Linear Equations of order 2

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0$$

if $P_0(a) \neq 0$ at $x=a$ $\Rightarrow$ $x=a$ is an ordinary point then $y(x)=\sum_{n=0}^{\infty} a_n (x-a)^n$

and $y = c_1 y_1(x) + c_2 y_2(x)$ $\quad y_1(x) = \sum_{n=0}^{\infty} A_n (x-a)^n$

$y_2(x) = \sum_{n=0}^{\infty} B_n (x-a)^n$

$y_1(x)$ and $y_2(x)$ are linearly independent & analytic at $x=a$

IF $P_0(a) = 0$ $\Rightarrow$ $x=a$ is a singular point

- a is a regular singular point if $\frac{P_1(x)}{P_0(x)}(x-a)$ and $\frac{P_2(x)}{P_0(x)}(x-a)^2$ can be expanded in Power series about $x=a$

  $\sum_{n=0}^{\infty} C_n (x-a)^n \qquad \sum_{n=0}^{\infty} D_n (x-a)^n$

- a is an irregular singular point if cannot be expanded in series in terms of $x-a$

radius of Convergence is $|x| = a$

- Example Regular singular point

$$(1+x)y'' + 2xy' - 3y = 0 \quad \text{at } x = -1$$

$$\frac{P_1}{P_0}(x+1) = 2x = 2(x+1) - 2 \qquad C_0 = -2,\ C_1 = 2,\ C_2 \dots C_\infty =$$

$$\frac{P_2}{P_0}(x+1)^2 = -3(x+1) \qquad D_0 = 0 \quad D_1 = -3 \quad D_2 \cdots D_\infty = 0$$

- FOR $x=a$ BEING AN IRREGULAR SINGULAR PT – A SOLUTION IN Power series form may or may not exist.

- FOR REGULAR SINGULAR PTS USE METHOD OF FROBENIUS (ABOUT $x=0$)
  - COEFF OF FIRST TERM IN POWER SERIES EXPANSION = INDICIAL EQN
  - INDICIAL EQUATION GIVES m (FOR 2nd order m_1 & m_2)
  - IF $m_1 \neq m_2$ and $|m_1 - m_2|$ is not integer

    $\Rightarrow$ 2 DISTINCT SOLNS EACH OF FORM $x^{m_1}\left[\sum_{n=0}^{\infty} A_n x^n\right]$ & $x^{m_2}\left[\sum_{n=0}^{\infty} B_n x^n\right]$

    where A_n depends on m_1 & B_n depends on m_2
  - IF $m_1 \neq m_2$ and $|m_1 - m_2|$ IS an integer
    - $\Rightarrow$ LARGER ROOT ALWAYS GIVES SOLN $m = m_1$ $\quad y_1 = x^{m_1}\left[1+\sum_{n=1}^{\infty} A_n x^n\right]$

    $m=m_2$ (2nd) can yield a soln no solution or a general soln

    - $$y_2(x) = A\, y_1(x) \ln x + \bar{y}_2(x) \qquad \bar{y}_2 = x^{m_2}\sum_{n=0}^{\infty} B_n x^n$$
      $$y(x) = C_1 y_1(x) + C_2 y_2(x)$$
  - IF $m_1 = m_2$

    $$y_2(x) = y_1(x) \ln x + \bar{y}(x) \qquad \bar{y} = x^{m_1}\sum_{n=0}^{\infty} B_n x^n$$
    $$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

  - AN EASIER FORM IS IF you know $y_1(x)$

    $$y_2(x) = \left.\frac{\partial y_1}{\partial m}\right|_{m=m_1}$$

- WHAT IF $a \neq 0$ let $t = x - a$ $\Rightarrow$ $x = a$ $\quad t = 0$

  $$\frac{d}{dx}(\;) = \frac{d}{dt}(\;)\cdot\frac{dt}{dx} = \frac{d}{dt}(\;)$$

- WHAT IF $a = \infty$ let $t = \frac{1}{x}$ $\Rightarrow$ $x \to \infty$ $\quad t \to 0$

Regular Singular

<u>Example</u> $\quad xy'' + y' - y = 0 \qquad \frac{P_1}{P_0}x = \frac{1}{x}\cdot x = 1 \qquad \frac{P_2}{P_0}\cdot x^2 = \frac{-1}{x}\cdot x^2 = -x$

let $y = x^m \sum_{n=0}^{\infty} A_n x^n$

$$y' = m x^{m-1}\sum_{n=0}^{\infty} A_n x^n + x^m \sum_{n=1}^{\infty} A_n n x^{n-1}$$
$$y'' = m(m-1)x^{m-2}\sum A_n x^n + 2m x^{m-1}\sum A_n n x^{n-1} + x^m \sum A_n n(n-1) x^{n-2}$$

$$xy'' + y' - y = m^2 A_0 x^{m-1} + [(m+1)^2 A_1 - A_0]x^m + [(m+2)^2 A_2 - A_1]x^{m+1} + \dots + [(m+n)^2 A_n - A_{n-1}]x^{m+n-1} + \dots = 0$$

$$\Rightarrow \quad A_n = \frac{A_{n-1}}{(m+n)^2} \qquad n = 1 \dots \infty$$

$\Rightarrow \quad A_0 = 0$ or $m = 0$ equal roots

$$A_n = \frac{A_{n-1}}{(m+n)^2} = \frac{A_{n-2}}{(m+n)^2(m+n-1)^2} = \dots \frac{A_0}{[(m+n)(m+n-1)\cdots(m+1)]^2}$$

$$\therefore \quad y_1 = x^m A_0 \sum \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2(m+2)^2} + \dots\right] = x^m \bar{y}_1$$

$$\frac{x^3}{(m+1)^2(m+2)^2(m+3)^2}$$

for $m=0$ $\quad y_1 = x^m A_0 \sum_{n=0}^{\infty} \frac{x^n}{(n!)^2} = x^m \bar{y}_1$

$$y_2 = \frac{\partial y_1}{\partial m} = x^m \ln x \, \bar{y}_1 + x^m A_0 \sum \left[\frac{-2x}{(m+1)^3} - \left\{\frac{2}{(m+1)^3(m+2)^2} + \frac{2}{(m+1)^2(m+2)^3}\right\} x^2\right.$$

@ $m=0$ $\quad y_2 = \bar{y}_1 \ln x + A_0 \sum \left[\frac{-2x}{1} - \left[\frac{2}{4} + \frac{2}{8}\right]x^2 - \left[\frac{2}{6} + \frac{2}{12} + \frac{2}{18}\right]x^3 + \dots\right.$

$$= \bar{y}_1 \ln x - 2A_0\left[x + \frac{1}{2!}\left(1+\frac{1}{2}\right)x^2 + \frac{1}{3!}\left(1+\frac{1}{2}+\frac{1}{3}\right)x^3 + \dots\right]$$

$$y = C_1 y_1(x) + C_2 y_2(x)$$

- Find a soln to $y'' = xy$ about $x = 1$ ordinary

- if x & x^2 are solns to a differential eqn ~~what~~ how can we define the d.e. about $x=0$ what about $x=\infty$
  what is the radius of convergence

- How can you classify $2(x-2)^2 x y'' + 3xy' + (x-2)y = 0$