5.3 ROMBERG INTEGRATION[1]

Romberg's method of integration is basically Richardson's extrapolation procedure.[2] Romberg's name is attached to the method because, according to Davis and Rabinowitz, he was the first to describe the algorithm in recursive form.[3]

We recall that the error estimate for trapezoidal integration can be expressed as

$$E_T = Ch^2 \tag{5.24}$$

in which h can take on variable values. It can be shown that, with the inclusion of higher order terms, the exact error can be expressed in the form

$$E_T = C_1h^2 + C_2h^4 + C_3h^6 + C_4h^8 + \cdots. \tag{5.25}$$

For small h, the first term dominates. Equation (5.25) can also be written as

$$E_T = C_1h^2 + O(h^4) \tag{5.26}$$

where $O(h^4)$ is a quantity of "order h^4," that is, the h^4 term dominates the higher order terms as $h \to 0$. If we obtain an integral value with a particular strip width, and then halve the strip width to get a second integral value and apply Equation (5.23), the C_1h^2 term is removed from the error. This leaves an error $O(h^4)$, which might also be expressed as $C_2h^4 + O(h^6)$. If we now determine a third integral value by again halving the strip width, we can combine it with the second to obtain another estimate containing an error expressed by $C_2h^4 + O(h^6)$. From the two values containing errors of $C_2h^4 + O(h^6)$, we can compute [from an equation similar to Equation (5.23)] a new value in which the C_2h^4 term is removed. [A derivation parallel to that for Equation (5.23), but using Ch^4 instead of Ch^2, gives an equation similar to Equation (5.23) except that the exponent 2 becomes 4.][4] This process can continue, next removing the C_3h^6 term from the error, until the desired accuracy is obtained. An example will clarify the procedure, but let us first convert Equation (5.23) to a form more commonly used in Romberg integration,

$$I_{\text{improved}} = \frac{I_{h_2}\left[\left(\dfrac{h_1}{h_2}\right)^2 - 1\right] + I_{h_2} - I_{h_1}}{\left(\dfrac{h_1}{h_2}\right)^2 - 1} = \frac{I_{h_2}\left(\dfrac{h_1}{h_2}\right)^2 - I_{h_1}}{\left(\dfrac{h_1}{h_2}\right)^2 - 1}. \tag{5.27}$$

If the second integration is obtained by halving the strip width, $h_1/h_2 = 2$, and Equation (5.27) can be written as

$$I_{\text{improved}} = \frac{2^2 I_{h_2} - I_{h_1}}{2^2 - 1}$$

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[1] W. Romberg, "Vereinfacte numerische Integration," *Norske Vidensk, Selskab Forhandlinger*, vol. 28, no. 7 (1955), pp. 30–36.

[2] L. F. Richardson and J. A. Gaunt, "The Deferred Approach to the Limit," *Trans. R. Soc., London*, vol. 226A (1927), p. 300.

[3] P. J. Davis and P. Rabinowitz, *Numerical Integration* (Waltham, Mass.: Blaisdell, 1967), p. 166.

[4] See the error analysis for Simpson's rule in Section 5.4.

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or

$$I_{\text{improved}} = \frac{2^2 I_{\text{more accurate}} - I_{\text{less accurate}}}{2^2 - 1}. \tag{5.28}$$

Equation (5.28) is used for what is called first-order extrapolation. For second-order extrapolation, the equation becomes

$$I_{\text{improved}} = \frac{2^4 I_{\text{more accurate}} - I_{\text{less accurate}}}{2^4 - 1}. \tag{5.29}$$

For nth-order extrapolation, the exponent on the 2 is equal in value to the exponent on the h in the error term removed by the extrapolation, and the nth-order extrapolation equation can be written as

$$I_{\text{improved}} = \frac{2^{(2n)} I_{\text{more accurate}} - I_{\text{less accurate}}}{2^{(2n)} - 1}. \tag{5.30}$$

A slight change in form is useful in writing a computer program for Romberg integration. It is

$$I_{\text{improved}} = \frac{4^n I_{\text{more accurate}} - I_{\text{less accurate}}}{4^n - 1} \tag{5.31}$$

where n is the order of the extrapolation and takes on values 1, 2, ...

■ EXAMPLE 5.1

Using a pocket calculator, let us determine

$$\int_0^{\pi/2} \sin x \, dx$$

by Romberg integration. We will use up to third-order extrapolation, or until the value from the latest extrapolation completed differs from the most accurate value of the previous extrapolation by less than an ε value of 0.000001.

Solution. We begin by calculating trapezoid rule results for a single strip, and for two strips, and form the following table:

Number of strips	*1*	*2*
Integral values from trapezoid rule	0.7853981634	0.948059449
First-order extrapolation value	1.002279878	

The extrapolated value shown is calculated from

$$\frac{4(0.948059449) - 0.7853981634}{4 - 1} = 1.002279878.$$

Comparing 1.002279878 with 0.948059449, we find a difference whose absolute value exceeds the prescribed ε. Therefore, we find the integral value for four strips and expand our table to the following:

Number of strips	*1*		*2*		*4*
Integral values	0.7853981634		0.948059449		0.987115801
First-order extrapolation values		1.002279878		1.000134585	
Second-order extrapolation value			0.9999915655		

The second-order extrapolation value is found from

$$\frac{4^2(1.000134585) - 1.002279878}{4^2 - 1} = 0.9999915655.$$

The value 0.9999915655 differs from 1.000134585 by more than ε, so one more extrapolation is taken. This will be the last extrapolation, regardless of the accuracy check. An integration value by the trapezoid rule with eight strips is calculated and the table expanded as follows:

Number of strips	*1*		*2*		*4*		*8*
Integral values	0.7853981634		0.948059449		0.987115801		0.9967851719
First-order extrapolation values		1.002279878		1.000134585		1.000008296	
Second-order extrapolation values			0.9999915655		0.9999998771		
Third-order extrapolation value				1.000000009			

The third-order extrapolation value is obtained from

$$\frac{4^3(0.9999998771) - 0.9999915655}{4^3 - 1} = 1.000000009.$$

This value is taken as our final answer. We note that it is an excellent result for using a maximum of only eight strips in our trapezoid integration.

In the computer, the table shown might be stored as a two-dimensional array in the form

$$\begin{array}{llll} R_{11} & R_{12} & R_{13} & R_{14} \\ R_{21} & R_{22} & R_{23} & \\ R_{31} & R_{32} & & \\ R_{41} & & & \end{array}$$

in which R_{41} is the final answer. ■

Computer Program for Romberg Integration

Before proceeding with the discussion of the computer program ROMBERG.FOR, it is suggested that the reader turn to the listing of the subroutine subprogram ROMBRG, shown in that program, and become familiar with the definitions of the arguments and other important variable and array names.