

Solving self similar solution using non-constant boundary condition

$$\begin{aligned}
 \nu \frac{\partial^2 u}{\partial y^2} &= \frac{\partial u}{\partial t} \\
 u(y=0, t) &= at^b \\
 u(y, t) &\rightarrow 0 \text{ as } y \rightarrow \infty \\
 u(y, t=0) &= 0 \\
 \eta &= \frac{By}{t^n} \quad u = At^m f(\eta) \\
 \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = At^m f'(\eta) \left(\frac{B}{t^n} \right)
 \end{aligned}$$

Defining the self similar parameter as usual and generalizing function of u to have the form of the BC at $y=0$

$$\begin{aligned}
 u(y, t) &\rightarrow 0 \text{ as } y \rightarrow \infty \\
 u(y, t=0) &= 0 \\
 \eta &= \frac{By}{t^n} \quad u = At^m f(\eta) \\
 \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial y} = At^m f'(\eta) \left(\frac{B}{t^n} \right) \\
 \frac{\partial^2 u}{\partial y^2} &= \frac{\partial}{\partial y} \left(\frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial y} \right) \cdot \frac{\partial \eta}{\partial y} = At^m f''(\eta) \left[\frac{B}{t^n} \right]^2
 \end{aligned}$$

Want to minimize the number of variables, meaning convert the y term to $-n\eta/t$. Now put what you have into original PDE. Here we replace terms involving y and t , if possible, with terms involving η only:

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \eta} \cdot \frac{\partial \eta}{\partial t} \\ &= A m t^{m-1} f(\eta) + A t^m f'(\eta) \cdot \frac{-n\eta}{t} \\ \underline{\underline{v}} \underline{\underline{A t^m}} f'' \left[\frac{B^2}{t^{2n}} \right] &= \underline{\underline{A m t^{m-1}}} f + \underline{\underline{A t^{m-1}}} f' (-n\eta) \\ v f'' \cdot B^2 t^{-2n} &= t^{-1} [m f - n \eta f'] \\ -2n &= -1 \Rightarrow n = \frac{1}{2}\end{aligned}$$

Plugging what we have back into the PDE and ensuring the ODE for f does not have t or y in the equation

$$\begin{aligned}&= A m t^{m-1} f(\eta) + A t^m f'(\eta) \cdot \frac{-n\eta}{t} \\ \rightarrow & \underline{\underline{v}} \underline{\underline{A t^m}} f'' \left[\frac{B^2}{t^{2n}} \right] = \underline{\underline{A m t^{m-1}}} f + \underline{\underline{A t^{m-1}}} f' (-n\eta) \\ u &= A t^m f(\eta) \\ A t^m f'(\eta) \left(\frac{B}{t^n} \right) & \\ = \frac{\partial}{\partial \eta} \left(\frac{\partial u}{\partial y} \right) \cdot \frac{\partial \eta}{\partial y} &= A t^m f''(\eta) \left[\frac{B}{t^n} \right]^2 \\ v f'' \cdot B^2 t^{-2n} &= t^{-1} [m f - n \eta f'] \\ -2n &= -1 \Rightarrow n = \frac{1}{2} \\ v B^2 f'' &= m f - \frac{1}{2} \eta f' \\ u(y=0, t) &= a t^b = A t^m f(0)\end{aligned}$$

From the BC at $y=0$ we get the screen below. Now choose $f(0)$ in the simplest form so not to complicate matters:

Handwritten notes on a whiteboard:

$$a = Af(0)$$

$$b = m$$

$$\text{let } f(0)=1 \Rightarrow A=a$$

$$(-\eta\eta) \quad 2\nu B^2 f'' + \eta f' - 2bf = 0$$

$$\text{choose } 2\nu B^2 = 1 \quad B = \frac{1}{\sqrt{2\nu}}$$

$$f'' + \eta f' - 2bf = 0$$

$$\text{let's assume } b = \frac{1}{2}$$

$$f'' + \eta f' - f = 0$$

Here we have chosen the value of B to simplify the differential equation. In this particular case we look at $b=1/2$. We can see that $f(\eta)=\eta$ is a solution of the ODE. We can build the second solution using methods learned at start of semester, namely variation of parameters where $f_2=f_1 \cdot g$, and g is an unknown function of η

Handwritten notes on a whiteboard:

$$f_1(\eta) = \eta$$

$$f_2(\eta) = f_1(\eta)g(\eta)$$

$$\frac{g''}{g'} = -\frac{(\eta f_1 + 2f_1')}{f_1}$$

$$= -\left(\eta + 2 \frac{f_1'}{f_1}\right)$$

We can get the diff eqn that g satisfies and solve for g. Next build $f_2 = f_1 * g$. Lastly build $f = c_1 * f_1 + c_2 * f_2$

$$f_1(\eta) = \eta$$

$$f_2(\eta) = f_1(\eta)g(\eta)$$

$$\frac{g''}{g'} = -\frac{(\eta f_1 + 2f_1')}{f_1}$$

$$= -\left(\eta + 2\frac{f_1'}{f_1}\right) = -\left(\eta + \frac{2}{\eta}\right)$$

$$\ln g' = -\frac{\eta^2}{2} - 2\ln \eta$$

$$g' = \frac{1}{\eta^2} e^{-\frac{\eta^2}{2}}$$

$$g(\eta) = \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\frac{\sigma^2}{2}} d\sigma$$

$$f_2(\eta) = \eta \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\frac{\sigma^2}{2}} d\sigma$$

$$f(\eta) = C_1 \eta + C_2 \eta \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\frac{\sigma^2}{2}} d\sigma$$

Now applying information we know,

$$f_1(\eta) = \eta$$

$$f_2(\eta) = f_1(\eta)g(\eta)$$

$$\frac{g''}{g'} = -\frac{(\eta f_1 + 2f_1')}{f_1}$$

$$= -\left(\eta + 2\frac{f_1'}{f_1}\right) = -\left(\eta + \frac{2}{\eta}\right)$$

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$$f_2(\eta) = \eta \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\frac{\sigma^2}{2}} d\sigma$$

$$f(\eta) = C_1 \eta + C_2 \eta \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\frac{\sigma^2}{2}} d\sigma$$

$$f(0) = 1$$

$$\infty > \sigma > \eta$$

$$\infty > \sigma^2 > \eta^2 > \eta$$

$$0 < \frac{1}{\sigma^2} < \frac{1}{\eta^2} < \frac{1}{\eta}$$

From our IC and the far field BC implies $f \rightarrow 0$. The first term $C_1 \eta$ will blow up; so for bounded solutions $C_1 = 0$

Handwritten mathematical derivation on a video screen:

$$0 < \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\frac{\sigma^2}{2}} d\sigma < \int_{\infty}^{\eta} \frac{1}{\eta} e^{-\frac{\sigma^2}{2}} d\sigma = \frac{1}{\eta} \int_{\infty}^{\eta} e^{-\frac{\sigma^2}{2}} d\sigma$$

$f(\eta) =$

$$u(0,t) = at^b = At^m f(\eta=0) \quad \begin{matrix} A=a \\ m=b \\ f(0)=1 \end{matrix}$$

$$u(y,0) = 0$$

$$u(y,t) \rightarrow 0 \text{ as } y \rightarrow \infty \quad At^m f(\eta) \rightarrow 0 \text{ as } y \rightarrow \infty$$

$$\Rightarrow f(\eta \rightarrow \infty) \rightarrow 0$$

$$\Rightarrow C_1 = 0$$

Video player interface at the bottom shows a timestamp of 1:11:42 and a date of 8/14/2020.

So the solution leads to

Handwritten mathematical derivation on a video screen:

$$f(\eta) = C_2 \eta \int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\frac{\sigma^2}{2}} d\sigma$$

let $\frac{1}{\sigma^2} d\sigma = dv \quad e^{-\frac{\sigma^2}{2}} = u$

$$-\frac{1}{\sigma} = v \quad -\sigma dv = du$$

$$\int_{\infty}^{\eta} \frac{1}{\sigma^2} e^{-\frac{\sigma^2}{2}} d\sigma = -\frac{1}{\sigma} e^{-\frac{\sigma^2}{2}} \Big|_{\infty}^{\eta} - \int_{\infty}^{\eta} \left(-\frac{1}{\sigma}\right)(-\sigma) e^{-\frac{\sigma^2}{2}} d\sigma$$

$$= -\frac{1}{\eta} e^{-\frac{\eta^2}{2}} - \int_{\infty}^{\eta} e^{-\frac{\sigma^2}{2}} d\sigma$$

Video player interface at the bottom shows a timestamp of 1:19 PM and a date of 8/14/2020.

$$f(\eta) = -C_2 e^{-\eta^2/2} - C_2 \eta \int_{\infty}^{\eta} e^{-\sigma^2/2} d\sigma$$

$$1 = f(0) = -C_2 \cdot 1 + 0$$

$$\boxed{C_2 = -1}$$

$$f(\eta) = e^{-\eta^2/2} + \eta \int_{\infty}^{\eta} e^{-\sigma^2/2} d\sigma$$

$$\eta = \frac{By}{t^n} = \frac{y}{\sqrt{2\nu t}}$$

$$f_1(\eta) = \eta$$

$$f_2(\eta) = f_1(\eta) g(\eta)$$

$$\frac{g''}{g'} = -\left(\frac{\eta f_1 + 2f_1'}{f_1} \right)$$

$$= -\left(\eta + 2 \frac{f_1'}{f_1} \right)$$

$$\ln g' = -\eta^2/2 - 2 \ln \eta$$

$$g' = \frac{1}{\eta^2} e^{-\eta^2/2}$$

Since we know η in terms of B and n , we define η . Using the definition of the $\text{erfc}(\cdot)$ we finally get

$$\boxed{C_2 = -1}$$

$$f(\eta) = e^{-\eta^2/2} + \eta \int_{\infty}^{\eta} e^{-\sigma^2/2} d\sigma$$

$$\eta = \frac{By}{t^n} = \frac{y}{\sqrt{2\nu t}}$$

$$f(\eta) = e^{-\eta^2/2} + \eta \sqrt{\frac{\pi}{2}} \text{erfc}\left(\frac{\eta}{\sqrt{2}}\right)$$

$$\frac{g}{g'} =$$

$$\ln g' =$$

$$g' =$$

$\int_0^{\infty} e^{-\sigma/2} d\sigma$
 $f'' + \eta f' - 2bf = 0$
 $\int e^{-\sigma^2/2} d\sigma$

We started with this equation and then looked at the specific case where $b=1/2$. If b is not $1/2$, then you would have to find solutions, possibly using series solutions in the form $f(\eta) = \sum A_i \eta^i$ in order to find the two solution functions $f(\eta)$, that satisfy the condition that $f(0)=1$ and $f(\eta \rightarrow \infty) \rightarrow \infty$ and those solutions must reduce to what we found when $b=1/2$.