5 Looking at 2^{nd} order systems

- Analytical solution of linear 2^{nd} order ODE

$$y'' + p(x)\, y' + q(x)\, y = g(x)$$

$$a(x) y'' + b(x) y' + c(x) y = h(x) \Rightarrow p(x) = \frac{b(x)}{a(x)} \; ; \; q(x) = \frac{c(x)}{a(x)} \; ; \; g(x) = \frac{h(x)}{a(x)}$$

- Will use variation of parameters to define solution

- For homogeneous equation $y'' + p(x) y' + q(x) y = 0$
  - Assume a solution $y_1(x)$ to the equation is known
  - Assume also $y_2(x)$ another linearly independent solution

  - $y_2(x)$ can be written as $y_2(x) = v(x)\, y_1(x)$

  - $v(x)$ satisfies the 1^{ST} order OLDE   Reduction in Order

$$v(x) \underbrace{\left[\cancel{y_1'' + p(x) y_1' + q(x) y_1}\right]}_{0} + y_1 v'' + (2y_1' + p\, y_1)\, v' = 0 \quad \text{or} \quad \frac{v''}{v'} = -\frac{2y_1'}{y_1} - p$$

$$\frac{dv'}{v'} = -\frac{2dy_1}{y_1} - p\,dx$$

$$\ln v' = -2 \ln y_1 - \int p\,dx$$

$$v' = \frac{1}{y_1^2} \cdot e^{-\int p\,dx}$$

and $$v(x) = \int^x \frac{1}{y_1^2(s)}\, e^{-\int^s p(t)\,dt}\, ds$$

- $$y_2(x) = v y_1 = y_1(x) \int^x \frac{1}{y_1^2(s)}\, e^{-\int^s p(t)\,dt}\, ds$$

so that $$y_h = C_1\, y_1(x) + C_2\, y_2(x)$$

satisfies $$y_h'' + p\, y_h' + q\, y_h = 0$$

- To solve and obtain the ~~particular~~ total solution to $y'' + p y' + q y = g$

  - Assume $y_p = u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$ Method of Variation of parameters

- If we assume $u_1' y_1 + u_2' y_2 = 0 \Rightarrow y_p' = u_1 y_1' + u_2 y_2'$
- This leads to $u_1' y_1' + u_2' y_2' = g$

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- Thus $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix}$
  - u_1' & u_2' can be found via Cramer's Rule

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{-g y_2}{y_1 y_2' - y_1' y_2} \Rightarrow u_1(x) = \int^x \frac{-g(t) y_2(t)\,dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} + \cancel{C_1}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{y_1 g}{y_1 y_2' - y_1' y_2} \Rightarrow u_2(x) = \int^x \frac{g(t) y_1(t)\,dt}{y_1(t)y_2'(t) - y_1'(t)y_2(t)} + \cancel{C_2}$$

- Note $y_1 y_2' - y_1' y_2 = W(y_1, y_2)$: Wronskian
  - if $W(y_1, y_2) \neq 0$ then y_1 & y_2 are linearly independent & $y_p = u_1 y_1 + u_2 y_2$
  - if at some pt the Wronskian is <u>zero</u> then it must be zero everywhere and y_1 & y_2 are <u>not</u> linearly independent.

- <u>Example</u> $x^2 y'' - 2xy' + 2y = 4x^2$ for $x > 0$ and $y_1(x) = x$

  $y_1'(x) = 1$

  $y_1''(x) = 0$

  1) Check if y_1 solves $x^2 y_1'' - 2xy_1' + 2y_1 = 0$

  $x^2 \cdot 0 - 2x \cdot 1 + 2x \equiv 0$ ✓

  2) Find $v(x) \Rightarrow$ $y'' - \frac{2}{x} y' + \frac{2}{x^2} y = 0 \Longrightarrow p = -2/x \quad q = 2/x^2$

$$v(x) = \int^x \frac{1}{s^2} e^{-\int^s -\frac{2}{t} dt} ds = \int^x \frac{1}{s^2} e^{2\ln s} ds = \int^x \frac{1}{s^2} s^2 ds = \underline{x}$$

  3) Find $y_h(x)$

$$y_2(x) = v(x)\, y_1(x) = x^2$$

$$\therefore y_h(x) = C_1 y_1(x) + C_2 y_2(x) = C_1 x + C_2 \cdot x^2$$

  4) Now find total solution

$$y_p(x) = u_1(x) \cdot x + u_2(x) \cdot x^2$$

7 5) find $u_1(x)$ & $u_2(x)$ $\qquad x^2y''-2xy'+2y=4x^2 \Rightarrow y''-\frac{2}{x}y'+\frac{2}{x^2}y=\underset{g(x)}{4}$

$$u_1(x)=\int^x \frac{-4\cdot t^2}{t\cdot 2t-1\cdot t^2}\,dt+C_1=\int^x -4\,dt+\cancel{C_1}$$

$$=-4x+\cancel{C_1} \qquad W(y_1,y_2)=2x^2-x^2=x^2$$

$$u_2(x)=\int^x \frac{4\cdot t}{t^2}\,dt+\cancel{C_2}=\int^x \frac{4}{t}\,dt+\cancel{C_2}$$

$$=4\ln x+\cancel{C_2}$$

6) $$y_p=u_1y_1+u_2y_2=(-4x+\cancel{C_1})\,x+(4\ln x+\cancel{C_2})\,x^2$$

$$=\underbrace{-4x^2+4x^2\ln x}_{\text{particular}}+\underbrace{\cancel{C_1x+C_2x^2}}_{\text{homog.}}$$

$$y_T=y_p+y_h=-4x^2+4x^2\ln x+C_1x+C_2x^2$$

<u>HW</u> Find y_2, u_1, u_2, y given $x^2y''+7xy'+5y=x \quad x>0 \quad y_1(x)=\frac{1}{x}$

$x^2y''+xy'+(x^2-\frac{1}{4})y=3x^{3/2}\sin x \quad x>0 \quad y_1(x)=\frac{\sin x}{\sqrt{x}}$