Series solutions

First Order Equations

- Given $y' = f(x,y)$ A solution exists if $f(x,y)$ is continuous &
  - single valued over the region of interest
  - $\partial f/\partial y$ exists & is continuous

- if so we can assume $y = \sum_{n=0}^{\infty} A_n x^n$ and all the A_n's can be determined in terms of A_0 & A_0 can be found if an initial value is given. if $y = y_0$ at $x = x_0$ use $y = \sum_{n=0}^{\infty} A_n (x-x_0)^n$ & $A_0 = y_0$

series $s_i = \sum_{j=0}^{i} a_j$ converges to s

$\lim_{i\to\infty} s_i = s$ if $\sum |a_i|$ converges & if $\left|\frac{a_{i+1}}{a_i}\right| < 1$

here $a_i = A_i (x-x_0)^i$

- Radius of convergence $\lim_{n\to\infty} \left|\frac{A_{n+1}}{A_n}\right| |x - x_0| < 1$
- example $y' = x - y$ $\quad y(x=0) = 1$ $\qquad f(x,y) = x - y$ $\quad \frac{\partial f}{\partial y} = -1$

let $y = \sum_{n=0}^{\infty} A_n (x-x_0)^n$ $\qquad x_0 = 0 \qquad y_0 = 1$

@ $x = x_0 = 0 \quad y = y_0 = 1 = A_0$

$$y' = \sum_{n=1}^{\infty} A_n n (x-x_0)^{n-1}$$

$$y' - x + y = A_1 + 2A_2 x + 3A_3 x^2 + \dots - x + [A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \dots] = 0$$

Collect terms in powers of x $= (A_1 + A_0) + (2A_2 - 1 + A_1)x + (A_2 + 3A_3)x^2 + \dots + (nA_n + A_{n-1})x^n + \dots = 0$

$= 0 + 0 \cdot x + 0 \cdot x^2$

Since RHS $= 0$

$$A_1 = -A_0$$

$$A_2 = \frac{1 - A_1}{2} = \frac{1}{2} + \frac{A_0}{2} = \frac{1}{2}(1 - A_1)$$

$$A_3 = -\frac{A_2}{3} = -\frac{1}{3}\cdot\frac{1}{2}(1 - A_1)$$

$$A_4 = -\frac{A_3}{4} = +\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{1}{2}(1 - A_1)$$

$$\vdots$$

$$A_n = -\frac{A_{n-1}}{n} = (-1)^n \cdot \frac{1}{n!}(1 - A_1)$$

$$y = \left[+\frac{x^2}{2} - \frac{x^3}{3!} + \dots\right](1 - A_1) + A_0 - A_0 x \qquad 1 - A_1 = 1 + A_0 = 2$$

$$= 1 - x + 2\left[\frac{x^2}{2} - \frac{x^3}{3!} + \dots\right] = -(1-x) + 2\left[1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right]$$

$$y = 2e^{-x} - 1 + x$$

radius of convergences $\left|\frac{A_{n+1}}{A_n}\right| |x - x_0| < 1 \Rightarrow \frac{1}{(n+1)!} \Big/ \frac{1}{n!} \; |x| \to 0 \Rightarrow |x| <$

HW $\quad (1-x)y' = 2x - y \qquad y = y_0$ when $x = 0$