Chapter 2

SELF-SIMILAR SOLUTIONS

2.1 Characteristic Scales; Scale-Similar Problems

It is often convenient to present the solution to a PDE problem in non-dimensional form. This makes the results independent of the size of the system for which the solution was obtained as well as independent of any choice of dimensional system. Non-dimensionalization is usually accomplished by choosing some length and time scales characterizing the problem, and then defining non-dimensional independent variables based on these scales. For example, the solution for fluid flow in a rotating sphere might be expressed non-dimensionally in terms of the dimensionless radius, $R = r/r_o$, where r_o is the radius of the sphere. Here r_o is the characteristic length scale of the problem. If the fluid is initially at rest, and at time zero it is put into rotation at angular velocity ω, then the period of rotation is $\tau = 2\pi/\omega$, and τ would be the characteristic time scale. Then a suitable dimensionless time would be $T = t/\tau$. Note that one of the characteristic scales for the independent variables (r_o) came from the geometry of the system, and the other (τ) from the boundary conditions.

The dependent variables also can be represented non-dimensionally. For example, in the rotating sphere problem the equatorial velocity is $u_o = \omega r_o$ and may be used as a characteristic velocity in the dimensionless velocity $\underline{U} = \underline{u}/u$.

The problem may also contain some parameters, such as the kinematic viscosity ν. The parameters also can be reduced to non-dimensional form, and in the case of viscosity it is customary to use a reciprocal dimensionless viscosity called the Reynolds number, $Re = u_o r_o/\nu$.

The solution for the velocity within the rotating sphere could then be expressed non-dimensionally as

$$\underline{U} = \underline{U}(R,T;Re)$$

This says that the dimensionless velocity (a vector) $\underline{U}$ will be a function of the dimensionless radial coordinate R, the dimensionless time coordinate T, and the parameter Re. It might also happen that the flow depends upon the polar angular coordinates ϕ and θ, which are additional non-dimensional independent variables.

Problems which have natural characteristic scales for the <u>independent</u> variables (here r_o and τ) are called <u>scale-similar</u>. Scale-similar solutions for systems of different size will have the same non-dimensional solution, provided that the two problems also have the same values of the dimensionless parameters and dimensionless boundary and initial conditions.

2.2 Self-Similarity

There are a few very interesting and important PDE problems for which no natural characteristic scales for the independent variables exist in the problem formulation. For example, consider the case of heat conduction in a semi-infinite slab initially at uniform temperature, subjected to a step increase in the surface temperature at time zero (Fig. 2.2.1). The appropriate PDE is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{2.2.1}$$

where α is a constant parameter called the thermal diffusivity of the medium. The initial condition is

$$T(x,0) = T_i \qquad x > 0 \tag{2.2.2}$$

The boundary condition at the surface is

$$T(0,t) = T_s \tag{2.2.3}$$

The temperature field must fall off to the initial temperature T_i as $x \to \infty$, giving a second boundary condition

$$T(x,t) \to T_i \qquad \text{as} \quad x \to \infty \tag{2.2.4}$$

There are no characteristic scales for either length or time in this problem. This fact is the clue that a self-similar solution must exist. Since the solution to all physical problems must be expressible in dimensionless form (nature is unaware of the length of a meter), there must be some way to non-dimensionalize the solution to this problem. The only possible way is for the variables to appear together in a non-dimensional group. Looking at the denominators in (2.2.1), it is readily apparent that x^2 and αt have the same dimensions, and therefore the quantity $x^2/(\alpha t)$ is dimensionless. Somehow the solution must be expressible in terms of this quantity, in order to have dimensionless form. Solutions made non-dimensional by combinations of the independent variables, rather than by characteristic scales imposed by the geometry, boundary, or initial conditions, are called self-similar solutions.

There is a characteristic temperature for this problem, namely the step increase in temperature $T_s - T_i$. Therefore, one might guess that the non-dimensional form of the solution is

$$\frac{T - T_i}{T_s - T_i} = f\left(\frac{x^2}{\alpha t}\right) \tag{2.2.5}$$

As we shall see, this guess is correct. In a moment we shall develop a systematic way of discovering the forms of self-similar solutions.

If (2.2.5) is indeed correct, then another fully equivalent form would be

$$\frac{T - T_i}{T_s - T_i} = g(x/\sqrt{\alpha t}) \tag{2.2.6}$$

and another would be

$$\frac{T - T_i}{T_s - T_i} = \frac{x}{\sqrt{\alpha t}}\, h(x/\sqrt{\alpha t}) \tag{2.2.7}$$

All of these solutions would really be the same, but the functions f, g, and h would be different.

In terms of the similarity variable, $\eta = x/\sqrt{\alpha t}$, the family of temperature profiles existing at different times will collapse to a single curve (Fig. 2.2.1b). This is the essence of self-similarity; the solution does not scale on the size of the system, instead it scales on itself.

At first glance, it may appear disadvantageous to seek a solution in terms of the non-linear combination of variables $\eta = x/\sqrt{\alpha t}$. However, note that a single function $g(\eta)$ would be involved, and therefore one would only have to deal with an ordinary differential equation (ODE). This is the practical advantage of a self-similar problem in two independent variables. The existence of self-similarity will always reduce the number of independent variables by one.

To summarize, self-similar solutions exist when a problem is not scale-similar, i.e. when characteristic scales for the independent variables do not exist in the problem formulation. In problems with two independent variables, self-similar solutions represent a collapse of the family of solutions as functions of the two variables to a single function of the similarity variable. The governing PDE is thereby reduced to an ODE, which may be solved by some appropriate analytical or numerical method. The proper form of the transformation depends upon the equation, the initial conditions, and the boundary conditions. The transformation can be discovered systematically, as we shall now illustrate by some examples.

2.3 Example with Constant Boundary Conditions

Consider the transient heat transfer problem discussed in section 2.2 The differential equation, boundary conditions, and initial conditions are (2.2.1)-(2.2.4). The solution must be expressible in terms of some similarity variable, which must be non-dimensional. Let's assume that the similarity variable is of the form

$$\eta = Ax/t^n \tag{2.3.1}$$

where A and n are constants to be chosen in a manner that reduces the PDE problem to an ODE problem. Now, suppose we assume that the dimensionless solution has the form

$$\frac{T - T_i}{T_s - T_i} = f(\eta) \tag{2.3.2}$$

This is suggested by the observation that the significant aspect is the difference between the temperature at any point $T(x,y)$ and the initial temperature T_i .* The form of η is suggested by the fact that the solution for $t=0$ and $x=\infty$ must give the same value of T, and hence must correspond to the same value of f, and hence to the same value of η. Now, we could have taken $\eta = Ax^m/t^n$ but this is no more general than the (2.3.1), since this η is just a power of the other η. Also, we could have taken $\eta = At/x^n$, which also is no more general. However, we will have to differentiate twice with respect to x, and only once with respect to t, and we will find our work easier if we keep the x-dependence of η as simple as possible. For this reason, we make η linear in x, and then divide by t to a power (to be chosen later).

The next step is to transform the PDE. Using the chain rule,

$$\frac{\partial T}{\partial x} = (T_s - T_i)\frac{df}{d\eta}\frac{\partial \eta}{\partial x} = (T_s - T_i)\, f' \cdot \frac{A}{t^n} \tag{2.3.3a}$$

$$\frac{\partial^2 T}{\partial x^2} = (T_s - T_i)\frac{A}{t^n}\frac{df'}{d\eta}\frac{\partial \eta}{\partial x} = (T_s - T_i)\frac{A}{t^n} f'' \cdot \frac{A}{t^n} \tag{2.3.3b}$$

$$\frac{\partial T}{\partial t} = (T_s - T_i)\frac{df}{d\eta}\frac{\partial \eta}{\partial t} = (T_s - T_i)\, f' \cdot \left(-\frac{Anx}{t^{n+1}}\right) \tag{2.3.3c}$$

Then, substituting in (2.2.1), we obtain

$$(T_s - T_i)\frac{A^2}{t^{2n}} f'' = -\frac{1}{\alpha}(T_s - T_i)\frac{Anx}{t^{n+1}} f'$$

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*We could instead take

$$\frac{T}{T_s - T_i} = g(\eta\ ;\ T_s/T_i) \tag{2.3.2x}$$

The student should work through the problem with this starting assumption to verify that the same solution is obtained.

which simplifies to

$$f'' + \frac{1}{\alpha A^2} Anx\ t^{n-1} f' = 0 \tag{2.3.4}$$

Now, this is supposed to be an ODE for $f(\eta)$. Therefore, it can only contain f , f' , f'' , and η ; somehow we must make x and t disappear. To do this, we first replace x using (2.3.1), $x = t^n \eta / A$, and find

$$f'' + \frac{n}{\alpha A^2} t^{2n-1} \eta f' = 0 \tag{2.3.5}$$

Next, we can select the proper value of n as that which drops out t , namely $n = 1/2$. With this choice, (2.3.5) reduces to

$$f'' + \frac{1}{2\alpha A^2} \eta f' = 0 \tag{2.3.6}$$

This is an ODE, as desired. We still are free to choose A any way we like. To make (2.3.6) as simple as possible, let's pick

$$A = 1/\sqrt{2\alpha} \tag{2.3.7}$$

which reduces our ODE to

$$f'' + \eta f' = 0 \tag{2.3.8}$$

Note that η is a dimensionless variable. Now we have

$$\eta = x/\sqrt{2\alpha t} \tag{2.3.9}$$

We must also be able to express the boundary and initial conditions in terms of $f(\eta)$ in order to complete the self-similar transformation. Eqs. (2.2.2) and (2.2.4) both require

$$f(\eta) \to 0 \quad \text{as} \quad \eta \to \infty \tag{2.3.10}$$

And, (2.2.3) requires

$$f(0) = 1 \tag{2.3.11}$$

Eqs. (2.3.8), (2.3.10), and (2.3.11) define the ODE problem that we must solve.

Eqn. (2.3.8) can be written as

$$\frac{df'}{f'} = -\eta d\eta \tag{2.3.12}$$

Integrating,

$$\ell n\ f' = -\frac{\eta^2}{2} + C_0$$

or,

$$f' = C_1\, e^{-\eta^2/2} \tag{2.3.13}$$

Integrating again,

$f = C_1 \int_0^{\eta} e^{-\sigma^2/2} d\sigma + C_2$

$C_2 = 1$

$f = 1 + C_1 \int_0^{\eta} e^{-\sigma^2/2} d\sigma$ let $z = \sigma/\sqrt{2}$

$1 + C_1\sqrt{2}\int^{\eta/\sqrt{2}} e^{-z^2} dz$

$$f = C_1 \int_{\infty}^{\eta} e^{-\sigma^2/2}\, d\sigma + C_2 \tag{2.3.14}$$

The lower limit is arbitrary, and ∞ is a good choice. We must be careful not to confuse the limit of integration (η) with the variable of integration, and therefore have introduced σ as the "dummy variable" of integration.

The boundary condition (2.3.10) requires $C_2 = 0$. The boundary condition (2.3.11) requires

$0 = 1 + C_1\sqrt{2}\cdot\frac{\sqrt{\pi}}{2} \Rightarrow C_1 = -\sqrt{2/\pi}$

$f = 1 - \frac{2}{\sqrt{\pi}}\int -$

$= \mathrm{erfc}(\eta/\sqrt{2})$

$$1 = C_1 \int_{\infty}^{0} e^{-\sigma^2/2}\, d\sigma \tag{2.3.15}$$

Hence, we can write the solution as

$$f = \int_{\eta}^{\infty} e^{-\sigma^2/2}\, d\sigma \Bigg/ \int_0^{\infty} e^{-\sigma^2/2}\, d\sigma \tag{2.3.16}$$

We can express the solution in terms of known special functions by letting $z = \sigma/\sqrt{2}$. Then, $d\sigma = \sqrt{2}\, dz$, and

$$f = \int_{\eta/\sqrt{2}}^{\infty} e^{-z^2}\, dz \Bigg/ \int_0^{\infty} e^{-z^2}\, dz \tag{2.3.17}$$

$\frac{\sqrt{\pi}}{2}$

The denominator has the value $\sqrt{\pi}/2$. The numerator is $\sqrt{\pi}/2 \ \text{erfc}\ (\eta/\sqrt{2})$, where erfc is the complementary error function.* Hence, the solution is

$$\frac{T - T_i}{T_s - T_i} = \text{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \qquad (2.3.18)$$

2.4 Example with Variable Boundary Conditions

The motion of a viscous fluid, initially at rest, over an infinite plate that is set into motion at time zero is described by (Fig. 2.4.1)

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t} \qquad (2.4.1)$$

where u is the velocity tangential to the plate, and ν is the (constant) kinematic viscosity. Suppose the boundary condition at the plate $y=0$, is

$$u(0,t) = at^b \qquad (2.4.2)$$

where a and b are fixed parameters. The other boundary condition is

$$u(y,t) \to 0 \quad \text{as} \quad y \to \infty \qquad (2.4.3)$$

The initial condition is

$$u(y,0) = 0 \qquad (2.4.4)$$

There are no characteristic length or time scales in either the domain or boundary conditions of this problem, hence, we expect a self-similar solution. Suppose we assume

$$u = A\, f(\eta) \ , \quad \eta = By/t^n \qquad (2.4.5)$$

where A, B, and n are parameters that we will try to select to produce an ODE problem. The form of η is suggested by (2.4.3) and (2.4.4), which require

*See HMF, Section 7.1.

that the solution have the same behavior for large y as for small t. However, when we try to fit the boundary condition (2.4.2) with this form, we get

$$A\, f(0) = at^b \tag{2.4.6}$$

Since A and $f(0)$ will be constants, (2.4.6) can't be true except for the special case $b=0$ (which reduces this example to the previous one). Hence, (2.4.5) will not work.

We need to allow additional freedom. If we expect the curves of Fig. (2.4.1a) to collapse on a single non-dimensional curve, the value of the fluid velocity must somehow scale on the instantaneous wall velocity. This suggests that we try

$$u = A\, t^m f(\eta) \qquad \eta = By/t^n \tag{2.4.7}$$

Where now A, m, B, and n may be chosen to give us the desired self-similar solution.*

We can immediately determine m using (2.4.2),

$$u(0,t) = A\, t^m f(0) = at^b \tag{2.4.8}$$

Hence, we <u>must</u> choose $m=b$. We <u>may</u> choose A any way we like. If we choose $A=a$, then we <u>must</u> impose the boundary condition

$$f(0) = 1 \tag{2.4.9}$$

Now, we have

$$u = a\, t^b f(\eta) \qquad \eta = By/t^n \tag{2.4.10}$$

which <u>will</u> fit the boundary conditions.

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*We could have used $u = A\, y^k t^m g(\eta)$, or $u = A\, y^m h(\eta)$. These forms are equivalent to (2.4.7), with different functions f, g, and h. Eq. (2.4.7) is the simplest, since we must take two y derivatives and only one t derivative.

Next, we substitute (2.4.10) in the differential equation (2.4.1), and find $(f' = df/d\eta \ , \ f'' = d^2f/d\eta^2)$

$$\nu aB^2 t^{b-2n} f'' \quad = \quad abt^{b-1} f - at^{b-n-1} nByf' \tag{2.4.11}$$

As an ODE in $f(\eta)$, this may contain only f and its derivatives, η, and constants; y and t may not appear. So, we will replace y by

$$y \quad = \quad t^n \eta / B \tag{2.4.12}$$

Then, (2.4.11) reduces to

$$\nu aB^2 t^{b-2n} f'' \quad = \quad abt^{b-1} f - at^{b-1} n\eta f' \tag{2.4.13}$$

In order that t drop out, we must choose n such that

$$b-2n \quad = \quad b-1 \qquad \text{or} \qquad n \quad = \quad 1/2$$

With this choice, our ODE becomes

$$\nu B^2 \ f'' \quad = \quad bf - \frac{1}{2} \eta \ f' \tag{2.4.14}$$

Let's choose B such that $\nu B^2 = \frac{1}{2}$, or $B = 1/\sqrt{2\nu}$. Then we have

$$f'' + \eta f' - 2bf \quad = \quad 0 \tag{2.4.15}$$

and our similarity variable η is

$$\eta \quad = \quad y/\sqrt{2\nu t} \tag{2.4.16}$$

The boundary conditions on (2.4.15) are, from (2.4.9),

$$f(0) \quad = \quad 1 \tag{2.4.17a}$$

and, from (2.4.3),

$$f(\eta) \to 0 \qquad \text{as} \quad \eta \to \infty \tag{2.4.17b}$$

To complete the problem, we must solve (2.4.15) subject to (2.4.17). This will provide a good review of some ODE solution methods and will introduce us to some special functions.

In order to solve (2.4.15), one must be specific about the value of b. Let's first take $b = 1/2$, for which (2.4.15) becomes

$$f'' + \eta f' - f = 0 \tag{2.4.18}$$

The general solution will be of the form

$$f = C_1 f_1 + C_2 f_2 \tag{2.4.19}$$

where f_1 and f_2 are two linearly-independent solutions. For this case, $f_1 = \eta$ is one obvious solution; when the first solution to a second-order linear ODE is known, the second can always be constructed by setting

$$f_2(\eta) = f_1(\eta) \cdot g(\eta) \tag{2.4.20}$$

So, we assume

$$f_2(\eta) = \eta\, g(\eta)$$

Differentiating, and substituting in (2.4.18), we find

$$\eta g'' + 2g' + \eta(\eta g' + g) - \eta g = 0 \tag{2.4.21}$$

The zero-derivative terms cancel, which is why this methods works. So, we have

$$\eta g'' + (2+\eta^2) g' = 0 \tag{2.4.22}$$

which is really a <u>first</u>-order ODE for g'; separating the variables,

$$\frac{dg'}{g'} = -\left(\frac{2}{\eta} + \eta\right) d\eta \qquad (2.4.23)$$

Integrating, and taking the exponential,*

$$g' = \exp\left(-2 \ln \eta - \frac{\eta^2}{2}\right) = \frac{1}{\eta^2} e^{-\eta^2/2} \qquad (2.4.24)$$

Integrating again,

$$g(\eta) = \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\sigma^2/2} \, d\sigma \qquad (2.4.25)$$

The lower limit choice is arbitrary, except that zero will cause problems; infinity is an "artistic" choice. So, we now have the <u>general</u> solution to (2.4.18) as

$$f = C_1\eta + C_2\eta \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\sigma^2/2} \, d\sigma \qquad (2.4.26)$$

Note that again we were careful not to confuse the limit of integration (η) with the variable of integration (σ) .

We now apply the boundary condition (2.4.17b), which will require $C_1 = 0$ <u>if</u> we can show that the second solution f_2 is bounded as $\eta \to \infty$. We have

$$f_2(\eta) = \eta \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\sigma^2/2} \, d\sigma < \eta \int_{\infty}^{\eta} \frac{1}{\eta} e^{-\sigma^2/2} \, d\sigma = \int_{\infty}^{\eta} e^{-\sigma^2/2} \, d\sigma$$

$$(\text{for } \eta > 1) \qquad (2.4.27)$$

So, clearly $f_2(\eta) \to 0$ as $\eta \to \infty$. Therefore, C_1 is indeed zero.

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*We choose the constant of integration to be 0. Any $g(\eta)$ will do since we can use any second solution.

The behavior of f_2 at $\eta = 0$ can be clarified through use of one of the most powerful tools of analysis-integration by parts.* With it, f_2 can be rewritten as

$$f_2 = \eta\left[-\frac{1}{\sigma}\, e^{-\sigma^2/2}\Big|_{\infty}^{\eta} - \int_{\infty}^{\eta}\left(-\frac{1}{\sigma}\right)(-\sigma)\, e^{-\sigma^2/2}\, d\sigma\right]$$

$$= -\, e^{-\eta^2/2} - \eta\int_{\infty}^{\eta} e^{-\sigma^2/2}\, d\sigma \qquad (2.4.28)$$

Now it is clear that $f_2(0) = -1$. Since (2.4.17a) requires that $f(0) = 1$, $C_2 = -1$. Therefore, the final solution is

$$f(\eta) = e^{-\eta^2/2} + \eta\int_{\infty}^{\eta} e^{-\sigma^2/2}\, d\sigma \qquad (2.4.29)$$

Using the charge of variables, $z = \sigma/\sqrt{2}$, this can be written as

$$f(\eta) = e^{-\eta^2/2} - \eta\sqrt{\frac{\pi}{2}}\, \mathrm{erfc}(\eta/\sqrt{2}) \qquad (2.4.30)$$

(for $b = 1/2$)

Next, let's consider the case $b = n/2$, where n is an integer. Eqn. (2.4.15) is then

$$f'' + \eta f' - nf = 0 \qquad (2.4.31)$$

If we let $z = \eta/\sqrt{2}$, then (2.4.31) becomes

$$\frac{d^2f}{dz^2} + 2z\,\frac{df}{dz} - 2nf = 0 \qquad (2.4.32)$$

*Recall that $\int u\,dv = uv - \int v\,du$; this is called integration by parts; become adept at doing it, because it is tremendously useful and important.

The two linearly independent solutions of this equation are repeated integrals of the error function,*

$$f = C_1 i^n \mathrm{erfc}(z) + C_2 i^n \mathrm{erfc}(-z) \tag{2.4.33}$$

where the function $i^n\mathrm{erfc}(x)$ is**

$$i^n \mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \frac{(t-x)^n}{n!} e^{-t^2} dt \tag{2.4.34}$$

Hence, our solution is

$$f = C_1 i^n \mathrm{erfc}(\eta/\sqrt{2}) + C_2 i^n \mathrm{erfc}(-\eta/\sqrt{2}) \tag{2.4.35}$$

The boundary condition $f(\infty) = 0$ requires $C_2 = 0$, since $i^n\mathrm{erfc}(-\infty)$ is a constant. The boundary condition $f(0) = 1$ fixes C_1 as***

$$C_1 = \frac{1}{i^n \mathrm{erfc}(0)} = 2^n \, \Gamma(\tfrac{n}{2}+1) \tag{2.4.36}$$

where $\Gamma(x)$ is the <u>Gamma function</u>,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{2.4.37}$$

Hence, the solution is

$$f(\eta) = 2^n \, \Gamma(\tfrac{n}{2}+1) \, i^n \mathrm{erfc}(\eta/\sqrt{2}) \tag{2.4.38}$$

(for $b = n/2$)

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*HMF Section 7.2.2.

**The student should verify (2.4.33) by substitution in (2.4.32). Integration by parts will be required.

***See HMF Section 7.2.7.

2.5 Example with Integral Constraint

Consider the problem of diffusion of a contaminant deposited at time zero at the surface of a semi-infinite slab (Fig. 2.5.1). The diffusion process is described by

$$\alpha \frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t} \tag{2.5.1}$$

where $c(x,t)$ is the concentration per unit volume, and α is the diffusion coefficient for the contaminant. The initial condition is

$$c(x,0) = 0 \qquad x > 0 \tag{2.5.2}$$

The boundary condition for large x is

$$c(x,t) \to 0 \qquad \text{as} \quad x \to \infty \tag{2.5.3}$$

The total amount of contaminant contained in the slab is fixed. This gives an integral constraint,

$$\int_0^\infty c\,dx = Q \tag{2.5.4}$$

This problem has no natural characteristic length or time scales and, hence, we expect a self-similar solution.

Let's try to construct the solution in the form*

$$c = At^n f(\eta) \qquad \eta = Bx/t^m \tag{2.5.5}$$

where A , n , B , and m are constants to be chosen. The integral constraint (2.5.4) immediately tells us something about n

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*Again, the similar boundary condition (2.5.3) and initial condition (2.5.2) suggest the form of η .

$$Q = \int_0^\infty At^n f(\eta) dx = \int_0^\infty At^n f(\eta) \cdot \frac{t^m}{B} d\eta$$

$$= \frac{A}{B} t^{n+m} \int_0^\infty f(\eta)\, d\eta = \text{constant} \qquad (2.5.6)$$

The integral will be some number. Therefore, for Q to be constant $n = -m$. We will later use (2.5.6) to help determine other constraints.

Next, we substitute (2.5.5), with $n = -m$, in (2.5.1), and obtain

$$\alpha AB^2 t^{-3m} f'' = -mt^{-m-1} Af - m\eta At^{-m-1} f' \qquad (2.5.7)$$

Note we have already eliminated x in favor of η. For this to be an ODE, t must drop out, hence $-3m = -m-1$, or $m = 1/2$. We pick $\alpha B^2 = 1/2$, $B = 1/\sqrt{2\alpha}$, and then our ODE becomes

$$f'' + \eta f' + f = 0 \qquad (2.5.8)$$

The boundary condition (2.5.3) requires

$$f(\eta) = 0 \quad \text{as} \quad \eta \to \infty \qquad (2.5.9)$$

We have the freedom to match this integral constraint with the choice of A. Hence, let's choose

$$f(0) = 1 \qquad (2.5.10)$$

Eqs. (2.5.8)-(2.5.10) define the ODE problem to be solved.

Our task becomes easy when we recognize that (2.5.8) is expressible as

$$(f')' + (\eta f)' = 0 \qquad (2.5.11)$$

Integrating,

$$f' + \eta f = C_1 \qquad (2.5.12)$$

Since the boundary condition (2.5.9) requires $f(\eta) \to 0$ as $\eta \to \infty$, $f'(\eta) \to 0$ as $\eta \to \infty$, and hence C_1 will have to be zero unless $\eta f \to$ constant as $\eta \to \infty$. Let's assume (subject to later verification) that $\eta f \to 0$ as $\eta \to \infty$, and hence that $C_1 = 0$. Separating the variables and integrating again,

$$f = C_2 \, e^{-\eta^2/2} \tag{2.5.13}$$

Note that indeed $\eta f \to 0$ as $\eta \to \infty$, as assumed. Our choice $f(0) = 1$ requires $C_2 = 1$. Hence,

$$f(\eta) = e^{-\eta^2/2} \tag{2.5.14}$$

To complete the solution we need*

$$\int_0^\infty f(\eta)\,d\eta = \int_0^\infty e^{-\eta^2/2}\,d\eta = \int_0^\infty e^{-\sigma^2}\sqrt{2}\,d\sigma = \sqrt{\frac{\pi}{2}} \tag{2.5.15}$$

Using this in (2.5.6), we find

$$A = \frac{Q}{\sqrt{\pi\alpha}} \tag{2.5.16}$$

Hence, the final solution is

$$c = \frac{Q}{\sqrt{\pi\alpha t}} \exp\left(-\frac{x^2}{4\alpha t}\right) \tag{2.5.17}$$

Note that the concentration at $x = 0$ is infinite at $t = 0$. This reflects a modest deficiency in the model, namely we assumed that we could place a finite amount of contaminant in a zero thickness layer at time zero. Thus, the solution is not useful for very small times. Fig. 2.5.1 shows the form of this solution.

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*See HMF Section 7.1.

2.6 A Non-Linear Problem

The laminar boundary layer over a flat plate is described by

$$\psi_y \psi_{xy} - \psi_x \psi_{yy} = \nu \psi_{yyy} \tag{2.6.1}$$

where ν is the kinematic viscosity and $\psi(x,y)$ is the stream function, which must satisfy the boundary conditions

$$\psi_x = 0 \quad \text{at} \quad y = 0 \tag{2.6.2a}$$

$$\psi_y = 0 \quad \text{at} \quad y = 0 \tag{2.6.2b}$$

$$\psi_y \to U_o \quad \text{as} \quad x \to 0 \tag{2.6.3a}$$

$$\Psi_y \to U_o \quad \text{as} \quad y \to \infty \tag{2.6.3b}$$

Students of fluid mechanics should look up the derivation of this problem; others may treat it simply as a mathematical example.

Since there are no characteristic scales in the problem, we look for a self-similar solution of the form

$$\psi(x,y) = Ax^n f(\eta) \quad , \quad \eta = By/x^m \tag{2.6.4}$$

Note that we will need three y derivatives, and only one x derivative, so we chose a form that keeps the y dependence simple.

Substituting (2.6.4) in (2.6.3),

$$\psi_y = ABx^{n-m} f'(\eta) \to U_o \quad \text{as} \quad (\eta \to \infty) \tag{2.6.5}$$

Now, $f'(\infty)$ will be a number; hence, for this to be constant, $m = n$. We will make the arbitrary choice $f'(\infty) = 1$. Then, we will have to choose A and B such that $AB = U_o$. With these choices, (2.6.5) will be satisfied for all x .

Next, we substitute (2.6.4) in (2.6.1), using $m = n$. This produces

$$-AB\, f' \cdot ABx^{-1} m\eta\, f'' - Bx^{m-1} m(f-\eta f') BA^2 x^{-m}\, f'' = \nu B^3 A x^{-2m}\, f''' \tag{2.6.6}$$

Note that we have already replaced y by $\eta x^m/B$. For x to drop out, $-2m = -1$, or $m = 1/2$. Then, if we pick $A^2B^2 = \nu B^3 A$, the equation reduces to

$$f''' + \frac{1}{2} f\, f'' = 0 \tag{2.6.7}$$

We have already chosen $f'(\infty) = 1$, which led us to $AB = U_o$. Hence,

$$A = \sqrt{U_o/\nu} \qquad B = \sqrt{U_o \nu} \tag{2.6.8}$$

The boundary conditions are, from (2.6.2a)

$$f(0) = 0 \tag{2.6.9a}$$

and from (2.6.2b)

$$f'(0) = 0 \tag{2.6.9b}$$

Eqs. (2.6.3) will be safisfied by our choice of constants if

$$f'(\eta) \to 1 \qquad \text{as } \eta \to \infty \tag{2.6.9c}$$

Eqn. (2.6.7) must now be solved, subject to the boundary conditions (2.6.9). The solution will introduce you to two useful ideas; rescaling, and numerical solution as an initial value problem.

In problems of this sort, it is often possible to use a "rescaling technique" to convert the two-point boundary value problem to a one-point initial value problem. The advantage of this is that the initial value problem can be solved numerically with a single-pass technique. To rescale, we let

$$z = C\eta \qquad f(\eta) = C^n g(z) \tag{2.6.10}$$

The idea is to pick an n such that we can solve the g equation <u>without</u> knowing the value of the constant C, which will be determined <u>after</u> the g equation has been solved. Substituting (2.6.10) in (2.6.7), one finds ($g' = dg/dz$, etc) .

$$C^{n+3} g''' + \frac{1}{2} C^{2n+2} g g'' = 0$$

Now, if we pick $n+3 = 2n+2$, i.e. $n = 1$, the g equation is

$$g''' + \frac{1}{2} g g'' = 0 \qquad (2.6.11)$$

The boundary conditions on g are, from (2.6.9a and b),

$$g(0) = 0 \qquad (2.6.12a)$$
$$g'(0) = 0 \qquad (2.6.12b)$$

We replace the outer boundary condition by a third condition at $z = 0$. Let's use

$$g''(0) = 1 \qquad (2.6.12c)$$

If we can solve (2.6.11), subject to (2.6.12), we can choose C to produce an f satisfying (2.6.9c), and the solution will be complete.

So now we go to the local computer center, and use a program that solves <u>systems</u> of <u>first</u> order ODE's by a marching method. These methods deal with systems of the form

$$\frac{dy_i}{dx} = A_i(x,\underline{y}) \qquad (2.6.13)$$

with the "initial" ($x = x_o$) values of the solution vector y_i prescribed. We define the three variables as

$$y_1 = g \tag{2.6.14a}$$

$$y_2 = g' \tag{2.6.14b}$$

$$y_3 = g'' \tag{2.6.14c}$$

Then, (2.6.11) is the <u>first</u> order equation

$$y_3' = -\frac{1}{2} y_1 y_3 \tag{2.6.15a}$$

The other two equations are, from the definitions,

$$y_1' = y_2 \tag{2.6.15b}$$

$$y_2' = y_3 \tag{2.6.15c}$$

The initial conditions are

$$y_1(0) = 0 \tag{2.6.16a}$$

$$y_2(0) = 0 \tag{2.6.16b}$$

$$y_3(0) = 1 \tag{2.6.16c}$$

It takes only a few lines of program to tell the general purpose program that we want it to solve (2.6.15), subject to (2.6.16), over a range from $x = 0$ to some large x (perhaps 20). We execute, and print y_1, y_2, and y_3 at different values of x. If all has gone well, at large x y_1 will be growing linearly, $y_2 = g'$ will be constant, and $y_3 = g''$ will be very small.

Knowing the value of $g'(z)$ as $z \to \infty$, we go back to the rescaling transformation (2.6.10) and the outer boundary condition (2.6.9c)

$$f'(\infty) = C^2 g'(\infty) = 1$$

Hence, $C = 1/\sqrt{g'(\infty)}$. We can now calculate and plot $f(\eta)$ for $0 \leq \eta < \infty$, and the problem is finished.

2.7 An Example in More Dimensions

The transient heat conduction in a quarter-infinite block (Fig. 2.7.1) is described by

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (2.7.1)$$

where the quantities are as defined in §2.3. Suppose that the initial condition is

$$T(x,y,0) = T_i \qquad \text{for} \quad x > 0\ , \quad y > 0 \qquad (2.7.2)$$

the boundary conditions are

$$T(x,0,t) = T_s \qquad (2.7.3a)$$

$$T(0,y,t) = T_s \qquad (2.7.3b)$$

Let's seek a self-similar solution in terms of two similarity variables,*

$$\xi = Ax/t^n \qquad \eta = Ay/t^n \qquad (2.7.4)$$

Following the example in §2.3, we assume

$$\frac{T - T_i}{T_s - T_i} = F(\xi,\eta) \qquad (2.7.5)$$

Substituting in (2.7.1) ,

$$A^2 t^{-2n}(F_{\xi\xi}+F_{\eta\eta}) = -\frac{nt^{-1}}{\alpha}(\xi F_\xi + \eta F_\eta) \qquad (2.7.6)$$

We choose $n = 1/2$ to reduce (2.7.6) to a PDE in just ξ and η . Then, with $A = 1/\sqrt{\alpha}$,

*Because the problem is symmetric in x and y , we have no reason to use different powers or coefficients in the two similarity variables.

$$F_{\xi\xi} + F_{\eta\eta} + \xi F_{\xi} + \eta F_{\eta} = 0 \qquad (2.7.7)$$

Note that the self-similar transformation has reduced the number of independent variables by one.

The boundary and initial conditions produce

$$F(0,\eta) = 1 \qquad (2.7.8a)$$

$$F(\xi,0) = 1 \qquad (2.7.8b)$$

Now, as $\xi \to \infty$, the solution should approach that of the semi-infinite solid (see § 2.3), so

$$F \to \hat{g}(\eta) = \text{erfc}(\eta) \quad \text{as} \quad \xi \to \infty \qquad (2.7.8c)$$

Similarly,

$$F \to \hat{f}(\xi) = \text{erfc}(\xi) \quad \text{as} \quad \eta \to \infty \qquad (2.7.8d)$$

The PDE for F can be solved by the method of separation of variables, discussed in the next three chapters. Following the approach to be presented there, we assume

$$F(\xi,\eta) = \hat{f}(\xi) + \hat{g}(\eta) + H(\xi,\eta) \qquad (2.7.9)$$

Since $\hat{f}'' + \xi\hat{f}' = 0$ and $\hat{g}'' + \eta\hat{g}' = 0$ (see 2.3.8), H also satisfies (2.7.7). Now, we assume a separable solution for H

$$H(\xi,\eta) = f(\xi) \cdot g(\eta) \qquad (2.7.10)$$

Substituting (2.7.9) in (2.7.7), and dividing by H, one finds

$$\frac{f'' + \xi f'}{f} = -\left(\frac{g'' + \eta g'}{g}\right) \qquad (2.7.11)$$

Since the left-hand side is independent of η, and the right-hand side is independent of ξ, both must be constant, and

$$\frac{f'' + \xi f'}{f} = C$$

$$\frac{g'' + \eta g'}{g} = -C$$

or,

$$f'' + \xi f' = Cf \tag{2.7.12a}$$

$$g'' + \eta g' = -Cg \tag{2.7.12b}$$

The boundary conditions on H are, from (2.7.8),

$$H \to 0 \quad \text{as} \quad \xi \to \infty \tag{2.7.13a}$$

$$H \to 0 \quad \text{as} \quad \eta \to \infty \tag{2.7.13b}$$

$$H(0,\eta) = -\hat{g}(\eta) \tag{2.7.13c}$$

$$H(\xi,0) = -\hat{f}(\xi) \tag{2.7.13d}$$

By symmetry, f and g must be the same function, hence C = 0. So if we take

$$C = 0 \qquad f(\xi) = i\hat{f}(\xi) \qquad g(\eta) = i\hat{g}(\eta) \tag{2.7.14}$$

(2.7.12) will be satisfied, and the boundary conditions (2.7.13) are all satisfied. Hence, the solution is

$$F(\xi,\eta) = -\text{erfc}(\xi/\sqrt{2})\ \text{erfc}(\eta/\sqrt{2}) + \text{erfc}(\xi/\sqrt{2}) + \text{erfc}(\eta/\sqrt{2}) \tag{2.7.15}$$

2.8 Summary

We have seen that self-similar solutions arise when there are no natural characteristic scales for the independent variables in the problem formulation. The self-similar transformation will always reduce the number of independent variables by one, so that in a problem with two independent variables the PDE will become an ODE. The steps used to systematically develop the self-similar solution are as follows:

(1) Assume a general form for the transformation, guided by the initial and boundary conditions. Use a form in which the variable that appears in the most complex way in the equations appears as simply as possible in the transformation.

(2) Express the boundary and initial conditions in terms of the similarity transformation, and verify that they can be satisfied by the

assumed transformation. If they can not, add additional degrees of freedom.

(3) Remove one (or more) of the independent variables using the similarity variable. Then, determine the parameters of the transformation necessary to reduce the PDE order by one.

(4) Express the boundary and initial conditions for the reduced problem, and solve by appropriate methods.

In all of the examples worked here, the similarity variable involved forms like $y/\sqrt{x}$. The square-root behavior occurs frequently, but not exclusively. Some of the problems at the end of this chapter will require other powers in the similarity variable.

For Further Reading on Similarity Solutions

- Kline, S. J., Similitude and Approximation Theory, McGraw-Hill Book Co., New York, 1965.
- Hansen, A. G., Similarity Analysis of Boundary Value Problems in Engineering, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.
- Sedov, L. I., Similarity and Dimensional Methods in Mechanics, Academic Press, New York, 1959.

Exercises:

2.1 The temperature field $T(x,t)$ in a semi-infinite slab with a constant heat flux is described by

$$\alpha \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t} \quad ; \quad T(x,0) = T_i$$

$$T(x,t) \rightarrow T_i \quad \text{as} \quad x \rightarrow \infty \quad ; \quad -k \frac{\partial T}{\partial x} = q \quad \text{at} \quad x = 0$$

Solve for the temperature field for $x \geq 0$, $t \geq 0$.

2.2 The temperature field in the thermal boundary layer that grows within a hydrodynamic boundary layer at a step in wall temperature is described by

$$\alpha \frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \quad ; \quad T(0,y) = T_\infty \quad y > 0$$

$$T(x,y) \rightarrow T_\infty \quad \text{as} \quad y \rightarrow \infty \quad ; \quad T(x,0) = T_w \; ;$$

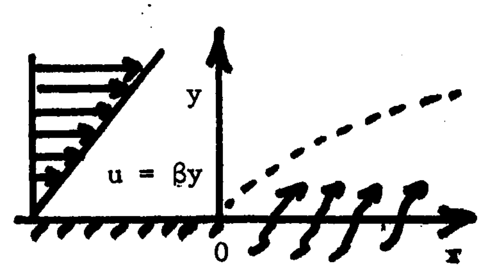
Solve for the temperature field for $x \geq 0$, $y \geq 0$.

2.3 A device for measuring the velocity gradient in flows is shown in the figure. It consists of a heated plate at the wall, over which a thermal boundary layer grows. As long as the thermal boundary layer is confined to the region where the flow velocity u is linear $(u = \beta y)$, the problem is described by

$$\alpha \frac{\partial^2 T}{\partial y^2} = \beta y \frac{\partial T}{\partial x} \quad ; \quad T(0,y) = T_\infty \quad y > 0$$

$$T(x,y) \rightarrow T_\infty \quad \text{as} \quad y \rightarrow \infty \quad ; \quad -k \frac{\partial T}{\partial y} = q \quad \text{at} \quad y = 0$$

Derive an expression relating the local wall temperature, $T_w(x)$, to the flow parameters and x . Evaluate any constants in this expression. Hint: Γ .



2.4 The diffusion of a contaminant deposited along a line within an infinite medium is described by

$$\alpha \frac{\partial}{\partial r}\left(r \frac{\partial c}{\partial r}\right) = r \frac{\partial c}{\partial t} \quad ; \quad c(r,t) \rightarrow 0 \text{ as } r \rightarrow \infty$$

$$c(r,0) = 0 \quad r > 0 \quad ; \quad 2\pi \int_0^\infty c r \, dr = Q$$

Solve this problem, and give an expression for $c(0,t)$.

2.5 The diffusion of a contaminant deposited at a point in an infinite medium is described by

$$\alpha \frac{\partial}{\partial r}\left(r^2 \frac{\partial c}{\partial r}\right) = r^2 \frac{\partial c}{\partial t} \quad ; \quad c(r,t) \rightarrow 0 \text{ as } r \rightarrow \infty$$

$$c(r,0) = 0 \quad r > 0 \quad ; \quad 4\pi \int_0^\infty c r^2 \, dr = Q$$

Solve this problem, and give an expression for $c(0,t)$.

2.6 Consider a non-linear diffusion problem described by

$$\frac{\partial}{\partial x}\left[\alpha(1+\beta c) \frac{\partial c}{\partial x}\right] = \frac{\partial c}{\partial t} \quad ; \quad c(x,0) = 0 \quad x > 0$$

$$c(0,t) = 1 \quad ; \quad c(x,t) \rightarrow 0 \text{ as } x \rightarrow \infty$$

Derive the similarity transform and associated ODE. Solve the problem numerically for $\beta = -0.5$, 0 , and 0.5 . Use the $\beta = 0$ case to check the numerical solution against the exact solution, and to guide the starting and direction of numerical marching.

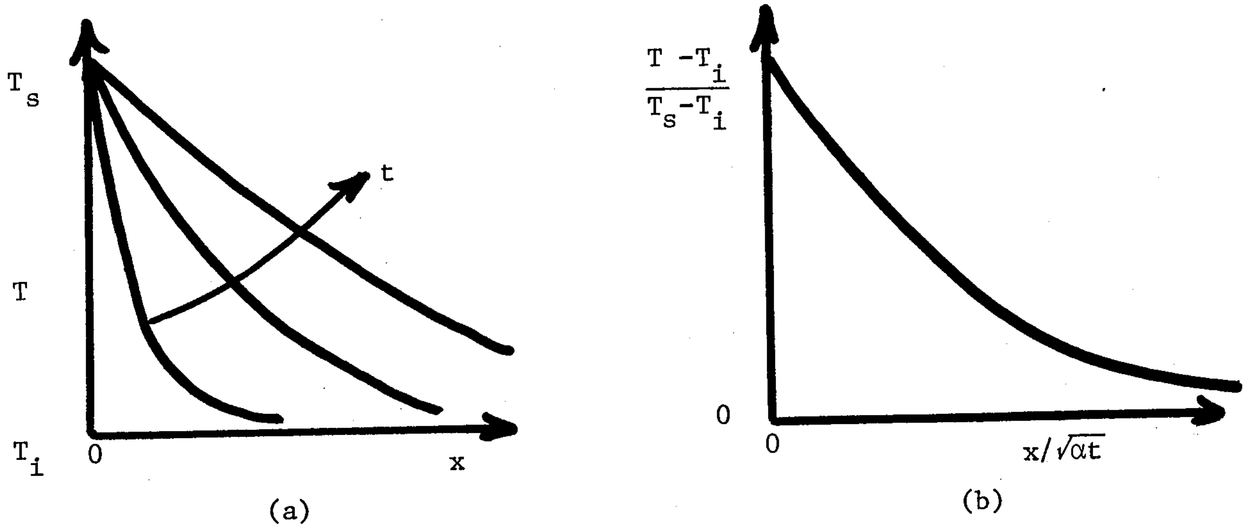
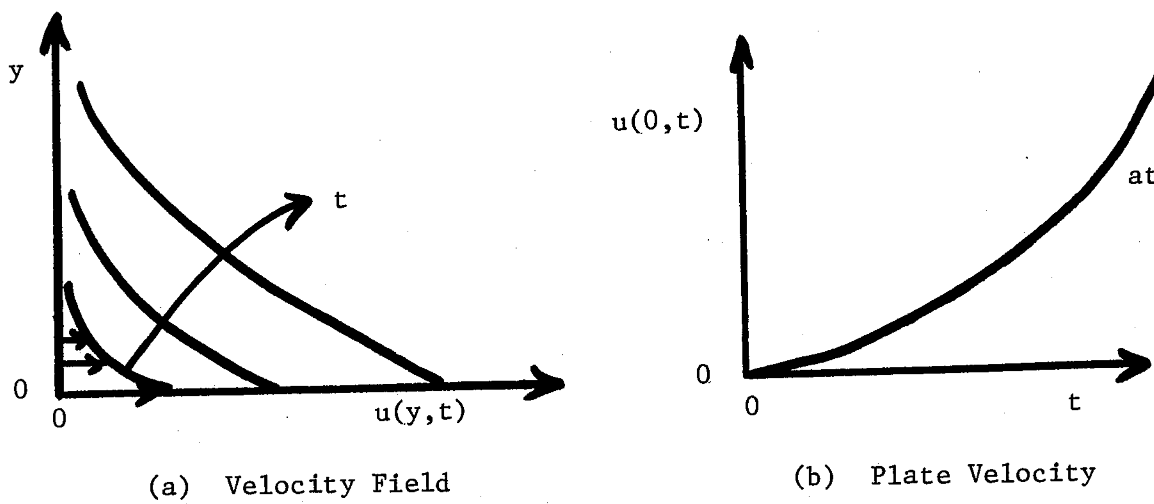


Fig. 2.2.1 Temperature Field in a Semi-Infinite Slab



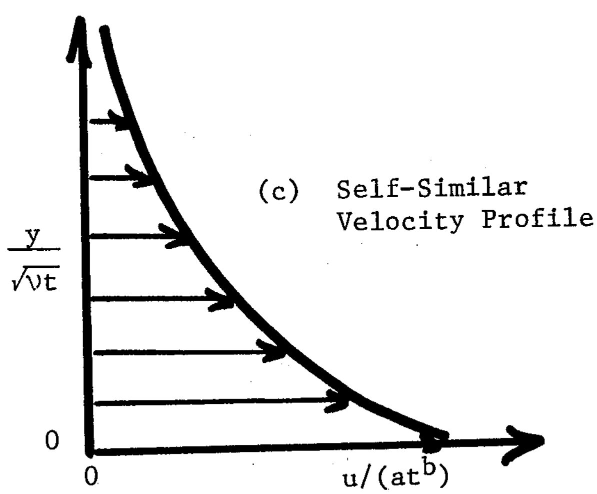
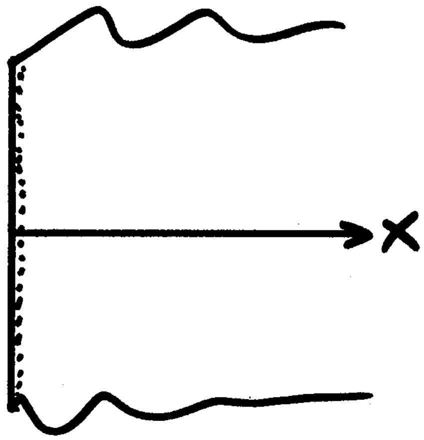
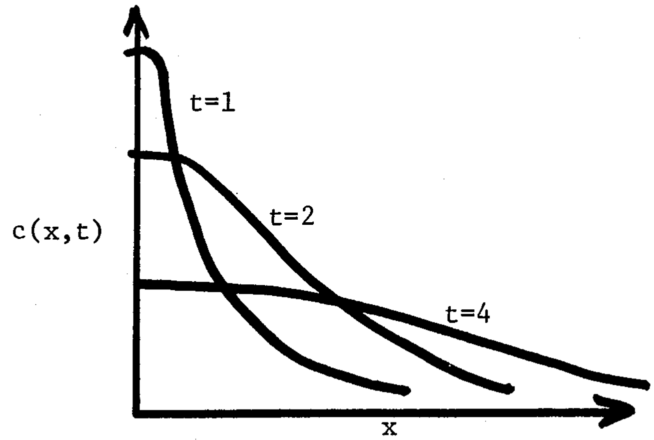


Fig. 2.4.1 Velocity Field in Viscous Flow over A Moving Plate

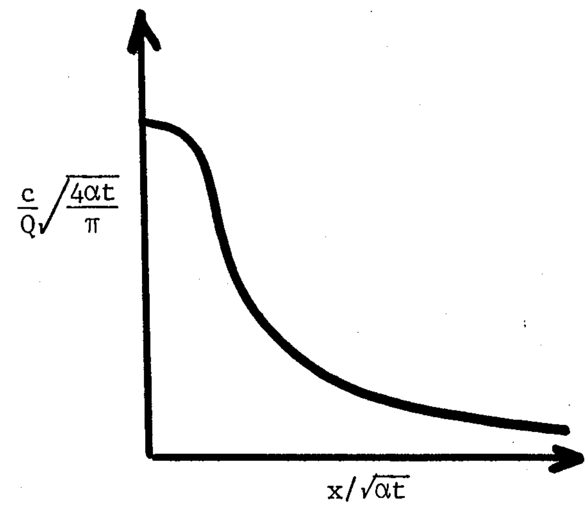


(a) The System



(b) Concentration Profiles

Fig. 2.5.1.



(c) Self-Similar Profile

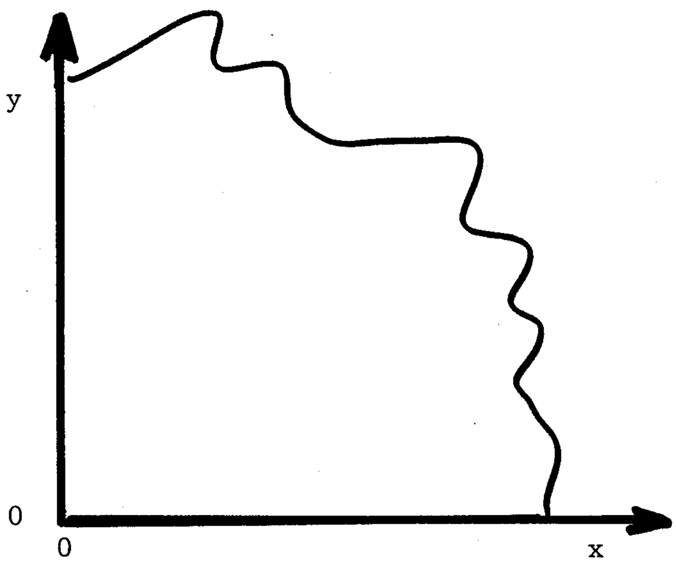


Fig. 2.7.1. Geometry for Analysis of Heating of a Corner