Concept of superposition

The effect of a compound cause, say a loading configuration, is the sum of the effects of the individual causes.

Motivation: solve problems involving complex load configurations based on simpler solutions.

Caveat: Superposition depends on linearity, both material and geometrical. Geometrical nonlinear situations include large deformation of a bar in bending, and contact between two spheres.

Safety factor

$$\text{Safety factor SF} = \frac{\text{failure load}}{\text{working load}} .$$

Failure does not necessarily mean fracture. It may mean excessive deformation, damage, or any effect which causes the structure or structural element to no longer function as intended.

Example: What if airplane wings could be made of an infinitely strong but not infinitely stiff material?

§2.1 Plane elasticity

Method of solution.

Recall in the mechanics of materials method, we began with an assumption about the deformation field. We only checked the stresses to make sure they agreed with the applied loads in terms of *resultants* .

In the elasticity method, one must simultaneously satisfy:
1. equilibrium conditions in a continuum sense at each point,
2. continuity of the displacement field,
3. boundary conditions at the surface.

If they are all satisfied exactly, we have an exact solution.

The theory of elasticity permits one to deal with problems which are not necessarily geometrically simple.

Few new elasticity solutions are now being discovered. Even so, study of elasticity aids in the development of physical insight.

Stress-strain relations

Elementary form of **Hooke's law** for a linear, isotropic, elastic solid:

$$\varepsilon_{xx} = \frac{1}{E}\ \{\sigma_{xx} - \nu\sigma_{yy} - \nu\sigma_{zz}\}$$

$$\varepsilon_{yy} = \frac{1}{E}\ \{\sigma_{yy} - \nu\sigma_{xx} - \nu\sigma_{zz}\},$$

$$\varepsilon_{zz} = \frac{1}{E}\ \{\sigma_{zz} - \nu\sigma_{xx} - \nu\sigma_{yy}\}.$$

These three are complete, but sometimes a shear relation is also presented,

$$\gamma_{xy} = \frac{1}{G}\ \tau_{xy} .$$

Plane stress in xy plane means σ_{zz}, σ_{yz}, and σ_{zx} are zero. Then,

$$\varepsilon_{xx} = \frac{1}{E} \{\sigma_{xx} - \nu\sigma_{yy}\}$$

$$\varepsilon_{yy} = \frac{1}{E} \{\sigma_{yy} - \nu\sigma_{xx}\}$$

$$\gamma_{xy} = \frac{1}{G} \tau_{xy} .$$

Although they are simpler in the compliance formulation, one can solve for stress and present them in the modulus formulation, for plane stress.

$$\sigma_{xx} = \frac{E}{1 - \nu^2} \{\varepsilon_{xx} + \nu\sigma_{yy}\}$$

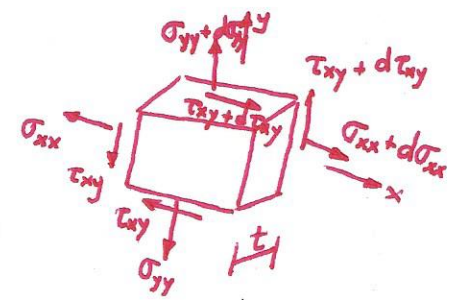
$$\sigma_{yy} = \frac{E}{1 - \nu^2} \{\varepsilon_{yy} + \nu\sigma_{xx}\}$$

$$\tau_{xy} = G \gamma_{xy} .$$

§2.2 Equilibrium equations, boundary conditions, Saint Venant's principle

We are familiar with the application of Newton's first law of equilibrium to macroscopic objects. In solving problems on a continuum scale, a differential form of the equations of **equilibrium** is needed.

Consider a free-body diagram of a differential element of thickness t of material.



From sum of forces in the x direction,

$$-t\,\sigma_{xx}\,dy - \tau_{xy}\,t\,dx + \{\sigma_{xx} + \frac{\partial\sigma_{xx}}{\partial x}dx\}t\,dy + \{\tau_{xy} + \frac{\partial\tau_{xy}}{\partial y}dy\}\,t\,dx = 0$$

From sum of forces in the y direction,

$$-t\,\tau_{xy}\,dy - \sigma_{yy}\,t\,dx + \{\tau_{xy} + \frac{\partial\tau_{xy}}{\partial x}dx\}t\,dy + \{\sigma_{yy} + \frac{\partial\sigma_{yy}}{\partial y}dy\}\,t\,dx = 0$$

Simplifying, the equilibrium equations are:

$$\frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0 \text{ in x direction}$$

$$\frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} = 0 \text{ in y direction.}$$

The equilibrium relations in the index notation are as follows for force and moment respectively: The Einstein summation convention assumed in which repeated indices are summed over. The comma represents differentiation with respect to the spatial coordinate corresponding to the index after the comma.

σ_{ji} is stress
G_i is a body force, or force per unit volume.
e_{ijk} is the permutation symbol
m_{ji} is a moment per unit area or couple stress. It is neglected in classical elasticity.
C_i is a body moment, or couple per unit volume.

$$\sigma_{ji,j} + G_i = 0 \qquad (1)$$

$$e_{ijk}\,\sigma_{jk} + m_{ji,j} + C_i = 0 \qquad (2)$$

Body forces arise due to gravitation.
Body moments arise due to electromagnetic interactions in magnetic materials.

Couple stresses represent a distributed average of moments upon fibers, ribs, layers, or other structural elements in composite materials.

In classical elasticity, in the absence of body couples or surface couples, Eq. 2 reduces to
$\sigma_{jk} = \sigma_{kj}$,

that is, the stress is symmetric.
If body couples or surface couples are permitted, the stress can become asymmetric.

Boundary conditions.

Boundary conditions entail prescription of stress or displacement upon the surface of the object in question. In many problems, the surface tractions (stresses at the surface) are zero over much of the surface.

Saint Venant's principle

Saint-Venant's principle is important in the application of elasticity solutions in many practical situations in which boundary conditions are satisfied in the sense of **resultants** rather than pointwise. For example, a bending moment may be applied to a beam via a complex array of bolted joints, which generate a locally complex stress pattern. In view of Saint-Venant's principle, one expects to observe bending type stresses far from the ends.

Saint-Venant's principle states that a localized **self-equilibrated** load system produces stresses which decay with distance more rapidly than stresses due to forces and moments. It is applicable in many situations of interest in engineering.

Demonstration: stress fields for concentrated loads which give rise to compression or bending, as seen with a photoelastic demonstrator.

There are some counter-examples. Consider a sandwich panel with rigid face sheets and an elastic material of Poisson's ratio ν sandwiched between them. For Poisson's ratios in the vicinity of 0.5, stresses applied to the end will decay with distance z as $\sigma(z) \propto e^{-\gamma z}$. The decay rate is

$$\gamma \propto \sqrt{\frac{3(1 - 2\nu)}{3 - 4\nu}} \; .$$

The distance $1/\gamma$, over which there is significant stress, diverges as Poisson's ratio approaches $\frac{1}{2}$.

In some thin-walled structures, localized self-equilibrated loads may propagate a significant distance. Saint-Venant's principle is inapplicable for such structures.

Constitutive relations

We mostly deal with linear isotropic elastic materials in this class. Many other possibilities exist.

Anisotropic: Dependent upon direction, referring to the material properties of composites, aggregates, single crystals, and oriented polycrystalline materials.

Creep: Time dependent strain in response to step stress; a manifestation of viscoelastic behavior.

Cubic: A type of anisotropic symmetry in which the unit cells are cube shaped. There are three independent elastic constants. Material is invariant to 90 degree rotations.

Elastic: Stress-strain path for loading is identical to the path for unloading, with immediate recovery to zero upon unloading.

Elastic-perfectly plastic: Elastic up to yield point after which strain increases with no increase in stress.

Elastic-plastic with work hardening: Beyond yield point, stress increases with strain.

Hexagonal: A type of anisotropic symmetry in which the unit cells are hexagonally shaped. Material is invariant to 60 degree rotations about an axis. There are five independent elastic constants. **Transverse isotropy** is mechanically equivalent to hexagonal although the structure may be random in the transverse direction.

Homogeneous: Material properties are identical at every point in the body. Concept of symmetry is expressed here as translational symmetry: material is invariant to translations. Homogeneous materials may be isotropic or anisotropic. At the atomic scale all materials are heterogeneous, but for many engineering applications we may view them as continuous media.

Isotropic: Independent of direction, referring to material properties. There are two independent elastic constants for a linearly elastic material. Engineering constants are E, G, B, ν, but they are interrelated.

Linear: Stress is proportional to strain, assuming all other variables upon which stress or strain might depend are held constant.

Orthotropic: A type of anisotropic symmetry in which the unit cells are shaped like rectangular parallelepipeds. In crystallography, this is called orthorhombic. There are nine independent elastic constants. Principal directions are mutually orthogonal. Material is invariant to reflections in two or three orthogonal planes.

Piezoelectric: In some crystalline or polycrystalline materials which lack a center of symmetry, there is coupling in which both stress and electric field contribute to the strain.

Thermoelastic: In all materials with a nonzero coefficient of thermal expansion, there is thermoelastic coupling in which both stress and temperature changes contribute to the strain.

Triclinic: A type of anisotropic symmetry in which the unit cells are oblique parallelepipeds with unequal sides and angles. There are 21 independent elastic constants.

Viscoelastic: Relation between stress and strain depends upon time or upon frequency.

Principal stress

Principal stresses are normal stresses which act on mutually perpendicular planes. They include the absolute largest and smallest normal stresses at a given point.

Recall Mohr's circle is a tool for 2-D transformations of stress and special 3-D transformations. It can be used to determine principal stresses under those special circumstances. It is not applicable to general 3-D transformations.

Consider a free body diagram of a cut corner of a unit cube. This is called a Cauchy tetrahedron.