and hence I_3 can also be evaluated in terms of boundary quantities. Thus, the A_n can be found without recourse to any numerical integration! This is often the case; the key is always integration by parts.

4.7 Splitting

We have seen that problems with linear PDEs and BCs can be solved by constructing linear combinations of the eigensolutions for appropriate homogeneous partial problems. We also saw that in transient problems the inhomogeneities can be "removed" by "splitting" the solution into steady-state and transient parts. The concept of problem splitting can also be used to "remove inhomogeneities" in other problems.

To illustrate the idea, consider the problem shown in Fig. 4.7.1. The PDE is the inhomogeneous Laplace equation,

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = h(x,y) \tag{4.7.1}$$

The domain is the rectangle shown, and the boundary conditions specify ϕ around the boundary, in terms of the functions shown. Note that all of these boundary conditions are inhomogeneous.

To use the methods developed in this chapter, we can "split" the problem into the five problems shown in Fig. 4.7.1. Problem (p) will take care of the inhomogeneity in the PDE. The solution $\phi^{(p)}$ is any particular solution of the PDE, without regard for boundary conditions. It will yield the values of $\phi^{(p)}$ on the boundaries denoted by the functions $g_1 - g_4$. We shall discuss means for finding the particular solution shortly. The four problems $\phi^{(1)} - \phi^{(4)}$ involve homogeneous PDEs and nearly completely homogeneous boundary conditions. Therefore, for each the eigensolutions of the homogeneous partial problem can be found, and then a linear combination of these eigenfunctions taken to construct a solution satisfying the remaining inhomogeneous boundary condition. Note that the sum

$$\phi = \phi^{(p)} + \sum_{k=1}^{4} \phi^{(k)} \tag{4.7.2}$$

satisfies the inhomogeneous PDE and inhomogeneous boundary conditions. This type of splitting can, of course, only be done in linear problems.

Let's presume that we have the particular solution $\phi^{(p)}$, and are ready to solve problems $\phi^{(1)} - \phi^{(4)}$. We will do the $\phi^{(1)}$ problem; the other three are done in the same way.

The $\phi^{(1)}$ PDE is, dropping the superscript (1),

$$\phi_{xx} + \phi_{yy} = 0 \qquad (4.7.3)$$

and the boundary conditions are

$$\phi = 0 \quad \text{on} \quad y = 0 \qquad (4.7.4)$$

$$\phi = 0 \quad \text{on} \quad x = 0 \qquad (4.7.5)$$

$$\phi = 0 \quad \text{on} \quad x = a \qquad (4.7.6)$$

$$\phi = f_1(x) - g_1(x) = q(x) \quad \text{on} \quad y = b \qquad (4.7.7)$$

We look for eigensolutions to the homogeneous partial problem (4.7.3) - (4.7.6) in the form

$$\phi = X(x)\, Y(y) \qquad (4.7.8)$$

and, from (4.7.3), find

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2 \qquad (4.7.9)$$

Hence,

$$X'' + \lambda^2 X = 0 \qquad (4.7.10)$$

$$Y'' - \lambda^2 Y = 0 \qquad (4.7.11)$$

The decision to name the separation constant $-\lambda^2$ was dictated by the recognition that the X-solutions must oscillate in X in order to match the boundary conditions. The X solution is

$$X = C_1 \sin(\lambda x) + C_2 \cos(\lambda x) \tag{4.7.12}$$

The BC (4.7.5) gives $C_2 = 0$. Then, the BC (4.7.6) requires $\sin(\lambda a) = 0$. Hence,

$$\lambda_n a = n\pi \tag{4.7.13}$$

The Y equation solution is

$$Y = C_3 \sinh(\lambda y) + C_4 \cosh(\lambda y) \tag{4.7.14}$$

The BC (4.7.4) requires $C_4 = 0$. Hence, the eigensolutions are (apart from a scaling constant)

$$\phi_n(x,y) = \sin(n\pi x/a) \sinh(n\pi y/a) \tag{4.7.15}$$

Finally, we seek the solution satisfying the inhomogeneous condition (4.7.7) as an expansion in the eigenfunctions,

$$\phi = \sum_{n=1}^{\infty} A_n \phi_n \tag{4.7.16}$$

Thus, at $y = b$,

$$\phi(b,x) = q(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \sinh(n\pi b/a) \tag{4.7.17}$$

The orthogonality property for the X_n eigenfunctions is*

$$\int_0^a X_n X_m dx = 0 \qquad n \neq m \tag{4.7.18}$$

*Developed in the usual way.

So, multiplying (4.7.17) by $\sin(m\pi x/a)$, and integrating

$$A_m = \frac{\int_0^a q(x)\ \sin(m\pi x/a)}{\sinh(m\pi b/a) \int_0^a \sin^2(m\pi x/a)dx} \tag{4.7.19}$$

Given $q(x)$, we could compute the A_n . Hence, the $\phi^{(1)}$ solution is completely known.

The $\phi^{(2)}$, $\phi^{(3)}$, and $\phi^{(4)}$ problems could be handled in much the same way. In the $\phi^{(3)}$ problem, the Y equations would again be (4.7.11), and $Y(b) = 0$. Hence, rather than (4.7.14), a "more artistic" form of the Y solution is

$$Y = C_5 \sinh[\lambda(y-b)] + C_6 \cosh[\lambda(y-b)] \tag{4.7.20}$$

because C_6 will have to be zero for $Y(b) = 0$.

Let's now discuss the particular solution. If h depends upon only one of the independent variables, say x , the particular solution may be developed by assuming

$$\phi^{(p)} = F(x) \tag{4.7.21}$$

The inhomogeneous PDE is then

$$F'' = h(x) \tag{4.7.22}$$

which has the solution (by double integration)

$$F = \int_0^x \int_0^\xi h(\sigma)\ d\sigma\ d\xi \tag{4.7.23}$$

If $h = h(x,y)$, the particular solution can be obtained by expanding h in a Fourier series in either x or y. If we choose to do it in x, we would write

$$h(x,y) = \sum_{n=o}^{\infty} a_n(y) \cos(2n\pi x/a) + \sum_{n=1}^{\infty} b_n(y) \sin(2n\pi x/a) \tag{4.7.24}$$

The coefficients a_n and b_n are determined using the orthogonality property of the sine and cosine functions;

$$a_o = \frac{1}{a}\int_0^a h \, dx \tag{4.7.25a}$$

$$a_m = \frac{2}{a}\int_0^a h \cos(2m\pi x/a)dx \tag{4.7.25b}$$

$$b_m = \frac{2}{a}\int_0^a h \sin(2m\pi x/a)dx \tag{4.7.25c}$$

Next, one would look for a particular solution in the form

$$\phi^{(p)} = \sum_{n=o}^{\infty} F_n(y) \cos(2n\pi x/a) + \sum_{n=1}^{\infty} G_n(y) \sin(2n\pi x/a) \tag{4.7.26}$$

Substituting into the PDE, and equating coefficients of the sines and cosines, one finds

$$F_o'' = a_o \qquad (4.7.27a)$$

$$F_n'' - \left(\frac{2n\pi}{a}\right)^2 F_n = a_n \qquad (4.7.27b)$$

$$G_n'' - \left(\frac{2n\pi}{a}\right)^2 G_n = b_n \qquad (4.7.27c)$$

Particular solutions to these three ODEs can be obtained by standard methods (e.g., the method of separation of variables).

In this problem, the corners would be singular points. The series solutions would converge everywhere, except at the corners, where the solutions $\phi^{(1)} - \phi^{(4)}$ would all be zero because of the method of solution.

4.8 Some Generalizations

While some problems fall into the Sturm-Liouville form, others do not. However, the same general ideas can be used with the help of a new concept, adjoint operations.

Suppose that the SOV process in a linear, homogeneous PDE problem produces the ODE

$$Lu = Mu + \lambda Nu = 0 \qquad (4.8.1)$$

where L, M, and N are linear operators. Suppose that the linear, homogeneous boundary conditions are a set of equations of the form

$$\{B_i u = 0\} \quad \text{at} \quad x = a \text{ or } b \qquad (4.8.2)$$

where the B_i are also linear operators. The eigenvalues λ are those values for which non-trivial solutions to (4.8.1), and (4.8.2) exist. The adjoint operators L^*, M^*, N^*, and B_i^* are defined by the requirement that

$$\int_a^b vLu\,dx = \int_a^b uL^*v\,dx \qquad (4.8.3)$$

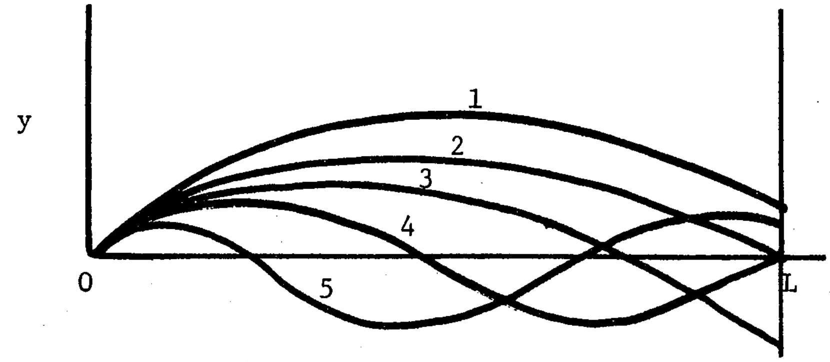

FIGURE 4.2.1

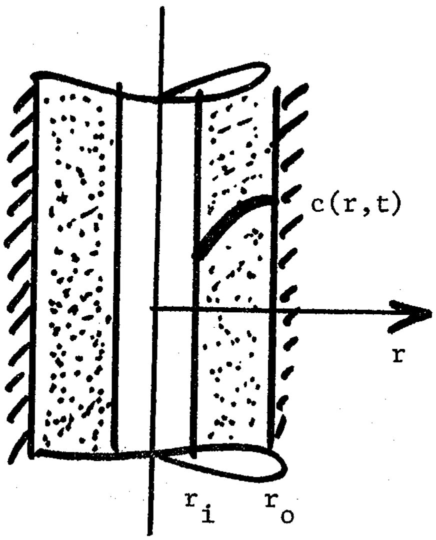

Fig. 4.6.1

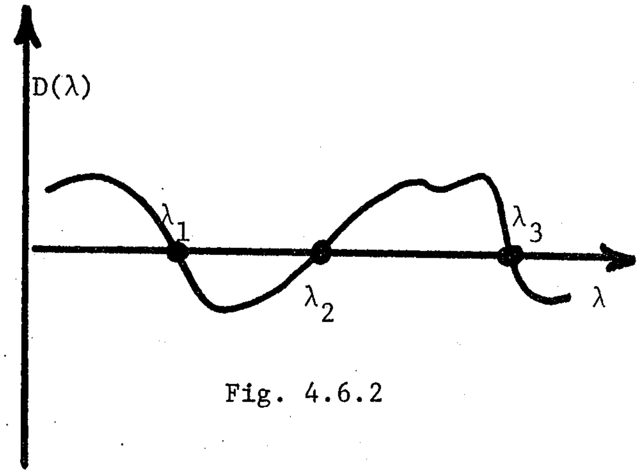

Fig. 4.6.2

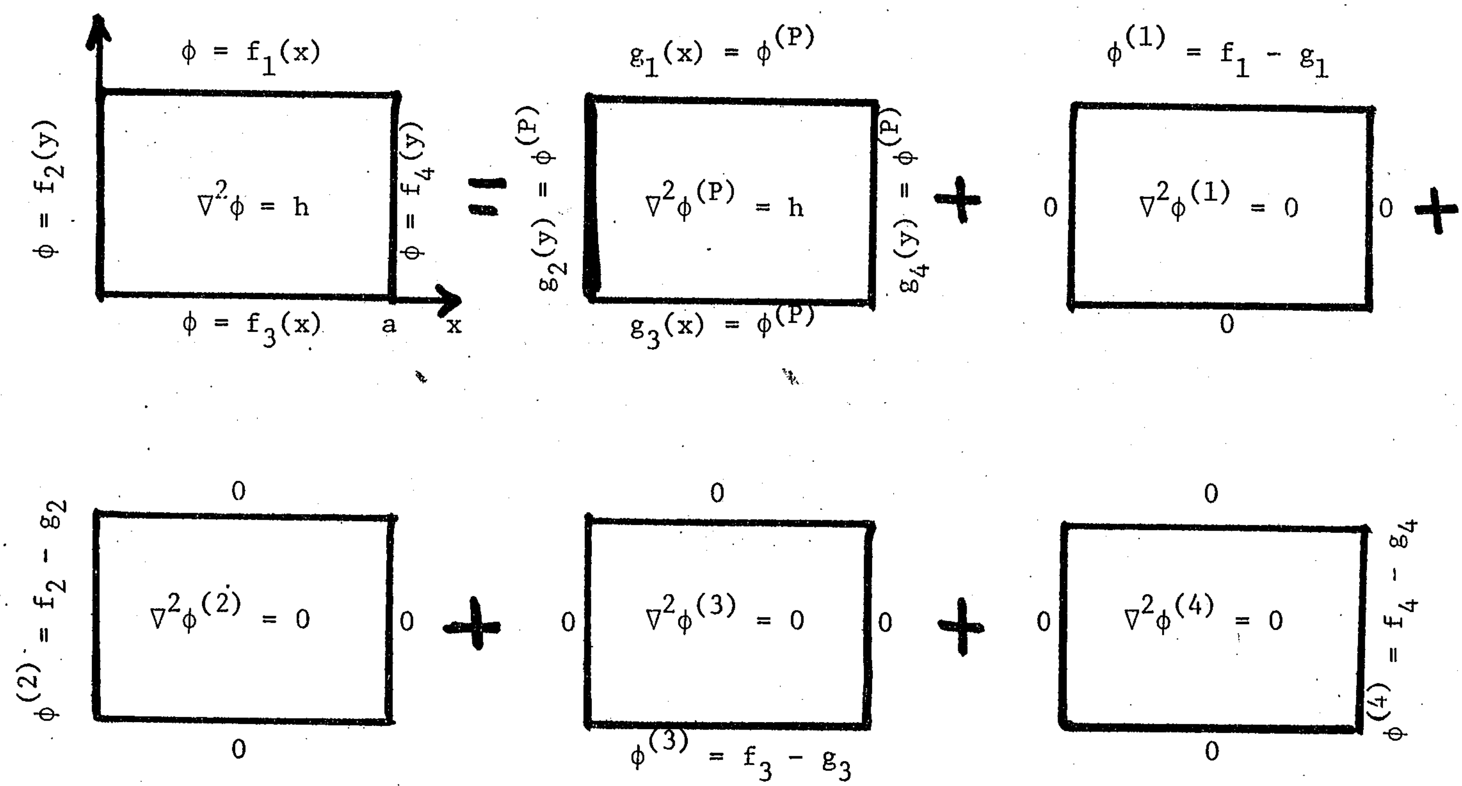

Fig. 4.7.1. Splitting