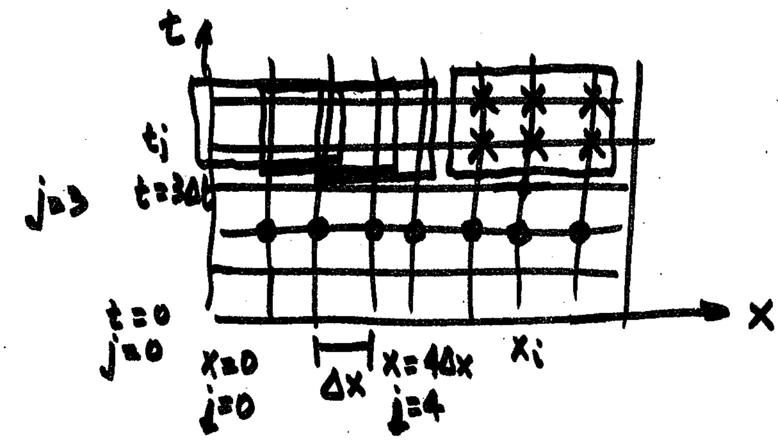
CRANK-NICHOLSON SCHEME

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \qquad t = t_j + \frac{\Delta t}{2} \ \& \ x_i$$

CD in time $\qquad$ CD in space $\qquad O(\Delta t^2, \Delta x^2)$

$$\frac{u_{i\,j+1} - u_{ij}}{2\left(\Delta t/2\right)} = \frac{\alpha}{2}\left[\frac{u_{i+1\,j+1} - 2u_{i\,j+1} + u_{i-1\,j+1}}{\Delta x^2} + \frac{u_{i+1\,j} - 2u_{ij} + u_{i-1\,j}}{\Delta x^2}\right]$$



$$R = \frac{\alpha \Delta t}{\Delta x^2}$$

$$\begin{bmatrix} 2+2R & -R & 0 & \cdots & 0 \\ -R & 2+2R & R & 0 \cdots & 0 \\ & \ddots & \ddots & \ddots & R \\ & & & -R & 2+2R \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}_{j+1} = \begin{bmatrix} 2-2R & C & 0 & \cdots & 0 \\ R & 2-2R & R & 0 \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & R \\ & & & R & 2-2R \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_N \end{bmatrix}_{j} + \text{B.C.}$$

$$A\,\underset{\sim}{u}^{(j+1)} = B\,\underset{\sim}{u}^{(j)}$$

$$\underset{\sim}{u}^{(j+1)} = A^{-1} B\,\underset{\sim}{u}^{(j)} \longrightarrow = \left(A^{-1}B\right)^{j+1} \underset{\sim}{u}^{(0)}$$

$$\lambda_n \text{ for } A^{-1}B = \frac{2 - 4R \sin^2\left(\frac{n\pi}{2(N-1)}\right)}{2 + 4R \sin^2\left(\frac{n\pi}{2(N-1)}\right)} \le 1$$

LAX EQUIVALENCE THEOREM

how do know that the numerical scheme converges to the solution of the PDE as Δt & $\Delta x \to 0$

States that if a linear difference equation is consistent with a properly posed linear Initial Value Problem (PDE + IC + BC), then stability is necessary & sufficient condition for convergence.

<u>Well Posed Problem</u>

1) solution is unique if it exists
2) solution depends continuously on the initial data
3) sol'n always exists for initial data that is arbitrarily close to initial data for which no solution exists.

let u_{ij} the numerical solution, U_{ij} the exact solution

$$e_{ij} = U_{ij} - u_{ij}$$

$$u_{ij} = U_{ij} - e_{ij}$$

Example: the explicit scheme $O(\Delta t, \Delta x^2)$

$$u_{ij+1} = R(u_{i+1j} + u_{i-1j}) + (1-2R)u_{ij}$$

$$e_{ij+1} = R(e_{i+1j} + e_{i-1j}) + (1-2R)e_{ij} - R\left[U_{i+1j} + U_{i-1j}\right] - (1-2R)U_{ij} + U_{ij+}$$

$$U_{i+1j} = U(x_{i+1}, t_j) = U_{ij} + \frac{\partial U}{\partial x}\Big|_{ij}\Delta x + \frac{\partial^2 U}{\partial x^2}(\xi_1, t_j)\frac{\Delta x^2}{2} \qquad x_i \le \xi_1 < x_{i+1}$$

$$U_{i-1j} = U(x_{i-1}, t_j) = U_{ij} - \frac{\partial U}{\partial x}\Big|_{ij}\Delta x + \frac{\partial^2 U}{\partial x^2}(\xi_2, t_j)\frac{\Delta x^2}{2} \qquad x_{i-1} < \xi_2 < x_i$$

$$U_{ij+1} = U(x_i, t_{j+1}) = U_{ij} + \frac{\partial U}{\partial t}(x_i, \eta) \cdot \Delta t \qquad t_j < \eta \le t_{j+1}$$

$$e_{ij+1} = R\left(e_{i+1j} + e_{i-1j}\right) + (1-2R)e_{ij} + R\left[\frac{\partial^2 U}{\partial x^2}(\ )\frac{\Delta x^2}{2} + \frac{\partial^2 U}{\partial x^2}(\ )\frac{\Delta x^2}{2}\right] + \Delta t \frac{\partial U}{\partial t}(\ )$$

$$R = \alpha \frac{\Delta t}{\Delta x^2}$$

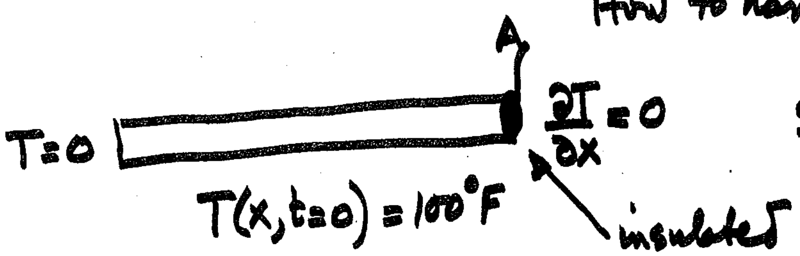
$$e_{ij+1} = R\left(e_{i+1j} + e_{i-1j}\right) + (1-2R)e_{ij} + \Delta t\left\{-\frac{\alpha}{2}\left[\frac{\partial^2 U}{\partial x^2}(\ ) + \frac{\partial^2 U}{\partial x^2}(\ )\right] + \frac{\partial U}{\partial t}(\ )\right\}$$

if the errors $\rightarrow 0$ as $\Delta x, \Delta t \rightarrow 0$

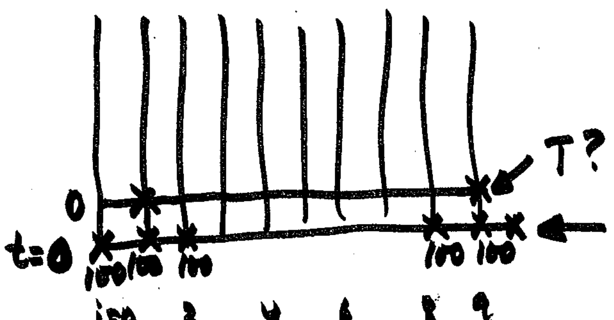
$$0 = -\alpha \frac{\partial^2 U}{\partial x^2} + \frac{\partial U}{\partial t}$$

Assignment show that L.E.T also works for the <u>implicit scheme</u>
Due on 10/20

How to handle derivative bc's



$$q = -kA\frac{\partial T}{\partial x}$$



$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}$$

explicit scheme $O(\Delta t, \Delta x^2)$

$$T_{i1} = R\left(T_{i+1,0} + T_{i-1,0}\right) + (1-2R)T_{i0}$$

let $i=1$ $\quad T_{11} = R(\underset{100}{T_{20}} + \underset{100}{T_{00}}) + (1-2R)\underset{100}{T_{10}}$

$$T_{91} = R\left(\underset{100}{T_{10,0}} + \underset{100}{T_{8,0}}\right) + (1-2R)\underset{100}{T_{90}}$$

$O(\Delta x^2)$

$$\left.\frac{\partial T}{\partial x}\right|_{ij} = \frac{T_{i+1j} - T_{i-1j}}{2\Delta x} = 0 \Rightarrow T_{i+1j} = T_{i-1j} \Rightarrow T_{10,0} = T_{8,0}$$

$$= \frac{q}{-kA} \qquad + \frac{q \cdot 2\Delta x}{-kA} \qquad + \frac{q \cdot 2\Delta x}{-kA}$$

3 TIME LEVEL SCHEME FOR $\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}$

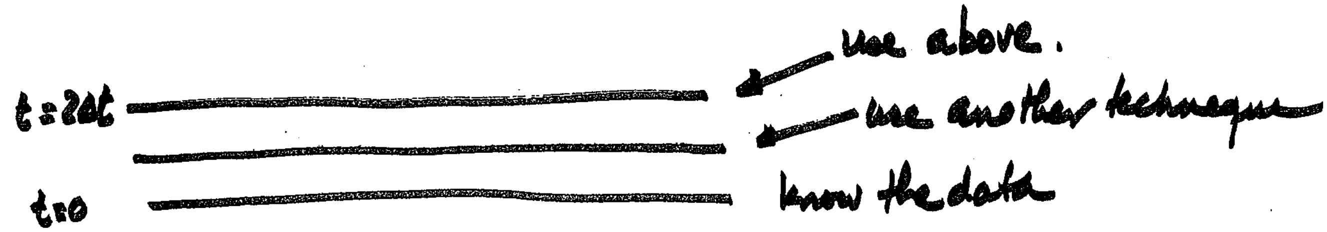
ONLY DO IT TO ACHIEVE AN ADVANTAGE OVER THE 2 TIME LEVEL SCHEME

1) SMALLER LOCAL TRUNCATION ERROR
2) GREATER STABILITY
3) TRANSFORMATION OF A NONLINEAR PROB TO A LINEAR PROBLEM

$$\left.\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2}\right|_{x_i, t_{j+1}} \quad \begin{array}{l} \text{BD } O(\Delta t^2) \text{ in } t \\ \text{CD } O(\Delta x^2) \text{ in } x \end{array}$$

$$\left[\frac{3}{2}\left(u_{ij+1} - u_{ij}\right) - \frac{1}{2}\left(u_{ij} - u_{ij-1}\right)\right] / \Delta t = \alpha \frac{\left(u_{i+1\,j+1} - 2u_{ij+1} + u_{i-1\,j+1}\right)}{\Delta x^2}$$

- THIS MAINTAINS DISCONT. IN INITIAL DATA
- THIS IS VERY GOOD IN THE CASE OF DATA THAT RAPIDLY VARIES IN X



$$\frac{\partial u}{\partial t} = \alpha\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \qquad \begin{array}{l} i \rightarrow x \\ j \rightarrow y \\ k \rightarrow t \end{array}$$

Explicit

$$\frac{u_{ijk+1} - u_{ijk}}{\Delta t} = \alpha\left\{\frac{u_{i+1jk} - 2u_{ijk} + u_{i-1jk}}{\Delta x^2} + \frac{u_{ij+1k} - 2u_{ijk} + u_{ij-1k}}{\Delta y^2}\right\}$$

error $O(\Delta t, \Delta x^2, \Delta y^2)$

$$\alpha \Delta t \left\{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}\right\} \leq \frac{1}{2}$$