First Central-Difference Expressions $O(\Delta x^2)$

$$
\begin{aligned}
y_i' &= \frac{y_{i+1} - y_{i-1}}{2(\Delta x)} \\
y_i'' &= \frac{y_{i+1} - 2y_i + y_{i-1}}{(\Delta x)^2} \\
y_i''' &= \frac{y_{i+2} - 2y_{i+1} + 2y_{i-1} - y_{i-2}}{2(\Delta x)^3} \\
y_i'''' &= \frac{y_{i+2} - 4y_{i+1} + 6y_i - 4y_{i-1} + y_{i-2}}{(\Delta x)^4}
\end{aligned}
\tag{5-38}
$$

Second Central-Difference Expressions $O(\Delta x^4)$

$$
\begin{aligned}
y_i' &= \frac{-y_{i+2} + 8y_{i+1} - 8y_{i-1} + y_{i-2}}{12(\Delta x)} \\
y_i'' &= \frac{-y_{i+2} + 16y_{i+1} - 30y_i + 16y_{i-1} - y_{i-2}}{12(\Delta x)^2} \\
y_i''' &= \frac{-y_{i+3} + 8y_{i+2} - 13y_{i+1} + 13y_{i-1} - 8y_{i-2} + y_{i-3}}{8(\Delta x)^3} \\
y_i'''' &= \frac{-y_{i+3} + 12y_{i+2} - 39y_{i+1} + 56y_i - 39y_{i-1} + 12y_{i-2} - y_{i-3}}{6(\Delta x)^4}
\end{aligned}
\tag{5-39}
$$

First Forward-Difference Expressions $O(\Delta x)$

$$
\begin{aligned}
y_i' &= \frac{y_{i+1} - y_i}{(\Delta x)} \\
y_i'' &= \frac{y_{i+2} - 2y_{i+1} + y_i}{(\Delta x)^2} \\
y_i''' &= \frac{y_{i+3} - 3y_{i+2} + 3y_{i+1} - y_i}{(\Delta x)^3} \\
y_i'''' &= \frac{y_{i+4} - 4y_{i+3} + 6y_{i+2} - 4y_{i+1} + y_i}{(\Delta x)^4}
\end{aligned}
\tag{5-40}
$$

Second Forward-Difference Expressions $O(\Delta x^2)$

$$
\begin{aligned}
y_i' &= \frac{-y_{i+2} + 4y_{i+1} - 3y_i}{2(\Delta x)} \\
y_i'' &= \frac{-y_{i+3} + 4y_{i+2} - 5y_{i+1} + 2y_i}{(\Delta x)^2} \\
y_i''' &= \frac{-3y_{i+4} + 14y_{i+3} - 24y_{i+2} + 18y_{i+1} - 5y_i}{2(\Delta x)^3} \\
y_i'''' &= \frac{-2y_{i+5} + 11y_{i+4} - 24y_{i+3} + 26y_{i+2} - 14y_{i+1} + 3y_i}{(\Delta x)^4}
\end{aligned}
\tag{5-41}
$$

First Backward-Difference Expressions $O(\Delta x)$

$$
\begin{aligned}
y_i' &= \frac{y_i - y_{i-1}}{(\Delta x)} \\
y_i'' &= \frac{y_i - 2y_{i-1} + y_{i-2}}{(\Delta x)^2} \\
y_i''' &= \frac{y_i - 3y_{i-1} + 3y_{i-2} - y_{i-3}}{(\Delta x)^3} \\
y_i'''' &= \frac{y_i - 4y_{i-1} + 6y_{i-2} - 4y_{i-3} + y_{i-4}}{(\Delta x)^4}
\end{aligned}
\tag{5-42}
$$

Second Backward-Difference Expressions $O(\Delta x^2)$

$$
\begin{aligned}
y_i' &= \frac{3y_i - 4y_{i-1} + y_{i-2}}{2(\Delta x)} \\
y_i'' &= \frac{2y_i - 5y_{i-1} + 4y_{i-2} - y_{i-3}}{(\Delta x)^2} \\
y_i''' &= \frac{5y_i - 18y_{i-1} + 24y_{i-2} - 14y_{i-3} + 3y_{i-4}}{2(\Delta x)^3} \\
y_i'''' &= \frac{3y_i - 14y_{i-1} + 26y_{i-2} - 24y_{i-3} + 11y_{i-4} - 2y_{i-5}}{(\Delta x)^4}
\end{aligned}
\tag{5-43}
$$

EXAMPLE 5-3

In Example 3-4 the Newton-Raphson method was used to determine the output lever angles of a crank-and-lever 4-bar linkage system for each 5° of rotation of the input crank. Now we shall determine the angular velocity and the angular acceleration of the output lever of the same type of mechanism for each 5° of rotation of the input crank, with the latter rotating at a uniform angular velocity of 100 radians/sec.

We can determine the output lever positions ϕ, corresponding to each 5° of crank rotation θ, by utilizing Freudenstein's equation and the Newton-Raphson method, as was done in Example 3-4. Such a set of values, in effect, gives us a series of points on the ϕ versus θ curve, and the ϕ values are stored in memory to provide data for the differentiation processes which follow. The slope of the ϕ-θ curve may be related to the angular velocity of the output lever $d\phi/dt$ if we realize that, with the crank rotating at a constant ω, its angular position is given by

$$\theta = \omega t$$

so that

$$\frac{d\phi}{d\theta} = \frac{1}{\omega}\frac{d\phi}{dt}$$

$$y(t_0) = y_0$$

$$y_1 = y(t_1) = y(t_0+\Delta t) = y_0 + y_0'\Delta t + y_0''\frac{\Delta t^2}{2!} + y_0'''\frac{\Delta t^3}{3!} + y_0^{IV}\frac{\Delta t^4}{4!} + \cdots$$

$$y_2 = y(t_2) = y(t_0+2\Delta t) = y_0 + y_0'\cdot 2\Delta t + y_0''\frac{(2\Delta t)^2}{2!} + y_0'''\frac{(2\Delta t)^3}{3!} + y_0^{IV}\frac{(2\Delta t)^4}{4!} + \cdots$$

$$y_3 = y(t_3) = y(t_0+3\Delta t) = y_0 + y_0'\cdot 3\Delta t + y_0''\frac{(3\Delta t)^2}{3!} + y_0'''\frac{(3\Delta t)^3}{3!} + y_0^{IV}\frac{(3\Delta t)^4}{4!}$$

$$Ay_0 + By_1 + Cy_2 + Dy_3 = (A+B+C+D)y_0 + (B+2C+3D)\Delta t\, y_0' + (B+4C+9D)\frac{\Delta t^2}{2!}y_0'' + (B+8C+27D)\frac{\Delta t^3}{3!}y_0''' + (B+16C+81D)\frac{\Delta t^4}{4!}y_0^{IV}$$

$$y_0'' + O(\Delta t^2)$$

$$\left.\begin{aligned} &A+B+C+D=0 \\ &(B+2C+3D)\Delta t=0 \\ &(B+4C+9D)\frac{\Delta t^2}{2!}=1 \\ &(B+8C+27D)\frac{\Delta t^3}{3!}=0 \end{aligned}\right\} \quad \begin{aligned} A&=\frac{2}{\Delta t^2} \\ B&=\frac{-5}{\Delta t^2} \\ C&=\frac{4}{\Delta t^2} \\ D&=\frac{-1}{\Delta t^2} \end{aligned}$$