special equation $y' = \lambda y$, λ constant, is usually considered sufficient, however, to give an indication of the stability of a method.

We consider first the Adams-Bashforth fourth-order method. If in (8.47) we set $f(x, y) = \lambda y$ we obtain

$$y_{n+1} - y_n - \frac{h\lambda}{24}(55y_n - 59y_{n-1} + 37y_{n-2} - 9y_{n-3}) = 0 \qquad (8.73)$$

The characteristic equation for this difference equation is

$$\beta^4 - \beta^3 - \frac{h\lambda}{24}(55\beta^3 - 59\beta^2 + 37\beta - 9) = 0$$

The roots of this equation are of course functions of $h\lambda$. It is customary to write the characteristic equation in the form

$$\rho(\beta) + h\lambda\sigma(\beta) = 0 \qquad (8.74)$$

where $\rho(\beta)$ and $\sigma(\beta)$ are polynomials defined by

$$\rho(\beta) = \beta^4 - \beta^3$$

$$\sigma(\beta) = -\frac{1}{24}(55\beta^3 - 59\beta^2 + 37\beta - 9)$$

We see that as $h \to 0$, (8.74) reduces to $\rho(\beta) = 0$, whose roots are $\beta_1 = 1$, $\beta_2 = \beta_3 = \beta_4 = 0$. For $h \neq 0$, the general solution of (8.73) will have the form

$$y_n = c_1\beta_1^n + c_2\beta_2^n + c_3\beta_3^n + c_4\beta_4^n$$

where the β_i are solutions of (8.74). It can be shown that β_1^n approaches the desired solution of $y' = \lambda y$ as $h \to 0$ while the other roots correspond to extraneous solutions. Since the roots of (8.74) are continuous functions of h, it follows that for h small enough, $|\beta_i| < 1$ for $i = 2, 3, 4$, and hence from the definition of stability that the Adams-Bashforth method is strongly stable. All multistep methods lead to a characteristic equation in the form (8.74) whose left-hand side is sometimes called the **stability polynomial**. The definition of stability can be recast in terms of the stability polynomial. A method is **strongly stable** if all the roots of $\rho(\beta) = 0$ have magnitude less than one except for the simple root $\beta = 1$.

We investigate next the stability properties of Milne's method (8.64*b*) given by

$$y_{n+1} = y_{n-1} + \frac{h}{3}(f_{n+1} + 4f_n + f_{n-1}) \qquad (8.75)$$

Again setting $f(x, y) = \lambda y$ we obtain

$$y_{n+1} - y_{n-1} - \frac{h\lambda}{3}(y_{n+1} + 4y_n + y_{n-1}) = 0$$

and its characteristic equation becomes

$$\rho(\beta) + h\lambda\sigma(\beta) = 0 \qquad (8.76)$$

with

$$\rho(\beta) = \beta^2 - 1$$

$$\sigma(\beta) = \beta^2 + 4\beta + 1$$

This time $\rho(\beta) = 0$ has the roots $\beta_1 = 1$, $\beta_2 = -1$, and hence by the definition above, Milne's method is not strongly stable. To see the implications of this we compute the roots of the stability polynomial (8.76). For h small we have

$$\beta_1 = 1 + \lambda h + \mathcal{O}(h^2)$$

$$\beta_2 = -(1 - \lambda h/3) + \mathcal{O}(h^2) \qquad (8.77)$$

Hence the general solution of (8.75) is

$$y_n = c_1(1 + \lambda h + \mathcal{O}(h^2))^n + c_2(-1)^n(1 - \lambda h/3 + \mathcal{O}(h^2))^n$$

If we set $n = x_n/h$ and let $h \to 0$, this solution approaches

$$y_n = c_1e^{\lambda x_n} + c_2(-1)^n e^{-\lambda x_n/3} \qquad (8.78)$$

In this case stability depends upon the sign of λ. If $\lambda > 0$ so that the desired solution is exponentially increasing, it is clear that the extraneous solution will be exponentially decreasing so that Milne's method will be stable. On the other hand if $\lambda < 0$, then Milne's method will be unstable since the extraneous solution will be exponentially increasing and will eventually swamp the desired solution. Methods of this type whose stability depends upon the sign of λ for the test equation $y' = \lambda y$ are said to be **weakly stable**. For the more general equation $y' = f(x, y)$ we can expect weak stability from Milne's method whenever $\partial f/\partial y < 0$ on the interval of integration.

In practice all multistep methods will exhibit some instability for some range of values of the step h. Consider, for example, the Adams-Bashforth method of order 2 defined by

$$y_{n+1} = y_n + \frac{h}{2}\{3f_n - f_{n-1}\}$$

If we apply this method to the test equation $y' = \lambda y$, we will obtain the difference equation

$$y_{n+1} - y_n - \frac{h\lambda}{2}\{3y_n - y_{n-1}\} = 0$$

and from this the stability polynomial

$$\beta^2 - \beta - \frac{h\lambda}{2}\{3\beta - 1\}$$

or the equation

$$\beta^2 - \left(1 + \frac{3h\lambda}{2}\right)\beta + \frac{h\lambda}{2} = 0$$

If $\lambda < 0$, the roots of this quadratic equation are both less than one in magnitude provided that $-1 < h\lambda < 0$. In this case we will have absolute stability since errors will not be magnified because of the extraneous solution. If, however, $|h\lambda| > 1$, then one of these roots will be greater than one in magnitude and we will encounter some instability. The condition that $-1 < h\lambda < 0$ effectively restricts the step size h that can be used for this method. For example, if $\lambda = -100$, then we must choose $h < 0.01$ to assure stability. A multistep method is said to be **absolutely stable** for those values of $h\lambda$ for which the roots of its stability polynomial (8.74) are less than one in magnitude. Different methods have different regions of absolute stability. Generally we prefer those methods which have the largest region of absolute stability. It can be shown, for example, that the Adams-Moulton implicit methods have regions of stability that are more than 10 times larger than those for the Adams-Bashforth methods of the same order. In particular, the second-order Adams-Moulton method given by

$$y_{n+1} = y_n + h\left(f_{n+1} - \tfrac{1}{2}f_n + \tfrac{1}{2}f_{n-1}\right)$$

is absolutely stable for $-\infty < h\lambda < 0$ for the test equation $y' = \lambda y$ with $\lambda < 0$.

For equations of the form $y' = \lambda y$ where $\lambda > 0$, the required solution will be growing exponentially like $e^{h\lambda}$. Any multistep method will have to have one root, the principal root, which approximates the required solution. All other extraneous roots will then have to be less in magnitude than this principal root. A method which has the property that all extraneous roots of the stability polynomial are less than the principal root in magnitude is said to be **relatively stable**. Stability regions for different multistep methods are discussed extensively in Gear [30].

EXERCISES

8.10-1 Show that the corrector formula based on the trapezoidal rule (8.52) is stable for equations of the form $y' = \lambda y$ (see Exercise 8.8-1).

8.10-2 Show that the roots of the characteristic equation (8.76) can be expressed in the form (8.77) as $h \to 0$, and that the solution of the difference equation (8.75) approaches (8.78) as $h \to 0$.

8.10-3 Write a computer program to find the roots of the characteristic equation (8.73) for the Adams-Bashforth formula. Take $\lambda = -1$ and $h = 0(0.1)\bar{h}$. Determine an approximate value of $\bar{h}$ beyond which one or more roots of this equation will be greater than one in magnitude. Thus establish an upper bound on h, beyond which the Adams-Bashforth method will be unstable.

8.10-4 Solve Eq. (8.67) by Milne's method (8.64) from $x = 0$ to $x = 6$ with $h = \frac{1}{2}$. Take the starting values from Table 8.1. Note the effect of instability on the solution.

*8.11 ROUND-OFF-ERROR PROPAGATION AND CONTROL

In Sec. 8.4 we defined the discretization error e_n as

$$e_n = y(x_n) - y_n$$

where $y(x_n)$ is the true solution of the differential equation, and y_n is the exact solution of the difference equation which approximates the differential equation. In practice, because computers deal with finite word lengths, we will obtain a value $\tilde{y}_n$ which will differ from y_n because of round-off errors. We shall denote by

$$r_n = y_n - \tilde{y}_n$$

the **accumulated round-off error**, i.e., the difference between the exact solution of the difference equation and the value produced by the computer at $x = x_n$. At each step of an integration, a round-off error will be produced which we call the **local round-off error** and which we denote by ε_n. In Euler's method, for example, ε_n is defined by

$$\tilde{y}_{n+1} = \tilde{y}_n + hf(x_n, \tilde{y}_n) + \varepsilon_n$$

The accumulated round-off error is not simply the sum of the local round-off errors, because each local error is propagated and may either grow or decay as the computation proceeds. In general, the subject of round-off-error propagation is poorly understood, and very few theoretical results are available. The accumulated roundoff depends upon many factors, including (1) the kind of arithmetic used in the computer, i.e., fixed point or floating point; (2) the way in which the machine rounds; (3) the order in which the arithmetic operations are performed; (4) the numerical procedure being used.

As shown in Sec. 8.10, where numerical instability was considered, the effect of round-off propagation can be disastrous. Even with stable methods, however, there will be some inevitable loss of accuracy due to rounding errors. This was illustrated in Chap. 7, where the trapezoidal rule was used to evaluate an integral. Over an extended interval the loss of accuracy may be so serious as to invalidate the results completely.

It is possible to obtain estimates of the accumulated rounding error by making some statistical assumptions about the distribution of local round-off errors. These possibilities will not be pursued here. We wish to consider here a simple but effective procedure for reducing the loss of accuracy due to round-off errors when solving differential equations.

Most of the formulas discussed in this chapter for solving differential equations can be written in the form

$$y_{n+1} = y_n + h\,\Delta y_n$$

where $h\,\Delta y_n$ represents an increment involving combinations of $f(x, y)$ at selected points. The increment is usually small compared with y_n itself. In