio: Designated by ν, where $\nu = -\epsilon_t/\epsilon_a$, and ϵ_t and ϵ_a are respectively the transverse and axial strains produced by a uniaxial stress σ_a below the proportional limit.

Principal stress: A normal stress σ, acting on an area A (or dA) when A (or dA) is free of shear stress. In this book, numerical subscripts on principal stresses indicate algebraic ordering, maximum to minimum; that is, $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

Prismatic member: A straight bar with identical cross sections. In other words, a uniform straight member; the solid generated by translating a plane shape along a straight axis normal to its plane.

Proportional limit: The largest uniaxial normal stress for which stress is directly proportional to strain. (Compare *yield strength.*)

Safety factor (*SF*): The number by which the working load (the maximum load anticipated in normal service) must be multiplied to produce the design load (the load that causes failure). If the loading has more than one component force or moment, all components must change proportionally if this definition is to apply. If stress is the quantity indicative of failure, and if stress is directly proportional to applied load, then *SF* can also be regarded as the number by which the stress that causes the material to fail must be divided in order to obtain the allowable stress, which is the maximum stress to be allowed in service. Typically, design codes prescribe allowable stresses. The number chosen for *SF* is influenced by uncertainties about loads, material properties, quality of fabrication, and accuracy of design procedures; by the cost of failure; and by the cost of adopting a large *SF.*

Saint-Venant's principle: The proposition that two statically equivalent loadings, applied (separately) to the same region of a body, each produces essentially the same state of stress and deformation in the body at distances from the loaded region greater than the larger dimension of the loaded region. (*Caution:* This principle is not reliable for thin-walled construction or for some orthotropic materials.)

Shaft: An elongated member, usually slender and straight, intended to resist torsional loads.

Shear modulus: The ratio of shear stress τ to shear strain γ; $G = \tau/\gamma$. Restricted to a linear relation between τ and γ. Also called *modulus of rigidity.*

Simply supported: A boundary condition in which lateral displacements are prevented but rotations are allowed. A simple support applies no moment to a structure. A simple support may also be called *pinned* or *hinged.*

Static indeterminacy: A condition in which one is unable to calculate all support reactions, or all internal forces or stresses, by use of only the conditions of static equilibrium. (Deformations must also be considered in order to obtain a complete solution.)

Static load: A load that does not vary with time. A more precise term would be "quasi-static load," because a truly static load could be neither applied nor removed.

Superposition: The principle that two or more static loads, applied sequentially in any order, produce the same final result as obtained by applying all loads simultaneously. The principle is not applicable in instances of nonlinearity of response, under either an individual load or combinations of loads.

Transverse: Across. Thus, for load or deflection, the same as *lateral.*

Truss: A structure built of bars in which each bar is idealized as a two-force member, as if ends of bars were connected together by frictionless pins. (Compare *frame.*)

Yield strength: The maximum uniaxial tensile stress that can be applied without exceeding a specified permanent set upon release of load. It may also be called *yield stress.* The specified permanent set is often taken as an axial strain of 0.002. In a metal, numerical values of the elastic limit, proportional limit, and yield strength are usually quite similar.

1.3 PROPERTIES OF A PLANE AREA

Properties of a plane area are often needed, particularly for beam problems. The more essential properties and manipulations are reviewed here.

Definitions. Consider a plane area A, with rectangular Cartesian coordinates st in the same plane, Fig. 1.3-1a. By definition,

$$I_s = \int_A t^2\, dA \qquad I_t = \int_A s^2\, dA \qquad I_{st} = \int_A st\, dA \tag{1.3-1}$$

I_s and I_t are *moments of inertia,* about s and t axes respectively. I_{st} is the *product of inertia.* I_s and I_t are always positive, but I_{st} can be positive, negative, or zero. Contributions $st\, dA$ are positive for areas dA in the first and third quadrants and negative for areas dA in the second and fourth quadrants (Fig. 1.3-1b). If s or t is a symmetry axis of A, then $I_{st} = 0$. The argument is shown in Fig. 1.3-1b. Each contribution $+st\, dA$ is matched by a contribution $-st\, dA$. Summing over A, we obtain $I_{st} = 0$.

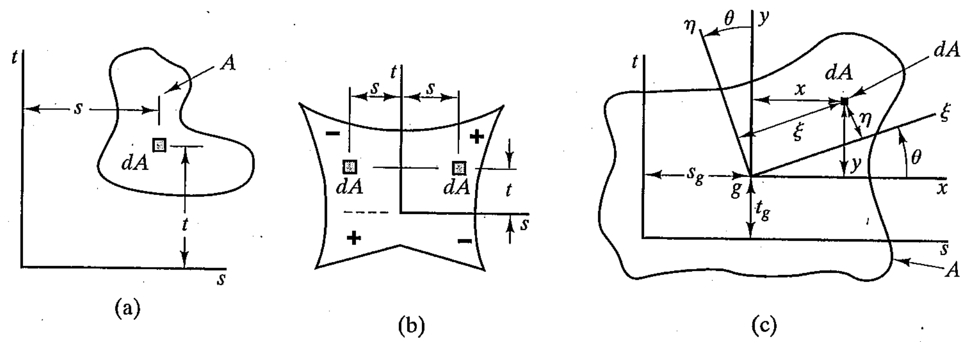


FIGURE 1.3-1 (a) Arbitrary plane area A. (b) Plane area symmetric about the t axis. Quadrants bear signs corresponding to their contribution to I_{st}. (c) Plane area with centroidal axes xy and $\xi\eta$.

Parallel Axis Theorems. These theorems relate quantities in Eq. 1.3-1 to corresponding quantities referred to *parallel* axes in the plane of A whose origin is at the *centroid of area* A. In Fig. 1.3-1c, let x and y be rectangular centroidal axes of A, respectively parallel to axes s and t, and in the plane of A. The parallel axis theorems are

$$I_s = I_x + At_g^2 \qquad I_t = I_y + As_g^2 \qquad I_{st} = I_{xy} + As_g t_g \tag{1.3-2}$$

where $I_x = \int y^2\, dA$, $I_y = \int x^2\, dA$, and $I_{xy} = \int xy\, dA$. Distances s_g and t_g are the coordinates of centroid g in the st system. These distances carry algebraic signs (both are positive in Fig. 1.3-1c). The argument for the last of Eqs. 1.3-2 is as follows. Substitute $s = s_g + x$ and $t = t_g + y$ into Eq. 1.3-1, and note that $\int x\, dA$ and $\int y\, dA$ both vanish because the xy system is centroidal. Thus

$$\begin{aligned} I_{st} &= \int_A (x + s_g)(y + t_g) dA = \int_A xy\, dA + 0 + 0 + s_g t_g \int_A dA \\ &= I_{xy} + As_g t_g \end{aligned} \tag{1.3-3}$$

The remaining two theorems in Eqs. 1.3-2 are proved in similar fashion.

Centroidal Principal Axes. In general, equations for principal axes do not require that axes be centroidal. However, in what follows, the origin of coordinates is placed at the centroid of area A because centroidal coordinates are the most useful.

Consider Fig. 1.3-1c. Systems xy and $\xi\eta$ are both rectangular, centroidal, and coplanar with A. The orientation of system xy can be chosen for convenience; for example, parallel to straight sides if area A happens to have them. System $\xi\eta$ is oriented at arbitrary angle θ with respect to system xy. Coordinates of a point in the rotated system $\xi\eta$ are $\xi = y \sin\theta + x \cos\theta$ and $\eta = y \cos\theta - x \sin\theta$. Thus we can obtain the following expressions by integration and substitution of trigonometric identities for $\sin^2\theta$, $\cos^2\theta$, and $\sin\theta\cos\theta$ (see Eqs. 1.10-1).

$$I_\xi = \int_A \eta^2\, dA \quad \text{yields} \quad I_\xi = \frac{1}{2}(I_x + I_y) + \frac{1}{2}(I_x - I_y)\cos 2\theta - I_{xy} \sin 2\theta \tag{1.3-4a}$$

$$I_{\xi\eta} = \int_A \xi\eta \, dA \qquad \text{yields} \qquad I_{\xi\eta} = \frac{1}{2}(I_x - I_y)\sin 2\theta + I_{xy}\cos 2\theta \tag{1.3-4b}$$

One can select θ so that the moment of inertia of A becomes a maximum about either the ξ axis or the η axis. If I_ξ is the maximum I, it happens that I_η is the minimum I, and vice versa. The maximum I and the minimum I are called *principal moments of inertia* and their corresponding axes are called *principal axes.* The value of θ that maximizes (or minimizes) I_ξ is called θ_p. It is determined from the equation $dI_\xi/d\theta = 0$, which yields

$$\tan 2\theta_p = \frac{2I_{xy}}{I_y - I_x} \tag{1.3-5}$$

Angle θ_p has two values, $\pi/2$ apart, one for $I_{\max}$, the other for $I_{\min}$. By using Eq. 1.3-5 in Eq. 1.3-4a, we obtain the *principal moments of inertia:*

$$I_{\max,\min} = \frac{I_x + I_y}{2} \pm \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2} \tag{1.3-6}$$

Substitution of Eq. 1.3-5 into Eq. 1.3-4b yields $I_{\xi\eta} = 0$. That is, *the product of inertia is zero for principal axes.* The converse is also true: if $I_{\xi\eta} = 0$, then axes ξ and η are principal. Therefore, *if ξ or η is an axis of symmetry, then ξ and η are principal axes.*

From Eq. 1.3-6, we see that $I_{\max} + I_{\min} = I_x + I_y$. This relation can be useful in calculation, for example to determine $I_{\min}$ when $I_{\max}$, I_x, and I_y have already been calculated. It may be physically obvious which of the two angles in Eq. 1.3-5 refers to the $I_{\max}$ axis, as in Fig. 1.3-2b. Otherwise the candidate angle can be substituted into Eq. 1.3-4a to see if I_ξ turns out to be $I_{\max}$ or $I_{\min}$. Or, adapting a formula developed for stress transformation (see below Eq. 2.2-5), the counterclockwise angle θ_p from the x axis to the axis about which I is maximum is given by $\tan\theta_p = (I_x - I_{\max})/I_{xy}$.

If $I_{\max} = I_{\min}$, angle θ does not matter. Then all centroidal axes yield the same I, and the product of inertia is zero for all these axes (Fig. 1.3-2c).

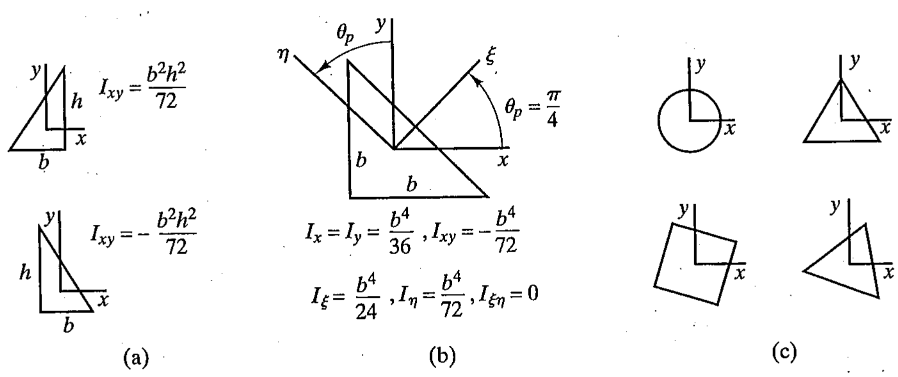


FIGURE 1.3-2 Various plane areas with centroidal axes xy. (a) Right triangles. (b) Isosceles right triangle. (c) Circle, square, and equilateral triangles. For each, $I_x = I_y$ and $I_{xy} = 0$.

Polar Moment of Inertia J. Let r be the distance from the origin of xy coordinates to an element of area dA. Then $r^2 = x^2 + y^2$, and with respect to the "pole" at $x = y = 0$,

$$J = \int_A r^2\,dA \quad \text{yields} \quad J = \int_A (x^2 + y^2)\,dA \quad \text{or} \quad J = I_y + I_x \tag{1.3-7}$$

The latter formula may be useful as a calculation device. Also, J is used in the torsional analysis of bars of circular cross section.

EXAMPLE

For the plane area in Fig. 1.3-3 we will determine I_x, I_y, I_{xy}, locate the principal centroidal axes, and determine the principal moments of inertia.

The centroid of A, at $x = y = 0$, has already been located, by means of calculations explained in textbooks about statics. For convenience in the following calculations, the cross section is arbitrarily divided into parts 1 and 2, as shown. Centroids of these parts are at $x = y = -15$ mm for part 1, and $x = y = 25$ mm for part 2. Equation 1.3-2 yields

$$I_x = \left[\frac{20(100)^3}{12} + 2000(-15)^2\right] + \left[\frac{60(20)^3}{12} + 1200(25)^2\right] \tag{1.3-8}$$

where the two bracketed expressions come from parts 1 and 2, respectively. I_y is obtained from a similar calculation and I_{xy} is

$$I_{xy} = [0 + 2000(-15)(-15)] + [0 + 1200(25)(25)] \tag{1.3-9}$$

Collecting results, we have

$$I_x = 2.907(10^6)\ \text{mm}^4 \qquad I_y = 1.627(10^6)\ \text{mm}^4 \qquad I_{xy} = 1.200(10^6)\ \text{mm}^4 \tag{1.3-10}$$

From Eq. 1.3-5, we calculate the orientation of a principal axis.

$$\tan 2\theta_p = \frac{2(1.200)}{1.627 - 2.907} \quad \text{which yields} \quad \theta_p = -31.0° \tag{1.3-11}$$

which is the clockwise angle shown in Fig. 1.3-3. The other possible angle, $\theta_p + 90°$, is a 59° counterclockwise angle from the x axis to the η axis. From Eq. 1.3-6,

$$I_{\max} = 3.627(10^6)\ \text{mm}^4 \qquad I_{\min} = 0.907(10^6)\ \text{mm}^4 \tag{1.3-12}$$

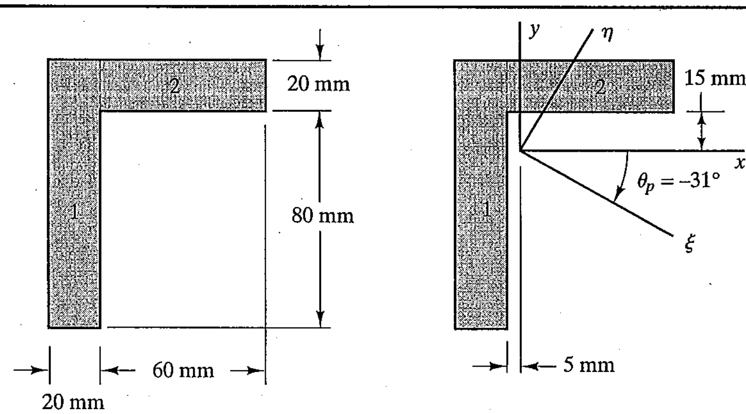


FIGURE 1.3-3 A plane area. Axes xy are centroidal. Axes $\xi\eta$ are centroidal and principal.

In this example it is clear by inspection of Fig. 1.3-3 that $I_\xi = I_{max}$ rather than $I_\eta = I_{max}$. Angle $\theta_p = -31°$ to the I_{max} axis, shown in Fig. 1.3-3, is verified by the formula $\tan\theta_p = (I_x - I_{max})/I_{xy}$.

1.4 AXIAL LOADING. PRESSURE VESSELS

Straight Bars. Consider a prismatic bar loaded by centroidal axial force P, Fig. 1.4-1a. The basic assumption about deformation is that plane cross sections remain plane when load P is applied. Thus any two cross sections a distance dx apart increase their separation an amount du (Fig. 1.4-1b), and axial strain is $\epsilon = du/dx$ at all points in a cross section. If the same stress-strain relation prevails throughout a cross section (that is, if the material is homogeneous), then axial stress σ is also the same at all points in a cross section. Equilibrium of axial forces requires that $\sigma A = P$. Thus the stress formula becomes $\sigma = P/A$. This result is not valid close to points of load application, where it is obvious that plane cross sections do not remain plane. According to Saint-Venant's principle, $\sigma = P/A$ should be an accurate formula at distances greater than ℓ from the loaded points, where ℓ is shown in Fig. 1.4-1c. The resultant force provided by a uniform stress distribution acts at the centroid of a cross section. For any cross section, load P must be collinear with this resultant. Therefore, if σ is to be uniformly distributed over a cross section, load P must be directed through centroids of cross sections. Accordingly, the bar cannot be curved. Taper, if not pronounced, causes little departure from the basic assumption; then σ is almost uniform over a cross section and is a function of axial coordinate x.

In uniaxial stress, a linearly elastic material has the stress-strain-temperature relation

$$\epsilon = \frac{\sigma}{E} + \alpha\,\Delta T \tag{1.4-1}$$

where α is the coefficient of thermal expansion and ΔT is the temperature change. From the strain expression $\epsilon = du/dx$, an increment of axial displacement is $du = \epsilon\,dx$. Combining this expression with Eq. 1.4-1 and integrating, we obtain

$$u = \int_0^L \left(\frac{\sigma}{E} + \alpha\,\Delta T\right) dx \tag{1.4-2}$$

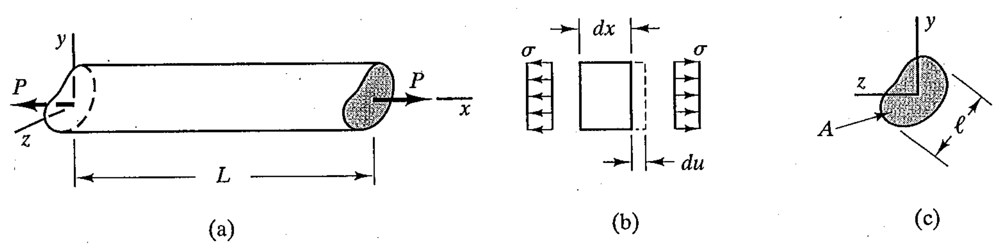


FIGURE 1.4-1 (a) Prismatic bar under centroidal axial load P. (b) Axial deformation, and axial stress σ. (c) Typical cross section.

as the axial deformation over a length L. (The symbol u is used in preference to δ or Δ in order to agree with notation in subsequent chapters, where u, v, and w denote displacement components in x, y, and z directions respectively.) Any of the quantities in parentheses in Eq. 1.4-2 may be a function of x. For the uniform bar in Fig. 1.4-1, with $\Delta T = 0$, Eq. 1.4-2 reduces to the familiar expression $u = PL/AE$. The presence of E in this formula—or in any other formula—makes it obvious that the formula is restricted to linearly elastic conditions.

Pressure Vessels. Let the cylindrical tank in Fig. 1.4-2 be thin walled, which customarily means that $r_i \approx 10t$ or more. Internal pressure causes points to displace radially but not circumferentially. Radial displacement u, greatly exaggerated, is shown in Fig. 1.4-2b. The initial length of arc CD is $r_i\, d\theta$. Its final length, after radial displacement u, is $(r_i + u)d\theta$. Its change in length is therefore $u\, d\theta$, and its circumferential strain is $\epsilon = (u\, d\theta)/(r_i\, d\theta) = u/r_i$. It is reasonable to assume that all points through the thickness have almost the same radial displacement u. Therefore, because all points also have almost the same radius, circumferential strain is almost uniform through the vessel wall. If the material is homogeneous, uniform strain implies uniform stress. Hence, summing forces in the direction of pressure p in Fig. 1.4-2c, we obtain

$$p(2r_i\, dx) = 2(\sigma t\, dx) \qquad \text{from which} \qquad \sigma = \frac{pr_i}{t} \tag{1.4-3}$$

In similar fashion one can obtain axial stress $pr_i/2t$ in the cylindrical tank and stress $pr_i/2t$ in any surface-tangent direction in a spherical tank. These formulas are not reliable, even for thin-walled pressure vessels, near changes in geometry such as AA and BB in Fig. 1.4-2a, which are circles where end caps are connected to the cylindrical vessel.

If the vessel were thick walled, we could not conclude that circumferential strains are almost uniform through the vessel wall. Imagine, for example, that $t = r_i$. Then, for circumferential strain to be the same both inside and outside, radial displacement of the outer surface would have to be twice that of the inner surface. This conclusion is unreasonable. In fact, the inside displaces somewhat more than the outside. Thus, if the wall is thick, the inner surface carries higher strain and therefore higher stress than the outer surface. Considerations from theory of elasticity are needed to obtain expressions for stresses in a thick-walled cylinder under internal pressure.

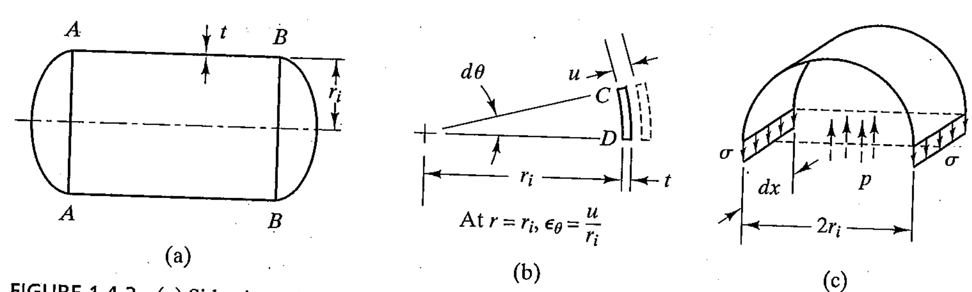


FIGURE 1.4-2 (a) Side view of a thin-walled cylindrical pressure vessel. (b) Deformation of the vessel wall due to internal pressure, viewed axially. (c) Circumferential stress, exposed by a cutting plane that contains the axis of the cylinder.

1.5 TORSION

Consider a prismatic bar of circular cross section, whose material is homogeneous and isotropic. The geometry of deformation, which may be established by experiment or by symmetry arguments, is that initially plane cross sections remain plane when the bar is twisted. Also, radial straight lines remain straight, and rotate about the axis. The diameter and length of the bar do not change. From all this one deduces that radial, circumferential, and axial normal strains are absent, and that shear strain γ varies linearly with distance r from the axis but is independent of the circumferential and axial coordinates. If a rectangular grid is drawn on the surface of the bar, one finds that twisting produces the deformed grid shown in Fig. 1.5-1a. All right angles of the grid change by the same amount. This amount is the value of shear strain γ at radius $r = c$.

Let the shear stress versus shear strain relation be linear, $\tau = G\gamma$. Then, since shear strain γ varies linearly with distance from the axis, so does shear stress τ: symbolically, $\tau = kr$, where k is a constant. To relate τ to the torque T that produces it, we consider equilibrium of moments about the axis of the bar. Thus, from Fig. 1.5-1b.

$$T = \int_A r(\tau\, dA) \qquad \text{or} \qquad T = k\int_A r^2\, dA = kJ \tag{1.5-1}$$

Hence $k = T/J$, and the expression $\tau = kr$ becomes $\tau = Tr/J$, which is the standard torsion formula. Note that τ acts on longitudinal planes as well as on transverse planes, as shown in Fig. 1.5-1c.

Figure 1.5-1c leads to a formula for θ, the angle of twist of one end of the bar relative to the other. Angles γ and $d\theta$ are small, so

$$ds = \gamma\, dx = r\, d\theta \qquad \text{hence} \qquad \theta = \int_0^L \frac{\gamma}{r}\, dx \tag{1.5-2}$$

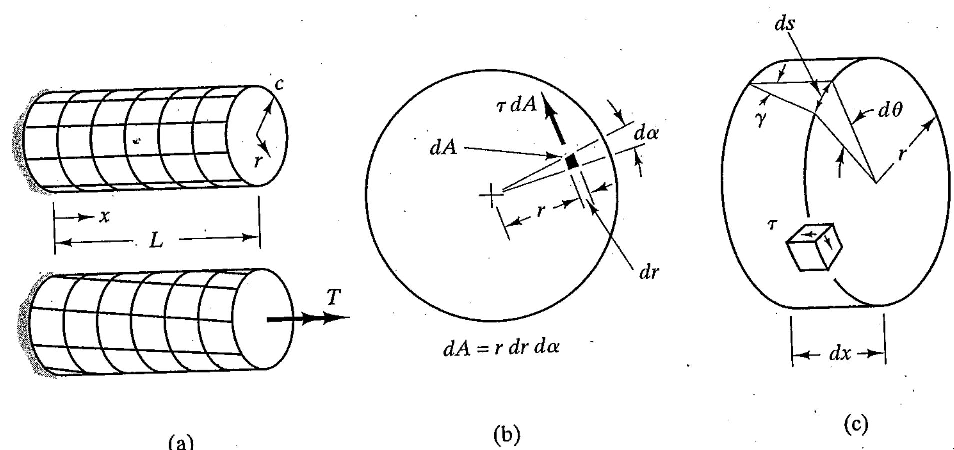


FIGURE 1.5-1 (a) Deformation produced by torque T applied to a bar of circular cross section. (b) Force increment $\tau\, dA$ produces torque increment $r(\tau dA)$. (c) Geometry of deformation that leads to a formula for angle of twist θ.

This result does not require that the material be linearly elastic. But if it is, we can substitute $\gamma = \tau/G = Tr/GJ$, whereupon the integrand in Eq. 1.5-2 becomes $T\,dx/GJ$. If T, G, and J are independent of x, we obtain the familiar expression $\theta = TL/GJ$. The presence of G in this formula makes it obvious that the formula is limited to linearly elastic conditions.

The manner of support or torsional load application, or the presence of stress raisers such as circumferential grooves, causes only local disturbances of stress, in accord with Saint-Venant's principle. These disturbances have little effect on the angle of twist. If the bar is tapered, then $J = J(x)$. The formula $\tau = Tr/J$ has little error provided the taper is slight. Changes that would invalidate our simple formulas, and the reasons why, are as follows. Orthotropy, unless it is polar about the axis of the bar, would make γ and τ depend on the circumferential coordinate as well as on r. The same effect would be produced by material properties that vary circumferentially, and by a noncircular cross section (see Section 7.11). A sharply curved geometry, as for the coil of a massive helical spring, would make γ larger toward the inside of the coil (see Section 6.1).

1.6 BEAM STRESSES

Bending. Consider a prismatic beam, whose material is homogeneous and isotropic. We require that the beam have a plane of symmetry, and that the beam be bent to an arc in this plane (Fig. 1.6-1). The geometry of deformation can be established by experiment or by symmetry arguments: Initially plane cross sections remain plane when bending moment is applied. Arbitrary cross sections AB and CD have the relative rotation $d\theta$. At coordinate y, axial strain is $-\epsilon$ and axial displacement is $-\epsilon\,dx$, negative because ϵ is compressive when y is positive. With ρ the radius of curvature, the small angle $d\theta$ can be expressed in two ways.

$$d\theta = \frac{-\epsilon\,dx}{y} \quad \text{and} \quad d\theta = \frac{dx}{\rho} \quad \text{hence} \quad \epsilon = -\frac{y}{\rho} \tag{1.6-1}$$

Thus we see that ϵ varies linearly with y. It is reasonable to assume a uniaxial state of stress. If the stress-strain relation is linear, then axial stress σ is $\sigma = ky$, where k is a constant. Two equilibrium conditions are applicable: The stress distribution provides zero axial force and bending moment M. That is,

$$0 = \int_A \sigma\,dA \quad \text{hence} \quad 0 = k\int_A y\,dA \tag{1.6-2a}$$

$$M = -\int_A y(\sigma\,dA) \quad \text{hence} \quad M = -k\int_A y^2\,dA = -kI \tag{1.6-2b}$$

Equation 1.6-2a demands that $\int y\,dA = 0$, which means that the z axis, at $y = 0$, passes through the centroid of the cross section. From Eq. 1.6-2b we obtain $k = -M/I$, where I is the moment of inertia of cross-sectional area A about its centroidal axis z. Hence the expression $\sigma = ky$ becomes $\sigma = -My/I$, which is the standard flexure formula. Typically we write simply $\sigma = My/I$, because the algebraic sign of σ at a given y is obvious from the direction of the bending moment.

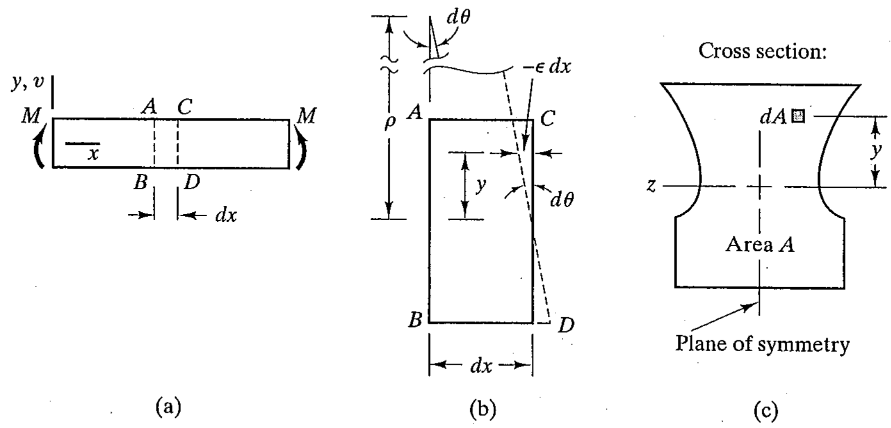


FIGURE 1.6-1 (a) Beam bent in the xy plane. (b) Deformations in the xy plane. (c) Arbitrary (but symmetric) cross section of area A. The symmetry plane of the cross section is normal to the paper.

Common situations to which the flexure formula is not applicable, or applicable only after modification, are as follow. If there is no symmetry plane, we cannot presume that axial strain ϵ is independent of z (axis z is shown in Fig. 1.6-1c). That is, Eq. 1.6-1 is no longer correct. This situation is called unsymmetric bending and is discussed in Chapter 10. If the beam has pronounced initial curvature before load is applied, plane cross sections still remain plane, but we cannot conclude that ϵ varies linearly with y (see Chapter 6). If the material is not linearly elastic, then $\sigma \neq ky$, and the latter forms of Eqs. 1.6-2 no longer apply. Similarly, if the material is not homogeneous, then $\sigma \neq ky$. Therefore we cannot use $\sigma = My/I$ to analyze a reinforced concrete beam. Finally, if the cross section is wide we must consider that the body is a plate rather than a beam (Chapter 12).

Transverse Shear. If bending moment M is not constant, a transverse shear force V exists in a straight beam. Force V produces transverse shear stress, which acts on transverse planes and on longitudinal planes. Formulas for shear flow and shear stress are derived from the flexure formula and are therefore subject to the same restrictions. From the shear flow formula, usually written as $q = VQ/I$, we obtain the *average* transverse shear stress $\tau = q/t = VQ/It$, where t is a thickness measured in the plane of the cross section. This average shear stress may be quite accurate or quite inaccurate, depending on circumstances. For example, in Fig. 1.6-2b, shear stress on plane AB is small because t in $\tau = VQ/It$ is large (here t is the width of the flange). Moreover, if plane AB is moved very close to the inner flange surface, τ must approach zero on that portion of the inner flange where the adjacent surface is free of stress. On the portion of plane AB immediately adjacent to the web, τ approaches the transverse shear stress on plane CD, where t is the web thickness and $\tau = VQ/It$ is accurate. The largest transverse shear stress in the flange is exposed by a vertical cutting plane such as EF, where t in VQ/It is the flange thickness. Details of these matters, and of how to use the formula VQ/It, appear in textbooks of elementary mechanics of materials.

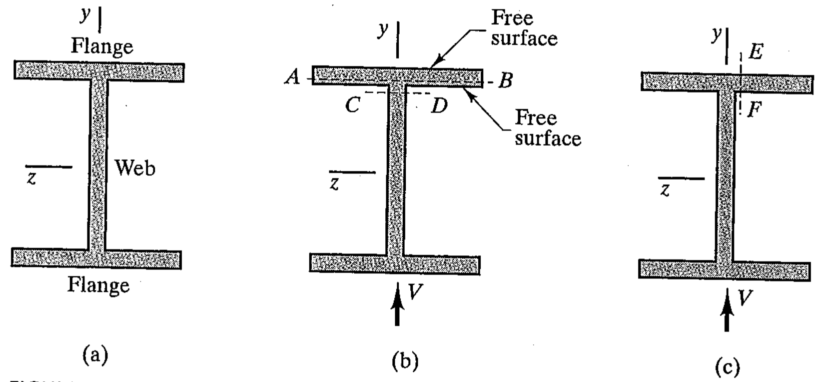


FIGURE 1.6-2 Cross section of a beam that carries transverse shear force V. Cutting planes AB, CD, and EF are normal to the yz plane.

1.7 BEAM DEFLECTIONS

Briefly, the formula that relates lateral deflection v to bending moment M is developed as follows. We use notation in Fig. 1.6-1. If $|dv/dx| \ll 1$, as is usual in practical beams, then the curvature of the deformed beam can be written as $1/\rho = d^2v/dx^2$. Also, for a linearly elastic material, Eq. 1.6-1 and the flexure formula $\sigma = -My/I$ yield another expression for curvature: $1/\rho = -\epsilon/y = -(\sigma/E)/y = -(-My/EI)/y = M/EI$. Equating the two expressions for curvature, we obtain

$$\frac{d^2v}{dx^2} = \frac{M}{EI} \tag{1.7-1}$$

Restrictions on this formula include those on the flexure formula. Also, deflections must be sufficiently small that slope $\theta = dv/dx$ of the deformed beam is everywhere much less than unity in magnitude. Transverse shear deformation has been neglected. Equation 1.7-1 actually says that M/EI is equal to the *change* in curvature. This viewpoint may become important for a beam having initial curvature before load is applied. For a beam initially straight and then bent to radius ρ, the initial curvature is zero, and the change in curvature is $(1/\rho - 0) = 1/\rho$.

An alternative form of Eq. 1.7-1 can be written, as follows. Equations of static equilibrium, applied to Fig. 1.7-1a, yield $dM/dx = V$ and $dV/dx = q$, where q is the intensity per unit length of distributed lateral load. Hence $d^2M/dx^2 = q$. For M we can substitute $EI(d^2v/dx^2)$ from Eq. 1.7-1. Thus

$$\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) = q \qquad \text{or} \qquad EI\frac{d^4v}{dx^4} = q \qquad \text{if } EI \text{ is independent of } x \tag{1.7-2}$$

The latter form will be useful in subsequent chapters.

One can determine beam deflections (or solve statically indeterminate beam problems) by integrating Eq. 1.7-1 and making use of support conditions to evaluate constants of integration (and redundant reactions). Usually it is easier to solve these problems by use of tabulated beam formulas and the superposition principle. Indeed,

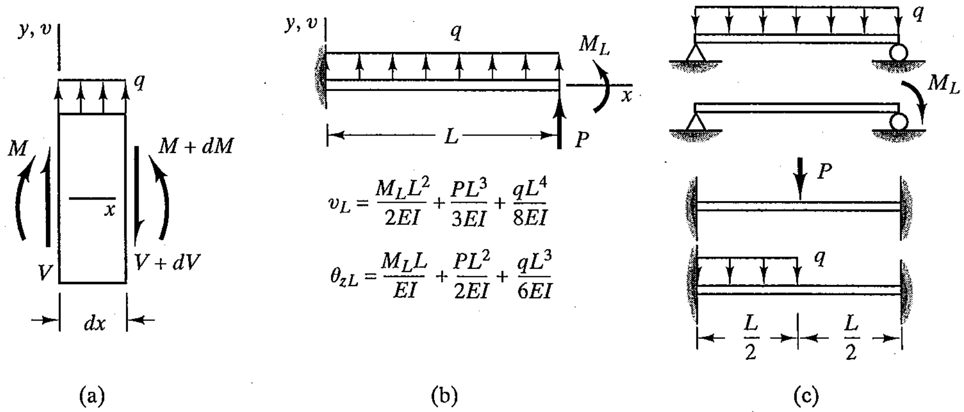


FIGURE 1.7-1 (a) Loads on a differential element of a beam. (b) Formulas for tip deflection and tip rotation of a uniform cantilever beam. (c) Problems of deflection, rotation, or static indeterminacy solvable by use of formulas in (b).

the few formulas in Fig. 1.7-1b are sufficient to solve most common problems of straight beams, including all those shown in Fig. 1.7-1c. An example problem is solved in Section 1.8. Like Eq. 1.7-1, formulas in Fig. 1.7-1b require that $|\theta| \ll 1$ throughout the beam.

1.8 SYMMETRY CONSIDERATIONS. STATIC INDETERMINACY

Symmetry Considerations. Sometimes one can exploit symmetry to obtain internal forces, determine support conditions, or reduce the effort required for analysis. For example, consider the simply supported beams in Fig. 1.8-1. Both have symmetry of geometry, elastic properties, and support conditions with respect to a plane normal to the beam axis at its center. The beams differ only in loading. In Fig. 1.8-1a, a mirror reflection of either half in the symmetry plane yields the other half in geometry, elastic properties, loading, support reactions, deformations, and internal forces at the symmetry plane. For antisymmetric loading, Fig. 1.8-1b, one half yields the other half after reflection and *reversal* of loading, support reactions, deformations, and internal forces at the symmetry plane. These considerations, in combination with the action-reaction nature of internal forces exposed by cutting open the beam, preclude the existence of shear forces V_C for symmetric loading and bending moments M_C for antisymmetric loading. Thus in either case the number of unknowns is immediately reduced by half.

The same considerations can be used in three dimensions. The semicircular beams in Fig. 1.8-2a lie in the xy plane. For each, there is symmetry of geometry, elastic properties, and support conditions about the yz plane. For symmetric loading, Fig. 1.8-2a, symmetry considerations dictate that at midpoint C there is no x direction displacement, no rotation about the y axis or the z axis, no transverse shear force in the y direction or the z direction, and no torque about the x axis. These conditions are listed in Fig. 1.8-2a. Unknowns at C are displacements v and w, rotation θ_x about the x axis, axial force F_x, and bending moments M_y and M_z about the y and z axes. These unknowns could be

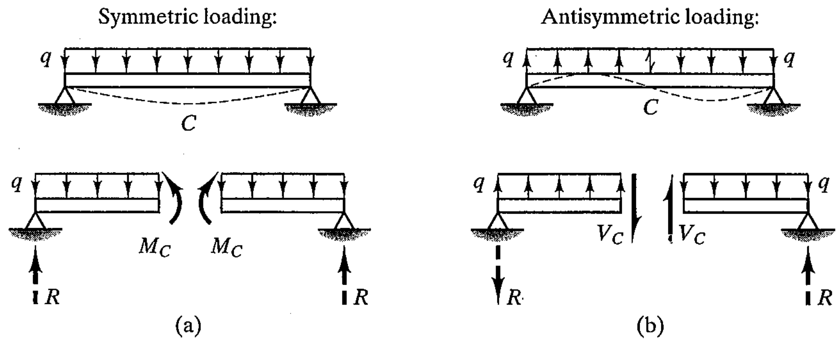


FIGURE 1.8-1 Uniform simply supported beams, showing internal moments and forces at center C. Supports apply negligible horizontal force if deflections are small.

determined by analysis of either half of the semicircular beam. In Fig. 1.8-2b two of the forces P and Q are reversed, so the load is antisymmetric. Symmetry considerations dictate the zero quantities listed in Fig. 1.8-2b. Again, analysis of either half of the beam is sufficient to determine the unknown quantities at C, which are $u, \theta_y, \theta_z, V_y, V_z$, and T_x.

The foregoing arguments are not immediately obvious. The reader is urged to consider these examples patiently, and to make supplementary sketches that show internal forces and moments.

Static Indeterminacy. The term is defined in Section 1.2. Calculations are illustrated by the following examples.

The stepped bar in Fig. 1.8-3a is all of the same material. It is to be uniformly heated from its stress-free temperature while confined between rigid walls. Statics tells us only that the walls apply forces P of equal magnitude. To determine them we must use a compatibility condition, which here is that the bar has no net change in length from end to end. Thus, taking P positive in tension and presuming that conditions are linearly elastic, we write

$$\alpha L\,\Delta T + \alpha L\,\Delta T + \frac{PL}{AE} + \frac{PL}{(2A)E} = 0 \tag{1.8-1}$$

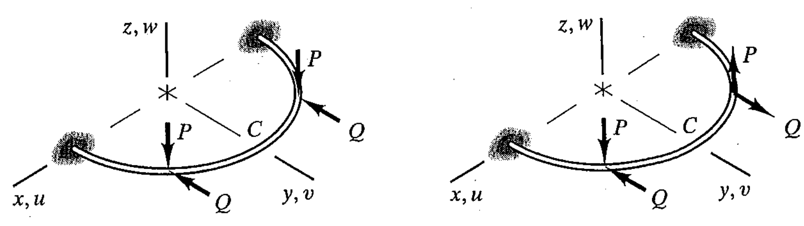
where α is the coefficient of thermal expansion and ΔT is the temperature change. Solving for P and then for stresses $\sigma_1 = P/A$ and $\sigma_2 = P/2A$, we obtain

$$P = -\frac{4EA\alpha\,\Delta T}{3} \qquad \sigma_1 = -\frac{4E\alpha\,\Delta T}{3} \qquad \sigma_2 = -\frac{2E\alpha\,\Delta T}{3} \tag{1.8-2}$$

Note that axial strains are not zero, even though the overall change in length is zero. For example, in part 1, $\epsilon_1 = (\sigma/E) + \alpha\,\Delta T = -\alpha\,\Delta T/3$. Note also that modest temperature change can produce large stress. In the present example, if the bar is steel and $\Delta T = 100°\text{C}$, then σ_1 is about 320 MPa in magnitude.

As a second example, consider the beam in Fig. 1.8-3b. It is statically indeterminate to the second degree. Symmetry considerations can be used to reduce the degree of indeterminacy. Imagine that M_C is applied as two couples $M_C/2$, an infinitesimal dis-

Symmetric loading: Antisymmetric loading:



Zero at C: u, θ_y, θ_z, V_y, V_z, T_x

(a)

Zero at C: v, w, θ_x, F_x, M_y, M_z

(b)

FIGURE 1.8-2 Uniform semicircular beams in the xy plane.

tance apart, and straddling point C. The loading is antisymmetric, so at C there is a transverse shear force V_C but zero bending moment and zero vertical displacement (Fig. 1.8-3c). Using formulas in Fig. 1.7-1b to state that the transverse displacement is zero at C, we solve for V_C and then for moment M_B at the wall.

$$-\frac{(M_C/2)a^2}{2EI}+\frac{V_C a^3}{3EI}=0 \qquad \begin{cases} V_C=\dfrac{3M_C}{4a} \\ M_B=V_C a-\dfrac{M_C}{2}=\dfrac{M_C}{4} \end{cases} \tag{1.8-3}$$

Finally, having resolved the indeterminacy, we can use Fig. 1.7-1b again to determine the rotation at C.

$$\theta_C=\frac{(M_C/2)a}{EI}-\frac{V_C a^2}{2EI} \qquad \text{hence} \qquad \theta_C=\frac{M_C a}{8EI} \tag{1.8-4}$$

Problems such as those in Fig. 1.8-3 are probably called to mind by the term "statically indeterminate analysis." However, the term is also appropriate for the derivation of conventional stress formulas such as $\sigma = My/I$: an equilibrium equation, such as the first of Eqs. 1.6-2b, yields the second only when it is known how stress varies over the cross section. The variation is obtained by consideration of displacements.

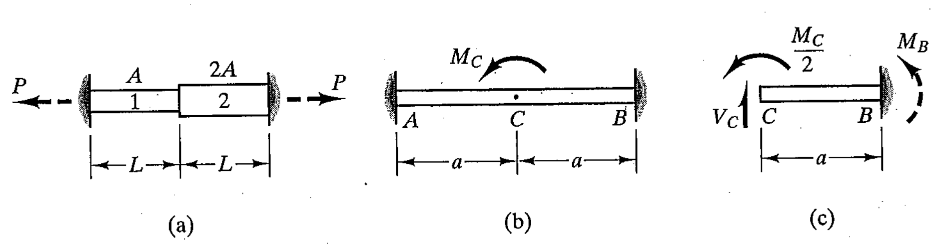


FIGURE 1.8-3 (a) Stepped bar held by rigid walls. (b) Statically indeterminate beam. (c) Right half of the beam, with symmetry considerations exploited.

1.9 PLASTIC DEFORMATION. RESIDUAL STRESS

Here we review the problem of plastic torsion for a shaft of solid circular cross section. Other instances of plastic action are considered in subsequent chapters.

The stress-strain relation in Fig. 1.9-1a is linearly elastic up to stress τ_Y and flat-topped thereafter. This idealized behavior is called "elastic–perfectly plastic" and is appropriate for low-carbon steel. Because strain hardening is ignored, calculations provide a maximum or "fully plastic" torque T_{fp} that is less than the actual maximum torque. We now ask for T_{fp} and the pattern of residual stress upon unloading.

As twist increases, yielding eventually begins. It spreads from the outer surface toward the axis of the shaft. To calculate T_{fp}, we assume that twist is sufficiently great that practically all the material has yielded. Thus, shear stress is the constant value τ_Y throughout, and the first of Eqs. 1.5-1 provides

$$T_{fp} = \tau_Y \int_0^{2\pi} \int_0^c r(r\,dr\,d\alpha) = \tau_Y \frac{2\pi c^3}{3} \qquad \text{hence} \qquad T_{fp} = \frac{4}{3}\left(\tau_Y \frac{\pi c^3}{2}\right) \tag{1.9-1}$$

where the latter expression in parentheses is the torque that initiates yielding, obtained from the torsion formula for linearly elastic conditions; that is, $\tau = Tr/J$ with $\tau = \tau_Y$ at $r = c$. This result shows that torque can be increased 33% after yielding begins.

Unloading can be accomplished by superposing on T_{fp} a torque of equal magnitude but reversed in direction. Anticipating that unloading will be elastic, we obtain the stress distribution in Fig. 1.9-1c from the reversed torque $T = T_{fp}$ and the elastic stress formula $\tau = Tr/J$. At first glance this calculation may appear wrong because the largest stress exceeds τ_Y. However, stresses in Fig. 1.9-1c always appear in *combination* with stresses in Fig. 1.9-1b. In combination, τ never exceeds τ_Y in magnitude, so unloading does not produce further yielding. If torque T_{fp} is again applied, residual stresses combine with the reverse of stresses in Fig. 1.9-1c to produce again the fully plastic stress pattern of Fig. 1.9-1b, but without renewed yielding.

The residual angle of twist after unloading cannot be calculated because we have not specified how much the shaft was twisted in producing T_{fp}. An *infinite* angle of twist would be required to bring inelastic strains all the way to $r = 0$.

What is the range of torque for which conditions are linearly elastic? If there are no residual stresses, a torque $T = \tau_Y J/c = \tau_Y \pi c^3/2$ could be applied in either direction without yielding, for an elastic range of $\tau_Y \pi c^3$. If the residual stresses in Fig. 1.9-1d pre-

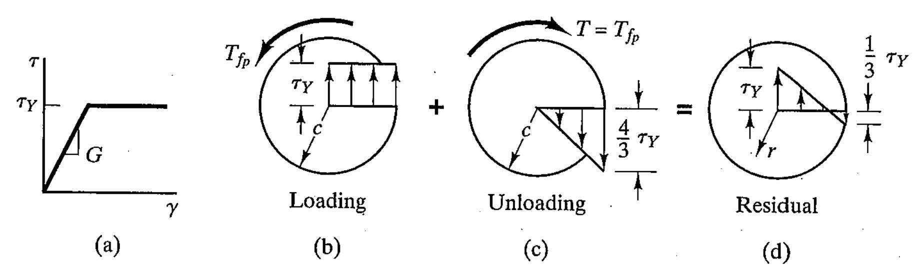


FIGURE 1.9-1 (a) Elastic–perfectly plastic material. (b,c,d) Stress distributions corresponding to fully plastic torque, unloading (reversed elastic) torque, and resultant (zero) torque.

vail, we could apply a torque T_{fp} in the original direction or $(2/3)(\tau_Y\ J/c)$ in the reversed direction without renewed yielding, for an elastic range of $\tau_Y \pi c^3$. Thus the magnitude of the elastic range has not changed.

1.10 OTHER REMARKS

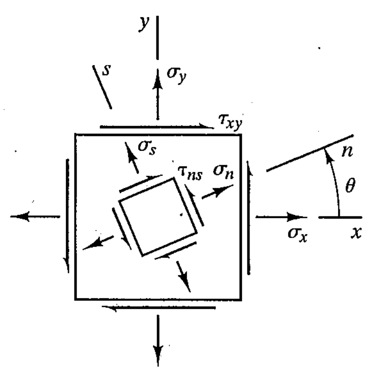
Stress Transformation. For reference, and for use in chapters that follow, two-dimensional stress transformation equations are shown in Fig. 1.10-1. These equations may be restated in other forms, for which the following trigonometric identities are useful.

$$\sin 2\theta = 2 \sin\theta \cos\theta \qquad \cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta \qquad \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta) \tag{1.10-1}$$

Dimensional Homogeneity. In the calculation of stresses and deflections, it is often best to obtain a numerical result as the final step of solution, by substitution of data into a symbolic result. Thus we avoid manipulating numbers for some quantities that may cancel if manipulated as symbols. A more important reason is that a symbolic result permits a partial check on the correctness of the solution. A valid result is dimensionally homogeneous. For example, in Fig. 1.7-1b let $[v_L]$ and $[\theta_L]$ denote the respective dimensions of deflection and rotation. With F and L used here to denote dimensions of force and length respectively, dimensions of terms that contain M_L in the formulas of Fig. 1.7-1b are

$$[v_L] = \frac{(FL)L^2}{(F/L^2)L^4} = L \quad \text{and} \quad [\theta_L] = \frac{(FL)L}{(F/L^2)L^4} = 1 \tag{1.10-2}$$

These dimensions are correct: length units for v_L and dimensionless (radians in this case) for θ_L. This result does not prove the formulas to be correct, but had we obtained any other dimensions we would know for sure that the result is wrong.



$$\sigma_n = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\,\tau_{xy} \sin\theta \cos\theta$$

$$\sigma_s = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\,\tau_{xy} \sin\theta \cos\theta$$

$$\tau_{ns} = \tfrac{1}{2}(\sigma_y - \sigma_x)\sin 2\theta + \tau_{xy} \cos 2\theta$$

FIGURE 1.10-1 Transformation of stresses in a plane.

Units. Example problems and homework problems serve as vehicles to convey concepts, principles, and procedures. Accordingly, the system of units used for numerical problems is of little importance. SI units are used in this book. Note that the average stress due to a 1 MN force on a square meter can be written in the following forms:

$$\sigma = \frac{10^6\text{N}}{1\text{ m}^2} = 10^6\text{ Pa} = 1\text{ MPa} \quad \text{or} \quad \sigma = \frac{10^6\text{N}}{(1000\text{ mm})^2} = 1\text{ MPa} \qquad (1.10\text{-}3)$$

The latter form, which is used in subsequent chapters, avoids the conversion factor of 10^6. That is, forces in newtons, dimensions in millimeters, and stresses and moduli in megapascals form a consistent set of units, without need for conversion factors. However, if mass must be considered, as for inertia force loading, it will be easier to use meters rather than millimeters.

Classification by Problem Geometry. A slender member is usually called a *bar, beam,* or *shaft,* depending on whether the load is axial, lateral, or torsional. These problems are called one-dimensional, even though stress varies over a cross section as well as axially under bending or twisting load. A flat body whose thickness is much less than its other dimensions provides a two-dimensional problem. It is usually called a *plane* problem if loads have no lateral (thickness-direction) component, and a *plate* or *plate bending* problem if they do. In general, stresses in plane and plate problems vary with both of the in-plane directions. Stresses also vary in the thickness direction of a plate under lateral load. A floor slab is a familiar example. A *shell* is like a plate, but curved; familiar examples include an egg shell and a water tank. A shell can carry both surface-tangent and surface-normal loads. Many shells, and many solids too thick to be called shells, are symmetric about an axis and have loading that is also axisymmetric. Then nothing varies in the circumferential direction and analysis is simplified. Such a body is called a *shell of revolution* if it is thin-walled or a *solid of revolution* if it is not. An example of the latter is a turbine disk of strongly varying thickness that rotates at constant speed.

Connections. In this book, as in most other books about stress analysis, we may simply state that members are connected together, without saying how, and perhaps even disregarding stress concentrations associated with the connection. Thus we limit the scope of the book. Unfortunately the reader may then infer that connections are unimportant, which is far from the case. The behavior of a real structure may depend as much on its connections as on its individual members. *Connections are often the weakest parts of a stucture.*

Bolted or riveted connections are sometimes analyzed in a first course in mechanics of materials. One learns that several modes of failure are possible and that analysis can be tedious, despite simplifying assumptions that neglect stress concentrations, friction and possible slipping, making and breaking of contacts, misalignment, initial stresses, and damage to the material from cutting, bending, and punching holes. Other complexities arise if we consider gluing, welding, and shrink fits. Practical analysis and design of connections may be done using accepted codes and procedures that differ according to type of joint, and which vary considerably with type of indus-

try. The study of connections is an important specialty in stress analysis. References include [1.1–1.5].

Handbooks. Many useful formulas for stress analysis do not appear in textbooks but may be found in handbooks or their computer software equivalents. The existence of this information does not erase the need for ability in stress analysis. Formulas can be used successfully only if the engineer understands the physical problem well enough to know what sort of formula to seek, understands the assumptions that underlie a formula, and is able to judge whether an answer produced by the formula is reasonable. Useful handbooks include [1.6, 1.7] for widespread coverage of stress and deflection, [1.8] for pressure vessels and the ASME code for them, [1.9] for buckling of bars, frames, plates, and shells, and [1.10] for modes and frequencies of vibration.

Codes. Engineering societies have produced codes that mandate allowable stresses, design procedure, and methods for testing, construction, operation, and maintainance of plants and equipment. Much of this information has grown out of experience with costly failures [1.11, 1.12]. Codes and specifications may receive little mention in engineering education, but it would be shortsighted to ignore them. Indeed, the engineer is often legally bound to follow one or more codes. Also, in situations where a code is applicable, it is likely to be the easiest route to an acceptable design. Students of structural engineering are probably familiar with design specifications of the American Institute of Steel Construction. There are a great many other codes and specifications, so many that space does not permit us to list them all.

The value of codes is illustrated by the history of boiler accidents. About the year 1900, on average, one boiler explosion occurred every day in the United States. Subsequently, codes for the design, construction, and operation of boilers were written and widely adopted. Today boiler explosions are rare despite a fifteen-fold increase in operating pressure since 1900 [1.13].

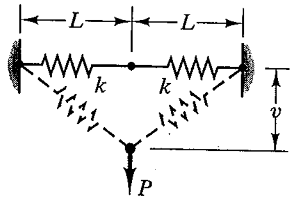
PROBLEMS

The following problems can be solved using the review material presented in this chapter, although many of the problems are less familiar or more challenging than those usually seen in an elementary textbook. Assume that materials are linearly elastic unless a nonlinear stress-strain relation is provided. State results symbolically in terms of loads, dimensions, properties of cross sections, and material constants, unless a numerical answer is required or other instructions are given.

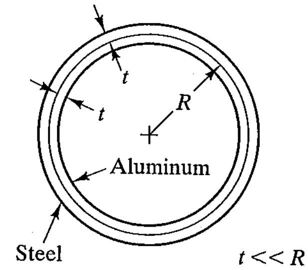
1.4-1. A prismatic bar is loaded by an axial force P. Show that P must be directed through centroids of cross sections if axial stress σ is not to vary over a cross section.

1.4-2. Springs in the structure shown are linear and are unstressed when displacement v is zero. Determine an expression for v without assuming that $v \ll L$. With $L = 100$ mm and $k = 20$ N/mm, obtain numerical values of P for displacements v of 10 mm, 40 mm, and 50 mm. Show that superposition using the first two results does not yield the third. Plot P versus v.

1.4-3. Two slender rings, one aluminum and the other steel, just fit together at temperature $T = 0°C$, as shown. What is the contact pressure between them when $T > 0°C$?

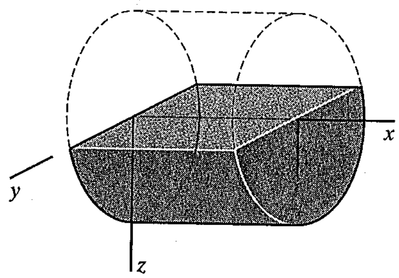


PROBLEM 1.4-2

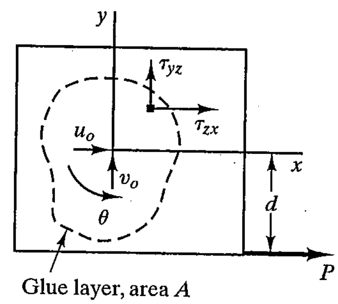


PROBLEM 1.4-3

1.5-1. A shaft of solid circular cross section is loaded by torque T. Consider a half-cylinder cut from the shaft by three cutting planes (see sketch). Show that stresses exposed by the cutting planes keep the half-cylinder in static equilibrium.



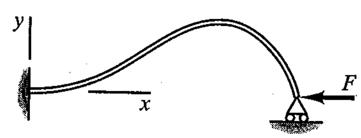
PROBLEM 1.5-1



PROBLEM 1.5-2

1.5-2. A flat plate is attached to a flat surface by a thin layer of glue of arbitrary shape and comparatively low modulus. An x-parallel load P is applied to the plate (see sketch). Axes xy are centroidal axes of the glue layer. What are shear stresses τ_{yz} and τ_{zx} in the glue layer? (Suggestion: Assume that these stresses are proportional to displacement components of the plate, and that the plate has rotation θ and translation components u_o and v_o at $x = y = 0$. Area A and its properties will appear in the solution.)

1.6-1. The sketch shows the post-buckling shape of a slender bar that was initially straight. Load F is known, and the shape $y = f(x)$ of the buckled bar is accurately known. What is the easiest way to determine support reactions at ends of the bar?



PROBLEM 1.6-1

1.6-2. In the beam of Fig. 1.6-1, it is proposed that flexural stress has the form $\sigma = ky$. Show that the cross-sectional area must have a zero product of inertia if this equation is to be correct.

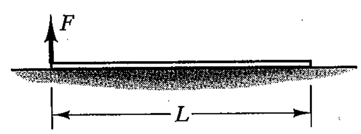
1.6-3. **(a)** Consider a prismatic beam, and conjecture that plane cross sections do *not* remain plane when bending moment is applied. Without equations, devise arguments that refute the conjecture.

(b) Similarly, consider a prismatic bar of circular cross section. Refute the conjectures that cross sections warp and radial lines become curved when torque is applied.

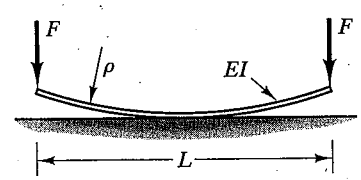
(c) The flexure formula $\sigma = My/I$ follows from the condition that plane cross sections remain plane in pure bending. A cantilever beam under transverse tip load experiences transverse shear deformation, and plane cross sections do *not* remain plane. Yet the flexure formula loses no accuracy. How can this be?

1.6-4. Let a prismatic beam have a rectangular cross section, b units wide and h units deep. The material has elastic moduli E_t in tension and E_c in compression. Derive expressions that relate stress to bending moment. The expressions should reduce to the conventional flexure formula if $E_t = E_c$.

1.6-5. The uniform beam shown has weight q per unit length. It rests on a rigid horizontal surface. If one end is lifted by a force $F < qL/2$, what is the maximum bending moment in the beam in terms of F and q?



PROBLEM 1.6-5



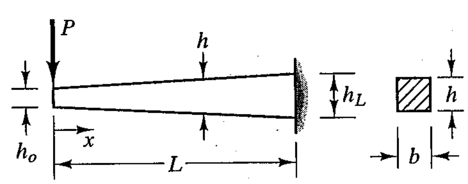
PROBLEM 1.6-6

1.6-6. When not loaded, the uniform beam shown has constant radius of curvature ρ, where $\rho \gg L$. Downward forces F are then applied to the ends.

(a) What value of F reduces curvature at the center of the beam to zero?

(b) For larger F, a central portion of length s becomes flat. Obtain an expression for s.

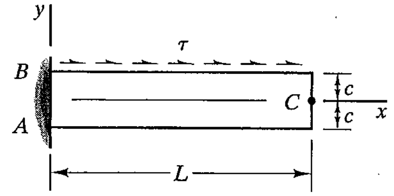
1.6-7. The beam shown has a slight taper. For what value of h_L/h_o does the largest flexural stress appear at $x = L/2$? What then is the ratio of flexural stress at $x = L/2$ to flexural stress at $x = L$?



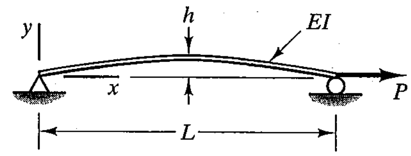
$$h = h_o + \frac{h_L - h_o}{L} x$$

PROBLEM 1.6-7

1.7-1. A cantilever beam is loaded by uniform shear stress τ applied to its upper surface only, as shown. Obtain expressions for x-direction normal stress at A and at B. Neglect stress concentration effects. Also determine the deflection components of point C.



PROBLEM 1.7-1

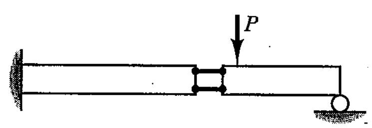


PROBLEM 1.7-2

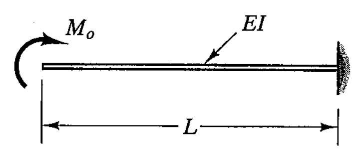
1.7-2. The slender bar shown is initially curved, so that with no load its axis has the equation $y = (4h/L^2)(Lx - x^2)$. What center deflection v_c is produced by force P? Assume that $v_c \ll h \ll L$.

1.7-3. Let the cantilever beam of Problem 1.7-1 be thermally loaded, such that the temperature varies linearly from ΔT on the lower surface to $-\Delta T$ on the upper surface. Obtain an expression for the deflection of point C due to ΔT.

1.7-4. It is proposed that a beam be constructed with a joint consisting of two horizontal links, as shown, so that the joint will transmit bending moment but no transverse shear force. Will this construction work as intended when load P is applied? Explain.



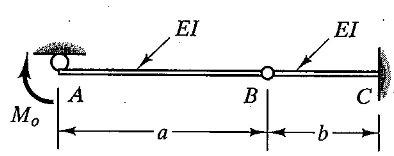
PROBLEM 1.7-4



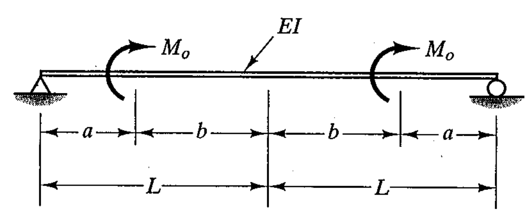
PROBLEM 1.7-5

1.7-5. The cantilever beam shown is so slender that its material remains linearly elastic even when displacements are large. Obtain expressions for the horizontal and vertical displacement components of the tip. Show that these expressions reduce to the expected small-deflection results when M_oL/EI is small.

1.7-6. For what value of a/b will the two parts of the beam shown have the same slope at hinge B when moment M_o is applied at A?



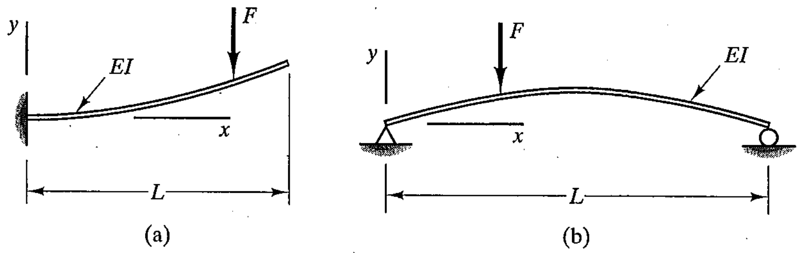
PROBLEM 1.7-6



PROBLEM 1.7-7

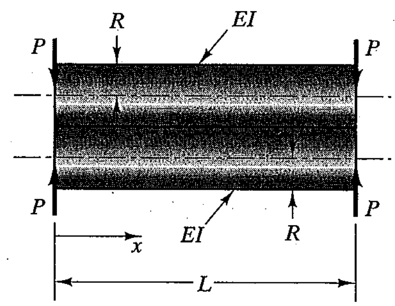
1.7-7. It is desired that both ends of a uniform beam remain horizontal when moments M_o are applied as shown. For what value of a/b will this be so?

1.7-8. Each of the beams shown is to be made with a small initial curvature, such that a load F moving across the beam will have no vertical displacement. What should be the initial shape $y = f(x)$ of each beam?

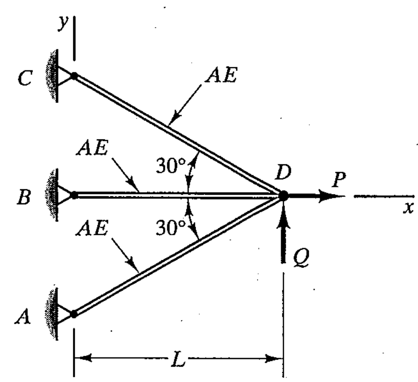


PROBLEM 1.7-8

1.7-9. Two identical rollers of average radius R are to be pushed together by end forces P, as shown. It is desired that the contact force between them be uniformly distributed along length L. Thus the rollers should not be quite cylindrical. How should R vary with x? Assume that the rollers are compact rather than quite slender, but that transverse shear deformation can be neglected.



PROBLEM 1.7-9

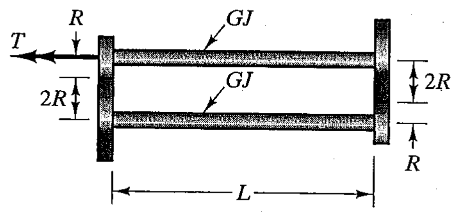


PROBLEM 1.8-1

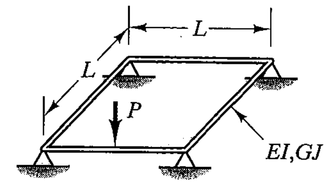
1.8-1. Members of the three-bar truss shown are identical except for length. Determine the displacement of joint D due to each of the following loadings. (a) $P = 0$, $Q > 0$. (b) $P > 0$, $Q = 0$. (c) $P = Q = 0$; all bars uniformly heated an amount ΔT.

1.8-2. Let gears of different sizes be fastened to either end of a prismatic shaft of circular cross section. Let there be two such shafts, set parallel so that gears of radius R and $2R$ engage in the manner shown. Frictionless bearings, not shown, ensure that the shafts twist without bending. What torsional stiffness T/θ is seen by torque T?

1.8-3. A square frame is made by welding together four identical slender bars of circular cross section. The frame is placed horizontally atop corner supports that can exert vertical



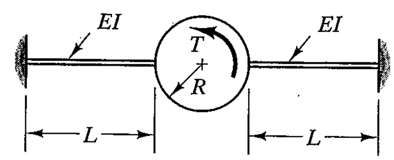
PROBLEM 1.8-2



PROBLEM 1.8-3

force but no moment. A vertical load P is applied to the middle of one side, as shown. What is the displacement of the loaded point and of the point opposite it? Let $G = E/2$; thus $EI = GJ$.

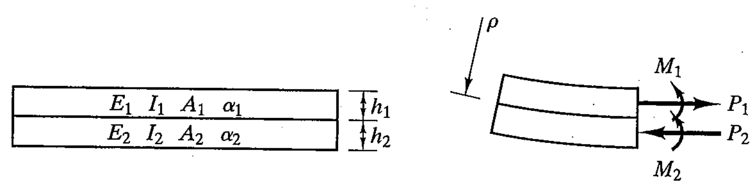
1.8-4. Two slender beams are built-in to a rigid disk and to rigid walls, as shown. Through what angle does the disk rotate if a small torque T is applied?



PROBLEM 1.8-4

1.8-5. Use formulas in Fig. 1.7-1b to determine the center deflection of each beam in Fig. 1.7-1c. In the second case, determine also the rotation at the right end.

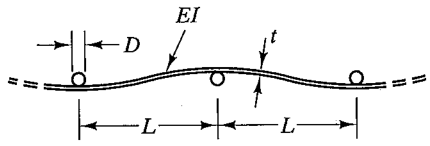
1.8-6. A bimetal beam is constructed by bonding together two slender beams of rectangular cross section. Material properties of the component beams differ, including thermal expansion coefficients α_1 and α_2. With $\alpha_1 < \alpha_2$, uniform heating an amount ΔT causes the deformation shown. Write, but do not solve, sufficient equations to determine radius of curvature ρ, internal forces P_1 and P_2, and internal bending moments M_1 and M_2. Also, write expressions for axial stresses at upper and lower surfaces of the composite beam in terms of the internal forces and moments.



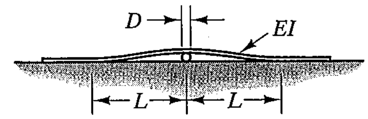
PROBLEM 1.8-6

1.8-7. Let several vertical posts of diameter D be arrayed in a straight line with distance L between them. A long slender beam is woven between the posts, as shown. Determine the maximum flexural stress in the beam.

1.8-8. A long straight beam has weight q per unit length. The beam is laid atop a small cylinder, as shown. Over what span $2L$ is the beam not in contact with the horizontal rigid floor?

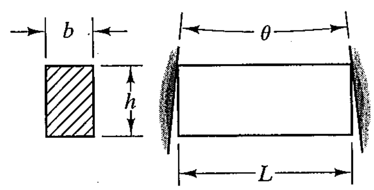


PROBLEM 1.8-7



PROBLEM 1.8-8

1.8-9. Two flat rigid walls include a small angle θ between them. A beam of rectangular cross section is just in contact with the walls, as shown. What uniform temperature increase ΔT is sufficient to place both ends of the beam in full contact with the walls? Express ΔT in terms of θ, h, α, and L.



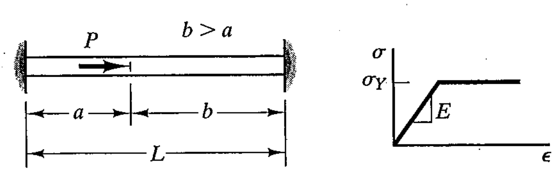
PROBLEM 1.8-9

1.9-1. The bar shown has uniform cross-sectional area A and is fixed at both ends. An idealized stress-strain relation is also shown. Assume that the relation is valid in compression as well as in tension.

(a) Determine the value of load P that initiates yielding.

(b) Determine the fully plastic load P_{fp}.

(c) Determine the state of residual stress after load P_{fp} is removed.



PROBLEM 1.9-1

1.9-2. Let the bar in Fig. 1.8-3a have the stress-strain relation used in Problem 1.9-1. Ends are fixed to the walls. Starting from the stress-free state, lower the temperature of the entire bar 1.5 times the amount ΔT that initiates plastic action.

(a) What then are the axial stresses? Express answers in terms of σ_Y.

(b) What are the residual stresses, and the residual displacement at the step, if the temperature is restored to its original value? Express answers in terms of σ_Y, L, and E.

1.9-3. For the three-bar truss of Problem 1.8-1, let $Q = 0$ and let the stress-strain relation be as depicted in Problem 1.9-1. Determine the fully plastic load P_{fp}. Also construct a dimensionless plot of P versus the horizontal displacement u_D of point D, using $P/A\sigma_Y$ as ordinate and $Eu_D/L\sigma_Y$ as abscissa.

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Instead of prescribed loading, the end conditions may prescribe displacements. In certain cases the stress will then have singularities at the corners $x = 0$, $y = \pm c$, and it becomes important to distinguish the character of the singular terms[1] and, if possible, to represent them in closed form so that the series part of the solution is asked to represent only a nonsingular part. An example occurs in the problem of the strip with one end clamped to zero displacements, and loaded in tension, which has been solved in this manner.[2] The problem of the compound tension strip, having elastic constants in the part $x > 0$ different from those in the part $x < 0$, has also been investigated.[3]

PROBLEMS

1. Investigate what problem of plane stress is solved by the stress function

$$\phi = \frac{3F}{4c}\left(xy - \frac{xy^3}{3c^2}\right) + \frac{P}{2}y^2$$

2. Investigate what problem is solved by

$$\phi = -\frac{F}{d^3}xy^2(3d - 2y)$$

applied to the region included in $y = 0$, $y = d$, $x = 0$, on the side x positive.

3. Show that

$$\phi = \frac{q}{8c^3}\left[x^2\left(y^3 - 3c^2y + 2c^3\right) - \frac{1}{5}y^3\left(y^2 - 2c^2\right)\right]$$

is a stress function, and find what problem it solves when applied to the region included in $y = \pm c$, $x = 0$, on the side x positive.

4. The stress function

$$\phi = s\left(\frac{1}{4}xy - \frac{xy^2}{4c} - \frac{xy^3}{4c^2} + \frac{ly^2}{4c} + \frac{ly^3}{4c^2}\right)$$

is proposed as giving the solution for a cantilever ($y = \pm c$, $0 < x < l$) loaded by uniform shear along the lower edge, the upper edge and the end $x = l$ being free from load. In what respects is this solution imperfect? Compare the expressions for the stresses with those obtainable from elementary tension and bending formulas.

5. In the cantilever problem of Fig. 26, the support conditions at $x = l$ are given as

At $x = l$, $y = 0$: $\quad u = v = 0$

At $x = l$, $y = \pm c$: $\quad u = 0$

Show that the deflection is now

$$(v)_{\substack{x=0\\y=0}} = \frac{Pl^3}{3EI}\left[1 + \frac{1}{2}(4 + 5\nu)\frac{c^2}{l^2}\right]$$

[1] This requires separate consideration of the corner region as in Chap. 4, Art. 42.
[2] See Benthem, *op. cit.*
[3] K. T. S. Iyengar and R. S. Alwar, *Z. Angew. Math. Phys.*, vol. 14, pp. 344–352, 1963; and *Z. Angew. Math. Mech.*, vol. 43, pp. 249–258, 1963.

Sketch the deformed shape of the supported end ($x = l$), and indicate on the sketch how this mode of support could be realized (hinges? rollers bearing on fixed planes?).

6. The beam of Fig. 28 is loaded by its own weight instead of the load q on the upper edge. Find expressions for the displacement components u and v. Find also an expression for the change of the (originally unit) thickness.

7. The cantilever of Fig. 26, instead of having a narrow rectangular cross section, has a wide rectangular cross section, and is maintained in plane strain by suitable forces along the vertical sides. The load is P per unit width on the end.

   Justify the statement that the stresses σ_x, σ_y, τ_{xy} are the same as those found in Art. 21. Find an expression for the stress σ_z, and sketch its distribution along the sides of the cantilever. Write down expressions for the displacement components u and v when a horizontal element of the axis is fixed at $x = l$.

8. Show that if V is a plane harmonic function, i.e., it satisfies the Laplace equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

then the functions xV, yV, $(x^2 + y^2)V$ will each satisfy Eq. (a) of Art. 18, and so can be used as stress functions.

9. Show that

$$(Ae^{\alpha y} + Be^{-\alpha y} + Cye^{\alpha y} + Dye^{-\alpha y}) \sin \alpha x$$

is a stress function.

   Derive series expressions for the stresses in a semi-infinite plate, $y > 0$, with normal pressure on the straight edge ($y = 0$) having the distribution

$$\sum_{m=1}^{\infty} b_m \sin \frac{m\pi x}{l}$$

Show that the stress σ_x at a point on the edge is a compression equal to the applied pressure at that point. Assume that the stress tends to disappear as y becomes large.

10. Show that (a) the stresses given by Eqs. (e) of Art. 24 and (b) the stresses in Prob. 9 satisfy Eq. (b) of Art. 17.

$x) \sin \gamma = 0$ (c)

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10. Show that (a) the stresses given by Eqs. (e) of Art. 24 and (b) the stresses in Prob. 9 satisfy Eq. (b) of Art. 17.

Florida International University
Department of Mechanical and Materials Engineering

EGM 5615 EXAMINATION B 1 November 2010

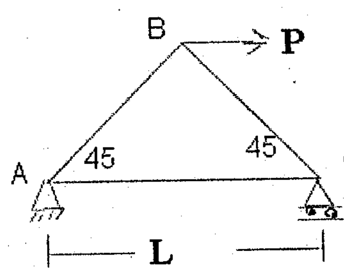
This examination will be a 75 minute exam. This exam allows you to use your notes only.

Please sign the following:

I certify that I will neither receive nor give unpermitted aid on this examination. Violation of this may result in failure of the exam.

PRINT NAME SIGN NAME

40% Problem 1 . For the following truss, find the displacement in the direction of the bar BC caused by the load P using Castigliano's second theorem. Assume all bars have the same E and A. Give your answer in terms of P, L, A and E.



40% Problem 2. Given the following stress tensor at a point, determine the principal stresses and determine the directional cosines for the second highest principal stress.
Hint: Get determinant first. If you factor determinant correctly, you don't have to use Newton-Raphson method.

$$\sigma = \begin{bmatrix} 50 & 0 & 0 \\ 0 & -20 & 15 \\ 0 & 15 & 45 \end{bmatrix} \text{MPa}$$

20% Problem 3. Determine the octahedral stress for the stress tensor data of Problem 2. What is the hydrostatic pressure for that stress tensor.

Florida International University
Department of Mechanical and Materials Engineering

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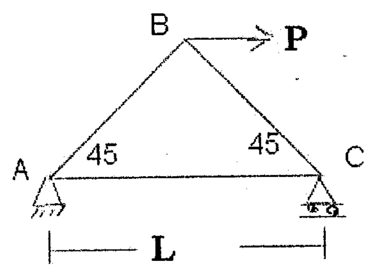
PRINT NAME SIGN NAME

40% Problem 1 a. For the following truss, find the displacement in the direction of the bar BC caused by the load P using Castigliano's second theorem. Assume all bars have the same E and A. Give your answer in terms of P, L, A and E.

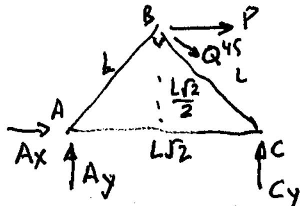


10% b). If L= √2 m, E=200 GPa, the cross-sectional area A= 5.1 cm^2, and P=100KN, what is the displacement in the direction of the load P

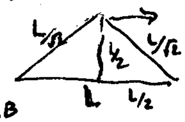
50% Problem 2. Given the following stress tensor at a point, determine the principal stresses and determine the directional cosines for the second highest principal stress.
Hint: Get determinant first. If you factor determinant correctly, you don't have to use Newton-Raphson method.

$$\sigma = \begin{bmatrix} 50 & 0 & 0 \\ 0 & -20 & 15 \\ 0 & 15 & 45 \end{bmatrix} \text{MPa}$$

a)

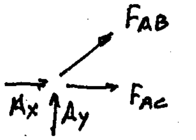


$$A_x + P + Q\cos 45 = 0$$
$$A_x = -P - Q\cos 45$$
$$A_y + C_y = Q\sin 45 = \frac{Q\sqrt{2}}{2}$$
$$\Sigma M_A = C_y \cdot L\sqrt{2} - P \cdot \frac{L\sqrt{2}}{2} + Q\cos 45 \cdot \frac{L\sqrt{2}}{2} - Q\sin 45 \frac{L\sqrt{2}}{2} = 0$$

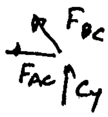


$$C_y = \frac{P}{\sqrt{2}}\frac{\sqrt{2}}{2} + \frac{Q}{2\sqrt{2}} + \frac{Q}{2\sqrt{2}} = \frac{P}{2} + \frac{Q}{\sqrt{2}}$$
$$A_y = -\frac{P\sqrt{2}}{2\sqrt{2}}$$

$$C_y \cdot \cancel{L} - P \cdot \frac{\cancel{L}}{2} - \frac{Q\sqrt{2}}{2}\frac{\cancel{L}}{2} - \frac{Q\sqrt{2}}{2}\frac{\cancel{L}}{2}$$
$$C_y = \frac{P}{2} + \frac{Q\sqrt{2}}{2}$$
$$A_y = -\frac{P}{2}$$



$$F_{AB}\sin 45 + A_y = 0 \qquad F_{AB} = -A_y\frac{2}{\sqrt{2}} = +\frac{P}{\sqrt{2}}$$
$$A_x + F_{AB}\cos 45 + F_{AC} = 0 \qquad -P - Q\cdot\frac{\sqrt{2}}{2} + \frac{P}{\cancel{\sqrt{2}}}\frac{\cancel{\sqrt{2}}}{2} + F_{AC} = 0 \qquad F_{AC} = \frac{Q\sqrt{2}}{2} + \frac{P}{2}$$



$$C_y + F_{BC}\sin 45 = 0 \qquad F_{BC} = -C_y\cdot\frac{2}{\sqrt{2}} = -\frac{P}{\sqrt{2}} - Q$$

$$U_c = \frac{\sum F_i^2 L_i}{2AE} = \frac{1}{2AE}\left[\left(\frac{P}{\sqrt{2}}\right)^2 \cdot L + \left(-\frac{P}{\sqrt{2}} - Q\right)^2 \cdot L + \left(\frac{Q\sqrt{2}}{2} + \frac{P}{2}\right)^2 \cdot L\sqrt{2}\right]$$

$$\left.\frac{\partial U_c}{\partial Q}\right|_{Q=0} = \frac{1}{2AE}\left[2\left(-\frac{P}{\sqrt{2}} - \cancel{Q}\right)(-1)L + 2\left(\cancel{\frac{Q\sqrt{2}}{2}} + \frac{P}{2}\right)\cdot\frac{\sqrt{2}}{2}\cdot L\sqrt{2}\right]$$
$$= \frac{1}{AE}\left(\frac{PL}{\sqrt{2}} + \frac{P\sqrt{2}}{4}\cdot L\sqrt{2}\right)$$
$$\frac{1}{AE}\frac{PL(\sqrt{2}+1)}{2}$$

b)

IF $Q = 0$
$$U_c = \frac{1}{2AE}\left[\left(\frac{P}{\sqrt{2}}\right)^2 \cdot L + \left(-\frac{P}{\sqrt{2}}\right)^2 L + \left(\frac{P}{2}\right)^2 \cdot L\sqrt{2}\right]$$
$$\frac{\partial U_c}{\partial P} = \frac{1}{AE}\left[\frac{P}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}L + \left(-\frac{P}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}L\right) + \frac{P}{2}\left(\frac{1}{2}\cdot L\sqrt{2}\right)\right]$$
$$\frac{P}{2}L + \frac{P}{2}L + \frac{PL\sqrt{2}}{4}$$

$L = 1$ here
$$\frac{PL}{AE}\left(1 + \frac{\sqrt{2}}{4}\right) = 1.3536\frac{PL}{AE} \quad \text{or } .957\frac{PL}{AE} \text{ if you use } L = \overline{AC}$$

for 100KN $L = 1$ $A = 5.1\times10^{-4}\ m^2$ $E = 200\ GPa$

$\delta = 1.326$ mm

$$\begin{array}{r} \lambda^3 + 25\lambda^2 + 1125\lambda \\ 50\lambda^2 - 1250\lambda - 56250 \\ \hline \lambda^3 - 75\lambda^2 + 125\lambda + 56250 \end{array}$$

$$\begin{bmatrix} 50-\lambda & 0 & 0 \\ 0 & -20-\lambda & 15 \\ 0 & 15 & 45-\lambda \end{bmatrix} = (50-\lambda)\left\{(-20-\lambda)(45-\lambda) - 15^2\right\}$$
$$(50-\lambda)\left\{(\lambda+20)(\lambda-45) - 225\right\}$$

$$\lambda^2 - 25\lambda - 1125$$
$$\lambda = \frac{25 \pm \sqrt{625 + 4(1125)}}{2}$$
$$\frac{25 \pm \sqrt{5125}}{2} = \frac{25 \pm 70.89}{2}$$

$71 \cdot 71 = 4900 + 141 = 5041$
$72 \cdot 72 = 4900 + 284 = 5184$
70.98
$(70+2)(70+2)$

$(71 - .1)(71 - .1)$
$5041 - 14.2 + .01$

$(71.5)(71.5)$
$5041 + 71 + .25$
5112

$$\frac{25 + 70.89}{2} = \frac{96.89}{2} = 48.93$$
$$\frac{25 - 70.89}{2} = -23.93$$

$$\begin{pmatrix} 1.7 & 0 & 0 \\ 0 & -68.\cancel{93} & 15 \\ 0 & 15 & -3.\cancel{93} \end{pmatrix} \begin{pmatrix} 1 \\ a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

~~2.05~~

$-68.\cancel{93}\,\cancel{a} + 15\cancel{b} = 0$

$15\cancel{a} \quad -3.\cancel{93}\,\cancel{b} = 0$

$\rightarrow a, b = 0$

$\Rightarrow a = 1$

$0 \quad 0\cdot\cancel{a} + 0\cancel{b} = 0$

$0 \quad -67.95b + 15c = 0$

$b = \frac{1}{4.5}c$

$$\begin{pmatrix} 0 \\ \frac{1}{4.5}c \\ c \end{pmatrix}$$

$0 + \left(\frac{1}{4.5}c\right)^2 + c^2 = 1$

$\left(\frac{1}{4.5}^2 + 1\right)c^2 = 1$

$\frac{1+4.5^2}{4.5^2}c^2 = 1$

$c = \frac{(4.5)}{\sqrt{4.5^2+1}}$

$C = .9767$

$b = .2145$

hydrostatic $\frac{50-20+45}{3} = 25$ MPa = hydrostatic press

dircos $\begin{pmatrix} 0 \\ .9767 \\ .2145 \end{pmatrix}$

Florida International University
Department of Mechanical Engineering

EGM 5615 EXAMINATION 8 March 2004

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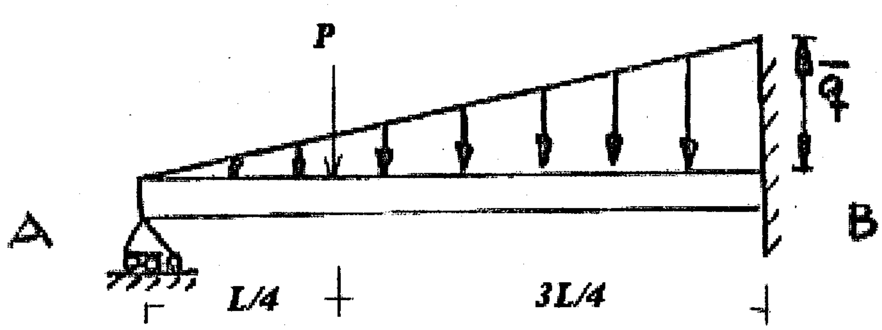
This examination consists of **five problems with several parts to one of the problems. Do all problems.** Read each question carefully. Show all work!!!!!

1. For the beam and loads shown in the diagram below, find the support forces and moments at B using energy methods.

$$q = \bar{q}\frac{x}{L}$$



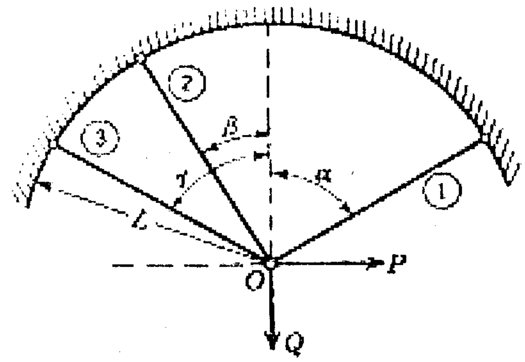
2. The components of stress at a point in a body referred to a rectangular Cartesian system of coordinates are given by

σx= 5 Mpa	τxy= 5 MPa	τxz = 8 MPa
τyx = 5 Mpa	σy = 0 MPa	τyz = -7.5 MPa
τzx = 8 MPa	τzy = -7.5 Mpa	σz = -3 MPa

a) Find the principal stresses σ1, σ2, and σ3 and
b) the directions that accompany these principal stresses;

c) Find the octahedral and absolute maximum shear stress
d) The deviatoric stresses

3. If the angles of the plane truss problem illustrated in the figure are $\alpha = 45°$, $\beta = 30°$, $\gamma = 45°$, determine the forces in its bars and the horizontal and vertical displacements u and v of joint O. Assume the bars have the same cross-sectional area A and the same elastic modulus E.



4. For the given stress state in the body, derive expressions for the displacement components u(x,y) and v(x,y), where c is a constant. Remember $\varepsilon x=\partial u/\partial x$ and $\varepsilon y=\partial v/\partial y$

$$\begin{bmatrix} cy^2 & 0 \\ 0 & -cx^2 \end{bmatrix}$$

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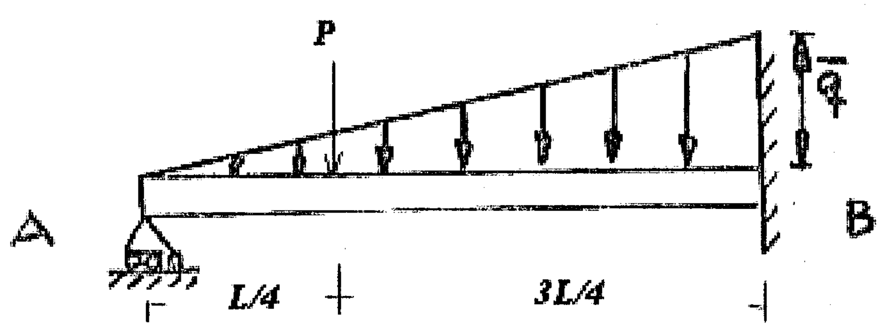

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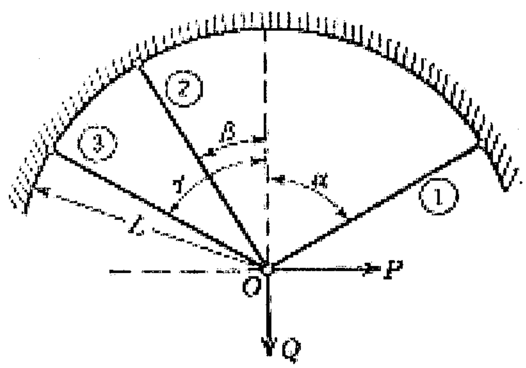


4. For the given stress state in the body, derive expressions for the displacement components u(x,y) and v(x,y), where c is a constant. Remember $\varepsilon x = \partial u/\partial x$ and $\varepsilon y = \partial v/\partial y$

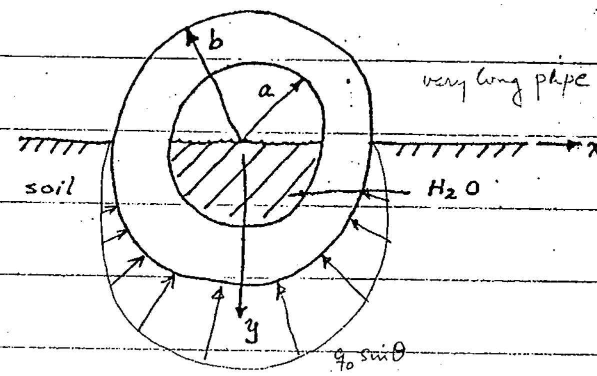
$$\begin{bmatrix} cy^2 & 0 \\ 0 & -cx^2 \end{bmatrix}$$

when $x > c$. Show that when $x = c + \epsilon$ (where ϵ is an arbitrarily small distance)

$$\sigma_{yy}(x, y=0) \sim \frac{K_I}{\sqrt{2\pi\epsilon}}$$

where $K_I = \sigma\sqrt{\pi c}$. K_I is known as the stress intensity factor.

2. A conduit is buried as shown and is half full of water. Ignore the weight of the conduit and assume the soil reaction varies from a maximum at the deepest part of the conduit to zero at ground level; a soil reaction of the form $q_0 \sin\theta$ might be a reasonable assumption to make. Assuming the conduit to be isotropic and linear elastic, solve for the stresses in the conduit.



3. The plane strain solution to Lame's problem of a cylinder under internal and external pressure requires the presence of an axial stress σ_{zz}. By superimposing an appropriate simple solution show that σ_{zz} may be reduced to zero. What does this say about the plane stress Lamé solution?