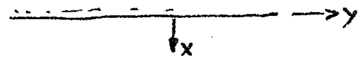
Diffusion of contaminant deposited at $t=0$ on surface of semi infinite slab $\alpha C_{xx} = C_t$



initially $c(x,t=0)=0 \quad x>0$
for $x \to \infty \quad c(x,t) \to 0$
$\int_0^\infty c(x,t)\,dx = Q$ constant total amount

Take L.T. of both sides of integ condition

$$\int_0^\infty c(x,t)\,dx = Q$$

$$\int_0^\infty e^{-st}\int_0^\infty c(x,t)\,dx\,dt = \int_0^\infty Q e^{-st}\,dt = -\frac{Q}{s} e^{-st}\Big|_0^\infty = \frac{Q}{s}.$$

Change order of integration

$$\int_0^\infty\int_0^\infty e^{-st} c(x,t)\,dt\,dx = \int_0^\infty C(x;s)\,dx = \frac{Q}{s}$$

$$\alpha \frac{d^2 C}{dx^2} = sC - c(x,t=0) = sC$$

$$C'' - \frac{s}{\alpha} C = 0 \quad \Longrightarrow \quad C = c_1 e^{\sqrt{\frac{s}{\alpha}}x} + c_2 e^{-\sqrt{\frac{s}{\alpha}}x}$$

as $x \to \infty \quad c \to 0 \qquad x \to \infty \quad C \to 0 \quad \Rightarrow \quad c_1 = 0$

$$c_2 \int_0^\infty e^{-\sqrt{\frac{s}{\alpha}}x}\,dx = \frac{Q}{s}$$

$$c_2\left(-\sqrt{\frac{\alpha}{s}}\, e^{-\sqrt{\frac{s}{\alpha}}x}\right)\Big|_0^\infty = c_2\sqrt{\frac{\alpha}{s}} = \frac{Q}{s} \qquad \therefore \quad c_2 = \frac{Q}{\sqrt{\alpha s}}$$

$$\therefore \quad C = \frac{Q}{\sqrt{\alpha}} \underbrace{\frac{1}{\sqrt{s}} e^{-\frac{x}{\sqrt{\alpha}}\cdot\sqrt{s}}} \qquad \mathcal{L}^{-1}\left\{\frac{1}{\sqrt{s}} e^{-\frac{x}{\sqrt{\alpha}}\sqrt{s}}\right\} = \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4\alpha t}}$$

29.3.84

$$c = \frac{Q}{\sqrt{\alpha}} \frac{1}{\sqrt{\pi t}} e^{-\frac{x^2}{4\alpha t}} = \frac{Q}{\sqrt{\pi\alpha t}} e^{-x^2/4\alpha t}$$

WORK FROM EXTRA CLASS:
4-5PM M-W
5^{30}-7^{30} TH

$$\left.\begin{aligned} &\frac{\partial^2 u}{\partial x^2} = \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2} \\ &u(x,t=0) = f(x) \\ &\frac{\partial u}{\partial t}(x,t=0) = g(x) \\ &u(x=0,t) = 0 \\ &u(x=L,t) = 0 \end{aligned}\right\}$$

Let $U(x;s) = \int_0^\infty u(x,t)e^{-st}dt$

$$\therefore \frac{d^2}{dx^2}U(x;s) = \frac{1}{a^2}\left[s^2U - su(x,t=0) - \frac{\partial u}{\partial t}(x,t=0)\right]$$
$$= \frac{1}{a^2}\left[s^2U - sf(x) - g(x)\right]$$

or $$\frac{d^2U}{dx^2} - \frac{s^2}{a^2}U = -[sf+g] = -G(x;s)$$

$$U_h = \mathscr{C}_1 \sinh\left(\tfrac{s}{a}x\right) + \mathscr{C}_2 \cosh\left(\tfrac{s}{a}x\right) = \mathscr{C}_1 U_1 + \mathscr{C}_2 U_2$$

using variation of parameters
$$U_p = U_1\int_0^x \frac{G(\bar{x};s)\cosh\left(\frac{s}{a}\bar{x}\right)d\bar{x}}{s/a} - U_2\int_0^x \frac{G(\bar{x};s)\sinh\left(\frac{s}{a}\bar{x}\right)d\bar{x}}{s/a}$$
$$= \int_0^x \frac{a}{s}G(\bar{x};s)\sinh\left[\tfrac{s}{a}(x-\bar{x})\right]d\bar{x}$$

now $$U_{TOT} = U_h + U_p = \mathscr{C}_1\sinh\left(\tfrac{s}{a}x\right) + \mathscr{C}_2\cosh\left(\tfrac{s}{a}x\right) + \int_0^x \frac{a}{s}G(\bar{x};s)\sinh\left[\tfrac{s}{a}(x-\bar{x})\right]d\bar{x}$$

putting in the B.C.'s $u(x=0,t)=0 \Rightarrow \mathscr{C}_2 = 0$

$$u(x=L,t)=0 \Rightarrow \mathscr{C}_1 = -\frac{\int_0^L \frac{a}{s}G(\bar{x};s)\sinh\left[\frac{s}{a}(L-\bar{x})\right]d\bar{x}}{\sinh\frac{L}{a}s}$$

$$\therefore\ U(x;s) = -\int_0^L \frac{a}{s}G(\bar{x};s)\sinh\left[\tfrac{s}{a}(L-\bar{x})\right]d\bar{x}\left(\frac{\sinh\frac{s}{a}x}{\sinh\frac{s}{a}L}\right) + \int_0^x \frac{a}{s}G(\bar{x};s)\sinh\left[\tfrac{s}{a}(x-\bar{x})\right]d\bar{x}$$

Residue Theorem: we can use $u(x,t) = \lim_{s\to a}(s-a)e^{st}U(x;s)$ if $U(x;s) = \dfrac{Q(x;s)}{s-a}$

the first & second integrals appear to have $s=0$ & $s = \frac{a}{L}n\pi i$ as poles

since $\sinh is = i\sin s$ & $\sin s = 0$ if $s = n\pi$

@ $s\to 0$ $$U(x;s) \to -\int_0^L \frac{a}{s}[sf+g]\,\frac{s}{a}(L-\bar{x})\,d\bar{x}\left(\frac{\frac{s}{a}x}{\frac{s}{a}L}\right) + \int_0^x \frac{a}{s}[sf+g]\left[\tfrac{s}{a}(x-\bar{x})\right]d\bar{x}$$

since $\sinh x = \dfrac{e^x - e^{-x}}{2} = x + \dfrac{x^3}{3!} + \cdots$

- when $s=0$ $U(x;s)$ tends to a constant not $\dfrac{Q(x;s)}{s}$ $\therefore$ $s=0$ is not a pole and so the residue $=0$

- when $s = \frac{a}{L}n\pi i$ $\sinh\frac{Ls}{a} = i\sin n\pi$ and

$$U(x;s) \to -\int_0^L \frac{a}{\frac{a}{L}n\pi i}\left[\tfrac{a}{L}n\pi i\, f + g\right]\sinh\frac{n\pi i}{L}(L-\bar{x})\,d\bar{x}\left(\frac{\sinh\frac{n\pi i}{L}x}{\sinh i n\pi}\right)$$
$$+ \int_0^x \frac{a}{\frac{a}{L}n\pi i}\left[\tfrac{a}{L}n\pi i f + g\right]\sinh i\frac{n\pi}{L}(x-\bar{x})\,d\bar{x}$$

the 2^{nd} integral is constant & gives no residue the first integral will give a residue at $s = \frac{a}{L}n\pi i$ for each & every n except $n=0$ (this is same as $s=0$)

$$\text{Thus} \sum_{s \to \frac{a}{L}n\pi i} \left(s - \tfrac{a}{L}n\pi i\right) e^{\overset{s}{\downarrow}{\frac{a}{L}n\pi i t}} \left\{ -\int_0^L \frac{L}{n\pi i}\left[\tfrac{a}{L}n\pi i f + g\right] \sinh \frac{n\pi i}{L}(L-\bar{x})\,d\bar{x} \;\; \frac{\sinh \frac{n\pi i}{L}x}{\sinh i n\pi} \right\}$$

$$\sum e^{\frac{a}{L}n\pi i t} \left\{ -\int_0^L \frac{L}{n\pi i}\left[\tfrac{a}{L}n\pi i f + g\right] i \sin \frac{n\pi}{L}(L-\bar{x})\,d\bar{x} \cdot \frac{i \sin \frac{n\pi x}{L}}{} \;\; \frac{(s - \frac{a}{L}n\pi i)}{\sinh \frac{L}{a}s} \right\}$$

$$\lim_{s \to \frac{a}{L}n\pi i}$$

using L'Hôpital's Rule

$$= \sum_{\substack{n=-\infty \\ n\neq 0}}^{n=\infty} e^{\frac{a}{L}n\pi i t}\left\{ + \int_0^L \frac{\cancel{L}}{n\pi i}\left[\tfrac{a}{L}n\pi i f + g\right] \sin \frac{n\pi}{L}(L-\bar{x})\,d\bar{x} \; \sin \frac{n\pi x}{L} \cdot \frac{1}{\frac{\cancel{L}}{a}\underbrace{\cosh n\pi i}_{=\cos n\pi = (-1)^n}} \right\}$$

$$= \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{\frac{a}{L}n\pi i t}\left\{ \int_0^L \frac{a}{n\pi i}\left[\tfrac{a}{L}n\pi i f + g\right] \sin \frac{n\pi}{L}(L-\bar{x})\,d\bar{x} \;\; \sin \frac{n\pi x}{L} \;\; (-1)^n \right\}$$

$$\text{but} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} = \sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty}$$

$$\text{now if } n=-m \qquad \sum_{n=-\infty}^{-1} = \sum_{m=1}^{\infty} e^{\frac{a}{L}(-m)\pi i t} \left\{ \int_0^L \frac{a}{(-m)\pi i}\left[\tfrac{a}{L}(-m)\pi i f + g\right]\left(-\sin \frac{m\pi}{L}(L-\bar{x})\right) d\bar{x} \left(-\sin\frac{m\pi x}{L}\right)(-1)^{-m}$$

since m is dummy index

$$= \sum_{m=1}^{\infty} e^{-\frac{a}{L}n\pi i t} \left\{ -\int_0^L \frac{a}{n\pi i}\left[-\tfrac{a}{L}(n\pi i) f + g\right] \sin \frac{n\pi}{L}(L-\bar{x})\,d\bar{x} \;\; \left(\sin \frac{n\pi x}{L}\right)(-1)^n \right\}$$

now by adding the two sums

$$\sum_{n=1}^{\infty} \int_0^L \frac{a}{\cancel{n\pi i}}\left[e^{\frac{a}{L}n\pi i t} + e^{-\frac{a}{L}n\pi i t}\right] \cdot \frac{a}{L}\cancel{(n\pi i)} f(\bar{x}) \sin \frac{n\pi}{L}(L-\bar{x})\,d\bar{x} \left(\sin \frac{n\pi x}{L}\right)(-1)^n$$

$$+ \int_0^L \frac{a}{n\pi i}\left[e^{a/L\, n\pi i t} - e^{-a/L\, n\pi i t}\right] g(\bar{x}) \sin \frac{n\pi}{L}(L-\bar{x})\,d\bar{x} \left(\sin\frac{n\pi x}{L}\right)(-1)^n$$

$$e^{a/L\, n\pi i t} + e^{-a/L\, n\pi i t} = 2\cos \tfrac{a}{L} n\pi t$$

$$e^{a/L\, n\pi i t} - e^{-a/L\, n\pi i t} = 2i \sin a/L\, n\pi t$$

thus

$$u(x,t) = \sum_{n=1}^{\infty} \int_0^L \frac{2a^2}{L} \cos \tfrac{a}{L}n\pi t \;\; f(\bar{x}) \sin \frac{n\pi}{L}(L-\bar{x})\,d\bar{x} \cdot \sin \frac{n\pi x}{L} \; (-1)^n$$

$$+ \sum_{n=1}^{\infty} \int_0^L \frac{2a}{n\pi} \sin \tfrac{a}{L}n\pi t \;\; g(\bar{x}) \sin \frac{n\pi}{L}(L-\bar{x})\,d\bar{x} \cdot \sin\frac{n\pi x}{L}\; (-1)^n$$

$$\boxed{\begin{array}{l} \text{or} \displaystyle\sum_{n=1}^{\infty} \sin \frac{n\pi x}{L}\left\{ A_n \cos \tfrac{a}{L}n\pi t + B_n \sin \tfrac{a}{L} n\pi t \right\} \quad \text{where} \\ \displaystyle A_n = \frac{2a^2}{L}\int_0^L f(\bar{x}) \sin \frac{n\pi}{L}(L-\bar{x})\,d\bar{x} \cdot (-1)^n \\ \displaystyle B_n = \frac{2a}{n\pi}\int_0^L g(\bar{x}) \sin \frac{n\pi}{L}(L-\bar{x})\,d\bar{x}\; (-1)^n \end{array}}$$