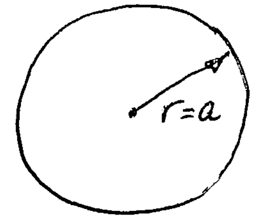
9/18

ON THURSDAY / WE SOLVED

$$\nabla^2 W - \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = 0$$ IN A CIRCULAR REGION



with $W(r,\theta,t) = W$

AND $W(r=a,\theta,t) = 0$

By writing $W = w(r,\theta)\cdot T(t)$ we can separate spatial & temporal functions

$$\nabla^2 W = \frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2}$$

THUS $$\nabla^2 W - \frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = T\nabla^2 w - \frac{1}{c^2} w \ddot{T} = 0$$

or $$\frac{c^2\nabla^2 w}{w} = \frac{\ddot{T}}{T} = -\omega^2 \implies T = \mathcal{E}_1 \cos\omega t + \mathcal{E}_2 \sin\omega t$$

ω = frequency of vibration

$$\implies \nabla^2 w + \left(\frac{\omega}{c}\right)^2 w = 0 \qquad \text{let } \frac{\omega}{c} = \lambda$$

or $\nabla^2 w + \lambda^2 w = 0$ this is an eigenvalue problem
Helmholtz Equation

TO SOLVE EIGENVALUE PROBLEM, LET $w(r,\theta) = R(r)\,\Theta(\theta)$

$$\therefore \nabla^2 w + \lambda^2 w = (R'' + \tfrac{1}{r}R')\Theta + \tfrac{1}{r^2}\Theta'' + \lambda^2 R\Theta = 0$$

$$\implies \frac{r^2(R'' + \frac{1}{r}R') + \lambda^2 r^2}{R} = -\frac{\Theta''}{\Theta} = +k^2$$ SINCE Θ FUNCTION MUST BE PERIODIC

$$\implies \Theta = A\cos k\theta + B\sin k\theta$$

$$\implies r^2 R'' + rR' + (\lambda^2 r^2 - k^2)R = 0$$

BESSEL EQN OF FORM $x^2y'' + xy' + (x^2 - p^2)y = 0$ of order $p(k)$

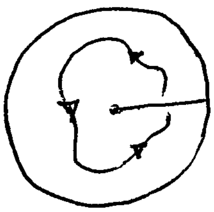
IF $x = \lambda r$
$R = y$
$p = k$

SOLUTIONS ARE

$$y = C_1 J_p(x) + C_2 J_{-p}(x) \qquad \text{IF } p \neq 0 \text{ or an integer}$$

$$y = C_1 J_n(x) + C_2 Y_n(x) \qquad \text{IF } p \text{ is zero or an integer}$$

- TO DETERMINE IF k IS integer $W(r,\theta,t) = W(r,\theta+2\pi,t) \Rightarrow w(r,\theta) = w(r,\theta+2\pi) \Rightarrow$



$\Theta \Rightarrow \Theta(\theta + 2k\pi) = \Theta(\theta) \Rightarrow \boxed{k=n} = 0, 1, 2, 3, 4 \cdots$

($n = -1, -2$ etc. not linearly independent)

$\therefore \; R(\lambda r) = C_1 J_n(\lambda r) + C_2 Y_n(\lambda r)$

— STOPPED HERE THURSDAY

- NOW $J_n(\lambda r)$ IS BOUNDED AT $r=0$

  $Y_n(\lambda r)$ IS <u>NOT</u> BOUNDED AT $r=0$

PHYSICAL PROBLEM DICTATES THAT $W(r,\theta,t)$ IS BOUNDED AT $r=0$

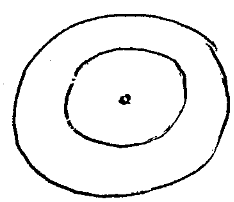
$\Longrightarrow$ MUST TAKE $\boxed{C_2 = 0}$ SINCE $Y_n(\lambda r)$ CONTAINS $\log(\lambda r)$ TERM

$\Longrightarrow$ NOTE $J_{-p}(x)$ IS NOT BOUNDED AT $x=0$ EITHER

$\Longrightarrow$ FOR AN ANNULAR MEMBRANE

ORIGIN <u>NOT</u> INCLUDED THUS

$Y_n(\lambda r)$ <u>IS</u> kept



- LET'S LOOK AT HOW TO HANDLE $W(r=a,\theta,t)=0$

  $W(r=a,\theta,t) = w(r=a,\theta)\,T(t) = R(\lambda r=\lambda a)\Theta(\theta)T(t) = 0$ IRRESPECTIVE OF t & θ

  $\Longrightarrow R(\lambda r = \lambda a) = 0$ THIS IS THE WAY TO FIND THE λ's

SO FAR $w(r,\theta) = J_n(\lambda r)\left[\bar{A}\cos n\theta + \bar{B}\sin n\theta\right]$

SINCE TRUE FOR ANY n & EQ $(\nabla^2 w + \lambda^2 w = 0)$ IS LINEAR

SO $w(r,\theta)$ DEPENDS ON $n \Rightarrow w_n(r,\theta)$

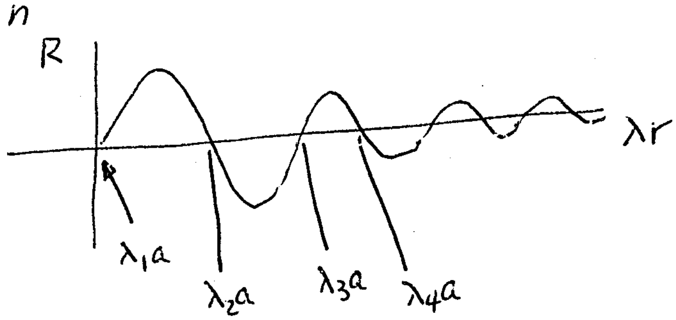
AND

$$w(r,\theta) = \sum_n w_n(r,\theta) = \sum_n J_n(\lambda r)\left[\bar{A}_n\cos n\theta + \bar{B}_n \sin n\theta\right]$$

$\Longrightarrow$ SINCE $R(\lambda a) = 0 \Rightarrow J_n(\lambda a) = 0$

FOR any n

TALK ABOUT TABLE



$\Longrightarrow$ THERE ARE AN INFINITE NO. OF VALUES FOR $J_0(\lambda r)$ & $J_1(\lambda r)$, & $J_2(\lambda r)$ ...

$\therefore$ WE MUST NUMBER THE ZEROES OF J_0, J_1, J_2 etc. AND PUT IN ORDER OF INCREASING MAGNITUDE

THUS

$$w(r,\theta) = \sum_m \sum_n J_n(\lambda_{nm} r)\left[\bar{A}_n \cos n\theta + \bar{B}_n \sin n\theta\right]$$

AND

$$W(r,\theta,t) = \sum_m \sum_n J_n(\lambda_{nm} r)\left[\bar{A}_n \cos n\theta + \bar{B}_n \sin n\theta\right]\left[\mathcal{C}_{1_{mn}} \cos \omega_{mn} t + \mathcal{C}_{2_{mn}} \sin \omega_{mn} t\right]$$

$$= \sum_m \sum_n J_n(\lambda_{nm} r)\left[\bar{C}_n \cos(n\theta + \psi_n)\right]\left[D_{mn} \cos(\omega_{mn} t + \phi_{mn})\right]$$

$\Longrightarrow$ ω_{mn} has DOUBLE SUBSCRIPT SINCE $\frac{\omega}{c} = \lambda$ & λ DEPENDS ON m & n

$\Longrightarrow$ $\bar{C}_n$, ψ_n, D_{mn} & ϕ_{mn} CANNOT BE FOUND WITHOUT IC's FOR $T(t)$ & BC's ON θ

Table 3.5.1 gives the first few values of these frequencies in dimensionless form. The solution for the membrane displacement u_{nm} in the vibration eigenmode n,m is then

$$u_{nm}(r,\theta,t) = A_{nm}J_n(\lambda_{nm}r)\cos(n\theta-\psi)\cos(\omega_{nm}t-\phi)$$

$$\lambda_{nm} = j_{n,m}/r_o \qquad \omega_{nm} = \lambda_{nm}a \tag{3.5.14}$$

The phase angles ϕ and ψ, and the amplitude A remain undetermined.

The lowest frequency occurs for the 0,1 mode. Note that for $n = 0$ the motion is axisymmetric, and has no nodes. The next higher frequency occurs for the 1,1 mode. This mode has one diametral node along which the

TABLE 3.5.1

DIMENSIONLESS MEMBRANE FREQUENCIES

n	m	$j_{n,m} = \omega_{nm}r_o/a$ ≈ c
0	1	2.40483
1	1	3.83171
2	1	5.13562
0	2	5.52008
3	1	6.38016
1	2	7.01559
4	1	7.58834

membrane does not move. (The phase angle of this node cannot be determined without initial conditions). The third mode is the 2,1 mode, which has <u>two</u> diametral nodes, and the fourth is the 0,2 mode, with one circular node at the point where $J_0(\lambda_{02}r) = 0$, i.e. at $\lambda_{02}r = j_{0,1} = 2.40483$. Figure 3.5.3 shows the nodal lines for the first several modes.