MOTION OF A VISCOUS FLUID OVER AN ∞ PLATE

$$\nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$

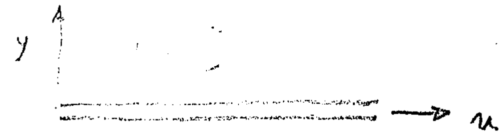

PLATE MOVES

ν kinematic viscosity

let $u = u(y,t)$ $\qquad u(y=0,t) = a t^b \qquad a, b$ fixed (1)

BC $\quad u(y,t) \to 0 \quad$ as $\quad y \to \infty$ (2)

IC $\quad u(y,0) = 0$ (3)

let $\quad \eta = B \frac{y}{t^n} \qquad u = A f(\eta) \qquad A, B, n$ constant

- for behavior of $u \to 0$ for y large & for t small $\Rightarrow \eta \to \infty$ for y large, t small

  note that (2) & (3) are collapsed into 1 condition

- Check $u(y=0,t) = A f(0) = a t^b \qquad y=0 \Rightarrow \eta = 0$

  impossible

Must add additional degree of freedom : let $u = A t^m f(\eta)$

pick t^m since $u(y=0,t)$ involves t^b

$\therefore \quad u(y=0,t) = A t^m f(0) = a t^b$ must pick $m = b$

$A f(0) = a$ $\qquad$ may pick $A = a \Rightarrow f(0) = 1$

Guidance: try to get fns $f(0)$, $f(\infty)$ etc to be either $1, 0, \infty$ but not 2.735, 15.2 etc.

$\therefore \quad u = a t^b f(\eta) \qquad \eta = By/t^n$

$$\frac{\partial u}{\partial y} = a t^b f'(\eta) \frac{d\eta}{dy} = a t^b f' \cdot \frac{B}{t^n}$$

$$\partial^2 u/\partial y^2 = a t^b f'' \, B^2/t^{2n}$$

$$\frac{\partial u}{\partial t} = a b t^{b-1} f + a t^b f' \cdot \frac{d\eta}{dt}$$

$$= a b t^{b-1} f + a t^b f' (-nBy/t^{n+1})$$

$$\frac{\partial u}{\partial t} = abt^{b-1} f + at^{b-1} - n\eta f' \; ; \quad \text{put into} \quad \nu \frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$$

$$\therefore B^2 \nu a t^{b-2n} f'' = at^{b-1} [bf + f'(-n\eta)]$$

if ODE, only f, f', f'' & η appears. So...

let $2n = 1 \quad \therefore n = 1/2$ $\qquad at^{b-1}$ cancel

$$B^2 \nu f'' + \frac{1}{2} \eta f' - bf = 0$$

$$2B^2 \nu f'' + \eta f' - 2bf = 0 \qquad \text{let } 2B^2\nu = 1 \implies B = \frac{1}{\sqrt{2\nu}}$$

$$\therefore \quad \eta = \frac{By}{t^n} = \frac{y}{\sqrt{2\nu t}}$$

$$f'' + \eta f' - 2bf = 0$$

from $u(y=0,t) = at^b = at^b f(0) \implies f(0) = 1$

irrespective of t: $u(y,t) \to 0$ as $y \to \infty \implies f(\eta \to \infty) \to 0$

in order to solve $f'' + \eta f' - 2bf = 0$ we need to know b

Suppose $b = 1/2$ $\qquad f'' + \eta f' - f = 0 \implies$ 2nd order ODE $\quad f = c_1 f_1 + c_2 f_2$

Use method of reduction in order - if you know a solution f_1

then $f_2 = f_1(\eta) g(\eta) \qquad f_2' = f_1' g + f_1 g' \quad f_2'' = f_1'' g + 2f_1' g' + f_1 g''$

PUT INTO ODE

$$f_1'' g + 2f_1' g_1' + f_1 g'' + \eta [f_1' g + f_1 g'] - f_1 g = 0$$

$$\overset{\checkmark 0}{(f_1'' + \eta f_1' - f_1)} g + f_1 g'' + (\eta f_1 + 2f_1') g' = 0$$

$$\therefore \quad \underbrace{g''/g'}_{\frac{d}{d\eta}(\ln g')} = -\left(\frac{\eta f_1 + 2f_1'}{f_1}\right) = -\left(\eta + 2\frac{f_1'}{f_1}\right) = -\frac{d}{d\eta}\left(\eta^2/2 + 2\ln f_1\right)$$

FIRST ORDER EQN in g'

NOTICE $f_1=\eta$ solves , $f_1'=1$ $f_1''=0$

$$\frac{dg'}{g'} = -\left(\eta + \frac{2}{\eta}\right)d\eta \Rightarrow \ln g' = -\eta^2/2 - 2\ln\eta$$

$$g' = \frac{1}{\eta^2}e^{-\eta^2/2} \qquad g(\eta) = \int_\infty^\eta \frac{1}{\sigma^2}e^{-\sigma^2/2}d\sigma$$

PICK ∞ as the lower limit since g' is bounded

$$f = C_1\eta + C_2\eta\int_\infty^\eta \frac{1}{\sigma^2}e^{-\sigma^2/2}d\sigma$$

: $f(\eta)\to 0$ as $\eta\to\infty$:

$$\infty > \sigma > \eta \qquad \frac{1}{\sigma^2} < \frac{1}{\eta^2} < \frac{1}{\eta}$$

$$\infty > \sigma^2 > \eta^2 > \eta$$

$$0 < \int_\infty^\eta \frac{1}{\sigma^2}e^{-\sigma^2/2}d\sigma < \int_\infty^\eta \frac{1}{\eta}e^{-\sigma^2/2}d\sigma$$ and $\frac{1}{\eta}\int_\infty^\eta e^{-\sigma^2/2}d\sigma$ is bounded

$$f_2 = \eta\, g(\eta) < \int_\infty^\eta e^{-\sigma^2/2}d\sigma \qquad \text{as } \eta\to\infty \quad \int \to 0$$

$\therefore$ $f_2 \to 0$ as $\eta\to\infty$ but $f_1=\eta\to\infty$ as $\eta\to\infty$ $\therefore C_1 = 0$

and $$f = C_2\eta\int_\infty^\eta \frac{1}{\sigma^2}e^{-\sigma^2/2}d\sigma$$

→ **INTEGRATE BY PARTS**

let $\frac{1}{\sigma^2}d\sigma = dv$ $\quad w = e^{-\sigma^2/2}$ $\quad v = -\frac{1}{\sigma}$ $\quad dw = e^{-\sigma^2/2}\cdot(-\sigma\, d\sigma)$

: $f(\eta=0) = 1$

$$C_2\eta\left[-\frac{1}{\sigma}e^{-\sigma^2/2}\Big|_\infty^\eta - \int_\infty^\eta\left(-\frac{1}{\sigma}\right)(-\sigma)e^{-\sigma^2/2}d\sigma\right]$$

$$\left[-\frac{1}{\eta}e^{-\eta^2/2} - \int_\infty^\eta e^{-\sigma^2/2}d\sigma\right]$$

$$f = -C_2e^{-\eta^2/2} - C_2\eta\int_\infty^\eta e^{-\sigma^2/2}d\sigma$$

at $\eta=0$ $\quad f = -C_2e^0 - C_2\cdot 0\cdot\int_\infty^0 e^{-\sigma^2/2}d\sigma = 1 \qquad C_2 = -1$

$$\therefore \quad f(\eta) = e^{-\eta^2/2} + \eta\int_\infty^\eta e^{-\sigma^2/2}d\sigma$$

let $z = \sigma/\sqrt{2}$ $\quad z^2 = \sigma^2/2$ $\quad dz = \frac{d\sigma}{\sqrt{2}}$

$$\eta \cdot \sqrt{2} \int_{\infty}^{\eta/\sqrt{2}} e^{-z^2} dz \qquad \int_0^{1} = \int_0^{\infty} - \int_1^{\infty}$$

$$- \frac{\sqrt{\pi}}{2} \operatorname{erfc} \frac{\eta}{\sqrt{2}} \qquad \int_1^{\infty} = \int_0^{\infty} - \int_0^{1}$$

$$f(\eta) = e^{-\eta^2/2} - \eta \sqrt{\pi/2} \operatorname{erfc}\left(\eta/\sqrt{2}\right) \qquad \operatorname{erfc} = 1 - \operatorname{erf}$$