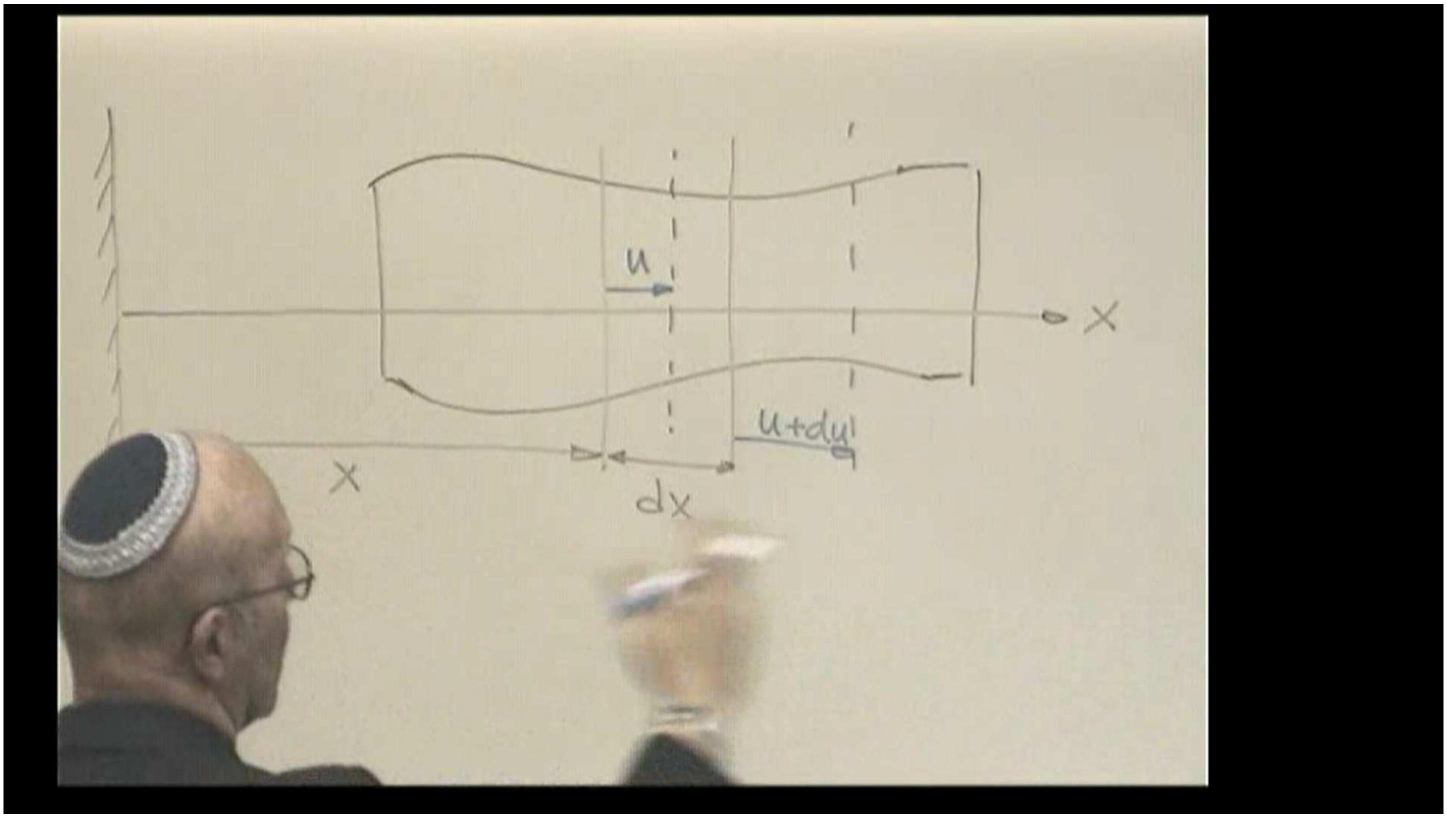
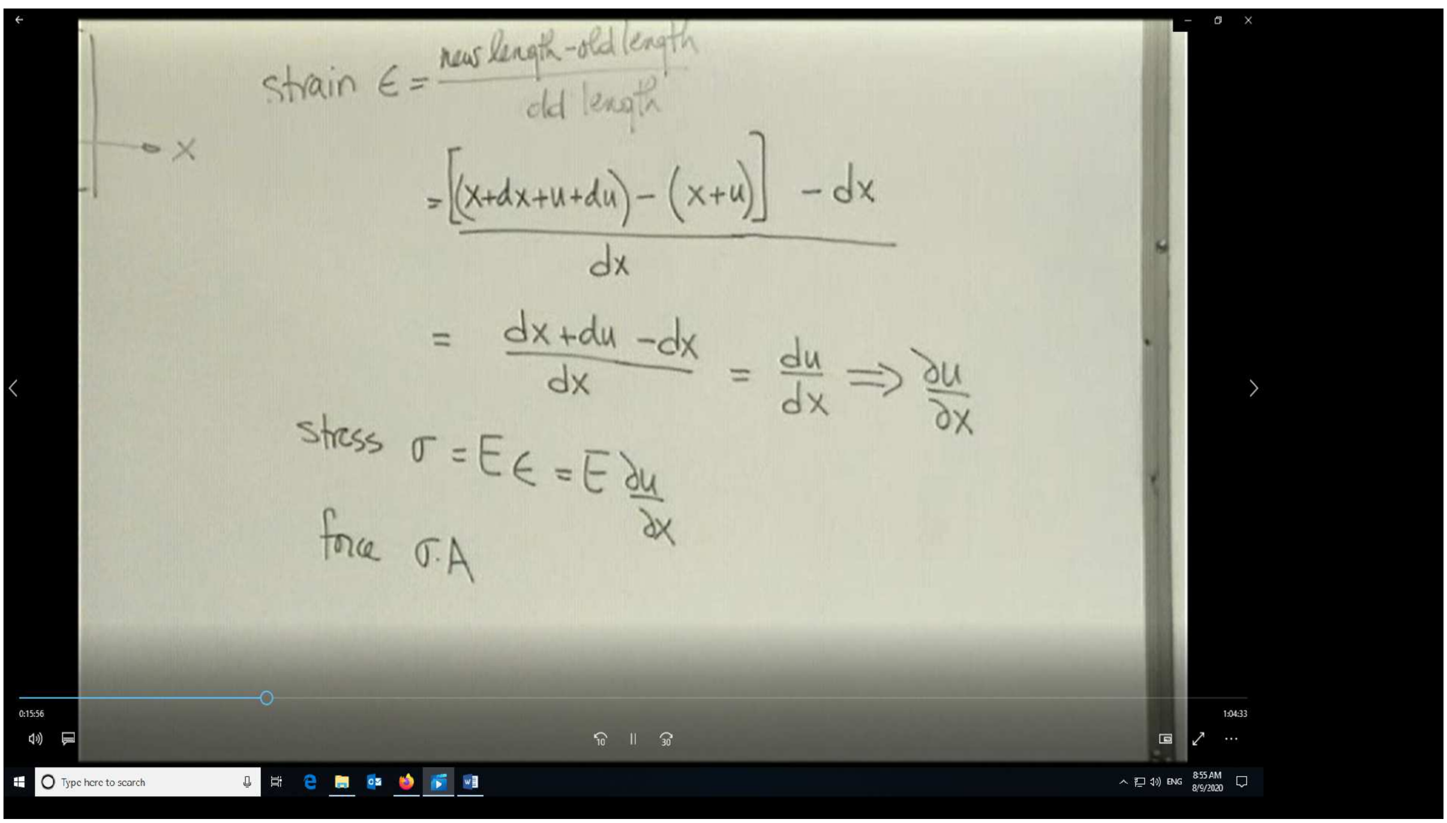
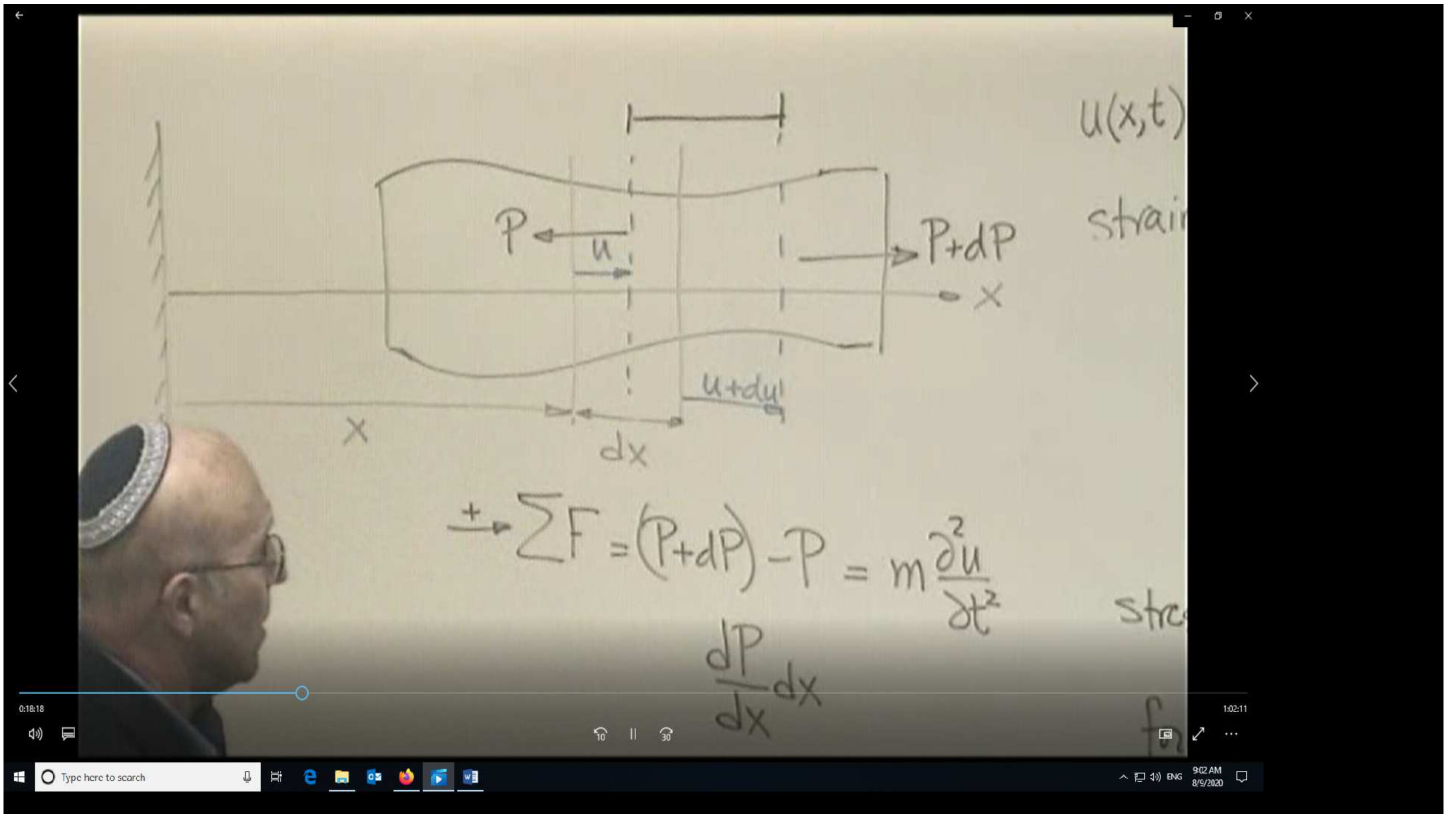
The derivation of the wave equation. Let u(x,t) represent the displacement of a cross-section of the bar at x



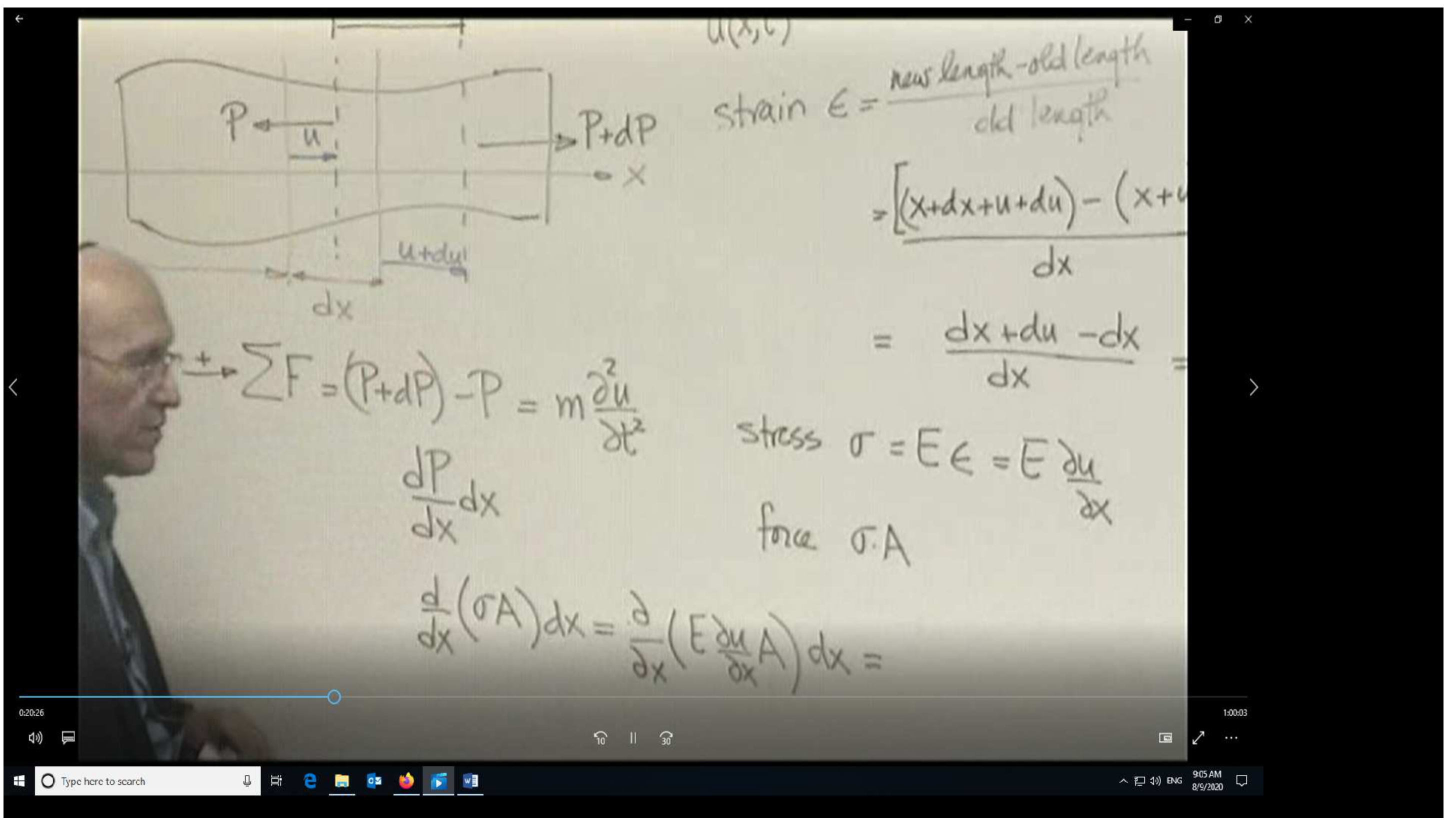


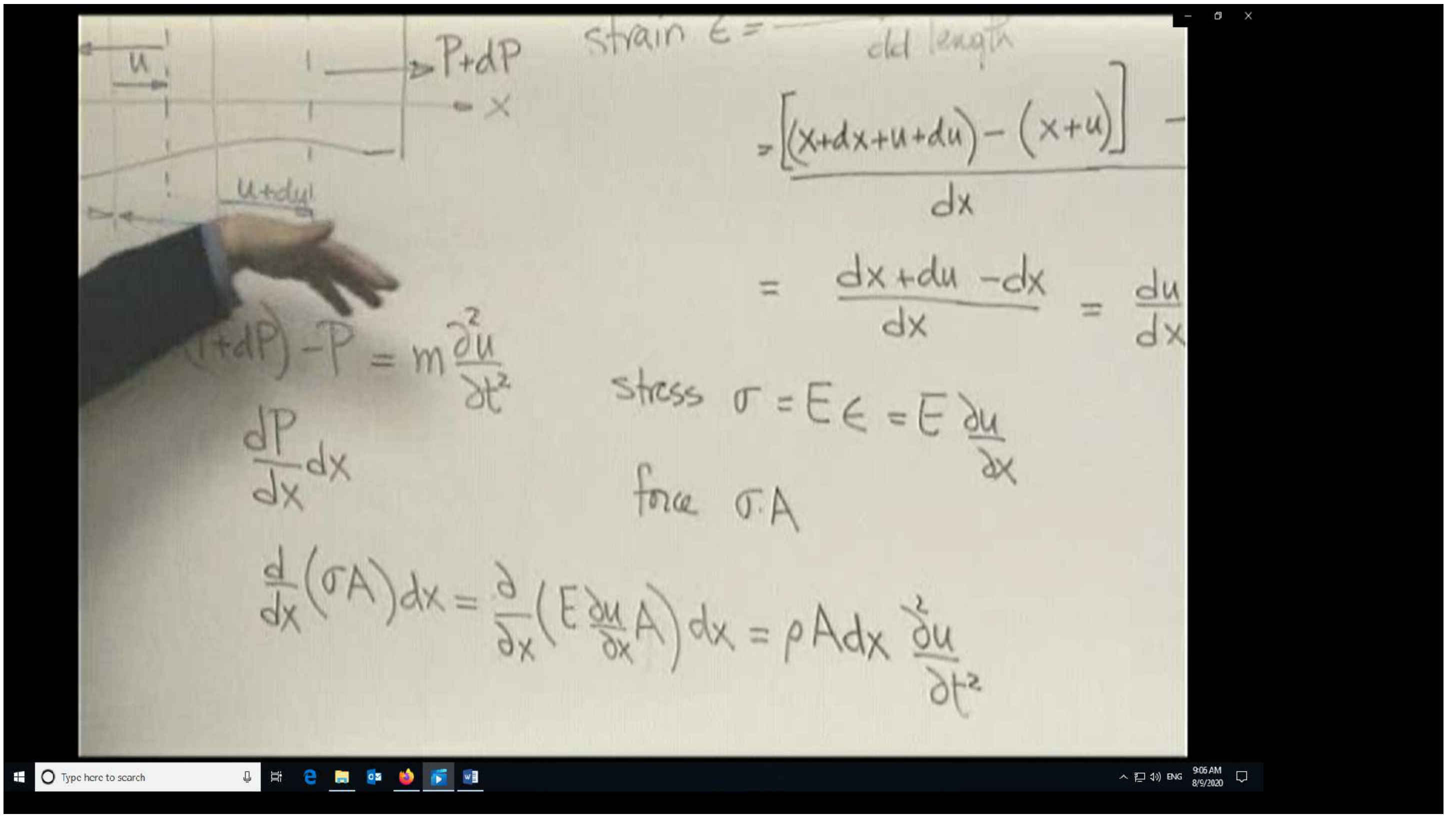
Force at any cross-section is σ•A (A the cross-sectional area)=P. In 1-D we assume stress is the average across the cross-section.

Now application of the ∑ F = mass x acceleration gives

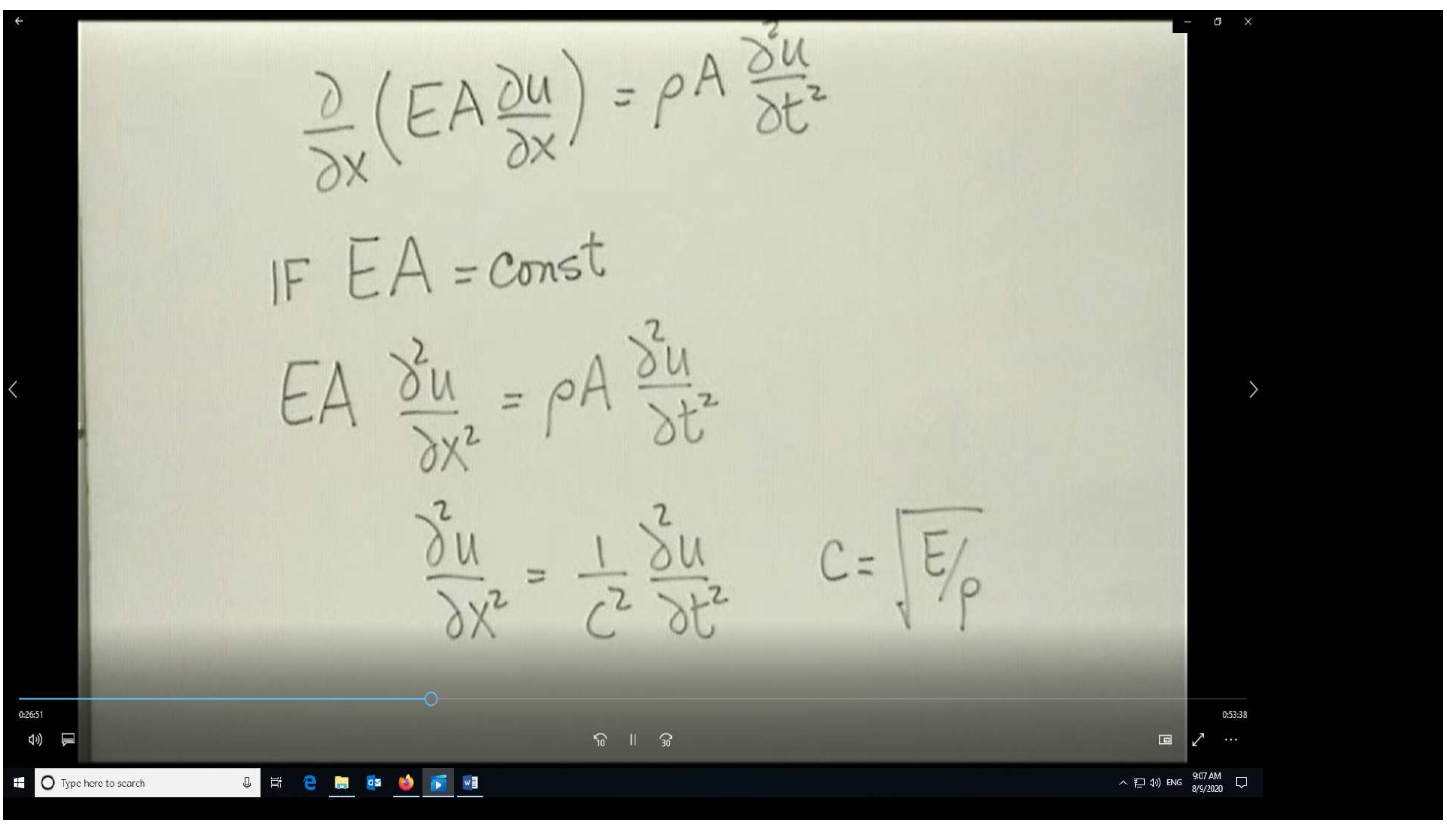


Since P depends on stress and stress depends on variation of u with respect to x, then P is a function of x

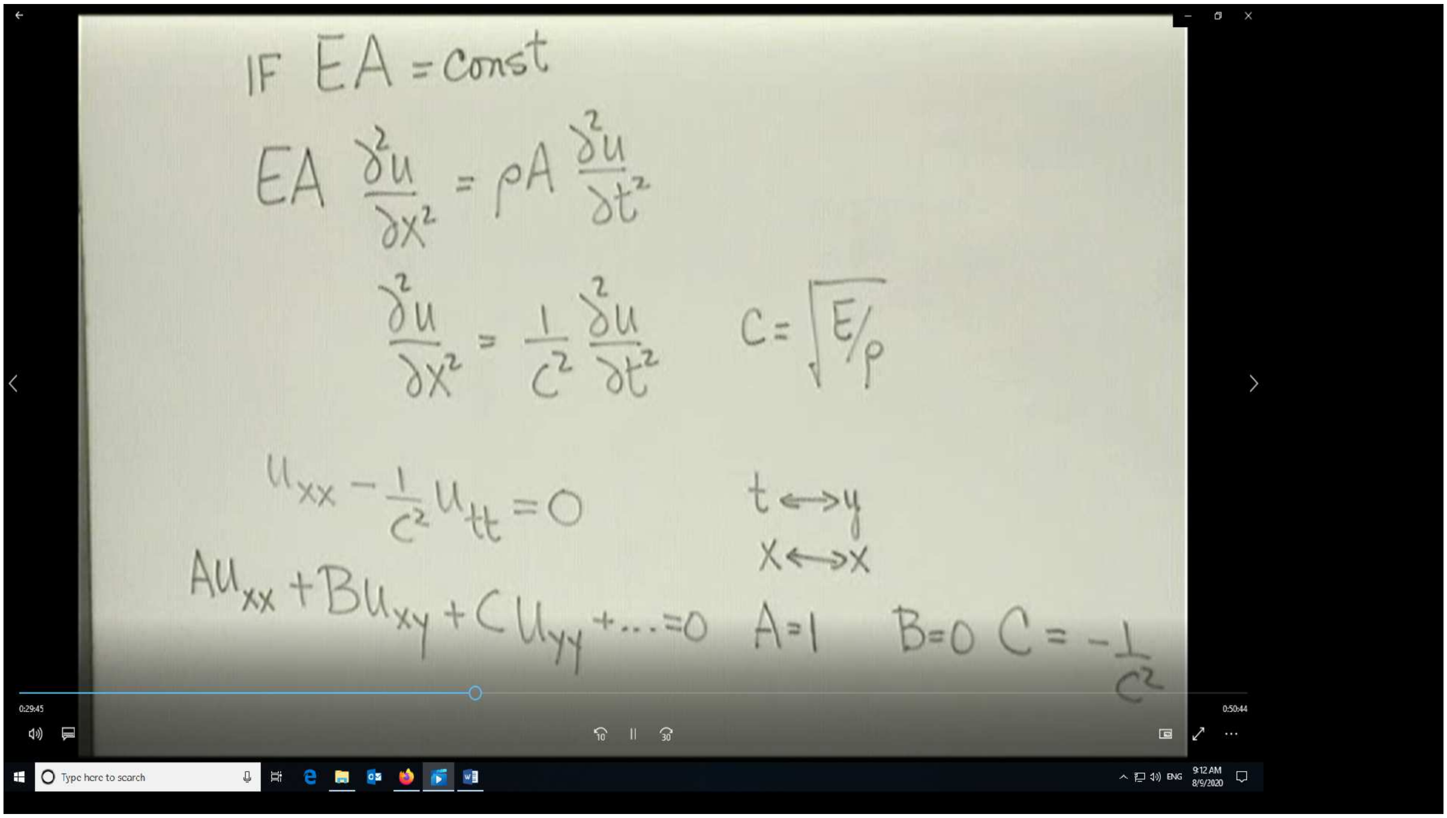




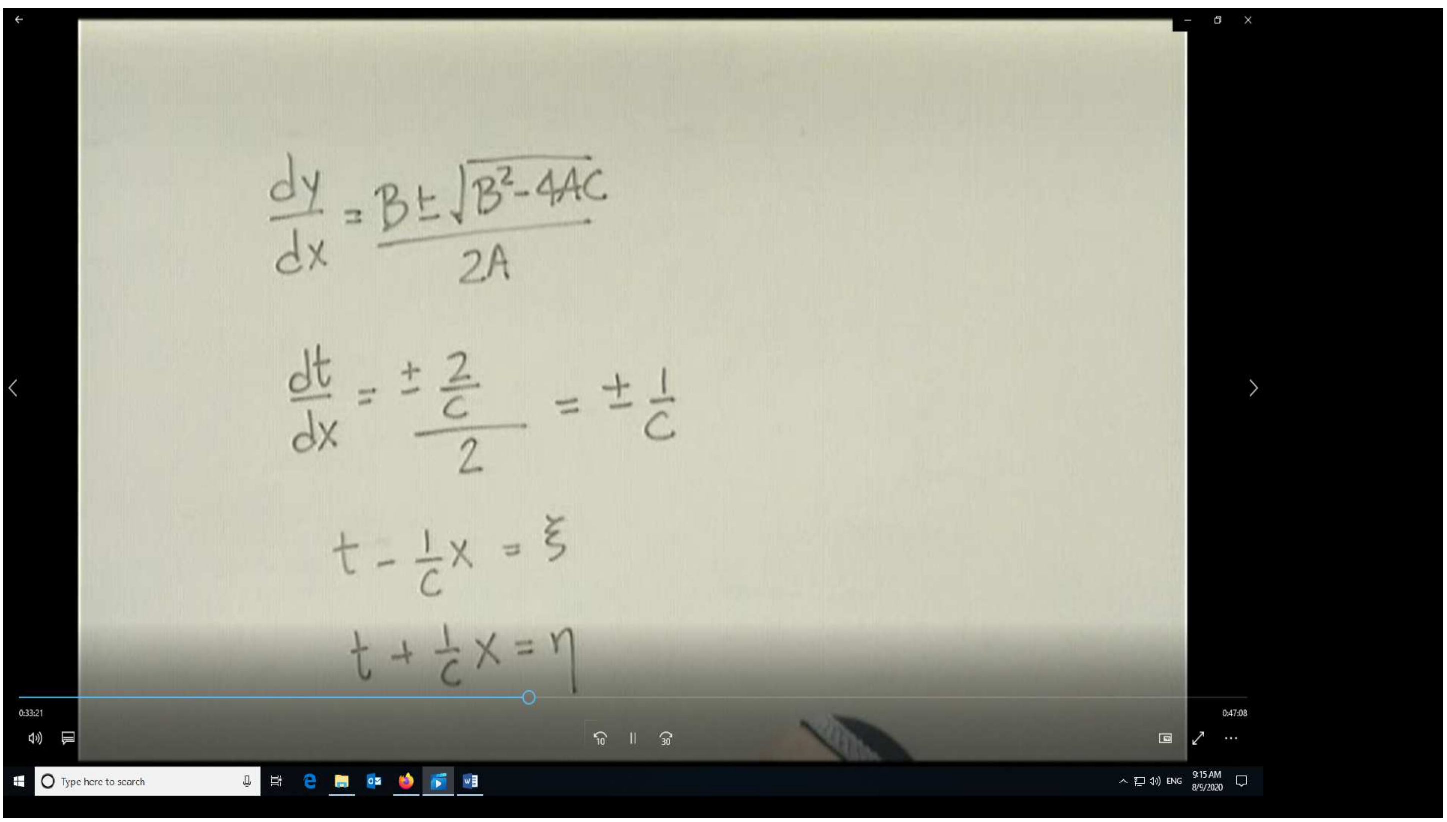
In the case where the extensional stiffness EA is a constant, each can change at every cross-section but their product is constant, then



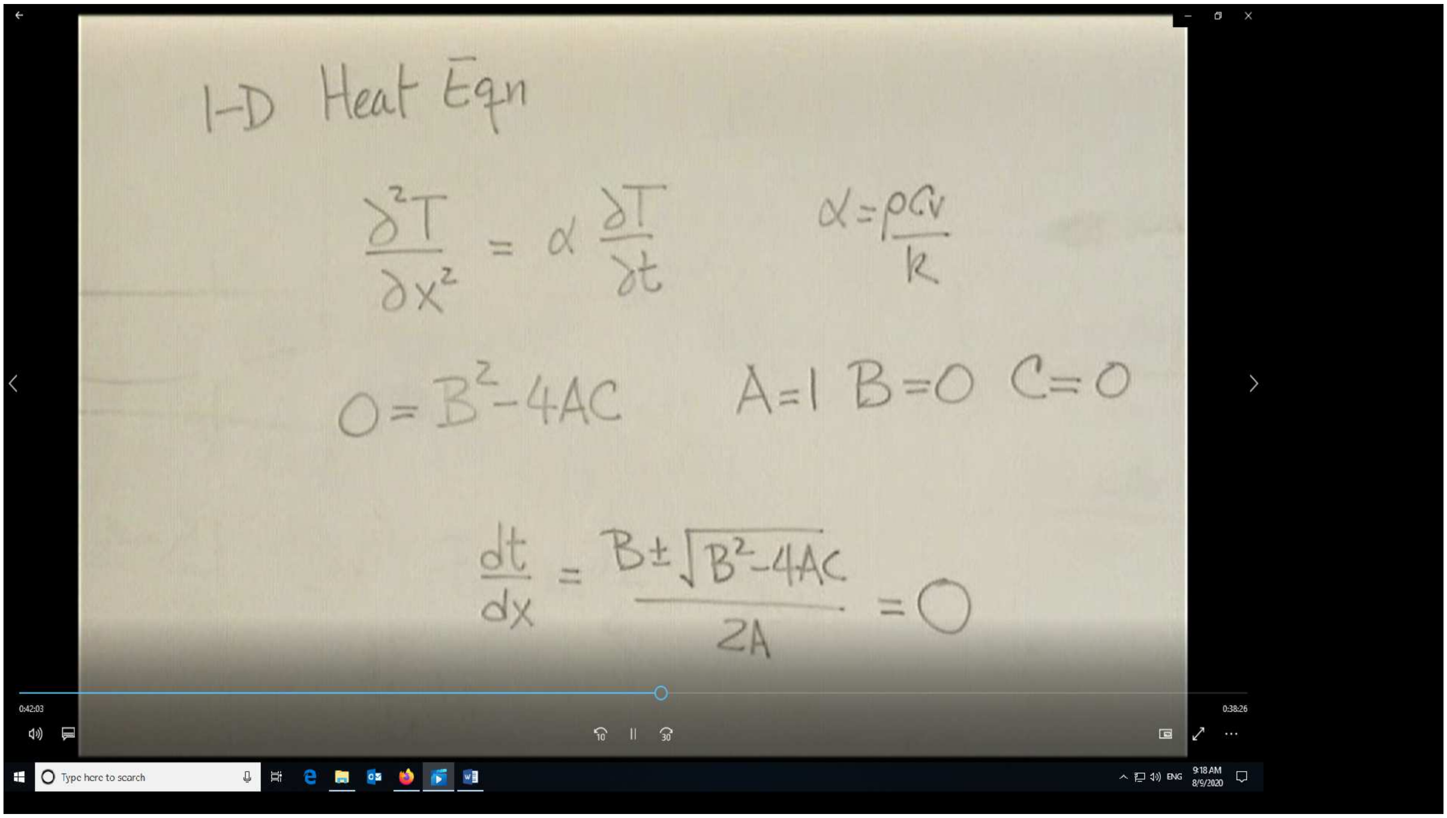
Here c is the bar speed. ρ is the density, in this case mass/unit length. E has units of load/unit area.



To characterize this equation to what we have learned previously

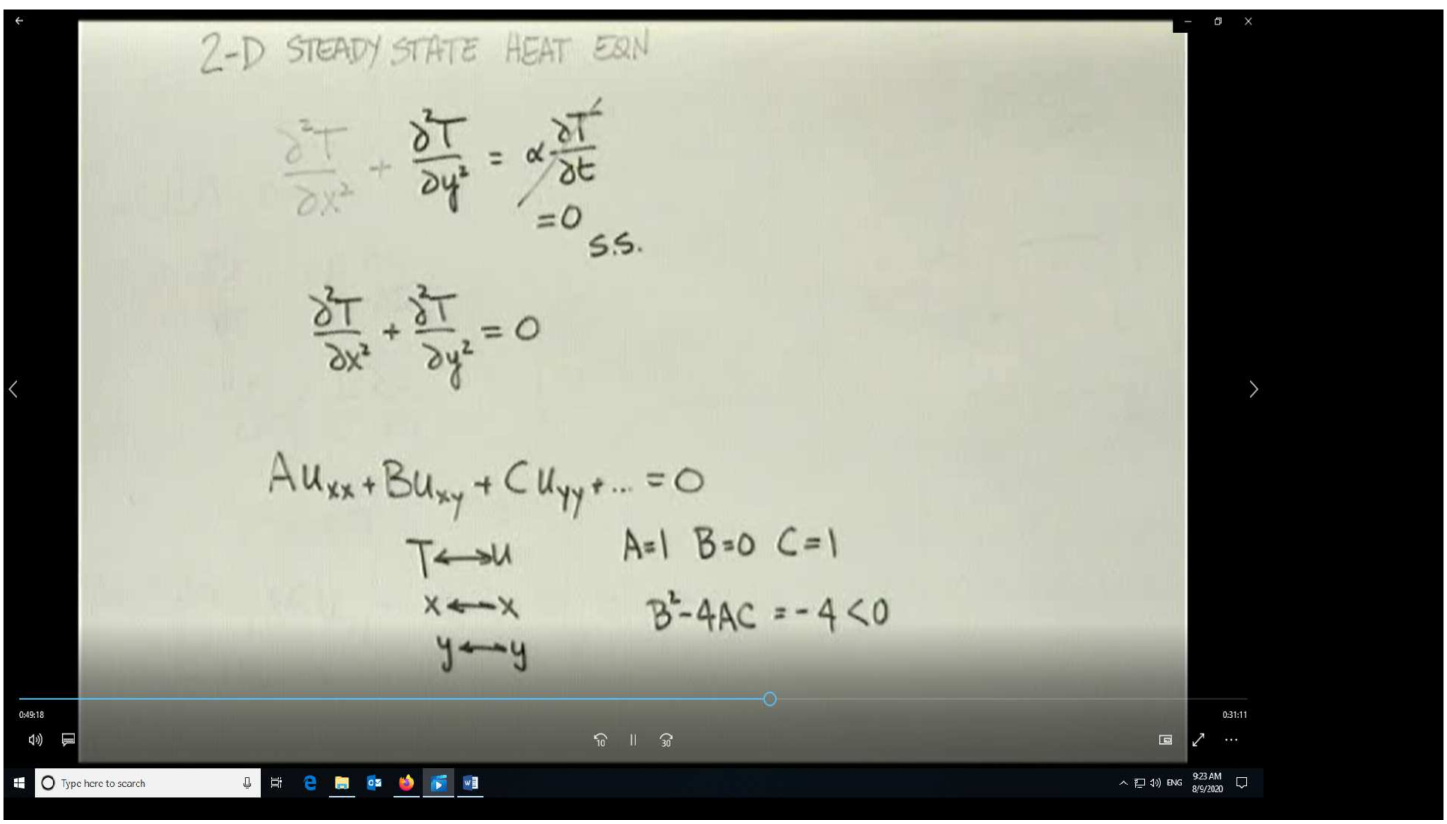


Now let us look at the 1-D heat equation and characterize it using the information we have previously learned

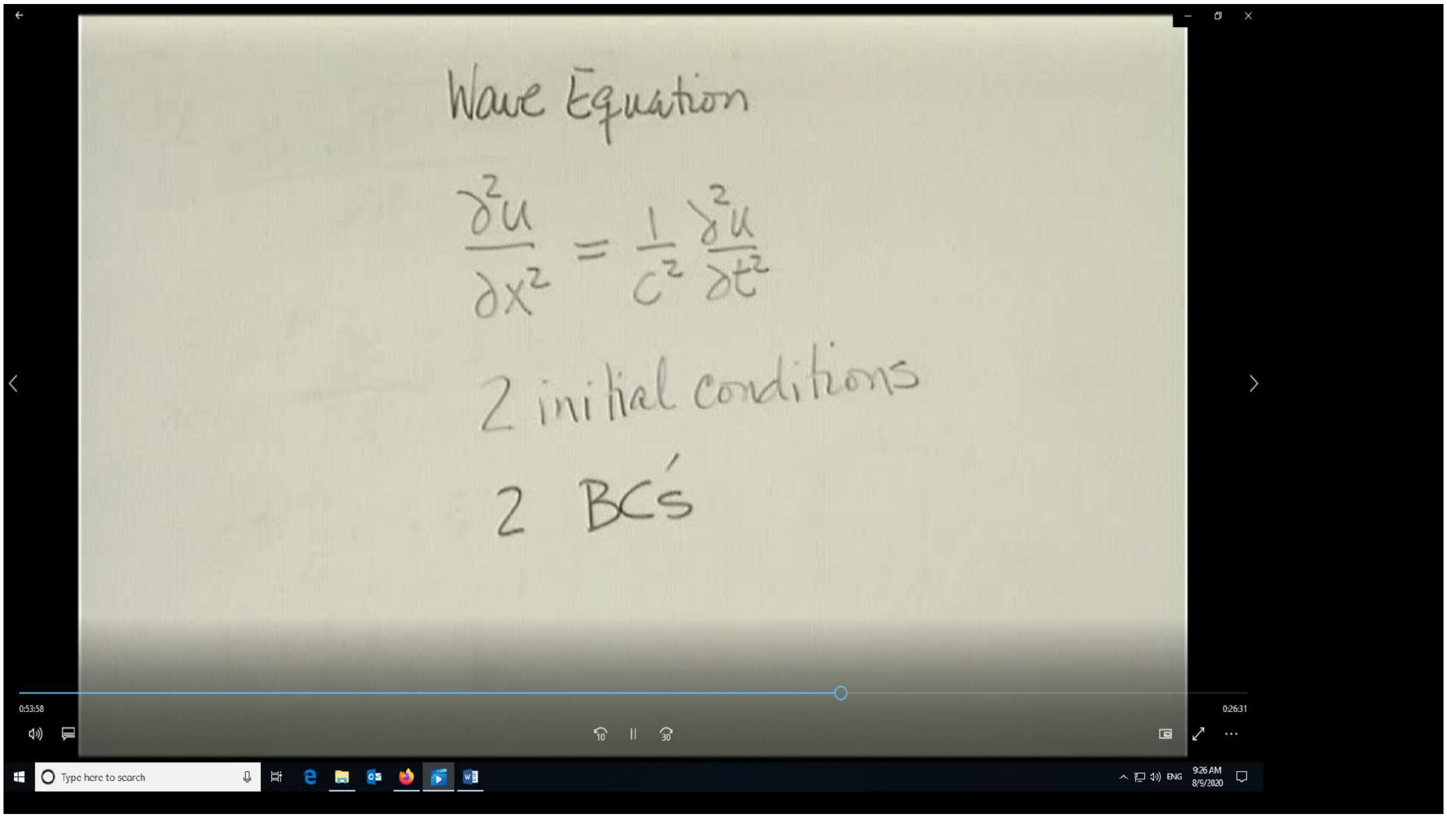


Since dt/dx=0, then t=constant, let's call it ξ and the other characteristic is any line that crosses it like η=constant=x. Hence, the 1-D Heat equation is an example of a parabolic PDE.

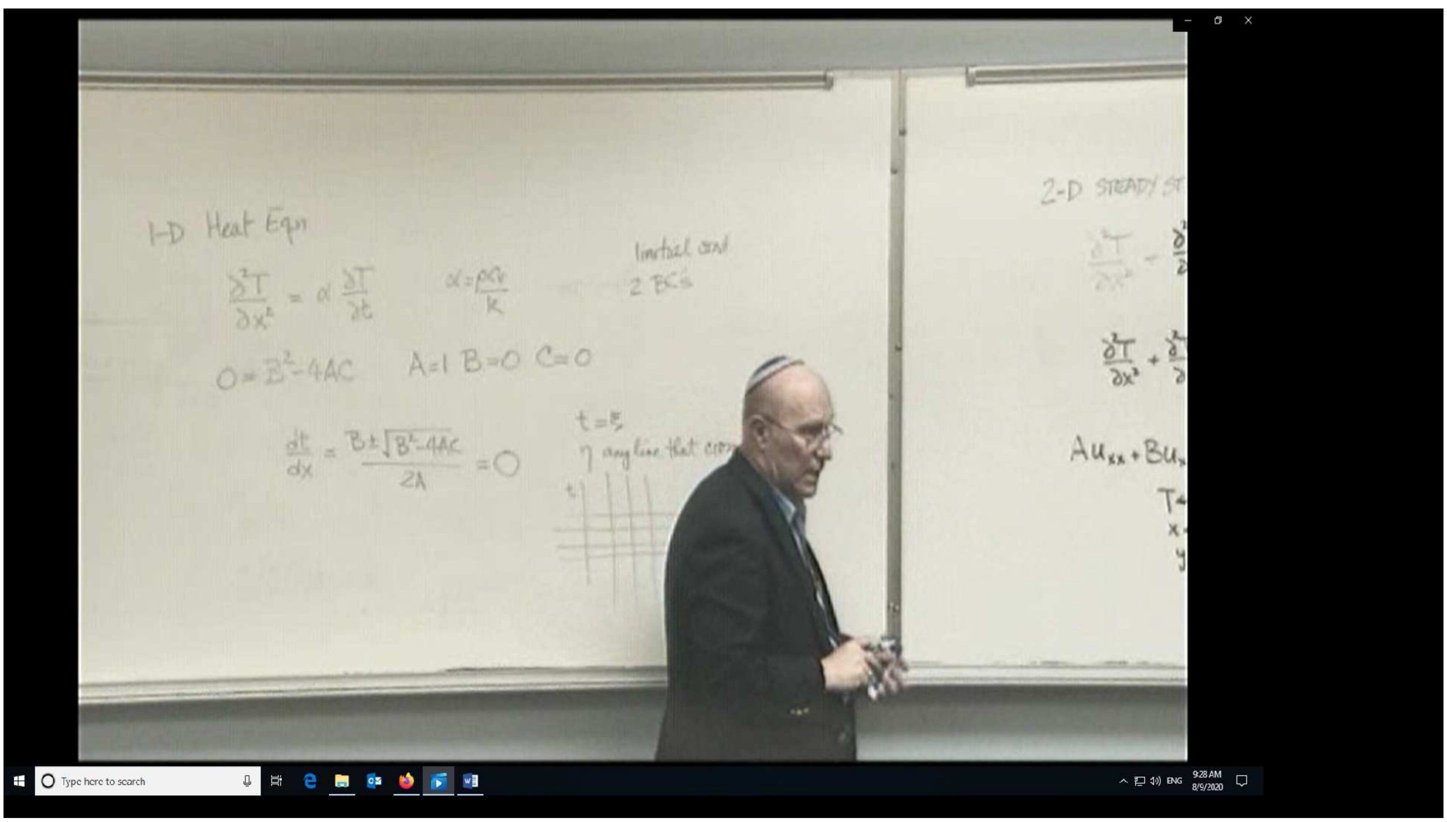
Now look at the steady state 2-D heat equation



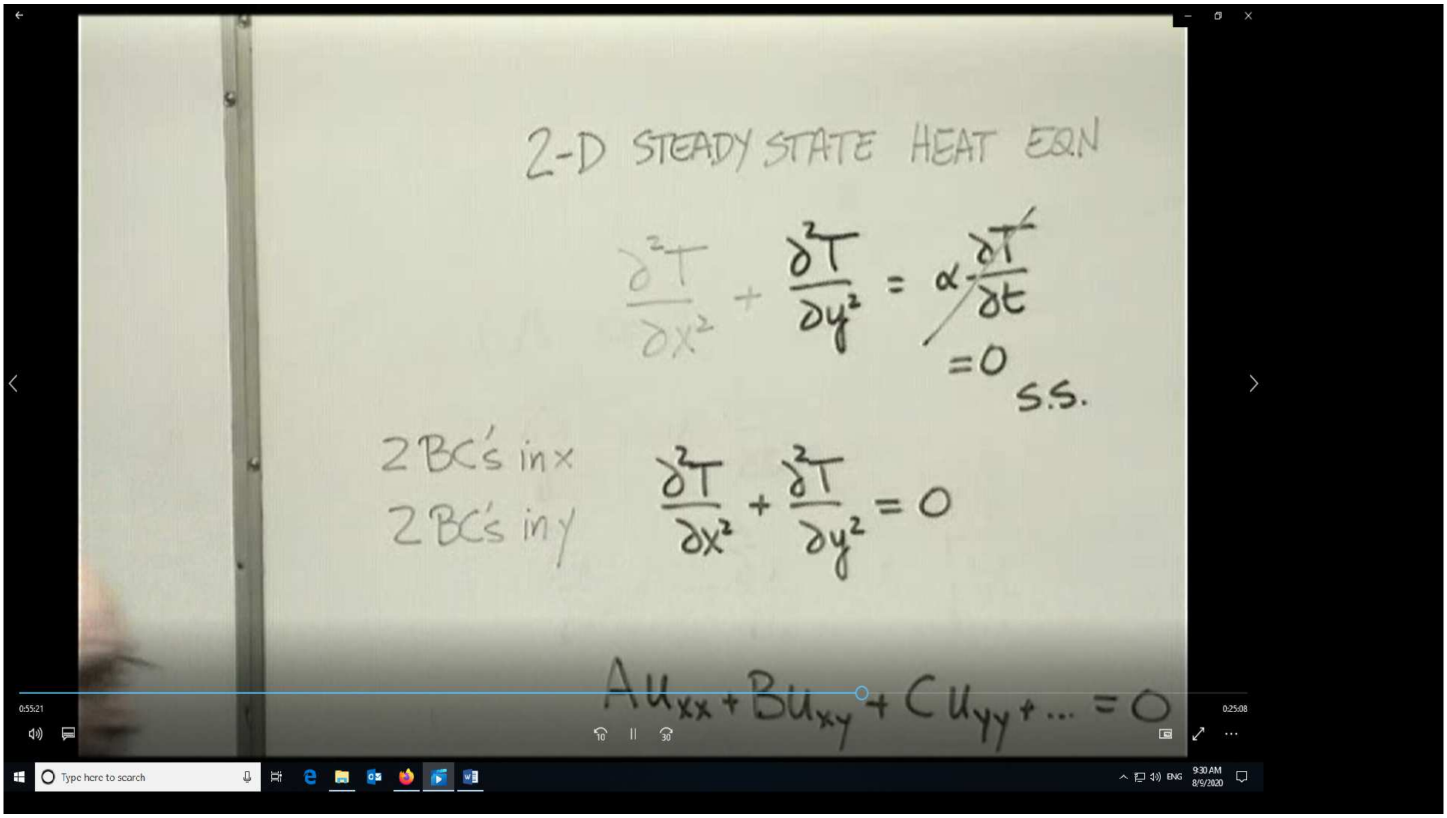
Since the discriminant is less than zero, this is an example of an elliptic PDE



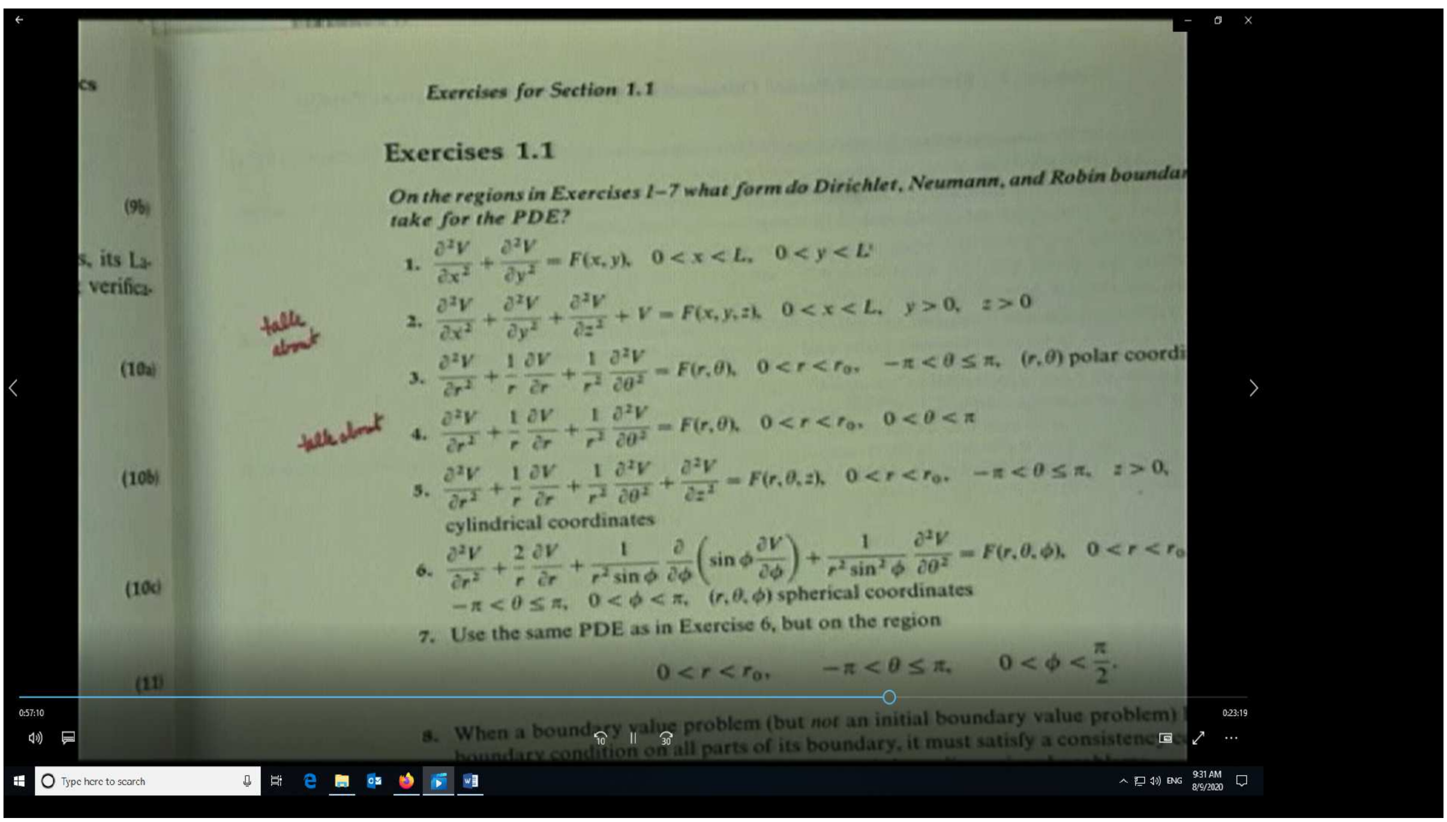
For a well posed wave-equation type problem, these are the basic requirements



For a well posed parabolic PDE, you need 1 IC and 2 BCs



For elliptic conditions you need 4 BCs, in this case 2 in x and 2 in y

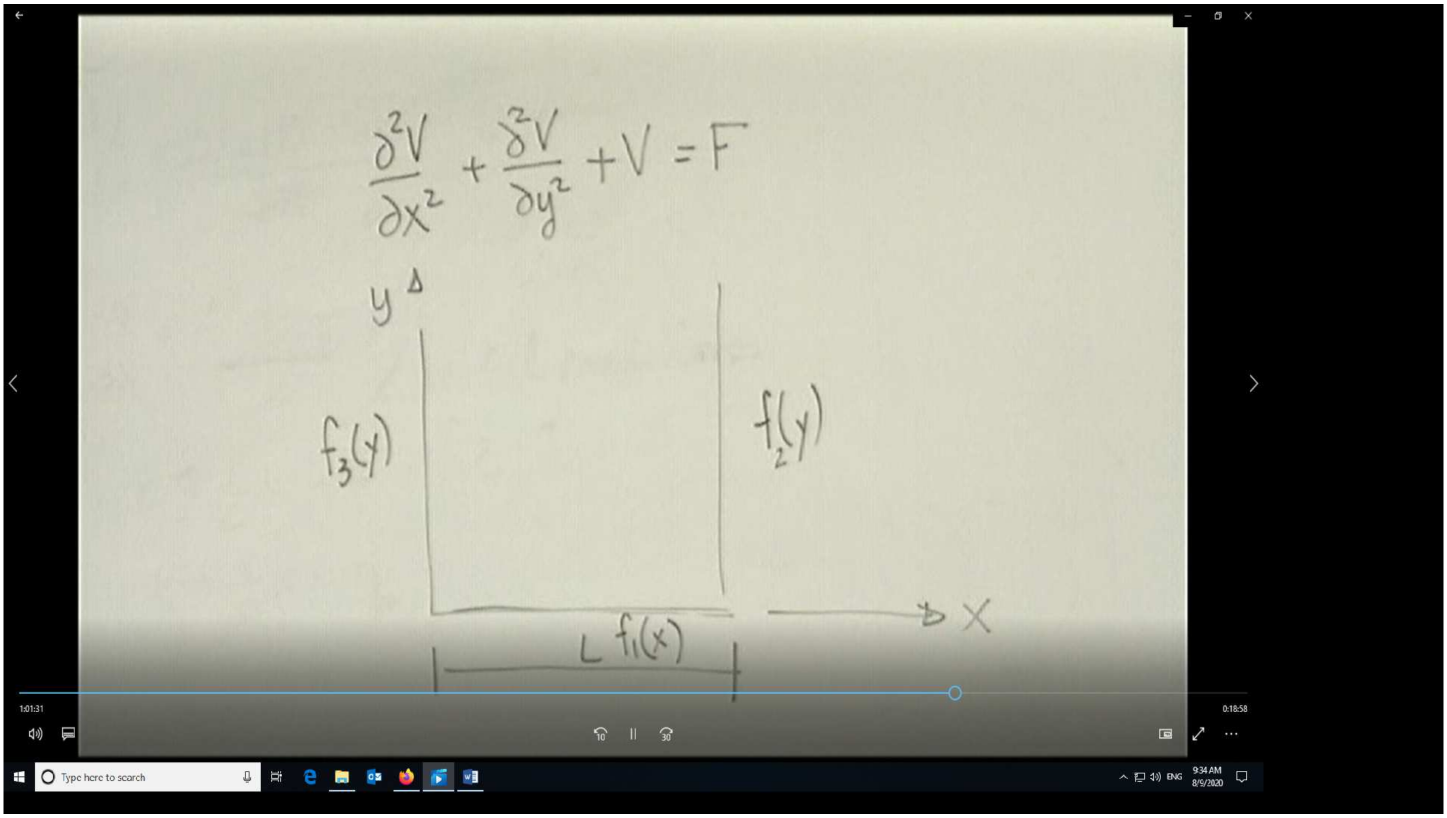

Exercises for Section 1.1

Exercises 1.1

On the regions in Exercises 1–7 what form do Dirichlet, Neumann, and Robin bounda[ry conditions] take for the PDE?

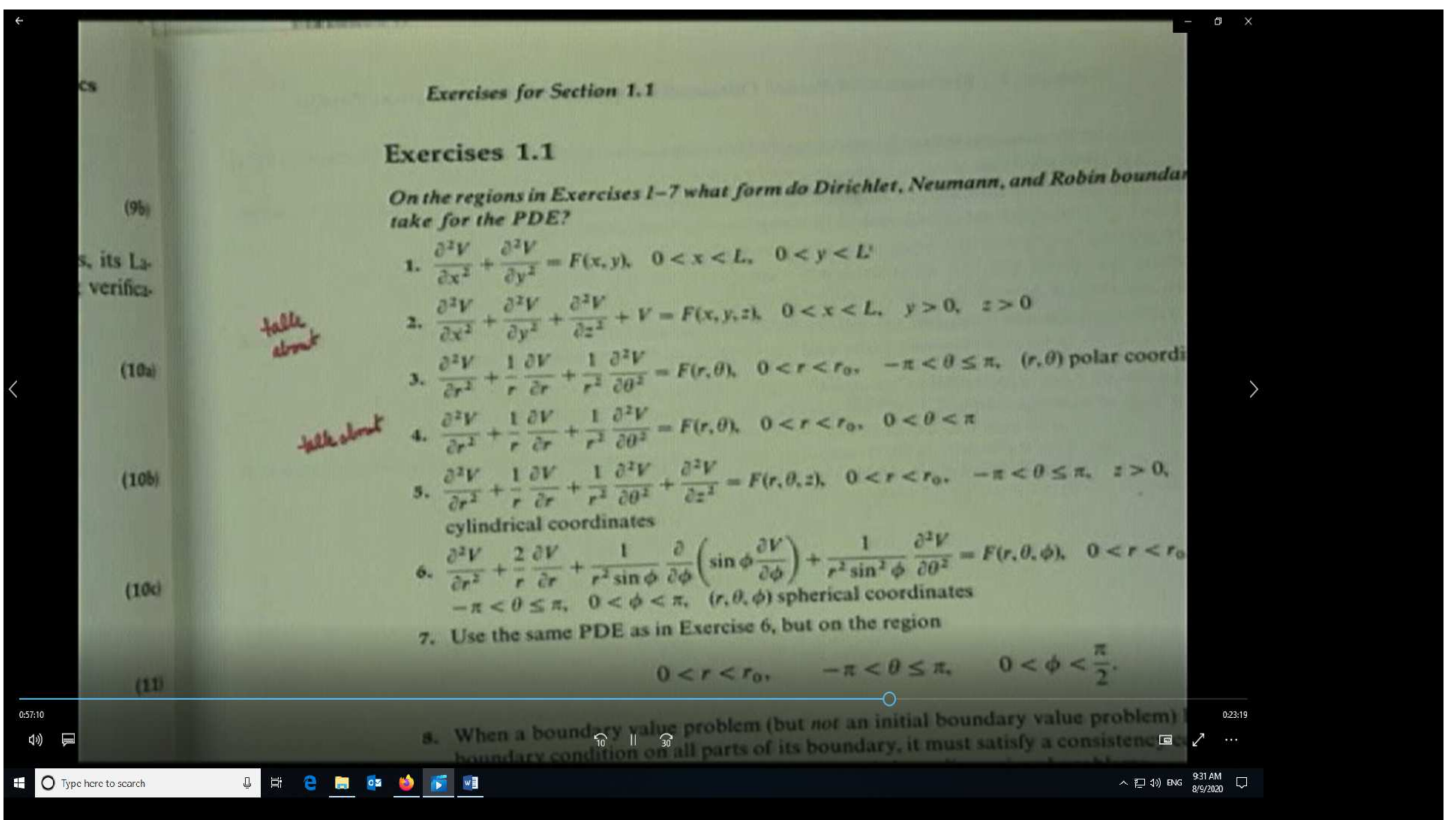
1. $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = F(x,y),\quad 0 < x < L,\quad 0 < y < L'$
2. $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} + V = F(x,y,z),\quad 0 < x < L,\quad y > 0,\quad z > 0$
3. $\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = F(r,\theta),\quad 0 < r < r_0,\quad -\pi < \theta \le \pi,$ (r,θ) polar coordi[nates]
4. $\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = F(r,\theta),\quad 0 < r < r_0,\quad 0 < \theta < \pi$
5. $\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} = F(r,\theta,z),\quad 0 < r < r_0,\quad -\pi < \theta \le \pi,\quad z > 0,$ cylindrical coordinates
6. $\frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial V}{\partial \phi}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 V}{\partial \theta^2} = F(r,\theta,\phi),\quad 0 < r < r_0$ $-\pi < \theta \le \pi,\quad 0 < \phi < \pi,$ (r,θ,ϕ) spherical coordinates
7. Use the same PDE as in Exercise 6, but on the region $0 < r < r_0,\quad -\pi < \theta \le \pi,\quad 0 < \phi < \frac{\pi}{2}.$
8. When a boundary value problem (but *not* an initial boundary value problem) [...] boundary condition on all parts of its boundary, it must satisfy a consistenc[y ...]

From Trim's book, Section 1.1 Exercise 1.1. Look at the 2-D equivalent of problem 2

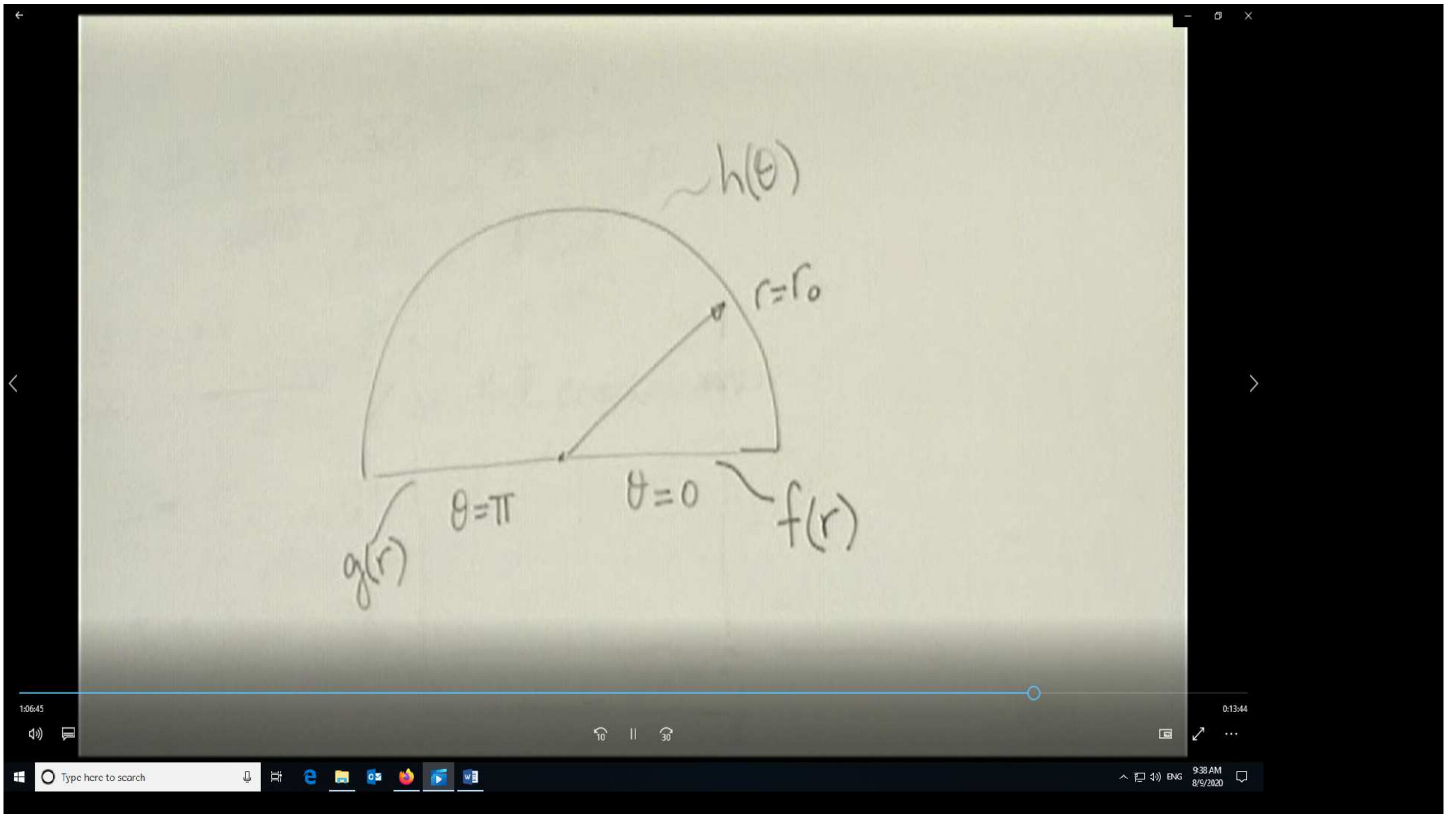


In this semi-infinite problem, at y=∞ the solution BC is that it must remain bounded. On the x=constant, the BCs can at most be functions of y, since x is fixed. On the y=constant line the BC can be at most a function of x.

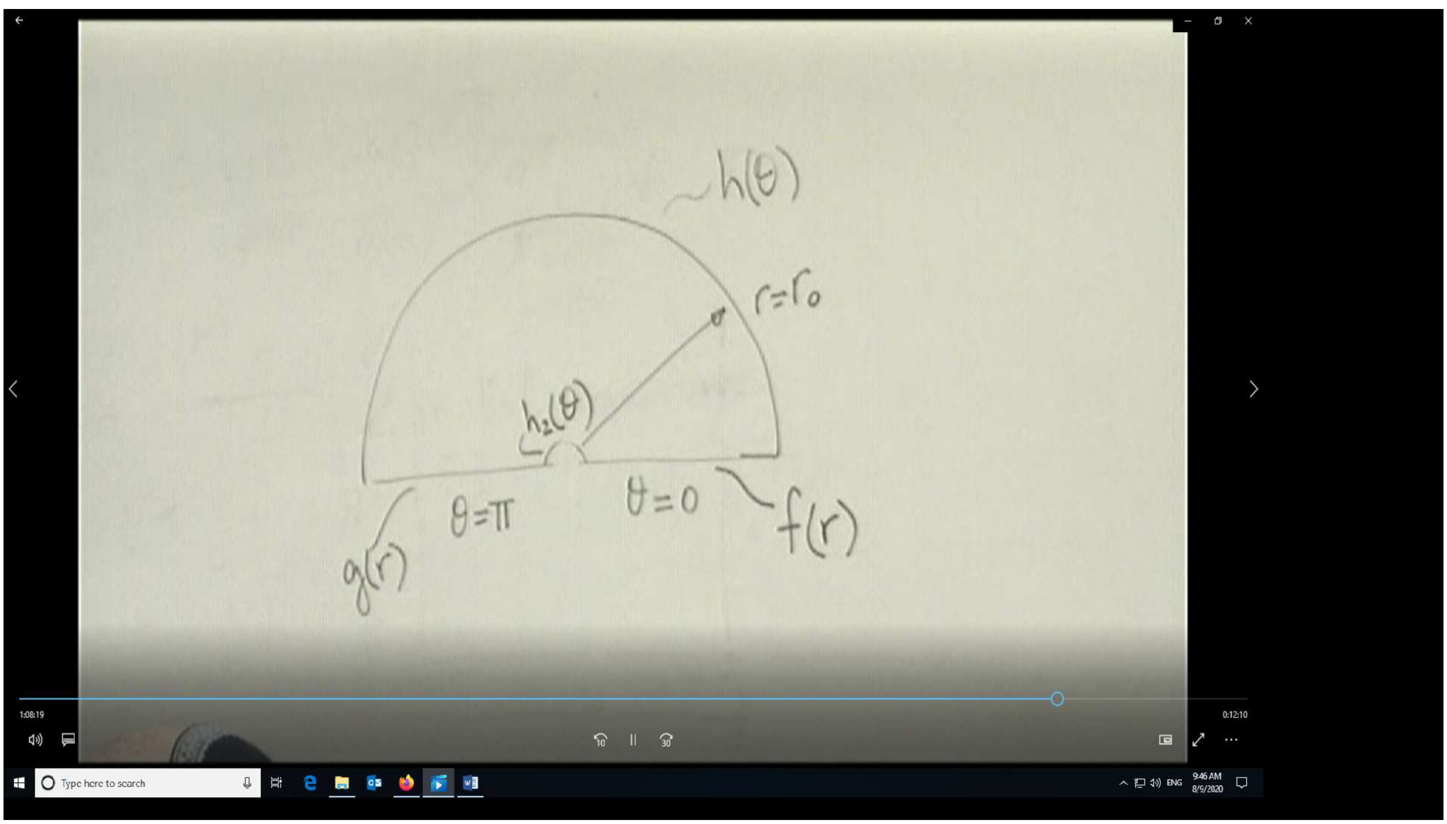
Now looking at problem 4



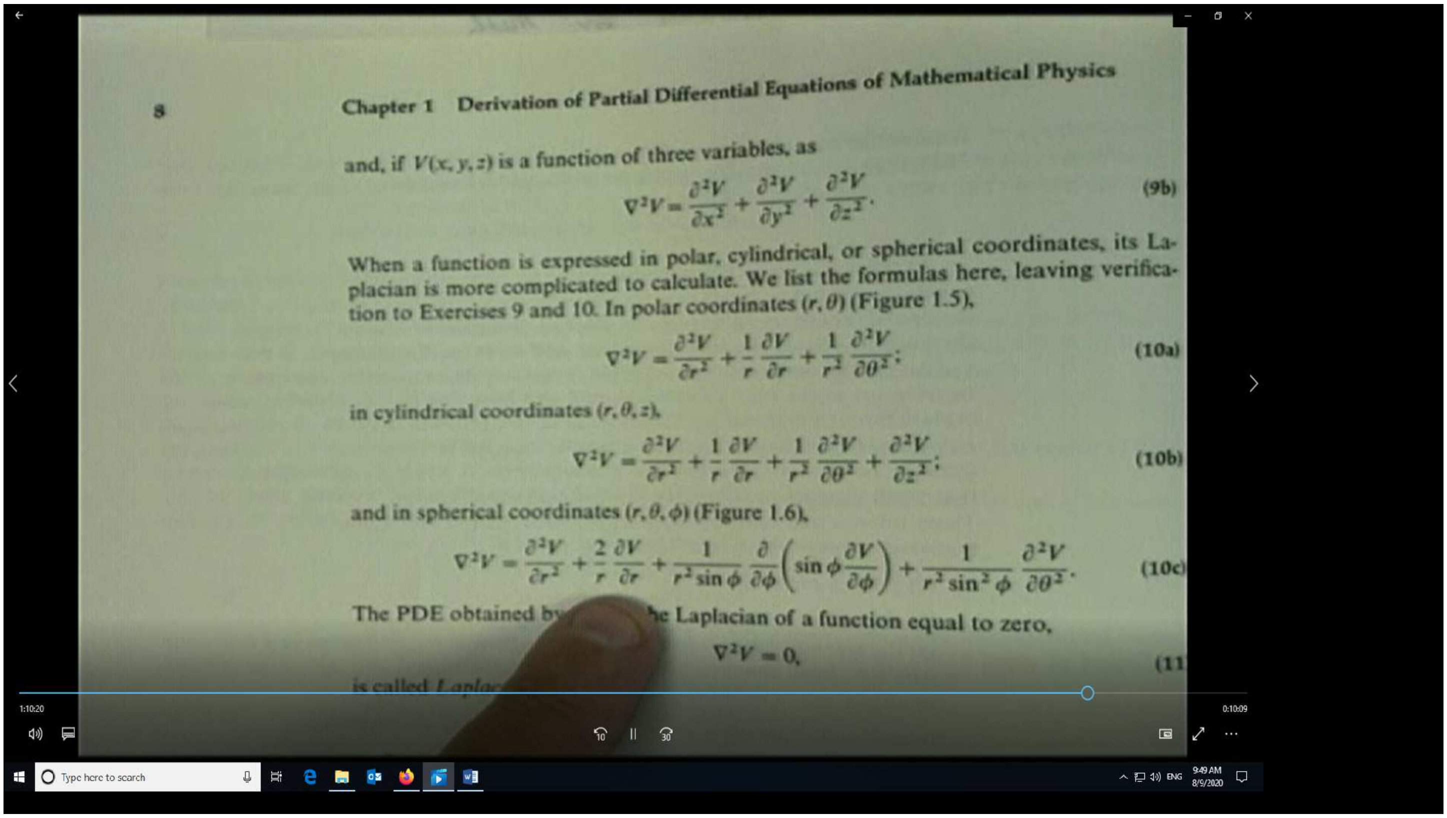
This is a semi circular region. The four boundaries are at θ=0, θ=π, r=ro and at r=0



r=0 is a limiting case of r=ε where epsilon is a small quantity which is made smaller and smaller until it reaches zero. Namely, the semi-circle is a limiting case of a thick-walled cylinder where the inner radius is made so small as to collapse to a point.



In the above, for θ=constant lines, the BCs can be at most functions of r; similarly, on constant r lines, the BCs can be at most functions of theta.



8 Chapter 1 Derivation of Partial Differential Equations of Mathematical Physics

and, if $V(x, y, z)$ is a function of three variables, as

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}. \quad (9b)$$

When a function is expressed in polar, cylindrical, or spherical coordinates, its Laplacian is more complicated to calculate. We list the formulas here, leaving verification to Exercises 9 and 10. In polar coordinates (r, θ) (Figure 1.5),

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2}; \quad (10a)$$

in cylindrical coordinates (r, θ, z),

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2}; \quad (10b)$$

and in spherical coordinates (r, θ, ϕ) (Figure 1.6),

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial V}{\partial \phi}\right) + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 V}{\partial \theta^2}. \quad (10c)$$

The PDE obtained by ... the Laplacian of a function equal to zero,

$$\nabla^2 V = 0, \quad (11)$$

is called *Laplac*...

Here we show the definition of the Laplacian in different coordinate systems, Cartesian, cylindrical and spherical.

Examples of types of BCs: first type of Dirichlet (on u), second type or Neumann (on derivative of u), third type or mixed/Robin (on combination of u and its derivative)