SELF - SIMILAR SOLUTIONS

1. SOMETIMES WE WANT TO FIND SOLUTION IN DIMENSIONLESS FORM

WHY ?

2. RESULTS ARE INDEPENDENT OF SIZE OF SYSTEM

3. CHOOSE SOME LENGTH OR TIME SCALE THAT CHARACTERIZE PROBLEM IN TERMS OF INDEPENDENT VARIABLES.

4. $\frac{x}{L} = \chi$ LENGTH OF PROBLEM

WHERE DO WE GET L, time

5. LENGTH OR TIME SCALE CAN COME FROM B.C. OR GEOMETRY

6. PROBLEMS WITH NATURAL CHARACTERISTIC SCALES ARE CALLED SCALE-SIMILAR.

SCALE - SIMILAR SOLUTIONS FOR SYSTEMS OF DIFFERENT SIZES WILL HAVE SAME NONDIM. SOL. IF THEY HAVE SAME DIMENSIONLESS PARAM, BC & IC.

GIVE PDE HANDOUT

- CERTAIN PROBLEMS DO NOT HAVE NATURAL SCALE FOR INDEPENDENT VARIABLE ...

FOR EXAMPLE

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad T(x,0) = T_i \quad x>0$$

$$T(0,t) = T_s$$

and $x \to \infty \quad T \to T_i$

IF FOR INDEPENDENT VARIABLES OF THE PROBLEM $\nexists$ CLUE TO EXISTENCE

$$\left.\begin{array}{l}\cdot \text{ NO CHARACTERISTIC LENGTH} \\ \text{ " \quad " \quad TIME}\end{array}\right\} \Rightarrow \text{SELF-SIMILAR SOLUTION.}$$

- SOLUTION OF ALL PHYSICAL PROBLEMS MAY BE EXPRESSED IN DIMENSIONLESS FORM

~~FROM INDEPENDENT VARIABLES~~

- $\Rightarrow$ t, x must form a dimensionless group

FROM PDE : $x^2 = \alpha t$ or $\dfrac{x^2}{\alpha t}$, $\dfrac{\alpha t}{x^2}$, or $\dfrac{x}{\sqrt{\alpha t}}$ or $\dfrac{\sqrt{\alpha t}}{x}$

- IF SOLN MADE DIMENSIONLESS BY COMBO OF INDEPENDENT VARIABLES INSTEAD OF GEOMETRY, BC OR IC. – PROBLEM IS SELF SIMILAR

- THERE IS A CHARACTERISTIC TEMP TO THE PROBLEM $T_s - T_i$

could guess

$$\frac{T - T_i}{T_s - T_i} = f\left(\frac{x^2}{\alpha t}\right) = g\left(\frac{x}{\sqrt{\alpha t}}\right) = \frac{x}{\sqrt{\alpha t}}\, h\left(\frac{x}{\sqrt{\alpha t}}\right)$$

- Could choose $\dfrac{T}{T_s - T_i} = f\left(\dfrac{x^2}{\alpha t}\right) = g\left(\dfrac{x}{\sqrt{\alpha t}}\right) = \cdots = h\left(\dfrac{\alpha t}{x^2}\right)$

- define $\eta = \dfrac{x}{\sqrt{\alpha t}}$ SIMILARITY VARIABLE

$\Rightarrow$ • ALL TEMPERATURE PROFILES FALL ONTO ONE GRAPH

- CHOOSING a solution involving η
  - REDUCES PDE TO AN ODE WHICH IS A FN OF η ONLY

- NOTE x, t (2 indep var.) now becomes η (1 indep var)

  $\Rightarrow$ SELF SIMILARITY REDUCE # OF INDEPENDENT VAR. BY 1

METHOD OF APPROACH

- $$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad w/ \quad T(0,t) = T_s$$
  $$T(x,0) = T_i$$
  $$T(x,t) \to T_i \quad \text{as } x \to \infty$$

<u>IN GENERAL</u>

Choose $\eta = \frac{Ax}{t^n}$ $\qquad A, n$ picked to reduce eqn. to ODE

let $\frac{T - T_i}{T_s - T_i} = f(\eta)$

- NOTE: SINCE $T_s - T_i$ is a basic aspect of problem

  FORM OF η : SINCE $t = 0$ & $x = \infty$ give T_i

CHOOSE FORM OF η : NUMERATOR – INDEP VAR DIFFERENTIATED MOST OFTEN

DENOMINATOR – LEAST OFTEN DIFFERENTIATED

$$T = T_i + (T_s - T_i) f(\eta)$$

- $$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial x} \; ; \qquad \frac{\partial \eta}{\partial x} = \frac{A}{t^n} \; ; \qquad \frac{\partial T}{\partial \eta} = (T_s - T_i)\frac{df}{d\eta}$$

$$\therefore \frac{\partial T}{\partial x} = (T_s - T_i) f' \cdot \left(\frac{A}{t^n}\right)$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial \eta}\left[(T_s - T_i) f'(\eta)\left(\frac{A}{t^n}\right)\right] \cdot \frac{\partial \eta}{\partial x} = (T_s - T_i)\left(\frac{A^2}{t^{2n}}\right) f''(\eta)$$

$$\frac{\partial T}{\partial t} = \frac{\partial T}{\partial \eta}\frac{\partial \eta}{\partial t} = (T_s - T_i) f' \cdot \frac{-nAx}{t^{n+1}} = (T_s - T_i) f' \left[-\frac{n\eta}{t}\right]$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$

$$(T_s - T_i)\frac{A^2}{t^{2n}} f''(\eta) = \frac{1}{\alpha}(T_s - T_i)\left(\frac{-\eta n}{t}\right) f'$$

$$(T_s - T_i)\frac{A}{t^{2n}}\left[f'' + \frac{n\eta}{\alpha A^2} t^{2n-1} f'\right] = 0$$

FOR ODE ; η, f, f', f'' *only* $\Rightarrow$ $t^{2n-1} = 1$ $\Rightarrow n = \frac{1}{2}$.

$$\therefore \quad f'' + \frac{1}{2\alpha A^2}\eta f' = 0 \qquad \text{pick } A \Rightarrow f'' + \eta f' = 0$$

$\therefore$ choose $A = \frac{1}{\sqrt{2\alpha}}$

$$\therefore \quad \eta = \frac{x}{\sqrt{2\alpha t}}$$

at $x = 0$ $\quad T = T_s \quad \Rightarrow \eta = 0 \quad \frac{T_s - T_i}{T_s - T_i} = 1 = f(\eta = 0)$

collapse of 2 conditions to 1

$t = 0 \quad T = T_i \quad \Rightarrow \eta = \infty \quad T = T_i \Rightarrow 0 = f(\eta = \infty)$

$x \to \infty \quad T \to T_i \quad \Rightarrow \eta \to \infty \quad T \to T_i \quad 0 \leftarrow f(\eta \to \infty)$

note $f'' + \eta f' = 0$ (2nd order ODE)

$$\frac{df'}{d\eta} + \eta f' = 0 \quad \Rightarrow \quad \frac{df'}{d\eta} = -\eta f' \quad \text{or} \quad \frac{df'}{f'} = -\eta\, d\eta$$

$$\ln f' = -\eta^2/2 + \ln C_1$$

$$f' = C_1 e^{-\eta^2/2}$$

$$df = C_1 e^{-\eta^2/2} d\eta \quad \text{or} \quad f = C_1 \int_0^\eta e^{-\sigma^2/2} d\sigma + C_2$$

when $\eta = 0$ $\quad f(\eta = 0) = 1 = C_2$

$f \to 0$ as $\eta \to \infty$ $\qquad$ NOW $\int_0^\infty e^{-\sigma^2/2} d\sigma = \sqrt{2}\int_0^\infty e^{-z^2} dz = \sqrt{\frac{\pi}{2}}$

$$0 = C_1 \cdot \sqrt{\frac{\pi}{2}} + 1 \qquad C_1 = -\sqrt{\frac{2}{\pi}} \qquad \therefore \quad f = 1 - \sqrt{\frac{2}{\pi}}\int_0^\eta e^{-\sigma^2/2} d\sigma$$

$$= 1 - \frac{2}{\sqrt{\pi}}\int_0^{\eta/\sqrt{2}} e^{-z^2} dz$$

$$\frac{T - T_i}{T_s - T_i} = f(\eta) = \mathrm{erfc}(\eta/\sqrt{2})$$

$$\frac{T-T_i}{T_s-T_i} = \operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) = 1-\operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

1. ALWAYS REMEMBER THAT IF THE PROBLEM IS INDEPENDENT OF LENGTH OR TIME SCALES — WE HAVE SELF SIMILAR SOLUTION

2. SOLUTION WILL REDUCE # OF INDEPENDENT VARIABLES BY 1 & $\underbrace{\text{PDE}_{\text{ }}\ \text{BECOMES ODE}}_{\text{OF 2 INDEP. VARIABLES}}$

3. ASSUME GENERAL FORM OF TRANSFORMATION BASED ON B.C.'s & I.C.

4. IN SIMILARITY PARAMETER η, MOST DIFFERENTIATED VARIABLE SHOULD APPEAR IN NUMERATOR. ie $A\frac{x}{t^n}$

5. TRANSFORM BC & IC USING SIMILARITY PARAMETER AND INSURE THEY ARE SATISFIED

IF NOT ADD ADDITIONAL DEGREES OF FREEDOM

6. CONVERT DE INTO ONE THAT CONTAINS f & ITS DERIV, η and only one of the independent variables t

Determine parameters of η to reduce PDE order by one

7. Express BC & IC for reduced problem & solve.