- WE WILL USE THE METHOD OF CHARACTERISTICS TO FIND SOLUTIONS TO PDE

  - IDEA : TO TRANSFORM EQN SO THAT ALONG CERTAIN LINES - ONLY DERIV ALONG THESE LINES EXIST & CAN BE INTEGRATED AS IF THE EQN WERE ODE. LINES ARE CHARACTERISTICS

- IN LINEAR PROBLEMS ~~SOLUTION~~ CHARACTERISTICS DEPEND ON COEFF a, b, c
- NON LINEAR CAN ALSO DEPEND ON SOLUTION, u, ITSELF

- LOOK AT SIMPLE FIRST ORDER PROBLEM

$$A\,u_x + B u_t + C = 0 \qquad u = u(x,t) \qquad A, B, C \text{ are fn of } x, t, u$$

- FIND TRANSFORMATION $\xi = \xi(x,t)$ $\quad \eta = \eta(x,t) \Rightarrow x = x(\eta,\xi)$ $t = t(\eta,\xi)$

  SO THAT EQN ~~ONLY~~ HAS DERIVATIVES OF EITHER ξ (OR η) ONLY

- e.g. $$u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi \xi_x + u_\eta \eta_x$$

- $$u_t = u_\xi \xi_t + u_\eta \eta_t$$

  PUT INTO $A u_x + B u_t + C = 0$ & COLLECT TERMS IN u_ξ & u_η

- $$\Rightarrow (A\xi_x + B\xi_t)\,u_\xi + (A\eta_x + B\eta_t)\,u_\eta + C = 0$$

- WE WANT COEFF OF u_ξ (OR u_η) $= 0$ along const ξ (or η)

- $\Rightarrow$ e.g., $A\,\xi_x + B\xi_t = 0$ along constant ξ line $\qquad$ (1)

- ALONG ANY ξ line $\quad d\xi = \xi_x\,dx + \xi_t\,dt$
- ALONG CONSTANT ξ line $\quad d\xi = 0 = \xi_x\,dx + \xi_t\,dt$ $\qquad$ (2)

THUS (1) & (2) YIELD
$$\begin{bmatrix} A & B \\ dx & dt \end{bmatrix}\begin{pmatrix} \xi_x \\ \xi_t \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- NON TRIVIAL SOLUTION IS $A\,dt - B\,dx = 0$ or $\frac{dx}{dt} = \frac{A}{B}$

- THIS GIVES SLOPE OF CHARACTERISTICS & $x = \int^t \frac{A}{B}\,dt' + \text{constant}$

$$\therefore \quad x - \int^t \frac{A}{B}\,dt' = \underset{\text{LET IT BE } \xi}{\text{constant}} \implies x - \int^t \frac{A}{B}\,dt' = \xi$$

THIS IS ONE OF CHARACTERISTI

- TO FIND η PICK ANY LINE THAT <u>INTERSECTS</u> ξ FOR EXAMPLE
  $\eta = t$

- THUS $\eta_t = 1$ $\eta_x = 0$ & $(A\eta_x + B\eta_t)\,u_\eta + C = 0 \implies Bu_\eta + C = 0$

  - WE CAN THEN INTEGRATE THIS ALONG THE CHARACTERISTIC ξ

$$u_\eta = -\frac{C}{B} \qquad u = \int^\eta -\frac{C}{B}\,d\bar{\eta} + f(\xi)$$

Exercise 7.1

- EXAMPLE $\frac{\partial T}{\partial t} + V\frac{\partial T}{\partial x} = \bar{A}\sin(\pi x/L)$

in our $Au_x + Bu_t + C = 0$
$A = V$ $B = 1$ $C = -\bar{A}\sin($
$u = T$

$$\frac{dx}{dt} = \frac{A}{B} = V \qquad \therefore \qquad x - Vt = \xi; \text{ choose } \eta = t;$$

$$x = \xi + Vt = \xi + V\eta$$

$$Bu_\eta + C = 1 \cdot T_\eta - \bar{A}\sin\left(\frac{\pi}{L}(\xi + V\eta)\right) = 0; \quad \text{INTEGRATE WRT } \eta$$

$$\therefore \quad T = \int^\eta \bar{A}\sin\frac{\pi}{L}(\xi + V\eta')\,d\eta' + f(\xi)$$

$$T = -\frac{\bar{A}L}{V\pi}\cos\frac{\pi}{L}(\xi + V\eta) + f(\xi)$$

$$\therefore \quad T(x,t) = -\frac{\bar{A}}{V}\frac{L}{\pi}\cos\frac{\pi}{L}(x - vt + vt) + f(x - vt)$$

$$= -\frac{\bar{A}L}{V\pi}\cos\frac{\pi x}{L} + f(x - vt)$$

- FOR 2^{nd} order quasilinear PDE $Au_{xx} + Bu_{xy} + Cu_{yy} + D = 0$
  A, B, C, D are fns of x, y, u, u_x, u_y

For 2^{nd} order quasilinear PDE $\quad A u_{xx} + B u_{xy} + C u_{yy} + D = 0$

- let $\quad V = u_x \qquad W = u_y$

$\Rightarrow \quad A V_x + B V_y + C W_y + D = 0 \quad (*) \qquad$ note $u_{xy} = V_y$

also $\quad V_y - W_x = 0 \quad (**) \quad \Rightarrow u_{xy} = u_{xy}$

- 2 eqns for V & W

- NEXT TRANSFORM EQNS USING $\xi = \xi(x,y) \qquad \eta = \eta(x,y)$

$\Rightarrow \quad V_x = V_\xi \xi_x + V_\eta \eta_x \qquad W_x = W_\xi \xi_x + W_\eta \eta_x$

$\qquad V_y = V_\xi \xi_y + V_\eta \eta_y \qquad W_y = W_\xi \xi_y + W_\eta \eta_y$

- PUT INTO $(*)$ & $(**)$. TAKE $C_1 \cdot$ TRANSFORMED 1 $+ C_2 \cdot$ TRANSFORMED 2 $= 0$
- Collect terms involving V_ξ, V_η, W_ξ, W_η

- $V_\xi \left[(A\xi_x + B\xi_y) C_1 + (\xi_y) C_2 \right] + W_\xi \left[(C\xi_y) C_1 + (-\xi_x) C_2 \right]$

$+ V_\eta \left[(A\eta_x + B\eta_y) C_1 + (\eta_y) C_2 \right] + W_\eta \left[(C\eta_y) C_1 + (-\eta_x) C_2 \right] + C_1 D = 0$

- AS BEFORE WANT COEFF OF DERIVS WRT EITHER ξ (OR η) TO VANISH ALONG CONSTANT ξ (OR η)

- LOOK ALONG CONSTANT η $\Rightarrow$ V_η & W_η TERMS MUST VANISH

$$(A\eta_x + B\eta_y) C_1 + (\eta_y) C_2 = 0$$
$$(C\eta_y) \; C_1 + (-\eta_x) \; C_2 = 0$$

- for non zero solutions $\Rightarrow$ determinant of coeffs of C_1 & $C_2 = 0$

$$\begin{vmatrix} A\eta_x + B\eta_y & \eta_y \\ C\eta_y & -\eta_x \end{vmatrix} = 0$$

or $\quad -A\eta_x^2 - B\eta_x\eta_y - C\eta_y^2 = 0 \qquad$ and $\quad C_2 = C \dfrac{\eta_y}{\eta_x} C_1$

- ALONG constant η lines $\quad d\eta = \eta_x dx + \eta_y dy = 0 \quad$ or $\quad \eta_x = -\eta_y y'$

$\quad \Rightarrow -\eta_y^2 \left[A (y')^2 - B y' + C \right] = 0 \qquad$ if $\eta_y \neq 0$

- $\dfrac{dy}{dx} = y' = \dfrac{B \pm \sqrt{B^2 - 4AC}}{2A} \qquad$ slope of characteristics

$\Rightarrow f_1(x,y) = \text{const} = \eta; \; f_2(x,y) = \text{const} = \eta \quad$ or $\quad \xi, \eta$

- IF $B^2 - 4AC > 0$ 2 distinct real characteristics : HYPERBOLIC PROBLEM
- IF $B^2 - 4AC = 0$ 1 real characteristic : parabolic problem
- IF $B^2 - 4AC < 0$ no real characteristics : elliptic problem.

$$\left.\begin{aligned} u_{xx} - u_{tt} &= 0 \\ u_{xx} - u_t &= 0 \\ u_{xx} + u_{tt} &= 0 \end{aligned}\right\} \text{examples}$$

- WE THEN HAVE $V_\xi [\qquad] + W_\xi [\qquad] + C, D = 0$

- AS BEFORE IF WE SUBSTITUTE FOR $\xi_x, \xi_y, \eta_x, \eta_y$ WE CAN INTEGRATE TO FIND V AS A FN OF W

- SOLUTIONS CAN BE OBTAINED FOR LINEAR PROBLEMS – CONSTANT COEFFICIENT
  - WITH THE TRANSFORMATIONS $\xi = \xi(x,y)$ $\eta = \eta(x,y)$

  we can transform the general equation

$$A u_{xx} + B u_{xy} + C u_{yy} + b_1 u_x + b_2 u_y + cu + f(x,y) = 0$$

using the characteristics

$$y - \left[\frac{B + \sqrt{B^2 - 4AC}}{2A}\right] x = \xi \qquad y - \left[\frac{B - \sqrt{B^2 - 4AC}}{2A}\right] x = \eta$$

into the following basic forms.

$$u_{\xi\xi} + u_{\eta\eta} + \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \tilde{c} u + \tilde{f} = 0 \qquad \text{elliptic type}$$

$$\left.\begin{aligned} &u_{\xi\eta} - \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \tilde{c} u + \tilde{f} = 0 \\ &u_{\xi\xi} - u_{\eta\eta} + \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \tilde{c} u + \tilde{f} = 0 \end{aligned}\right\} \qquad \text{hyperbolic type}$$

$$u_{\xi\xi} + \tilde{b}_1 u_\xi + \tilde{b}_2 u_\eta + \tilde{c} u + \tilde{f} = 0 \qquad \text{parabolic}$$