29. Laplace Transforms

29.1. Definition of the Laplace Transform

One-dimensional Laplace Transform

29.1.1 $$f(s)=\mathscr{L}\{F(t)\}=\int_0^\infty e^{-st}F(t)dt$$

$F(t)$ is a function of the real variable t and s is a complex variable. $F(t)$ is called the original function and $f(s)$ is called the image function. If the integral in **29.1.1** converges for a real $s=s_0$, i.e.,

$$\lim_{\substack{A\to 0\\ B\to\infty}}\int_A^B e^{-s_0t}F(t)dt$$

exists, then it converges for all s with $\mathscr{R}s>s_0$, and the image function is a single valued analytic function of s in the half-plane $\mathscr{R}s>s_0$.

Two-dimensional Laplace Transform

29.1.2 $$f(u,v)=\mathscr{L}\{F(x,y)\}=\int_0^\infty\int_0^\infty e^{-ux-vy}F(x,y)dxdy$$

Definition of the Unit Step Function

29.1.3 $$u(t)=\begin{cases}0 & (t<0)\\ \tfrac{1}{2} & (t=0)\\ 1 & (t>0)\end{cases}$$

Heaviside step function

In the following tables the factor $u(t)$ is to be understood as multiplying the original function $F(t)$.

29.2. Operations for the Laplace Transform[1]

	Original Function $F(t)$	Image Function $f(s)$
29.2.1	$F(t)$	$\int_0^\infty e^{-st}F(t)dt$
	Inversion Formula	
29.2.2	$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{ts}f(s)ds$	$f(s)$
	Linearity Property	
29.2.3	$AF(t)+BG(t)$	$Af(s)+Bg(s)$
	Differentiation	
29.2.4	$F'(t)$	$sf(s)-F(+0)$
29.2.5	$F^{(n)}(t)$	$s^nf(s)-s^{n-1}F(+0)-s^{n-2}F'(+0)-\ldots-F^{(n-1)}(+0)$
	Integration	
29.2.6	$\int_0^t F(\tau)d\tau$	$\frac{1}{s}f(s)$
29.2.7	$\int_0^t\int_0^\tau F(\lambda)d\lambda d\tau$	$\frac{1}{s^2}f(s)$
	Convolution (Faltung) Theorem	
29.2.8	$\int_0^t F_1(t-\tau)F_2(\tau)d\tau=F_1*F_2$	$f_1(s)f_2(s)$
		Differentiation
29.2.9	$-tF(t)$	$f'(s)$
29.2.10	$(-1)^nt^nF(t)$	$f^{(n)}(s)$

[1] Adapted by permission from R. V. Churchill, Operational mathematics, 2d ed., McGraw-Hill Book Co., Inc., New York, N.Y., 1958.

	Original Function $F(t)$	Image Function $f(s)$
		Integration
29.2.11	$\frac{1}{t}F(t)$	$\int_s^\infty f(x)dx$
		Linear Transformation
29.2.12	$e^{at}F(t)$	$f(s-a)$
29.2.13	$\frac{1}{c}F\left(\frac{t}{c}\right) \quad (c>0)$	$f(cs)$
29.2.14	$\frac{1}{c}e^{(b/c)t}F\left(\frac{t}{c}\right) \quad (c>0)$	$f(cs-b)$
	Translation	
29.2.15	$F(t-b)u(t-b) \quad (b>0)$	$e^{-bs}f(s)$
	Periodic Functions	
29.2.16	$F(t+a)=F(t)$	$\dfrac{\int_0^a e^{-st}F(t)dt}{1-e^{-as}}$
29.2.17	$F(t+a)=-F(t)$	$\dfrac{\int_0^a e^{-st}F(t)dt}{1+e^{-as}}$
	Half-Wave Rectification of $F(t)$ in 29.2.17	
29.2.18	$F(t)\sum_{n=0}^{\infty}(-1)^n u(t-na)$	$\dfrac{f(s)}{1-e^{-as}}$
	Full-Wave Rectification of $F(t)$ in 29.2.17	
29.2.19	$\|F(t)\|$	$f(s)\coth\frac{as}{2}$
	Heaviside Expansion Theorem	
29.2.20	$\sum_{n=1}^{m}\frac{p(a_n)}{q'(a_n)}e^{a_n t}$	$\frac{p(s)}{q(s)},\ q(s)=(s-a_1)(s-a_2)\ldots(s-a_m)$ $p(s)$ a polynomial of degree $<m$
29.2.21	$e^{at}\sum_{n=1}^{r}\frac{p^{(r-n)}(a)}{(r-n)!}\frac{t^{n-1}}{(n-1)!}$	$\frac{p(s)}{(s-a)^r}$ $p(s)$ a polynomial of degree $<r$

29.3. Table of Laplace Transforms[2,3]

For a comprehensive table of Laplace and other integral transforms see [29.9]. For a table of two-dimensional Laplace transforms see [29.11].

	$f(s)$	$F(t)$
29.3.1	$\frac{1}{s}$	1
29.3.2	$\frac{1}{s^2}$	t

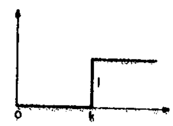
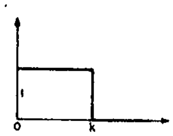
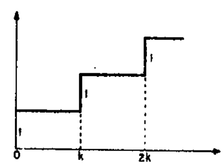
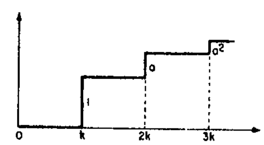
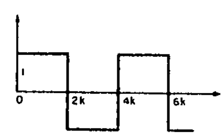
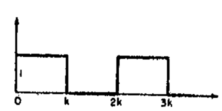
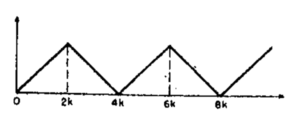
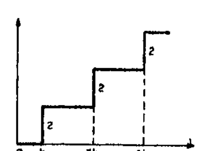
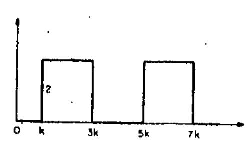
[2] The numbers in bold type in the $f(s)$ and $F(t)$ columns indicate the chapters in which the properties of the respective higher mathematical functions are given.

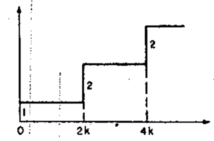
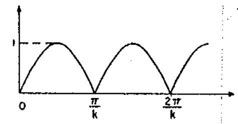
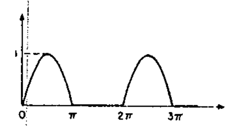
[3] Adapted by permission from R. V. Churchill, Operational mathematics, 2d. ed., McGraw-Hill Book Co., Inc., New York, N. Y., 1958.

	$f(s)$		$F(t)$
29.3.3	$\frac{1}{s^n} \quad (n=1, 2, 3, \ldots)$		$\frac{t^{n-1}}{(n-1)!}$
29.3.4	$\frac{1}{\sqrt{s}}$		$\frac{1}{\sqrt{\pi t}}$
29.3.5	$s^{-3/2}$		$2\sqrt{t/\pi}$
29.3.6	$s^{-(n+\frac{1}{2})} \quad (n=1, 2, 3, \ldots)$		$\frac{2^n t^{n-\frac{1}{2}}}{1\cdot 3\cdot 5 \ldots (2n-1)\sqrt{\pi}}$
29.3.7	$\frac{\Gamma(k)}{s^k} \quad (k>0)$	6	t^{k-1}
29.3.8	$\frac{1}{s+a}$		e^{-at}
29.3.9	$\frac{1}{(s+a)^2}$		te^{-at}
29.3.10	$\frac{1}{(s+a)^n} \quad (n=1, 2, 3, \ldots)$		$\frac{t^{n-1}e^{-at}}{(n-1)!}$
29.3.11	$\frac{\Gamma(k)}{(s+a)^k} \quad (k>0)$	6	$t^{k-1}e^{-at}$
29.3.12	$\frac{1}{(s+a)(s+b)} \quad (a\neq b)$		$\frac{e^{-at}-e^{-bt}}{b-a}$
29.3.13	$\frac{s}{(s+a)(s+b)} \quad (a\neq b)$		$\frac{ae^{-at}-be^{-bt}}{a-b}$
29.3.14	$\frac{1}{(s+a)(s+b)(s+c)}$ (a, b, c distinct constants)		$-\frac{(b-c)e^{-at}+(c-a)e^{-bt}+(a-b)e^{-ct}}{(a-b)(b-c)(c-a)}$
29.3.15	$\frac{1}{s^2+a^2}$		$\frac{1}{a}\sin at$
29.3.16	$\frac{s}{s^2+a^2}$		$\cos at$
29.3.17	$\frac{1}{s^2-a^2}$		$\frac{1}{a}\sinh at$
29.3.18	$\frac{s}{s^2-a^2}$		$\cosh at$
29.3.19	$\frac{1}{s(s^2+a^2)}$		$\frac{1}{a^2}(1-\cos at)$
29.3.20	$\frac{1}{s^2(s^2+a^2)}$		$\frac{1}{a^3}(at-\sin at)$
29.3.21	$\frac{1}{(s^2+a^2)^2}$		$\frac{1}{2a^3}(\sin at-at\cos at)$

	$f(s)$	$F(t)$	
29.3.22	$\frac{s}{(s^2+a^2)^2}$	$\frac{t}{2a}\sin at$	
29.3.23	$\frac{s^2}{(s^2+a^2)^2}$	$\frac{1}{2a}(\sin at+at\cos at)$	
29.3.24	$\frac{s^2-a^2}{(s^2+a^2)^2}$	$t\cos at$	
29.3.25	$\frac{s}{(s^2+a^2)(s^2+b^2)} \quad (a^2\neq b^2)$	$\frac{\cos at-\cos bt}{b^2-a^2}$	
29.3.26	$\frac{1}{(s+a)^2+b^2} = \frac{1}{s^2+2as+(a^2+b^2)}$	$\frac{1}{b}e^{-at}\sin bt$	
29.3.27	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$	
29.3.28	$\frac{3a^2}{s^3+a^3}$	$e^{-at}-e^{\frac{1}{2}at}\left(\cos\frac{at\sqrt{3}}{2}-\sqrt{3}\sin\frac{at\sqrt{3}}{2}\right)$	
29.3.29	$\frac{4a^3}{s^4+4a^4}$	$\sin at\cosh at-\cos at\sinh at$	
29.3.30	$\frac{s}{s^4+4a^4}$	$\frac{1}{2a^2}\sin at\sinh at$	
29.3.31	$\frac{1}{s^4-a^4}$	$\frac{1}{2a^3}(\sinh at-\sin at)$	
29.3.32	$\frac{s}{s^4-a^4}$	$\frac{1}{2a^2}(\cosh at-\cos at)$	
29.3.33	$\frac{8a^3s^2}{(s^2+a^2)^3}$	$(1+a^2t^2)\sin at-at\cos at$	
29.3.34	$\frac{1}{s}\left(\frac{s-1}{s}\right)^n$	$L_n(t)$	22
29.3.35	$\frac{s}{(s+a)^{3/2}}$	$\frac{1}{\sqrt{\pi t}}e^{-at}(1-2at)$	
29.3.36	$\sqrt{s+a}-\sqrt{s+b}$	$\frac{1}{2\sqrt{\pi t^3}}(e^{-bt}-e^{-at})$	
29.3.37	$\frac{1}{\sqrt{s}+a}$	$\frac{1}{\sqrt{\pi t}}-ae^{a^2t}\operatorname{erfc} a\sqrt{t}$	7
29.3.38	$\frac{\sqrt{s}}{s-a^2}$	$\frac{1}{\sqrt{\pi t}}+ae^{a^2t}\operatorname{erf} a\sqrt{t}$	7
29.3.39	$\frac{\sqrt{s}}{s+a^2}$	$\frac{1}{\sqrt{\pi t}}-\frac{2a}{\sqrt{\pi}}e^{-a^2t}\int_0^{a\sqrt{t}}e^{\lambda^2}d\lambda$	7
29.3.40	$\frac{1}{\sqrt{s}(s-a^2)}$	$\frac{1}{a}e^{a^2t}\operatorname{erf} a\sqrt{t}$	7

	$f(s)$		$F(t)$	
29.3.41	$\frac{1}{\sqrt{s}(s+a^2)}$		$\frac{2}{a\sqrt{\pi}} e^{-a^2t}\int_0^{a\sqrt{t}} e^{\lambda^2}d\lambda$	7
29.3.42	$\frac{b^2-a^2}{(s-a^2)(b+\sqrt{s})}$		$e^{a^2t}[b-a \operatorname{erf} a\sqrt{t}]-be^{b^2t} \operatorname{erfc} b\sqrt{t}$	7
29.3.43	$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$		$e^{a^2t} \operatorname{erfc} a\sqrt{t}$	7
29.3.44	$\frac{1}{(s+a)\sqrt{s+b}}$		$\frac{1}{\sqrt{b-a}} e^{-at} \operatorname{erf} (\sqrt{b-a}\sqrt{t})$	7
29.3.45	$\frac{b^2-a^2}{\sqrt{s}(s-a^2)(\sqrt{s}+b)}$		$e^{a^2t}\left[\frac{b}{a} \operatorname{erf} (a\sqrt{t})-1\right]+e^{b^2t} \operatorname{erfc} b\sqrt{t}$	7
29.3.46	$\frac{(1-s)^n}{s^{n+\frac{1}{2}}}$		$\frac{n!}{(2n)!\sqrt{\pi t}} H_{2n}(\sqrt{t})$	22
29.3.47	$\frac{(1-s)^n}{s^{n+\frac{3}{2}}}$		$\frac{n!}{(2n+1)!\sqrt{\pi}} H_{2n+1}(\sqrt{t})$	22
29.3.48	$\frac{\sqrt{s+2a}}{\sqrt{s}}-1$		$ae^{-at}[I_1(at)+I_0(at)]$	9
29.3.49	$\frac{1}{\sqrt{s+a}\sqrt{s+b}}$		$e^{-\frac{1}{2}(a+b)t}I_0\left(\frac{a-b}{2}\, t\right)$	9
29.3.50	$\frac{\Gamma(k)}{(s+a)^k(s+b)^k}$	$(k>0)$ 6	$\sqrt{\pi}\left(\frac{t}{a-b}\right)^{k-\frac{1}{2}} e^{-\frac{1}{2}(a+b)t}I_{k-\frac{1}{2}}\left(\frac{a-b}{2}\, t\right)$	10
29.3.51	$\frac{1}{(s+a)^{\frac{1}{2}}(s+b)^{\frac{3}{2}}}$		$te^{-\frac{1}{2}(a+b)t}\left[I_0\left(\frac{a-b}{2}\, t\right)+I_1\left(\frac{a-b}{2}\, t\right)\right]$	9
29.3.52	$\frac{\sqrt{s+2a}-\sqrt{s}}{\sqrt{s+2a}+\sqrt{s}}$		$\frac{1}{t} e^{-at}I_1(at)$	9
29.3.53	$\frac{(a-b)^k}{(\sqrt{s+a}+\sqrt{s+b})^{2k}}$	$(k>0)$	$\frac{k}{t} e^{-\frac{1}{2}(a+b)t}I_k\left(\frac{a-b}{2}\, t\right)$	9
29.3.54	$\frac{(\sqrt{s+a}+\sqrt{s})^{-2\nu}}{\sqrt{s}\sqrt{s+a}}$	$(\nu>-1)$	$\frac{1}{a^\nu} e^{-\frac{1}{2}at}I_\nu(\frac{1}{2}at)$	9
29.3.55	$\frac{1}{\sqrt{s^2+a^2}}$		$J_0(at)$	9
29.3.56	$\frac{(\sqrt{s^2+a^2}-s)^\nu}{\sqrt{s^2+a^2}}$	$(\nu>-1)$	$a^\nu J_\nu(at)$	9
29.3.57	$\frac{1}{(s^2+a^2)^k}$	$(k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}}J_{k-\frac{1}{2}}(at)$	6, 10

	$f(s)$	$F(t)$		
29.3.58	$(\sqrt{s^2+a^2}-s)^k \quad (k>0)$	$\frac{ka^k}{t} J_k(at)$		9
29.3.59	$\frac{(s-\sqrt{s^2-a^2})^\nu}{\sqrt{s^2-a^2}} \quad (\nu>-1)$	$a^\nu I_\nu(at)$		9
29.3.60	$\frac{1}{(s^2-a^2)^k} \quad (k>0)$	$\frac{\sqrt{\pi}}{\Gamma(k)}\left(\frac{t}{2a}\right)^{k-\frac{1}{2}} I_{k-\frac{1}{2}}(at)$		6, 10
29.3.61	$\frac{1}{s} e^{-ks}$	$u(t-k)$		
29.3.62	$\frac{1}{s^2} e^{-ks}$	$(t-k)u(t-k)$		
29.3.63	$\frac{1}{s^\mu} e^{-ks} \quad (\mu>0)$	$\frac{(t-k)^{\mu-1}}{\Gamma(\mu)} u(t-k)$		6
29.3.64	$\frac{1-e^{-ks}}{s}$	$u(t)-u(t-k)$		
29.3.65	$\frac{1}{s(1-e^{-ks})}=\frac{1+\coth \frac{1}{2}ks}{2s}$	$\sum_{n=0}^{\infty} u(t-nk)$		
29.3.66	$\frac{1}{s(e^{ks}-a)}$	$\sum_{n=1}^{\infty} a^{n-1}u(t-nk)$		
29.3.67	$\frac{1}{s} \tanh ks$	$u(t)+2\sum_{n=1}^{\infty} (-1)^n u(t-2nk)$		
29.3.68	$\frac{1}{s(1+e^{-ks})}$	$\sum_{n=0}^{\infty} (-1)^n u(t-nk)$		
29.3.69	$\frac{1}{s^2} \tanh ks$	$tu(t)+2\sum_{n=1}^{\infty} (-1)^n (t-2nk)u(t-2nk)$		
29.3.70	$\frac{1}{s \sinh ks}$	$2\sum_{n=0}^{\infty} u[t-(2n+1)k]$		
29.3.71	$\frac{1}{s \cosh ks}$	$2\sum_{n=0}^{\infty} (-1)^n u[t-(2n+1)k]$		

	$f(s)$	$F(t)$	
29.3.72	$\frac{1}{s}\coth ks$	$u(t)+2\sum_{n=1}^{\infty} u(t-2nk)$ 	
29.3.73	$\frac{k}{s^2+k^2}\coth\frac{\pi s}{2k}$	$\lvert\sin kt\rvert$ 	
29.3.74	$\frac{1}{(s^2+1)(1-e^{-\pi s})}$	$\sum_{n=0}^{\infty}(-1)^n u(t-n\pi)\sin t$ 	*
29.3.75	$\frac{1}{s}e^{-\frac{k}{s}}$	$J_0(2\sqrt{kt})$	9
29.3.76	$\frac{1}{\sqrt{s}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cos 2\sqrt{kt}$	
29.3.77	$\frac{1}{\sqrt{s}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi t}}\cosh 2\sqrt{kt}$	
29.3.78	$\frac{1}{s^{3/2}}e^{-\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}}\sin 2\sqrt{kt}$	
29.3.79	$\frac{1}{s^{3/2}}e^{\frac{k}{s}}$	$\frac{1}{\sqrt{\pi k}}\sinh 2\sqrt{kt}$	
29.3.80	$\frac{1}{s^{\mu}}e^{-\frac{k}{s}} \quad (\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}} J_{\mu-1}(2\sqrt{kt})$	9
29.3.81	$\frac{1}{s^{\mu}}e^{\frac{k}{s}} \quad (\mu>0)$	$\left(\frac{t}{k}\right)^{\frac{\mu-1}{2}} I_{\mu-1}(2\sqrt{kt})$	9
29.3.82	$e^{-k\sqrt{s}} \quad (k>0)$	$\frac{k}{2\sqrt{\pi t^3}}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.83	$\frac{1}{s}e^{-k\sqrt{s}} \quad (k\geq 0)$	$\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.84	$\frac{1}{\sqrt{s}}e^{-k\sqrt{s}} \quad (k\geq 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.85	$\frac{1}{s^{\frac{3}{2}}}e^{-k\sqrt{s}} \quad (k\geq 0)$	$2\sqrt{\frac{t}{\pi}}\exp\left(-\frac{k^2}{4t}\right)-k\operatorname{erfc}\frac{k}{2\sqrt{t}}=2\sqrt{t}\ \mathrm{i}\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.86	$\frac{1}{s^{1+\frac{1}{2}n}}e^{-k\sqrt{s}} \quad (n=0,1,2,\ldots;\ k\geq 0)$	$(4t)^{\frac{1}{2}n}\,\mathrm{i}^n\operatorname{erfc}\frac{k}{2\sqrt{t}}$	7
29.3.87	$s^{\frac{n-1}{2}}e^{-k\sqrt{s}} \quad (n=0,1,2,\ldots;\ k>0)$	$\frac{\exp\left(-\frac{k^2}{4t}\right)}{2^n\sqrt{\pi t^{n+1}}}H_n\left(\frac{k}{2\sqrt{t}}\right)$	22
29.3.88	$\frac{e^{-k\sqrt{s}}}{a+\sqrt{s}} \quad (k\geq 0)$	$\frac{1}{\sqrt{\pi t}}\exp\left(-\frac{k^2}{4t}\right)-ae^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\frac{k}{2\sqrt{t}}\right)$	7

*See page II.

	$f(s)$		$F(t)$	
29.3.89	$\dfrac{ae^{-k\sqrt{s}}}{s(a+\sqrt{s})}$	$(k\geq 0)$	$-e^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\dfrac{k}{2\sqrt{t}}\right)+\operatorname{erfc}\dfrac{k}{2\sqrt{t}}$	**7**
29.3.90	$\dfrac{e^{-k\sqrt{s}}}{\sqrt{s}(a+\sqrt{s})}$	$(k\geq 0)$	$e^{ak}e^{a^2t}\operatorname{erfc}\left(a\sqrt{t}+\dfrac{k}{2\sqrt{t}}\right)$	**7**
29.3.91	$\dfrac{e^{-k\sqrt{s(s+a)}}}{\sqrt{s(s+a)}}$	$(k\geq 0)$	$e^{-\frac{1}{2}at}I_0(\frac{1}{2}a\sqrt{t^2-k^2})u(t-k)$	**9**
29.3.92	$\dfrac{e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}}$	$(k\geq 0)$	$J_0(a\sqrt{t^2-k^2})u(t-k)$	**9**
29.3.93	$\dfrac{e^{-k\sqrt{s^2-a^2}}}{\sqrt{s^2-a^2}}$	$(k\geq 0)$	$I_0(a\sqrt{t^2-k^2})u(t-k)$	**9**
29.3.94	$\dfrac{e^{-k(\sqrt{s^2+a^2}-s)}}{\sqrt{s^2+a^2}}$	$(k\geq 0)$	$J_0(a\sqrt{t^2+2kt})$	**9**
29.3.95	$e^{-ks}-e^{-k\sqrt{s^2+a^2}}$	$(k>0)$	$\dfrac{ak}{\sqrt{t^2-k^2}}J_1(a\sqrt{t^2-k^2})u(t-k)$	**9**
29.3.96	$e^{-k\sqrt{s^2-a^2}}-e^{-ks}$	$(k>0)$	$\dfrac{ak}{\sqrt{t^2-k^2}}I_1(a\sqrt{t^2-k^2})u(t-k)$	**9**
29.3.97	$\dfrac{a^\nu e^{-k\sqrt{s^2+a^2}}}{\sqrt{s^2+a^2}(\sqrt{s^2+a^2}+s)^\nu}$	$(\nu>-1, k\geq 0)$	$\left(\dfrac{t-k}{t+k}\right)^{\frac{1}{2}\nu}J_\nu(a\sqrt{t^2-k^2})u(t-k)$	**9**
29.3.98	$\dfrac{1}{s}\ln s$		$-\gamma-\ln t\,(\gamma=.57721\ 56649\ \ldots$ Euler's constant)	
29.3.99	$\dfrac{1}{s^k}\ln s$	$(k>0)$	$\dfrac{t^{k-1}}{\Gamma(k)}[\psi(k)-\ln t]$	**6**
29.3.100	$\dfrac{\ln s}{s-a}$	$(a>0)$	$e^{at}[\ln a+E_1(at)]$	**5**
29.3.101	$\dfrac{\ln s}{s^2+1}$		$\cos t\operatorname{Si}(t)-\sin t\operatorname{Ci}(t)$	**5**
29.3.102	$\dfrac{s\ln s}{s^2+1}$		$-\sin t\operatorname{Si}(t)-\cos t\operatorname{Ci}(t)$	**5**
29.3.103	$\dfrac{1}{s}\ln(1+ks)$	$(k>0)$	$E_1\left(\dfrac{t}{k}\right)$	**5**
29.3.104	$\ln\dfrac{s+a}{s+b}$		$\dfrac{1}{t}(e^{-bt}-e^{-at})$	
29.3.105	$\dfrac{1}{s}\ln(1+k^2s^2)$	$(k>0)$	$-2\operatorname{Ci}\left(\dfrac{t}{k}\right)$	**5**
29.3.106	$\dfrac{1}{s}\ln(s^2+a^2)$	$(a>0)$	$2\ln a-2\operatorname{Ci}(at)$	**5**

	$f(s)$		$F(t)$	
29.3.107	$\frac{1}{s^2}\ln(s^2+a^2) \qquad (a>0)$		$\frac{2}{a}[at\ln a+\sin at-at\,\mathrm{Ci}(at)]$	5
29.3.108	$\ln\frac{s^2+a^2}{s^2}$		$\frac{2}{t}(1-\cos at)$	
29.3.109	$\ln\frac{s^2-a^2}{s^2}$		$\frac{2}{t}(1-\cosh at)$	
29.3.110	$\arctan\frac{k}{s}$		$\frac{1}{t}\sin kt$	
29.3.111	$\frac{1}{s}\arctan\frac{k}{s}$		$\mathrm{Si}(kt)$	5
29.3.112	$e^{k^2s^2}\operatorname{erfc} ks \qquad (k>0)$	7	$\frac{1}{k\sqrt{\pi}}\exp\left(-\frac{t^2}{4k^2}\right)$	
29.3.113	$\frac{1}{s}e^{k^2s^2}\operatorname{erfc} ks \qquad (k>0)$	7	$\operatorname{erf}\frac{t}{2k}$	7
29.3.114	$e^{ks}\operatorname{erfc}\sqrt{ks} \qquad (k>0)$	7	$\frac{\sqrt{k}}{\pi\sqrt{t}(t+k)}$	
29.3.115	$\frac{1}{\sqrt{s}}\operatorname{erfc}\sqrt{ks} \qquad (k\geq 0)$	7	$\frac{1}{\sqrt{\pi t}}u(t-k)$	
29.3.116	$\frac{1}{\sqrt{s}}e^{ks}\operatorname{erfc}\sqrt{ks} \qquad (k\geq 0)$	7	$\frac{1}{\sqrt{\pi(t+k)}}$	
29.3.117	$\operatorname{erf}\frac{k}{\sqrt{s}}$	7	$\frac{1}{\pi t}\sin 2k\sqrt{t}$	
29.3.118	$\frac{1}{\sqrt{s}}e^{\frac{k^2}{s}}\operatorname{erfc}\frac{k}{\sqrt{s}}$	7	$\frac{1}{\sqrt{\pi t}}e^{-2k\sqrt{t}}$	
29.3.119	$K_0(ks) \qquad (k>0)$	9	$\frac{1}{\sqrt{t^2-k^2}}u(t-k)$	
29.3.120	$K_0(k\sqrt{s}) \qquad (k>0)$	9	$\frac{1}{2t}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.121	$\frac{1}{s}e^{ks}K_1(ks) \qquad (k>0)$	9	$\frac{1}{k}\sqrt{t(t+2k)}$	
29.3.122	$\frac{1}{\sqrt{s}}K_1(k\sqrt{s}) \qquad (k>0)$	9	$\frac{1}{k}\exp\left(-\frac{k^2}{4t}\right)$	
29.3.123	$\frac{1}{\sqrt{s}}e^{\frac{k}{s}}K_0\left(\frac{k}{s}\right) \qquad (k>0)$	9	$\frac{2}{\sqrt{\pi t}}K_0(2\sqrt{2kt})$	9
29.3.124	$\pi e^{-ks}I_0(ks) \qquad (k>0)$	9	$\frac{1}{\sqrt{t(2k-t)}}[u(t)-u(t-2k)]$	
29.3.125	$e^{-ks}I_1(ks) \qquad (k>0)$	9	$\frac{k-t}{\pi k\sqrt{t(2k-t)}}[u(t)-u(t-2k)]$	

	$f(s)$			$F(t)$
29.3.126	$e^{as}E_1(as)$	$(a>0)$	5	$\frac{1}{t+a}$
29.3.127	$\frac{1}{a}-se^{as}E_1(as)$	$(a>0)$	5	$\frac{1}{(t+a)^2}$
29.3.128	$a^{1-n}e^{as}E_n(as)$	$(a>0; n=0, 1, 2, \ldots)$	5	$\frac{1}{(t+a)^n}$
29.3.129	$\left[\frac{\pi}{2}-\mathrm{Si}(s)\right]\cos s+\mathrm{Ci}(s)\sin s$		5	$\frac{1}{t^2+1}$

29.4. Table of Laplace-Stieltjes Transforms [4]

	$\phi(s)$		$\Phi(t)$
29.4.1	$\int_0^\infty e^{-st}d\Phi(t)$		$\Phi(t)$
29.4.2	e^{-ks}	$(k>0)$	$u(t-k)$
29.4.3	$\frac{1}{1-e^{-ks}}$	$(k>0)$	$\sum_{n=0}^{\infty} u(t-nk)$
29.4.4	$\frac{1}{1+e^{-ks}}$	$(k>0)$	$\sum_{n=0}^{\infty} (-1)^n u(t-nk)$
29.4.5	$\frac{1}{\sinh ks}$	$(k>0)$	$2\sum_{n=0}^{\infty} u[t-(2n+1)k]$
29.4.6	$\frac{1}{\cosh ks}$	$(k>0)$	$2\sum_{n=0}^{\infty} (-1)^n u[t-(2n+1)k]$
29.4.7	$\tanh ks$	$(k>0)$	$u(t)+2\sum_{n=1}^{\infty} (-1)^n u(t-2nk)$
29.4.8	$\frac{1}{\sinh (ks+a)}$	$(k>0)$	$2\sum_{n=0}^{\infty} e^{-(2n+1)a}u[t-(2n+1)k]$
29.4.9	$\frac{e^{-hs}}{\sinh (ks+a)}$	$(k>0, h>0)$	$2\sum_{n=0}^{\infty} e^{-(2n+1)a}u[t-h-(2n+1)k]$
29.4.10	$\frac{\sinh (hs+b)}{\sinh (ks+a)}$	$(0<h<k)$	$\sum_{n=0}^{\infty} e^{-(2n+1)a}\{e^b u[t+h-(2n+1)k]$ $-e^{-b}u[t-h-(2n+1)k]\}$
29.4.11	$\sum_{n=0}^{\infty} a_n e^{-k_n s}$	$(0<k_0<k_1<\ldots)$	$\sum_{n=0}^{\infty} a_n u(t-k_n)$

For the definition of the Laplace-Stieltjes transform see [29.7]. In practice, Laplace-Stieltjes transforms are often written as ordinary Laplace transforms involving Dirac's delta function $\delta(t)$. This "function" may formally be considered as the derivative of the unit step function, $du(t)=\delta(t)dt$, so that $\int_{-\infty}^{x} du(t)=\int_{-\infty}^{x}\delta(t)dt=\begin{cases}0 & (x<0)\\ 1 & (x>0).\end{cases}$ The correspondence **29.4.2**, for instance, then assumes the form $e^{-ks}=\int_0^\infty e^{-st}\delta(t-k)dt$.

[4] Adapted by permission from P. M. Morse and H. Feshbach, Methods of theoretical physics, vols. 1, 2, McGraw-Hill Book Co., Inc., New York, N.Y., 1953.