10. Solve Exercise 9 for every contour that does not touch the half $x \geqq 0$ of the real axis. *Ans.* $-\pi i$.

11. Note the single-valued function

$$f(z) = z^{\frac{1}{2}} = \sqrt{r} \exp \frac{i\theta}{2} \qquad \left(r > 0, -\frac{\pi}{2} \leqq \theta < \frac{3\pi}{2}\right),$$
$$f(0) = 0$$

is continuous throughout the half plane $0 \leqq \theta \leqq \pi$, $r \geqq 0$. Let C denote the entire boundary of the half disk $r \leqq 1$, $0 \leqq \theta \leqq \pi$, where C is described in the positive direction. Show that

$$\int_C f(z)\, dz = 0$$

by computing the integrals of f over the semicircle and over the two radii on the x axis. Why does the Cauchy-Goursat theorem not apply here?

12. *Nested Intervals.* An infinite sequence of closed intervals $a_n \leqq x \leqq b_n$ $(n = 0, 1, 2, \ldots)$ is determined according to some rule of selecting half intervals, so that the interval (a_1,b_1) is either the left-hand or right-hand half of a given interval (a_0,b_0); then (a_2,b_2) is one of the two halves of (a_1,b_1), and so on. Prove that there is a point x_0 which belongs to every one of the closed intervals (a_n,b_n).

Suggestion: Note that the left-hand end points a_n represent a bounded nondecreasing sequence of numbers, since $a_0 \leqq a_n \leqq a_{n+1} < b_0$; hence they have a limit A as $n \to \infty$. Show likewise that the end points b_n have a limit B; then that $B = A = x_0$.

13. *Nested Squares.* A square σ_0: $a_0 \leqq x \leqq b_0$, $c_0 \leqq y \leqq d_0$, where $b_0 - a_0 = d_0 - c_0$, is divided into four equal squares by lines parallel to the coordinate axes. One of those four smaller squares σ_1: $a_1 \leqq x \leqq b_1$, $c_1 \leqq y \leqq d_1$, where $b_1 - a_1 = d_1 - c_1$, is selected according to some rule, and it is divided into four equal squares, one of which, σ_2, is selected, etc. (Sec. 47). Prove that there is a point (x_0,y_0) which belongs to every one of the closed regions of the infinite sequence $\sigma_0, \sigma_1, \sigma_2, \ldots$.

Suggestion: Apply the results of Exercise 12 to each of the sequences $a_n \leqq x \leqq b_n$ and $c_n \leqq y \leqq d_n$ $(n = 0, 1, 2, \ldots)$.

51. The Cauchy Integral Formula.

Another fundamental result will now be established.

Theorem. *Let f be analytic everywhere within and on a closed contour C. If z_0 is any point interior to C, then*

analytic means no poles within & on C

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz, \tag{1}$$

important

where the integral is taken in the positive sense around C.

Formula (1) is *Cauchy's integral formula.* It shows that the value of a function that is analytic in a region is determined throughout the region by its values on the boundary. Thus there is no choice of ways in which the function can be defined at points away from the boundary once the function is defined on the boundary. Every alteration of values of the function at interior points must be accompanied by a change of its values on the boundary, if the function is to remain analytic. We shall see further evidence of this *organic* character of analytic functions as we proceed.

According to the Cauchy integral formula, for example, if C is the circle $|z| = 2$ described in the positive sense, then, taking z_0 to be $-i$, we can write

$$\int_C \frac{z\, dz}{(9 - z^2)(z + i)} = 2\pi i \frac{-i}{9 - i^2} = \frac{\pi}{5},$$

since the function $f(z) = z/(9 - z^2)$ is analytic within and on C.

To prove the theorem, let C_0 be a circle about z_0,

$$|z - z_0| = r_0,$$

whose radius r_0 is small enough that C_0 is interior to C (Fig. 38). The function $f(z)/(z - z_0)$ is analytic at all points within and on C except the point z_0.

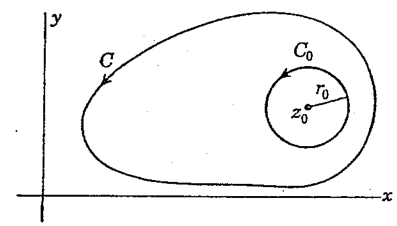

FIG. 38

Hence its integral around the boundary of the ring-shaped region between C and C_0 is zero, according to the Cauchy-Goursat theorem; that is,

$$\int_C \frac{f(z)\, dz}{z - z_0} - \int_{C_0} \frac{f(z)\, dz}{z - z_0} = 0,$$

where both integrals are taken counterclockwise.

Since the integrals around C and C_0 are equal, we can write

$$\int_C \frac{f(z)\, dz}{z - z_0} = f(z_0) \int_{C_0} \frac{dz}{z - z_0} + \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0}\, dz. \tag{2}$$

But $z - z_0 = r_0 e^{i\theta}$ on C_0 and $dz = ir_0 e^{i\theta}\, d\theta$, so that

$$\int_{C_0} \frac{dz}{z - z_0} = i \int_0^{2\pi} d\theta = 2\pi i, \tag{3}$$

for every positive r_0. Also, f is continuous at the point z_0. Hence, if we select any positive number ϵ, then a positive number δ exists such that

$$|f(z) - f(z_0)| < \epsilon \quad \text{whenever } |z - z_0| \leqq \delta.$$

We take r_0 equal to that number δ. Then $|z - z_0| = \delta$, and

$$\left| \int_{C_0} \frac{f(z) - f(z_0)}{z - z_0}\, dz \right| \leqq \int_{C_0} \frac{|f(z) - f(z_0)|}{|z - z_0|}\, |dz| < \frac{\epsilon}{\delta}(2\pi\delta) = 2\pi\epsilon.$$

The absolute value of the last integral in equation (2) can therefore be made arbitrarily small by taking r_0 sufficiently small. But since the other two integrals in that equation are independent of r_0, in view of equation (3), this one must be independent of r_0 also. Its value must therefore be zero. Equation (2) then reduces to the formula

$$\int_C \frac{f(z)\, dz}{z - z_0} = 2\pi i f(z_0),$$

and the theorem is proved.

52. Derivatives of Analytic Functions. A formula for the derivative $f'(z_0)$ can be written formally by differentiating the integral in Cauchy's integral formula

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\, dz \tag{1}$$

with respect to z_0, inside the integral sign. Thus,

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2}\, dz. \tag{2}$$ ← important

As before, we assume that f is analytic within and on the closed contour C and that z_0 is within C. To establish formula (2), we first note that, according to (1),

$$\frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0} = \frac{1}{2\pi i\, \Delta z_0} \int_C \left(\frac{1}{z - z_0 - \Delta z_0} - \frac{1}{z - z_0} \right) f(z)\, dz$$
$$= \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0 - \Delta z_0)(z - z_0)}.$$

The last integral approaches the integral

$$\int_C \frac{f(z)\, dz}{(z - z_0)^2}$$

as Δz_0 approaches zero; for the difference between that integral and this one reduces to

$$\Delta z_0 \int_C \frac{f(z)\, dz}{(z - z_0)^2 (z - z_0 - \Delta z_0)}.$$

Let M be the maximum value of $|f(z)|$ on C and let L be the length of C. Then, if d_0 is the shortest distance from z_0 to C and if $|\Delta z_0| < d_0$, we can write

$$\left| \Delta z_0 \int_C \frac{f(z)\, dz}{(z - z_0)^2 (z - z_0 - \Delta z_0)} \right| < \frac{ML|\Delta z_0|}{d_0^2 (d_0 - |\Delta z_0|)},$$

and the last fraction approaches zero when Δz_0 approaches zero. Consequently,

$$\lim_{\Delta z_0 \to 0} \frac{f(z_0 + \Delta z_0) - f(z_0)}{\Delta z_0} = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^2},$$

and formula (2) is established.

If we differentiate both members of equation (2) and assume that the order of differentiation with respect to z_0 and integration with respect to z can be interchanged, we find that

$$f''(z_0) = \frac{2!}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^3}. \tag{3}$$

This formula can be established by the same method that was used to establish formula (2). For it follows from formula (2) that

$$2\pi i \frac{f'(z_0 + \Delta z_0) - f'(z_0)}{\Delta z_0}$$
$$= \int_C \left[\frac{1}{(z - z_0 - \Delta z_0)^2} - \frac{1}{(z - z_0)^2} \right] \frac{f(z)\, dz}{\Delta z_0}$$
$$= \int_C \frac{2(z - z_0) - \Delta z_0}{(z - z_0 - \Delta z_0)^2 (z - z_0)^2} f(z)\, dz.$$

Following the same procedure that was used before, we can show that the limit of the last integral, as Δz_0 approaches zero, is

$$2 \int_C \frac{f(z)\, dz}{(z - z_0)^3},$$

and formula (3) follows at once.

We have now established the existence of the derivative of the function f' at each point z_0 interior to the region bounded by the curve C.

We recall our definition that a function f is analytic at a point z_1 if and only if there is a neighborhood about z_1 at each point of which $f'(z)$ exists. Hence f is analytic in some neighborhood of the point. If the curve C used above is a circle $|z - z_1| = r_1$ in that neighborhood, then $f''(z)$ exists at each point inside the circle, and therefore f' is analytic at z_1. We can apply the same argument to the function f' to conclude that its derivative f'' is analytic at z_1, etc. Thus the following fundamental result is a consequence of formula (3).

Theorem. *If a function f is analytic at a point, then its derivatives of all orders, f', f'', . . . , are also analytic functions at that point.*

Since f' is analytic and therefore continuous, and since

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y},$$

it follows that the partial derivatives of $u(x,y)$ and $v(x,y)$ of the first order are continuous. Since $f''(z)$ is analytic and

$$f''(z) = \frac{\partial^2 u}{\partial x^2} + i\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial x\,\partial y} - i\frac{\partial^2 u}{\partial x\,\partial y},$$

etc., it follows that the partial derivatives of u and v of all orders are continuous functions of x and y at each point where f is analytic. This result was anticipated in Sec. 20, for the partial derivatives of the second order, in the discussion of harmonic functions.

The argument used in establishing formulas (2) and (3) can be applied successively to obtain a formula for the derivative of any given order. But mathematical induction can now be applied to establish the general formula

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)\,dz}{(z - z_0)^{n+1}} \qquad (n = 1, 2, \ldots). \tag{4}$$ important

That is, if we assume that this formula is true for any particular integer $n = k$, we can show by proceeding as before that it is true if $n = k + 1$. The details of the proof can be left to the reader, with the suggestion that in the algebraic simplifications he retain the difference $(z - z_0)$ throughout as a single term.

The closed contour C here, as well as in Cauchy's integral formula, can be replaced by *the oriented boundary B of a multiply connected closed region R* of the type described in the theorem in Sec. 49, when f is analytic in R and z_0 is any interior point of R. Our derivations of the Cauchy integral formula and its extensions (4) are still valid when C is replaced by B.

53. Morera's Theorem. In Sec. 50 we proved that the derivative of the function

$$F(z) = \int_{z_0}^{z} f(z')\,dz'$$

exists at each point of a simply connected domain D, in fact, that

$$F'(z) = f(z).$$

We assumed there that f is analytic in D. But in our proof we used only two properties of the analytic function f, namely, that it is continuous in D and that its integral around every closed contour interior to D vanishes. Thus, when f satisfies those two conditions, the function F is analytic in D.

We proved in Sec. 52 that the derivative of every analytic function is analytic. Since $F'(z) = f(z)$, it follows that f is analytic. The following theorem, due to E. Morera (1856–1909), is therefore established.

Theorem. *If a function f is continuous throughout a simply connected domain D and if, for every closed contour C interior to D,*

$$\oint_C f(z)\,dz = 0,$$ important

then f is analytic throughout D.

Morera's theorem serves as a converse of the Cauchy-Goursat theorem.

54. Maximum Moduli of Functions. Let f be analytic at a point z_0. If C_0 denotes any one of the circles $|z - z_0| = r_0$ within and on which f is analytic, then, according to Cauchy's integral formula,

$$f(z_0) = \frac{1}{2\pi i}\int_{C_0} \frac{f(z)\,dz}{z - z_0}.$$

It follows that

$$|f(z_0)| \leqq \frac{1}{2\pi r_0}\int_{C_0} |f(z)|\,|dz| = A_0, \tag{1}$$